Uniqueness and Stability in the Cauchy Problem for Maxwell' and elasticity systems

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Abstract

While the unique continuation problem is fairly well understood for scalar equations, for systems it is considerably less explored. The aim of this article is to consider the unique continuation problem for several systems of interest. We start with a weakly coupled hyperbolic system. Then we take a look at Maxwell's equations and we conclude with isotropic elasticity. All these systems share the common feature that they can be uncoupled for the principal part, modulo an elliptic component.

1 Preliminaries

By $x = (x_0, x_1, ..., x_n)$ we denote the coordinates in $R \times R^n$. For the differentiation operators we set $\partial_j = \partial/\partial x_j$ and $D_j = \frac{1}{i}\partial/\partial x_j$. Sometimes we shall call x_0 the time coordinate, and use the alternate notation t for it.

Given a positive scalar function a in R^n we define the scalar associate wave operator in $R \times R^n$ by

$$\Box_a = a\partial_t^2 - \Delta.$$

By H^s we denote the classical Sobolev spaces, with the norm denoted $\|.\|_s$. Often we shall use on these spaces the weighted norms,

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$$\|u\|_{s;\tau}^2 = (2\pi)^{-n} \int (\tau^2 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \ . \tag{1}$$

These norms can also be defined in a bounded domain Ω in a standard manner.

2 Pseudoconvexity and Carleman estimates

The aim of this section is to review the concept of pseudoconvexity and state several kinds of Carleman estimates for scalar equations. The Carleman estimates have been the main tool in the study of unique continuation problems; see [5]. While we cannot transfer the unique continuation results directly from equations to systems, it turns out that sometimes we can transfer the Carleman estimates to systems; these in turn imply unique continuation results for systems.

Thus, let $P(x, D)$ be a partial differential operator with real principal part whose principal symbol we denote by $p(x, \xi)$. We will assume that the coefficients of the principal part of P are in C^1 while the other coefficients are bounded and measurable.

Let S be a smooth oriented surface in R^{n+1}. Then we can represent it as a level set $S = \{\phi = 0\}$ of a smooth function ϕ so that $\nabla\phi \neq 0$ on S. The function ϕ is uniquely determined up to multiplication by smooth positive functions.

To a smooth function ϕ we associate the operator

$$P_\phi(x, D, \tau) = P(x, D + i\tau\nabla\phi)$$

which is the conjugate of the operator P with respect to the exponential weight $e^{\tau\phi}$,

$$P_\phi(x, D, \tau) = e^{\tau\phi} P(x, D) e^{-\tau\phi}$$

The principal symbol of the conjugated operator P_ϕ is

$$p_\phi(x, \xi, \tau) = p(x, \xi + i\tau\nabla\phi)$$

The Poisson bracket of two symbols P, q is defined as

$$\{p, q\} = \nabla_\xi p \cdot \nabla_x q - \nabla_x p \cdot \nabla_\xi q$$

Let Γ be a conical subspace of the cotangent bundle T^*R^{n+1} . Now we are ready to define pseudoconvex functions and surfaces.

Definition 2.1 *a) A smooth function ϕ is pseudo-convex with respect to P on Γ if*

$$\{p,\{p,\phi\}\}(x,\xi) > 0 \quad \forall\ (x,\xi) \in \Gamma, \xi \neq 0,\ p(x,\xi) = 0 \tag{2}$$

b) A smooth oriented surface $S = \{\phi = 0\}$ is called pseudo-convex with respect to P on Γ if

$$\{p,\{p,\phi\}\}(x,\xi) > 0 \quad \forall\ (x,\xi) \in \Gamma_S,\ \xi \neq 0,\ p(x,\xi) = \{p,\phi\}(x,\xi) = 0 \tag{3}$$

Definition 2.2 *a) A smooth function ϕ is strongly pseudo-convex with respect to P on Γ if it is pseudoconvex and*

$$\frac{1}{i\tau}\{\bar{p_\phi}, p_\phi\}(x,\xi) > 0 \quad \forall\ (x,\xi) \in \Gamma,\ \xi \neq 0,\ \tau > 0,\ p_\phi(x,\xi) = \ 0\,. \tag{4}$$

b) A smooth oriented surface S is strongly pseudo-convex with respect to P on Γ if it is pseudoconvex and

$$\frac{1}{i\tau}\{\bar{p_\phi}, p_\phi\}(x,\xi) > 0\ \forall\ (x,\xi) \in \Gamma_S,\ \begin{array}{l} \xi \neq 0,\ \tau > 0, \\ p_\phi(x,\xi) = \{p_\phi, \phi\}(x,\xi) = 0. \end{array} \tag{5}$$

The relation between pseudoconvex functions and surfaces is the obvious one,

Remark 2.3 *a) An oriented surface S is (strongly) pseudoconvex with respect to P on Γ iff it is a level set of a function ϕ which is (strongly) pseudoconvex with respect to P on Γ, so that $\nabla\phi \neq> 0$ on Γ.*

b) Moreover, if the oriented surface $S = \{\phi = 0\}$ is (strongly) pseudoconvex with respect to P on Γ then $e^{\lambda\phi}$ is (strongly) pseudoconvex with respect to P on Γ for large enough λ.

c) The (strong) pseudoconvexity condition for both functions and surfaces is stable with respect to small C^2 perturbations.

For second order operators the condition (5) is void for noncharacteristic surfaces, therefore

Proposition 2.4 *Let P be a second order operator with real principal symbol. If a noncharacteristic oriented surface S is pseudoconvex with respect to P on Γ then it is strongly pseudoconvex with respect to P on Γ.*

Now we discuss the corresponding Carleman estimates. Since this is all we use later on, in the sequel we only refer to second order partial differential operators $P(x, D)$ with real principal symbol. The classical result (see [4]) is

Theorem 2.5 *Let Ω be a compact subset of R^{n+1}. If ϕ is strongly pseudoconvex with respect to P in Ω then*

$$\tau \|e^{\tau\phi} u\|_{1;\tau}^2 \leq C \|e^{\tau\phi} P u\|_0^2 \tag{6}$$

whenever $u \in H^1$ is supported in Ω so that the RHS is finite.

The substitution $v = e^{\tau\phi} u$ reduces the estimate to

$$\tau \|v\|_{1;\tau}^2 \leq C \|P_\phi v\|_0^2$$

which is essentially a subelliptic estimate for P_ϕ. The strong pseudoconvexity condition therefore expresses the subellipticity of P_ϕ in terms of its symbol.

If instead Ω is a bounded domain with smooth boundary and u is a function in Ω, then the appropriate estimate should include the Cauchy data of u on the boundary, see Tataru [11].

