Null controllability for the dissipative semilinear heat equation

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Abstract

We consider the exact null controllability problem for the semilinear heat equation with a dissipative nonlinearity in a bounded domain of $\mathbb{R}^n$. The main result of the article asserts that if the nonlinearity is even mildly superlinear then global null controllability in arbitrarily short time fails; instead we provide sharp estimates for the controllability time in terms of the size of the initial data.

Keywords. Null controllability, heat equation, nonlinear parabolic equation

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We consider the internal exact null controllability problem for a semilinear heat equation

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with Dirichlet boundary condition

$$\begin{cases} y_t - \Delta y + f(x,y) = m(x)u & \text{in} \quad \Omega \times (0,T) \\ y = 0 & \text{in} \quad \partial\Omega \times (0,T) \\ y(0) = y_0 & \text{in} \quad \Omega \ . \end{cases} \tag{1}$$

Here $T > 0$ and m is the characteristic function of a nonempty open subset $\omega \subset \Omega$. The function f is locally Lipschitz continuous and $f(x,0) = 0$.

Definition 1. *a) An initial data $y_0 \in L^2(\Omega)$ is exactly null controllable in time T if there is a control $u \in L^2(\omega \times [0,T])$ so that the solution y to (1) satisfies $y \in C([0,T]; L^2(\Omega))$ and $y(T) = 0$.*

b) Global exact null controllability at time T holds for the equation (1) if any initial data $y_0 \in L^2(\Omega)$ is exact null controllable in time T.

For the corresponding linear equation local and global controllability are equivalent and hold for any $T > 0$; see Lebeau and Robbiano [8], Fursikov and Imanuvilov [6] and Tataru [10]. For the semilinear equation the same result was first proved by Fursikov and Imanuvilov [6] for sublinear f and later by Barbu [3] and Fernández-Cara [4] for weakly superlinear f, namely when

$$\lim_{|y| \to +\infty} \frac{|f(y)|}{|y| \, (\ln |y|)^{\frac{3}{2}}} = 0 \ .$$

For strongly nonlinear f one can still apply the linear result to the linearization of the equation (1) near 0 and establish local exact null controllability for arbitrarily short time. However, there seems to be little n reason to assume that the global behavior would mirror the local one unless f satisfies some structure conditions. Indeed, Fursikov and Imanuvilov [6] and later Fernández-Cara and Zuazua [5] produced examples of nonlinearities f for which there is no global null controllability. Their idea is to take (superlinear) f for which the solutions to the uncontrolled equation can blow up in an arbitrarily short time, and then show that the blow up still occurs for the controlled equation. The growth of f at infinity in these examples can be as little as

$$|f(y)| \approx |y| \, (\ln |y|)^p \,, \quad p > 2 \ .$$

The aim of this article is to study the case when the nonlinear term f is dissipative, i.e. $yf(x,y) \geq 0$. Blow-up cannot occur in this case. Furthermore, such a nonlinearity produces energy decay for the uncontrolled equation, therefore naively one may be led to believe that it can only help in steering the solution to 0 in an arbitrarily short time. Our results shows that this is false, more precisely that for "strongly" superlinear f one needs a sufficiently large time in order to bring the solution to 0. An intuitive explanation for this is that the nonlinearity is also damping the effect of the control as it expands from the controlled region into the uncontrolled region. Our methods are robust and can be easily expanded to variable coefficients, arbitrary (linear) lower order terms and various boundary conditions.

In the sequel we assume for simplicity that f is nondecreasing as a function of y. This insures that for any $y_0 \in L^2(\Omega)$ and $u \in L^2(\omega \times [0,T])$ the equation (1) has a unique solution $y \in C([0,T]; L^2(\Omega))$ (see Barbu [2]). This condition can also be relaxed somewhat, without any significant change in the results. The notation $A \lesssim B$ means $A \leq cB$ with c depending only on Ω, ω, the Lipschitz constant for f in $[-1,1]$ and, for Theorems 3 and 4, on the parameters C, z_0, p in (2), (3). We say that $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

Given an initial data $y_0 \in L^2(\Omega)$ denote

$$T(y_0) = \inf\{T > 0; y_0 \text{ is exactly null controllable in time } T\} \ .$$

For positive R we introduce the nonnegative nondecreasing function

$$T(R) = \sup\{T(y_0);\ \|y_0\|_{L^2(\Omega)} \leq R\}$$

and note that the results quoted above show that if f is "weakly" superlinear then $T(R) = 0$ for all R. To set up the stage for our results we begin with a consequence of the local theory:

Theorem 2. *Assume that f is locally Lipschitz continuous, nondecreasing in the second variable y and $f(x,0) = 0$. Set*

$$\Phi(R) = \begin{cases} (1 + |\ln R|)^{-1}, & R \leq 1 \\ 1 + |\ln R|, & R > 1 \ . \end{cases}$$

Then for all $R > 0$ we have

$$T(R) \lesssim \Phi(R) \ .$$

Are these bounds sharp for "strongly" superlinear f ? Our next result shows that the answer to this question is affirmative only for small R:

Theorem 3. *Let f be locally Lipschitz continuous, nondecreasing in y with $f(x,0)=0$. Set*

$$\Psi(R) = \begin{cases} (1+|\ln R|)^{-1}, & R \leq 1 \\ \\ 1, & R > 1 \ . \end{cases}$$

a) Assume that there exists $C>0$ such that at least one of the following two inequalities holds:

$$f(x,z) \geq Cz\,(\ln|z|)^p\,, \qquad z > z_0 > 1, \quad p > 2, \tag{2}$$

$$f(x,z) \leq Cz\,(\ln|z|)^p\,, \qquad z < z_0 < -1, \quad p > 2. \tag{3}$$

Then

$$T(R) \gtrsim \Psi(R).$$

b) Assume that both (2) and (3) hold. Then

$$T(R) \lesssim \Psi(R).$$

Comparing this with the results of Barbu [3] and Fernández-Cara [4] one can see that the problem remains open in the case when f behaves like $s\,(\ln|s|)^p$ with $3/2 < p \leq 2$. We conjecture that the natural threshold is at $p=2$, in other words that our negative results here are sharp.

The second example we consider is a worst case scenario where the damping is effective in a neighborhood of the controlled region but not elsewhere. As it turns out, in this case the bound in Theorem 2 is sharp.

Theorem 4. *Let f be locally Lipschitz, nondecreasing in y with $f(x,0)=0$. Assume in addition that either (2) and (3) hold for x in a neighborhood of ω, while $f(x,y)=0$ for x in an open set $\tilde{\omega}$. Then the controllability time satisfies*

$$T(R) \approx \Phi(R)$$

In the next three sections we give the proofs of Theorems 2, 3, 4. In the last section we discuss several extensions of these results. We shall use the standard notions for the Sobolev spaces $H^k(\Omega)$, $H_0^1(\Omega)$ and the L^k-spaces, $1 \leq k \leq +\infty$. For the definitions and basic properties of these spaces we refer to Adams [1] and for basic results concerning the parabolic boundary value problems in L^k frame we refer to [7] and [9].

2 Proof of Theorem 2

The first step of the proof is to consider the case when the initial data is sufficiently small in $L^\infty(\Omega)$. For this we use the exact null controllability result for the linear heat equation with a bounded potential, which can be stated as follows:

Lemma 5. *Let $0 < T < 1$. Let V be a bounded potential satisfying $|V(x,t)| \leq L$ in $\Omega \times [0,T]$. Then there exists a constant $c > 0$ depending on L such that for any $y_0 \in L^2(\Omega)$ there is a control $u \in L^\infty(\omega \times [0,T])$ satisfying*

$$\|u\|_{L^\infty(\omega\times[0,T])} \leq e^{\frac{c}{T}} \|y_0\|_{L^2(\Omega)} \tag{4}$$

and

$$y(T) = 0 \quad a.e. \quad x \in \Omega, \tag{5}$$

where y is the solution to

$$\begin{cases} y_t - \Delta y + Vy = m(x)u & in\ \Omega \times [0,T] \\ y = 0 & in\ \partial\Omega \times [0,T] \\ y(0) = y_0 & in\ \Omega\ . \end{cases} \tag{6}$$

The proof follows as in [5].

We use this to prove its local nonlinear counterpart, namely

Lemma 6. *Let $0 \leq T \leq 1$. Then there exists a constant $c > 0$ such that for any initial data y_0 satisfying*

$$\|y_0\|_{L^2(\Omega)} \leq e^{-\frac{c}{T}}, \qquad \|y_0\|_{L^\infty(\Omega)} \leq 1 \tag{7}$$

there is a control $u \in L^\infty(\omega \times [0,T])$ satisfying (4) so that the solution y to (1) satisfies $y(T) = 0$.

