Math 1B, Fall 2008, Wilkening

Midterm 2, version 1,3,5

0. (1 point) write your name, your GSI's name, and your section number at the top of your exam.

1. (3 points or 0 points) Suppose $|\cos x| \neq 1$. Evaluate $\sum_{n=0}^{\infty} (\cos x)^{2n}$.

a. $\cot x$

b. $\csc^2 x$

c. $\cosh x$

d. $\frac{x}{\sqrt{1-x^2}}$

e. none of the above

2. (3 points or 0 points) Describe the behavior of the sequence $a_1 = 0, \quad a_{n+1} = \dfrac{a_n^2 + 3}{4}$

a. a_n increases monotonically and converges to 1

b. a_n increases monotonically and converges to 3

c. a_n increases monotonically to ∞

d. a_n decreases monotonically to $-\infty$

e. a_n is not monotonic

3. (3 points or 0 points) Supppose $0 < a_n < 1$, $a_n < b_n$, and $\sum b_n$ is convergent. Circle all the statements that are necessarily true:

a. $\sum b_n^2$ converges and $\sum b_n^2 < \sum b_n$

b. $\sum \sqrt{b_n}$ converges and $\sum \sqrt{b_n} < \sum b_n$

c. $\sum a_n^2$ converges and $\sum a_n^2 < \sum a_n$

d. $\sum \sqrt{a_n}$ converges and $\sum \sqrt{a_n} < \sum a_n$

e. if $p > 0$ then $\sum (-1)^n a_n^p$ is convergent

4a. (3 points) Let $f(x) = e^{-x^2}$.
Write down the Maclaurin series for $f(x)$ and evaluate $f^{(99)}(0)$ and $f^{(100)}(0)$.

4b. (4 points) Find all x that satisfy the equation $\displaystyle\sum_{n=1}^{\infty} n x^n = \frac{1}{2}$.

5. (2 points each) For each of the following series, determine whether the series is absolutely convergent (AC), conditionally convergent (CC), or divergent (D). Show some work, but do not spend excessive time justifying all your steps.

$$\sum_{n=1}^{\infty}(-1)^n[\sin(1/n^2)]^{2/3}$$

$$\sum_{n=1}^{\infty} \ln \cos \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{(2n)!}{3^n(n!)^2}$$

$$\sum_{n=0}^{\infty} \binom{5}{n} 3^n$$

6a. (2 pts) Is the following statement True or False? Justify your answer with a proof or counterexample. (Obviously it's true if $a_n \geq 0$ and $b_n \geq 0$, so don't assume this).

If $\sum a_n$ is divergent and $\sum b_n$ is divergent, then $\sum(a_n + b_n)$ is also divergent.

6b. (3 points) Suppose $\sum c_n x^n$ has radius of convergence 2 while $\sum d_n x^n$ has radius of convergence 5. What is the radius of convergence of the series $\sum(c_n + d_n)x^n$? Explain.

7a. (3 points) Prove that $e \geq \left(1 + \frac{1}{k}\right)^k$ for $k \geq 1$. $\left(\textit{Hint: } \ln(1+x) = x - \frac{x^2}{2} + \cdots\right)$

7b. (3 points) Use part (a) and mathematical induction to prove the following crude version of Stirling's approximation:

$$e^n n! \geq n^n \qquad \text{for all } n \geq 1.$$

Math 1B, Fall 2008, Wilkening

Midterm 2, version 2,4,6

0. (1 point) write your name, your GSI's name, and your section number at the top of your exam.

1. (3 points or 0 points) Suppose $|\sin x| \neq 1$. Evaluate $\sum_{n=0}^{\infty} (\sin x)^{2n}$.

a. $\tan x$

b. $\sec^2 x$

c. $\sinh x$

d. $\frac{x}{\sqrt{1-x^2}}$

e. none of the above

2. (3 points or 0 points) Describe the behavior of the sequence $a_1 = 2, \quad a_{n+1} = \dfrac{a_n^2 + 3}{4}$

a. a_n increases monotonically and converges to 3

b. a_n decreases monotonically and converges to 1

c. a_n increases monotonically to ∞

d. a_n decreases monotonically to $-\infty$

e. a_n is not monotonic

3. (3 points or 0 points) Supppose $0 < a_n < 1$, $a_n < b_n$, and $\sum b_n$ is convergent. Circle all the statements that are necessarily true:

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c. $\sum b_n^2$ converges and $\sum b_n^2 < \sum b_n$

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e. if $p > 0$ then $\sum (-1)^n a_n^p$ is convergent

4a. (3 points) Let $f(x) = e^{-x^3}$.
Write down the Maclaurin series for $f(x)$ and evaluate $f^{(99)}(0)$ and $f^{(100)}(0)$.

