GEOMETRIC REDUCTIVE DUAL PAIRS AND A MOD p THETA CORRESPONDENCE

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Abstract. This paper aims to recast the theory of theta correspondence in the geometric context of abelian schemes and nondegenerate line bundles, which replace the role of symplectic vector spaces, continuing our previous work. We start by formulating the notion of reductive dual pairs which fit well with our notion of Weil representations and metaplectic group functors constructed in the earlier work. When specialized to the case of base scheme $\mathrm{Spec}\,\overline{\mathbb{F}}_p$, our work provides an experimental framework for a mod p theta correspondence for p-adic reductive groups. In the case of type II pairs, two results are obtained on the structure of mod p Weil representations. First, the Weil representation turns out to have few quotients, which tells us that the naive analogue of the classical theta correspondence is hopeless and would demand a new approach. Second, a weak analogue of the unramified version of Howe's conjecture still holds.

1. Introduction

1.1. **Basic ingredients.** [1] In §1.1 of [Shi], we mentioned that the following are needed to formulate the classical theta correspondence. That paper studied (i), (ii) and (iii) in the context of abelian schemes and nondegenerate line bundles. Once (iii) is obtained from (i) and (ii) and realized on explicit models, it is often harmless to forget (i) and (ii).

(i) a p-adic Heisenberg group arising from a symplectic vector space $(V, \langle\cdot,\cdot\rangle)$ over $\mathbb{Q}_p$,
(ii) Stone Von Neumann theorem and Schur's lemma for representations of the Heisenberg group,
(iii) the Weil representations of the p-adic metaplectic group $\mathrm{Mp}(V, \langle\cdot,\cdot\rangle)$, and
(iv) reductive dual pairs in $\mathrm{Sp}(V, \langle\cdot,\cdot\rangle)$.

Our first goal is to give a definition of (iv), in a general setting, which goes well with our earlier construction of (i), (ii) and (iii). This will get us ready for exploring the "theta correspondence" in a broader context, in particular for representations of p-adic reductive groups on $\overline{\mathbb{F}}_p$-vector spaces.

In fact when formulating the theta correspondence, one also needs a lemma amounting to the classical fact that two elements in the metaplectic group commute if and only if their images in the symplectic group commute. In the study of type II correspondence (§§5.4-5.5) the lemma is easy to prove. Although we conjecture that the lemma is true in general, we have not been able to confirm it. (See §5.3 for detail.)

1.2. **Mod l and mod p theta correspondences.** There have been a few attempts (e.g. Minguez ([Min08b]) to extend the local theta correspondence for p-adic groups to $\overline{\mathbb{F}}_l$-vector spaces (rather than $\mathbb{C}$-vector spaces) when $l \neq p$. In fact, the basic objects like (i)-(iv) above all carry over from $\mathbb{C}$ to $\overline{\mathbb{F}}_l$ rather easily, basically by replacing $\mathbb{C}$ with $\overline{\mathbb{F}}_l$ everywhere. Nevertheless, finding the correct formulation of mod l theta correspondence may be a hard task. Minguez observed that already in the type II case, the naive analogue of Howe's conjecture (§5.1) on the bijectivity of the local theta correspondence does not hold (if l is not a "banal" prime). Still, one has a good starting point for investigating a suitable mod l theta correspondence. By contrast, perhaps there has not been much prospect for a mod p theta correspondence for p-adic groups as (i)-(iv) were missing to our knowledge (before [Shi]).

1.3. **Reductive dual pairs.** We are naturally led to build a new definition of reductive dual pairs (marked as (iv) above) which fits well into the geometric picture and coincides with the usual notion in the classical case ($S = \mathrm{Spec}\,\mathbb{C}$). Recall that the usual way to construct a reductive dual pair (of type I) is as follows: a

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Hermitian space $(W_1, \langle\cdot,\cdot\rangle_1)$ and a skew-Hermitian space $(W_2, \langle\cdot,\cdot\rangle_2)$, say over $\mathbb{Q}_p$, give rise to a symplectic space $(W_1 \otimes_{\mathbb{Q}_p} W_2, \langle\cdot,\cdot\rangle)$. Then the natural embedding

$$\mathrm{Aut}(W_1, \langle\cdot,\cdot\rangle_1) \times \mathrm{Aut}(W_2, \langle\cdot,\cdot\rangle_2) \hookrightarrow \mathrm{Sp}(W_1 \otimes W_2, \langle\cdot,\cdot\rangle)$$

realizes $(\mathrm{Aut}(W_1, \langle\cdot,\cdot\rangle_1), \mathrm{Aut}(W_2, \langle\cdot,\cdot\rangle_2))$ as a reductive dual pair in the symplectic group. Our idea is basically to consider (A_2, L_2) instead of $(W_2, \langle\cdot,\cdot\rangle_2)$, where A_2 is an abelian scheme and L_2 is a nondegenerate line bundle over A_2. Or more precisely, $(W_2, \langle\cdot,\cdot\rangle_2)$ is replaced with $(V_pA_2, \widehat{e}^{L_2})$, consisting of the rational p-adic Tate module of A and the L_2-Weil pairing. Once we make sense of $(A, L) := (W_1 \otimes A_2, \langle\cdot,\cdot\rangle_1 \otimes L_2)$ as an abelian scheme with a line bundle, we will be able to think of $\mathrm{Aut}(W_1, \langle\cdot,\cdot\rangle_1)$ and $\mathrm{Aut}(V_pA_2, \widehat{e}_p^{L_2})$ as a reductive dual pair in $\mathrm{Sp}(V_pA, \widehat{e}_p^L)$.