Theorem 2.6 *Let Ω be a compact subset of R^n. If ϕ is strongly pseudoconvex with respect to P in Ω then*

$$\tau \|e^{\tau\phi} u\|_{1;\tau,\Omega}^2 \leq C(\|e^{\tau\phi} P u\|_{(0)}^2 + \tau(\|e^{\tau\phi} u\|_{1;\tau,\partial\Omega}^2 + \|e^{\tau\phi} \frac{\partial u}{\partial \nu}\|_{0,\partial\Omega}^2)) \tag{7}$$

whenever $u \in H^1$ is supported in Ω so that the RHS is finite.

In effect such an estimate might hold even with only one boundary data on the RHS, provided that the corresponding boundary operator satisfies a strong Lopatinskii boundary condition with respect to $d\phi$, see [11].

If the coefficients of P are analytic with respect to some variables then the unique continuation results can be improved. We state here the Carleman estimate corresponding to the simplest result of this type, namely for the wave equation with time independent coefficients. In this case, we set $\Gamma = \{\xi_0 = 0\}$, where ξ_0 is the time Fourier variable and consider functions and operators in R^{n+1}.

Theorem 2.7 *Let P be a second order hyperbolic operator with time independent coefficients. Furthermore, let ϕ be a quadratic function in $\bar{\Omega}$ which is strongly pseudo-convex with respect to P on Γ. Then there exist $d > 0$ and $C > 0$ such that*

$$\tau \| e^{-\frac{\epsilon}{2\tau} D_0^2} e^{\tau\phi} u \|_{1;\tau}^2 \leq C \Big(\| e^{-\frac{\epsilon}{2\tau} D_0^2} e^{\tau\phi} P(x, D) u \|_2^2 + \| e^{\tau(\phi - d\epsilon)} u \|_{1;\tau}^2 \Big) \qquad (8)$$

for all $u \in H^1$, supported in Ω, provided τ is large enough and ϵ is sufficiently small.

This result was proved in [10]. In effect it holds even if we only assume that the coefficients of P are analytic in time and that ϕ is analytic. However, the proof is substantially simpler in the special case considered above. The critical point is the conjugation argument, which needs to be carried through also with respect to the Gaussian.

With the substitution $w = e^{\tau\phi} u$ the estimate (8) reduces to

$$\tau \| e^{-\frac{\epsilon}{2\tau} D_0^2} w \|_{1;\tau}^2 \leq C \Big(\| e^{-\frac{\epsilon}{2\tau} D_0^2} P(x, D + i\tau\nabla\phi) w \|_2^2 + \| e^{-d\epsilon\tau} w \|_{1;\tau}^2 \Big) \qquad (9)$$

Furthermore

$$e^{-\frac{\epsilon}{2\tau} D_0^2} t = (t + i\frac{\epsilon}{\tau} D_0) e^{-\frac{\epsilon}{2\tau} D_0^2}$$

Since $\nabla\phi$ is linear, this implies that

$$e^{-\frac{\epsilon}{2\tau} D_0^2} P(x, D + i\tau\nabla\phi) = P(x, D + i\tau\nabla\phi - \epsilon\partial_t(\nabla\phi) D_t) e^{-\frac{\epsilon}{2\tau} D_0^2} > \qquad (10)$$

Hence if we set $v = e^{-\frac{\epsilon}{2\tau} D_0^2} w$ then (9) reduces to

$$\tau \| v \|_{1;\tau}^2 \leq C \| P_{\epsilon,\phi} v \|_2^2 + \| e^{-d\epsilon\tau} w \|_{1;\tau}^2 \Big)$$

with

$$P_{\epsilon,\phi} = P(x, D + i\tau\nabla\phi - \epsilon\partial_t(\nabla\phi) D_t)$$

If ϵ is small enough then $P_{\epsilon,\tau}$ is a small perturbation of P_ϕ. Then the pseudoconvexity condition for ϕ implies some subellipticity for $P_{\epsilon,\phi}$ in the region $|\xi_0| \leq c\tau$. Outside this region, the Gaussian yields exponential decay in τ and leads to the last RHS term.

3 Uniqueness and stability for principally scalar systems

Here we apply the results in the previous section to the systems of the type

$$P_j u_j + b_j(x,t;\nabla u) + c_j(x,t;u) = f_j,\ j = 1,...,m \tag{11}$$

Here P_j are second order operators with real principal part and C^1 coefficients and other coefficients in L^∞_{loc},b_j, c_j are linear functions of $\nabla u, u$ with L^∞_{loc}-coefficients, $u = (u_1, ..., u_m)$.

First we obtain the Carleman estimates for such systems.

Theorem 3.1 *Let ϕ be a smooth function which is strongly pseudoconvex with respect to all P_j in a given compact set K. Then*

$$\tau \|e^{\tau\phi} u\|^2_{1,\tau} \le c\|e^{\tau\phi} f\|^2, \qquad \tau > \tau_0 \tag{12}$$

whenever u is supported in K and solves (11).

Proof: First we apply the Carleman estimates in Theorem 2.5 to u_j with respect to P_j. This yields

$$\tau \|e^{\tau\phi} u_j\|^2_{1,\tau} \le c\|e^{\tau\phi} P_j u_j\|^2,$$

and further

$$\tau \|e^{\tau\phi} u_j\|^2_{1,\tau} \le c\|e^{\tau\phi} u_j\|^2_{1,\tau} + \|e^{\tau\phi} f_j\|^2_{1,\tau}$$

Summing with respect to j, the first terms on the right are absorbed on the left for sufficiently large τ and we obtain (12).

Next we consider the unique continuation problem for the corresponding homogeneous problem

$$P_j v_j + b_j(x,t;\nabla v) + c_j(x,t;v) = 0,\ j = 1,...,m \tag{13}$$

Theorem 3.2 *Let $S = \{\phi = 0\}$ be an oriented surface which is strongly pseudoconvex with respect to P_j for all j. Then we have unique continuation across S for H^1_{loc} solutions u to (13). More precisely, given any $x_0 \in S$ and $v \in H^1_{loc}$ solving (13) near x_0 which is identically zero in $\{\phi > 0\}$ near S, this implies that $v = 0$ near x_0.*

Proof:

Without any restriction in generality we assume that $x^0 = 0$. Since S is strongly pseudo-convex, by Remark 2.3 we can also assume that ϕ is strongly pseudoconvex with respect to P_j.

Consider the modified function $\psi(x,t) = \phi(x,t) + \epsilon(r^2 - |x|^2)$. For small ϵ, ψ is still strongly pseudoconvex with respect to all P_j near x_0. Furthermore,

$$\{\psi > 0\} \cap \{\phi < 0\} \subset B(0,r)$$

Hence, by choosing r sufficiently small we insure that (13) holds in $\{\psi > 0\}$.

