Unique continuation problems for partial differential equations

Daniel Tataru*
Department of Mathematics
University of California, Berkeley

March 4, 2002

1 Introduction

The aim of this article is to give an overview of the main problems and results in unique continuation. Broadly speaking, an unique continuation result is any statement of the following type:

> *Given a linear partial differential operator P and two regions $A \subset B$, a solution u to $Pu = 0$ is uniquely determined in the larger set B by its values (behavior) in the smaller set A.*

A quantitative version of such a statement is called a stability result:

> *If two solutions u_1 and u_2 to $Pu = 0$ are close in A then they must be close in B.*

To this setup one can also associate a corresponding existence result,

> *A solution u to $Pu = 0$ in A extends to a solution in B.*

If both the uniqueness and the existence results hold then we say that the solvability problem for P (in this setup) is well-posed. However, most interesting unique continuation problems arise in contexts where the corresponding existence result does not hold. For this reason unique continuation problems are often referred to as "uniqueness in ill-posed problems".

This presentation is focused more on the geometrical aspects of the problems and less on the many relevant technical issues. Consequently, we choose to limit our scope in two ways:

(i) As a method for proving unique continuation results we only discuss the L^2 Carleman estimates.

(ii) We do not aim for sharp statements concerning the regularity of the coefficients and of the solution.

*Research partially supported by NSF grant DMS-9622942

2 The classical unique continuation problem.

Consider a partial differential operator $P(x, D)$ of order m in $\mathbb{R}^n$. Let Σ be an oriented hypersurface in $\mathbb{R}^n$, which we represent as a nondegenerate level set of a smooth function, $\Sigma = \{\phi = 0\}$. The sign of ϕ away from Σ is determined by the orientation of Σ. Denote the two sides of Σ by $\Sigma^+ = \{\phi > 0\}$ and $\Sigma^- = \{\phi < 0\}$. Then we define the unique continuation property across Σ for solutions to $P(x, D)u = 0$ as follows:

Definition 1. *We say that unique continuation property across Σ holds for the operator $P(x, D)$ if for each $x_0 \in \Sigma$ and V a neighborhood of x_0 the following holds: Let u be a solution for $P(x, D)u = 0$ in V so that $u = 0$ in $\Sigma^+ \cap V$. Then $u = 0$ near x_0.*

In other words, the values of a solution u to $Pu = 0$ on one side of Σ (i.e. in Σ^+) uniquely determine its values on the other side (i.e. in Σ^-) near Σ. Note that the choice of the orientation is important; in effect, as we shall see later, in most cases if unique continuation holds in one direction then it does not hold in the opposite direction. If the surface Σ is noncharacteristic then we can also introduce the Cauchy problem:

Definition 2 (Cauchy problem). *Given functions $(u_0, u_1, \cdots, u_{m-1})$ on Σ, find u in Σ^- so that:*

$$\begin{cases} P(x, D)u = 0 & \text{in } \Sigma^- \\ u = u_0, \dfrac{\partial u}{\partial \nu} = u_1, \cdots, \dfrac{\partial^{m-1} u}{\partial \nu^{m-1}} = u_{m-1} & \text{on } \Sigma \end{cases}$$

It is easy to see that the unique continuation property is equivalent to (local) uniqueness in the Cauchy problem.

Roughly speaking the unique continuation property says that the information about a solution u to $Pu = 0$ in Σ^- can be retrieved from information about u in Σ^+. Thus one may hope to gain some intuition about the unique continuation problem by looking at special solutions u to $Pu = 0$.

For instance, if P has a real symbol and is not elliptic, then one can consider high frequency solutions which are localized near null bicharacteristic rays of P, given by

$$\dot{x} = p_\xi, \qquad \dot{\xi} = -p_x$$

on $\{p(x, \xi) = 0\}$ where $p(x, \xi)$ is the principal symbol of P. Then it is natural to ask that all such rays which intersect a small neighborhood of Σ in Σ^- must also intersect Σ^+. A strong form of this condition can be expressed in the form

$$\{p, \phi\} = p = 0 \implies \{p, \{p, \phi\}\} > 0 \qquad \text{on } \Sigma \tag{1}$$

where $\{\cdot, \cdot\}$ is the Poisson bracket of two symbols, $\{p, q\} = p_\xi q_x - p_x q_\xi$. The Poisson bracket of p and q is the derivative of q along the bicharacteristic (Hamilton) flow of p. Then (1) says that ϕ is strictly convex on null bicharacteristics of

P near its critical points. In geometrical terms, this means that all null bicharacteristics which are tangent to Σ must curve away from Σ toward Σ^+. One can think of (1) as some sort of convexity condition for Σ^-, not with respect to all straight lines but instead with respect to the null bicharacteristic flow of P.

Even in the case when P is elliptic, a second obstruction for unique continuation could arise if we were able to construct highly localized approximate solutions u to $Pu = 0$ in Σ^- which decay exponentially toward Σ. This corresponds to replacing real phase functions with complex phase functions, where the imaginary part of the phase is a positive multiple of the function ϕ. To study such solutions one needs to complexify the symbol of p in the direction of the inner normal, i.e. to look at $p_\phi = p(x, \xi + i\tau\nabla\phi)$ where $\tau > 0$. It is not possible to simply look at the geometry of the bicharacteristics for this new symbol, because it is complex valued. Instead this is superseded by the (necessary) local solvability condition for p_ϕ, which has the form

$$\{\Re p_\phi, \Im p_\phi\} \leq 0 \qquad \text{on } \{p_\phi = 0\}$$

If this is violated then we get a sufficient condition which roughly guarantees the absence of such exponentially decreasing solutions, namely

$$\{\Re p_\phi, \Im p_\phi\} > 0 \quad \text{on } T^*\Sigma \cap \{p_\phi = 0\},\ \tau > 0 \tag{2}$$

In addition observe that ϕ is only determined up to multiplication by a smooth positive function, while the above inequality is not invariant with respect to such a transformation. As we need (2) to hold only for some selection of ϕ, a short algebraic computation leads to the relaxed condition

$$\{\Re p_\phi, \Im p_\phi\} > 0 \quad \text{on } T^*\Sigma \cap \{p_\phi = \{p_\phi, \phi\} = 0\},\ \tau > 0 \tag{3}$$

It also interesting to note that as a limit of (3) as $\tau \to 0$ one retrieves (1) (with "$>$" replaced by "$\geq$", though).

