

Midterm 1, version 0

0. (1 point) write your name, your GSI's name, and your section number on your exam.

1. (5 points) Evaluate the integral: $\int x \sec^2 x \tan x \, dx$

2. (5 points) Find a and b : $\int_1^{\sqrt{2}} \frac{\sqrt{x^2 - 1}}{x} \, dx = a - \frac{\pi}{b}$

3. (5 points) Evaluate the integral: $\int \frac{x^4 + 3}{x^3 - x} \, dx$

4a. (2 points) Write down the equation of the parabola $p(x)$ such that

$$p(1) = 0, \quad p(2) = 0, \quad p(3) = 1.$$

4b. (2 points) With $p(x)$ as above, evaluate $\int_1^3 p(x) \, dx$

4c. (3 points) Let $f(x) = (x + 1)e^{-x}$. Find K_2 such that $|f''(x)| \leq K_2$ for $1 \leq x \leq 3$.

5a. (3 points) Prove that

$$\cos x \geq 1 - \frac{2}{\pi}x \quad \text{for } 0 \leq x \leq \frac{\pi}{2}$$

5b. (3 points) Use part (a) to show that the following integral is convergent:

$$\int_0^{\pi/2} \frac{1}{\sqrt{\cos x}} \, dx$$

6. (3 points for correct answer, -1 for wrong answer, 0 for blank)

Let $F(x) = \int_0^x e^{-t^2} \, dt$. Evaluate $\int_0^\infty F(x) \, dx$.

(a) $-1/2$

(b) $1/2$

(c) 1

(d) ∞

(e) cannot be expressed in terms of elementary functions

Midterm 1, version 1

0. (1 point) write your name, your GSI's name, and your section number on your exam.

1. (5 points) Evaluate the integral: $\int x[\cos^2 x - \sin^2 x] dx$

2. (5 points) Find a and b : $\int_1^2 \frac{\sqrt{x^2 - 1}}{x} dx = a - \frac{\pi}{b}$

3. (5 points) Evaluate the integral: $\int \frac{x^4 + 2}{x^3 - x} dx$

4a. (2 points) Write down the equation of the parabola $p(x)$ such that

$$p(1) = 0, \quad p(2) = 1, \quad p(3) = 0.$$

4b. (2 points) With $p(x)$ as above, evaluate $\int_1^3 p(x) dx$

4c. (3 points) Let $f(x) = (x + 1)e^{-x}$. Find K_2 such that $|f''(x)| \leq K_2$ for $0 \leq x \leq 1$.

5a. (3 points) Prove that

$$\cos x \geq 1 - \frac{2}{\pi}x \quad \text{for } 0 \leq x \leq \frac{\pi}{2}$$

5b. (3 points) Use part (a) to show that the following integral is convergent:

$$\int_0^{\pi/2} \frac{1}{\sqrt{\cos x}} dx$$

6. (3 points for correct answer, -1 for wrong answer, 0 for blank)

Let $F(x) = \int_x^\infty e^{-\frac{t^2}{2}} dt$. Evaluate $\int_0^\infty F(x) dx$.

(a) $-1/2$

(b) $1/2$

(c) 1

(d) ∞

(e) cannot be expressed in terms of elementary functions

Midterm 1, version 2

0. (1 point) write your name, your GSI's name, and your section number on your exam.

1. (5 points) Evaluate the integral: $\int x \sin x \cos x dx$

2. (5 points) Find a and b : $\int_1^{\frac{2}{\sqrt{3}}} \frac{\sqrt{x^2 - 1}}{x} dx = a - \frac{\pi}{b}$

3. (5 points) Evaluate the integral: $\int \frac{x^4 + 1}{x^3 - x} dx$

4a. (2 points) Write down the equation of the parabola $p(x)$ such that

$$p(1) = 1, \quad p(2) = 0, \quad p(3) = 0.$$

4b. (2 points) With $p(x)$ as above, evaluate $\int_1^3 p(x) dx$

4c. (3 points) Let $f(x) = (x + 2)e^{-x}$. Find K_2 such that $|f''(x)| \leq K_2$ for $0 \leq x \leq 3$.