From the representation theoretic viewpoint, it may not be always desirable that $\mathrm{Sp}(V_pA, \widehat{e}_p^L)$ (which is viewed as a sheaf on the base scheme S of A) varies when A moves in a family of abelian varieties, though this may be an interesting phenomenon (especially in mixed characteristic). Thus we introduce a level structure which is an isomorphism from V_pA onto a fixed object with compatible symplectic structure. In particular, it induces an isomorphism from $\mathrm{Sp}(V_pA, \widehat{e}_p^L)$ onto a fixed symplectic group. When A is over $\mathbb{C}$, a level structure provides a way to recover the classical notion of (i)-(iv) from our construction. When A is over $\overline{\mathbb{F}}_p$, a level structure is an isomorphism from V_pA onto $V_p\Sigma$ for a fixed p-divisible group Σ. The group $\mathrm{Sp}(V_pA, \widehat{e}_p^L)$ is now different from the usual symplectic group. (For instance, if A is ordinary then $\mathrm{Sp}(V_pA, \widehat{e}_p^L)$ is isomorphic to a general linear group.)

In fact level structures alone are usually insufficient. To consider all possible reductive dual pairs, we need to also consider an endomorphism structure for A as well. In other words, the formalism of reductive dual pairs naturally requires a PEL structure[2] on A, modulo the fact that we prefer a nondegenerate line bundle to a polarization. Often the geometric literature restricts attention to ample line bundles, but it is worth pointing out that we do need to treat nondegenerate line bundles in order to deal with all reductive dual pairs. The reason is that L may not be ample even if L_2 is ample, in the above notation. As an aside, we recall that Howe asked in [How79, Rem §5.(c)] whether the reductive dual pairs have something do to with the data defining PEL Shimura varieties. Our paper provides a partial answer to his question by pulling them close to each other.

This completes a geometric construction of the basic objects which are needed to consider the theta correspondence, which recovers the classical objects if we work over $\mathrm{Spec}\,\mathbb{C}$. As a consequence, we broaden the scope of the theta correspondence. When the base scheme is $\mathrm{Spec}\,\overline{\mathbb{F}}_p$, this gets us ready for exploring a mod p theta correspondence. (See §1.4 below.)

1.4. **A mod p theta correspondence for p-adic groups.** There would be two approaches to the theta correspondence for $\overline{\mathbb{F}}_p$-representations of p-adic groups. The first way is to realize the classical theta correspondence on $\overline{\mathbb{Q}}_p$-representations (whenever possible), prove that certain $\overline{\mathbb{Z}}_p$-structure is preserved, and then take modulo p. However, this may not be the most natural approach. In the context of the p-adic Langlands program, it is more natural to consider the mod p of a unitary Banach representation than the mod p of a classical admissible representation of a p-adic group. Unfortunately we do not have a p-adic Banach version of the Weil representation on a p-adic vector space, and this approach cannot be taken.

The second approach is based on the mod p Weil representation constructed in this article, which is natural from the geometric viewpoint. Although we have the notion of reductive dual pairs, the difficulty here is that it is not even clear how to formulate useful and plausible conjectures. The mod p Weil representations have few quotients, and the naive analogue of Howe's conjecture breaks down completely. Nevertheless, we also prove a positive result, namely that a weak analogue of the unramified theta correspondence can be shown for certain type II pairs. See §§5.4-5.5 for more detail.

It is worth recalling from [Shi] that the mod p versions of the Heisenberg group and the Heisenberg/Weil representation can be constructed from a p-divisible group with a symplectic pairing, without any use of (A, L). When the p-divisible group is ordinary, the Schrodinger model exists and provides a shortcut to the Weil representation and the theta correspondence (Example 4.2), with few prerequisites. We use the ordinary Schrodinger model in §5.

[2]P=Polarization, E=Endomorphism, L=Level structure

1.5. **Further developments and speculations.** The geometric interpretation of the reductive dual pairs may be of some use in the classical theta correspondence. We mentioned some idea in [Shi, §1.5]. As for a conjectural mod p theta correspondence for p-adic groups, it is imperative to come up with a sensible statement replacing Howe's conjecture. To do so, a first step would be to collect many computational examples. We restricted ourselves to type II pairs and ordinary p-divisible groups in §5 but the story would be even more exciting for type I pairs and supersingular p-divisible groups. Eventually the proposed mod p correspondence for p-adic groups should be compatible with the global theta correspondence for mod p automorphic forms (whenever it becomes available) and interact reasonably with the mod p Langlands program. Still not many tools are available in the mod p representations theory of p-adic groups. It would be nice if our work eventually adds another weapon in constructing and classifying representations.

1.6. **Organization.** The organization of the paper is simple. Section 2 recollects basic materials on Hermitian and skew-Hermitian pairings (for which [MVW87] is a comprehensive reference) and adapt them to rational Tate modules. In §3 and §4 we define geometric analogues of classical reductive dual pairs of type I and II and give examples. The last section 5 reviews Howe's conjecture in two versions, tests the validity of their analogues in the setting of type II mod p theta correspondence, and ends with a brief speculation.

1.7. **Acknowledgments.** As this paper is a natural continuation of [Shi], I express my gratitude again to all the people in the acknowledgments of that paper. Part of this work was completed during my stay at the Institute for Advanced Study. I thank for its generous support.

1.8. **Notation and Convention.** When S is a scheme, denote by (Sch/S) the category of S-schemes. Most of the time we work with schemes and ind schemes over S (often with group structure), in which case $\times$ always means a fiber product over S. Underlined notation such as $\underline{\mathrm{Hom}}$, $\underline{\mathrm{Aut}}$ and $\underline{\mathrm{End}}$ is used to denote a sheaf or a functor whereas Hom, Aut and End denote the corresponding set, group and a ring. When X is an abelian variety or a p-divisible group, we write $\mathrm{End}^0(X)$ for $\mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ (equivalently the endomorphism ring in the category in which isogenies are inverted).

