The Structure of 2D Semi-simple Field Theories

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February 8, 2010

Abstract

I classify the cohomological 2D field theories based on a semi-simple complex Frobenius algebra A. They are controlled by a linear combination of κ-classes and by an extension datum to the Deligne-Mumford boundary. Their effect on the Gromov-Witten potential is described by Givental's Fock space formulae. This leads to the *reconstruction of Gromov-Witten (ancestor) invariants* from the quantum cup-product at a *single* semi-simple point, confirming Givental's higher-genus reconstruction conjecture. This in turn implies the Virasoro conjecture for manifolds with semi-simple quantum cohomology. The classification uses the Mumford conjecture, proved by Madsen and Weiss [MW].

Introduction

This paper studies structural properties of *topological field theories* (TFT's), a notion introduced by Atiyah and Witten [W] and inspired by Segal's axiomatisation of Conformal Field Theory. A TFT extracts the topological information which is implicit in quantum field theories defined over space-time manifolds more general than Euclidean space. The first non-trivial example is in 2 dimensions, a setting which has been the focus of much interest in relation to *Gromov-Witten theory*: the latter captures the expected count of pseudo-holomorphic curves in a compact symplectic target manifold. The result proved here, the classification of semi-simple theories, shows that an important property of these invariants is a formal consequence of the underlying structure, rather than a reflection of geometric properties of the target manifold. Loosely stated, the property in question is that a count of rational curves with three marked points, encoded in the *quantum cohomology* of the target, determines the answer to enumerative questions about curves of all genera, when the quantum cohomology ring is semi-simple.[1]

(0.1) A 2-*dimensional topological field theory* over a ring k is a strong symmetric monoidal functor Z from the 2-dimensional oriented bordism category to the tensor category of finitely generated projective k-modules. This means that Z assigns to every closed oriented 1-manifold X a k-module $Z(X)$, and to any compact oriented surface Σ, with independently oriented boundary $\partial\Sigma$, a linear "propagator"

$$Z(\Sigma) : Z(\partial_-\Sigma) \to Z(\partial_+\Sigma).$$

The sign $\pm$ of a boundary component compares the orientation induced from Σ with the independent one on $\partial\Sigma$; we call $\partial_-\Sigma$ the *incoming* boundary and $\partial_+\Sigma$ the *outgoing* one. The above definition requires that

[1] To be precise, this is true of the *ancestor* Gromov-Witten invariants. The complete, *descendent* invariants require some additional genus zero information (the *J-function*).

(i) Z is multiplicative under disjoint unions, $Z(X_1 \amalg X_2) = Z(X_1) \otimes Z(X_2)$;
(ii) Sewing boundary components leads to the composition of maps;
(iii) The cylinder "$=$" with an incoming and an outgoing end represents the identity.

Pat (i) is the "monoidal" condition, while (ii) is the functorial poperty. Condition (iii) compensates for the absence of identity morphisms in (the most naïve definition of) the bordism category. Without it, $Z(=)$ would only need to be a projector: $Z(=) \circ Z(=) = Z(=)$.

(0.2) A folk theorem (with non-trivial proof, see [A1]) ensures that Z is equivalent to the datum of a *commutative Frobenius k-algebra* structure on the space $A := Z(S^1)$. This last notion comprises a commutative k-algebra structure on A, together with an A-module isomorphism $\iota : A \xrightarrow{\sim} A^* := \mathrm{Hom}_k(A, k)$. The Frobenius structure on $Z(S^1)$ can be read from the functor Z as follows:

- the multiplication map $A \otimes A \to A$ is defined by the trinion with two incoming circles and an outgoing one;
- the unit in A is defined by the disk with outgoing boundary, $Z(\supset) : k \to A$;
- the disk $\subset$ with incoming boundary determines the vector $\theta := \iota(1) \in A^*$.

The pictures represent the outlines of surfaces, with their boundaries omitted. (Also, the reader will have noticed that incoming boundaries are on the right and outgoing ones on the left, matching the ordering convention for operator composition.) The form θ, in turn, determines a symmetric pairing $\beta : A \times A \to k$, $\beta(a \times b) = \theta(ab)$, which is the partial adjoint to ι in one of the variables. Non-degeneracy of β — equivalently, the isomorphy condition on ι — is enforced by *Zorro's lemma*: a diagram wherein a "Z"-shaped identity cylinder is factored into a "right elbow" $\Supset$, a cylinder with two outgoing ends, sewed on to a left elbow $\Subset$ at one of its outputs: $Z(\Subset)$ represents β, and $Z(\Supset)$ provides an inverse co-form.

(0.3) An easy but important special case concerns *semi-simple* algebras A over $k = \mathbb{C}$. As algebras, these are isomorphic to $\bigoplus_i \mathbb{C} \cdot P_i$ for projectors P_i, uniquely determined up to reordering. From the definition and non-degeneracy of β, the projectors are β-orthogonal and their θ-values $\theta_i = \theta(P_i)$ must be non-zero complex numbers. Up to isomorphism, A is classified by the (unordered) collection of the θ_i. The TFT is easy to describe in the *normalised canonical basis* of rescaled projectors $p_i := \theta_i^{-1/2} P_i$, as follows. For a connected surface Σ with m incoming and n outgoing boundaries, the matrix of $Z(\Sigma)$ has entry $\theta_i^{\chi(\Sigma)/2}$ linking $p_i^{\otimes m}$ to $p_i^{\otimes n}$, while all entries involving mixed tensor monomials in the p_i are null.

(0.4) Key definition and example. A Frobenius algebra contains a distinguished vector, the *Euler class* α, defined as the output of a torus with one outgoing boundary. When A is the cohomology ring of a closed oriented manifold with coefficients in a field, and β the Poincaré duality pairing, α is the usual Euler class. (Of course, A will be a *skew*-commutative Frobenius algebra if there is odd cohomology.) By contrast, in the semi-simple case, α is the *invertible* element $\sum_i \theta_i^{-1} P_i$. The endomorphism of A defined by a two-holed surface of genus g is the multiplication by α^g: in matrix form, $\mathrm{diag}[\theta_i^{-g}]$. This observation will allow the recovery of low-genus Z from high genus, and will play a key role in the paper.

There is actually a converse: invertibility of α implies semi-simplicity of A. (The trace on A of the operator of multiplication by x is $\theta(\alpha x)$, so Tr_A defines a non-degenerate bilinear form on A; it follows that, over any residue field of the ground ring k, A is a sum of separable field extensions.) This, and the importance of an invertible α may have been first flagged by Abrams, also in connection with quantum cohomology; the reader is referred to the nice paper [A2].

(0.5) In this paper, I give an algebraic classification for *family TFTs* (FTFTs), in which the surfaces vary in families and the functor Z takes values in the cohomology of the parameter spaces, with coefficients in tensor powers of A. These theories are variants of the *Cohomological Field Theories* (CohFT's) introduced by Kontsevich and Manin [KM1]. "Families" consisting of single surfaces recover the previous TFT notion and detect the underlying Frobenius algebra A. My classification applies whenever A is semi-simple and k is a field of characteristic zero; I use $\mathbb{C}$ for simplicity.

(0.6) The theories of greatest interest involve nodal surfaces, the *stable curves* of algebraic geometry, and come from *Gromov-Witten invariants*. In this setting, I provide a structure formula for the Gromov-Witten invariants of manifolds whose quantum cohomology is generically semi-simple. Such theories have additional structure, the grading that stems from the fact that spaces of stable maps have topologically determined (expected) dimensions. This structure limits the freedom of choice considerably: the full FTFT is determined by the Frobenius algebra and the grading information. This affirms a conjecture of Givental's [G1] on the reconstruction of higher-genus invariants, and in particular, as pointed out in [G3], the Virasoro conjecture for such manifolds. Verification of this conjecture involves tracing through Givental's construction, with an improvement to the formulation which (I think) is originally due to M. Kontsevich, reviewed in §6.

(0.7) With different starting hypotheses, a vast extension of my classification has been reached by Kontsevich and collaborators in the framework of *open-closed* FTFTs (see [KKP] and sequels in preparation). From that perspective, I show that any semi-simple (closed string) CohFT may be assumed to come from an open-closed FTFT with a semi-simple category of boundary states. In Gromov-Witten theory, this statement would even follow from a sufficiently optimistic formulation of Homological Mirror Symmetry: semi-simplicity of quantum cohomology suggests a Landau-Ginzburg B-model mirror with isolated Morse critical points of the potential, since (in the case of isolated singularities) the quantum cohomology ring is meant to be isomorphic to the Jacobian ring of the potential. In this situation, the mirror category of boundary states (B-branes) is also semi-simple. Assuming all this, we could then invoke Kontsevich's classification.

However, while it seems clear that the open-closed framework (or some related 2-categorical approach) is the right setting, Gromov-Witten theory is not quite ready for it, as the requisite assumptions on the Fukaya category of boundary states have only been checked in special cases; whereas the CohFT axioms are well-established. Examples of varieties with generically semi-simple quantum cohomology include: toric manifolds, most Fano three-folds with no odd Betti numbers [Ci], as well as blow-ups of such varieties at any number of points [B]. Of these, only for the toric ones does the open-closed theory seem to be in convincing shape, thanks to work by Fukaya and collaborators [FOOO].

Acknowledgements

I have lectured on successive approximations of this work since the fall of 2004. Comments from colleagues were essential in correcting and re-shaping the results, and it is a pleasure to thank them here. Early on, K. Costello suggested a link between my classification and Givental's conjecture; T. Coates, Y. Eliashberg, S. Galatius, E. Getzler, M. Kontsevich, Y. Manin, J. Morava and D. Sullivan all contributed useful comments. Y.-P. Lee offered numerous helpful comments on the manuscript, as did the anonymous referee. Special thanks are due to A. Givental for patient explanation of his work. Substantial portions of this work were completed during the 2006 programme on "New Topological structures in physics" at MSRI and much of the initial writing was done during a visit to the Max Planck Institute in Bonn in August 2007. Partial support was provided by NSF grant DMS-0709448.

Contents

1. Summary of definitions and results

This section outlines the definition and classification of the various versions of FTFT's used throughout the paper, as well as the background of the two key results, Theorems 1 and 2. It is not possible to cover all the details in the space suited to an opening section, and the reader will often be referred to later paragraphs for clarification. For instance, a classifying spaces of surface bundles are discussed in §2; a refresher on $\kappa-$ and $\psi-$classes is found in §2.15; and the list of axioms for a DMT is only truly completed by spelling out the 'nodal relations' in §4.4.

(1.1) Functorial definition. Family TFT's admit a categorical definition in the style described in the introduction. I give it here for logical completeness; its meaning and use, in the several variants outlined in §1.3 below, will be spelt out in more detail in Section 2.

Consider the following two contra-functors $\mathcal{C}$ and $\mathcal{F}$, defined over the category of topological spaces and continuous maps, and taking values in symmetric monoidal categories. At a topological space X, the first category $\mathcal{C}(X)$ has as objects bundles of closed oriented 1-manifolds over X, and as morphisms bundles of compact oriented 2-bordisms, modulo homeomorphisms over X (which fix the boundaries). Objects of the second category $\mathcal{F}(X)$ are flat complex vector bundles over X, while

$$\mathrm{Hom}_{\mathcal{F}(X)}(V,W) = H^\bullet\left(X;\mathrm{Hom}_X(V,W)\right).$$

A FTFT is a symmetric monoidal transformation Z from $\mathcal{C}$ to $\mathcal{F}$. Variants of this notion are obtained by varying the defining features of $\mathcal{C}$: we can require all circles in $\mathcal{C}(X)$ to be parametrised (§1.3.i) or not (§1.3.ii), allow Lefschetz fibrations as morphisms (§1.3.iii), and finally, impose the Deligne-Mumford stability condition on such fibrations (§1.3.iv).

1.2 Remark. The objects of $\mathcal{C}$ and $\mathcal{F}$ form *sheaves* over the site of topological spaces, but the morphisms do not. Morphisms of $\mathcal{F}$ are the cohomologies of a differential-graded version of $\mathcal{F}$, in which the objects are complexes of local coefficient systems over X and the morphisms are cochains. There is a similar enhancement of $\mathcal{C}$ to a sheaf of categories enriched over topological spaces: the 1-morphisms are classifying spaces of the groupoids of surface bundles and their homeomorphisms. A (symmetric monoidal) natural transformation between these sheaves of categories is a possible definition of *chain-level FTFT's*, and is closely related to Segal's definition of *topological conformal field theory* [S, Ge]; but we will not use this notion in the paper.

(1.3) FTFT variants and their classification. We will consider several versions of family field theories, and their classification increases in complexity. All four variants below are relevant to the eventual focus of interest, semi-simple cohomological field theories.

(i) In the simplest model, the surfaces have parametrised boundaries. These theories are controlled by a single, group-like class $\tilde{Z}^+$ in the A-valued cohomology of the stable mapping class group of surfaces (§2.20). Such a class has the form $\exp\{\sum_{j>0} a_j\kappa_j\}$, with elements $a_j \in A$ coupled to the Morita-Mumford classes κ_j. The class $\tilde{Z}$ associated to a surface bundle with acts diagonally on tensor monomials of the normalised canonical basis: if $a_j = \sum a_{ij}P_i$, $a_{ij} \in \mathbb{C}$, then the entry $\theta_i^{\chi/2}$ of the propagator matrix (0.3) of a stationary connected surface is now multiplied by the factor $\exp\{\sum_{j>0} a_{ij}\kappa_j\}$. Since $\chi = -\kappa_0$, we can account for the θ_i by including in the sum a term $j = 0$, with $a_0 = \frac{1}{2}\log\alpha$.

(ii) Allowing the boundaries to rotate freely introduces a new classification datum, a $\mathbb{C}$-linear map $E : A \to A[\![z]\!]$ with $E \equiv \mathrm{Id} \pmod z$. Within the functorial definition 1.1, this is a trivialization of the local system $A = Z(S^1)$ over $\mathbb{CP}^\infty$ defined by the universal circle bundle. Here, $(-z) \in H^2(\mathbb{CP}^\infty)$ is the universal Euler class.[2] Thus, if A is the cohomology of a space[3] X with an action of the circle $\mathbb{T}$, then the inputs and outputs at free boundaries belong naturally to the $\mathbb{T}$-equivariant cohomology of X, and E is here to enforce a splitting of the latter as $A \otimes \mathbb{C}[\![z]\!]$. A free boundary theory is determined by E and the earlier $\tilde{Z}^+$, as follows: twist the incoming states by E^{-1} and the outputs by E, with z specialised to the sign-changed Euler classes of the respective boundary circle bundles. In-between, the fixed-boundary propagator of (i) applies.

(iii) Next in line are the *Lefschetz theories*, where surfaces are allowed to degenerate nodally into the Lefschetz fibrations of algebraic geometry. Now, a nodal surface can be deformed uniquely to a smooth one; the cohomological nature keeps Z unchanged under this deformation, so *single* nodal surfaces carry no new information. Things are different in a *family*: up to homotopy, the automorphism group of the "nodal propagator" $\supset\!\subset$, an incoming-outgoing pair of crossing disks, is the product $\mathbb{T} \times \mathbb{T}$ of the two independent circle rotation groups. This provides a new datum $Z(\supset\!\subset)$, an $\mathrm{End}(A)$-valued formal series $D(-\omega_+, \omega_-)$ in the Euler classes $\omega_\pm$ of the two universal disk bundles.

Keeping only the diagonal rotation, we can deform the node $\supset\!\subset$ into a rotating cylinder. Since $Z(=) = \mathrm{Id}$ for a fixed cylinder, and it must remain a projector when the cylinder rotates, we conclude that $D = \mathrm{Id} \mod (\omega_+ - \omega_-)$. In addition, we will find a symmetry constraint relating D and E; see §4. These are all the data and constraints: an involved, but explicit formula for the Lefschetz theory classes is given in §4.7 from $\tilde{Z}, E$ and D, as a kind of "time-ordered exponential integral" along the surfaces in any family.

(iv) Lastly, we are interested in *Deligne-Mumford theories* (DMT's): these are Lefschetz theories involving only *stable* nodal surfaces, the Deligne-Mumford stable curves of algebraic geometry. Excluding cylinders and disks (which are unstable) brings about the need for two additional axioms, the *nodal factorisation rules* and *vacuum axiom* (see §2.4), which come for free in a Lefschetz theory.

The best-known DMT's are the Cohomological Field theories à la Kontsevich and Manin, which satisfy $D = \mathrm{Id}$. It is more customary to state their structure in terms of surfaces with

[2]The awkward sign is reluctantly adopted here to avoid worse later; it stems from a sign mismatch between Euler and ψ classes for inbound circles.

[3]Or a complex: in an open-closed field theory the Hochschild complex of the category of boundary states.

inputs only, but that is a matter of convenience. In CohFT's, the compatibility constraint on E becomes *Givental's symplectic constraint* $E(z) \circ E^*(-z) = \mathrm{Id}$. The main examples of CohFT's are the Gromov-Witten cohomology theories of compact symplectic manifolds, see §1.5 below. Those carry even more structure and constraints (§1.7).

Functors of the types (i), (ii) and (iii,iv) shall be denoted by $\tilde{Z}$, Z and $\bar{Z}$, respectively.

(1.4) Idea of proof. For the first two types of FTFTs, the classification is an easy consequence of the Mumford conjecture, proved by Madsen and Weiss [MW]. (We will also use an older result of Looijenga's on ψ-classes, [L].) In the limit of large genus surfaces, the sewing axiom becomes an equation in the complex cohomology of the stable mapping class group. The latter is a power series ring in the tautological classes (see §2.15), and we solve the equation there. Semi-simplicity of A lets us retrieve the low-genus answer from high genus thanks to invertibility of the Euler class α, as in §0.4.

DMT's require an additional argument. The universal families of stable nodal surfaces are classified by orbifolds with a normal-crossing stratification. The argument above determines the classes Z on each stratum, but there could be ambiguities and obstructions in patching these classes together. However, the Euler classes of certain boundary strata involving large-genus surfaces are *not* zero-divisors in low-degree cohomology. This ensures the unique gluing of cohomology classes over suitably chosen strata. We find enough strata to cover all Deligne-Mumford moduli orbifolds, and prove the unique patching of the Z-classes to a global class $\overline{Z}$. This observation is the key contribution of the paper; the remainder falls in the "known to experts" category.

A more natural resolution of the gluing ambiguity involves the use of *chains*, instead of homology classes. This point of view, pioneered by Kontsevich in the context of homological mirror symmetry, fits naturally with the notion of *open-closed field theories* and their A_∞-categories, and was successfully developed by Costello, leading to a beautiful classification result [C]. It also ties in nicely with the *string topology* example of Chas and Sullivan [Su]. From this angle, my result shows that the semi-simple case is considerably easier: open strings and chain-level structures are not needed.

(1.5) Example: Gromov-Witten theory. Here, A is the quantum cohomology of a compact symplectic manifold X at some chosen point $u \in H^{ev}(X)$. To apply the classification, we must assume the existence of a u where this ring is semi-simple. This can be the generic point, which may be the only choice, if the series defining the quantum cup-product turn out to diverge. Semi-simplicity confines $H^\bullet(X)$ to even degrees, because odd classes are necessarily nilpotent. (More is true: it turns out that semi-simplicity of the *even part* H^{ev} of the quantum cohomology ring forces the vanishing of odd cohomology, see [HMT].)