5a. (3 points) Prove that

$$\cos x \geq 1 - \frac{2}{\pi}x \quad \text{for } 0 \leq x \leq \frac{\pi}{2}$$

5b. (3 points) Use part (a) to show that the following integral is convergent:

$$\int_0^{\pi/2} \frac{1}{\sqrt{\cos x}} dx$$

6. (3 points for correct answer, -1 for wrong answer, 0 for blank)

Let $F(x) = \int_x^\infty e^{-t^2} dt$. Evaluate $\int_0^\infty F(x) dx$.

(a) $-1/2$

(b) $1/2$

(c) 1

(d) ∞

(e) cannot be expressed in terms of elementary functions

results of first midterm:

students: 366

total possible: 32 points

average: 18.64

standard dev: 5.96

rough grading scale (worry about +/- later)

raw score	grade	# of students
25-32	A	74
20-24	B	101
15-19	C	95
10-14	D	71
1-9	F	25

curve for comparison to other midterm and final:

curved score = 25 + 2.5 * raw score

histogram:

raw score	# students	running total	
1	0	0	
2	0	0	
3	1	1	#
4	2	3	##
5	3	6	###
6	1	7	#
7	5	12	#####
8	12	24	#####
9	1	25	#
10	11	36	#####
11	9	45	#####
12	18	63	#####
13	16	79	#####
14	17	96	#####
15	21	117	#####
16	17	134	#####
17	21	155	#####
18	19	174	#####
19	17	191	#####
20	19	210	#####
21	30	240	#####
22	15	255	#####
23	12	267	#####
24	24	291	#####
25	22	313	#####
26	25	338	#####
27	12	350	#####
28	8	358	#####
29	4	362	####
30	1	363	#
31	1	364	#
32	2	366	##

the answers below were written up by Matthias Goerner. Thank you, Matthias!

$$1) \int \tan x \sec^2 x dx = \int u du = \frac{u^2}{2} = \frac{\tan^2 x}{2} + C$$

$$u = \tan x \\ du = \sec^2 x dx$$

$$A) \int \underbrace{x}_{u} \underbrace{\sec^2 x \tan x dx}_{dv} = \frac{x \tan^2 x}{2} - \int \frac{\tan^2 x}{2} dx$$

$$du = dx \quad v = \tan^2 x / 2 = \frac{x \tan^2 x}{2} - \int \frac{\sec^2 x - 1}{2} dx = \frac{x \tan^2 x - \tan x + x}{2} + C$$

$$B) \int x \sin x \cos x dx = \frac{1}{2} \int \underbrace{x}_{u} \underbrace{\sin(2x) dx}_{dv} = \frac{-x \cos(2x)}{4} + \frac{\sin(2x)}{8} + C$$

$$du = dx \quad v = -\cos(2x)/2$$

$$C) \int x [\cos^2 x - \sin^2 x] dx = \int x \cos 2x dx = \frac{x \sin(2x)}{2} + \frac{\cos(2x)}{4} + C$$

$$2) \int_a^b \frac{\sqrt{x^2-1}}{x} dx = \int_{\sec^{-1}a}^{\sec^{-1}b} \frac{\tan \theta}{\sec \theta} \sec \theta \tan \theta d\theta = \int_{\sec^{-1}a}^{\sec^{-1}b} \tan^2 \theta d\theta$$

$$x = \sec \theta \\ dx = \sec(\theta) \tan(\theta) d\theta$$

$$= [\tan \theta - \theta]_{\sec^{-1}a}^{\sec^{-1}b}$$

$$= \sqrt{b^2-1} - \sqrt{a^2-1} - \sec^{-1}b + \sec^{-1}a$$

$$A) \int_1^{\sqrt{13}} \frac{\sqrt{x^2-1}}{x} dx = \sqrt{4/3-1} - \sqrt{12-1} - \pi/6 + 0 = \sqrt{1/3} - \pi/6 \quad | \text{ Answer } a = \sqrt{1/3}, b = 6$$

$$B) \int_1^2 \frac{1}{\sqrt{x^2-1}} dx = \sqrt{3} - \pi/3 \quad | \text{ Answer: } a = \sqrt{3}, b = 3$$

$$C) \int_1^4 \frac{1}{\sqrt{x^2-1}} dx = 1 - \pi/4 \quad | \text{ Answer: } a = 1, b = 4$$

