1) Let $f(x) = x^{50}$. Then $f'(x) = 50x^{49}$.

$$\lim_{x\to 1} \frac{x^{50}-1}{x-1} = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} \underset{\uparrow}{=} f'(1) = 50(1)^{49} = 50$$

definition of $f'(1)$

2) $$\left.\frac{d}{dx}\right|_{x=1} \frac{x^{7/3}}{x^{1/3}+x^{4/3}}$$

Quotient rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} =$$

$$= \frac{(x^{1/3}+x^{4/3})\left(\frac{7}{3}x^{4/3}\right) - x^{7/3}\left(\frac{1}{3}x^{-2/3} + \frac{4}{3}x^{1/3}\right)}{(x^{1/3}+x^{4/3})^2}$$

$$\left.\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)\right|_{x=1} = \frac{(1+1)\left(\frac{7}{3}\right) - 1\left(\frac{1}{3}+\frac{4}{3}\right)}{(1+1)^2} = \frac{\frac{14}{3} - \frac{5}{3}}{4} = \frac{3}{4}$$

Alternatively: $$\frac{x^{7/3}}{x^{1/3}+x^{4/3}} = \frac{x^{7/3}\cdot x^{-1/3}}{(x^{1/3}+x^{4/3})\cdot x^{-1/3}} = \frac{x^2}{1+x}$$

$$\frac{d}{dx}\left(\frac{x^2}{1+x}\right) = \frac{(1+x)(2x) - x^2(1)}{(1+x)^2} = \frac{x^2+2x}{(1+x)^2}$$

$$\left.\frac{d}{dx}\right|_{x=1}\left(\frac{x^2}{1+x}\right) = \frac{1^2+2}{(1+1)^2} = \frac{3}{4}$$

3) i) $f(x) = \text{even}$ ie $f(-x) = f(x)$

$g(x) = \text{odd}$ ie $g(-x) = -g(x)$

$f \circ g$ is <u>even</u>

$$\text{proof: } (f \circ g)(-x) = f(g(-x)) \overset{g \text{ is odd}}{=} f(-g(x)) \overset{f \text{ is even}}{=} f(g(x)) = (f \circ g)(x)$$

ii) $g \circ f$ is <u>even</u>

$$\text{proof: } (g \circ f)(-x) = g(f(-x)) \overset{f \text{ is even}}{=} g(f(x)) = (g \circ f)(x)$$

iii) fg is <u>odd</u>

$$\text{proof: } (fg)(-x) = f(-x)\,g(-x) = (f(x))(-g(x)) = -f(x)g(x)$$
$$= -(fg)(x)$$

iv) $f+g$ is <u>neither</u>

eg: $f(x) = x^2$ $g(x) = x$ $(f+g)(x) = x^2 + x$

$$\left(\begin{array}{l} (f+g)(-1) = f(-1) + g(-1) = (-1)^2 + (-1) = 1 - 1 = 0 \\ (f+g)(1) = f(1) + g(1) = 1^2 + 1 = 2 \\ \\ \qquad\qquad 0 \neq 2 \text{ or } -2 \end{array} \right)$$

4) Let $f(x) = x\cos x$

Show there is a point c in $(0, \pi/2)$ st the tangent line at $(c, f(c))$ is horizontal.

I.e. Show there is a point c in $(0, \pi/2)$ st $f'(c) = 0$

$$f'(x) = x(-\sin x) + \cos x$$

$$f'(0) = 0 + 1 = 1 \qquad f'\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{2}(-1) + 0 = -\tfrac{\pi}{2}$$

$f'(x)$ is continuous, so because $f'(0) > 0$ and $f'\left(\frac{\pi}{2}\right) < 0$, by the Intermediate value theorem, there must be a c between 0 and $\frac{\pi}{2}$ such that $f'(c) = 0$

5) Suppose $\lim_{x\to\infty} (f(x) - 2x) = 3$

Then
$$\lim_{x\to\infty}\left(\frac{f(x)}{x}\right) = \lim_{x\to\infty}\left(\frac{f(x)}{x} - 2 + 2\right)$$

$$= \lim_{x\to\infty}\left(\frac{f(x) - 2x}{x} + 2\right)$$

$$= \lim_{x\to\infty}\left(\frac{1}{x}(f(x) - 2x) + 2\right) = 0 \cdot 3 + 2 = 2$$

6) $\lim_{t\to 0} \dfrac{t^2}{1-\cos 5t} = \lim_{t\to 0} \dfrac{t^2(1+\cos 5t)}{1-\cos^2 5t}$

$= \lim_{t\to 0} \dfrac{t^2(1+\cos 5t)}{\sin^2 5t} = \lim_{t\to 0} \dfrac{1}{25}\left(\dfrac{5t}{\sin 5t}\right)^2 (1+\cos 5t)$

$= \dfrac{1}{25}(1)^2(1+1) = \dfrac{2}{25}$

7) To show: $\lim_{x\to 1} \sqrt{x} = 1$

Let $\varepsilon > 0$

Choose $\delta = \varepsilon$

Then, if $|x-1| < \delta$

then $\left|\sqrt{x}-1\right| = \left|\sqrt{x}-1\right| \cdot \left|\dfrac{\sqrt{x}+1}{\sqrt{x}+1}\right|$

$= \left|\dfrac{x-1}{\sqrt{x}+1}\right| < \dfrac{\delta}{\sqrt{x}+1} \leq \delta = \varepsilon$

(note: $\sqrt{x}+1 \geq 1$ for all x in the domain of f
so $\dfrac{1}{\sqrt{x}+1} \leq 1$ for all x in the domain of f)