The Gromov-Witten theory of X is constructed as follows. Denote by $X^n_{g,\delta}$ the space of Kontsevich stable maps to X with genus g, degree $\delta \in H_2(X)$, and n marked points. We obtain maps

$$GW^n_{g,\delta} : H^\bullet(X)^{\otimes n} \to H^\bullet\left(\overline{M}^n_g\right)$$

to the cohomology of Deligne-Mumford spaces $\overline{M}^n_g$ by pulling back cohomology classes on X via the evaluation map $X^n_{g,\delta} \to X^n$, and then integrating along the forgetful map $X^n_{g,\delta} \to \overline{M}^n_g$. This last step uses the virtual fundamental class of $X^n_{g,\delta}$, and the degree of each map GW_δ, in the natural grading on $H^\bullet(X)$, is determined by the relative (virtual) dimension of moduli spaces:

$$\deg GW_{g,\delta} = 2(\dim_{\mathbb{C}} M^n_g - \dim_{\mathbb{C}} X^n_{g,\delta}) = 2(g-1)\dim_{\mathbb{C}} X - 2\langle c_1(X)|\delta\rangle. \tag{1.6}$$

Summing over homology degrees δ yields a class

$$GW_g^n := \sum_\delta GW_{g,\delta}^n \cdot \mathrm{e}^\delta,$$

with coefficients in (a completion of) the group ring $\mathbb{Q}[H_2]$, called the *Novikov ring* of X. For a fixed $u \in H^2(X;\mathbb{C})$, sending $\mathrm{e}^\delta \mapsto \exp\langle u|\delta\rangle$ furnishes a ring homomorphism $\mathbb{Q}[H_2] \to \mathbb{C}$, and subject to convergence we get a u-dependent family of complex cohomology classes $GW_{u,g}^n$. We recall in Section 2 below why this is equivalent to a family of DMT's GW_u, in the sense of 1.3.iv. It is no accident that we obtain an entire family of DMT's: a deformation construction (Definition 7.1 below) produces a family parametrised by all u in (an open, or possibly formal subset of) $H^{ev}(X)$. Example 7.17 spells this out in Gromov-Witten cohomology.

(1.7) Gromov-Witten cohomology constraints. The theories GW just described meet three additional constraints. They are specifically traced to the use of *ordinary* cohomology (for instance, they do not apply in this form to the exotic Gromov-Witten theories of Coates and Givental [CG]).

(i) The *Cohomological Field Theory* (CohFT) condition $D = \mathrm{Id}$;

(ii) The *flat vacuum* condition: inserting the identity $\mathbf{1} \in A$ as the first input in GW_u^n leads to the pull-back of GW_u^{n-1} along the first forgetful morphism $\overline{M}_g^n \to \overline{M}_g^{n-1}$;

(iii) *Homogeneity* of the family GW_u ($u \in H^{ev}$) under the *Euler vector field* ξ. We will discuss this in §7.15 below (see also [M, §I.3]); along $H^2(X)$, ξ is the constant vector field $c_1(X)$, but in general
$$\xi_u := c_1(X) + \sum (i-1)u_{2i} \quad \text{at} \quad u = \sum u_{2i} \in \bigoplus\nolimits_i H^{2i}(X).$$

In GW theory, condition (i) reflects the factorisation of the (virtual) fundamental class of $X_{g,\delta}^n$ at the boundary of Deligne-Mumford space [G2]. Condition (ii) is the base change formula in the square of forgetful morphisms[4]

$$\begin{array}{ccc} X_g^n & \to & X_g^{n-1} \\ \downarrow & & \downarrow \\ \overline{M}_g^n & \to & \overline{M}_g^{n-1} \end{array}$$

Readers may know that (ii) implies the *flatness of the identity* in the associated Frobenius manifold [M, III]. Finally, (iii) encodes the degrees (1.6) of the maps $GW_{g,\delta}^n$ (Example 7.17).

These constraints can be axiomatised in the setting of abstract DMT's, and imposing them narrows down the classification of semi-simple theories. In CohFT's, the operator E of §1.3.ii satisfies $E(z) \circ E^*(-z) = \mathrm{Id}$. The flat vacuum condition determines the $\tilde{Z}^+$ of §1.3.i from E (Proposition 3.14). Confirming a prediction of Givental's [G1], we will see that semi-simple CohFT's are determined by their *genus-zero part*, the restriction to families of genus zero curves, save for an ambiguity related to the *Hodge bundle*. (See §8.6 for the precise statement.) Homogeneous theories (iii) have no such ambiguity, and we can then give an explicit *reconstruction procedure* from the Frobenius algebra A alone and the homogeneity constraint, as we explain after reviewing the following example.

(1.8) Example: the Manin-Zograf conjecture. A simple illustration of the classification concerns the cohomological field theories of rank one[5]: $\dim A = 1$, so A is necessarily semi-simple. In this

[4] This is not altogether trivial, because the square is not Cartesian, due to the contractions of the universal curve.

[5] There is a tensor structure on the category of CohFT's for which rank one theories are the units.

case, my classification affirms an older conjecture of Manin and Zograf [MZ]: $\overline{Z}$ is an exponentiated linear combination of κ- and μ-classes, the latter being the Chern character components of the Hodge bundle. The coefficients of the μ-classes are easily related to those of $\log E(z)$: This illustrates nicely the ambiguities in reconstruction. Genus zero CohFT's of rank one are described using κ-classes alone, [M, §III.6]: the Hodge bundle is trivial in genus zero and so the μ-classes are invisible. On the other hand, the flat vacuum CohFT's are precisely those involving μ-classes only (Proposition 8.10). Imposing all three conditions in §1.7 leads to $\tilde{Z}^+ = 1$ and $E = \mathrm{Id}$, leaving only one choice: the Frobenius algebra structure on A, determined by the single complex number $\theta(1)$.

(1.9) Reconstruction from genus zero. We now outline the reconstruction result of semi-simple homogeneous CohFTs from their underlying Frobenius algebra; full details are given in §7 and §8.

For any a CohFT $\overline{Z}$, a formal construction (Definition 7.1) produces a family $\overline{Z}_u$ of CohFT's, parametrised by $u \in U$, an open (or formal) nieghbourhood of $0 \in A$. In Gromov-Witten theory, the H^2 part of this family was described in §1.5. The Frobenius algebra structure on A varies in this family and leads to a so-called *Frobenius manifold* structure on U; see §7.3 below, or [D, M, LP] for an extensive discussion. A reconstruction theorem [M] determines the genus-zero part of the CohFT from this Frobenius manifold. This fact has no known analogue for the higher-genuspart of the theory.

However, for semi-simple theories, Givental [G1] conjectured a formula for the classifying datum E from genus-zero information.[6] Specifically, he characterised E by a system of linear ODE's on U (Dubrovin's *first structure connection*), and the the homogeneity constraint §1.7.iii led to a unique solution. In the final sections of this paper, I verify the ODE's for E in the abstract setting of CohFT's (along with a companion ODE for $\tilde{Z}$) and conclude

Theorem (1)**.** *A semi-simple cohomological Field theory satisfying the homogeneity constraint 1.7.iii is uniquely and explicitly reconstructible from genus zero data. For homogeneous theories with flat vacuum, the Euler vector field and the Frobenius algebra structure suffice for reconstruction.*

Reconstruction takes the form of a recursion for the Taylor coefficients of $E(z) = \sum_k E_k z^k$. We spell this out in Gromov-Witten cohomology, when $A = H^\bullet(X)$ with the quantum cup-product at some point $u \in H^{ev}(X)$, assumed to be semi-simple. Denote by μ the linear operator $(\deg - \dim_{\mathbb{C}}(X))/2$ on A, and by $(\xi \cdot_u)$ that of quantum multiplication by the Euler vector ξ_u at u. Then, the recursion

$$[(\xi \cdot_u), E_{k+1}] = (\mu + k) \cdot E_k$$

determines $E(z)$ uniquely form $E_0 = \mathrm{Id}$. (See the proof of Theorem 8.15.) Thus, all Gromov-Witten classes $GW^n_{g,\delta} \in H^\bullet(\overline{M}^n_g)$ are constructible from $c_1(X)$ and the quantum multiplication operator $(\xi \cdot_u)$ at a single point u.

1.10 Remark. The series $E(z)$ has an interpretation already flagged by Dubrovin [D]. Namely, the formal expression $E(z) \cdot \exp(-\xi \cdot_u / z)$ gives the asymptotics at $z = 0$ of solutions of an ODE with irregular (quadratic) singularities there (see §8.1). In the case of quantum cohomology, genuine solutions have unipotent, but non-trivial monodromy around 0. (The monodromy logarithm is the operator of classical multiplication by $c_1(X)$, cf. [D], and this does not vanish for manifolds with semi-simple quantum cohomology.) Because the asymptotic formula is single-valued, it cannot represent a genuine solution and so the series $E(z)$ cannot converge. This makes the prospect of expressing E in terms of immediate geometric data of the symplectic manifold problematic, and this last question is open.

[6] The conjecture was framed in the sightly different setting of potentials, described in §1.11 below.

(1.11) Potential of a DMT. Let $\overline{Z}_g^n : A^{\otimes n} \to H^\bullet(\overline{M}_g^n)$ be the class associated by the DMT to the universal stable curve over the Deligne-Mumford space $\overline{M}_g^n$. The *primary invariants* are the integrals of the $\overline{Z}$'s on the $\overline{M}_g^n$'s. However, $\overline{M}_g^n$ also carries the Euler classes $\psi_1, \ldots, \psi_n$ of the cotangent lines to the universal curve at the marked points. More information about $\overline{Z}$ is recovered by including ψ's before integration; the resulting numbers are encoded in a generating series, the *potential*, a function of a series $x(z) = x_0 + x_1 z + \cdots \in A[\![z]\!]$:

$$\mathcal{A}(x) = \exp\left\{\sum_{g,n} \frac{\hbar^{g-1}}{n!} \int_{\overline{M}_g^n} \overline{Z}_g^n \left(x(\psi_1), \ldots, x(\psi_n)\right)\right\}; \tag{1.12}$$

the sum excludes the values $(g, n) = (0, 0), (0, 1), (0, 2)$ and $(1, 0)$ for which $\overline{M}$ is not an orbifold. While the series in (1.12) need not converge analytically, it is at least formally convergent as a power series in $\{\hbar, x^3/\hbar\}$; so its exponential is well-defined in *some* space of functions.

1.13 Example. The *trivial* 1-dimensional theory has $A = \mathbb{C}$ and $\overline{Z} = 1$ for all g and n; the integrand is $x(\psi_1) \wedge \cdots \wedge x(\psi_n)$ and $\mathcal{A}$ is the *τ-function* of Kontsevich and Witten.

More generally, any Frobenius algebra A can be coupled to the trivial cohomological field theory, by letting each ${}^q\overline{Z}_g^p$ be the degree zero class specified by any single curve of that type. The potential is then expressible in terms of Kontsevich integrals.

The potentials $\mathcal{A}_u$ corresponding to the family GW_u of Gromov-Witten cohomology theories of a compact symplectic manifold are parametrised by $u \in A = H^{ev}(X)$ (or a formal version thereof, since the convergence question seems open in general). They are known as the *ancestor GW potentials* of X. Their relation to the more customary *descendent potential*, defined using the ψ-classes and integration over the spaces $X_{g,\delta}^n$, was determined[7] by Kontsevich and Manin [KM2]. The ancestor-descendent relation was reframed by Givental in the setting of loop group actions, which we now recall.

(1.14) Givental's loop group conjecture. For clarity, let us focus here on Cohomological Field Theories ($D = \mathrm{Id}$), postponing discussion of the general case until §6. Let $\mathbf{F}(\!(\hbar)\!)$ be the space of ($\mathbb{C}(\!(\hbar)\!)$-valued) polynomials on $A[\![z]\!]$; the potentials $\mathcal{A}$ in (1.12) live in a completion of this. (The choice of completion is not important, as our constructions reduce to recursively defined operations on power series coefficients; see §6.) Define a symplectic form on the space $A(\!(z)\!)$ of formal Laurent series,

$$\Omega(x, y) = \oint \beta\left(x(-z), y(z)\right) dz,$$

using the Frobenius bilinear form β. We view $\mathbf{F}(\!(\hbar)\!)$ as a Fock representation of the Heisenberg group $\mathbf{H}$ built on $\{A(\!(z)\!), \hbar\Omega\}$. The symplectic group $\mathbf{Sp}$ on $A(\!(z)\!)$ acts (projectively) on (completions of) $\mathbf{F}(\!(\hbar)\!)$ by the intertwining metaplectic representation. The Laurent series loop group $\mathrm{GL}(A)(\!(z)\!)$ acts point-wise on $A(\!(z)\!)$. Consider the following subgroups of $\mathbf{Sp}$:

- $\mathbf{Sp}_L := \mathbf{Sp} \cap \mathrm{GL}(A)(\!(z)\!)$, the symplectic part of $\mathrm{GL}(A)(\!(z)\!)$;
- $\mathbf{Sp}_L^+ := \mathbf{Sp} \cap (\mathrm{Id} + z \cdot \mathrm{End}(A)[\![z]\!])$.

The term "symplectic loop group" is sometimes used for $\mathbf{Sp}_L$, but it really is the *twisted form* of the loop group of $\mathrm{GL}(A)$. The subgroup $\mathbf{Sp}_L^+$ contains the matrix series $E(z)$ of §1.7. In [G1, G3], Givental described the Kontsevich-Manin relation between descendent and ancestor potentials

[7] For clarification, I recall that the descendent potential carries additional information from the 1-point, or J- function, which is not contained in the CohFT, as defined.

of Gromov-Witten cohomology theory in terms of the action of $\mathbf{Sp}_L$ (without assuming semi-simplicity). In addition, he proposed (and proved, for toric Fano manifolds) a formula for the value of the GW ancestor potential $\mathcal{A}_u$ at a semi-simple point u of quantum cohomology. This was formulated in terms of the action $\mathbf{Sp}_L^+$, using ingredients from Dubrovin's isomonodromy description [D] of semi-simple Frobenius manifolds.

Call A^{DM} the subspace of those vectors in the cohomology $\prod_{g,n} H^\bullet(\overline{M}_g^n; (A^*)^{\otimes n})$ of all Deligne-Mumford spaces which are invariant under the symmetric groups. A DM field theory $\overline{Z}$ defines a vector in A^{DM}, and is in turn completely determined by it. There is a vector $I_A \in \prod H^0$ representing the trivial theory based on A. Let $\mathbf{H}^+, \mathbf{H}^{++}$ denote the subspaces $zA[\![z]\!]$ and $z^2A[\![z]\!]$ in the Heisenberg group $\mathbf{H}$, acting on $\mathbf{F}((\hbar))$ by translation. In §6, we describe an action of $\mathbf{Sp}_L^+ \ltimes \mathbf{H}^+$ on A^{DM}, which lifts the metaplectic and translation actions on potentials. (A construction along similar lines was alluded to in [CKS].) Let T_x denote the translation by $x \in \mathbf{H}^+$, $(T_x\mathcal{F})(y) = \mathcal{F}(y-x)$, and write T_z short for $T_{z\mathbf{1}}$, for the unit $\mathbf{1} \in A$. The classification of DMT's will imply the following.

Theorem (2). *The CohFT's with underlying semi-simple Frobenius algebra A constitute the $\mathbf{Sp}_L^+ \ltimes \mathbf{H}^{++}$-orbit of the trivial theory I_A. The theories with flat vacuum form the orbit of the subgroup $T_z \circ \mathbf{Sp}_L^+ \circ T_z^{-1}$.*

The element of $\mathbf{Sp}_L^+ \ltimes \mathbf{H}^{++}$ taking I_A to the theory with data $\{A, E(z), \tilde{Z}^+\}$ in the classification of §1.3 is $E(z) \cdot \zeta$, with

$$\zeta = z \exp\left(-\sum\nolimits_{j>0} a_j z^j\right) - z \in \mathbf{H}^{++}.$$

This formula is closely related to the coordinate changes studied by Kabanov and Kimura[8] [KK].

Note that our ζ contains no z-linear term. Adding a term $\zeta_1 z$, with $\zeta_1 = \sum_i \zeta_{i1} P_i$ turns out to change the structure constants θ_i of A, scaling them by $(1+\zeta_{i1})^2$ (Proposition 6.13). Every complex semi-simple Frobenius algebra can be obtained in this way from a sum of copies of the trivial one, $\mathbb{C}$ with $\theta(1) = 1$. It is tempting to say that all semi-simple CohFT of the same rank constitute a single $\mathbf{Sp}_L^+ \ltimes \mathbf{H}^+$-orbit, except that there is trouble when some $\zeta_{i1} = -1$: in other words, the action of the linear modes $zA \in \mathbf{H}^+$ on A^{DM} has some singularities, so this re-formulation of the first part of Theorem 2 requires some care.

Translation by z is the *dilaton shift* of the literature; it encodes the expression of ζ from E in flat vacuum theories. With a general vacuum $\mathbf{v}(z)$ (as in §3.12), we are instead looking at the set $T_{z\mathbf{v}(z)} \circ \mathbf{Sp}_L^+ \circ T_z^{-1}(I_A)$; cf. §6.18. Even more generally, abandoning the CohFT condition to allow $D \neq \mathrm{Id}$ enlarges the space of DM theories to the orbit of a larger subgroup $\mathbf{Sp}^+ \subset \mathbf{Sp}$; this requires a slightly different setup and will be discussed in §6, where the proof of Theorem 2 is completed.

1.15 Remark. The translation action of $\mathbf{H}^+$ on the space of CohFT's has an analogue for the zero-modes $A \in A[\![z]\!]$: this leads to the Frobenius manifold mentioned in §1.9. The interaction with the group $\mathbf{Sp}_L^+ \ltimes \mathbf{H}^+$ is rather complicated, given by a system of ODE's we derive in §7.4. For instance, A-translations and $\mathbf{H}^+$-translations do not commute. In the setting of open-closed theories, translation along the Frobenius manifold and that by $\mathbf{H}^+$ correspond to deformations of the TFT coming from independent sources: to wit, deformation of the category of boundary states, versus deformation of the cyclic trace.

2. Field Theories from universal classes

We now review the definitions of FTFTs and reframe them from the point of view of moduli spaces of oriented surfaces. In discussing the latter, we may switch between topological, smooth, metric

[8] I am grateful to V. Tonita for pointing this out.

and Riemann surfaces as convenience dictates, because these structures are related by contractible spaces of choices (the spaces of Riemannian metrics, or metrics up to conformal equivalence), so their classifying spaces — the bases of universal surface bundles — are homotopy equivalent. Similarly, we can describe boundary circles (parametrised or not) more economically as follows.

(2.1) Points versus boundaries. Call a surface (m,n)-pointed if it carries a set of $m+n$ distinct unordered points, separated into *m incoming* and *n outgoing* ones. Given a vector space A, the base X of an (m,n)-pointed surface bundle Σ_X carries local systems $A^{(m)}, A^{(n)}$ with fibres $A^{\otimes m}, A^{\otimes n}$, permuted by the monodromy in the base. Removing open disks centred at the special points shows that, up to homotopy in the family X, the points contain the same information as un-parametrised boundary circles. Moreover, since $\mathrm{Diff}_+(S^1)$ is homotopy equivalent to its subgroup of rigid rotations, we may capture the parametrisation information, up to a contractible space of choices, by specifying unit tangent vectors. More precisely, there is a torus bundle $\tilde{X} \twoheadrightarrow X$ with fibre $\mathbb{T}^m \times \mathbb{T}^n$, the product of unit tangent spaces at the special points.[9] Up to homotopy, $\tilde{X}$ parametrises the surfaces in the family Σ_X, together with all parametrisations of their boundary circles.

(2.2) FTFT's reviewed. Let us spell out the functorial definition of FTFT's. We will then convert the information to a collection of cohomology classes of the classifying spaces of surface bundles. This is especially necessary for DMT's, since we have yet to formulate the two additional axioms mentioned in §1.3.iv, the nodal factorisation and vacuum axioms.