$$3. A) \int \frac{x^4+3}{x^2-x} dx = \int x + \frac{x^2+3}{x^2-x} dx$$

$$\frac{x^2+3}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$x^2+3 = A(x^2-1) + B(x^2+x) + C(x^2-x)$$

$$A = -3$$

$$B = C = 2$$

$$\int = \int x + \frac{-3}{x} + \frac{2}{x-1} + \frac{2}{x+1} dx = \frac{x^2}{2} - 3 \ln|x| + 2 \ln|x^2-1| + C$$

$$B) \int \frac{x^4+2}{x^2-x} dx = \dots = \frac{x^2}{2} - 2 \ln|x| + \frac{3}{2} \ln|x^2-1| + C$$

$$C) \int \frac{x^4+1}{x^2-x} dx = \dots = \frac{x^2}{2} - \ln|x| + \ln|x^2-1| + C$$

$$= \frac{x^2}{2} + \ln\left|\frac{x^2-1}{x}\right| + C$$

4a)

A) $p(1) = p(2) = 0$ $p(3) = 1$

↓
symmetric
around $x = 3/2$

$$p(x) = A(x - 3/2)^2 - B \quad \rightarrow \quad \begin{matrix} A = 1/2 \\ B = -1/8 \end{matrix}$$

$$p(x) = 1/2(x - 3/2)^2 - 1/8$$

or roots at 1 and 2

$$p(x) = k(x-1)(x-2) \quad \rightarrow \quad k = 1/2$$

$$p(x) = \frac{(x-1)(x-2)}{2}$$

or $p(x) = Ax^2 + Bx + C$

$$p(1) = A + B + C = 0$$

$$p(2) = 4A + 2B + C = 0$$

$$p(3) = 9A + 3B + C = 1$$

$$3A + B = 0$$

$$B = -3A$$

$$8A + 2B = 1$$

$$8A - 6A = 1$$

$$A = 1/2$$

$$B = -3/2$$

$$C = 1$$

$$p(x) = x^2/2 - 3/2x + 1$$

B)

$$p(1) = 1 \quad p(2) = 0 \quad p(3) = 0$$

$$p(x) = \frac{(x-2)(x-3)}{2}$$

C) $p(1) = 0$ $p(2) = 1$ $p(3) = 0$

$$p(x) = -(x-1)(x-3) = 1 - (x-2)^2$$

4b)

$$\int_1^2 p(x) dx = \frac{3-1}{2} \cdot \frac{1}{3} (p(1) + 4p(2) + p(3)) = \begin{matrix} A), B) & 1/3 \\ C) & 4/3 \end{matrix}$$

↑
Simpson's rule

4c) A) $f(x) = (x+1)e^{-x}$ $0 \leq x \leq 1$

$f''(x) = (x-1)e^{-x}$ $|x-1| \leq 1$ $|e^{-x}| \leq 1$

$|f''(x)| \leq 1 = K_2$

B) $f(x) = (x+1)e^{-x}$ $1 \leq x \leq 3$

$K_2 = 2/e$

$|x-1| \leq 2$ $|e^{-x}| \leq 1/e$

C) $f(x) = (x+2)e^{-x}$ $0 \leq x \leq 3$

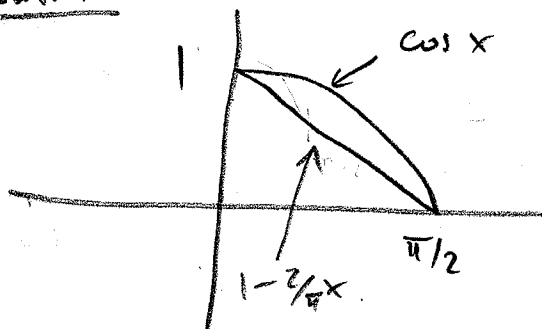
$f''(x) = xe^{-x}$

$|f''(x)| \leq |3| |1| = 3 = K_2$

5a) $\cos x \geq 1 - \frac{2}{\pi}x$ for $0 \leq x \leq \pi/2$

Notice $(\cos x)'' = -\cos x \leq 0$ for $0 \leq x \leq \pi/2$ $\cos x$ concave downward

Solution 1:



The graph of $1 - \frac{2}{\pi}x$ is the secant line segment from 0 to $\pi/2$.

