Sample Midterm 1

1. (2 points) What is the domain of the function $f(x) = \dfrac{\sqrt{x-2}+\sqrt{7-x}}{x-5}$?

Answer:
$[2,5) \cup (5,7]$
x must be ≥ 2 for $\sqrt{x-2}$ to be defined. It must be ≤ 7 for $\sqrt{7-x}$ to be defined, and it must $\neq 5$ so that $x-5 \neq 0$

2. (3 points) Compute the following derivative:

$$\frac{d}{d\theta}\bigg|_{\theta=0} \tan\theta \, \cos\theta$$

Answer:

$$\frac{d}{d\theta}\bigg|_{\theta=0} \tan\theta \, \cos\theta = \frac{d}{d\theta}\bigg|_{\theta=0} \frac{\sin\theta}{\cos\theta} \, \cos\theta = \frac{d}{d\theta}\bigg|_{\theta=0} \sin\theta = \cos 0 = 1$$

3. (5 points) Find the equation of the tangent line to the curve $y = 1/(1+x^2)$ at the point $\left(3, \frac{1}{10}\right)$.

Answer:

$$\frac{dy}{dx} = \frac{-2x}{(1+x^2)^2}$$

so at $\left(3, \frac{1}{10}\right)$, the slope of the tangent line is $\frac{-2*3}{(1+3^2)^2} = \frac{-3}{50}$.
Putting the line into point-slope format gives $y - \frac{1}{10} = \frac{-3}{50}(x-3)$ or $y = \frac{-3}{50}x + \frac{7}{25}$

4. (5 points) Compute $\displaystyle\lim_{x\to\infty} \frac{(x-1)(2x-3)}{(4x+1)(7x+1)}$

Answer:

$$\lim_{x\to\infty} \frac{(x-1)(2x-3)}{(4x+1)(7x+1)} = \lim_{x\to\infty} \frac{(1-1/x)(2-3/x)}{(4+1/x)(7+1/x)} = \frac{(1-0)(2-0)}{(4+0)(7+0)} = \frac{1}{14}$$

5. (5 points) Prove that $f(x) = 1 - x^5$ has a fixed point (i.e. there is a number c such that $f(c) = c$).

Answer:
Let $g(x) = f(x) - x = 1 - x^5 - x$
Then, $g(0) = 1 > 0$ and $g(1) = -1 < 0$, so by the Intermediate Value Theorem, there is some c in $(0, 1)$ such that $g(c) = 0$, and therefore, for that same c, $f(c) = c$

6. (5 points) Evaluate the limit

$$\lim_{\theta \to 0^-} \frac{\sin\theta}{\sqrt{1 - \cos\theta}}$$

.

Answer:
Note first that when $\theta < 0$ but θ is close to 0, $\sin\theta < 0$
so $\sin\theta = -\sqrt{\sin^2\theta} = -\sqrt{1 - \cos^2\theta}$, thus we get

$$\lim_{\theta \to 0^-} \frac{\sin\theta}{\sqrt{1 - \cos\theta}} = \lim_{\theta \to 0^-} \frac{-\sqrt{1 - \cos^2\theta}}{\sqrt{1 - \cos\theta}} = \lim_{\theta \to 0^-} -\sqrt{\frac{1 - \cos^2\theta}{1 - \cos\theta}} = \lim_{\theta \to 0^-} -\sqrt{1 + \cos\theta} = -\sqrt{2}$$

7. (5 points) Use the δ-ε definition of the limit to prove **one** of the following:

(a) $\lim_{x \to 2} 1/x = 1/2$.

Answer:
Let $\varepsilon > 0$. Choose $\delta = \min(2\varepsilon, 1)$. Then, if $0 < |x - 2| < \delta$, then
$|1/x - 1/2| = |(2 - x)/2x| = |x - 2|/|2x| < \delta/|2x| <^1 \delta/2 \leq \varepsilon$

[1]: note that $\delta \leq 1$, so if $0 < |x - 2| < \delta$ then $x > 1$ so $\delta/|2x| < \delta/2$

(b) Suppose $\lim_{x \to 0} f(x) = L$. Define $g(x) = f(-x)$. Then $\lim_{x \to 0} g(x) = L$.

Answer:
Let $\varepsilon > 0$. Choose δ so that if $0 < |x - 0| < \delta$ then $|f(x) - L| < \varepsilon$
(such a δ exists because $\lim_{x \to 0} f(x) = L$)
Then, for the chosen δ, if $0 < |x - 0| < \delta$ then $0 < |(-x) - 0| < \delta$ so
$|g(x) - L| = |f(-x) - L| < \varepsilon$

Another Sample Midterm 1

1. (2 points) Determine whether f is even, odd or neither: $f(x) = \dfrac{x}{1+x^2}$.