4b. (4 points) Find all x that satisfy the equation $\displaystyle\sum_{n=1}^{\infty} n x^n = 1$.

5. (2 points each) For each of the following series, determine whether the series is absolutely convergent (AC), conditionally convergent (CC), or divergent (D). Show some work, but do not spend excessive time justifying all your steps.

$$\sum_{n=1}^{\infty} [\sin(1/n^2)]^{1/3}$$

$$\sum_{n=1}^{\infty} (-1)^n \ln \cos \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{(2n)!}{5^n (n!)^2}$$

$$\sum_{n=0}^{\infty} \binom{5}{n} (-3)^n$$

6a. (2 pts) Is the following statement True or False? Justify your answer with a proof or counterexample. (Obviously it's true if $a_n \geq 0$ and $b_n \geq 0$, so don't assume this).

If $\sum a_n$ is divergent and $\sum b_n$ is divergent, then $\sum (a_n + b_n)$ is also divergent.

6b. (3 points) Suppose $\sum c_n x^n$ has radius of convergence 2 while $\sum d_n x^n$ has radius of convergence 1. What is the radius of convergence of the series $\sum (c_n + d_n) x^n$? Explain.

7a. (3 points) Prove that $e \geq \left(1 + \frac{1}{k}\right)^k$ for $k \geq 1$. $\left(\textit{Hint: } \ln(1+x) = x - \frac{x^2}{2} + \cdots\right)$

7b. (3 points) Use part (a) and mathematical induction to prove the following crude version of Stirling's approximation:

$$e^n n! \geq n^n \qquad \text{for all } n \geq 1.$$

results of second midterm:

students: 345

total possible: 36 points
max score: 32 points
average: 14.09
standard dev: 5.93

rough grading scale (worry about +/- later)

raw score	grade	# of students
19-32	A	88
14-18	B	88
10-13	C	81
6-9	D	71
1-5	F	17

curve for comparison to other midterm and final:

curved score = 35 + 2.75 * raw score

histogram:

raw score	# students	running total	
1	1	1	#
2	0	1	
3	5	6	#####
4	5	11	#####
5	6	17	######
6	19	36	###################
7	18	54	##################
8	16	70	################
9	18	88	##################
10	14	102	##############
11	21	123	#####################
12	27	150	###########################
13	19	169	###################
14	21	190	#####################
15	17	207	#################
16	15	222	###############
17	18	240	##################
18	17	257	#################
19	18	275	##################
20	21	296	#####################
21	13	309	#############
22	5	314	#####
23	9	323	#########
24	9	332	#########
25	5	337	#####
26	1	338	#
27	1	339	#
28	2	341	##
29	2	343	##
30	0	343	
31	1	344	#
32	1	345	#

the answers below were written up by Peter Mannisto and James Tener.

1, 3, 5

Name:
GSI's Name: $\boxed{\text{Solution}}$
Section:

Midterm 2
Math 1B, Fall 2008
Wilkening

0	1	
1	3	
2	3	
3	3	
4	7	
5	8	
6	5	
7	6	
total	36	

0. (1 point) write your name, your GSI's name, and your section number at the top of your exam.

1. (3 points or 0 points) Suppose $|\cos x| \neq 1$. Evaluate $\sum_{n=0}^{\infty} (\cos x)^{2n}$.

a. $\cot x$

b. $\csc^2 x$

c. $\cosh x$

d. $\frac{x}{\sqrt{1-x^2}}$

e. none of the above

$$\sum_{0}^{\infty} (\cos^2 x)^n = \frac{1}{1-\cos^2 x} = \frac{1}{\sin^2 x} = \csc^2 x$$

2. (3 points or 0 points) Describe the behavior of the sequence $a_1 = 0, \quad a_{n+1} = \dfrac{a_n^2+3}{4}$

a. a_n increases monotonically and converges to 1

b. a_n increases monotonically and converges to 3

c. a_n increases monotonically to ∞

d. a_n decreases monotonically to $-\infty$

e. a_n is not monotonic

$$a_n < a_{n+1} < 1$$

$$\Rightarrow a_n^2 < a_{n+1}^2 < 1$$

$$\Rightarrow a_n^2 + 3 < a_{n+1}^2 + 3 < 4$$

$$\Rightarrow \frac{a_n^2+3}{4} < \frac{a_{n+1}^2+3}{4} < 1$$

(increasing)

(bounded above by 1)

$$L = \frac{L^2+3}{4}$$

$$4L = L^2 + 3 \qquad L^2 - 4L + 3 = 0 \qquad (L-3)(L-1) = 0 \qquad L = 1 \qquad \cancel{3}$$