Based on this analysis we introduce the pseudoconvexity condition for surfaces,

Definition 3. *We say that the surface Σ is strongly pseudoconvex with respect to P if either*

a) P is elliptic and (3) holds.

b) P has real principal symbol and (1), (3) hold.

Now we are ready to state the main result (see [5] and references therein):

Theorem 4 (Hörmander). *Suppose that the oriented surface Σ is strongly pseudoconvex with respect to P. Then unique continuation across Σ holds for P.*

This is nicely complemented by the following negative result, proved in [1]:

Theorem 5 (Alihnac). *Suppose that the condition (1) for the oriented surface Σ strictly fails at some $(x_0, \xi) \in T^*\Sigma$. Then there is a function u supported in Σ^- and a smooth function a Q so that $x_0 \in$ supp u but $x_0 \notin$ supp $(P + a)u$.*

These two results show that the notion of pseudoconvexity is the right concept to use for unique continuation problems.

We now sketch the proof of the positive result. This is based on a one parameter family of weighted estimates, called Carleman estimates. The proper set of weights to use is tied to the notion of pseudoconvex functions.

Definition 6. *We say that the function ϕ is strongly pseudoconvex with respect to P if either*

a) P is elliptic and (2) holds.

b) P has real principal symbol and (1), (2) hold.

Obviously the nondegenerate level sets of pseudoconvex functions are pseudoconvex surfaces. There is a weaker converse to this statement, namely

Remark 7. *Suppose that Σ is strongly pseudoconvex with respect to P, and let ϕ be a defining function for Σ. Then for sufficiently large λ the function $e^{\lambda\phi}$ is strongly pseudoconvex with respect to P near Σ.*

Now we can state the estimates:

Theorem 8 (Carleman estimates). *Suppose that ϕ is strongly pseudoconvex with respect to P in some compact region $\Omega \subset \mathbb{R}^n$. Then there are $c, \tau_0 > 0$ so that for all functions u supported in Ω we have:*

(a) P is elliptic:

$$\tau^{-1}\|e^{\tau\phi}u\|^2_{H^m_\tau} \leq c\|e^{\tau\phi}P(x,D)u\|^2_{L^2}, \quad \tau \geq \tau_0 \tag{4}$$

(b) P has real principal symbol:

$$\tau\|e^{\tau\phi}u\|^2_{H^{m-1}_\tau} \leq c\|e^{\tau\phi}P(x,D)u\|^2_{L^2}, \quad \tau \geq \tau_0 \tag{5}$$

Here the H^m_τ norms are weighted Sobolev norms,

$$\|v\|_{H^m_\tau} = \|(D^2+\tau^2)^{\frac{m}{2}}v\|_{L^2}$$

Proof of Theorem 8. We first eliminate the exponential weight from (4). If we set $v = e^{\tau\phi}u$ then

$$e^{\tau\phi}P(x,D)u = e^{\tau\phi}P(x,D)e^{-\tau\phi}v = P_\phi(x,D,\tau)v$$

where the conjugated operator P_ϕ has the form

$$P_\phi(x,D,\tau) = e^{\tau\phi}P(x,D)e^{-\tau\phi} = P(x, D + i\tau\nabla\phi)$$

Hence (4) for instance reduces to

$$\tau^{-1}\|v\|^2_{H^m_\tau} \leq c\|P_\phi(x,D,\tau)v\|^2_{L^2}, \quad \tau \geq \tau_0 \tag{6}$$

Given the definition of the H^m_τ norms, it follows that all the lower order terms in P are controlled by the left hand side for large τ, therefore they are negligible. We decompose P_ϕ into a part with real symbol and one with imaginary symbol,

$$P_\phi = P^r_\phi + iP^i_\phi$$

and compute

$$\begin{aligned}\|P_\phi v\|^2_{L^2} &= \|P^r_\phi v\|^2_{L^2} + \|P^i_\phi v\|^2_{L^2} + 2\Im\langle P^r_\phi v, P^i_\phi v\rangle \\ &= \|P^r_\phi v\|^2_{L^2} + \|P^i_\phi v\|^2_{L^2} + \langle Qv, v\rangle\end{aligned}$$

where

$$Q = i((P^r_\phi)^* P^i_\phi - (P^i_\phi)^* P^r_\phi) = i(((P^r_\phi)^* - P^r_\phi)P^i_\phi - ((P^i_\phi)^* - P^i_\phi)P^r_\phi + [P^r_\phi, P^i_\phi])$$

where the notation $[\cdot,\cdot]$ stands for the commutator of two operators. Since both P^r_ϕ and P^i_ϕ have real principal symbols $\Re P_\phi$, respectively $\Im P_\phi$, it follows that the principal symbol of Q is a homogeneous polynomial of degree $2m-1$ in ξ, τ which has the form

$$q(x,\xi,\tau) = (\Re p_\phi)_{x\xi} \Im p_\phi - (\Im p^i_\phi)_{x\xi} \Re p_\phi + \{\Re p_\phi, \Im p_\phi\}$$

Now we consider separately the two cases:

a) If P is elliptic then

$$p_\phi = 0 \implies |\xi| \leq c\tau$$

From (2) and the above relation we get

$$\tau q(x,\xi,\tau) > 0 \qquad \text{on} \quad \{p_\phi = 0\}$$

therefore for large enough C we have

$$\tau q(x,\xi,\tau) + C|p_\phi|^2 > c(\tau^2 + \xi^2)^m$$

Hence Garding's inequality implies that

$$\tau\langle Qv, v\rangle + C(\|P^r_\phi v\|^2 + \|P^i_\phi v\|^2) \geq c\|v\|^2_{H^m_\tau} - d\|v\|_{H^{m-1}_\tau}$$

The second term on the right is negligible for large enough τ, therefore (6) follows.