Answer:
f is odd.
$f(-x) = \dfrac{(-x)}{1+(-x)^2} = -\dfrac{x}{1+x^2} = -f(x)$

2. (3 points) Let $f(x) = \dfrac{x^{3/2} + 2\sqrt{x}}{x^5}$. Evaluate $f'(1)$.

Answer:
$f(x) = \dfrac{x^{3/2} + 2\sqrt{x}}{x^5} = x^{-7/2} + 2x^{-9/2}$ so $f'(x) = (-7/2)x^{-9/2} - 9x^{-11/2}$

3. (5 points) Find all the points on the curve $y = x^3 + 3x^2 + 3x + 1$ where the tangent line is horizontal.

Answer:
The tangent line is horizontal whenever the derivative is zero.
$y' = 3x^2 + 6x + 3 = 3(x+1)^2$ so $y' = 0$ when $x = -1$,
also, when $x = -1$, $y = (-1)^3 + 3(-1)^2 + 3(-1) + 1 = 0$,
so the only point with a horizontal tangent line is $(-1, 0)$

4. (5 points) Find the value a such that the following limit exists and evaluate the limit:

$$\lim_{x\to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

Answer: As $x \to -2$, the denominator $\to 0$, so for the limit to exist, the numerator must also $\to 0$, so we must have $0 = 3(-2)^2 + a(-2) + a + 3 = 15 - a$, ie $a = 15$

$$\lim_{x\to -2} \frac{3x^2 + 15x + 15 + 3}{x^2 + x - 2} = \lim_{x\to -2} \frac{3(x+2)(x+3)}{(x+2)(x-1)} = \lim_{x\to -2} \frac{3(x+3)}{x-1} = \frac{3((-2)+3)}{(-2)-1} = -1$$

5. (5 points) Compute $\lim_{x\to\infty} x \tan\left(\frac{\pi}{x}\right)$. Hint: $\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f(1/x)$.

Answer:
$$\lim_{x\to\infty} x \tan\left(\frac{\pi}{x}\right) = \lim_{x\to 0^+} \frac{1}{x} \tan\left(\frac{\pi}{1/x}\right) = \lim_{x\to 0^+} \frac{\tan(\pi x)}{x} = \lim_{x\to 0^+} \pi \frac{\sin(\pi x)}{\pi x} \frac{1}{\cos(\pi x)} = \pi$$

6. (5 points) Evaluate the limit

$$\lim_{x\to-\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2+1}\right)$$

Answer:

$$\lim_{x\to-\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2+1}\right) = \lim_{x\to-\infty}\frac{\left(\sqrt{x^2+x+1}-\sqrt{x^2+1}\right)\left(\sqrt{x^2+x+1}+\sqrt{x^2+1}\right)}{\sqrt{x^2+x+1}+\sqrt{x^2+1}} =$$

$$\lim_{x\to-\infty}\frac{(x^2+x+1)-(x^2+1)}{\left(\sqrt{x^2+x+1}+\sqrt{x^2+1}\right)} = \lim_{x\to-\infty}\frac{x}{\left(\sqrt{x^2+x+1}+\sqrt{x^2+1}\right)} =$$

$$\lim_{x\to-\infty}\frac{x/\sqrt{x^2}}{\left(\sqrt{1+1/x+1/x^2}+\sqrt{1+1/x^2}\right)} = \frac{-1}{\left(\sqrt{1+0+0}+\sqrt{1+0}\right)} = -\frac{1}{2}$$

7. (5 points) A stone is dropped from height H on a planet with gravitational constant g. The equation of motion of the stone is $y(t) = H - \frac{1}{2}gt^2$. Show that the instantaneous velocity of the stone when it hits the ground is twice the average velocity during its fall.

Answer:
Let T be the time that the stone hits the ground. Then $0 = y(T) = H - \frac{1}{2}gT^2$,
so $H = \frac{1}{2}gT^2$ and thus $T = \sqrt{2H/g}$
The average velocity, v_{avg} is (the total distance traveled)/(the total time traveled),
$v_{avg} = -H/T = -H/(\sqrt{2H/g}) = -\sqrt{Hg/2}$
(the velocity is negative because the stone traveled down)

The velocity at time t is given by $y'(t) = -gt$, therefore the final velocity v_{fin} is $y'(T)$
$v_{fin} = -gT = -g\sqrt{2H/g} = -\sqrt{2Hg} = -2\sqrt{Hg/2} = 2v_{avg}$