Proof. The function f is Lipschitz continuous on $[-2,2]$, i.e. there exists $L \geq 0$ such that

$$|f(x,y) - f(x,z)| \leq L \cdot |y - z|\,, \quad \forall\, y,z \in [-2,2]\,,\ x \in \Omega\,.$$

We first prove the result under the additional assumption that f is differentiable at 0 with respect to y. Define the continuous function $a : \Omega \times [-2,2] \to \mathbb{R}$ by

$$a(x,y) = \begin{cases} \dfrac{f(x,y)}{y}, & y \in [-2,2] \setminus \{0\} \\ f_y(x,0) & \text{otherwise.} \end{cases}$$

which satisfies $0 \leq a(x,y) \leq L$. Consider the set

$$K_T = \left\{ w \in L^2\left(\Omega \times [0,T]\right); \quad \|w\|_{L^\infty(\Omega \times [0,T])} \leq 2 \right\}.$$

which is closed and convex in $L^2\left(\Omega \times [0,T]\right)$.

For each $w \in K_T$ we denote by $\Gamma\left(w\right) \subset L^2(\Omega \times [0,T])$ the set of all solutions y to (6) with $V = a(x,w)$ such that (4) and (5) hold. Observe that if the solution y to (1) is in K_T then it also solves (6) with the potential $V = a(x,y)$. Hence in order to prove the Lemma it suffices to show that Γ has a fixed point in K_T.

By Lemma 5 we know that $\Gamma\left(w\right) \neq \emptyset$ for each $w \in K_T$. Using the comparison result for the solutions of the parabolic equations it follows that for any $y_0 \in L^\infty\left(\Omega\right)$ the solution y to (6) (with $V = a(x,y)$) satisfies

$$\|y\|_{L^\infty(\Omega \times [0,T])} \leq \|y_0\|_{L^\infty(\Omega)} + T\,\|u\|_{L^\infty(\omega \times [0,T])}\ .$$

Hence, by (4) and (7) we get

$$\|y\|_{L^\infty(\Omega \times [0,T])} \leq 2$$

therefore Γ maps K_T into K_T. We use Kakutani's fixed point theorem with respect to the $L^2(\Omega \times [0,T])$ topology to show that Γ has a fixed point in K_T.

Note first that for any $w \in K_T$, $\Gamma(w)$ is a closed and convex subset of $L^2\left(\Omega \times [0,T]\right)$. By the Aubin compactness theorem $\Gamma\left(K_T\right)$ is relatively compact in $L^2\left(\Omega \times [0,T]\right)$. It remains to verify that Γ is upper semicontinuous in $L^2\left(\Omega \times [0,T]\right) \times L^2\left(\Omega \times [0,T]\right)$.

Indeed, let $w_n \in K_T$ $(n \in \mathbb{N}^*)$, $w_n \to w$ in $L^2(\Omega \times [0,T])$ and $y_n \to y$ in $L^2(\Omega \times [0,T])$, where $y_n \in \Gamma(w_n)$. Denote by u_n the corresponding controls, which satisfy (4). On a subsequence we have

$$w_n \to w \quad \text{a.e. in } \Omega \times [0,T]$$

The functions y_n are in K_T, have the same initial data and $(\partial_t - \Delta)y_n$ are uniformly bounded in $L^\infty(\Omega \times [0,T])$. This implies that there is a function y in K_T so that on a subsequence

$$\begin{array}{lc} y_n \to y & \text{strongly in } C([0,T];L^2(\Omega)) \\ y_n \to y & \text{a.e. in } \Omega \times [0,T] \\ y_n - y \to 0 & \text{weakly in } L^2(0,T;H_0^1(\Omega) \cap H^2(\Omega)) \cap W^{1,2}([0,T];L^2(\Omega)) \end{array}$$

The functions u_n satisfy (4); hence, there is u satisfying (4) so that on a subsequence

$$u_n \to u \quad \text{weakly in } L^2(\omega \times [0,T])$$

The relations above imply that

$$a(x,w_n)\,y_n \to a(x,w)\,y \textit{ a.e. in } \Omega \times [0,T].$$

Then we can let $n \to +\infty$ in

$$\begin{cases} (y_n)_t - \Delta y_n + a(x,w_n)\,y_n = m(x)u_n & \text{in } \Omega \times [0,T] \\ y_n = 0 & \text{in } \partial\Omega \times [0,T] \\ y_n(0) = y_0 & \text{in } \Omega \end{cases}$$

and conclude that $y \in \Gamma(w)$.

This proves the lemma in the case when f is differentiable at 0 with respect to y. If this is not the case then we approximate f uniformly with functions f_n which have the same Lipschitz constant in $[-2,2]$ and are differentiable at 0. For these we find controls u_n which steer the corresponding solution y_n to 0. Then one can argue as above to pass to the limit on a subsequence and prove the Lemma for f.