The same notation as in the previous paper [Shi] will also be used here. For the reader's convenience, we recall some of the notation with reference points *in that paper*.

- A is an abelian scheme over S, and L is a nondegenerate line bundle.
- $\lambda_L : A \to A^\vee$ is the morphism given by $x \mapsto T_x^* L \otimes L^{-1}$ on points. (§2)
- TA (resp. VA) are (resp. ind-) group scheme versions of the (resp. rational) Tate module for A. (§3.1)
- $\widehat{e}^{L,\mathrm{Weil}} : VA \times VA \to V\mathbb{G}_m$ is the L-Weil pairing. (§3.4)
- $\widehat{e}^{L} : VA \times VA \to \mathbb{G}_m$ is the commutator pairing of the Heisenberg group, satisfying $\flat \circ \widehat{e}^{L,\mathrm{Weil}} = \widehat{e}^{L,\mathrm{Weil}}$ via the natural map $\flat : V\mathbb{G}_m \to \mathbb{G}_m$. (§3.4)
- $\underline{\mathrm{Mp}}(VA, \widehat{e}^L)$ and $\underline{\mathrm{Sp}}(VA, \widehat{e}^L)$ are metaplectic and symplectic groups, defined as group functors on (Sch/S). (§5.1)

2. Hermitian and skew-Hermitian pairings

This section contains preliminaries to be used later as a reference. It is recommended that the reader skip §2 and come back as needed. Throughout this section,

- F_0 is a field,
- D is a finite dimensional division algebra over F_0, so that $E := Z(D) \supset F_0$,
- $*$ is an involution on D acting as the identity on F_0,
- $F := E^{*=1}$, so that $F_0 \subset F \subset E$ and $1 \le [E : F] \le 2$.

If $E = F$ (resp. $E \supsetneq F$) then $*$ is an involution of the **first** (resp. **second**) kind.

2.1. Basic definitions and properties.

Definition 2.1. Let $\epsilon \in \{\pm 1\}$. Let V (resp. W) be a left (resp. right) D-module. We say that a nondegenerate bilinear map $\langle \cdot, \cdot \rangle : V \times V \to D$ a **left** $(D, *)$-linear ϵ-**Hermitian** pairing if

$$\langle v', v \rangle = \epsilon \cdot \langle v, v' \rangle^* \qquad \langle dv, d'v' \rangle = d \langle v, v' \rangle (d')^*, \qquad d, d' \in D,\ v, v' \in D. \tag{2.1}$$

Similarly, a nondegenerate bilinear map $\langle\cdot,\cdot\rangle : W\times W\to D$ is a **right** $(D,*)$-linear ϵ-**Hermitian** pairing if

$$\langle w',w\rangle=\epsilon\cdot\langle w,w'\rangle^*,\qquad \langle dw,d'w'\rangle=d^*\langle w,w'\rangle d',\qquad d,d'\in D,\ w,w'\in D.$$

Often we say Hermitian (resp. skew-Hermitian) in place of ϵ-Hermitian if $\epsilon=1$ (resp. $\epsilon=-1$).

With the above notation, define an F-group $\mathrm{Aut}(V,\langle\cdot,\cdot\rangle)$ by the rule

$$\mathrm{Aut}(V,\langle\cdot,\cdot\rangle)(R)=\{g\in\mathrm{End}_R(V\otimes_F R)|\langle gv,gv'\rangle=\langle v,v'\rangle,\ \forall v,v'\in V\}$$

for each F-algebra R. An F-group $\mathrm{Aut}(W,\langle\cdot,\cdot\rangle)$ is defined similarly. When the context makes it clear whether $(W,\langle\cdot,\cdot\rangle)$ is unitary, symplectic or orthogonal, we will often write U, Sp, O in place of Aut.

Definition 2.2. Let ϵ, V and W be as above. A nondegenerate F_0-bilinear map $(\cdot,\cdot):V\times V\to F_0$ is a **left** $(D,*)$-ϵ-**Hermitian** F_0-valued pairing on V if

$$(v',v)=\epsilon\cdot(v,v')^*\qquad (dv,v')=(v,d^*v'),\qquad d\in D,\ v,v'\in D. \tag{2.2}$$

A **right** $(D,*)$-ϵ-**Hermitian** F_0-valued pairing on W is defined similarly. When F_0 is clear from the context, we will omit the reference to F_0. (Note that different brackets are used in (2.1) and (2.2).)

Define an F_0-group

$$\mathrm{Aut}(V,(\cdot,\cdot))(R)=\{g\in\mathrm{End}_R(V\otimes_{F_0}R)|(gv,gv')=(v,v'),\ \forall v,v'\in V\}$$

for F_0-algebras R. Set $\mathrm{tr}_{D/F_0}:=\mathrm{tr}_{E/F_0}\circ\mathrm{tr}_{D/E}$ where $\mathrm{tr}_{D/E}$ is the reduced trace. Let Res_{F/F_0} denote the Weil restriction of scalars. The following lemma is obvious.

Lemma 2.3. *If $\langle\cdot,\cdot\rangle:V\times V\to D$ (resp. $\langle\cdot,\cdot\rangle:W\times W\to D$) is a left (resp. right) $(D,*)$-linear ϵ-Hermitian pairing then $\mathrm{tr}_{D/F_0}\circ\langle\cdot,\cdot\rangle$ is a left (resp. right) $(D,*)$-ϵ-Hermitian F_0-valued pairing. Further, there is a canonical isomorphism*

$$\mathrm{Res}_{F/F_0}(\mathrm{Aut}(V,\langle\cdot,\cdot\rangle))\simeq\mathrm{Aut}(V,\mathrm{tr}_{D/F_0}\circ\langle\cdot,\cdot\rangle).$$

Remark 2.4. In fact the lemma is true if tr_{D/F_0} is replaced by any nondegenerate F_0-linear pairing t_{D/F_0} : $D\times D\to F_0$.