b) If P has real principal symbol then we want to argue as above, but in addition we need to consider what happens when τ is small compared to $|\xi|$. In this case for small τ we have

$$\Re p_\phi = p + O(\tau^2), \qquad \Im p_\phi = \tau\{p, \phi\} + O(\tau^3)$$

Then we use (1) for $\tau \ll |\xi|$ and (2) otherwise to conclude that for large enough C we have

$$\tau^{-1} q(x,\xi,\tau) + C(\tau^{-2}|\Im p_\phi|^2 + (\tau^2+\xi^2)^{-1}|\Re p_\phi|^2) \geq c(\xi^2+\tau^2)^{m-1}$$

Therefore Garding's inequality shows that

$$\tau^{-1}\langle Qv, v\rangle + C(\|P^r_\phi v\|^2_{H^{-1}_\tau} + \tau^{-2}\|P^i_\phi v\|^2_{L^2}) \geq c\|v\|^2_{H^{m-1}_\tau} - d\|v\|_{H^{m-2}_\tau}$$

which in turn yields the counterpart of (5) after the conjugation, namely

$$\tau\|v\|^2_{H^{m-1}_\tau} \leq c\|P_\phi(x, D, \tau)v\|^2_{L^2}, \quad \tau \geq \tau_0 \tag{7}$$

□

Proof of Theorem 4. Here we show how to use the Carleman estimates in order to obtain the unique continuation result. By Remark 7, we can represent Σ as a level set of a strongly pseudoconvex function ϕ, $\Sigma = \{\phi = 0\}$. We do not use the Carleman estimates with respect to the function ϕ but with respect to a modified function, namely

$$\psi(x) = \phi(x) - 2\delta^3 + \delta|x - x_0|^2, \qquad \delta > 0$$

For small δ this is a small C^2 perturbation of ϕ, therefore it remains strongly pseudoconvex with respect to P. Observe also that for small δ

$$\{\psi > -\delta^3\} \cap \Sigma^- \subset B(x_0, \delta) \subset V$$

We choose a smooth cutoff function χ supported in $\{\psi > -\delta^3\}$ which equals 1 in $\{\psi > 0\}$. We apply the Carleman estimate (say (4)) to χu to get

$$\tau^{-1}\|e^{\tau\psi}\chi u\|^2_{H^m_\tau} \leq c\|e^{\tau\psi}P(x, D)\chi u\|^2_{L^2}, \quad \tau \geq \tau_0$$

According to our hypothesis on u, it follows that $P(\chi u)$ is supported in $\{\psi \leq 0\}$. Then we further get

$$\tau^{-1}\|e^{\tau\psi}\chi u\|^2_{H^m_\tau} \leq c\|P(x, D)\chi u\|^2_{L^2}, \quad \tau \geq \tau_0$$

Letting $\tau \to \infty$ in this relation we get $\chi u = 0$ in $\{\psi > 0\}$, i.e. $u = 0$ in $\{\psi > 0\}$. But this region contains a neighborhood of x_0. □

Example 1. Consider a second order elliptic operator $P = a^{ij}(x)\partial_i\partial_j$. Then

$$p_\phi(x, \xi, \tau) = -a^{ij}(x)(\xi_i + i\tau\partial_i\phi)(\xi_j + i\tau\partial_j\phi)$$

Hence $\Im p_\phi = -2a^{ij}(x)\xi_i\partial_j\phi$ therefore $\{\Im p_\phi, \phi\} = -2a^{ij}\partial_i\phi\partial_j\phi \neq 0$. Then (3) is always satisfied, which implies that any smooth surface is pseudoconvex with respect to P.

Example 2. Consider the wave operator, $P = \partial_t^2 - \Delta_x$. Arguing as above we see that the condition (3) is always satisfied, at least as long as Σ is noncharacteristic. However, now (1) is nontrivial. Since P has constant coefficients, its bicharacteristics are straight therefore any convex surface is also pseudoconvex. However, there are also nonconvex surfaces which are still pseudoconvex, for instance the positive level sets of the function $\phi = x^2 - t^2$.

3 Strong unique continuation

As seen before, in the case of second elliptic equations any surface is pseudoconvex. Hence, if a solution vanishes in an open set then it must vanish everywhere. Shrinking the open set to a point one can deduce that the same conclusion follows provided the solution decays rapidly enough near that point. What is the precise meaning of "rapidly enough" ? The answer to this was provided in dimension 2 by Carleman [3] in 1939 (this is where the Carleman estimates were first introduced), and in higher dimension by Aronszajn [2]. See also [7] for up to date references.

Definition 9. *Let $x_0 \in \mathbb{R}^n$. We say that a function $u \in L^2_{loc}$ vanishes of infinite order at x_0 if for all integers N we have*

$$\lim_{r \to 0} r^{-N} \int_{B(x_0,r)} |u|^2 dx = 0$$

Definition 10. *We say that strong unique continuation holds for an operator P if for each x_0 and each solution u to $Pu = 0$ near x_0 the following statement is true. If u vanishes of infinite order at x_0 then u is zero near x_0.*

Theorem 11. *Strong unique continuation holds for second order elliptic operators whose principal part has real coefficients.*

The proof of this theorem uses a family of Carleman estimates with weights which blow up polynomially at x_0. To simplify the statement, set $x_0 = 0$. If $P = \Delta$ then one can verify directly that the function $\phi(x) = -\ln|x|$ satisfies a degenerate pseudoconvexity condition, which easily leads to weaker Carleman estimates of the form

$$r^{-1}(\tau \| |x|^{-\tau-1} u\|_{L^2} + \| |x|^{-\tau} \nabla u\|_{L^2}) \leq \| |x|^{-\tau} \Delta u\|_{L^2} \quad \text{supp } u \subset B(0,r)$$

In order to work with a general second order elliptic operator P or to get stronger estimates one has to add more convexity to this weight. A sample result follows.