- A *family TFT with fixed boundaries* and coefficients in A assigns to each family $\Sigma_X \mapsto X$ of closed oriented (m,n)-pointed surfaces a class

  $$\tilde{Z}(\Sigma_X) \in H^\bullet(\tilde{X}; \mathrm{Hom}(A^{(m)}, A^{(n)})).$$

  This must be functorial in $\tilde{X}$ and subject to the condition that sewing together any collection of incoming-outgoing boundary pairs gives the corresponding composition of linear maps.
- In a *free boundary FTFT*, the class $Z(\Sigma_X)$ lives in $H^\bullet(X; \mathrm{Hom}(A^{(m)}, A^{(n)}))$, is functorial in X, and the sewing condition must hold for *any* given identification over X of an incoming-outgoing boundary pair.
- A *Lefschetz* FTFT assigns such $\overline{Z}$'s functorially to (chiral) Lefschetz fibrations of closed oriented pointed surfaces.
- Finally, a *Deligne-Mumford FTFT* is a Lefschetz FTFT for *stable* surfaces, satisfying a *nodal factorisation rule* and a *vacuum axiom*, which we describe in §2.9 and §2.13 below, after introducing the universal classes ${}^p\overline{Z}^q$.

2.3 Remark. (i) Single surfaces define a commutative Frobenius algebra structure on A.

(ii) "Sewing" of pointed surfaces in a family is well-defined, up to homotopy, from an identification of tangent spaces at the matched points.

(iii) As usual, nodes and special points must avoid each other.

(iv) *Stability* of surfaces is needed for an *orbifold* description of the moduli of nodal surfaces. We include it mainly to connect with the standard Cohomological Field Theory approach. The classification of semi-simple theories remains unchanged for Lefschetz theories, which allow *pre-stable curves*. The benefit of including pre-stable curves is unclear, except as a conceptual bridge between smooth surface theories and DMT's.

[9] $\tilde{X}$ is a *principal* bundle only if there is no monodromy, that is, the special points can be ordered over X.

(2.4) Nodal factorisation rule. Every Lefschetz theory carries a *nodal factorisation* rule, which describes the class $\overline{Z}$ for the nodal surface family obtained by attaching two surfaces at marked points. This rule is a consequence of the smooth surface sewing condition: simply cut out the pair of crossing disks near the node. For a pair of marked points of opposite type, the relevant operator in the universal family is the nodal propagator $Z(\supset\subset) = D(-\omega_+, \omega_-)$ of §1.3.iii. Similarly, the effect of attaching two marked points of the same type into a node is controlled by a bilinear form B on A with values in $k[\![\omega_\pm]\!]$ (outgoing disks case), respectively by a co-form $C \in (A \otimes A)[\![\omega_\pm]\!]$ (incoming case). See §2.7 below for the factorisation rules spelt out. The tensors B, C and D are not independent: a formal game with connecting elbows shows in fact that each of them determines the other two; we will write out the formulae in §4.4 below.

The pair of crossing disks is an unstable surface, so the cutting argument above is *not* permitted in a DMT. Therefore, the nodal factorisation rules form an additional axiom for a DMT: B, C and D must be specified, and must determine each other by the formulae in §4.4. We spell out the axiom for B in terms of Deligne-Mumford spaces in (2.11) below.

(2.5) Reformulation using universal classes. Let ${}^qM_g^p$ denote the classifying space of the universal surface with $p+q$ distinct ordered points, and denote by ${}^q\tilde{M}_g^p$ (or alternatively, ${}_qM_{g,p}$, as is common in the literature) the principal torus bundle defined by the choices unit tangent vectors at those points. Functoriality reduces a fixed-boundary FTFT to the specification of universal classes

$${}^q\tilde{Z}_g^p \in H^\bullet_{\mathfrak{S}_p \times \mathfrak{S}_q}\left({}^q\tilde{M}_g^p; \mathrm{Hom}(A^{\otimes p}; A^{\otimes q})\right),$$

where the symmetric groups $\mathfrak{S}_p, \mathfrak{S}_q$ act on ${}^qM_g^p$ by permuting marked points and simultaneously on $A^{\otimes p,q}$ by permuting the factors. Over $\mathbb{C}$, equivariance under finite groups simply means invariance. With free boundary theories, we obtain classes ${}^qZ_g^p$ over ${}^qM_g^p$, and in the case of DM theories, ${}^q\overline{Z}_g^p$ over the Deligne-Mumford compactifications ${}^q\overline{M}_g^p$. The FTFT axioms translate into the sewing, or nodal factorisation rules for these classes, described in a moment.

2.6 Remark. The classifying space for the universal Lefschetz fibration has a less familiar model of a finite-dimensional Artin stack ${}^q\overline{A}_g^p$ of infinite type, classifying nodal curves with arbitrary chains and trees of rational curves. This has an infinite descending normal-crossing stratification, reflecting the unlimited bubbling that can occur in families.

(2.7) Sewing condition. Sewing two specified boundary components together defines maps (with $x = x' + x''$ for $x = g, p, q$)

$$s : {}^{q'}\tilde{M}_{g'}^{p'} \times {}^{q''}\tilde{M}_{g''}^{p''} \to {}^{q-1}\tilde{M}_g^{p-1}, \tag{2.8}$$

and similar maps where several pairs of boundaries are simultaneously identified. (There are also self-sewing maps for single surfaces, but these can be expressed by sewing on elbows.) The FTFT sewing condition is

$$s^*\left({}^{q-1}\tilde{Z}^{p-1}\right) = {}^{q'}\tilde{Z}^{p'} \circ {}^{q''}\tilde{Z}^{p''},$$

with composition of the appropriate entries.

Free boundary FTFT's are different, in that the sewing maps (2.8) does not descend to the base moduli spaces M, M', M'' for surfaces with free boundaries: sewing requires an identification of the boundaries. A natural circle bundle $\pi : \partial N \twoheadrightarrow M' \times M''$ represents the space of possible identifications. This ∂N is also (the pull-back to $M' \times M''$ of) the circular neighbourhood of the divisorial boundary stratum $b_2(M' \times M'')$ in in $\overline{M}$. Functoriality stipulates now that the pull-back class $\pi^*({}^{q'}Z_{g'}^{p'} \times {}^{q''}Z_{g''}^{p''})$, after contracting the $A \otimes A^*$ factor from the two sewing indices, agrees with the restriction of ${}^{q-1}Z_g^{p-1}$ to ∂N. There is a matching condition for b_1.

(2.9) Deligne-Mumford factorisation rules. On Deligne-Mumford spaces, a composition law similar to the maps (2.8) is provided by the following boundary morphisms, defined by identifying two marked points on distinct surfaces (respectively, on the same surface) into a node:

$$\begin{aligned} b_2 &: {}^{q'}\overline{M}_{g'}^{p'} \times {}^{q''}\overline{M}_{g''}^{p''} \to {}^{q-1}\overline{M}_{g}^{p-1}, &\qquad b_1 &: {}^{q}\overline{M}_{g}^{p} \to {}^{q-1}\overline{M}_{g+1}^{p-1}, \\ b_2' &: {}^{q'}\overline{M}_{g'}^{p'} \times {}^{q''}\overline{M}_{g''}^{p''} \to {}^{q}\overline{M}_{g}^{p-2}, &\qquad b_1' &: {}^{q}\overline{M}_{g}^{p} \to {}^{q}\overline{M}_{g+1}^{p-2}, \\ b_2'' &: {}^{q'}\overline{M}_{g'}^{p'} \times {}^{q''}\overline{M}_{g''}^{p''} \to {}^{q-2}\overline{M}_{g}^{p}, &\qquad b_1'' &: {}^{q}\overline{M}_{g}^{p} \to {}^{q-2}\overline{M}_{g+1}^{p}. \end{aligned} \tag{2.10}$$

The morphisms differ from each other only in the types of attaching points. DMT's must satisfy a twisted nodal factorisation rule for the $\overline{Z}$, which involves contraction with $D(\psi',\psi'')$, $C(\psi',\psi'')$, respectively $B(\psi',\psi'')$, with the two ψ-classes at the node. Thus, for $b_2'' : {}^{1}\overline{M}_{g'}^{n'} \times {}^{1}\overline{M}_{g''}^{n''} \to \overline{M}_{g}^{n}$,

$$b_2''^{*}\overline{Z}_g^n = B(\psi',\psi'')\left({}^{1}\overline{Z}_{g'}^{n'} \otimes {}^{1}\overline{Z}_{g''}^{n''}\right), \tag{2.11}$$

and similarly for the other maps. Such factorisation rules appear in generalised-cohomology Gromov-Witten theory [CG], although the dependence on ψ',ψ'' has a very special form there (B, C and D are scalars). In the more familiar case of CohFT's, we impose $D = \mathrm{Id}$, $B = \beta$, and C is the inverse co-form.

2.12 Remark. (i) This discussion also applies to the stacks $\overline{A}$ classifying Lefschetz fibrations.

(ii) The nodal factorisation law, when lifted from $M' \times M''$ to ∂N, becomes precisely the smooth surface sewing axiom, by virtue of the identity $D(-\omega,\omega) = \mathrm{Id}$. In particular, Lefschetz theories restricted to smooth surfaces give free boundary FTFT's.

(2.13) Vacuum axiom. The final DMT axiom is the existence of a "vacuum" vector $\mathbf{v}(z) \in A[\![z]\!]$; in a Lefschetz theory, this is defined by the universal sphere with a single output. This vector must satisfy the following condition: let $\varphi : \overline{M}_g^{n+1} \to \overline{M}_g^n$ be the morphism of Deligne-Mumford spaces forgetting the last marked point.

$$\overline{Z}_g^{n+1}\big(x_1,\ldots,x_n,\mathbf{v}(\psi_{n+1})\big) = \varphi^*\overline{Z}_g^n.$$

We will concentrate later on the special class of theories with flat vacuum, when $\mathbf{v} = \mathbf{1}$.

(2.14) PROP description. The sewing maps (2.8) assemble to a so-called *PROP structure* on the spaces ${}^{q}\tilde{M}_g^p$, which carries over to their homology. In this language, an FTFT structure $\tilde{Z}$ on A is equivalent to that an algebra structure over the homology PROP. Similarly, the Deligne-Mumford boundary morphisms (2.10) give a PROP structure on $H_*(\overline{M}_g)$; self-sewing of single surfaces enhances this to a *wheeled PROP* (a notion introduced in [MMS]). Cohomological field theories are algebras over the associated homology PROP of DM spaces; to capture the entire CohFT structure, we must add a *cyclic* structure, permuting inputs and outputs. (We lack the ability to switch inputs and outputs by means of elbows.) DMT's with general D are algebras over a twisted form of the DM homology PROP.

Free boundary FTFT's do not fit into PROP language, for the reason explained earlier.

(2.15) Tautological classes. The classification will describe the various field theories in terms of the *tautological classes* on the moduli of surfaces. We briefly recall the generating tautological classes on $\overline{M}_g^n$; those on $M_g^n, \tilde{M}_g^n$ are obtained by restriction. Let $\varphi : \overline{M}_g^{n+1} \to \overline{M}_g^n$ be the map forgetting

the last marked point. The marked points define n sections σ_i of φ, with smooth divisors $[\sigma_i]$ as their images. Let T^*_φ be the relative cotangent complex of φ and define

$$\psi_i := \sigma_i^* c_1(T^*_\varphi), \qquad \kappa_j = \varphi_*\left(\psi_{n+1}^{j+1}\right),$$

(where ψ_{n+1} on $\overline{M}_g^{n+1}$ is defined using σ_{n+1} and $\overline{M}_g^{n+2}$). These classes satisfy the relations

$$\begin{gathered}[\sigma_i]\cdot\psi_i = [\sigma_i]\cdot\psi_{n+1} = 0,\\ \psi_i^k = \varphi^*\psi_i^k + \sigma_{i*}(\psi_i^{k-1}), \qquad \kappa_j = \varphi^*\kappa_j + \psi_{n+1}^j.\end{gathered}$$

The correction term $\sigma_{i*}(\psi_i^{k-1})$ in the first relation is only visible on $\overline{M}_g^n$, but the one for κ also appears on M_g^n. Thus defined, the κ_j are *primitive*: that is, under the boundary maps (2.10),

$$b_2^*(\kappa_j) = \kappa_j' + \kappa_j'', \qquad b_1^*(\kappa_j) = \kappa_j.$$

Additional tautological classes on Deligne-Mumford spaces arise by the recursive pushing forward polynomials in the $\kappa-$ and $\psi-$classes from boundary divisors.

(2.16) The stability theorems. The key to the classification are two stability theorems, due to Harer [H] (later improved by Ivanov [I]), and to Madsen and Weiss [MW], respectively. For the first theorem, let $M_{g,m}^n$ be the base family of the universal surface of genus g with $m+n$ ordered points, equipped with unit tangent vectors at the first m special points.

2.17 Theorem ("Harer stability" [H, I]). *The maps $M_{g,m}^n \to M_{g,m-1}^n$ and $M_{g,m}^n \to M_{g+1,m}^n$, defined by sewing in a disk, respectively by sewing on a two-holed torus, induce homology isomorphisms in degree less than $(g-1)/2$.*

An important consequence describes the homological effect of adding marked points:

2.18 Corollary (Looijenga, [L]). *In the stable range of total degree $< (g-1)/2$, we have*

$$H^\bullet(M_g^n;\mathbb{Q}) \cong H^\bullet(M_g;\mathbb{Q})[\psi_1,\dots,\psi_n].$$

We reproduce the easy proof. The circle bundle $\pi : M_{g,1} \twoheadrightarrow M_g^1$ presents $H^\bullet(M_{g,1};\mathbb{Q})$ as the cohomology of the differential graded ring $\{H^\bullet(M_g^1;\mathbb{Q})[\eta], d\}$, with $\deg\eta = 1$ and $d\eta = \psi_1$. Now, thanks to Harer, $\pi^* : H^\bullet(M_g^1;\mathbb{Q}) \to H^\bullet(M_{g,1};\mathbb{Q})$ must surject in the stable range, since the forgetful pull-back $H^\bullet(M_g;\mathbb{Q}) \to H^\bullet(M_g^1;\mathbb{Q})$ splits it there. But then, ψ_1 is not a zero-divisor in that range: if $\psi_1 x = 0$, then ηx is a class and is not in the image of π^*. Repeat for the other ψ.

2.19 Theorem ("Mumford conjecture" [MW]). *In the stable range, we have*

$$H^\bullet(M_g;\mathbb{Q}) = \mathbb{Q}[\kappa_j], \quad j = 1,2,\dots.$$

(2.20) Primitive and group-like classes. We conclude by spelling out the role of κ-classes in our context. Genus-stabilisation $M_{g,m}^n \to M_{g+1,m}^n$ defines a limiting homotopy type $M_{\infty,m}^n$. This agrees with the classifying space of the *stable mapping class group* $\Gamma_{\infty,m}^n$ of a surface with m fixed and n free boundaries. Harer stability makes the fixed boundaries invisible in the homology of the classifying space, while the homological effect of free boundaries is described by Corollary 2.18; so

we focus on $M_{\infty,1}$. Sewing two surfaces, with one fixed boundary each, into a fixed pair of pants defines a map

$$m : M_{g,1} \times M_{h,1} \to M_{g+h,1}, \tag{2.21}$$

which gives a homotopy-commutative monoidal structure on $\coprod_g M_{g,1}$ and, in the limit, on $M_{\infty,1}$. The latter becomes a group-like topological monoid, and its cohomology $H^\bullet(M_{\infty,1};\mathbb{Q})$ acquires a (commutative and co-commutative) Hopf algebra structure. By the Milnor-Moore theorem, this must be the free power series algebra in the *primitive* cohomology classes, that is, the classes x satisfying $m^*(x) = x \otimes 1 + 1 \otimes x$. The κ's do have that property (§2.15), so the Madsen-Weiss theorem has the following important consequence.

2.22 Corollary. *All primitive rational cohomology classes on $M_{\infty,1}$ are linear combinations of the κ's.*

2.23 Remark. Corollary 2.22 is equivalent to the rational Mumford conjecture. Madsen and Weiss prove an *integral* version, identifying the homotopy type of the group completion of the topological monoid $\amalg M_g$ with the infinite loop space $\Omega^\infty\mathbb{CP}^\infty_{-1}$ of the *Madsen-Tillmann spectrum* [MT]. An integral, in fact spectrum version of Looijenga's theorem was found earlier by Bödigheimer and Tillmann [BT].

Another important notion is that of a *group-like* class $X \in H^\bullet(M_{\infty,1};\mathbb{Q})$, a non-zero class for which $m^*X = X \otimes X$. It is easy to see that the group-like classes are precisely those of the form $\exp(x)$, with primitive x.

3. Smooth surface theories

Armed with the boundary maps between the M_g and the tautological classes, we proceed to classify FTFT's of the first two types, involving smooth surfaces with parametrised or with free boundaries. This might be the place to confess to a minor gap in the classification: the definitions do not seem to determine the value of the universal $\tilde{Z}_g$ without marked points, although a valid choice can always be made from my data. For free boundaries, the same ambiguity applies to Z_1^0. The genus one problem persists for Lefschetz theories, but not for DMT's, since $\overline{M}_1$ does not exist.

(3.1) Fixed boundary theories. With $g = g' + g''$, consider the effect on the class ${}^1\tilde{Z}_{g'} \in H^\bullet({}^1\tilde{M}_{g'};A)$ of the operation of sewing onto the general surface of genus g' a fixed 2-holed surface of genus g'':

$${}^1\tilde{Z}_g = \alpha^{g''} \cdot {}^1\tilde{Z}_{g'} \quad \text{on } {}^1\tilde{M}_{g'}$$

where $\alpha \in A$ is the Euler class of §0.4 and the left-side class has been restricted to ${}^1\tilde{M}_{g'}$. When α is invertible, it follows that $\alpha^{-g} \cdot {}^1\tilde{Z}_g$ stabilises, as $g \to \infty$, to a class $\tilde{Z}^+ \in H^\bullet(M_\infty;A)$. (The superscript "+" is here to flag the fact that we have omitted the κ_0-contribution to $\tilde{Z}$, as in §1.3.i.) The sewing axiom, applied to the multiplication map (2.21) and corrected by the same power $\alpha^{-(g+h)}$ on both sides, implies that $\tilde{Z}^+$ is *group-like*. It follows that

$$\tilde{Z}^+ = \exp\left\{\textstyle\sum_{j>0} a_j\kappa_j\right\}, \quad \text{for certain } a_j \in A.$$

3.2 Remark. Integrally, $\tilde{Z}^+$ would be a group-like class in the A-valued cohomology of $\Omega^\infty\mathbb{CP}^\infty_{-1}$. Additively, there exist additional primitive classes, the Dyer-Lashof descendants of the κ's [T]; quite likely, the analogue holds for group-like classes as well. The new classes could be ruled out by imposing the FTFT axioms at *chain level*; then, $\tilde{Z}^+$ would be a class in the cohomology of the spectrum $\mathbb{CP}^\infty_{-1}$ with generalized coefficients (GL_1 of cohomology with values in A).

Clearly, ${}^1\tilde{Z}_g$ is the restriction to ${}_1M_g$ of $\alpha^g \cdot \tilde{Z}^+$; let us find ${}^m\tilde{Z}^n_g$. Sewing on large genus surfaces to one boundary lets us assume g is large without loss of information. Map now ${}^1\tilde{M}_g$ to ${}^m\tilde{M}^n_g$ by sewing on to the universal surface a fixed sphere with $n+1$ inputs and m outputs. This sphere represents the map ${}_mS_{n+1} : A^{\otimes(n+1)} \to A^{\otimes m}$, multiplication to A followed by co-multiplication $A \to A^{\otimes m}$. Thanks to Harer, the map ${}^1\tilde{M}_g \to {}^m\tilde{M}^n_g$ is a homology equivalence in a range, so we detect ${}^m\tilde{Z}^n_g$ by pulling back to ${}^1\tilde{M}_g$, where we see the result of feeding ${}^1\tilde{Z}_g$ as one of the inputs in ${}_mS_{n+1}$: thus,

$$ {}^m\tilde{Z}^n_g = {}_mS_1(\alpha^g \cdot \tilde{Z}^+ \cdot {}_1S_n) $$

and we conclude the desired classification, with $a_j = \sum_i a_{ij}P_i$:

3.3 Proposition. *If m and n do not both vanish, then, in the tensor monomials of the normalised canonical basis, ${}^m\tilde{Z}^n_g$ is diagonal, with all entries null save for the ones relating $p_i^{\otimes n}$ to $p_i^{\otimes m}$, which are*

$$ \theta_i^{\chi/2} \cdot \exp\left\{\sum\nolimits_{j>0} a_{ij}\kappa_j\right\}. $$

3.4 Remark. The argument fails when $m = n = 0$, and the axioms don't seem to determine $\tilde{Z}_g$ for closed surfaces, except in the stable range of homology (we can detect that by lifting to $M_{g,1}$). One valid choice is given by summing the classes in (3.3) over i.