□

We now complete the proof of Theorem 2. Consider first the case when the initial data is small in $L^2(\Omega)$: $\|y_0\|_{L^2(\Omega)} = R \le e^{-c}$. The parabolic $L^2 \to L^\infty$ inequality

$$\|y(t)\|_{L^\infty(\Omega)} \lesssim t^{-\frac{n}{4}} \|y_0\|_{L^2(\Omega)}, \qquad t \lesssim 1$$

for the uncontrolled equation shows that it takes at most a time $t \approx R^{\frac{4}{n}}$ to get the solution into the set

$$\{\|y(t)\|_{L^2(\Omega)} \leq R, \quad \|y(t)\|_{L^\infty} \leq 1\} .$$

From this point we use Lemma 6 to steer the solution to 0 into a time no larger than $-c(\ln R)^{-1} \leq 1$. Summing up, to get to zero we need at most

$$O(R^{\frac{4}{n}}) - c(\ln R)^{-1} \lesssim \Phi(R) .$$

Now consider the case of large data, $\|y_0\|_{L^2(\Omega)} = R > e^{-c}$. Then we use the L^2 exponential decay of solutions to the uncontroled equation,

$$\|y(t)\|_{L^2(\Omega)} \leq e^{-\lambda_1 t}\|y_0\|_{L^2(\Omega)},$$

where λ_1 is the first eigenvalue of the Dirichlet Laplacian in Ω. Hence we need a time of at most $\frac{c+\ln R}{\lambda_1}$ to get into the set $\{\|y(t)\|_{L^2(\Omega)} \leq e^{-c}\}$. To get to zero from here we need a time of $O(1)$, given by the small data case.

3 Proof of Theorem 3

a) Suppose that (3) holds. Without any restriction in generality assume that $0 < R \leq 1$. Select a ball $B\left(x_0;r\right) \subset\subset \Omega \setminus \overline{\omega}$ and choose initial data $y_0 \in L^2\left(\Omega\right)$ so that

$$y_0(x) = R, \qquad x \in B\left(x_0;r\right) .$$

The result will follow from pointwise bounds from below on the solution y to (1) which are independent of the control u. Such bounds are obtained from the maximum principle, provided we construct appropriate subsolutions which equal $-\infty$ outside the ball $B\left(x_0;r\right)$. More precisely, the result in part (a) follows by comparing the solution y to the equation (1) with the subsolution z given by the next lemma:

Lemma 7. *There is a locally Lipschitz subsolution z to (1) in $B(x_0,r) \times \mathbb{R}^+$ so that*

(i) $lim_{d(x,\partial B(x_0,r))\to 0} z(x,t) = -\infty$ uniformly in $t \geq 0$,
(ii) $z(x,0) \leq R$ in $B(x_0,r)$,
(iii) $z(x_0,T) > 0$ for $0 \leq T \ll \Psi(R)$.

Proof. Without any restriction in generality we assume that $x_0 = 0$. We begin by constructing a negative time independent subsolution which satisfies the condition (i). Set

$$\zeta(x) = -Ae^{(r^2-|x|^2)^{-\alpha}} .$$

Then one can verify that for large enough A, α we have

$$-\Delta\zeta + C\zeta(\ln|\zeta|)^p < 0$$

which by (3) shows that ζ is a subsolution for (1).

We produce the function z by modifying the function ζ. Choose $0 < r_1 < r$ and define z as follows:

$$z(x,t) = \begin{cases} \zeta(x), & r_1 \le |x| < r \\ \mu(x,t), & |x| \le r_1 \end{cases}$$

where μ solves the original equation within the smaller ball,

$$\begin{cases} \mu_t - \Delta\mu + f(x,\mu) = 0, & |x| \le r_1 \\ \mu(x,t) = \zeta(r_1), & |x| = r_1 \\ \mu(x,0) = R, & |x| \le r_1 . \end{cases}$$

Since $\mu(x,0) \ge \zeta(x)$, the comparison principle shows that $\mu(x,t) \ge \zeta(x)$. This in turn implies that z is a subsolution for (1). It remains to verify (iii). For this we need bounds from below for μ. Since $\mu \le R \le 1$ it follows that $f(x,\mu) \le LR$, where $L > 0$ is the Lipschitz constant for f with respect to y in $[0,1]$. Then μ satisfies

$$\begin{cases} \mu_t - \Delta\mu \ge -LR, & |x| \le r_1 \\ \mu(x,t) = \zeta(r_1), & |x| = r_1 \\ \mu(x,0) = R, & |x| \le r_1 . \end{cases}$$