Theorem 12. *Let $P(x,D)$ be a second order elliptic operator with real coefficients so that $P(0,D) = \Delta$. Then for sufficiently large λ the function $\phi(x) = -\ln(|x| + \lambda|x|^2)$ is strongly pseudoconvex with respect to P near 0, and the following estimate holds for all u supported near 0 which vanish of infinite order at 0.*

$$\begin{aligned} \tau^3 \|(|x| + \lambda|x|^2)^{-\tau-1} u\|^2_{L^2} + \tau \|(|x| + \lambda|x|^2)^{-\tau} \nabla u\|_{L^2} \leq \\ c\|(|x| + \lambda|x|^2)^{-\tau} \Delta u\|^2_{L^2} \end{aligned} \tag{8}$$

4 Operators with analytic coefficients

The unique continuation results described in the previous section can be considerably improved in the special case when one has some information about the analyticity of the coefficients. The classical example in this direction is Holmgren's theorem for noncharacteristic surfaces. Recall that a surface $\Sigma = \{\phi = 0\}$ is called noncharacteristic if $p(x, \nabla\phi) \neq 0$ for $x \in \Sigma$.

Theorem 13 (Holmgren). *Assume that P has analytic coefficients and that Σ is noncharacteristic. Then unique continuation across Σ holds for the operator P.*

However, it is equally interesting to consider also the case when the coefficients are analytic only with respect to some of the variables. A good example is the case of autonomous evolution equations, i.e. with time independent coefficients. A natural setup to consider is as follows. We split the coordinates $x = (x_a, x_b)$ and consider partial differential operators P whose coefficients are analytic as functions of x_a, but possibly nonanalytic as functions of x_b. We define the subset of the cotangent bundle $\Gamma = \{\xi_a = 0\}$, which one can think of as the conormal bundle of the foliation of $\mathbb{R}^n$ by the hyperplanes $x_b = \text{const}$, along which the coefficients are analytic.

Theorem 14. *Assume that the coefficients of P are analytic with respect to x_a. Let Σ be an oriented hypersurface so that the pseudoconvexity condition (3) for Σ with respect to P holds in Γ. Assume that either of the following two conditions holds:*

a) P is elliptic in Γ.

b) P has real symbol on Γ and Γ is invariant with respect to the null bicharacteristic flow of P (i.e. $p_{x_a} = 0$ on $\{p = 0, \xi_a = 0\}$) .

Then unique continuation across Σ holds for the operator P.

This result was first proved in Tataru [15] for a more restrictive class of operators, which was subsequently relaxed in [12], [4] and finally, in the present form, in [17].

Comparing this with Theorem 4, one can see that the assumptions are almost identical, with the only difference that now we are asking for these assumptions to be satisfied on a smaller subset of the cotangent bundle, namely on Γ.

Suppose we consider the case of operators whose coefficients are analytic with respect to all variables. Then $x = x_a$, $\Gamma = \{\xi = 0\}$ so we only need to ask for the pseudoconvexity condition (3) at a single point. There it reads

$$\Re p_x(x, \nabla\phi)\overline{p}_\xi(x, \nabla\phi) + p_\xi(x, \nabla\phi)\nabla^2\phi\, \overline{p}_\xi(x, \nabla\phi) > 0 \quad \text{on } \{p(x, \nabla\phi) = 0\}$$

If Σ is noncharacteristic then $p(x, \nabla\phi) \neq 0$ on Σ so this is satisfied by default and we retrieve Holmgren's theorem. But even if Σ is characteristic at some points, the above condition may still hold, so we get a slight improvement over Holmgren's theorem.

The key step in the proof of Theorem 14 is still a Carleman estimate, but this time the weight is a pseudodifferential weight.

Theorem 15. *Let P be as in Theorem 14. Let ϕ be a smooth function which is real analytic in x_a and is strongly pseudoconvex with respect to P on Γ within* *some compact region $\Omega \subset \mathbb{R}^n$. Then there are positive constants c, d so that for all functions u supported in Ω all $\epsilon > 0$ sufficiently small and all τ sufficiently large we have:*

(a) P is elliptic in Γ:

$$\tau^{-1} \|e^{-\frac{\epsilon D_a^2}{2\tau}} e^{\tau\phi} u\|_{H_\tau^m}^2 \leq c(\|e^{-\frac{\epsilon D_a^2}{2\tau}} e^{\tau\phi} P(x,D)u\|_{L^2}^2 + e^{-d\epsilon\tau} \|e^{\tau\phi} u\|_{H_\tau^m}^2) \tag{9}$$

(b) P has real symbol in Γ:

$$\tau \|e^{-\frac{\epsilon D_a^2}{2\tau}} e^{\tau\phi} u\|_{H_\tau^{m-1}}^2 \leq c(\|e^{-\frac{\epsilon D_a^2}{2\tau}} e^{\tau\phi} P(x,D)u\|_{L^2}^2 + e^{-d\epsilon\tau} \|e^{\tau\phi} u\|_{H_\tau^{m-1}}^2) \tag{10}$$

Proof. (Sketch) We first conjugate with respect to the exponential weight as before. Then (9) for instance reduces to

$$\tau^{-1} \|e^{-\frac{\epsilon D_a^2}{2\tau}} v\|_{H_\tau^m}^2 \leq c(\|e^{-\frac{\epsilon D_a^2}{2\tau}} P_\phi v\|_{L^2}^2 + e^{-d\epsilon\tau} \|v\|_{H_\tau^m}^2) \tag{11}$$

However, this time we also need to conjugate with respect to the Gaussian. This interacts with the coefficients of P_ϕ. A simple computation gives the conjugate of x_a,

$$e^{-\frac{\epsilon D_a^2}{2\tau}} x_a = (x_a + i\frac{\epsilon D_a}{\tau}) e^{-\frac{\epsilon D_a^2}{2\tau}}$$

It is slightly more difficult to do the same for powers of x_a. The result has a simple form if one uses the Weyl calculus:

$$e^{-\frac{\epsilon D_a^2}{2\tau}} x_a^\alpha = OP^w(x_a + i\frac{\epsilon \xi_a}{\tau}) e^{-\frac{\epsilon D_a^2}{2\tau}}$$