(3.5) Free boundaries and E. Restricting to surfaces with fixed boundaries determines a $\tilde{Z}$ as above. Let now ${}^1Z_{g,1}$ denote the lift of ${}^1Z^1_g$ to ${}^1M_{g,1}$. Recall that the latter is a circle bundle over ${}^1M^1_g$ and classifies surfaces with a fixed incoming boundary and a free outgoing one. Sewing a fixed surface of genus g'' into the fixed incoming boundary of the general surface over ${}^1M_{g',1}$ tells us that

$$ {}^1Z_{g,1} \in H^\bullet\left({}^1M_{g,1}; \mathrm{End}(A)\right) \quad \text{restricts to} \quad {}^1Z_{g',1} \circ (\alpha^{g''}\cdot\,) \in H^\bullet\left({}^1M_{g',1}; \mathrm{End}(A)\right), $$

with $(x\cdot\,)$ denoting the operator of multiplication by $x \in A$. Again, we get a stable class

$$ {}^1Z_1^+(\kappa, \psi_+) := {}^1Z_{g,1} \circ (\alpha^{-g}\cdot\,) \in H^\bullet\left({}^1M_{g,1}; \mathrm{End}(A)\right) \quad \text{as} \quad g \to \infty, \tag{3.6} $$

minding that the cohomology ring is freely generated by the κ_j ($j > 0$) and the class ψ_+ at the outgoing point. Similarly, switching the roles of the boundary circles defines a stable class

$$ {}_1Z^{+,1}(\kappa, \psi_-) := \lim_{g\to\infty} (\alpha^{-g}\cdot\,) \circ {}_1Z^1_g \tag{3.7} $$

Setting the κ's to 0 in (3.6) defines a formal Taylor series $E(-\psi) := {}^1Z_1^+(0, \psi) \in \mathrm{End}(A)[\![\psi]\!]$.

3.8 Lemma. *We have*

$$ {}^1Z_1^+(\kappa, \psi_+) = E(-\psi_+) \circ (\tilde{Z}^+(\kappa)\cdot\,) \quad \textit{and} \quad {}_1Z^{+,1}(\kappa, \psi_-) = (\tilde{Z}^+(\kappa)\cdot\,) \circ E(\psi_-)^{-1}. $$

More generally, in any genus g,

$$ {}^1Z^1_g = E(-\psi_+) \circ (\tilde{Z}^+(\kappa) \cdot \alpha^g\cdot\,) \circ E^{-1}(\psi_-). $$

Proof. Modify the sewing above by letting *both* surfaces vary, while the sewing circle rotates freely. This takes place over

$$ \partial N = {}^1M_{g',1} \underset{\mathbb{T}}{\times} {}_1M^1_{g''} $$

where the circle $\mathbb{T}$ simultaneously rotates the two boundaries being sewn together. The sewn surface is classified by a map $\partial N \to {}^1M^1_g$. Pull-backs to ∂N being understood, we have

$$ {}^1Z^1_{g'} \circ {}^1Z^1_{g''} = {}^1Z^1_g. \tag{3.9} $$

In a moment, we will proceed by fixing the incoming or outgoing boundaries, as convenient. In any case, $\partial N \to {}^1M^1_{g'} \times {}^1M^1_{g''}$ is a circle bundle with Chern class $-(\psi' + \psi'')$, using the ψ-classes at the node. On the total space ∂N, $\psi'' = -\psi'$, the common value representing the Euler class of the sewing circle. The Leray sequence and our knowledge of stable cohomology show that $H^\bullet(\partial N)$, below degree $g''/2$, is freely generated over $H^\bullet({}^1M^1_{g'})$ by the κ''_j. Similarly, it is freely generated over $H^\bullet({}^1M^1_{g''})$ by the κ'_j, below degree $g'/2$. Let now both g' and g'' be as large as needed, and lift (3.9) to ${}_1M_{g,1}$; we obtain from (3.6) and (3.7), after cancelling powers of α:

$$ {}_1Z^{+,1}(\kappa', \psi') \circ {}^1Z^+_1(\kappa'', -\psi') = \tilde{Z}^+(\kappa). \tag{3.10} $$

Using the relation $\kappa = \kappa' + \kappa''$ and the algebraic independence of κ', κ'', ψ', we obtain the second formula in the lemma by setting $\kappa'' = 0$, and from there, the first formula by setting $\kappa' = 0$.

For the final and more general formula, return to (3.9) and let only g'' be large. Fixing the incoming circle leads to

$$ {}^1Z^1_{g'} = {}^1Z_{g,1} \circ \left({}^1Z_{g'',1}\right)^{-1} $$

with both factors on the right now known. Minding that $\psi' = -\psi''$ gives the formula. □

3.11 Proposition. *For $(g, m, n) \neq (1, 0, 0)$, we obtain ${}^nZ^m_g$ as follows: each input is transformed by $E^{-1}(\psi)$, with the respective ψ class; the product of these is multiplied by $\alpha^g \cdot \tilde{Z}^+(\kappa)$, the result is co-multiplied out to $A^{\otimes n}$, where each factor is transformed by the respective $E(-\psi)$. The unit $\mathbf{1}$ substitutes for the empty of inputs, and the Frobenius trace θ is applied if there is no output.*

Proof. If there is at least one marked point, repeat the final argument above: for each output or input point, compose with a large-genus ${}_1Z^1_G$ or ${}^1Z_{G,1}$, respectively, to arrive at the known operator ${}_mZ_{>G,n}$. Since ${}^1Z^1_G$ is invertible and known, we are done. The case $m = n = 0$ needs an extra argument. Pull back Z_g along the forgetful map $\varphi : M^1_g \to M_g$. The closed surface bundle splits over M^1_g into an open surface and a disk sewed along their common (moving) boundary, and we can compute $\varphi^* Z_g$ from the known formulae to get the desired

$$ \varphi^* Z_g = \theta\left(\alpha^g \cdot \tilde{Z}^+(\varphi^*\kappa)\right), $$

from the primitivity of κ-classes. (More precisely, the κ-classes of the unpointed disk precisely undo the $\varphi^*\kappa$-correction of §2.15; see the discussion of the vacuum below for more help). Now, for any $g \neq 1$, φ^* is split in rational cohomology by integration down to M_g against ψ, so we recover Z_g as hoped. □

(3.12) The vacuum. The universal disk with outgoing boundary defines the *vacuum vector* $\mathbf{v}(z) \in A[\![z]\!]$, where we take z to be the *opposite* of the boundary Euler class (and of the ψ_{out} at the output, in the pointed sphere model). Capping a boundary in the universal surface with a disk shows that

$$ {}^mZ^{n+1}_g(\mathbf{v}(\psi), \dots) = \varphi^{*m}Z^n_g(\dots), \tag{3.13} $$

for the map $\varphi : {}^pM^{q+1}_g \to {}^pM^q_g$ forgetting the first input point, and with the ψ-class there.

Fixing the disk shows that $\mathbf{v} \equiv \mathbf{1} \pmod z$. We say that that Z has *flat vacuum* if in fact $\mathbf{v} = \mathbf{1}$.

3.14 Proposition. *In a semi-simple free boundary FTFT, the vacuum is given by*

$$\mathbf{v}(z) = E(z)\left(\exp\{-\textstyle\sum_{j>0} a_j z^j\}\right),$$

and so Z has a flat vacuum precisely when $\exp\left\{-\sum_{j>0} a_j z^j\right\} = E(z)^{-1}(\mathbf{1})$.

Proof. This is the formula in Lemma 3.8 together with the equality $\kappa_j = -(-\psi_{out})^j$ on ${}^1M_0$. One way to see the latter is to use the correction formula in §2.15 for the pull-back to ${}^1M_0^1$, on which space all κ's vanish and the two ψ-classes are opposite. □

3.15 Remark. (i) If we limit ourselves to *stable* surfaces, the discussion in this section continues to apply, except for the definition of the vacuum and the consequent determination of Z_g^0. In semi-simple theories, we can recover the value of $\mathbf{v}$ from large-genus surfaces, using the contraction formula (3.13) and invertibility of $\tilde{Z}^+$. This helps explain why there will be no classification distinction later between semi-simple Lefschetz and DM theories.

(ii) Unlike Harer stability and Looijenga's result on ψ-classes, the Mumford conjecture has not seriously been used: the κ's could have been replaced by the primitives in the Hopf algebra $H^\bullet(M_{\infty,1})$. However, later on, unknown primitive classes would break the argument for reconstruction from genus 0.

4. Lefschetz theories: construction

Restricting a Lefschetz theory to the open moduli spaces M_g^n gives a free-boundary theory and determines $\tilde{Z}^+$ and E as before, but a new parameter D arises from the universal pair of crossing disks; this controls the behaviour of classes at the boundaries of $\overline{M}_g^n$, and the analogous boundary strata of the stacks $\overline{A}_g^n$. In addition to the Frobenius algebra A, we thus have

(i) the class $\tilde{Z}^+ = \exp\left\{\sum_{j>0} a_j\kappa_j\right\}$ of §3.1,

(ii) the Taylor series $E(z) = \mathrm{Id} + zE_1 + z^2E_2 + \cdots \in \mathrm{End}(A)[\![z]\!]$ of §3.5,

(iii) the "nodal propagator" $\overline{Z}(\supset\subset) = D(-\omega_+, \omega_-)$ of §2.4.

Ingredients (ii) and (iii) are subject to constraints we will spell out in a moment.

I will construct a Lefschetz theory based on these parameters. Of course, restriction to stable curves gives a DMT. Unlike the proof of uniqueness in the next section, the construction does not assume the semi-simplicity of A. We switch from the use of Euler classes ω of boundary circles to the ψ-classes *at the node*, and must mind the signs: $\omega = -\psi$ for an outgoing disk, but $\omega = \psi$ at an incoming one. We use z's to denote universal ψ classes.

(4.1) Constraints on D and E. We have seen in §1.3.iii that $D(z_+, z_-) = \mathrm{Id}$ when $z_+ + z_- = 0$. In addition, the pairing

$$B : A \otimes A \to k[\![z_{1,2}]\!]$$

defined by two disks with incoming boundaries and crossing at their centres, must be symmetric under simultaneous swap of the A factors and nodal ψ-classes $z_{1,2}$. This pair of crossing disks can be constructed from $\Subset$ and $\supset\subset$, so B can be expressed from D, E and β. To simplify notation, use the Frobenius quadratic form β to express quadratic tensors in terms of endomorphisms and define B' by $\beta(a_1, B'(a_2)) = B(a_1 \otimes a_2)$. We then have

$$B'(z_1, z_2) = E^{-1}(-z_1)^* \circ E^{-1}(z_1) \circ D(z_1, z_2), \tag{4.2}$$

and this must satisfy $B'(z_2, z_1) = B'(z_1, z_2)^*$. The same symmetry constraint applies to the co-form $C \in (A \otimes A)[\![z_{1,2}]\!]$ defined by the outgoing crossing disks, but in fact this is equivalent to symmetry of B: C is expressed from B and two factors of $Z(\Supset)$, symmetrically applied to the two arguments. At any rate, defining C' by $\beta(a_1, C'(a_2)) = \beta^{\otimes 2}(a_1 \otimes a_2, C)$ leads to

$$C'(z_1, z_2) = D(z_2, z_1) \circ E(-z_1) \circ E^*(z_1). \tag{4.3}$$

Note that, while E is only defined in a semi-simple theory, the composition $E^{-1}(z)^* \circ E^{-1}(-z)$ is always defined in terms of $Z(\Subset)$ and the Frobenius bilinear form β.

(4.4) Relating B, C and D. In a Lefschetz theory, B, C and D determine each other without reference to E, which can be eliminated from (4.2) and (4.3) by means of the following identities:

$$E^{-1}(-z)^* \circ E^{-1}(z) = D^*(0, z) \circ D^{-1}(z, 0) = B'(z, -z) = C'(-z, z)^{-1}.$$

To see these relations, set one of the arguments to 0 and the other to z in (4.2), to get

$$B'(0, z) = D(0, z), \quad B'(z, 0) = E^{-1}(-z)^* E^{-1}(z) D(z, 0).$$

The first relation now comes from the symmetry of B. For the second, set $z = z_2 = -z_1$ in (4.2). The third identity arises from the same specialization of (4.3).

In a Deligne-Mumford theory, this argument is disallowed because the elbows relating B, C and D are unstable surfaces. Therefore, the relations between B, C, D (with E eliminated) must be imposed as axioms of the theory.

(4.5) Alternative parameters. The following description will be useful in our second construction of Lefschetz theories, in §6. Since $D(z, -z) \equiv \mathrm{Id}$, we can write

$$C'(z_1, z_2) = E\left(z_2\right) \circ \left(\mathrm{Id} + (z_1 + z_2) W'(z_1, z_2)\right) \circ E^*(z_1) \tag{4.6}$$

for a uniquely determined W' satisfying the more straight-forward constraint

$$W'(z_1, z_2)^* = W'(z_2, z_1),$$

corresponding to a symmetric $W \in (A \otimes A)[\![z_{1,2}]\!]$. Then, the triple $(\tilde{Z}^+, W, E)$ is an alternative set of parameters for the DMT, with symmetry of W as the only constraint.

(4.7) Construction of the theory. Given $\tilde{Z}$, E and B, here is a way to produce a field theory with these data. For a single curve, the smooth-surface and nodal factorisation rules leave no choice for $\overline{Z}$: we resolve the surface, viewing all nodal points as outgoing say, apply the free boundary formula to each component, and use B to contract the two factors of A at each node as prescribed by formula (2.11). Clearly, this method works in any family which does not vary the topological type of the curve, in particular over any stratum of $\overline{M}_g^n$. However, gluing strata together requires more comment.

The recipe for any boundary stratum also applies to a nearby smoothing of our nodal surface: we can cut the handle smoothing out the node and use the Euler class of the cutting circle, with the two choices of sign, in lieu of the two nodal ψ-classes. Let us call this the *nodal recipe*. This nodal recipe is unavailable as we move farther into the bulk of Deligne-Mumford space, where the cutting circle is lost; the *smooth recipe*, based on the true topology of the surface, must take over. Now, Condition 4.10 ensures that the smooth and nodal recipes agree at the level of cohomology where they are both defined, because of the smooth-surface sewing rule.

However, to ensure that we get a well-defined *cohomology* class on the $\overline{M}_g^n$, we must produce *cocycle-level representatives*, such as differential forms, for the local $\overline{Z}$-classes, and we must check their agreement on overlaps. For this purpose, choose differential forms representing the ψ's over $\overline{M}_g^{n+1}$ such that:

(i) ψ_{n+1} vanishes near the $[\sigma_i]$ and near the nodes of the universal curve

(ii) The two ψ-classes at a node are defined on a tubular neighbourhood of the node, disjoint from the support of ψ_{n+1}, and they agree near the boundary of that neighbourhood.

This is possible because the line bundle $\det \sigma_{n+1}^* T_\varphi^*$ is trivial near the $[\sigma_i]$ and flat near the nodes, so the curvature forms in any metric will do.

We can now apply the nodal recipe for $\overline{Z}$ with differential forms, using $\int_\varphi \psi_{n+1}^{j+1}$ for every occurrence of κ_j in the formula. The vanishing of ψ_{n+1} near the nodes ensures that equation (4.10) gives the algebraic cancellation that ensures a match on overlaps.

Another prescription for the classes $\overline{Z}$ will be given in §6, in terms of a group action on cohomology of the Deligne-Mumford spaces.

(4.8) The CohFT condition. Because of the relations of §4.4, $D = \mathrm{Id} \Leftrightarrow B' = \mathrm{Id} \Leftrightarrow C' = \mathrm{Id}$. In addition, (4.2) implies the *symplectic condition*

$$E^*(z) = E^{-1}(-z).$$

This says that the action of $E(z)$ on $A((z))$ preserves the symplectic form

$$\Omega(a_1, a_2) := \mathrm{Res}_{z=0}\, \beta\left(a_1(-z)a_2(z)\right) dz.$$

In terms of W, the CohFT constraint is

$$W'(z_1, z_2) = \frac{E^{-1}(z_2)E^{-1}(z_1)^* - \mathrm{Id}}{z_1 + z_2},$$

which can be met precisely for symplectic E.

4.9 Remark. If, instead, we fix a general symmetric B, then E falls subject to the constraint

$$B'(z, -z) = E^{-1}(-z)^* E^{-1}(z), \tag{4.10}$$

which determines E up to left multiplication by an $\mathrm{End}(A)$-valued Taylor series $F(z) = \mathrm{Id} + O(z)$ preserving the symplectic form on $A((z))$

$$\Omega_B(a_1, a_2) = \mathrm{Res}_{z=0}\, B(-z, z)\left(a_1(-z), a_2(z)\right) dz.$$

(4.11) The vacuum. Existence of a vacuum as in §2.13 follows from the Lefschetz theory sewing rule. In the theory constructed in §4.7, $\mathbf{v}(z)$ is given by the formula of Proposition 3.14, derived from the restricted free boundary theory. The *flat vacuum* condition $\mathbf{v}(z) = \mathbf{1}$ then amounts to

$$\exp\left\{-\textstyle\sum_{j>0} a_j z^j\right\} = E^{-1}(z)(\mathbf{1}). \tag{4.12}$$

In the semi-simple case, large genus surfaces detect the vacuum (Remark 3.15.i), so the restricted Deligne-Mumford theory will also have a flat vacuum precisely when (4.12) holds.

The final Gromov-Witten *homogeneity* condition in §1.7.iii requires a digression on Frobenius manifolds, and will be discussed in §7.

5. Deligne-Mumford theories: uniqueness

This section contains the key argument of the paper: we show that semi-simple DMT's are uniquely determined by the nodal propagator D and by the associated free-boundary theory on smooth curves.[10] The argument also applies to Lefschetz theories, but we focus on the DM case. A reformulation of the main result, suggested by one of the referees, is found in the appendix.

(5.1) Extending Z-classes over Deligne-Mumford strata. Let $j : S \hookrightarrow M$ be the divisor parametrising a (locally versal) nodal degeneration of a family $\Sigma_M \to M$ of marked Riemann surfaces. The normal bundle ν_S to S in M is the tensor product $L' \otimes L''$ of the complex tangent lines at the two exceptional points p', p'' of the resolved surface $\tilde{\Sigma}$ over S.