Lemma 2.5. *There is a natural bijection between the two sets consisting of*

(i) left $(D,*)$-linear ϵ-Hermitian pairings $\langle\cdot,\cdot\rangle$ on V and
(ii) left $(D,*)$-ϵ-Hermitian F_0-valued pairings $(\cdot,\cdot)$ on V, respectively,

induced by $\langle\cdot,\cdot\rangle\mapsto\mathrm{tr}_{D/F_0}\circ\langle\cdot,\cdot\rangle$. Moreover the same map induces a bijection on the sets of isomorphism classes of (i) and (ii).

Proof. The first bijectivity is deduced from the following claim: for any given $(\cdot,\cdot)$, there exists a unique $\langle\cdot,\cdot\rangle$ such that

$$(v,v')=\mathrm{tr}_{D/F_0}(\langle v,v'\rangle),\qquad v,v'\in V.$$

To prove the claim, for each pair $v,v'\in V$ define $\langle v,v'\rangle$ to be the unique element $\delta\in D$ such that $\mathrm{tr}_{D/F_0}(d\delta)=(dv,v')$ for all $d\in D$. (Such a δ uniquely exists since the trace pairing is nondegenerate.) Then it is a routine check that $\langle\cdot,\cdot\rangle$ is $(D,*)$-linear and ϵ-Hermitian. To verify the uniqueness of $\langle\cdot,\cdot\rangle$, it is enough to note that $\mathrm{tr}_{D/F_0}\circ\langle\cdot,\cdot\rangle=\mathrm{tr}_{D/F_0}\circ\langle\cdot,\cdot\rangle'$ implies $\langle\cdot,\cdot\rangle=\langle\cdot,\cdot\rangle'$ as the trace pairing is nondegenerate.

Let us show the bijectivity on the level of isomorphism classes. Let $\gamma\in\mathrm{Aut}_D(V)$. What need to be proved is the equivalence that

$$\mathrm{tr}_{D/F_0}\langle\gamma v,\gamma v'\rangle=\mathrm{tr}_{D/F_0}\langle v,v'\rangle,\ \forall v,v'\in V\qquad\Leftrightarrow\qquad\langle\gamma v,\gamma v'\rangle=\langle v,v'\rangle\ \forall v,v'\in V.$$

The implication $\Leftarrow$ is obvious. To see $\Rightarrow$, substitute dv for v to obtain

$$\mathrm{tr}_{D/F_0}(d\langle\gamma v,\gamma v'\rangle)=\mathrm{tr}_{D/F_0}(d\langle v,v'\rangle),\quad\forall v,v'\in V,\ \forall d\in D.$$

The latter implies $\langle\gamma v,\gamma v'\rangle=\langle v,v'\rangle$ as the trace pairing is nondegenerate. $\square$

So far F_0 has been any field. Let us put ourselves in the adelic situation in the case of $F_0=\mathbb{Q}$.

Lemma 2.6. *There is a bijection between the two sets consisting of*

(i) left $(D\otimes_{\mathbb{Q}}\mathbb{A}^\infty,*)$-linear ϵ-Hermitian pairings $\langle\cdot,\cdot\rangle$ on $V\otimes_{\mathbb{Q}}\mathbb{A}^\infty$ and

(ii) left $(D \otimes_{\mathbb{Q}} \mathbb{A}^\infty, *)$-$\epsilon$-Hermitian $\mathbb{A}^\infty$-valued pairings $(\cdot,\cdot)$ on $V \otimes_{\mathbb{Q}} \mathbb{A}^\infty$, respectively,

induced by $\langle\cdot,\cdot\rangle \mapsto \operatorname{tr}_{D/\mathbb{Q}} \circ \langle\cdot,\cdot\rangle$. The same map induces a bijection on the sets of isomorphism classes.

Proof. The method of proof is the same as for Lemma 2.5. ☐

2.2. **Hermitian pairings on VA.** Let $F_0 = \mathbb{Q}$ and $(D,*)$ be as before. Let A be an abelian scheme over S equipped with a D-action via a $\mathbb{Q}$-algebra morphism $i : D \hookrightarrow \operatorname{End}^0_S(A)$. Let $D \otimes_{\mathbb{Q}} V\mathbb{G}_m$ denote the ind-group scheme over S representing the group functor $T \mapsto D \otimes_{\mathbb{Q}} V\mathbb{G}_m(T)$ on (Sch/S). To show that it is representable, choose a $\mathbb{Q}$-basis $\{e_i\}_{i\in I}$ of D and write $e_j e_k = \sum_{i\in I} a_{ijk} e_i$ with $a_{ijk} \in \mathbb{Q}$. Then $D \otimes_{\mathbb{Q}} V\mathbb{G}_m$ is isomorphic as a group functor to the ind-scheme $\prod_{i\in I} V\mathbb{G}_m$ equipped with group law

$$\prod_{j\in I} V\mathbb{G}_m(T) \times \prod_{k\in I} V\mathbb{G}_m(T) \mapsto \prod_{i\in I} V\mathbb{G}_m(T)$$

$$((\gamma_j)_{j\in I}, (\gamma_k)_{k\in I}) \mapsto \left(\sum_{j,k} (\gamma_j\gamma_k)^{a_{ijk}}\right)_{i\in I}.$$