To conjugate the coefficients of P_ϕ we first extend them to smooth compactly supported functions in $\mathbb{C}^{n_a} \times \mathbb{R}^{n_b}$ which are holomorphic in x_a in a small (complex) neighborhood of Ω. Then a somewhat more involved computation shows that for v supported in Ω we have

$$e^{-\frac{\epsilon D_a^2}{2\tau}} P_\phi = P_{\phi,\epsilon} e^{-\frac{\epsilon D_a^2}{2\tau}} + R_{\phi,\epsilon}$$

where, assuming that P_ϕ is written as a Weyl operator, we have

$$P_{\phi,\epsilon} = OP^w(p_\phi(x_a + i\frac{\epsilon\xi_a}{\tau}, x_b, \xi, \tau))$$

and the remainder R is exponentially small,

$$\|Rv\|_{H_\tau^m} \leq e^{-c\tau} \|v\|_{H_\tau^m} \qquad \text{supp } v \subset \Omega$$

Hence the estimate (11) reduces to

$$\tau^{-1} \|w\|_{H_\tau^m}^2 \leq c(\|P_{\phi,\epsilon} w\|_{L^2}^2 + e^{-d\epsilon\tau} \|v\|_{H_\tau^m}^2) \qquad w = e^{-\frac{\epsilon D_a^2}{2\tau}} \tag{12}$$

The second right hand side term accounts for the part of w whose Fourier transform is supported in $|\xi_a| > d\tau$. On the other hand, if d is small enough then the region $|\xi_a| \leq d\tau$ is close to Γ therefore the pseudoconvexity condition holds there. If in addition ϵ is sufficiently small then $P_{\epsilon,\phi}$ is a small perturbation of P_ϕ therefore the proof of (11) is a straightforward modification of the proof of (6).

The case (b) raises some additional difficulties because when $\tau = 0$ the estimate (10) is degenerate while on the other hand $P_{\phi,\epsilon}$ no longer has real symbol (like P_ϕ). This is where the invariance of Γ with respect to the null bicharacteristic flow comes in, as it insures that $P_{\epsilon,\phi}$ satisfies a principal normality condition on Γ,

$$\{\Re p_{\epsilon,\phi}, \Im p_{\epsilon,\phi}\} \leq c|\xi|^{m-1}|p_{\epsilon,\phi}| \quad \text{on } \{\xi_a = 0,\ \tau = 0\}$$

This suffices in order to carry out the proof of the counterpart of (12), though it is a bit more difficult. $\square$

Proof of Theorem 14. This is done in the same manner as the proof of Theorem 4. The δ there must be chosen much smaller than ϵ, so that the second term on the right in the estimates (9), (10) can be neglected. The nontrivial part of the argument is summarized in the following

Lemma 16. *Assume that*

$$\|e^{-\frac{\epsilon D_a^2}{2\tau}} e^{\tau\psi} u\|_{L^2} \leq c \qquad \tau > \tau_0$$

Then $u = 0$ in $\{\psi > 0\}$.

To prove this take a function v with compactly supported Fourier transform and define the entire function

$$g(s) = \langle v, e^{is\psi} u\rangle$$

g is trivially bounded on the real axis, and by hypothesis it follows that it is also bounded on the positive imaginary axis. But g is of exponential type, therefore the Phragmen-Lindelof theorem implies that g is bounded in the upper half-plane. Then its Fourier transform,

$$\hat{g}(\theta) = \langle v, \delta_{\psi=\theta} u\rangle$$

is supported in $\mathbb{R}^-$. Hence

$$\langle v, 1_{\psi>0} u\rangle = 0$$

By density this must hold for all $v \in L^2$ therefore $u = 0$ in $\{\psi > 0\}$. $\square$

Example 3: Consider a wave operator with time independent coefficients, $P = \partial_t^2 - g^{ij}(x)\partial_i\partial_j$ with positive definite g^{ij}. Its symbol is $p(x, x, \xi) = -s^2 + g^{ij}(x)\xi_i\xi_j$ and $\Gamma = \{s = 0\}$. Then P is elliptic in Γ. On the other hand, as in Example 1, we can argue that (3) is satisfied by default for a surface Σ provided that Σ is noncharacteristic. Hence we conclude that for P we have unique continuation across any noncharacteristic surface. This was the first problem to be considered in this direction, and a partial result for it by Robbiano [11] opened the way for further developments.

5 Boundary value problems

Let Ω be an open subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Consider a boundary value problem in Ω,

$$\begin{cases} P(x,D)u = 0 & \text{in } \Omega \\ B(x,D)u = 0 & \text{in } \partial\Omega \end{cases} \tag{13}$$

Here $B = \{B_k\}_{k=1,k_0}$ with $k_0 \leq m-1$ are defined on the boundary $\partial\Omega$. We denote their orders by $m_k \leq m-1$.

Given a smooth oriented surface $\Sigma = \{\phi = 0\}$ one can still ask whether unique continuation across Σ holds for solutions to this equation. At points on Σ which are inside Ω one can disregard the boundary condition and look at this problem as before. However, at points on $\Sigma \cap \partial\Omega$ the boundary condition plays an essential role. Some additional condition must be imposed on the boundary operators in order for such a result to hold. By analogy with the similar condition arising in the study of elliptic or hyperbolic boundary value problems, such a condition is called a Lopatinskii condition. Then the main result is

Theorem 17. *Assume that Σ is strongly pseudoconvex with respect to P and that the pair (B,P) satisfies the Lopatinskii condition with respect to $d\phi$. Then unique continuation across Σ holds for solutions to (13).*

Of course what is missing in this statement is a definition of what the Lopatinskii condition is, and this is where things get a bit tricky.

On one hand, there is a strong form of the Lopatinskii condition, called the strong Lopatinskii condition. This is an algebraic condition connecting the symbols of P, B, the conormal direction ν to $\partial\Omega$ and the conormal direction $d\phi$ to Σ. The precise definition is somewhat lengthy, so we refer the reader to [14] for details. It is an analogue in this set-up of Sakamoto's strong Lopatinskii condition which guarantees the strong L^2 well-posedness of hyperbolic initial/boundary value problems.