Now let χ be a cutoff function which is 1 in $\psi > 0$ and 0 in $\psi < -\epsilon r^2$. Then the function $u = \chi v$ solves (11) with $f \in L^2$, supported in

$$\text{supp } \nabla\chi \subset \{\psi < 0\}.$$

Apply the Carleman estimates (12) to u. We obtain

$$\tau \|e^{\tau\psi}\chi f\|_{1,\tau} \leq c\|e^{\tau\psi} f\|^2$$

Since f is supported in $\{\psi < 0\}$, letting τ go to infinity yields

$$\lim_{\tau\to\infty} \|e^{\tau\psi}\chi v\| = 0$$

which implies that $v = 0$ in $\{\psi > 0\}$. Since $\psi(0) > 0$ this concludes the proof.

The next step is to establish the corresponding stability estimate. For this it is more interesting to work with boundary value problems. Thus, let Ω be a bounded domain in R^{n+1} with C^1 boundary $\partial\Omega$. Let S be a part of $\partial\Omega$ where one prescribes the Cauchy data for u

$$u = g_0, \quad \partial_\nu u = g_1 \quad \text{on } S \tag{14}$$

Given a smooth function ϕ let

$$\Omega_\epsilon = \Omega \cap \{\epsilon < \phi\}, \quad \partial\Omega_\epsilon = \partial\Omega \cap \{\epsilon < \phi\}$$

.

Theorem 3.3 *Let ϕ be strongly pseudo-convex with respect to all operators P_j on $\overline{\Omega}$ and $\partial\Omega_0 \subset S$. Let $u \in H^1(\Omega)$ solve the Cauchy problem (11),(14).*

Then with some positive constants C, $\lambda = \lambda(\epsilon) \in (0,1)$ we have

$$\|u\|_{1,\Omega_\epsilon} \leq C(\epsilon)(F + F^\lambda M^{1-\lambda})$$

where

$$F = \|g_0\|_{1,S} + \|g_1\|_{0,S} + \|f\|_{0,\Omega_0}, \quad M = \|u\|_{1,\Omega_0}.$$

Proof: We want to use the analogue of the Carleman estimate (12) but for boundary value problems,

$$\tau\|e^{\tau\phi}u\|^2_{1;\tau,\Omega} \leq C(\|e^{\tau\phi}f\|^2_{0,\Omega} + \tau(\|e^{\tau\phi}u\|^2_{1;\tau,\partial\Omega} + \|e^{\tau\phi}\frac{\partial u}{\partial\nu}\|^2_{0,\partial\Omega})) \tag{15}$$

for u solving (11) in Ω. This follows in the same way as (12) if we use (7) instead of (6).

To proceed we need to localize to Ω_0. Hence, introduce a cut-off function $\chi \in C^\infty(R^{n+1}), \chi = 1$ on $\Omega_{\epsilon/2}$ and $\chi = 0$ outside Ω_0. Then work with χu. After commuting we get

$$P_j(\chi u_j) = \chi f_j + A_{1;j}u$$

where $A_{1;j}$ is a linear differential operator of first order with measurable bounded coefficients depending on ϵ, supported in Ω_0.

Hence, applying (15) to χu we obtain the inequality

$$\begin{aligned} \tau\|e^{\tau\phi}\chi u\|^2_{1;\tau} &\leq C(\epsilon)(\|e^{\tau\phi}u\|^2_{1,\tau,\Omega_0} + \|e^{\tau\phi}\chi f_j\|^2_0 + \\ &+ \tau\sum_{k=0,1}\|e^{\tau\phi}\partial^k_\nu u_j\|^2_{1-j;\tau,S}) \end{aligned}$$

Since $\chi = 1$ on $\Omega_{\epsilon/2}$, this further gives

$$\begin{aligned} (\tau - C(\epsilon))\|e^{\tau\phi}u\|^2_{1;\tau,\Omega_{\epsilon/2}} &\leq C(\epsilon)\|e^{\tau\phi}u\|^2_{1,\tau,\Omega_0\backslash\Omega_{\epsilon/2}} + \|e^{\tau\phi}\chi f_j\|^2_0 + \\ &+ \tau\sum_{k=0,1}\|e^{\tau\phi}\partial^k_\nu u_j\|^2_{1-j;\tau,S}) \end{aligned}$$

For large enough τ this finally gives

$$\|u_j\|_{1,\tau,\Omega_\epsilon} \leq C(\epsilon)(Fe^{\tau(\Phi-\epsilon/2)} + Me^{-\tau\epsilon/2})$$

where Φ is the supremum of ϕ in Ω. This inequality holds for all sufficiently large τ, say $\tau \geq \tau_0(\epsilon)$. Increasing $C(\epsilon)$ we can take $\tau_0(\epsilon) = 0$. Hence we can minimize the right hand side with respect to τ. If $M \leq F$ then Theorem 3.3) follows if we simply set $\tau = 0$. Otherwise, we choose τ so that

$$Fe^{\tau(\Phi-\epsilon/2)} = Me^{-\tau\epsilon/2}$$

and then the right hand side equals

$$C(\epsilon)M^{1-\lambda}F^{\lambda}, \quad \lambda = \frac{\epsilon}{2\Phi}$$

and we obtain the conclusion of the theorem with $\lambda = \epsilon/(2\Phi)$.

If a portion of the boundary is strongly pseudoconvex, then the above result can be used locally near S. The corresponding uniqueness result follows easily from Theorem 3.2.

Corollary 3.4 *Assume that the surface $S \subset \partial\Omega$ is strongly pseudo-convex with respect to $P_j, j = 1, ..., m$. If $u \in H^1(\Omega)$ solves the system (13) with $f = 0$ and has zero Cauchy data (14) on S then $u = 0$ in some neighborhood of S in Ω.*

Another corollary gives an explicit description of an uniqueness domain when $S = \gamma \times (-T, T) \in C^2$ in the two following cases: 1) $\gamma = \partial\Omega'$, the space origin is in Ω' and 2) $\Omega' \subset \{-h < x_n < 0, x_1^2 + ... + x_{n-1}^2 < r^2\}, S = \partial\Omega' \cap \{x_n < 0\}$. Here we let $x' = x \in R^n$. To formulate the result we need the weight function $\psi(x,t) = x_1^2 + ... + x_{n-1}^2 + (x_n - \beta)^2 - \theta^2 t^2 - s$ to define $\Omega(\epsilon)$ as $\Omega \cap \{\psi > \epsilon\}$

Corollary 3.5 *Assume that the coefficients $a(j), j = 1, ..., m$ of the principal part $P_j = \Box_{a(j)}$ of the system (11) satisfy the following conditions*

$$\theta^2 a(j)(a(j) + \partial_t a(j)/2 + a^{-1/2}(j)|t\nabla a(j)|) < a(j) + 1/2x \cdot \nabla a(j) - 1/2\beta\partial_n a(j)$$

$$\theta^2 a(j) \leq 1 \qquad on\ \bar{\Omega}\ , a(j) \in C^1(\bar{\Omega}) \tag{16}$$

and $\Omega' \subset B(0; \theta T), \beta = s = 0$ in case 1) and $h(h + 2\beta) < \theta^2 T^2, \beta^2 + r^2 = s$ in case 2). Assume that $u \in H^1(\Omega)$ solves the system (11).