Since p' and p'' may be switched by the monodromy over S, we view them both as outgoing. Over S, and hence over a tubular neighbourhood N, $Z(\Sigma)$ is the contraction of $Z(\tilde{\Sigma}) \in H^\bullet(\partial N; A^{(2)})$ by $B(\psi', \psi'')$. The Mayer-Vietoris sequence

$$\cdots \to H^{\bullet-1}(\partial N) \xrightarrow{\delta} H^\bullet(M) \to H^\bullet(M \setminus N) \oplus H^\bullet(N) \to H^\bullet(\partial N) \xrightarrow{\delta} \cdots,$$

shows that cohomology classes over $M \setminus S$ and N patch into one over M, if they agree over the circular neighbourhood ∂N; but an ambiguity arises from the δ-image of $H^{\bullet-1}(\partial N)$. More precisely, if η is a connection form on the circle bundle $\partial N \to S$, then $H^\bullet(\partial N)$ is computed as the cohomology of the DGA $H^\bullet(S)[\eta]$, with differential $d\eta = \mathrm{eul}(\nu_S)$, the normal bundle Euler class. For $a \in H^{\bullet-1}(\partial N)$, $\delta(a)$ is given by the differential of any extension of a to N as a co-chain. This kills classes pulled back from S, while a class $b\eta$, with b from S, is sent to $j_*(b)$. Now, $b\eta$ is a co-cycle iff $b \cdot \mathrm{eul}(\nu_S) = 0$, so the patching ambiguity is precisely the Thom push-forward j_* of the annihilator of $\mathrm{eul}(\nu_S)$ in $H^{\bullet-2}(S)$.

This observation applies to Deligne-Mumford strata S of any co-dimension c: a class in $H^\bullet(M)$ with known restrictions to $M \setminus S$ and S is ambiguous only up to addition of some $j_*(b)$, with $b \in H^{\bullet-2c}(S)$ annihilated by $\mathrm{eul}(\nu_S)$. We see this from the long exact cohomology sequence

$$\cdots \to H^{\bullet-2c}(S) \xrightarrow{j_*} H^\bullet(M) \to H^\bullet(M \setminus S) \xrightarrow{\delta} H^{\bullet-2c+1}(S) \to \ldots,$$

(we have used the Thom isomorphism $j_* : H^{\bullet-2c}(S) \cong H^\bullet(M, M \setminus S)$) and from the fact that $j_*(b)|_S = \mathrm{eul}(\nu_S) \cdot b$. Note that $\mathrm{eul}(\nu_S)$ is the product of Euler factors for the Deligne-Mumford divisors containing S.

(5.2) Uniqueness for large genus: the main idea. If $M \setminus S$ is the universal family of smooth surfaces of large genus and S a boundary divisor in its DM compactification, Looijenga's theorem (2.18) ensures that $\mathrm{eul}(\nu_S) = -\psi' - \psi''$ is not a zero-divisor within a range of degrees, as one component of $\tilde{\Sigma}$ must have large genus. Classes then patch uniquely. This applies to strata of any co-dimension, and even if the family M includes nodal and reducible surfaces, the only requirement being that each node defining the degeneration to S should belong to at least *one* large genus component. This is the germ of an inductive proof of unique extension of $Z(\Sigma_M)$ to the Deligne-Mumford boundary. The induction requires a careful stratification of the Deligne-Mumford spaces $\overline{M}_g^n$.

(5.3) Stratification of $\overline{M}_g^n$. Assume that $n > 0$, and call the irreducible component of the universal curve containing the marked point n *special*. We now decompose $\overline{M}_g^n$ following the *topological type*

[10] In the context of chain-level theories, this fact is true without the semi-simplicity assumption; but in that situation, it can be made obvious with the right definitions.

τ of the special component. A partial ordering on the resulting strata is defined by stipulating that higher types can only degenerate to lower ones (plus extra components, which cease to be special). We extend this to some complete ordering; an example is the dictionary order on geometric genus, number of nodes and total number of marked points of the special component. (Nodes linking the special component to other components should be counted for this purpose as marked points, not nodes.) The smooth stratum M_g^n is by itself. Every stratum in the decomposition is isomorphic to $(M_\gamma^\nu \times \overline{M})/F$, where γ and ν pertain to the special component, while $\overline{M}$ parametrises the complementary components, and F is the group of symmetries of the modular graph describing the topological type our curves.

5.4 Example. With $g > 2$ and $n = 1$, if we split off an elliptic curve crossing the special component at two nodes, $\gamma = g-2, \nu = 2$, $\overline{M} = \overline{M}_1^2$ and $F = \mathbb{Z}/2$, switching the two crossings.

Our decomposition M_τ of $\overline{M}_g^n$ is not a stratification in the strict sense: it is not compatible with the dimensional ordering. However, we have the following:

(i) Each M_τ is a union of Deligne-Mumford strata.

(ii) Every descending union $\amalg_{\tau' \geq \tau} M_{\tau'}$ of strata is open.

(iii) Each M_τ is a closed sub-orbifold of $\amalg_{\tau' \geq \tau} M_{\tau'}$.

(iv) The normal bundle to M_τ is (locally) a sum of lines $L' \otimes L''$ for tangent line pairs at the nodes which belong to the special component (and possibly one other component).

Parts (i) and (ii) are clear by construction. To see (iv), choose a surface Σ in M_τ. It belongs to a DM stratum M_Σ, which is wholly contained within M_τ. The deformation space of Σ is smooth, and its tangent space is the sum of the lines $L' \otimes L''$, over *all* nodes, with the tangent space to M_Σ. The nodes which lie on the special component give deformations which change the topology of the special component, hence they represent normal directions to M_τ; whereas the other nodes correspond to deformations of the complement of the special component, which are tangent to M_τ. Since an automorphism of Σ preserves the special component, it cannot interchange tangent and normal lines. This shows that the symmetry group F, acting on the tubular neighbourhood of M_τ, preserves the decomposition into tangent and normal directions; so M_τ has no self-intersections, proving smoothness in (iii).

(5.5) Unique patching. Let us now prove uniqueness of the patched class on every $\overline{M}_g^n$ ($n > 0$). Attach to the marked input point n a moving smooth surface Σ_G of large genus G with an incoming point marked "$-$" and an outgoing one marked "$+$" (the latter attached to n). This embeds $S := \overline{M}_g^n \times {}^1M_G^1$ as part of the boundary of $\overline{M}_{g+G}^n$. Let, as before, N be a tubular neighbourhood of S and ∂N its boundary.

5.6 Lemma. *The projection $\partial N \to \overline{M}_g^n \times M_G^1$ forgetting the point $+$ gives the following isomorphism in degree less than $(G-1)/2$: $H^\bullet(\partial N) \cong H^\bullet(\overline{M}_g^n) \otimes H^\bullet(M_G)[\psi_-]$.*

Proof. The description of ∂N as a circle bundle over S gives the description of $H^\bullet(\partial N)$ in the stable range as the cohomology of the differential graded algebra

$$H^\bullet(\overline{M}_g^n) \otimes H^\bullet(M_G)[\psi_+, \psi_-, \eta] \quad \text{with} \quad d\eta = \psi_n + \psi_+,$$

which implies our statement. □

Now, S parametrises nodal degenerations at $n = +$ of those surfaces corresponding to the open union of U of DM strata in $\overline{M}^n_{g+G}$ which meet ∂N. We carry over our type decomposition of §5.3 to $U \subset \overline{M}^n_{g+G}$ with special point $-$, and observe that properties (i)–(iv) continue to hold. In addition, the special component now has geometric genus G or higher. All the normal Euler classes in (iv) are then products of free generators of the cohomology ring. The classes Z over the U_τ then patch uniquely. But each U_τ factors as $M^\nu_{G+\gamma} \times \overline{M}$, and $\overline{M}$ parametrises surfaces whose type is *strictly lower* than that of geometric genus g, with n marked points. We can inductively assume their Z-classes to be known; the factorisation rule gives the Z-class on each U_τ, therefore on all of U and then also on S. The class on S is $\overline{Z}^n_g \circ D(-\psi_n, \psi_+) \circ {}^1Z^1_G$, with D fed into the nth entry of $\overline{Z}^n$. Lifting to ∂N recovers $\overline{Z}^n_g$, by Lemma 5.6.

(5.7) Pre-stable surfaces. Restriction to stable surfaces may seem unnatural from the axiomatic point of view. There are Artin stacks $\overline{A}^n_g$ parametrising all *pre-stable* curves, nodal curves with no condition on the rational components: they arise from stable curves by inserting chains of $\mathbb{P}^1$'s at a node (leading to semi-stable curves) and trees of $\mathbb{P}^1$'s at smooth points. However, these stacks also have normal-crossing stratifications à la Deligne-Mumford, and the inductive argument applies as before, ensuring uniqueness of the extension to $\overline{A}^n_g$.

(5.8) Appendix: An infinite-genus Deligne-Mumford space. One referee observed that the splitting result of this section has a re-formulation in the guise of a homological splitting of a certain "infinite-genus Deligne-Mumford space" $\overline{M}^n_{n\cdot\infty}$ into its constituent strata. This space is a partial completion of the classifying space $B\Gamma^n_\infty$ of the infinite-genus mapping class group, and can be obtained by the addition of certain boundary strata. Roughly speaking, $\overline{M}^n_{n\cdot\infty}$ parametrizes infinite-genus nodal surfaces with n marked points such that each irreducible component which carries one of the marked points has infinite genus, but the other components have *finite* genus.

A geometric construction of the requisite DM space, as well as its moduli interpretation, require some effort; so I shall only outline the story here. While it is true that we need the spaces only up to homotopy in order to know their cohomology, we need to describe $\overline{M}^n_{n\cdot\infty}$ as a *stratified* homotopy type, with normal structure at the strata. In this format, the space can be assembled from its constituent strata, which are products of various M^k_g and factors of $B\Gamma^l_\infty$, in the manner in which $\overline{M}^n_g$ is assembled from its Deligne-Mumford strata, and with the same normal-crossing structure. Readers familiar with the structure of Deligne-Mumford boundary divisors should have no trouble supplying the details for this case.

A point in $\overline{M}^n_{n\cdot\infty}$ represents a nodal curve C; to this, we associate its stable graph $\tilde{\gamma}(C)$ in the usual way (a genus-labeled vertex for each component, an edge for each node, a labeled external edge for each marked point), and the modified graph $\gamma(C)$ which collapses all the edges which link vertices of finite genus. We now stratify $\overline{M}^n_{n\cdot\infty}$ according to the modified graph. (For this purpose, one must take care that the 'infinite' genera of components of the curve are really very large numbers, to be stabilized later; for instance, splitting of some finite genus piece from a large genus surface changes the graph. This book-keeping must be built into the construction of $\overline{M}^n_{n\cdot\infty}$.) For a single marked point, we recover the stratification of §5.3 by topological type of the special component (now stabilised to infinite genus). Call c_γ the complex co-dimension of M_γ.

There is a partial ordering on strata, compatible with degeneration of the infinite-genus components: $\gamma \geq \gamma'$ if the closure of the stratum M_γ contains $M_{\gamma'}$. (This happens as soon as the former meets the latter.) This gives an increasing filtration of $\overline{M}^n_{n\cdot\infty}$ by the open subsets $F_\gamma := \coprod_{\gamma' \geq \gamma} M_{\gamma'}$. The following proposition, suggested by the referee, has the same proof as Lemma 5.6.

5.9 Proposition. *(i) The cohomology spectral sequence associated to the filtration F_γ collapses at the first page:*

$$\mathrm{gr}H^\bullet(\overline{M}^n_{n\cdot\infty}) = \bigoplus H^\bullet(M_\gamma)[2c_\gamma].$$

(ii) Every cohomology class of $\overline{M}^n_{n\cdot\infty}$ is uniquely determined by its restrictions to all the strata M_γ. □

6. A group action on DM field theories

This section reformulates the classification of semi-simple DMT's in terms of the action of a subgroup of the symplectic group on the cohomology of Deligne-Mumford spaces. This construction, which lifts some of Givental's quadratic Hamiltonians, was perhaps first flagged by Kontsevich [CKS] (see also the recent [KKP]), and plays a substantial role in his study of deformations of open-closed field theories. Here, it is merely a convenient way to rephrase the classification. The context is more general than the Introduction: we allow $D \neq \mathrm{Id}$, so we must review the notation.

(6.1) Definitions. Let Δ be the completed second symmetric power of $A[\![z]\!]$; we may view it as the space of (symmetric) 2-variable Taylor series in $A^{\otimes 2}[\![z_{1,2}]\!]$. The group $\mathrm{GL}(A)[\![z]\!]$ acts on $V \in \Delta$ by $\mathrm{Ad}_g(V) = (g(z_1) \otimes g(z_2)) \circ V(z_{1,2})$. Let $\mathrm{GL}^+ \subset \mathrm{GL}(A)[\![z]\!]$ be the congruence subgroup $\equiv \mathrm{Id}$ (mod z), and define $\mathbf{Sp}^+ := \mathrm{GL}^+ \ltimes \exp(\Delta)$, the second factor denoting the vector Lie group with Lie algebra Δ. Call $\mathbf{F}$ the space polynomial functions on $A[\![z]\!]$, introduce a formal parameter $\hbar$ and consider, on the space $\mathbf{F}(\!(\hbar)\!)$

- the *translation action* of $A[\![z]\!]$: $(T_x\mathcal{F})(y) = \mathcal{F}(y - x)$;
- the *geometric action* of GL^+: $(g\mathcal{F})(x) = \mathcal{F}(g^{-1}x)$;
- the action of $\exp(\Delta)$, exponentiating the quadratic-differentiation action of $\hbar\Delta$.

They assemble to an action of $\mathbf{Sp}^+ \ltimes A[\![z]\!]$. When $A[\![z]\!]$ is doubled to a symplectic vector space and $\mathbf{F}$ is regarded as the Fock representation of its Heisenberg group $\mathbf{H}$, $\mathbf{Sp}^+$ is a subgroup of the symplectic group $\mathbf{Sp}$ acting on $\mathbf{H}$ and Δ is the "upper right corner" of the Lie algebra of $\mathbf{Sp}$. The (projective) metaplectic representation of $\mathbf{Sp}$ on $\mathbf{F}$ induces on $\mathbf{F}(\!(\hbar)\!)$ the action of $\mathbf{Sp}^+$ that we have just described. However, we are *not yet* committed to an identification of the symplectic space $A[\![z]\!] \oplus A[\![z]\!]^*$ with $(A(\!(z)\!), \hbar\Omega)$ as in §1.14. The geometric action of $g \in \mathrm{GL}^+$ does *not* agree with the metaplectic one coming from its point-wise action on $A(\!(z)\!)$, as in §1.14: rather, the latter comes from a *different* embedding of (part of) GL^+ in $\mathbf{Sp}^+$, see Proposition 6.17 below.

A Deligne-Mumford theory defines, and is determined by, a vector in the space of $\mathfrak{S}_n$-invariant cohomologies

$$A^{DM} := \prod_{g,n} H^\bullet\left(\overline{M}^n_g; (A^*)^{\otimes n}\right)^{\mathfrak{S}_n}.$$

There is a distinguished $I_A \in A^{DM}$, representing the trivial theory based on A. To any vector $\overline{Z} \in A^{DM}$, not necessarily one defining a DMT, we now assign as in §1.11 its "potential" $\mathcal{A}$,

$$\mathcal{A}(x) = \exp\left\{\sum_{g,n} \frac{\hbar^{g-1}}{n!} \int_{\overline{M}^n_g} \overline{Z}^n_g\left(x(\psi_1), \ldots, x(\psi_n)\right)\right\}, \tag{6.2}$$

in a completion of $\mathbf{F}(\!(\hbar)\!)$. It is a formal power series in $\hbar$ and $x^3/\hbar$, but of a restricted kind, thanks to the dimensions of the $\overline{M}^n_g$: for instance, the exponent is polynomial in the $z^2A[\![z]\!]$ directions. These facts explain why the translation action of $z^2A[\![z]\!]$, as well as that of $\hbar\Delta$, can be exponentiated to these potentials (but this will also be a consequence of the next construction). Let $\mathbf{H}^+$ and $\mathbf{H}^{++}$ be the lifts of $zA[\![z]\!], z^2A[\![z]\!]$ in $\mathbf{H}$. I will define an action of $\mathbf{Sp}^+ \ltimes \mathbf{H}^{++}$ on A^{DM} which lifts the action on potentials, and verify that the semi-simple DM theories of §4 constitute the $\mathbf{Sp}^+ \ltimes \mathbf{H}^{++}$-orbit of the trivial theory based on A.

The action extends infinitesimally to the larger group $\mathbf{Sp}^+ \ltimes \mathbf{H}^+$, but we will see (from the case of of semi-simple DMT's) that the exponentiated action of the linear modes $zA \subset \mathbf{H}^+$ has singularities. In addition, these linear modes will vary the algebra structure of A, scaling the projectors. The translation by zero-modes is more complicated and does *not* commute with the rest of $\mathbf{H}^+$, see §8.

(6.3) Translation. Let $\overline{Z} \in A^{DM}$ be any class. For $a(z) \in z \cdot A[\![z]\!]$, define a new class ${}_a\overline{Z}$ by setting

$$ {}_a\overline{Z}_g^n(x_1,\dots,x_n) =: \sum_{m\geq 0} \frac{(-1)^m}{m!} \int_{\overline{M}_g^{n+m}}^{\overline{M}_g^n} \overline{Z}_g^{n+m}(x_1,\dots,x_n,a(\psi_{n+1}),\dots,a(\psi_{n+m})). $$

All ψ-classes are on $\overline{M}_g^{n+m}$. With $a=0$, we recover $\overline{Z}$. For dimensional reasons, the sum is *finite* if $a \in z^2\cdot A[\![z]\!]$, but linear components zA can cause convergence problems and should *a priori* be treated formally. We will compute their effect explicitly below, for semi-simple DMT's.

We claim that ${}_a\left({}_b\overline{Z}\right) = {}_{a+b}\overline{Z}$: indeed, the second-order infinitesimal variation, capturing the linear effect of b-translation followed by that of a-translation, is

$$ \frac{\delta^2 \overline{Z}_g^n}{\delta a \delta b}(x_1,\dots,x_n) = \int_{\overline{M}_g^{n+1}}^{\overline{M}_g^n} \int_{\overline{M}_g^{n+2}}^{\overline{M}_g^{n+1}} \overline{Z}_g^{n+2}\left(x_1,\dots,x_n,a(\varphi^*\psi_{n+1}),b(\psi_{n+2})\right), \tag{6.4} $$

where φ is the morphism forgetting the point $n+2$. The difference $a(\psi_{n+1}) - a(\varphi^*\psi_{n+1})$ is a multiple of $[\sigma_{n+1}]$ (cf. §2.15), so it is killed by ψ_{n+2}, therefore also by $b(\psi_{n+2})$; so the right-hand side is symmetric in a,b.

The same argument, using the presence of ψ-classes in a, gives the expected binomial expansion

$$ \int_{\overline{M}_g^n} \overline{Z}_g^n\left(x+a(\psi_1),\dots,x+a(\psi_n)\right) = \sum\nolimits_k \binom{n}{k} \int_{\overline{M}_g^k} {}_a\overline{Z}_g^k\left(x,\dots,x\right); $$

defining a potential $\mathcal{A}_a$ from ${}_a\overline{Z}$ as in (1.12) leads to

$$ \mathcal{A}_a(x) = \mathcal{A}(x-a) \quad \text{for} \quad a \in zA[\![z]\!]. $$

In other words, $\overline{Z} \mapsto {}_a\overline{Z}$ lifts to DMT classes the translation action of a on $\mathbf{F}_\hbar$.

(6.5) The $\mathbf{Sp}^+$*-action.* It is clear how the action of elements $g(z) \in \mathrm{GL}(A)[\![z]\!]$ lifts to A^{DM}: the ith input of $\overline{Z}$ is transformed by $g^{-1}(\psi_i)$. The quadratic differentiations of Δ can be implemented by the addition of boundary terms, as I now describe.

Recall first that $\overline{M}_g^n$ has one boundary divisor parametrising irreducible nodal curves of genus $g-1$, and additional divisors corresponding to reducible nodal curves. These latter divisors are labelled by tuples (g',g'',n',n'',σ), where $(g',n')+(g'',n'')=(g,n)$ and the partitions σ of marked points range over co-sets in $\mathfrak{S}_n/(\mathfrak{S}_{n'}\times\mathfrak{S}_{n''})$. As usual, unstable degenerations with forbidden values of (g',n') or (g'',n'') are excluded. Our labelling double-counts the boundaries because of the interchange $(g',n') \leftrightarrow (g'',n'')$; in the case $g'=g''$ and $n'=n''$, this becomes an involution of the respective boundary stratum, interchanging the local components of the curve at the node. In other words, a label determines a boundary stratum together with an ordering of the two local components at the node. Denote by ψ',ψ'' the two ψ-classes at the node. Call Λ the set of labels for reducible degenerations and Θ_λ the Thom class of the boundary $\lambda \in \Lambda$.