This is necessary and sufficient in order for a strong form of the Carleman estimates to hold:

Theorem 18. *Assume that ϕ is strongly pseudoconvex with respect to P and that the pair (B,P) satisfies the strong Lopatinskii condition with respect to $d\phi$ in a compact subset K of Ω. Then for all u supported in K we have:*

(a) P is elliptic:

$$\begin{array}{c} \tau^{-1}\|e^{\tau\phi}u\|^2_{H^m_\tau(\Omega)} + \tau \displaystyle\sum_{j=0}^{m-1} \|e^{\tau\phi}\partial^j u\|^2_{H^{m-k-1}_\tau(\partial\Omega)} \leq \\ c(\|e^{\tau\phi}P(x,D)u\|^2_{L^2(\Omega)} + \tau \displaystyle\sum_k \|e^{\tau\phi}B_k u\|^2_{H^{m_k}_\tau(\partial\Omega)}) \end{array}, \quad \tau \geq \tau_0 \tag{14}$$

(b) P has real principal symbol:

$$\begin{array}{l} \tau\|e^{\tau\phi}u\|^2_{H^{m-1}_\tau(\Omega)} + \tau \sum_{j=0}^{m-1} \|e^{\tau\phi}\partial^j u\|^2_{H^{m-k-1}_\tau(\partial\Omega)} \leq \\ \qquad c(\|e^{\tau\phi}P(x,D)u\|^2_{L^2(\Omega)} + \tau \sum_k \|e^{\tau\phi}B_k u\|^2_{H^{m_k}_\tau(\partial\Omega)}) \end{array}, \quad \tau \geq \tau_0 \tag{15}$$

In both cases the constant c can be chosen not to depend on τ.

For the proof we refer the reader to [14].

On the other hand one also expects that in certain cases only a weaker form of these estimates may hold, which nevertheless imply the unique continuation property.

Theorem 19. *Assume that ϕ is strongly pseudoconvex with respect to P and that the pair (B, P) satisfies the weak Lopatinskii condition with respect to $d\phi$ in a compact subset K of Ω. Then for all u supported in K with $Bu = 0$ on $\partial\Omega$ we have:*

(a) P is elliptic:

$$\tau^{-1}\|e^{\tau\phi}u\|^2_{H^m_\tau(\Omega)} \leq c\|e^{\tau\phi}P(x,D)u\|^2_{L^2(\Omega)}, \quad \tau \geq \tau_0 \tag{16}$$

(b) P has real principal symbol:

$$\tau\|e^{\tau\phi}u\|^2_{H^{m-1}_\tau(\Omega)} \leq c\|e^{\tau\phi}P(x,D)u\|^2_{L^2(\Omega)}, \quad \tau \geq \tau_0 \tag{17}$$

In both cases the constant c can be chosen not to depend on τ.

This is only a quasi-theorem in that it is missing the proof and a clear definition of what the weak Lopatinskii condition is. One could conceivably take the estimates as a definition of the weak Lopatinskii condition and attempt to produce an algebraic characterization of it. However, this is a highly nontrivial endeavor even in the case of the L^2 well-posedness for second order hyperbolic initial boundary value problems. At the very least, one may keep this in mind as a general principle and prove it in special cases.

Example 4: Let $P = g^{ij}(x)\partial_i\partial_j$ be a second order elliptic operator. Let ν be the outer normal to the boundary $\partial\Omega$. N be the Neumann operator, $Nu = \nu_i g^{ij}\partial_j$. Then the strong Lopatinskii condition is satisfied if B is either Dirichlet, $Bu = u$ or Neumann, $Bu = Nu$, if and only if $N\phi < 0$ on $\partial\Omega$.

Example 5: Let $P = \partial_t^2 + g^{ij}(t,x)\partial_i\partial_j$ be a second order wave operator. Suppose that the boundary is time-like, and set $N = -\nu_0\partial_0 + \nu_i g^{ij}\partial_j$. Then

(i) The strong Lopatinskii condition is satisfied if B is Dirichlet, $Bu = u$, if and only if $N\phi < 0$ on $\partial\Omega$.

(ii) The weak Lopatinskii condition is satisfied if B is Neumann, $Bu = Nu$, if and only if $N\phi = 0$ on $\partial\Omega$.

6 Anisotropic unique continuation

In the classical unique continuation results one works with partial differential operators whose principal symbol is homogeneous of a certain order, and all lower order terms play no role in the geometry of the problem. Such a set-up fits some operators but does not fit others. Consider for instance the heat operator $\partial_t - \Delta$, the Schroedinger operator $\partial_t - i\Delta$ or the KdV operator $\partial_t - \partial_x^3$. In all these examples the time derivative is of lower order than the spatial derivatives, however one can only neglect it at the expense of completely changing the type of the operator. The natural conclusion is that one has to adapt the setup to the operator being studied. We call such operators, which require a different set-up, anisotropic. For the heat and Schroedinger operators one needs a calculus where one time derivative carries the same weight as two space derivatives. For the KdV operator, one time derivative must carry the same weight as three space derivatives. Such a calculus can be constructed in a very similar framework. However, to keep things simple we confine ourselves to the case which corresponds to both the heat and Schroedinger operators. To fix the notations suppose we work in $\mathbb{R} \times \mathbb{R}^n$, (t, x) are the physical variables and (s, ξ) are the Fourier variables. For a multiindex $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n)$ we set $\partial^\alpha = \partial_t^{\alpha_0} \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ and we define the order of this operator by

$$|\alpha| = 2\alpha_0 + \alpha_1 + \cdots + \alpha_n$$

For a partial differential operator we let the principal part be the part containing the highest order terms according to this method of counting.

A straightforward observation in this case is that one has to modify the definition of the Poisson bracket, and set

$$\{p(t, x, s, \xi), q(t, s, x, \xi)\} = p_\xi(t, x, s, \xi) q_x(t, x, s, \xi) - p_x(t, x, s, \xi) q_\xi(t, x, s, \xi)$$

Missing on the right are the terms which correspond to t and s derivatives, which in this context become lower order. This is because differentiation with respect to s reduces the order by 1 in s, which means by 2 with respect to the anisotropic definition of the order.

Unique continuation problems for operators of this type were considered as early as 1958, see Mizohata [9]. Using the modified Poisson bracket in the definition of the pseudoconvexity condition Isakov [6] proved that all the classical results in Section 2 can be transferred to the anisotropic case without any change. However, such a result is not entirely satisfactory as it is not properly adapted to this context.