Then

$$\|u\|_{1,\Omega_\epsilon} \leq C(F + M^{1-\lambda}F^{\lambda})$$

where F is $\|f\|_{0,\Omega} + \|u\|_{1,S} + \|\nabla u\|_{0,S}$ and $M = \|u\|_{1,S}$. In particular, if $u = 0, \partial_\nu u = 0$ on S then $u = 0$ in Ω_0.

Proof: It is known that under the conditions (16) with respect to the principal coefficients the function $\phi = exp(\sigma/2\psi)$ for large σ is strictly pseudoconvex in $\bar{\Omega}$ with respect to the wave operators $\square_{a(j)}$ [6]. Moreover the additional conditions on h, β, θ, r, T imply that $(\partial\Omega)_0 \subset S$. Applying Theorem 3.3 we complete the proof.

We observe that for constant $a(j)$ in case 1) the conditions (16) simplify to $\theta^2 a(j) < 1$ and $\Omega' \subset B(0; \theta T)$. Then Corollary 3.5 gives the sharp description of the uniqueness domain in the lateral Cauchy problem. Generally, when $\partial_n a(j) < 0$ on $\overline{\Omega}$ one can satisfy the first condition (16) by choosing β positive and large. The uniqueness domain Ω_0 in case 2) is discussed in [7], section 3.4.

While applicable to a wide class of systems, including semilinear ones, Theorem 3.2 can be substantially improved when the coefficients do not depend on the time variable.

Theorem 3.6 *Let S be a C^1-smooth surface which is non- characteristic with respect to the operators $P_j, j = 1, ..., m$. Assume that P_j are hyperbolic with respect to x_0. Assume that the coefficients of the system (13) do not depend on x_0. Then we have unique continuation for H^1 solutions v to (13) across S, in the sense that if $v = 0$ on one side of S then $v = 0$ near S.*

Proof: Denote the two sides of S by S^+, respectively S^-. Let ϕ be a C^1 function so that $S = \{\phi = 0\}$. Let $y \in S$ so that $u = 0$ in S^+ near y. Then we want to show that $\phi = 0$ in S^- near y. Without any restriction in generality we can assume that $y = 0$.

To achieve this we use an argument similar to the one in the proof of Theorem 3.2. We start by constructing a quadratic function ψ with the following properties:

i) The set $\psi > 0$ cuts a small neighbourhood of 0 in S^-. More precisely

$$\{\psi > 0\} \cap S^- \cap B(0, 4r) \subset B(0, 2r)$$

ii) $\psi(0) > 0$

iii) ψ is a small C^1 perturbation of ϕ in $B(0, 4r)$.

iv) ψ is strongly pseudoconvex with respect to all P_j in $\Gamma = \{\xi_0 = 0\}$.

Here r is an arbitrarily small parameter. To achieve this set

$$\psi(x) = x\nabla\phi(0) + \frac{2\lambda\epsilon}{r}(x\nabla\phi(0))^2 - \frac{2\delta}{r}(x^2 - r^2)$$

where λ is a fixed large parameter. The first term insures that $\nabla\psi \approx \nabla\phi(0)$ in $B(0,4r)$; the second term yields (i)-(iii); finally, the third term provides the strong pseudoconvexity condition on Γ while doing no harm to (i)-(iii).

For small positive ϵ both (ii) and (iii) are clearly fulfilled, so it remains to verify (i). It suffices to show that $\phi + \frac{4\lambda\delta}{r}\phi^2 > \psi$ in $B(0,4r) \setminus B(0,2r)$, or that

$$\phi(x) + \frac{4\lambda\delta}{r}\phi^2 + \delta|x| > x\nabla\phi(0) + \frac{2\lambda\delta}{r}(x\nabla\phi(0))^2 \qquad x \in B(0,4r)$$

Since

$$\phi(x) = x\nabla\phi(0) + o(|x|)|x|$$

this follows if r is sufficiently small.

It remains to verify the strong pseudoconvexity condition. We have

$$p_\psi(x,\xi,\tau) = p(x,\xi) - \tau^2 p(x,\nabla\psi) + 2i\tau p_\xi \nabla\psi$$

therefore

$$\begin{aligned} \frac{1}{i\tau}\{\bar{p_\psi}, p_\psi\} &= p_\xi(x,\xi)(\nabla^2\psi)p_\xi(x,\xi) + \tau^2 p_\xi(x,\nabla\psi)(\nabla^2\psi)p_\xi(x,\nabla\psi) + O(|\xi|^2+\tau^2) \\ &= \frac{\lambda\delta}{r}((p_\xi\nabla\psi(0))^2 + \tau^2|p(x,\nabla\psi(0))|^2) + O(\frac{\delta}{r}+1)(|\xi|^2+\tau^2) \end{aligned}$$

Since $p(x,\nabla\phi(0)) \neq 0$ and λ is large, the above expression is positive as long as τ is away from 0. But when τ is close to 0, p_ψ is close to p, which is elliptic on $\xi_0 = 0$ because of the hyperbolicity condition. This concludes the proof of (iv).

Now we can use the Carleman estimates in (8) with respect to the weight function ψ in $B(0,4r)$. If u solves (11) and is supported in $B(0,4r)$ then we can proceed as in Theorem 3.1 to write (8) for each u_j and sum them up. The coupling is lower order and, as in Theorem 3.1, can be absorbed into the left hand side after summation. In the end we get

$$\tau\|e^{-\frac{\epsilon}{2\tau}D_0^2}e^{\tau\phi}u\|_{1;\tau}^2 \leq C(\|e^{-\frac{\epsilon}{2\tau}D_0^2}e^{\tau\phi}f\|_2^2 + \|e^{\tau(\phi-d\epsilon)}u\|_{1;\tau}^2) \tag{17}$$

To use this for solutions u to (13) we need to truncate first. Let χ be a cut-off C^∞- function such that it is 1 on $B(0,2r)$ and 0 outside $B(0,4r)$. Then the function $u = \chi v$ is supported in $B(0,4r)$ and solves (11) with f in L^2, supported in $B(0,4r) \setminus B(0,2r)$.

Then (17) yields

$$\tau \| e^{-\frac{\epsilon}{2\tau} D_0^2} e^{\tau\phi} u \|_{1;\tau}^2 \le c e^{\gamma\tau}, \qquad \gamma = \max\{0, \Psi - d\epsilon\} \tag{18}$$

where Ψ is the supremum of ψ in the support of v.

To complete the proof it suffices to make use of the following result of Tataru [10], Proposition 4.1.

