

Week 10: The Infamous 1004913 Problem

For a finite group G and a prime p dividing $|G|$, let $n_p(G)$ denote the number of Sylow p -subgroups of G .

1. Let p be a prime and let $h \in S_n$ have order p . Suppose that $g \in S_n$ satisfies $ghg^{-1} = h^k$ for some integer k . Show that if $gh \neq hg$ then g fixes at most one point in each cycle of h .
2. Let G be a finite simple group of order $3^3 \cdot 7 \cdot 13 \cdot 409$. Show that

$$\begin{aligned}n_7(G) &= 3^2 \cdot 13 \cdot 409, \\n_{13}(G) &= 3^2 \cdot 7 \cdot 409, \\n_{409}(G) &= 3^2 \cdot 7 \cdot 13.\end{aligned}$$

3. Let P and Q be Sylow 3-subgroups of G with $|P \cap Q|$ maximal. Suppose for contradiction that $P \cap Q \neq 1$. Let $H = N_G(P \cap Q)$.
 - (a) Show that $P \cap H$ and $Q \cap H$ are distinct Sylow 3-subgroups of H .
 - (b) Show that $9 \mid |H|$ and that $|H| \mid 3^3 \cdot 7 \cdot 13$.
 - (c) Show that if $7 \mid |H|$ then $n_7(H) = 351$.
 - (d) Show that if $13 \mid |H|$ then $n_{13}(H) = 27$.
 - (e) Show that H has too many elements.

Thus, $P \cap Q = 1$ so any two distinct Sylow 3-subgroups of G intersect trivially.

4. Let P be a Sylow 3-subgroup of G . By letting P act by conjugation on the Sylow 3-subgroups of G , show that $n_3(G) \equiv 1 \pmod{27}$. Deduce that

$$n_3(G) = 7 \cdot 409.$$

5. Show that every nonidentity element of G has prime order.
6. Compute the class equation for G .
7. Let G act by conjugation on the Sylow 409-subgroups of G . Use Burnside's lemma to show that each element of G of order 3 fixes exactly 9 Sylow 409-subgroups of G .
8. Use the first problem to obtain a contradiction.