Note that $V\mathbb{G}_m(T)$ is a $\mathbb{Q}$-module (even an $\mathbb{A}^\infty$-module), thus $(\gamma_j\gamma_k)^{a_{ijk}}$ makes sense. There is an obvious trace map $\operatorname{tr}_{D/\mathbb{Q}} : D \otimes_{\mathbb{Q}} V\mathbb{G}_m \mapsto V\mathbb{G}_m$ given by

$$(D \otimes_{\mathbb{Q}} V\mathbb{G}_m)(T) = D \otimes_{\mathbb{Q}} V\mathbb{G}_m(T) \overset{\operatorname{tr}_{D/\mathbb{Q}}\otimes \mathrm{id}}{\to} \mathbb{Q} \otimes_{\mathbb{Q}} V\mathbb{G}_m(T) = V\mathbb{G}_m(T).$$

Let

$$\widetilde{e} : VA \times VA \to D \otimes_{\mathbb{Q}} V\mathbb{G}_m \quad \text{and} \quad e : VA \times VA \to V\mathbb{G}_m$$

be morphisms of S-group schemes which are bilinear in both arguments and nondegenerate in the obvious sense. We say that $\widetilde{e}$ is a $(D,*)$-linear ϵ-Hermitian pairing if (2.1) holds on scheme-valued points of VA (with $\langle\cdot,\cdot\rangle$ replaced by $\widetilde{e}$). Similarly e is said to be a $(D,*)$-ϵ-Hermitian pairing if (2.2) holds for e. (In our convention the D-action on an abelian scheme is always a left action, so the word "left" will be omitted.)

Lemma 2.7. *There is a bijection between the two sets consisting of isomorphism classes of*

(i) $(D,*)$-linear ϵ-Hermitian pairings $\widetilde{e} : VA \times VA \to D \otimes_{\mathbb{Q}} V\mathbb{G}_m$ and

(ii) $(D,*)$-ϵ-Hermitian pairings $e : VA \times VA \to V\mathbb{G}_m$, respectively,

induced by $\widetilde{e} \mapsto \operatorname{tr}_{D/\mathbb{Q}} \circ \widetilde{e}$.

Proof. The proof is omitted as it is an adaptation of the proof of Lemma 2.5 to the scheme-theoretic setting and the essential idea is unchanged. ☐

2.3. **Hermitian pairings on V_pA.** Now consider the p-adic case where $F_0 = \mathbb{Q}_p$. Thus D is a finite dimensional division $\mathbb{Q}_p$-algebra. We define $D \otimes_{\mathbb{Q}_p} V_p\mathbb{G}_m$ and the trace map $\operatorname{tr}_{D/\mathbb{Q}_p} : D \otimes_{\mathbb{Q}_p} V_p\mathbb{G}_m \to V_p\mathbb{G}_m$ analogously as before.

Lemma 2.8. *There is a bijection between the two sets consisting of isomorphism classes of*

(i) $(D,*)$-linear ϵ-Hermitian pairings $\widetilde{e} : V_pA \times V_pA \to D \otimes_{\mathbb{Q}_p} V_p\mathbb{G}_m$ and

(ii) $(D,*)$-ϵ-Hermitian pairings $e : V_pA \times V_pA \to V_p\mathbb{G}_m$, respectively,

induced by $\widetilde{e} \mapsto \operatorname{tr}_{D/\mathbb{Q}_p} \circ \widetilde{e}$.

Proof. Essentially the same as the proof of Lemma 2.7. ☐

3. Reductive dual pairs of Type I

To achieve our goal of formulating a candidate theta correspondence, we need to bring classical reductive dual pairs into our context. This will take up sections 3 and 4. Rather than pursuing an abstract definition, we give an explicit construction of reductive dual pairs. When the base S is $\operatorname{Spec}\mathbb{C}$ we essentially recover the classical dual pairs. In sections 3 and 4 we always assume that W_1, W_2 are nonzero D-modules and that A_2 has positive dimension unless specified otherwise.

3.1. **Classical reductive dual pairs of type I.** As the p-adic case is similar, we only recall the number field case. Let

- F_0 be a finite extension field of $\mathbb{Q}$,
- D, $*$, E, F be as at the start of §2.
- W be a finite dimensional F_0-vector space of even dimension,
- $\langle\cdot,\cdot\rangle : W \times W \to F_0$ be a non-degenerate alternating F_0-linear pairing.

Recall ([How79, §5], [MVW87, 1.19-1.20]) that a type I reductive dual pair arises from the following data.

- a right D-module W_1 and a left D-module W_2,
- a right $(D,*)$-linear Hermitian pairing $\langle\cdot,\cdot\rangle_1 : W_1 \times W_1 \to D$,
- a left $(D,*)$-linear skew-Hermitian pairing $\langle\cdot,\cdot\rangle_2 : W_2 \times W_2 \to D$

such that there is an F_0-vector space isomorphism $W_1 \otimes_D W_2 \simeq W$ under which we have

$$\langle w_1 \otimes w_2, w_1' \otimes w_2'\rangle = \mathrm{tr}_{D/F_0}(\langle w_1, w_1'\rangle_1^* \langle w_2, w_2'\rangle_2) \tag{3.1}$$

for all $w_1, w_1' \in W_1$ and $w_2, w_2' \in W_2$. For $i = 1, 2$, define $G_i := \mathrm{Aut}_D(W_i, \langle\cdot,\cdot\rangle_i)$. The groups G_1 and G_2 form a reductive dual pair in $\mathrm{Sp}_{F_0}(W, \langle\cdot,\cdot,\rangle)$. If $*$ is of the first kind, G_1 is an orthogonal group and G_2 a symplectic group. Otherwise G_1 and G_2 are unitary groups.

3.2. **Setup for a geometric analogue.** Consider the following data.