Lemma 3.7 *Let $u \in L^2$ be a function with compact support. Furthermore, let $\phi \in C^r$ and assume that*

$$\| e^{-\frac{\epsilon}{2\tau} D_0^2} e^{\tau\phi} u \|_2 \to 0 \qquad \textit{as } \tau \to \infty$$

Then $u = 0$ in $\{x \in R^n : \phi(x) > 0\}$.

Applying this in (18) we get $v = 0$ in $\{\psi > \gamma\}$. Then $\Psi \ge \gamma$, therefore by the definition of Γ we get $\Psi \le 0$, i.e. $u = v = 0$ in $\{\psi > 0\}$. The proof is complete.

Corollary 3.8 *Assume in addition that the principal part is in divergence form and the coefficients b_j are C^1. Then the same result holds for L^2 solutions u to (13).*

Proof: Let $\mu(t)$ be a smooth regularizing kernel supported in $[-1, 1]$, with integral 1. Set

$$\mu_\delta(t) = \delta^{-1}\mu(\frac{t}{\delta})$$

and look at the mollified functions

$$u_\delta = \mu_\delta * u$$

where the convolution is only with respect to t.

Since (11) has time independent coefficients and v solves it, it follows that the functions v_δ solve it as well. We claim that $v_\delta \in H^1$. Clearly $\partial_t^k u_\delta \in L^2$ for all k. Using this in (11) we get

$$P_j^* u_j \in L_t^2(H_{loc}^{-1})$$

where P_j^* contains the part of P_j without any time derivatives. Then P_j^* is elliptic, in divergence form, with C^1 coefficients, therefore the elliptic theory yields $u_j \in L_t^2(H_{loc}^1)$, which proves our claim.

On the other hand, supp $u \cap \{\psi \geq 0\} \subset B(0,2r)$ therefore supp $u_\delta \cap \{\psi \geq 0\} \subset B(0,2r)$ if δ is sufficiently small. Then the argument in the proof of the Theorem applies to u_δ and gives $u_\delta = 0$ in $\psi > 0$. Letting $\delta \to 0$ we get the same for u, q.e.d.

It is convenient to have a global version of the previous Theorem. To formulate the next result we assume that S_θ is a continuous family of C^1 regular non-characteristic mutually disjoint surfaces, $\theta \in [0,1]$. Set $\Omega_0 = \cup_{\theta \in [0,1]} S_\theta$

Corollary 3.9 *Under the conditions of Theorem 3.6 let u be an H^1 solution to the system (11) which is zero near S_1 so that supp $u \cap \Omega_0$ is compact. Then $u = 0$ in Ω_0. The same holds for L^2 solutions if the additional conditions in Corollary 3.8 are fulfilled.*

Proof: a) Let

$$s = \sup\{\theta \in [0,1];\ \text{supp } u \cap S_\theta \neq 0\}$$

Assume by contradiction that $s > 0$. Then supp $u \cap S_\theta = 0$ for $\theta > s$, while, by compactness, supp $u \cap S_s \neq 0$. If $x_0 \in$ supp $u \cap S_s$ then, by continuity, we conclude that u is 0 on one side of S_s near x. Then we can use the local argument above to conclude that $u = 0$ near x_0, which contradicts our earlier assumption, q.e.d.

4 Applications to the Maxwell system

The results of the previous can be applied to classical systems of mathematical physics which can be reduced to the systems of the form (11) by relatively simple transformations. First we consider the Maxwell system

$$\begin{cases} \partial_t(\epsilon E) - \text{curl } H = j \\ \partial_t(\mu H) + \text{curl } E = 0 \\ \text{div } (\epsilon E) = 4\pi\rho \\ \text{div } (\mu H) = 0 \end{cases} \tag{19}$$

for electric and magnetic fields $E = (E_1, E_2, E_3), H = (H_1, H_2, H_3)$ depending on $(x,t) \in \Omega \subset R^4$. Here $\epsilon(x,t), \mu(x,t)$, denote correspondingly electric permittivity, magnetic permeability, $j(x,t)$ is the density of electrical current and $\rho(x,t)$ is the electrical charge density. We consider the isotropic case when $\epsilon, \mu \in C^2$ are scalar functions.We are interested in the Cauchy problem for (19) when one prescribes the Cauchy data

$$E = E_0, H = H_0 \quad \text{on } S \tag{20}$$

where S is a C^2 surface.

Uniqueness and stability in the Cauchy problem for the system (19) will follow from the results of the previous section and from the following simple lemma.

Lemma 4.1 *For solutions to the Maxwell system (19) with $j = 0, \rho = 0$ we have*

$$\partial_t^2(\epsilon E) - \Delta(\mu^{-1}E) + \nabla(\nabla(\epsilon\mu)^{-1} \cdot \epsilon E) - \text{curl } (\nabla\mu^{-1} \times E) + \text{curl } (\frac{\mu_t}{\mu}H) = 0. \tag{21}$$

and

$$\partial_t^2(\mu H) - \Delta(\epsilon^{-1}H) + \nabla(\nabla(\epsilon\mu)^{-1} \cdot \mu H) - \text{curl } (\nabla\epsilon^{-1} \times H) - \text{curl } (\frac{\epsilon_t}{\epsilon}E) = 0. \tag{22}$$

Note that this computation requires only C^1 regularity for E, H. If in addition both E and H are at least C^2 then one can rewrite these equations as

$$(\epsilon\mu\partial_t^2 - \Delta)E = Q_1(E, H)$$
$$(\epsilon\mu\partial_t^2 - \Delta)H = Q_2(E, H)$$

where Q_1, Q_2 are first order operators with bounded coefficients.

Proof: To reduce the system (19) to (21) (22) we differentiate its first equation with respect to t and use the second equation (19) to replace $\partial_t(\mu H)$ by curl E. We obtain

$$\partial_t^2\epsilon E + \text{curl } (\mu^{-1}\text{curl } E) - \text{curl } (\frac{\mu_t}{\mu}H) = 0.$$

and further

$$\partial_t^2(\epsilon E) + \text{curl } (\text{curl } (\mu^{-1}E)) + \text{curl } (\nabla\mu^{-1} \times E) + \text{curl } (\frac{\mu_t}{\mu}H) = 0.$$

But

$$\text{curl curl } E = -\Delta E + \nabla \text{div } E$$

therefore

$$\partial_t^2(\epsilon E) - \Delta(\mu^{-1}E)) + \nabla \text{div } (\mu^{-1}E) - \text{curl } (\nabla\mu^{-1} \times E) + \text{curl } (\frac{\mu_t}{\mu}H) = 0.$$

Now using the third equation (19) we finally get

$$\partial_t^2(\epsilon E) - \Delta(\mu^{-1}E)) + \nabla(\nabla(\epsilon\mu)^{-1} \cdot \epsilon E) - \text{curl } (\nabla\mu^{-1} \times E) + \text{curl } (\frac{\mu_t}{\mu}H) = 0.$$

The equation for H can be obtained in a similar fashion.