3. (3 points or 0 points) Supppose $0 \le a_n < 1$, $a_n < b_n$, and $\sum b_n$ is convergent. Circle all the statements that are necessarily true:

a. $\sum b_n^2$ converges and $\sum b_n^2 < \sum b_n$

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d. $\sum \sqrt{a_n}$ converges and $\sum \sqrt{a_n} < \sum a_n$

e. if $p > 0$ then $\sum (-1)^n a_n^p$ is convergent

Counterexample for (e): $a_n = \begin{cases} 1/n^2 & n \text{ even} \\ 1/n^4 & n \text{ odd} \end{cases}$

$p = 1/2 \qquad (-1)^n a_n^p = \begin{cases} 1/n & n \text{ even} \\ -1/n^2 & n \text{ odd} \end{cases}$

4a. (3 points) Let $f(x) = e^{-x^2}$.
Write down the Maclaurin series for $f(x)$ and evaluate $f^{(99)}(0)$ and $f^{(100)}(0)$.

$$e^x = \sum_0^\infty \frac{x^n}{n!}$$

$$\Rightarrow e^{-x^2} = \sum_0^\infty \frac{(-1)^n x^{2n}}{n!}$$

$$\sum_0^\infty \frac{(-1)^n x^{2n}}{n!} = \sum_0^\infty \frac{f^{(k)}(0)}{k!} x^k$$

match up coefficients 99, 100

(no odd coefficients left, so $f^{(99)}(0) = 0$)

$$\frac{f^{(100)}(0)}{100!} = \frac{(-1)^{50}}{50!} \Longrightarrow f^{(100)}(0) = \frac{100!}{50!}$$

4b. (4 points) Find all x that satisfy the equation $\sum_{n=1}^{\infty} nx^n = \frac{1}{2}$.

$$\frac{1}{1-x} = \sum_0^\infty x^n$$

$$\frac{1}{(1-x)^2} = \sum_1^\infty n x^{n-1} \Rightarrow \frac{x}{(1-x)^2} = \sum_1^\infty n x^n$$

So set $\frac{x}{(1-x)^2} = \frac{1}{2}$

$$x = \frac{1}{2}(1-x)^2 = \frac{1}{2} - x + \frac{x^2}{2}$$

$$\frac{x^2}{2} - 2x + \frac{1}{2} = 0$$

$$x = \frac{2 \pm \sqrt{3}}{2 \cdot 1/2} = 2 \pm \sqrt{3}$$

need $|x| < 1$, so

$$x = 2 - \sqrt{3}$$

5. (2 points each) For each of the following series, determine whether the series is absolutely convergent (AC), conditionally convergent (CC), or divergent (D). Show some work, but do not spend excessive time justifying all your steps.

$$\sum_{n=1}^{\infty}(-1)^n[\sin(1/n^2)]^{2/3}$$

Check for absolute convergence:

$\sum_1^\infty \left(\sin \frac{1}{n^2}\right)^{2/3}$ $\quad \sin \frac{1}{n^2} \approx \frac{1}{n^2}$ for large n (small x)

lim comparison to $(1/n^2)^{2/3}$: $\lim_{n\to\infty} \frac{(\sin 1/n^2)^{2/3}}{(1/n^2)^{2/3}} = 1$

since $\sum_1^\infty 1/n^{4/3}$ conv abs, $\sum_1^\infty (-1)^n \sin\left(\frac{1}{n^2}\right)^{2/3}$ AC

$$\sum_{n=1}^{\infty} \ln \cos \frac{1}{n}$$

$\ln(\cos(1/n)) \sim \ln(1 - 1/n^2)$

$\sim -1/n^2$ (asymptotically)

more precisely, $\lim_{n\to\infty} \frac{\ln(\cos(1/n))}{-1/n^2} = 1$ (check by l'Hospital's)

$\therefore$ since $\sum_1^\infty -1/n^2$ converges absolutely, $\sum_1^\infty \ln(\cos 1/n)$ is AC

$$\sum_{n=1}^{\infty} \frac{(2n)!}{3^n(n!)^2}$$

Ratio test: $\frac{a_{n+1}}{a_n} = \frac{(2n+2)(2n+1)}{3(n+1)^2} \longrightarrow 4/3 > 1$

D

$$\sum_{n=0}^{\infty} \binom{5}{n} 3^n$$

Note: $\binom{5}{n} = 0$ for $n \geq 5$

So this is a <u>finite</u> sum $\Rightarrow$ AC

6a. (2 pts) Is the following statement True or False? Justify your answer with a proof or counterexample. (Obviously it's true if $a_n \geq 0$ and $b_n \geq 0$, so don't assume this).