$\cos x$ is concave, hence the graph of $\cos x$ is lying above the secant line

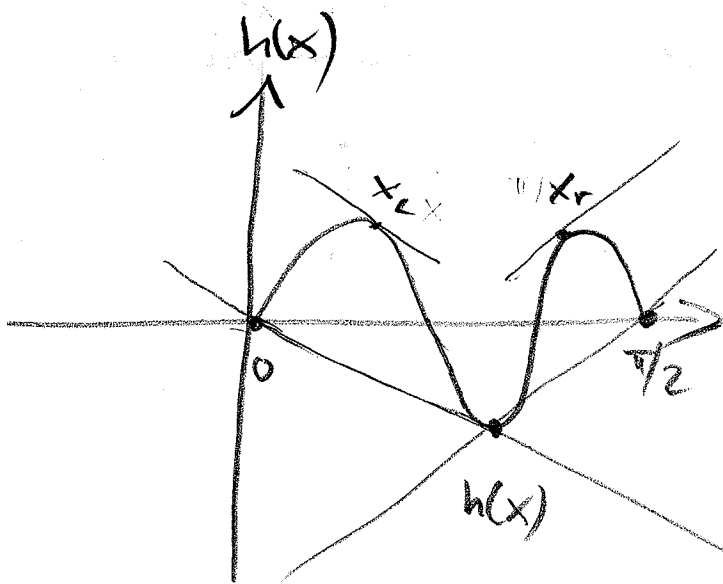
Solution 2: Look at $h(x) = \cos x - (1 - \frac{2}{\pi}x)$.

Notice $h(0) = h(\pi/2) = 0$ $h''(x) < 0$ for $0 < x < \pi/2$

\downarrow
 h has no minimum between 0 and $\pi/2$

Minimum $\stackrel{0}{=}$ archived at 0 and $\pi/2$, hence $h(x) \geq 0$,
 so $\cos x \geq 1 - \frac{2}{\pi}x$.

Solution 3: Take h as above. Assume there is an x s.t. $h(x) < 0$.



By the mean value thm, there is

$0 \leq x_l \leq x$ such that

$$h'(x_l) = \frac{h(x) - h(0)}{x} < 0$$

and

$x \leq x_r \leq \pi/2$ such that

$$h'(x_r) = \frac{h(\pi/2) - h(x)}{\pi/2 - x} > 0.$$

But $h'(x_r) > h'(x_l)$ contradicts $x_r > x_l$ the fact that h' is decreasing because $h'' < 0$.

56)

$$\int_0^{\pi/2} \frac{dx}{\sqrt{\cos x}} \leq \int_0^{\pi/2} \frac{dx}{\sqrt{1 - \frac{2}{\pi}x}} = \frac{1}{\pi/2} \int_0^1 \frac{du}{\sqrt{u}} = \frac{1}{\pi/2} [2\sqrt{u}]_0^1 = \pi$$

convergent!

$$0 \leq \frac{1}{\sqrt{\cos x}} \stackrel{\text{by 5a}}{\leq} \frac{1}{\sqrt{1 - \frac{2}{\pi}x}}$$

$$6. A) F(x) = \int_0^x e^{-t^2} dt$$

$$F(1) = \int_0^1 e^{-t^2} dt \geq \frac{1}{e}$$

$$F(x) \geq \frac{1}{e} \text{ for } x \geq 1$$

$$\int_1^+ F(t) dt \geq \frac{t-1}{e} \rightarrow \infty \text{ hence } \int_0^{\infty} F(t) dt = \infty$$

$$B) F(x) = \int_x^{\infty} e^{-t^2} dt$$

$$\int_0^{\infty} \underbrace{F(x)}_u \underbrace{dx}_{dv} = \underbrace{x F(x)}_0^{\infty} + \int_0^{\infty} x e^{-x^2} dx = \left. -\frac{e^{-x^2}}{2} \right|_0^{\infty} = \frac{1}{2}$$

$$C) F(x) = \int_x^{\infty} e^{-t/2} dt$$

$$\int_0^{\infty} F(x) dx = 0 + \int_0^{\infty} x e^{-x/2} dx = \left. -2e^{-x/2} \right|_0^{\infty} = 1$$