The equations (21),(22) form a principally scalar hyperbolic system, so from Theorems 3.2,3.3,3.6 and their corollaries we can derive uniqueness and stability results for the Cauchy problem.

The Carleman estimates for the Maxwell's equations have the form

$$\begin{aligned}\tau(\|e^{\tau\phi}E\|_{1,\tau}^2 + \|e^{\tau\phi}H\|_{1,\tau}^2) &\leq c(\|e^{\tau\phi}(\partial_t(\epsilon E) - \text{curl } H)\|_{1,\tau}^2 + \|e^{\tau\phi}(\partial_t(\mu H) - \text{curl } E)\|_{1,\tau}^2 \\ &+ \|\text{div } (\epsilon E)\|_{1,\tau}^2 + \|\text{div } (\mu H)\|_{1,\tau}^2\end{aligned}$$

In the presence of the boundary one needs to add the term $\tau\|e^{\tau\phi}(E,H)\|_{1,\tau,\partial\Omega}^2$; the conormal derivatives of both E and H can be derived from the equations. By Theorem 3.1, these estimates hold if the coefficients ϵ, μ are C^2, provided that ϕ is strongly pseudoconvex with respect to the wave operator $\square_{\epsilon\mu}$.

The main unique continuation result is

Theorem 4.2 *Assume that the oriented surface $S = \{\phi = 0\}$ is strongly pseudo-convex with respect to $\square_{\epsilon\mu}$. If $(E,H) \in H^1(\Omega)$ solve the system (19) and are zero on S then they are zero near S in $\{\phi > 0\}$.*

.

Correspondingly, from Lemma 4.1 and Theorem 3.3, we get the stability estimate

Theorem 4.3 *Let ϕ be strongly pseudo-convex with respect to $\square_{\mu\epsilon}$ on $\overline{\Omega}$ and $(\partial\Omega)_0 \subset S$.*

Then for a solution $(E,H) \in H^1(\Omega)$ to the Cauchy Problem (19), (20) one has

$$\|E\|_{1,\Omega_\delta} + \|H\|_{1,\Omega_\delta} \leq C(F + M^{1-\lambda}F^\lambda) \tag{23}$$

with some positive constants $C, \lambda < 1$ depending on δ. Here $F = \|E_0\|_{1,S} + \|H_0\|_{1,S}$, $M = \|E\|_{1,\Omega_0} + \|H\|_{1,\Omega_0}$.

Now we give explicit description of uniqueness domains when $\Omega = \Omega' \times (-T, T)$ and $S = \gamma \times (-T, T)$ in the cases considered in Corollary 3.5

Corollary 4.4 *Assume that the coefficients $a = \epsilon\mu$ satisfy the conditions (16) and that parameters h, β, θ, T, r also satisfy the conditions of Corollary 3.5. Then (23) holds.*

Finally we give a sharp description of the uniqueness domain in the Cauchy problem when ϵ, μ, σ do not depend on time; this follows from Theorem 3.6 and Corollary 3.8.

Theorem 4.5 *Let S be a C^1-smooth surface which is non-characteristic with respect to the operator $\square_{\epsilon\mu}$. Then we have unique continuation across S for L^2 solutions E, H to (13) , in the sense that if $v = 0$ on one side of S then $v = 0$ near S.*

5 Applications to the classical elasticity system

The uniqueness of the continuation for the static elasticity system under C^∞-assumptions and with arbitrary first order perturbations was proven by Dehman and Robbiano [3]. Recently, Ang,Ikehata,Trong, and Yamamoto [1] reduced a static system with zero order perturbations to a principally diagonal one and obtained sharp uniqueness of the continuation results under reduced smoothness assumptions.The time dependent classical elasticity with arbitrary first order perturbations was considered by Isakov [6] by pseudo-convexity methods. Recently Yamamoto [12] applied principal diagonalization to some inverse source problems for the Maxwell system. A similar reduction is available the classical elasticity system

$$\rho\partial_t^2 u - \mu(\Delta u + \nabla \text{div } u) - \nabla(\lambda \text{div } u) - \sum_{j=1}^{3} \nabla\mu \cdot (\nabla u_j + \partial_j u)e_j = 0 \quad \text{in } \Omega \quad (24)$$

for the displacement vector $u = (u_1, u_2, u_3)$ depending on $(x, t) \in \Omega$. We will assume that the the density $\rho \in C^1(\overline{\Omega})$ and the Lame parameters μ, λ are functions in $C^2(\overline{\Omega})$.We are interested in the Cauchy problem for this system when one prescribes the Cauchy data

$$u = u_0, \partial_\nu u = u_1 \quad \text{on } S \qquad (25)$$

We will extend the system (24) for three unknown functions u_1, u_2, u_3 to a new one for seven unknown functions by introducing two more vector functions $v = \text{div } u$ and $w = \text{curl } u$. Another, more economical reduction requires one to introduce just one auxiliary variable, namely div u. However, this requires some lengthy computations and is relegated to the appendix. For a particular case of this reduction we refer to [2].

To find differential equations for new functions we apply the divergence and the curl operators to the system (24).

Lemma 5.1 *Let $v = \text{div } u, w = \text{curl } u$.*

If u solves (24) then

$$\begin{aligned} \rho/\mu\partial_t^2 u - \Delta u + A_{1;1}(u, v) = 0 \\ \rho/(2\mu + \lambda)\partial_t^2 v - \Delta v + A_{2;1}(u, v, w) = 0 \quad \text{in } \Omega \\ \rho/\mu\partial_t^2 w - \Delta w + A_{3;1}(u, v, w) = 0 \end{aligned} \tag{26}$$

where $A_{j;1}$ are linear differential operators of first order with measurable and bounded coefficients in Ω. In addition, when the $\rho \in C^2(\overline{\Omega'})$, $\mu, \lambda \in C^3(\overline{\Omega'})$ do not depend on t, then the coefficients of $A_{j;1}$ do not depend on t and the coefficients of first order derivatives are in $C^1(\overline{\Omega'})$.