If $\sum a_n$ is divergent and $\sum b_n$ is divergent, then $\sum(a_n + b_n)$ is also divergent.

False

Countoex: $a_n = 1$ (for all n)
$b_n = -1$

6b. (3 points) Suppose $\sum c_n x^n$ has radius of convergence 2 while $\sum d_n x^n$ has radius of convergence 5. What is the radius of convergence of the series $\sum(c_n + d_n)x^n$? Explain.

It must be 2. For $|x|<2$, $\Sigma(c_n+d_n)x^n$ is the sum of 2 convergent series, hence convergent. For $2<|x|<5$, $\Sigma(c_n+d_n)x^n$ is the sum of a convergent series and a divergent series, hence divergent. This is enough to force $R=2$.

7a. (3 points) Prove that $e \geq \left(1+\frac{1}{k}\right)^k$ for $k \geq 1$. (*Hint:* $\ln(1+x) = x - \frac{x^2}{2} + \cdots$)

$$e \geq \left(1+\frac{1}{k}\right)^k$$

$$\iff \ln(e) = 1 \geq k\ln\left(1+\frac{1}{k}\right)$$

$$= k\left[\frac{1}{k} - \frac{1}{2k^2} + \frac{1}{3k^3} - \ldots\right]$$

$$= 1 - \frac{1}{2k} + \frac{1}{3k^2} - \ldots$$

this is an alternating series with decreasing absolute value of terms. Hence

$1 \geq 1 - \frac{1}{2k} + \frac{1}{3k^2} - \ldots$, completing the proof.

7b. (3 points) Use part (*a*) and mathematical induction to prove the following crude version of Stirling's approximation:

$$e^n n! \geq n^n \qquad \text{for all } n \geq 1.$$

Base case $n=1$: $e^1\, 1! \overset{?}{\geq} 1^1$

$(e \geq 1)$ ✓

induction step: suppose $e^k k! \geq k^k$. Then

$$e^{k+1}(k+1)! = e \cdot e^k \cdot (k+1)!$$

(by induction hyp.) $\geq e \cdot \frac{k^k}{k!}(k+1)!$

(by (a)) $\geq (1+1/k)^k k^k \frac{(k+1)!}{k!}$

$$= (k+1)^k (k+1)$$

$$= (k+1)^{k+1}$$ ✓

2, 4, 6

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a. $\tan x$

b. $\sec^2 x$

c. $\sinh x$

d. $\frac{x}{\sqrt{1-x^2}}$

e. none of the above

$$\sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2} \text{ when } |x|<1. \text{ When}$$

$|\sin x| \neq 1$, we must have $|\sin x| < 1$, so

$$\sum_{n=0}^{\infty} (\sin x)^{2n} = \frac{1}{1-\sin^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

2. (3 points or 0 points) Describe the behavior of the sequence $a_1 = 2, \quad a_{n+1} = \dfrac{a_n^2 + 3}{4}$

a. a_n increases monotonically and converges to 3

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4a. (3 points) Let $f(x) = e^{-x^3}$.
Write down the Maclaurin series for $f(x)$ and evaluate $f^{(99)}(0)$ and $f^{(100)}(0)$.

$$e^x = \sum_0^\infty \frac{x^n}{n!}$$

$$e^{-x^3} = \sum_0^\infty \frac{(-1)^n x^{3n}}{n!}$$

$$\sum_0^\infty \frac{(-1)^n x^{3n}}{n!} = \sum_0^\infty \frac{f^{(k)}(0)}{k!} x^k$$

match like coefficients:

$$\frac{f^{(99)}(0)}{99!} = \frac{(-1)^{33}}{33!}, \quad \frac{f^{(100)}(0)}{100!} = 0$$

4b. (4 points) Find all x that satisfy the equation $\sum_{n=1}^{\infty} nx^n = 1$.

$$\sum_0^\infty x^n = \frac{1}{1-x}$$

$$\sum_1^\infty nx^{n-1} = \frac{1}{(1-x)^2} \Rightarrow \sum_1^\infty nx^n = \frac{x}{(1-x)^2}$$

Solve $\frac{x}{(1-x)^2} = 1$

$$x = (1-x)^2$$
$$x = 1 - 2x + x^2$$
$$x^2 - 3x + 1 = 0$$
$$x = \frac{3 \pm \sqrt{5}}{2}$$

only $\frac{3-\sqrt{5}}{2}$ has $|x|<1$,

So $x = \frac{3-\sqrt{5}}{2}$

5. (2 points each) For each of the following series, determine whether the series is absolutely convergent (AC), conditionally convergent (CC), or divergent (D). Show some work, but do not spend excessive time justifying all your steps.