6.6 Definition. The infinitesimal action of $\delta V = v'z^p \otimes v''z^q + v''z^q \otimes v'z^p \in \Delta$ on $\overline{Z} \in A^{DM}$ is given by

$$\begin{aligned}\delta\overline{Z}_g^n(x_1,\ldots,x_n) = -\sum_{\lambda\in\Lambda} \Theta_\lambda \wedge \overline{Z}_{g'}^{n'+1}(x_{\sigma(1)},\ldots,x_{\sigma(n')},v') \wedge \psi'^p \\ \wedge \overline{Z}_{g''}^{n''+1}(x_{\sigma(n'+1)},\ldots,x_{\sigma(n)},v'') \wedge \psi''^q \\ - \Theta_{(g-1,n+2)} \wedge \overline{Z}_{g-1}^{n+2}(x_1,\ldots,x_n,v',v'') \wedge \psi'^p \wedge \psi''^q.\end{aligned}$$

To see that this gives an action of Δ, we must check that the effects of any two $\delta V, \delta W$ commute. Now, the second variation, in either order, is a sum over all boundary strata of co-dimension 2 in $\overline{M}_g^n$. These strata are labelled by stable curves with two nodes, and a stratum S contributes the following term: the Thom class of S, times the product of $\overline{Z}$-classes, one factor for each irreducible component of the curve, and with the pair of entries at each of the two nodes contracted with δV, respectively with δW. We are using the fact that the Thom classes and nodal ψ-classes of boundary strata restrict to their obvious counterparts on second boundaries. This is the desired symmetry of the second variation.

Let us now show that the actions just defined on A^{DM} assemble to an action of $\mathbf{Sp}^+ \ltimes \mathbf{H}^+$.

6.7 Proposition. *The action of* $\mathrm{GL}(A)[\![z]\!]$ *intertwines naturally with those of* $\mathbf{H}^+$ *and* Δ*, which commute with each other. Moreover, the resulting action of* $\mathbf{Sp}^+ \ltimes \mathbf{H}^+$ *lifts the metaplectic action on potentials.*

Proof. The statement about GL is clear, because it twists the input fields. Commutation of $\mathbf{H}^+$ with Δ can be checked infinitesimally. By definition, the derivative $\partial_a\overline{Z}_g^n$ in the direction $a \in \mathbf{H}^+$ is the integral along the universal curve of the a-contraction of $\overline{Z}_g^{n+1}$. Omitting the obvious symbols in Definition 6.6, we therefore have

$$\begin{aligned}\frac{\delta^2\overline{Z}}{\delta a\delta V} &= \sum \left(\Theta_\lambda \wedge \partial_a\big(\overline{Z}'\psi'^p\big) \wedge \overline{Z}''\psi''^q + \overline{Z}'\psi'^p \wedge \partial_a\big(\overline{Z}''\psi''^q\big)\right) + \Theta_{(g-1,n+2)} \wedge \partial_a\big(\overline{Z}_{g-1}\psi'^p\psi''^q\big),\\ \frac{\delta^2\overline{Z}}{\delta V\delta a} &= \sum \left(\Theta_\lambda \wedge \partial_a\big(\overline{Z}'\big)\psi'^p \wedge \overline{Z}''\psi''^q + \overline{Z}'\psi'^p \wedge \partial_a\big(\overline{Z}''\big)\psi''^q\right) + \Theta_{(g-1,n+2)} \wedge \partial_a\big(\overline{Z}_{g-1}\big)\psi'^p\psi''^q;\end{aligned}$$

and the first and second expression differ only through the meanings of ψ', ψ'': as the ψ-classes at the node on the universal curve, versus the lifts of the same from the base. However, the positive powers of ψ_{n+1} which are present in a kill the difference between the two.

Finally, let us compare this action with the metaplectic one. Translation was checked earlier. It is clear that the GL-action lifts the geometric action on $\mathbf{F}(\!(\hbar)\!)$. The analogue for the metaplectic action of Δ is seen in the following interpretation of $\mathcal{A}$: it is the integral over the moduli of *all, possibly disconnected* stable nodal surfaces, with individual components of the moduli space weighted down by the automorphisms of their topological type. In this expansion of the potential $\mathcal{A}$, differentiation in the input x involves replacing one x-entry in a $\overline{Z}$-factor in each term by the direction of differentiation, and summing over all choices of doing so. Quadratic differentiation is the same procedure, applied to all pairs of entries. Thanks to the Thom classes in formula (6.6), we can re-interpret the integral of $\delta\overline{Z}_g^n$ there over $\overline{M}_g^n$ as a sum of integrals over the relevant boundaries instead. Book-keeping confirms that we thus supply all requisite terms for the quadratic differentiation in the expansion of $\mathcal{A}$. $\square$

6.8 Proposition. *If* $\overline{Z}$ *defines a DMT, then so does any of its transforms under* $\mathbf{Sp}^+ \ltimes \mathbf{H}^+$*.*

Proof. For the action of GL^+, this is clear from first definitions. For Δ and $\mathbf{H}^+$, we will check that the infinitesimal action gives a first-order deformation of a field theory; we will also spell out its effect on the co-form C, and will do so first in the more delicate case of Δ.

More precisely, we claim that for the variation $\delta\overline{Z}$ resulting from δV, $\overline{Z} + \epsilon \cdot \delta\overline{Z}$ is a DMT over the ground ring $k[\epsilon]/\epsilon^2$, with nodal co-form $C + \epsilon\delta C$, where $\delta C(z_{1,2}) = (z_1 + z_2) \cdot \delta V(z_{1,2})$. Writing the DMT factorisation rule (2.10) at a splitting node corresponding to a boundary divisor D_{λ_0} (labelled by $\lambda_0 \in \Lambda$) as

$$b_2^* \overline{Z} = \overline{Z}' \dashv C(\psi', \psi'') \vdash \overline{Z}'',$$

where the two contractions $\dashv$ and $\vdash$ absorb the left and right factors of C into the nodal slots of Z', Z'', shows that the ϵ-linear part becomes the "Leibniz rule"

$$b_2^* \delta\overline{Z} = \delta\overline{Z}' \dashv C(\psi', \psi'') \vdash \overline{Z}'' + \overline{Z}' \dashv \delta C(\psi', \psi'') \vdash \overline{Z}'' + \overline{Z}' \dashv C(\psi', \psi'') \vdash \delta\overline{Z}''. \qquad (6.9)$$

To verify this identity for our $\delta\overline{Z}$, restrict formula (6.6) to D_{λ_0}. Noting that $b_2^*\Theta_{\lambda_0}$ is the Euler class $-(\psi' + \psi'')$ of D_{λ_0}, the term $\lambda = \lambda_0$ in the sum becomes

$$\epsilon(\psi' + \psi'') \wedge \overline{Z}_{g'}^{n'+1}(x_{\sigma(1)}, \ldots, x_{\sigma(n')}, v') \wedge \psi'^p \wedge \overline{Z}_{g''}^{n''+1}(x_{\sigma(n'+1)}, \ldots, x_{\sigma(n)}, v'') \wedge \psi''^q.$$

This is precisely the contribution to (6.9) of the variation δC posited above. On the other hand, using the nodal factorisation rule for $\overline{Z}$ shows that the $\lambda \neq \lambda_0$ and last terms in (6.6) restrict by b_2^* to give the $\overline{Z} \dashv C \vdash \delta\overline{Z} + \delta\overline{Z} \dashv C \vdash \overline{Z}$ terms in the Leibniz factorisation (6.9). A similar discussion applies to the boundary divisor $\overline{M}_{g-1}^{n+2}$, proving our claim.

For an infinitesimal translation by $a(z)$, the first variation $\delta\overline{Z}$ is the integral of $a(\psi) \dashv \overline{Z}$ along the universal curve, with the insertion and ψ-class at the new marked point. Restricting $\delta\overline{Z}_g^n$ to D_{λ_0}, we can split the integral into two terms, for the two irreducible components, to get $\overline{Z}' \dashv C(\psi', \psi'') \vdash \delta\overline{Z}'' + \delta\overline{Z}' \dashv C(\psi', \psi'') \vdash \overline{Z}''$, and there is no term to account for a δC contribution.

This very last argument argument conceals a subtlety: thanks to the presence of a ψ-factor, contraction with $a(\psi)$ kills the difference between the nodal ψ', ψ''-classes pulled back from D_{λ_0} and those on the universal curve, over which integration is taking place. $\square$

6.10 Remark. If $a(z)$ contains a constant term, and the co-form C carries a dependence on ψ', ψ'', there would be a δC-term accounting for the difference between nodal ψ', ψ''-classes on the curve and their pull-backs from D_{λ_0}. We will exploit this argument again in §7.4 below.

6.11 Scholium. *Upon transforming by* $\mathrm{e}^{V(z_{1,2})} \in \exp(\Delta)$, *the nodal co-form C of a DMT is changed to* $C(z_{1,2}) + (z_1 + z_2)V(z_{1,2})$, *whereas $\mathbf{H}^+$-translation does not change C.* $\square$

It is clear, on the other hand, that GL^+ has the obvious effect on C, via its action on Δ.

(6.12) The action on semi-simple DMT's. Let us now determine the action of a general group element $g \cdot e^V \cdot \zeta \in \mathrm{GL}^+ \ltimes (\exp\Delta \times \mathbf{H}^+)$ on semi-simple DMT's, in terms of their classification. The natural description involves the alternative parameters $(\tilde{Z}, W, E)$ of (4.6). We will meet a restriction on the z-linear term of ζ.

Write $\zeta = \sum_{j>0} \zeta_j z^j$. If $\zeta_1 = 0$, we will not change the algebra structure on A, and the reader can skip straight to the statement of the Proposition below, ignoring the primes. However, if $\zeta_1 \neq 0$, let A' be the Frobenius algebra which is identified with A as a vector space with quadratic form β, but with the multiplication re-defined in such a way that the new projectors are $P_i' = (1 + \zeta_1)P_i$. Thus, $x' \cdot' y' := x \cdot y \cdot (1 + \zeta_1)^{-1}$, the new identity is $\mathbf{1}' = \mathbf{1} + \zeta_1$, and the Euler class is now

$\alpha' = \alpha \cdot (\mathbf{1}+\zeta_1)^{-1}$. However, note that $(\alpha')^{1/2}$, with the square root in the prime algebra, agrees with the old $\alpha^{1/2}$. The construction breaks down when $(\mathbf{1}+\zeta_1)$ is *not* a unit in A, so we must exclude that case.

6.13 Proposition. *The trivial DMT I_A transforms under $g \cdot \mathrm{e}^V \cdot \zeta \in \mathrm{GL}^+ \ltimes (\exp(\Delta) \times \mathbf{H}^+)$ into the semi-simple theory based on the algebra A', with alternative parameters*

$$\tilde{Z} = \exp'\left\{\sum\nolimits_{j\geq 0} a'_j\kappa_j\right\},\ E(z) = g(z),\ W(z_{1,2}) = V(z_{1,2}).$$

Here, $\sum_{j\geq 0} a'_j z^j$ is the Taylor expansion of $\log'\alpha^{1/2} - \log'(\mathbf{1}+\zeta/z) \in A'[\![z]\!]$, and the logarithm and exponential are computed in A'.

6.14 Remark. Since $\log'(\mathbf{1}+\zeta_1) = \log'(\mathbf{1}') = 0$, we have $\exp' a'_0 = \alpha^{1/2}$. In the original algebra A, we can expand $\log\alpha^{1/2} - \log(\mathbf{1}+\zeta/z) = \sum_{j\geq 0} a_j z^j$; the relation $\exp' x' = (1+\zeta_1)\cdot\exp x$ for $x' = (1+\zeta_1)\cdot x$ shows that the Taylor coefficients are then related by $a'_j = (\mathbf{1}+\zeta_1)a_j$. The operators of multiplication by $\exp\left\{\sum_{j\geq 0} a_j\kappa_j\right\}$ on A and by $\exp'\left\{\sum_{j\geq 0} a'_j\kappa_j\right\}$ on A' coincide, when we identify the two vector spaces as above. (However, the customary relation $a_0 = \log\alpha^{1/2}$ is broken if $\zeta_1 \neq 0$ since involves the 'wrong' log.)

Proof. Note that E and W do not change the Frobenius algebra structure, which is determined by β and by the tensor $Z_0^3 : A^{\otimes 3} \to \mathbb{C}$. The effect of ζ will be checked in a moment. In particular, semi-simple theories remain semi-simple and we are merely looking for the change in parameters.

The effect of E is clear from its definition, while that of e^V was explained in (6.11) above: on a theory with $E = \mathrm{Id}$, $W \mapsto W + V$. To understand ζ, note first that translation cannot affect the E and W parameters of a DMT, because of the group law in $\mathbf{Sp}^+ \ltimes \mathbf{H}^+$. To find its effect on $\tilde{Z}$, it suffices to take $n = 1$ and compute its first-order variation over $\tilde{M}_g^1$ under $\delta\zeta$. This leads to a differential equation governing the action of ζ, which we solve. We omit the ζ-subscript from the notation for tidiness (so $\tilde{Z}$ should really be ${}_\zeta\tilde{Z}$, etc.) and let $C_{g,1} \to M_{g,1}$ denote the universal curve.

$$\delta\tilde{Z}(\kappa_j) = -\int_{C_{g,1}}^{M_{g,1}} \alpha^{-1/2}\cdot\tilde{Z}(\kappa_j)\cdot\delta\zeta(\psi_2) = -\alpha^{-1/2}\tilde{Z}(\kappa_j)\int_{C_{g,1}}^{M_{g,1}} \tilde{Z}(\psi_2^j)\delta\zeta(\psi_2),$$

where $\tilde{Z}(\kappa_j) = \exp\left\{\sum_{j\geq 0} c_j\kappa_j\right\}$ with the c_j as yet unknown, $\tilde{Z}(\psi_2^j) = \exp\left\{\sum_j c_j\psi_2^j\right\}$ and we have used the fact that κ_j inside the integral is κ_j outside plus ψ_2^j. Integration converts ψ_2^{j+1} to κ_j. Clearly, quadratic and higher terms in $\delta\zeta$ do not give rise to κ_0 and so do not affect the multiplication in A. Assuming first that $\zeta_1 = 0$, we specialise to $\kappa_j \mapsto z^j$:

$$\delta\tilde{Z}(z^j) = -\alpha^{-1/2}\tilde{Z}(z^j)^2\cdot\delta\zeta(z)/z\,,$$

solved by

$${}_\zeta\tilde{Z}(z^j) = \frac{\alpha^{1/2}}{\mathbf{1}+\zeta(z)/z}$$

since we know the initial value $\tilde{Z} = \alpha^{1/2}$. Now, $\log\tilde{Z}$ is linear homogeneous in the κ_j, so we recover the true $\tilde{Z}$ from our specialisation by substituting $z^j \mapsto \kappa_j$ in $\log\tilde{Z}$, and then exponentiating.

Finally, the effect of ζ_1-translation on the trivial A-theory can be determined directly from the formula

$$\int_{\overline{M}_g^{n+1}}^{\overline{M}_g^1} \psi_1\wedge\cdots\wedge\psi_n = (2g+n-2)\cdot\cdots\cdot(2g-1),$$

giving

$$\tfrac{1}{\zeta}\tilde{Z}_g = \alpha^g \sum_n \frac{(-\zeta_1)^n}{n!} \int_{\overline{M}_g^{n+1}}^{\overline{M}_g^1} \psi_1 \wedge \cdots \wedge \psi_n = \alpha^g \sum_n \binom{1-2g}{n} \zeta_1^n = \frac{\alpha^g}{(\mathbf{1}+\zeta_1)^{2g-1}}.$$

This introduces no higher κ-classes, but changes the multiplication on A in the manner claimed. □

(6.15) Cohomological Field theories. We now deduce Theorem 1 from Proposition 6.13 by identifying the subgroup of $\mathbf{Sp}^+ \ltimes \mathbf{H}^+$ which preserves the Cohomological Field theory constraint (1.7.i). Recall from §4.8 that this constraint takes the equivalent forms $B' = \mathrm{Id}$, $C' = \mathrm{Id}$ and $D = \mathrm{Id}$. In terms of E and W, we need the symplectic condition $E(z)^*E(-z) \equiv \mathrm{Id}$ of §4.8, together with

$$W(z_1, z_2) = W_E := \frac{E(z_2)^{-1}E(-z_1) - \mathrm{Id}}{z_1 + z_2}, \tag{6.16}$$

In §1.14, we denoted the subgroup of symplectic matrix series $E \in \mathrm{Id} + zA[\![z]\!]$ by $\mathbf{Sp}_L^+$. It follows from Scholium 6.11 that the group homomorphism $E(z) \mapsto E(z) \cdot e^{W_E(z_{1,2})}$ identifies $\mathbf{Sp}_L^+$ with the stabiliser of $C' = \mathrm{Id}$ in $\mathbf{Sp}^+$. We now use the symplectic form Ω of §1.14 to identify the symplectic double of $A[\![z]\!]$ with $A(\!(z)\!)$. The group GL^+ acts on $A(\!(z)\!)$, point-wise in z; its subgroup $\mathbf{Sp}_L^+$ preserves Ω and thus lies in $\mathbf{Sp}$. We write $E \mapsto \widehat{E}$ for this embedding of $\mathbf{Sp}_L^+$. We must compare the two resulting actions of $\mathbf{Sp}_L^+$ on $\mathbf{F}(\!(\hbar)\!)$.

6.17 Proposition. *The two embeddings of $\mathbf{Sp}_L^+$ into $\mathbf{Sp}$ agree: $\widehat{E} = E \cdot e^{W_E}$.*

Proof. We verify this on Lie algebras. Let $\delta E = \sum_{n>0} \delta E_n z^n$; then,

$$\delta W_E(z_{1,2}) = \frac{\delta E(-z_1) - \delta E(z_2)}{z_1 + z_2} = -\sum_{p,q} \delta E_{p+q+1}(-z_1)^p z_2^q.$$

In the monomial decomposition $\{z^n \cdot A\}_{n\in\mathbb{Z}}$ of $A(\!(z)\!) \cong A[\![z]\!] \oplus A[\![z]\!]^*$, the geometric action of δE is given by the operator with (p,q) blocks

$$O_{p,q} = \begin{cases} -\delta E_{p-q} & \text{for } p > q \geq 0 \\ (-1)^{p+q-1}\delta E^*_{p-q} & \text{for } 0 > p > q \\ 0 & \text{otherwise} \end{cases}$$

The symplectic condition is $(-1)^{p+q}\delta E^*_{p-q} = \delta E_{p-q}$. On the other hand, the operator corresponding via the symplectic form Ω to the quadratic differentiation operator $\delta W_E(z_{1,2})$ has blocks $-\delta E_{p-q}$ for $q < 0 \leq p$. This supplies precisely the missing $p \geq 0 > q$ blocks for the metaplectic action of the multiplication operator $\delta E(z) : A(\!(z)\!) \to A(\!(z)\!)$. Our statement follows. □

(6.18) Flat vacuum. Let us identify the vacuum vector (§2.13) of the theory in terms of the group element $\widehat{E} \cdot \zeta$. In particular, we will identify the subgroup of $\mathbf{Sp}_L^+ \ltimes \mathbf{H}^+$ whose action on I_A preserves the flat vacuum condition (1.7.ii) with the conjugate of $\mathbf{Sp}_L^+$ by the translation T_z by $z\mathbf{1}$. This will conclude the proof of Theorem 2.