- S is a locally noetherian scheme,
- a finite dimensional division algebra D with involution $*$ whose center is finite over $\mathbb{Q}$,
- F_0 is a field contained in $D^{*=1}$,
- a pair $(W_1, \langle\cdot,\cdot\rangle_1)$ where
  - W_1 is a finite right D-module,
  - $\langle\cdot,\cdot\rangle_1 : W_1 \times W_1 \to D$ is a $(D,*)$-linear Hermitian pairing and
- a triple (A_2, L_2, ι_2) where
  - A_2 is an abelian variety over S,
  - L_2 is a nondegenerate symmetric line bundle over A_2 of index $i(L_2)$,
  - $\iota_2 : D \hookrightarrow \mathrm{End}^0(A_2)$ such that $\ddagger_{\lambda_{L_2}}(\iota_2(d)) = \iota_2(d^*)$ for all $d \in D$, where $\ddagger_{\lambda_{L_2}}$ is the map $\phi \mapsto \lambda_L^{-1}\widehat{\phi}\lambda_L$.

The aim of §3.2 is to construct (A, L) with respect to which $(W_1, \langle\cdot,\cdot\rangle_1)$ and (A_2, L_2, ι_2) define a reductive dual pair in a suitable sense, in analogy with §3.1.

Remark 3.1. Note that the triple (A_2, L_2, ι_2) is almost the datum in a moduli problem for abelian varieties (cf. [Mum67, §9]). However we assume neither that $*$ is positive nor that L_2 is ample.

Lemma 3.2. $\widehat{e}^{L_2,\mathrm{Weil}} : VA_2 \times VA_2 \to V\mathbb{G}_m$ *is a* $(D \otimes_{\mathbb{Q}} \mathbb{A}^\infty, *)$*-skew-Hermitian pairing.*

Proof. It is a standard fact that $\widehat{e}^{L_2,\mathrm{Weil}}$ is $\mathbb{A}^\infty$-linear and alternating. So it is enough to verify that

$$\widehat{e}^{L_2,\mathrm{Weil}}(\iota_2(d)v, v') = \widehat{e}^{L_2,\mathrm{Weil}}(v, \iota_2(d^*)v'), \quad \forall d \in D,\ v, v' \in VA_2(T).$$

Let $e : VA \times VA^\vee \to V\mathbb{G}_m$ denote the canonical pairing. Then $\widehat{e}^{L_2,\mathrm{Weil}}(v, v') = e(v, \lambda_{L_2}v')$. Thus

$$\widehat{e}^{L_2,\mathrm{Weil}}(v, \iota_2(d^*)v') = e(v, \lambda_{L_2}(\lambda_{L_2}^{-1}\iota_2(d)^\vee\lambda_{L_2})v') = e(v, \iota_2(d)^\vee\lambda_{L_2}v') = e(\iota_2(d)v, \lambda_{L_2}v') = \widehat{e}^{L_2,\mathrm{Weil}}(\iota_2(d)v, v').$$

□

Corollary 3.3. *There exists a* $(D \otimes_{\mathbb{Q}} \mathbb{A}^\infty, *)$*-linear skew-Hermitian pairing*

$$\widetilde{e}^{L_2,\mathrm{Weil}} : VA_2 \times VA_2 \to D \otimes_{\mathbb{Q}} V\mathbb{G}_m$$

such that $\widehat{e}^{L_2,\mathrm{Weil}} = \mathrm{tr}_{D/\mathbb{Q}} \circ \widetilde{e}^{L_2,\mathrm{Weil}}$.

Proof. Immediate from the last lemma and Lemma 2.7. □

In order to proceed we make the following hypothesis on $(W_1, \langle\cdot,\cdot\rangle_1)$ from now on, which seems to be harmless in practice. (For instance the case of standard orthogonal pairing or hermitian pairing of signature (p, q) can be captured, and L has a simple form in this case as in (3.3) below.)

Hypothesis 3.4. There are a D-basis $\{\epsilon_i\}_{i\in I}$ of W_1 and $\{\alpha_{i,j}\in\mathbb{Q}|i,j\in I\}$, $\{\beta_{i,j}\in D|i,j\in I\}$ such that $W_1=\oplus_{i\in I}\epsilon_i\cdot D$ and $\langle\epsilon_i,\epsilon_j\rangle_1=\alpha_{i,j}\beta^*_{i,j}\beta_{i,j}$.

Since $\langle\epsilon_i,\epsilon_j\rangle_1=\langle\epsilon_j,\epsilon_i\rangle_1^*$, we may and will arrange that $\alpha_{i,j}=\alpha_{j,i}$ and $\beta_{i,j}=\beta_{j,i}$. By scaling $\{\epsilon_i\}_{i\in I}$ we may and will assume that $\alpha_{i,j}\in\mathbb{Z}$ and $\beta_{i,j}\in\mathcal{O}_D$.

We are ready to construct (A,L). Set

$$\begin{aligned}
A &:= \prod_{i\in I}A_2\\
L &:= \bigotimes_{i\in I}\left(p_i^*\iota_2(\beta_{i,i})^*L_2^{\otimes\alpha_{i,i}}\otimes\bigotimes_{\substack{j\in I\\ j\neq i}}p_i^*\iota_2(\beta_{i,j})^*L_2^{-\otimes\alpha_{i,j}}\right)\otimes\bigotimes_{\substack{i,j\in I\\ i<j}}(p_i,p_j)^*m^*\iota_2(\beta_{i,j})^*L_2^{\otimes\alpha_{i,j}} \qquad (3.2)\\
\iota &: F_0\hookrightarrow \mathrm{End}^0(A)\quad\text{induced by }\iota_2
\end{aligned}$$

Example 3.5. When $\beta_{i,j}=1$ for all $i,j\in I$ (which can be always achieved when $D=\mathbb{Q}$ for instance), (3.2) simplifies as

$$L := \bigotimes_{i\in I}\left(p_i^*L_2^{\otimes(\alpha_{i,i}-\sum_{j\neq i}\alpha_{i,j})}\right)\otimes\bigotimes_{\substack{i,j\in I\\ i<j}}(p_i,p_j)^*m^*L_2^{\otimes\alpha_{i,j}}.$$