One way to think about this is that equations such as the Schroedinger or the heat equation have an infinite speed of propagation in time. In other words, high frequency information propagate essentially along time slices (at least locally in space). This may lead one to believe that the unique continuation results should also apply along time slices, rather than in space-time domains.

A naive attempt would be to show that if a solution vanishes on some part of a time slice then it must also vanish on some other part. But this cannot work in general; consider for instance for the Schroedinger equation, where one can prescribe the initial data arbitrarily on a given time slice. The same example shows that even assuming that the solution vanishes of infinite order is insufficient. As it turns out, one needs an exponential rate of decay in order for such an argument to work.

To state the result we need to redefine the notion of pseudoconvexity. For this we consider oriented surfaces Σ within a fixed time-slice $\{t = t_0\}$.

Definition 20. *An oriented surface $\Sigma = \{\phi(x) = 0\} \subset \{t = t_0\}$ is strongly pseudoconvex with respect to P if either*

a) P is elliptic and (3) holds.

b) P has real principal symbol and (1), (3) hold.

This definition is seemingly identical with Definition 3 but one has to keep in mind that the principal part of P is now determined with respect to the anisotropic scaling. Also that we are using the anisotropic version of the Poisson bracket, which only requires x derivatives of ϕ.

Definition 21. *We say that anisotropic unique continuation across $\Sigma \subset \{t = t_0\}$ holds for the operator P if for each $\epsilon \neq 0$, $x_0 \in \Sigma$ and V a neighborhood of x_0 in $\mathbb{R}^n$ the following holds: Let u be a solution for $P(x,D)u = 0$ in $V \times [t_0, t_0 + \epsilon]$ so that*

$$\lim_{t \to t_0,\ t \in (t_0, t_0+\epsilon]} u(t,x) e^{\frac{M}{t}} = 0 \qquad \textit{for all } M > 0 \tag{18}$$

uniformly for $x \in \Sigma^+ \cap V$. Then the same holds for all x in a neighborhood of x_0.

Note that we only ask for one-sided information on the solution u, either for $t > t_0$ (when $\epsilon > 0$) or for $t < t_0$ (when $\epsilon < 0$). Then the following result is proved in Tataru [16].

Theorem 22. *Assume that the oriented surface $\Sigma \subset \{t = t_0\}$ is strongly pseudoconvex with respect to P. Then anisotropic unique continuation across Σ holds for the operator P.*

If we define strongly pseudoconvex functions in the same manner as strongly pseudoconvex surfaces, then the corresponding Carleman estimates have the form (see [16]):

Theorem 23. *Assume that the function ϕ is strongly pseudoconvex with respect to P in a compact region $\{t_0\} \times K$. Let g be a smooth positive function so that*

$$|g^{(j)}(t)| \leq c_j g(t)^j \qquad j \geq 1 \tag{19}$$

Then for small enough ϵ and for all functions u supported in $[t_0 - \epsilon, t_0 + \epsilon] \times K$ we have

(a) P is elliptic:

$$\tau^{-1} \| g^{-\frac{1}{2}} e^{\tau\phi(x)g(t)} u \|^2_{H^m_{\tau g}} \leq c \| e^{\tau\phi(x)g(t)} P(x,D) u \|^2_{L^2}, \quad \tau \geq \tau_0 \tag{20}$$

(b) P has real principal symbol:

$$\tau\|g^{\frac{1}{2}}e^{\tau\phi(x)g(t)}u\|^2_{H^{m-1}_{\tau g}} \leq c\|e^{\tau\phi(x)g(t)}P(x,D)u\|^2_{L^2}, \quad \tau\geq\tau_0 \tag{21}$$

In both cases the constant c can be chosen not to depend on τ and on g subject to (19).

Here the spaces $H^m_{\tau g}$ are defined as before but with τ replaced by τg. The function g depends on t but this does not cause problems in the definition of the norm because, by (19), the function g varies slowly enough.

As a limit of functions g which satisfy (19) uniformly we obtain functions g which blow up, for instance

$$g(t) = |t-t_0|^{-1} = \lim_{\delta\to 0}(\delta^2+(t-t_0)^2)^{-\frac{1}{2}}$$

Using this weight the (weaker) two sided version of the unique continuation result immediately follows as in the proof of Theorem 4. The only difference is that here it suffices to use the estimate for fixed τ, as opposed to letting $\tau\to\infty$.

It is somewhat more difficult, but one can also show that the corresponding one sided estimates with the weight $|t-t_0|^{-1}$ are also valid. The price one needs to pay for this is to increase the lower limit for the values of τ for which the estimate holds.

Example 6: Let P be the parabolic operator $P=\partial_t - g^{jk}\partial_j\partial_k$. Then P is elliptic, and its principal symbol is $p = is + g^{jk}\xi_j\xi_k$. Hence the symbol of the conjugated operator is $p_\phi = is + g^{ij}(\xi_i+i\partial_i\phi)(\xi_j+i\partial_j\phi)$. A short computation yields $\{\Im p_\phi,\phi\} = 2g^{ij}\partial_i\phi\partial_j\phi \neq 0$, therefore (3) always holds. Hence all surfaces Σ are strongly pseudoconvex with respect to P. This example also shows that Theorem 22 is sharp, in the sense that one cannot relax the decay requirement at t_0. Indeed, consider the parabolic initial value problem with initial data $u=u_0$ at time 0. If u_0 vanishes in some open set ω, then on compact subsets of ω the solution u decays like $e^{-\frac{c}{t}}$ as t approaches 0.

Example 7: Let P be the Schroedinger operator $P = i\partial_t - \Delta$. Then $p=-s+\xi^2$ and $p_\phi = -s+(\xi+i\tau\nabla\phi)^2$. Since $\{\Im p_\phi,\phi\} = 2|\nabla\phi|^2\neq 0$ it follow that (3) always holds. On the other hand, the bicharacteristic flow for P contains all straight lines on time slices, therefore (1) says that a surface Σ is strongly pseudoconvex if and only if it is convex.