By equation (4.12) and Proposition 6.13,

$$E^{-1}(z)(\mathbf{v}) = \exp'\left\{-\textstyle\sum_{j>0} a'_j z^j\right\} = \mathbf{1} + \zeta/z$$

so $\zeta = z(E^{-1}(z)(\mathbf{v}) - \mathbf{1})$. Clearly, the CohFTs with vacuum $\mathbf{v}$ constitute the set $T_{z\mathbf{v}(z)} \cdot \widehat{E} \cdot T_z^{-1}(I_A)$, with E ranging over the symplectic $\mathrm{End}(A)$-valued series considered. In particular, notice that changing the vacuum of a theory with fixed underlying algebra and E-parameter is accomplished by $\mathbf{H}^{++}$-translation.

7. Frobenius manifolds and homogeneity

We now enrich a given DMT $\overline{Z}$ into a family of DMT's parametrised by a (possibly formal) neighbourhood U of $0 \in A$. When starting with a cohomological field theory, the genus zero part of this family defines on U the structure of a *Frobenius manifold*, a notion introduced by Dubrovin [D]. The family of DMT's will allow us to incorporate the grading information of Gromov-Witten theory in the form of a *homogeneity* condition under a vector field on U. The reader may consult [M, §I] or [LP] for a broader account of the subject.

7.1 Definition. Given a DMT $\overline{Z}$, define for $u \in U$

$$ {}_u\overline{Z}_g^n(x_1, \ldots, x_n) := \sum_{m \geq 0} \frac{(-1)^m}{m!} \int_{\overline{M}_g^{n+m}}^{\overline{M}_g^n} \overline{Z}_g^{n+m}(x_1, \ldots, x_n, u, \ldots, u). $$

Restriction to U may be required for convergence, but for convenience we will treat u as a genuine parameter in our formulae. It is straightforward to verify the DMT axioms for ${}_u\overline{Z}$ from those for $\overline{Z}$; the construction is formally similar to the *translation* of §6.3, but in this case we are using the subspace $A \subset A[\![z]\!]$ of the Heisenberg group. However, while the effect of translation by $zA[\![z]\!]$ was easily expressed in terms of κ-classes, the structure resulting now is more complicated, because the new translation interacts with the boundary terms, and in fact fails to commute with $\mathbf{H}^+$. Microscopically, the absence of a ψ-factor in u breaks the calculations in the proof of Proposition 6.8. Conceptually, in the case of open-closed field theories, which are controlled by linear categories with a cyclic trace, the u-parameter is related to deformations of the category of boundary states, whereas translation by $\mathbf{H}^+$ is tied to the (easier) deformation of the trace.[11] There is, however, one easy fact to state.

7.2 Proposition. *If the DMT $\overline{Z}$ is actually a CohFT, then so is every ${}_u\overline{Z}$; moreover, the Frobenius bilinear form β remains unchanged.*

Sketch of proof. We must show that the nodal contraction form B, equal to the (ψ-independent) form β at $u = 0$, remains unchanged. The argument for this was given at the end of the proof of Proposition 6.8 (see also Remark 6.10): but this time, the co-form C has no dependence on the nodal ψ-classes, so that no correction terms appear under their forgetful pull-backs. □

(7.3) Frobenius manifold of a CohFT. The previous proposition does conceal something: the product and the Frobenius trace θ on A *will* vary with u. We obtain a u-dependent family of Frobenius algebra structures on A, viewed as a fixed vector space with bilinear form β. Spelt out, we get for $g = 0, n = 3$ a map

$$ {}_u\overline{Z}_0^3 : A^{\otimes 3} \to \mathbb{C}. $$

Converted to a map $A^{\otimes 2} \to A$ by means of β, this gives a u-dependent multiplication $\cdot_u$ on A. This multiplication is commutative, because of the symmetry of $\overline{Z}$, but turns out to be associative as well. (The requisite relation arises by applying the nodal factorisation rule to the several boundary

[11] Unfortunately, the author does not know of a written reference detailing this point of view.

restrictions of the map ${}_u\overline{Z}_0^4 : A^{\otimes 4} \to H^*\left(\overline{M}_0^4\right)$. Since $\overline{M}_0^4 = \mathbb{P}^1$ is connected, these restrictions define the same map $A^{\otimes 4} \to \mathbb{C}$, so that $\beta(a \cdot_u b, c \cdot_u d)$ is unchanged under permutation of the four variables. Of course, this is all implicit in the CohFT structure.)

We write A_u when referring to the algebra structure at u, and identify each A_u with the tangent space T_uU using the linear structure. The multiplications satisfy an integrability condition, which is captured by the observation that ${}_u\overline{Z}_0^3$ is the third total partial derivative of a function ${}_u\overline{Z}_0^0$. This function, the *potential* of the Frobenius manifold, is expressed by the series in definition 7.1 with $g = n = 0$, after omitting the $m \leq 2$ terms. This integrable family of Frobenius algebras on U, together with the (flat) metric β, is called a *Frobenius manifold* structure. The linear structure on $U \subset A$ is characterized by the *flat coordinates* under β.

We say that the Frobenius manifold has *flat identity* if the unit vector field $\mathbf{1}$ is flat in the metric (constant in flat coordinates). It is shown in [M, III] that this is implied by the flat vacuum condition on $\overline{Z}$; we will also verify that as part of Proposition 7.13 below. A Frobenius manifold is in fact equivalent to the datum of a genus-zero CohFT (the collection of classes $\overline{Z}_0^n$, satisfying the CohFT axioms), by a fairly explicit reconstruction [M].

(7.4) The basic differential equations. Semi-simplicity of A ensures that of the nearby A_u, so nearby theories are classified by u-dependent data $\tilde{Z}_u, E_u, B_u$. Assuming that $\overline{Z}$ is a CohFT, I describe the changes in $\tilde{Z}$ and E by means of differential equations.

To isolate the effect of the varying multiplication, we will express it in the (moving) normalised canonical basis $p_i = \theta_i^{-1/2}P_i$, in which the product can be computed entry-wise. Let $\Pi_u : A_0 \to A_u$ be the map identifying the normalised canonical bases in the two spaces. In the normalised canonical identification $\mathbb{C}^N \cong A_0$, this gives the normalised canonical framing of TU. Let $*$ denote the entry-wise multiplication of column vectors, and $\cdot_u$ the multiplication in A_u; we have

$$\Pi_u(x * y) = \alpha_u^{-1/2} \cdot_u \Pi_u(x) \cdot_u \Pi_u(y). \tag{7.5}$$

Also define the following column vector depending on u and on the κ-classes,

$$Y_u = Y_u(\kappa) := \Pi_u^{-1}(\alpha^{1/2}\tilde{Z}_u),$$

whose entries are the eigenvalues of multiplication by $\tilde{Z}_u$: that is, $\Pi \circ (Y_u *) \circ \Pi^{-1} = (\tilde{Z}_u \cdot)$. (The ith entry of Y is $\exp\{\sum_{j\geq 0} a_{ij}\kappa_j\}$, with u-dependent coefficients a_{ij}.) Write $Y_u(z)$ for the result of the substitution $\kappa_j \mapsto z^j$. Since $\log Y(\kappa)$ is linear homogeneous in the κ's, $Y(z)$ determines $Y(\kappa)$. We can now write the propagator ${}^1_uZ_g^n : A_u^{\otimes n} \to A_u$ for smooth curves of genus g, with incoming points $\{1, \ldots, n\}$ and one outgoing point labelled by 0, as follows:

$${}^1_uZ_g^n(x_1, \ldots, x_n) = E_u(-\psi_0)\Pi_u\left(Y_u(\kappa) * \Pi_u^{-1}E_u^{-1}(\psi_1)(x_1) * \cdots * \Pi_u^{-1}E_u^{-1}(\psi_n)(x_n)\right). \tag{7.6}$$

The contribution of n to $\kappa_0 = 2g + n - 1$ gives a factor of $\alpha^{n/2}$ in ${}_u\tilde{Z}$ and has the virtue of correcting the n operations $*$ into the multiplication $\cdot_u$, cf. (7.5). We now differentiate in u.

7.7 Proposition. *E_u and Y_u verify the following systems of ODE's in u, $\forall v \in T_uU$:*

$$\frac{\partial(E_u\Pi_u)}{\partial v}(z) \circ \Pi_u^{-1} = \left[E_u(z), \frac{(v\cdot_u)}{z}\right]; \tag{7.7.a}$$

$$\frac{\partial Y_u(z)}{\partial v} * Y_u(z)^{-1} = -Y_u(z) * \Pi_u^{-1}E_u(z)^{-1}\left(\frac{v}{z}\right) + Y_u(0) * \Pi_u^{-1}\left(\frac{v}{z}\right). \tag{7.7.b}$$

Before turning to the proof, the following comments might be helpful.

7.8 Remark. (i) We use the flat structure of TU to differentiate Π_u and E_u.

(ii) Since $E = \mathrm{Id} \pmod z$, the commutator in equation (7.7.a) is regular at $z = 0$, where we obtain, with $E_{u,1}$ denoting the z-linear term of E_u,

$$\partial_v \Pi_u \circ \Pi_u^{-1} = [E_{u,1}, (v\cdot_u)]\,.$$

By substituting this for the derivative of Π, (7.7.a) can be expressed as a *non-linear* ODE system in E alone; Π can then be recovered from E.

(iii) The second term on the right in equation (7.7.b) removes the pole present in the first term.

(iv) Let $C_g^1 := M_g^1 \times_{M_g} M_g^1$ be the universal curve over M_g^1 and note that $\int_{C_g^1}^{M_g^1} \psi^j = \kappa_{j-1}$, or zero if $j = 0$. Because $\partial_v Y(\kappa) * Y^{-1}$ is linear homogeneous in the κ's, we can write the ODE's for $Y_u(\kappa)$ explicitly:

$$\frac{\partial Y_u(\kappa)}{\partial v} * Y_u(\kappa)^{-1} = -\int_{C_g^1}^{M_g^1} Y_u(\psi) * \Pi^{-1}E^{-1}(\psi)(v). \tag{7.7.c}$$

Indeed, we will prove the equation in this form.

(v) A cleaner form of equation (7.7.b) is found in Proposition 7.13 below.

Proof. Proving the proposition will require us to find the variation of (7.6) with $n = 1$. However, to keep the formulas simple, we first write out the variation with $n = 0$. It will then be straightforward to describe the additional terms for general n. We also drop the u-subscript from the notation when no confusion arises.

From (7.6),

$$\partial_v(^1Z) = \partial_v(E\Pi)(-\psi_0)\,(Y(\kappa)) + E(-\psi_0)\Pi\,(\partial_v Y(\kappa))\,. \tag{7.9}$$

This same variation is also, by definition, an integral along the universal curve:

$$-\int_{C_g^1}^{M_g^1} E(-\psi_0)\Pi\left(Y(\kappa) * \Pi^{-1}E^{-1}(\psi)(v)\right) - v\cdot_u \frac{1-E(-\psi_0)}{\psi_0}\Pi\,(Y(\kappa))\,;$$

the second term is the boundary correction to $\overline{Z}$ on the diagonal section σ_0 of $M_g^1 \times_{M_g} M_g^1$. The requisite picture for this correction attaches a three-pointed $\mathbb{P}^1$ to C_g^1 at its output σ_0; this $\mathbb{P}^1$ absorbs v at the second input, and the output is read at the third point.

Using the familiar formula upstairs, $\kappa_j = \varphi^*\kappa_j + \psi^j$, the integral above (without sign) becomes

$$E(-\psi_0)\Pi\left(Y(\kappa) * \int Y(\psi) * \Pi^{-1}E^{-1}(\psi)(v)\right) + \frac{E(-\psi_0)-1}{\psi_0}\,(v\cdot_u \Pi(Y(\kappa)))\,;$$

the second term comes from the correction to ψ_0 on the diagonal σ_0, and all the κ's now live on the base M_g^1. All in all, we get

$$\partial_v(^1Z) = \left[(v\cdot_u), \frac{E(-\psi_0)}{\psi_0}\right] \circ \Pi\,(Y(\kappa)) - E(-\psi_0)\circ\Pi\left(Y(\kappa) * \int Y(\psi) * \Pi^{-1}E^{-1}(\psi)(v)\right), \tag{7.10}$$

and comparing with formula (7.9) *suggests* a separation into two identities, namely (7.7.a), with $z = -\psi_0$, and (7.7.c). However, in order to *prove* the proposition, we must:

- consider $n = 1$ in the variation of (7.6), in order to allow the insertion of arbitrary arguments in the first operator, in place of $\Pi(Y)$;
- justify the splitting of the one resulting identity into two pieces.

Taking $n = 1$ changes (7.10) as follows: $Y(\kappa)$ is replaced by $Y(\kappa) * \Pi^{-1}E^{-1}(\psi_1)(x_1)$, and an additional term,

$$E(-\psi_0)\left(\tilde{Z}\cdot_u\left[\frac{E^{-1}(\psi_1)}{\psi_1}, v\cdot_u\right]\right),$$

appears from the correction of ψ_1 along σ_1 and from the boundary contribution of σ_1 to $\overline{Z}$, just as explained in the case of ψ_0. Likewise, (7.9) changes by inserting $*\Pi^{-1}E^{-1}(\psi_1)(x_1)$ after $Y(\kappa)$ and $\partial_v Y(\kappa)$, and by the addition of

$$E(-\psi_0)\Pi\left(Y(\kappa) * \partial_v (E\Pi)^{-1}(\psi_1)(x_1)\right).$$

Splitting the identity into separate ones will now complete the proof. This is accomplished by setting the κ's or ψ's, which are now independent variables, selectively to zero. *A priori*, this leaves a constant term ambiguity; that, however, is resolved by noting that the constant term of the first ODE, $\partial_v\Pi\circ\Pi^{-1}$, is a skew matrix, whereas the operator $\partial_v Y*$ is purely diagonal; so there is no possible mixing of constant terms. □

(7.11) Flat vacuum preserved. If $\overline{Z}$ verifies the flat vacuum condition (1.7.ii), then the identity vector $\mathbf{1} \in A_0$ remains the identity in the algebra structure at all u: indeed, in the formula for ${}_u\overline{Z}_0^3(1, a, b)$ in Def. 7.1, all integrals with $m \neq 0$ vanish, because the integrand is lifted from the lower moduli space missing the first marked point:

$$\overline{Z}_0^{3+m}(\mathbf{1}, a, b, u, \dots) = \varphi^*\overline{Z}_0^{2+m}(a, b, u, \dots).$$

Moreover, each ${}_u\overline{Z}$ then satisfies the flat vacuum condition $\varphi^*{}_u\overline{Z}_g^n(x_1, \dots) = {}_u\overline{Z}_g^{n+1}(\mathbf{1}, x_1, \dots)$, because of the "base change" identity

$$\begin{aligned}\varphi^*\int_{\overline{M}_g^{n+m}}^{\overline{M}_g^n}\overline{Z}_g^{n+m}(x_1,\dots,x_n,u,\dots,u) &= \int_{\overline{M}_g^{n+1+m}}^{\overline{M}_g^{n+1}}\varphi^*\overline{Z}_g^{n+m}(x_1,\dots,x_n,u,\dots,u)\\ &= \int_{\overline{M}_g^{n+1+m}}^{\overline{M}_g^{n+1}}\overline{Z}_g^{n+1+m}(\mathbf{1},x_1,\dots,x_n,u,\dots,u)\end{aligned}$$

confirming condition (1.7.ii) term-by-term in the sum (7.1). Note that it is the *absence* of ψ in u which carries the argument here: the vacuum, of course, is not preserved by $\mathbf{H}^+$-translations.

(7.12) Vacuum differential equation. The ODE's for $Y(z)$ have a cleaner, equivalent form in terms of the vacuum vector $\mathbf{v}(z)$ of the theory (§3.12).

7.13 Proposition. $\dfrac{\partial\mathbf{v}(z)}{\partial v} = \dfrac{v}{z}\cdot_u(\mathbf{1} - \mathbf{v}(z))$.

Proof. $Y(z)$ and $\mathbf{v}(z)$ are related by $\mathbf{v}(z) = E(z)\Pi\big(Y(z)^{-1}\big)$ (Proposition 3.14). Direct computation gives (omitting the argument z, when not set to zero):

$$\begin{aligned}\frac{\partial E\Pi(Y^{-1})}{\partial v} &= \frac{\partial E\Pi}{\partial v}(Y^{-1}) + \frac{v - E\Pi\left(Y^{-1} * Y(0) * \Pi^{-1}(v)\right)}{z} \\ &= \frac{\partial E\Pi}{\partial v}(Y^{-1}) + \frac{v - E\left(\Pi(Y^{-1}) \cdot v\right)}{z} \\ &= \frac{E\left(v \cdot \Pi(Y^{-1})\right) - v \cdot E\Pi(Y^{-1}) + v - E\left(\Pi(Y^{-1}) \cdot v\right)}{z} \\ &= \frac{v - v \cdot \mathbf{v}}{z},\end{aligned}$$

having used (7.5) and the relation $\Pi(Y(0)) = \alpha$ to convert $*$ to the product in A_u. □

Proposition 7.13 provides the following formula for $\mathbf{v}(z)$ in terms of derivatives of $\mathbf{1}$. Let $\partial_{\mathbf{1}}$ be the operator of differentiation, in flat coordinates, along the vector field $\mathbf{1}$.

7.14 Corollary. $\mathbf{v}(z) = (1 + z\partial_{\mathbf{1}})^{-1}(\mathbf{1}) = \sum_k (-1)^k z^k \cdot \partial_{\mathbf{1}}^k(\mathbf{1})$. □

In particular, $\mathbf{v}$ is determined by the Frobenius manifold, and the latter has flat identity iff the CohFT has flat vacuum, $\mathbf{v}(z) \equiv \mathbf{1}$.

(7.15) Homogeneity and the Euler vector field. Assume that we are given a vector field ξ on our Frobenius manifold $U \subset A$, whose Lie derivative action on T_uU we denote by $\mathcal{L}$. We call U *homogeneous* (or *conformal*) of weight d with *Euler vector field* ξ if the (u-dependent) multiplication operator on T_uU and the quadratic form β are homogeneous with weights 1 and $2 - d$, respectively.

In flat coordinates x^j on A, ξ must be affine-linear,

$$\xi = \xi_0 - \mu^i_j \cdot x^j\partial_i + (1 - d/2)x^j\partial_j;$$

the matrix μ^i_j contributes an infinitesimal rotation about 0 in A, and the last term is the conformal scaling. The action of $\mathcal{L}$ on the flat frame of vector fields, commonly denoted ad_ξ, is given by $\mu + \left(\frac{d}{2} - 1\right)\mathrm{Id}$.

Following Dubrovin, we can reformulate this by viewing the space of sections $\Gamma(U; TU)$ as a Frobenius algebra over the ring $\mathbb{C}[U]$ of functions on U. Differentiation by ξ gives a derivation of $\mathbb{C}[U]$, and the shifted operator $\mathcal{L}^+ := \mathcal{L} + \mathrm{Id}$ defines a compatible derivation of the algebra $\Gamma(U; TU)$. The metric has $\mathcal{L}^+$-weight $(-d)$, and in general the $\mathcal{L}^+$-weights of the basic objects in A are eminently more reasonable than their $\mathcal{L}$-weights, cf. Table 1.

Viewing the ${}_u\overline{Z}^n_g \in H^\bullet\left(\overline{M}^n_g; (A^*)^{\otimes n}\right)$ as tensors on U with values in $H^\bullet(\overline{M}^n_g)$ and using the action of $\mathcal{L}^+$, we can lift the notion of homogeneity to the entire CohFT:

7.16 Definition. The CohFT ${}_u\overline{Z}$ is *homogeneous of weight d* under ξ if each $\overline{Z}^n_g : (TU)^{\otimes n} \to H^{2\bullet}(\overline{M}^n_g)$ is $\mathcal{L}^+$-homogeneous with weight $(g-1)d$. (The cohomology of $\overline{M}$ is weighted by half-degree.)