Proof:

Dividing the equations (24) by ρ and applying the divergence we obtain

$$\partial_t^2 v - (2\mu + \lambda)/\rho\Delta v - (\nabla(\mu/\rho) + \lambda\nabla(1/\rho) + 2/(\rho)\nabla\lambda + \nabla\mu/\rho) \cdot \nabla v$$

$$-(\nabla(\mu/\rho) + \nabla\mu/\rho) \cdot u - \text{div } (1/\rho\nabla\lambda)v - \sum_{j=1}^{3} \partial_j(\nabla\mu/\rho) \cdot (\nabla u_j + \partial_j u) = 0$$

or using the identity $\Delta u = \nabla\text{div } u - \text{curl curl } u$ write it as

$$\partial_t^2 v - (2\mu + \lambda)/\rho\Delta v - 2/(\lambda + 2\mu)\nabla((\lambda + 2\mu)^2/\rho) \cdot \nabla v +$$

$$1/\mu\nabla(\mu^2/\rho) \cdot \text{curl } w - \sum_{j\leq 3} \partial_j(\nabla\mu/\rho) \cdot (\nabla u_j + \partial_j u) - \text{div } (\nabla\lambda/\rho)v = 0$$

Similarly applying the curl and using the identity curl $(fu) = f\text{curl } u + \nabla f \times u$ leads to the equations

$$\partial_t^2 w - \mu/\rho\Delta w - (\nabla(\mu/\rho)) \times (\Delta u + \nabla v) - \nabla(1/\rho) \times \nabla(\lambda v) -$$

$$\sum_{j\leq 3}(((1/\rho)\partial_j\mu)\text{curl }\partial_j u + \nabla((\partial_j\mu)/\rho)\times(\partial_j u + \nabla u_j)) = 0$$

Using again the formula $\Delta u = \nabla v - \text{curl } w$ we arrive at

$$\partial_t^2 w - \mu/\rho\Delta w - \nabla(\mu/\rho)\times(\nabla v - \text{curl } w + \nabla v) -$$

$$\nabla(1/\rho)\times\nabla(\lambda v) - \sum_{j\leq 3}(1/\rho\partial_j\mu\partial_j w + \nabla(\partial_j\mu/\rho)\times(\partial_j u + \nabla u_j)) = 0$$

Summing up we obtain the extended principally hyperbolic system (26).

As for the Maxwell system this Lemma and our results on principally scalar systems imply the following uniqueness and stability theorems for the Cauchy problem (24),(25).

Theorem 5.2 *Let ϕ be strongly pseudo-convex with respect to the operators $\Box_{\rho/\mu}, \Box_{\rho/(2\mu+\lambda)}$ on $\overline{\Omega}$ and $(\partial\Omega)_0 \subset S$.*

Then for a solution $u \in H^2(\Omega)$ to the Cauchy problem (24),(25) one has

$$\|u\|_{1,\Omega_\epsilon} + \|\text{div } u\|_{1,\Omega_\epsilon} + \|\text{curl } u\|_{1,\Omega_\epsilon} \leq C(F + M^{1-\lambda}F^{\lambda}) \tag{27}$$

with some positive constants C, λ depending on ϵ. Here

$$F = \|u_0\|_{2,S} + \|u_1\|_{1,S}, \quad M = \|u\|_{2,\Omega}$$

. In particular, when $u_0 = u_1 = 0$ on S one has $u = 0$ on Ω_0.

This result follows from Theorem 3.3. If the part S of the boundary is itself strongly pseudoconvex then we get the following local result:

Corollary 5.3 *Assume that the surface S is strongly pseudo-convex with respect to the operators $\Box_{\rho/\mu}, \Box_{\rho/(2\mu+\lambda)}$ in $\overline{\Omega}$. If $u \in H^2(\Omega)$ solves the Cauchy problem (24),(25) with zero u_0, u_1 then u=0 near S on Ω.*

Now we give an explicit uniqueness domain when $\Omega = \Omega' \times (-T, T)$ and $S = \gamma \times (-T, T)$ in the cases considered in Corollary 3.5.

Corollary 5.4 *Assume that Ω, S are the domain and the part of its boundary described in Corollary 3.5. Let the functions $a = \rho/\mu$ and $a = \rho/2(\mu+\lambda)$ and the parameters h, β, θ, T, r satisfy the conditions (16). Let $u \in H^2(\Omega)$ solve the Cauchy problem (24),(25). Then (27) holds. In particular if the Cauchy data are zero then $u = 0$ on Ω_0.*

As for the Maxwell system we can improve these results and reduce regularity assumptions about u when the elastic parameters are time independent.

Theorem 5.5 *Assume that the coefficients $\rho \in C^2$,$\mu, \lambda \in C^3$ are time independent. Let S be a noncharacteristic surface with respect to both $\square_{\rho/\mu}, \square_{\rho/(2\mu+\lambda)}$. Then we have unique continuation across S for H^1_{loc} solutions u to (24).*

A A diagonalization of the classical elasticity system

Let

$$Lu = (\lambda + \mu)\nabla(\nabla \cdot u) + \mu\nabla u + (\nabla \cdot u)\nabla\lambda - \{\nabla u + (\nabla u)^T\}\nabla\mu$$

be the spatial part of the elasticity system, where $(\nabla u)^T$ denotes the transpose of ∇u. By a direct computation we obtain the following lemma:

Lemma A.1 *Let*

$$\gamma = \mu^{-\frac{1}{2}},\ \beta = \gamma(\lambda + \mu),\ \Theta = (\lambda + \mu)\nabla\gamma + \gamma\nabla\mu$$

and

$$\begin{array}{rcl} V &=& \nabla\lambda \otimes \nabla\gamma + \nabla\gamma \otimes \nabla\mu + (\lambda + \mu)\nabla^2\gamma + (\nabla\gamma \cdot \nabla\mu + \mu\Delta\gamma)I_n, \\ W &=& \gamma\{V - (\nabla\Theta)^T\} - \nabla\gamma \otimes \Theta. \end{array}$$

Then,

$$\gamma L(\gamma u) = \Delta u + \nabla(\gamma\beta\nabla \cdot u + \gamma\Theta \cdot u) - (\nabla \cdot u)\gamma\Theta + Wu.$$

Since

$$\nabla \cdot \{\gamma L(\gamma u)\} = \Delta\{\nabla \cdot u + \gamma\beta\nabla \cdot u + \gamma\Theta \cdot u\} - \nabla \cdot \{(\nabla \cdot u)\gamma\Theta\} + \nabla \cdot \{Wu\},$$

we have

$$\begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} \gamma L(\gamma u) = \left\{ \Delta M + \begin{pmatrix} \nabla^T & 0 \\ 0 & \nabla \end{pmatrix} \begin{pmatrix} -\gamma\Theta & W \\ \gamma\beta & \gamma\Theta^T \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\gamma\Theta & W \end{pmatrix} \right\} \begin{pmatrix} \nabla \cdot u \\ u \end{pmatrix}$$
$$= \left\{ \Delta I_{n+1} + \begin{pmatrix} \nabla^T & 0 \\ 0 & \nabla \end{pmatrix} V_1 + V_2 \right\} M \begin{pmatrix} \nabla \cdot u \\ u \end{pmatrix}$$

where I_k is the k times k identity matrix and

$$\begin{aligned} M &= \begin{pmatrix} 1+\gamma\beta & \gamma\Theta^T \\ 0 & I_n \end{pmatrix}, \\ V_1 &= \begin{pmatrix} -\dfrac{\gamma\Theta}{1+\gamma\beta} & \dfrac{\gamma^2}{1+\gamma\beta}\Theta \otimes \Theta + W \\ \dfrac{\gamma\beta}{1+\gamma\beta} & \dfrac{\gamma}{1+\gamma\beta}\Theta^T \end{pmatrix}, \\ V_2 &= \begin{pmatrix} 0 & 0 \\ -\dfrac{\gamma}{1+\gamma\beta}\Theta & \dfrac{\gamma^2}{1+\gamma\beta}\Theta \otimes \Theta + W \end{pmatrix}. \end{aligned}$$