Further, suppose that $\langle,\cdot,\cdot\rangle_1$ is a standard orthogonal or hermitian pairing in the following sense: there is a partition $I=I_1\coprod I_2$ such that $\langle\epsilon_i,\epsilon_j\rangle_1=0$ unless $i=j$, and $\langle\epsilon_i,\epsilon_i\rangle$ is equal to 1 (resp. -1) if $i\in I_1$ (resp. $i\in I_2$). Then one can take $\beta_{i,j}=1$ for all $i,j\in I$, $\alpha_{i,j}=0$ if $i\neq j$ and $\alpha_{i,i}=(-1)^{a-1}$ if $i\in I_a$. Then we simply have L is simply the exterior tensor product

$$L=(L_2)^{\boxtimes|I_1|}\boxtimes(L_2^{-1})^{\boxtimes|I_2|}. \qquad (3.3)$$

We define $\langle\cdot,\cdot\rangle_1^*\otimes\widetilde{e}^{L_2,\mathrm{Weil}}: VA\times VA=(\prod_{i\in I}VA_2)\times(\prod_{j\in I}VA_2)\to D\otimes V\mathbb{G}_m$ by

$$((v_i)_{i\in I},(v'_j)_{j\in I})\mapsto\prod_{i,j\in I}\langle\epsilon_i,\epsilon_j\rangle_1^*\cdot\widetilde{e}^{L_2,\mathrm{Weil}}(v_i,v'_j)$$

where $v_i,v'_i\in VA_2(T)$ for each S-scheme T. In fact there is no need for $*$ in the above formula as $\langle\epsilon_i,\epsilon_j\rangle_1^*=\langle\epsilon_i,\epsilon_j\rangle_1$. Nevertheless we keep $*$ in order to remember the correct D-linear action. (cf. (3.1))

Lemma 3.6. $\widehat{e}^{L,\mathrm{Weil}}=\mathrm{tr}_{D/\mathbb{Q}}(\langle\cdot,\cdot\rangle_1\otimes\widetilde{e}^{L_2,\mathrm{Weil}})$.

Proof. Let $v=(v_i)_{i\in I}$, $v'=(v'_j)_{j\in I}$ be in $VA(T)$. Then

$$\begin{aligned}
\widehat{e}^{L,\mathrm{Weil}}(v,v') &= \prod_{i\in I}\left(\widehat{e}^{L_2,\mathrm{Weil}}(\iota_2(\beta_{i,i})(v_i),\iota_2(\beta_{i,i})(v'_i))^{\alpha_{i,i}}\prod_{j\neq i}\widehat{e}^{L_2,\mathrm{Weil}}(\iota_2(\beta_{i,j})(v_i),\iota_2(\beta_{i,j})(v'_i))^{-\alpha_{i,j}}\right)\\
&\quad\times\prod_{i<j}\widehat{e}^{L_2,\mathrm{Weil}}(\iota_2(\beta_{i,j})(v_i+v_j),\iota_2(\beta_{i,j})(v'_i+v'_j))^{\alpha_{i,j}}\\
&= \prod_{i,j\in I}\widehat{e}^{L_2,\mathrm{Weil}}(\iota_2(\beta_{i,j})(v_i),\iota_2(\beta_{i,j})(v'_j))^{\alpha_{i,j}}\\
&= \prod_{i,j\in I}\mathrm{tr}_{D/\mathbb{Q}}\left(\widetilde{e}^{L_2,\mathrm{Weil}}(\iota_2(\beta^*_{i,j}\beta_{i,j})(v_i),(v'_j))^{\alpha_{i,j}}\right)\\
&= \prod_{i,j\in I}\mathrm{tr}_{D/\mathbb{Q}}\left((\alpha_{i,j}\beta^*_{i,j}\beta_{i,j})\cdot\widetilde{e}^{L_2,\mathrm{Weil}}(v_i,v'_j)\right)\\
&= \mathrm{tr}_{D/\mathbb{Q}}\left(\prod_{i,j\in I}\left(\langle\epsilon_i,\epsilon_j\rangle_1\cdot\widetilde{e}^{L_2,\mathrm{Weil}}(v_i,v'_j)\right)\right)
\end{aligned}$$

□

Remark 3.7. The definition of L certainly depends on the choice of basis $\{e_i\}_{i\in I}$, and possibly $\alpha_{i,j}$ and $\beta_{i,j}$ as well. The isogeny class of (A, L, ι) should be independent of such choices but we do not check it here.

Remark 3.8. It would have been more natural to have defined A as the S-group scheme representing $T \mapsto \mathcal{O}_D \otimes_{\mathbb{Z}} A_2(T)$. We did not take this approach because we have not found a natural definition of L to go with such a definition of A.

Lemma 3.9. *L is symmetric and nondegenerate.*

Proof. We have $(-1)^*L \simeq L$ because the multiplication by -1 commutes with m and p_i's. To see L is nondegenerate, it suffices to show that L_s is nondegenerate for each $s \in S$, so the proof is reduced to the case where $S = \operatorname{Spec} k$. We may even assume that $\overline{k} = k$. It is enough to verify that $\widehat{e}_l^{L_s,\mathrm{Weil}}$ is a nondegenerate pairing for a prime l, which we choose to be prime to $\mathrm{char}(k)$. Note that $\widehat{e}_l^{L_s}$ is a symplectic form on the constant group scheme V_lA, which can be thought of as a symplectic $\mathbb{Q}_l$-vector space. Since $\langle\cdot,\cdot\rangle_1$ and $\widetilde{e}^{L_2,\mathrm{Weil}}$ are nondegenerate by the initial assumption, Lemma 3.6 implies that $\widehat{e}^{L,\mathrm{Weil}}$, in particular $\widehat{e}_l^{L,\mathrm{Weil}}$, is nondegenerate. □

