problem 70a page 471.

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

claim: there exist polynomials $P_0(x)$, $P_1(x)$, $P_2(x)$, ...

such that

$$f^{(n)}(x) = \begin{cases} P_n\left(\frac{1}{x}\right) e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$\frac{f^{(n)}(0)}{n!}$

thus, $f(x)$ is not equal to its Maclaurin series $\sum_{n=0}^{\infty} 0 x^n = 0$

<u>proof by induction</u>:

<u>$n=0$:</u> $P_0(x) = 1$ works

<u>$k \Rightarrow k+1$:</u> if $x \neq 0$, then

$$f^{(k+1)}(x) = \frac{d}{dx} f^{(k)}(x) = \frac{d}{dx}\left[P_k\left(\frac{1}{x}\right) e^{-1/x^2}\right]$$

$$= P_k'\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) e^{-1/x^2} + \frac{2}{x^3} P_k\left(\frac{1}{x}\right) e^{-1/x^2}$$

$$\overset{*}{=} P_{k+1}\left(\frac{1}{x}\right) e^{-1/x^2}$$

if we define $P_{k+1}(x) = 2x^3 P_k(x) - x^2 P_k'(x)$,

which is also a polynomial.

If $x = 0$, then $f^{(k+1)}(0) = \lim_{h \to 0} \frac{f^{(k)}(h) - f^{(k)}(0)}{h}$

$$= \lim_{h \to 0} \frac{1}{h} P_k\left(\frac{1}{h}\right) e^{-1/h^2} = 0$$

$y = \frac{1}{h^2}$ l'Hospital, induction

since $\lim_{h \to 0} \left|\frac{1}{h^m} e^{-1/h^2}\right| = \lim_{y \to \infty} y^{m/2} e^{-y} = \lim_{y \to \infty} \frac{y^{m/2}}{e^y} = 0$ for any $m \geq 1$

problem 29 page 478.

Suppose r is a solution of $f(x)=0$

Suppose $|f''(x)| \leq M$ and $|f'(x)| \geq K > 0$ for all x in an interval $I=(a,b)$ containing r and x_n.

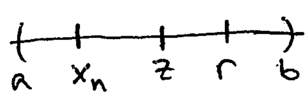
Let
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{(Newton's method)}$$

Show that
$$|x_{n+1} - r| \leq \frac{M}{2K} |x_n - r|^2$$

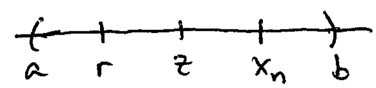
<u>Solution:</u> By Taylor's formula, there is a z between r and x_n such that

$$0 = f(r) = f(x_n) + f'(x_n)(r-x_n) + R_1(r)$$

where $|R_1(r)| = \left| \frac{f''(z)}{2}(r-x_n)^2 \right| \leq \frac{M}{2} |x_n - r|^2$



or



← Taylor series is centered at x_n not a

solving for r we obtain

$$r = x_n - \frac{f(x_n) + R_1(r)}{f'(x_n)} = x_{n+1} - \frac{R_1(r)}{f'(x_n)}$$

we conclude
$$|x_{n+1} - r| = \left| \frac{R_1(r)}{f'(x_n)} \right| \leq \frac{\frac{M}{2}|x_n - r|^2}{K} = \frac{M}{2K}|x_n - r|^2$$

as claimed.