$\sum_{n=1}^{\infty} [\sin(1/n^2)]^{1/3}$

$\sin 1/n^2 \sim 1/n^2$ for large n.

More precisely, $\lim_{n\to\infty} \frac{(\sin 1/n^2)^{1/3}}{(1/n^2)^{1/3}} = 1$

Since $\sum_{0}^{\infty} \frac{1}{n^{2/3}}$ diverges, $\sum_{1}^{\infty} \left(\sin \frac{1}{n^2}\right)^{1/3}$ is D

$\sum_{n=1}^{\infty} (-1)^n \ln \cos \frac{1}{n}$

$\ln \cos \frac{1}{n} \sim \ln(1 - 1/n^2)$

$\sim -1/n^2$, so

$\left|(-1)^n \ln \cos \frac{1}{n}\right| \sim \frac{1}{n^2}$ in the sense that

$\lim_{n\to\infty} \frac{|(-1)^n \ln \cos 1/n|}{1/n^2} = 1.$ Since $\sum_{1}^{\infty} 1/n^2$ converges, $\sum_{1}^{\infty} (-1)^n \ln \cos \frac{1}{n}$ is AC

$\sum_{n=1}^{\infty} \frac{(2n)!}{5^n (n!)^2}$

ratio test: $\frac{a_{n+1}}{a_n} = \frac{(2n+2)(2n+1)}{5(n+1)^2} \longrightarrow \frac{4}{5} < 1$

AC

$\sum_{n=0}^{\infty} \binom{5}{n} (-3)^n$

$\binom{5}{n} = 0$ for $n \geq 5$

$\therefore$ the sum is finite

AC

6a. (2 pts) Is the following statement True or False? Justify your answer with a proof or counterexample. (Obviously it's true if $a_n \geq 0$ and $b_n \geq 0$, so don't assume this).

If $\sum a_n$ is divergent and $\sum b_n$ is divergent, then $\sum(a_n + b_n)$ is also divergent.

False. If $a_n = \frac{1}{n}$ and $b_n = -\frac{1}{n}$, then both $\sum a_n$ and $\sum b_n$ diverge, but $a_n + b_n = 0$ so $\sum (a_n + b_n)$ converges.

6b. (3 points) Suppose $\sum c_n x^n$ has radius of convergence 2 while $\sum d_n x^n$ has radius of convergence 1. What is the radius of convergence of the series $\sum(c_n + d_n)x^n$? Explain.

If $|x| < 1$, then $\sum c_n x^n$ and $\sum d_n x^n$ converge, so $\sum (c_n + d_n) x^n$ converges.

If $1 < |x| < 2$, then $\sum c_n x^n$ converges and $\sum d_n x^n$ diverges, so $\sum (c_n + d_n) x^n$ diverges.

Because $\sum (c_n + d_n) x^n$ is a power series, and it diverges for $1 < |x| < 2$ it must diverge for $1 < |x|$.

So $R = 1$.

7a. (3 points) Prove that $e \geq \left(1 + \frac{1}{k}\right)^k$ for $k \geq 1$. $\left(\textit{Hint: } \ln(1+x) = x - \frac{x^2}{2} + \cdots\right)$

$$e \geq \left(1 + \frac{1}{k}\right)^k$$

$$\Leftrightarrow \quad \ln e = 1 \geq k \ln(1 + 1/k)$$

$$= k \left[1/k - 1/2k^2 + 1/3k^3 - \cdots \right]$$

$$= 1 - 1/2k + 1/3k^2 - \cdots$$

Since $1 - 1/2k + 1/3k^2 - \cdots$ is an alternating series with decreasing abs value of terms,

$1 \geq 1 - 1/2k + 1/3k^2 - \cdots$, completing the proof.

7b. (3 points) Use part (a) and mathematical induction to prove the following crude version of Stirling's approximation:

$$e^n n! \geq n^n \qquad \text{for all } n \geq 1.$$

Base case $n = 1$: $\quad e^1 \, 1! \overset{?}{\geq} 1^1$

$(e \geq 1)$ ✓

Induction step: suppose $e^k k! \geq k^k$.

Then $e^{k+1}(k+1)! = e^k \cdot e \cdot k! \cdot (k+1)$

(by induction hyp) $\geq k^k \cdot e \cdot (k+1)$

(by (a)) $\geq k^k (1 + 1/k)^k (k+1)$

$= (k+1)^k (k+1)$

$= (k+1)^{k+1}$