7.17 Example. In the Gromov-Witten theories of §1.5, the series

$$GW^n_{g,u} := \sum_{\delta \in H_2(X;\mathbb{Z})} \mathrm{e}^{\langle u|\delta\rangle} \cdot GW^n_{g,\delta} \tag{7.18}$$

object	$\mathcal{L}$-weight	$\mathcal{L}^+$-weight	reason
product	1	0	definition
β	$2-d$	$-d$	definition
$\mathbf{1} \in A$	-1	0	$\mathbf{1} \cdot x = x$
projector P	-1	0	$P \cdot P = P$
θ_i	$-d$	$-d$	$\beta(P,P)$
$\theta : A \to \mathbb{C}$	$1-d$	$-d$	$\beta(\mathbf{1}, .)$
α_u	$d-1$	d	$\theta(x \cdot \alpha) = \mathrm{Tr}_A(x \cdot)$
$(\alpha_u \cdot)$	d	d	

Table 1: Some basic weights

gives a (possibly formal) function on the group $H^2(X;\mathbb{C}^\times)$, expressed in the Fourier modes e^u. This group is a disjoint union of tori, each labelled by a character of the torsion subgroup of $H_2(X;\mathbb{Z})$. The *divisor equation* (see for instance [LP, G2])

$$\int GW_\delta^{n+1}(\ldots, u) = -\langle u|\delta\rangle \cdot GW_\delta^n(\ldots), \quad \text{for} \quad u \in H^2(X),$$

where we integrate along the last forgetful map, ensures that the family ${}_u\overline{Z} := GW_u$ is its own u-variation along the H^2 torus directions, in the sense of Definition 7.1. Near any chosen base-point, $H^2(X;\mathbb{C}^\times)$ can be identified with $U \cap H^2(X;\mathbb{C}) \subset A$ by means of a translated exponential map. Subject to convergence, we can extend the family GW_u to an open set U of $A = H^{ev}(X)$, starting from our base point. If convergence fails, we treat $H^2(X;\mathbb{C}^\times) \times H^{ev,\neq 2}(X)$ as a formal Frobenius manifold. The dimension formula (1.6) for the spaces of stable maps ensures that the family GW_u obtained from (7.18) is homogeneous of weight $d = \dim_{\mathbb{C}} X$ with respect to the Euler field

$$\xi_{GW} = c_1(X) + \sum\nolimits_j \left(1 - \frac{\deg(x^j)}{2}\right) \frac{\partial}{\partial x_j}$$

in a homogeneous basis x^j of $H^\bullet(X)$. Thus, $\mu = (\deg - d)/2$.

We conclude by describing the homogeneity condition in terms of the data E_u, $\tilde{Z}_u$.

7.19 Proposition. *In a homogeneous semi-simple CohFT, $E_u(z)$, $\tilde{Z}_u^+$ and $\mathbf{v}(z)$ are invariant under the shifted Lie action $\mathcal{L}^+$ of the Euler field ξ.*

Recall that z has weight 1, so we are saying that the z^jth Taylor coefficient in E_u has weight $(-j)$. The same applies to the coefficient a_j of κ_j in $\log \tilde{Z}^+$. It is not difficult to show that, for a vector field ξ of the form in §7.15, these conditions are also sufficient for homogeneity of $\overline{Z}$, but we will not use that fact.

Proof. The operator ${}_u^1Z_g^1$ for smooth surfaces must have weight $gd = (g-1)d + 2 + (d-2)$, the last term being the added weight of replacing an input by an output. In particular, ${}^1\tilde{Z}_g^1 = (\alpha^g \tilde{Z}^+ \cdot)$ has weight gd, whereas $(\alpha \cdot)$ has weight d; this settles $(\tilde{Z}_u^+ \cdot)$. Next, since ${}_u^1Z_{g,1} = E(-\psi_0) \circ {}^1\tilde{Z}_g^1$,

$$\mathcal{L}({}_u^1Z_{g,1}) = \mathcal{L}(E(-\psi_0)) \circ {}^1\tilde{Z}_g^1 + E(-\psi_0) \circ \mathcal{L}({}^1\tilde{Z}_g^1),$$

showing that the first term vanishes, so $\mathcal{L}(E(-\psi_0)) = 0$. The final statement follows from the relation

$$E(z)^{-1}(\mathbf{v}(z)) = (\tilde{Z}^{+})^{-1}\Big|_{\kappa_j = z^j}.$$

$\square$

8. Reconstruction

I now explain the reconstruction of semi-simple cohomological field theories from genus zero information, confirming a conjecture of Givental's for Gromov-Witten theory [G1]. In the case of homogeneous theories with flat vacuum, I also give a concrete variant which uses less input: the Euler vector field plus the Frobenius *algebra* at a single semi-simple point of the Frobenius manifold (Theorem 1). This more economical recipe is implicit in Dubrovin's paper [D]. The present section is largely a review and adaptation of Givental's relevant work.

(8.1) Reconstruction from the Frobenius manifold: Givental's conjecture. Let $\mathbf{u}$ be the vector of *canonical coordinates*, for which the associated vector fields $\partial/\partial u^i$ are the projectors P_i in the multiplication at the respective point. The existence of such coordinates, shown in [D], follows from the integrability conditions of §7.3. Clearly, the u_i are unique up to constant shifts. In the case of homogeneous Frobenius manifolds, a distinguished choice of canonical coordinates is given by the eigenvalues of multiplication by the Euler vector field ξ.

8.2 Proposition. *(i) The linear map $d\mathbf{u} : T_u U \to \mathbb{C}^N$ is given by $\Pi_u^{-1} \circ (\alpha_u^{-1/2} \cdot)$.*
(ii) The system of ODE's in (7.7.a) is equivalent to

$$\frac{\partial F}{\partial v} = -\frac{(v\cdot)}{z} \circ F, \quad \textit{with} \quad F(z) = E_u(z) \circ \Pi_u \circ \exp\left(-\frac{\mathbf{u}*}{z}\right).$$

Proof. The first part merely rewrites the defining property of $\mathbf{u}$: $d\mathbf{u}$ takes the projector frame to the standard frame of $\mathbb{C}^N$. For the second claim, use the chain rule and the relation $\Pi \circ (\frac{\partial \mathbf{u}}{\partial v} *) = (v\cdot) \circ \Pi$, which in turn is a consequence of part (i) and of formula (7.5). $\square$

8.3 Remark. (i) Usually, $E(z)$ does not converge, so $F(z)$ does not really belong to a "symplectic loop group", but to a thickened version of it.
(ii) Letting $\xi = \sum_i u_i \partial/\partial u_i$, in canonical coordinates, an alternative expression for F is

$$F(z) = E_u(z) \circ \exp\left(-\frac{(\xi \cdot_u)}{z}\right) \circ \Pi_u.$$

In the homogeneous case, ξ is the Euler vector field.

The system of ODE's in Proposition 8.2.ii is that of [G1, pp.1269–1270], with the change of notation $\Psi = \Pi$, $R(z) = \Pi^{-1} E(z) \Pi$. Recall:

8.4 Proposition ([D, G1])**.** *The system of Proposition 8.2.ii has solutions in which $R \equiv$ Id (mod z) satisfies the symplectic condition $R_u(z) R_u^*(-z) =$ Id. These solutions are unique up to right multiplication by a matrix series $H(z) = \exp\left(H_1 z + H_3 z^3 + \dots\right)$ with constant* diagonal *matrices H_{2i+1}. In the homogeneous case, there is a* unique *solution with R invariant under the Euler field.* $\square$

The proof of the proposition, for which we refer to Givental [G1], is closely related to the reconstruction procedure we will give below, in the homogeneous case. The ambiguity in R reflects the possibility of a z-dependent shift in the canonical coordinates, the parity constraint coming from the symplectic condition. In terms of E, this ambiguity is the right composition with the operator of multiplication by a "symplectic" unit in $A[\![z]\!]$. Note that Euler invariance of R and E are equivalent because of the relation $\mathcal{L}(\Pi) = (d/2-1)\Pi$.

8.5 Corollary. *A semi-simple homogeneous CohFT is determined from its Frobenius manifold by the unique Euler-invariant solution E of the ODE (7.7.a) and the vacuum* (7.14). $\square$

Of course, in a flat vacuum theory, $\mathbf{v}(z)$ and $\tilde{Z}$ are already determined by E, cf. (4.12), §6.18.

(8.6) Ambiguity for inhomogeneous theories. The various inhomogeneous theories for given semi-simple Frobenius manifold are related geometrically by *Hodge bundle twists*. More precisely, let $\mu_j = \mathrm{ch}_j\Lambda$ be the Chern components of the Hodge bundle Λ, with fibre the global fibre-wise differentials on the universal curve, with simple poles allowed along the marked sections σ_i. Recall that the classes μ_j vanish for even j. We can form a *Hodge-twisted theory* CohFT based on A from any odd series $h(z) := \sum_j h_{2j-1}z^{2j-1}$ in $A[\![z]\!]$ by setting

$$ {}^n\overline{Z}_g[h] = h(\psi_1) \wedge \cdots \wedge h(\psi_n) \wedge \exp\left\{\sum\nolimits_j h_{2j-1}\frac{(2j)!}{B_{2j}} \cdot \mu_{2j-1}\right\}, $$

with the Bernoulli numbers B_{2j}. In genus zero, this class is identically 1, because the Hodge bundle Λ is trivial. The other $\overline{Z}$-classes are determined from the CohFT axioms. The fact that this does give a CohFT with flat vacuum can be checked directly from the basic properties of the Hodge bundle: Λ is *primitive*[12] under restriction to boundaries of $\overline{M}_g^n$, and changes under forgetful pullback only by the addition of a trivial line. Givental's calculation in [G1, §2.3], summarised in Part (i) of the proposition below, identifies the theory for us.

8.7 Proposition. *(i) The theory $\overline{Z}[h]$ is the transform of the trivial A-theory by $T_z^{-1} \circ \exp h(z) \circ T_z$.*
(ii) All cohomological Field theories with flat vacuum based on the same *semi-simple Frobenius manifold are classified by matrices $E \circ \exp h(z)$, with arbitrary h but the* same *E. That is, they all are the transforms of a general Hodge-twisted theory based on A by the same group element E.* $\square$

It is amusing to revisit the flat vacuum condition (4.12). Restricting to M_g, we obtain from statement (i)

$$ \sum_j h_{2j-1}\frac{(2j)!}{B_{2j}} \cdot \mu_{2j-1} = \sum_j h_{2j-1}\kappa_{2j-1}, $$

which of course follows from the Riemann-Roch identities $\mu_{2j-1} = \frac{B_{2j}}{(2j)!} \cdot \kappa_{2j-1}$ over M_g.

(8.8) Rank one theories: a conjecture of Manin and Zograf. When A has rank 1, CohFT's are necessarily semi-simple. Moreover, $\mathrm{GL}(A)[\![z]\!]$ is abelian and we can give closed formulae for all possible $\overline{Z}$.

Taking logarithms converts the FTFT factorisation axiom for the classes $\overline{Z}_g^n$ into a *primitivity* condition. Manin and Zograf conjectured in [MZ] that the κ_j, $j \geq 0$ and the μ_j ($j > 0$, odd) of the Hodge bundle were the only primitive classes on the $\overline{M}_g^n$; consequently, they proposed that any rank 1 theory should have the form

$$ \overline{Z}_g^n = \exp\left\{\sum\nolimits_{j\geq 0} a_j\kappa_j + \sum\nolimits_{j>0} h_j\mu_j\right\} \tag{8.9} $$

[12]When there are no marked points, we ought to normalise the bundle to $\Lambda - \mathbb{C}$.

with constants $a_j, f_j \in \mathbb{C}$. This is also the 'matrix' form of $\overline{Z}$ in the canonical basis $p = \log a_0$.

8.10 Proposition. *Formula* (8.9) *describes all possible rank one CohFT's. Flat vacuum theories are those with $a_j = 0$ for $j > 0$.*

Proof. The symplectic condition says that the classifying element $E(z)$ has the form E_h above. Allowing for general translation ζ inserts an arbitrary κ-class combination as in (8.9), but the flat vacuum condition fixes the a_j to be zero, as per Proposition 8.7.i (but with redefined h_j's). $\square$

(8.11) Classification of homogeneous CohFT's. Since the family ${}_u\overline{Z}$ of theories is constructed from the theory at $u = 0$, we can describe the homogeneity condition in terms of the Euler field ξ and the classification datum E. As before, let $\xi = \xi_0 - \mu^i_j x^j \partial_i + (1 - d/2) x^i \partial_i$, with the constant vector field ξ_0. As always, $(\xi_0 \cdot)$ denotes the operator of multiplication by ξ_0 in A. We focus on the important special case of flat vacuum theories, and show that they are completely determined by the Frobenius algebra structure and the Euler field.

8.12 Proposition. *The CohFT $\overline{Z}$ with flat vacuum defined by E is homogeneous of weight d for ξ iff*

$$\mu(\mathbf{1}) = -\frac{d}{2} \cdot \mathbf{1} \quad \textit{and} \quad [(\xi_0 \cdot), E_{k+1}] + (\mu + k) E_k = 0.$$

8.13 Remark. (i) Without the flat vacuum assumption, the first equation must be replaced by

$$\left(\mu + \frac{d}{2}\right) \mathbf{v}(z) = \frac{\xi_0}{z} \cdot (\mathbf{v}(z) - \mathbf{1}).$$

At a generic point where ξ_0 is invertible in the algebra (away from the canonical coordinate axes), the Taylor coefficients of $\mathbf{v}$ are recursively determined by this equation.
(ii) The second recursion is equivalent to an ODE for the expression $F(z)$ of Remark 8.3.ii,

$$\frac{dF}{dz} + \frac{\mu}{z} \circ F = \frac{(\xi \cdot)}{z^2} \circ F.$$

(iii) For $k = 0$, we find $\mu = [E_1, (\xi \cdot)]$. When $(\xi \cdot)$ has repeated eigenvalues (on the big diagonal in canonical coordinates), solvability of this equation places constraints on μ. In a general Frobenius manifold, one can expect semi-simplicity to fail on the big diagonal. However, the requisite constraint on μ *must* hold at all *semi-simple* diagonal points of the Frobenius manifold, because the solution E_u exists there.

Proof. First, $\mathbf{1} = -\mathcal{L}(\mathbf{1}) = -\partial(\mathbf{1})/\partial\xi - \mu(\mathbf{1}) + (1 - d/2) \cdot \mathbf{1}$; flatness of $\mathbf{1}$, $\partial(\mathbf{1})/\partial\xi = 0$, gives the first relation. Next, $\mathcal{L}(E_k) = -kE_k$ from Proposition 7.19. But

$$\mathcal{L}(E_k) = \frac{\partial E_k}{\partial \xi} + \mu \circ E_k - E_k \circ \mu,$$

whereas according to equation (7.7.a),

$$\frac{\partial E_k}{\partial \xi} = [E_{k+1}, (\xi \cdot)] - E_k \circ \frac{\partial \Pi}{\partial \xi} \circ \Pi^{-1}.$$

The normal canonical frame Π scales with weight $(d/2 - 1)$ under the Euler flow; since

$$\mathcal{L}(\Pi) = \frac{\partial \Pi}{\partial \xi} + \mu \circ \Pi + (d/2 - 1)\Pi,$$

we have $\partial_\xi \Pi \circ \Pi^{-1} = -\mu$ and combining the equations proves necessity of the conditions:

$$-kE_k = [E_{k+1}, (\xi\cdot)] - E_k \circ \partial_\xi(\Pi) \circ \Pi^{-1} + \mu \circ E_k - E_k \circ \mu = [E_{k+1}, (\xi\cdot)] + \mu \circ E_k.$$

Conversely, the same calculations show that the two conditions imply the $\mathcal{L}$-homogeneity of ${}_u\overline{Z}_g^n$ at $u = 0$,

$$\mathcal{L}({}_u\overline{Z}_g^n)\Big|_{u=0} = (gd - d + n)\overline{Z}_g^n.$$

We now check that Euler homogeneity at any other point is a formal consequence. Recall from §7.15 the action ad_ξ of $\mathcal{L}$ on the flat frame of TU and its multi-linear extension to tensors. Also, denote by Δ half the degree operator on $H^\bullet(\overline{M})$; it was implicit in Definition 7.16. At a point u, ξ has the value $\xi_u = \xi_0 - \mathrm{ad}_\xi(u)$ and

$$(\mathcal{L} - \Delta)({}_u\overline{Z}_g^n) = \partial_{\xi_u}({}_u\overline{Z}_g^n) - {}_u\overline{Z}_g^n \circ \mathrm{ad}_\xi = \int_{\overline{M}_g^{n+1}}^{\overline{M}_g^n} \iota(\xi_0 - \mathrm{ad}_\xi(u))_u\overline{Z}_g^{n+1} - {}_u\overline{Z}_g^n \circ \mathrm{ad}_\xi.$$

Substitute now formula (7.1) for ${}_u\overline{Z}$, this becomes

$$-\sum_m \frac{(-1)^m}{m!} \int_{\overline{M}_g^{n+m+1}}^{\overline{M}_g^n} \left(\iota(u)^m \iota(\xi_0)\overline{Z}_g^{n+m+1} - \iota(u)^m \iota(\mathrm{ad}_\xi(u))\overline{Z}_g^{n+m+1}\right) - {}_u\overline{Z}_g^n \circ \mathrm{ad}_\xi,$$

and shifting the summation index $m \mapsto m+1$ in the second term of the sum converts this into

$$\sum_m \frac{(-1)^m}{m!} \int_{\overline{M}_g^{n+m}}^{\overline{M}_g^n} \iota(u)^m \left(\partial_{\xi_0}\overline{Z}_g^{n+m} - \overline{Z}_g^{n+m} \circ \mathrm{ad}_\xi\right)$$

By homogeneity at $u = 0$, the integrand is $\iota(u)^m(\mathcal{L} - \Delta)\overline{Z}_g^{n+m} = \iota(u)^m(gd - d + n + m - \Delta)\overline{Z}_g^{n+m}$. Pulling Δ through the integral gives $(gd - d + n - \Delta)_u\overline{Z}_g^n$, proving homogeneity at u. □

(8.14) GW invariants from quantum cohomology. As we now explain, Proposition 8.12 determines E from A, ξ_0 and μ. In Gromov-Witten theory, we have:

8.15 Theorem. *The Gromov-Witten classes $GW_{g,d}^n \in H^{ev}(\overline{M}_g^n)$ of a compact symplectic manifold are uniquely determined by its first Chern class and by the quantum multiplication law at any* single *semi-simple point.*

Proof. Assume first that the quantum multiplication operator $(\xi\cdot)$ has distinct eigenvalues. Working in the normal canonical basis, the second equation in Proposition 8.12 supplies the off-diagonal entries of E_k, once E_{k-1} is known. Next, since $(\xi\cdot)$ is a diagonal matrix, the diagonal entries of the commutator $[(\xi\cdot), E_{k+1}] = (\mu + k)E_k$ must vanish; since those of the skew matrix μ vanish as well, this fact determines the diagonal part of E_k from its off-diagonal part. Finally, $E_0 = \mathrm{Id}$.

In the general case, consider the block-decompositions of μ and of the E_k corresponding to the eigenspaces of $(\xi\cdot)$. The first equation $[(\xi\cdot), E_1] = \mu$ implies the vanishing of the diagonal blocks of μ. This is a constraint which *must* hold if A is semi-simple. Given that, the off-diagonal blocks of E_1 are determined from those of μ. The diagonal blocks are determined from the vanishing of those of $(\mu + \mathrm{Id})E_1$ — which must equal $[(\xi\cdot), E_2]$ — and in this way, the recursive determination of the E_k proceeds as before. □

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