This leads to

$$\mu(0,t) \ge R(1-Lt) - ce^{-\frac{r_1^2}{4t}}, \qquad 0 < t < 1 .$$

Since $R \le 1$, this leads to

$$\mu(0,t) > 0, \qquad 0 < t \lesssim (1 + |\ln R|)^{-1} .$$

b) Due to theorem 2, it suffices to consider only the case of large data. Furthermore, the result follows if we show that after an $O(1)$ time any solution for the uncontrolled problem reaches a fixed bounded set in $L^2(\Omega)$. Due to the parabolic regularizing effect, all solutions with L^2 initial data enter L^∞ instantly. Hence without any restriction in generality we can assume that the initial data is bounded. Then we can bound the solutions to the uncontrolled version of (1) from above with positive x independent solutions of the ode

$$w_t + g(w) = 0, \quad g(w) = \min_{x\in\Omega} f(x, w) \ .$$

Due to our assumption on f this blows up backwards in finite time, which implies that forward it reaches a bounded set in a fixed amount of time. Similarly we get negative bounds from below. □

4 Proof of Theorem 4

The small data case can be dealt with as in the proof of Theorem 3. It is the case of large data which is of interest here. The bound from above on $T(R)$ is provided by Theorem 2, so it remains to prove the bound from below. We use a similar strategy as in the proof of Theorem 3. Choose a ball $B(x_0, r) \subset \tilde{\omega}$ and for $R > 1$ take the initial data $y_0 = R$. Then we seek a pointwise bound from below for the solution y to (1) which is independent of the choice of the control u. The needed subsolution is provided by the following lemma:

Lemma 8. *There is a continuous subsolution $z : \Omega \times \mathbb{R}^+ \to \mathbb{R} \cup \{-\infty\}$ to (1) with the following properties:*
(i) $z(x, t) = -\infty$ for $(x, t) \in \omega \times \mathbb{R}^+$,
(ii) $z(x, 0) \leq R$ for $x \in \Omega$,
(iii) $z(x_0, T) > 0$ for $0 \leq T \ll \Psi(R)$.

Suppose the lemma is true. Then we can use the comparison principle to bound the solution y to (1) from below by z. Therefore part (iii) implies the desired bound from below on the controllability time.

Proof. We first produce a time independent negative subsolution ζ which satisfies the condition (i) while being constant outside the region where (3) holds. At the expense of slightly enlarging it we can assume that ω has a smooth boundary. Consider a smooth nonincreasing function $a : (0, \infty) \to \mathbb{R}$ so that $a(s) = 1$ for $s > \epsilon$ while $a(s) = e^{s^{-\alpha}}$ for small s. The parameter ϵ is chosen so that (3) holds in an ϵ neighborhood of ω. Now we define

$$\zeta(x) = \begin{cases} -\infty, & x \in \omega \\ -Aa(d(x, \partial\omega)), & \text{otherwise} \ . \end{cases}$$

One can use (3) to verify that for A, α sufficiently large ζ is a subsolution to (1).

Next we construct z by modifying ζ in $B(x_0, r)$. More precisely, let λ_1, w_1 be the first eigenvalue and the first positive eigenfunction of the Dirichlet Laplacian in $B(x_0, R)$ normalized so that $w_1(x_0) = 1$. Then set

$$z(x,t) = \begin{cases} \zeta(x) & \text{outside } B(x_0, r) \\ -A + (R + A)e^{-\lambda_1} w_1(x) & \text{in } B(x_0, r) \ . \end{cases}$$

It is easy to see that z is also a subsolution to (1). Furthermore,

$$z(x_0, t) > 0, \qquad t \lesssim 1 + \ln R$$

which gives (iii). $\square$

5 Further remarks

Here we discuss several ways in which one can generalize the setup of Theorems 2, 3, 4.

(i) (variable coefficients) The Laplacian can be replaced by any second order elliptic operator with Lipschitz continuous coefficients,

$$P(x, \partial) = -a^{ij}\partial_i\partial_j + b^i\partial_i + c$$

(ii) (boundary conditions) The Dirichlet boundary condition can be replaced by the Neuman boundary condition.

(iii) (well-posedness) The problem remain well-posed in L^2 if we only assume that

$$(f(x,y) - f(x,z))(y - z) \geq \beta(y - z)^2$$

for some real β.

(iv) (linear dissipation) In order to insure that the energy decays exponentially one should replace the condition $sf(s) \geq 0$ by

$$sf(s) \geq \gamma s^2, \qquad \gamma > -\lambda_1$$

where λ_1 is the first eigenvalue of P. When estimating λ_1 one should keep in mind that, because of the control, the coefficients of the lower order terms can be arbitrarily modified within ω.

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