Note that for $w = \gamma u$, we have

$$\begin{pmatrix} \nabla \cdot u \\ u \end{pmatrix} = Q \begin{pmatrix} \nabla \cdot w \\ w \end{pmatrix}$$

with

$$Q = \begin{pmatrix} \gamma^{-1} & (\nabla\gamma^{-1})^T \\ 0 & \gamma^{-1} I_n \end{pmatrix}.$$

Hence

$$\begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} \gamma L w = \left\{ \Delta I_{n+1} + \begin{pmatrix} \nabla^T & 0 \\ 0 & \nabla \end{pmatrix} V_1 + V_2 \right\} MQ \begin{pmatrix} \nabla \cdot w \\ w \end{pmatrix}.$$

Moreover, from

$$\rho\gamma{\partial_t}^2(\nabla \cdot w) + \nabla(\rho\gamma) \cdot {\partial_t}^2 w - \nabla^T(\gamma L w) = \nabla^T(\rho\gamma{\partial_t}^2 w - \gamma L w),$$

we obtain

$$\begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} \gamma\{\rho {\partial_t}^2 w - Lw\} = S {\partial_t}^2 \begin{pmatrix} \nabla \cdot w \\ w \end{pmatrix} - \begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} \gamma L w$$

with

$$S = \begin{pmatrix} \rho\gamma & (\nabla(\rho\gamma))^T \\ 0 & (\rho\gamma) I_n \end{pmatrix}.$$

Summing up, we have proved that

$$\begin{aligned} &\begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} \gamma\{\rho {\partial_t}^2 w - Lw\} = \\ &S \left[{\partial_t}^2 I_{n+1} - S^{-1} \left\{ \Delta I_{n+1} + \begin{pmatrix} \nabla^T & 0 \\ 0 & \nabla \end{pmatrix} V_1 + V_2 \right\} MQ \right] \begin{pmatrix} \nabla \cdot w \\ w \end{pmatrix}. \end{aligned} \tag{28}$$

A short computation yields

$$S^{-1} MQ = K$$

with

$$K = \rho^{-1} \begin{pmatrix} \lambda + 2\mu & 2(\nabla\mu)^T - \rho^{-1}\mu(\nabla\rho)^T \\ 0 & \mu I_n \end{pmatrix}.$$

Moreover, we have

$$D := H^{-1} K H = \rho^{-1} \begin{pmatrix} \lambda + 2\mu & 0 \\ 0 & \mu I_n \end{pmatrix}$$

with

$$H = \begin{pmatrix} 1 & 2(\nabla\mu)^T - \rho^{-1}\mu(\nabla\rho)^T \\ 0 & -(\lambda+\mu) I_n \end{pmatrix}.$$

Hence, (29) yields the following principally diagonal conjugate to the linear elasticity operator:

Theorem A.2

$$\begin{pmatrix} \nabla^T \\ I_n \end{pmatrix} \gamma\{\rho{\partial_t}^2 w - Lw\} = H\{{\partial_t}^2 I_{n+1} - (D\Delta I_{n+1} + P_1)\}H^{-1} \begin{pmatrix} \nabla \cdot w \\ w \end{pmatrix}$$

where

$$\begin{aligned} P_1 &= C[\Delta I_{n+1}, H] + A[\Delta I_{n+1}, B] + A\left(\begin{pmatrix} \nabla^T & 0 \\ 0 & \nabla \end{pmatrix} R + R'\right) \\ A &= (SH)^{-1} = (A_{ij})_{1\le i,j\le 2},\ B = MQ = (B_{ij})_{1\le i,j\le 2}, \\ C &= H^{-1}K = (C_{ij})_{1\le i,j\le 2},\ R = V_1BH = (R_{ij})_{1\le i,j\le 2}, \\ R' &= V_2BH = (R'_{ij})_{1\le i,j\le 2}. \end{aligned}$$

The explicit forms of these are the following :

$A_{11} = \mu^{\frac{1}{2}}\rho^{-1},\ A_{12} = \dfrac{1}{2}\mu^{-\frac{1}{2}}\rho^{-1}(\lambda+\mu)^{-1}(\lambda+5\mu)(\nabla\mu)^T - \mu^{\frac{1}{2}}\rho^{-2}(\lambda+\mu)^{-1}(\lambda+2\mu)(\nabla\rho)^T$

$A_{21} = 0,\ A_{22} = -(\lambda+\mu)^{-1}\mu^{\frac{1}{2}}\rho^{-1}I_n$

$B_{11} = \mu^{-\frac{1}{2}}(\lambda+2\mu),\ B_{12} = \dfrac{3}{2}\mu^{-\frac{1}{2}}(\nabla\mu)^T,\ B_{21} = 0,\ B_{22} = \mu^{\frac{1}{2}}I_n$

$C_{11} = \rho^{-1}(\lambda+2\mu),\ C_{12} = \rho^{-1}(2(\nabla\mu)^T - \rho^{-1}\mu(\nabla\rho)^T)$
$C_{21} = 0,\ C_{22} = -\mu(\lambda+\mu)^{-1}I_n$

$R_{11} = -\dfrac{1}{2}\mu^{-\frac{3}{2}}(\mu-\lambda)\nabla\mu$
$R_{12} = \dfrac{1}{2}\mu^{-\frac{3}{2}}(\lambda-3\mu)\nabla\mu\otimes\nabla\mu + \dfrac{1}{2}\rho^{-1}\mu^{-\frac{1}{2}}(\mu-\lambda)\nabla\mu\otimes\nabla\rho + \mu^{-\frac{1}{2}}(\lambda+\mu)\nabla^2\mu - (\lambda+\mu)(\dfrac{1}{4}\mu^{-\frac{3}{2}}|\nabla\mu|^2 - \dfrac{1}{2}\mu^{-\frac{1}{2}}\Delta\mu)I_n$
$R_{21} = \mu^{-\frac{1}{2}}(\lambda+\mu),\ R_{22} = \mu^{-\frac{1}{2}}(\lambda+\mu)(\nabla\mu)^T - \rho^{-1}\mu^{\frac{1}{2}}(\lambda+\mu)(\nabla\rho)^T$

$R'_{11} = R'_{12} = 0,\ R'_{21} = R_{11},\ R'_{22} = R_{12}$

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