7 Parabolic strong unique continuation

As we have seen in Example 6, the anisotropic unique continuation result is sharp for the forward heat equation, i.e. if we have information only for $t>t_0$. If, on the other hand, we also know that the function u is also a solution to the heat equation before the time t_0, then we can get a much stronger result, which is the parabolic counterpart of the elliptic strong unique continuation result.

A second order parabolic operator is an operator of the form

$$P=\partial_t - a^{ij}(t,x)\partial_i\partial_j, \qquad \{a^{ij}\}\text{ real, positive definite}$$

possibly with arbitrary lower order terms. Given $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, a backward parabolic neighborhood of (t_0, x_0) is a set containing a cylinder $[t_0 - r^2, t_0] \times B(x_0, r)$.

Definition 24. *We say that a function $u \in L^2_{loc}$ vanishes of infinite order at (t_0, x_0) if for all integers N we have*

$$\lim_{r\to 0} r^{-N} \int_{t_0-r^2}^{t_0} \int_{B(x_0,r)} |u|^2 dtdx = 0$$

Definition 25. *We say that parabolic strong unique continuation holds for an operator P if for each $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ and each solution u to $Pu = 0$ in a backward parabolic neighbourhood of (t_0, x_0) the following statement is true. If u vanishes of infinite order at (t_0, x_0) then $u(t_0, x)$ is zero for x near x_0.*

Theorem 26. *Parabolic strong unique continuation holds for second order parabolic operators with real coefficients.*

A result of this type was first proved by Lin [8] in a more restrictive setting, but a slightly weaker formulation has been known for much longer, see for instance ([13]). As stated the result was proved in [10].

One can prove this result using a class of Carleman estimates with weights which blow up polynomially at (t_0, x_0). Set $(t_0, x_0) = 0$. If $P = \partial_t - \Delta$ then we have a nice family of weights for which a degenerate estimate holds.

$$r^{-2}(\||t|^{-\tau} e^{-\frac{x^2}{4t}} u\|_{L^2} \leq \||t|^{-\tau} e^{-\frac{x^2}{4t}} (\partial_t - \Delta u)\|_{L^2} \quad \text{supp } u \subset [-r^2, 0] \times B(0, r)$$

In order to work with a general second order parabolic operator P or to get stronger estimates one needs to add more convexity to the weight.

Theorem 27. *Let P be a second order parabolic operator with real coefficients so that $P(0, D) = \partial_t - \Delta$. Then for sufficiently large λ the function $\phi(x) = -\ln(|x| + \lambda|x|^2)$ is strongly pseudoconvex with respect to P near 0, and the following estimate holds for all u supported in $[-c\tau^{-1}, 0]$ which vanish of infinite order at 0.*

$$\tau^{\frac{3}{2}} \|(|t| + \sqrt{\tau|t|^3})^{-\tau-\frac{3}{2}} e^{-\frac{x^2}{4t}} u\|_{L^2} + \tau^{\frac{1}{2}} \|(|t| + \sqrt{\tau|t|^3})^{-\tau-\frac{1}{2}} e^{-\frac{x^2}{4t}} \nabla_x u\|_{L^2} \leq \\ \|(|t| + \sqrt{\tau|t|^3})^{-\tau} e^{-\frac{x^2}{4t}} u\|_{L^2} \tag{22}$$

References

[1] S. Alinhac and M. S. Baouendi. A nonuniqueness result for operators of principal type. *Math. Z.*, 220(4):561–568, 1995.

[2] N. Aronszajn. A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order. *J. Math. Pures Appl. (9)*, 36:235–249, 1957.

[3] T. Carleman. Sur un problème d'unicité pur les systèmes d'équations aux dérivées partielles à deux variables indépendantes. *Ark. Mat., Astr. Fys.*, 26(17):9, 1939.

[4] Lars Hörmander. On the uniqueness of the Cauchy problem under partial analyticity assumptions. unpublished.

[5] Lars Hörmander. *The analysis of linear partial differential operators. IV.* Springer-Verlag, Berlin, 1985. Fourier integral operators.

[6] Victor Isakov. Carleman type estimates in an anisotropic case and applications. *J. Differential Equations*, 105(2):217–238, 1993.

[7] Herbert Koch and Daniel Tataru. Carleman estimates and unique continuation for second-order elliptic equations with nonsmooth coefficients. *Comm. Pure Appl. Math.*, 54(3):339–360, 2001.

[8] Fang-Hua Lin. A uniqueness theorem for parabolic equations. *Comm. Pure Appl. Math.*, 43(1):127–136, 1990.

[9] Sigeru Mizohata. Unicité du prolongement des solutions pour quelques opérateurs différentiels paraboliques. *Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math.*, 31:219–239, 1958.

[10] Chi-Cheung Poon. Unique continuation for parabolic equations. *Comm. Partial Differential Equations*, 21(3-4):521–539, 1996.

[11] Luc Robbiano. Théorème d'unicité adapté au contrôle des solutions des problèmes hyperboliques. *Comm. Partial Differential Equations*, 16(4-5):789–800, 1991.

[12] Luc Robbiano and Claude Zuily. Uniqueness in the Cauchy problem for operators with partially holomorphic coefficients. *Invent. Math.*, 131(3):493–539, 1998.

[13] Taira Shirota. A theorem with respect to the unique continuation for a parabolic differential equation. *Osaka Math. J.*, 12:377–386, 1960.

[14] D. Tataru. Carleman estimates and unique continuation for solutions to boundary value problems. *J. Math. Pures Appl. (9)*, 75(4):367–408, 1996.

[15] Daniel Tataru. Unique continuation for solutions to PDE's; between Hörmander's theorem and Holmgren's theorem. *Comm. Partial Differential Equations*, 20(5-6):855–884, 1995.

[16] Daniel Tataru. Carleman estimates, unique continuation and controllability for anizotropic PDEs. In *Optimization methods in partial differential equations (South Hadley, MA, 1996)*, pages 267–279. Amer. Math. Soc., Providence, RI, 1997.

[17] Daniel Tataru. Unique continuation for operators with partially analytic coefficients. *J. Math. Pures Appl. (9)*, 78(5):505–521, 1999.