Remark 3.10. Even if L_2 is ample, L may not be ample. To see this, suppose that $S = \operatorname{Spec}\mathbb{C}$, that $D = F_0 = \mathbb{Q}$, and that $(W_1, \langle\cdot,\cdot\rangle_1)$ is an orthogonal space. In the notation of [Mum74, §2], write $L_2 = L(H_2, \alpha_2)$ and $L = L(H, \alpha)$ for nondegenerate hermitian forms H_2 and H. They are related via $H = \langle\cdot,\cdot\rangle_1 \otimes H_2$. If L_2 is ample then H_2 is positive definite. However, unless $\langle\cdot,\cdot\rangle_1$ is positive definite (for instance unless I_2 is empty in Example 3.5), H is not positive definite and thus L is not ample. This remark explains one major reason why we did not restrict to ample line bundles in developing the theory of Heisenberg groups and representations in [Shi].

Keeping Remark 3.8 in mind, it is clear how to define an action of $\alpha \in \mathrm{End}_D(W_1)$ as a $\mathbb{Q}$-isogeny on A. For each $i \in I$, write $\alpha(\epsilon_i) = \sum_{j\in I} \epsilon_j \cdot d_{ji}$ for some $d_{ji} \in D$. Then α acts on $A = \prod_{i\in I} A_2$ by

$$(x_i)_{i\in I} \mapsto \left(\prod_{j\in I} \iota_2(d_{ji})(x_j))\right)_{i\in I}. \tag{3.4}$$

On the other hand, each $\beta \in \mathrm{End}(VA_2)$ acts on $VA = \prod_{i\in I} VA_2$ by diagonal action. This induces a map of ring-valued functors on (Sch/S)

$$\underline{\mathrm{End}}_D(W_1) \times \underline{\mathrm{End}}(VA_2) \to \underline{\mathrm{End}}(VA). \tag{3.5}$$

(As usual, $\underline{\mathrm{End}}_D(W_1)$ is viewed as the constant ring scheme over S associated with $\mathrm{End}_D(W_1)$.) By restricting to the automorphisms preserving pairings, we obtain the following lemma, whose proof is not difficult.

Lemma 3.11. *The map* (3.5) *induces a map of group functors on* (Sch/S)

$$\underline{\mathrm{Aut}}_D(W_1, \langle\cdot,\cdot\rangle_1) \times \underline{\mathrm{Aut}}(VA_2, \widetilde{e}^{L_2,\mathrm{Weil}}, \iota_2) \to \underline{\mathrm{Sp}}(VA, \widehat{e}^L, \iota) \hookrightarrow \underline{\mathrm{Sp}}(VA, \widehat{e}^L)$$

which is an injection on $\underline{\mathrm{Aut}}_D(W_1, \langle\cdot,\cdot\rangle_1) \times \{1\}$ *and* $\{1\} \times \underline{\mathrm{Aut}}(VA_2, \widehat{e}^{L_2}, \iota_2)$.

Proof. The injectivity is straightforward. In the rest of the proof we check that the image lands in $\underline{\mathrm{Sp}}(VA, \widehat{e}^L, \iota)$ (not just in $\underline{\mathrm{End}}(VA)$). Let $\beta \in \underline{\mathrm{Aut}}(VA_2, \widehat{e}^{L_2}, \iota_2)$, $v, v' \in VA$ and write $v = (v_i)_{i\in I}$ and $v' = (v'_j)_{j\in I}$. Then

$$\widehat{e}^{L,\mathrm{Weil}}(\beta(v), \beta(v')) = \mathrm{tr}_{D/\mathbb{Q}}\left(\prod_{i,j} \langle\epsilon_i, \epsilon_j\rangle_1 \cdot \widetilde{e}^{L_2,\mathrm{Weil}}((\beta(v_i))_{i\in I}, (\beta(v'_j))_{j\in I})\right)$$

$$= \mathrm{tr}_{D/\mathbb{Q}}\left(\prod_{i,j} \langle\epsilon_i, \epsilon_j\rangle_1 \cdot \widetilde{e}^{L_2,\mathrm{Weil}}((v_i)_{i\in I}, (v'_j)_{j\in I})\right) = \widehat{e}^{L,\mathrm{Weil}}(v, v').$$

Now consider $\alpha \in \mathrm{End}_D(W_1)$ and write $\alpha(\epsilon_i) = \sum_{j\in I} \epsilon_j \cdot d_{ji}$ as above. Let $k, l \in I$. It is enough to consider "basis elements" v, v' with $v_i = 0$ and $v'_j = 0$ for all $i \neq k$ and $j \neq l$. Then $\widehat{e}^{L,\mathrm{Weil}}(\alpha(v), \alpha(v'))$ equals

$$\mathrm{tr}_{D/\mathbb{Q}}\left(\prod_{i,j}\langle \epsilon_i, \epsilon_j\rangle_1^* \cdot \widetilde{e}^{L_2,\mathrm{Weil}}(d_{ik}v_k, d_{jl}v'_l)\right) = \mathrm{tr}_{D/\mathbb{Q}}\left(\prod_{i,j}\langle \epsilon_i, \epsilon_j\rangle_1^* d_{ik}\widetilde{e}^{L_2,\mathrm{Weil}}(v_k, v'_l)d_{jl}^*\right)$$

$$= \mathrm{tr}_{D/\mathbb{Q}}\left(\prod_{i,j}\langle \epsilon_i d_{ik}, \epsilon_j d_{jl}\rangle_1^* \cdot \widetilde{e}^{L_2,\mathrm{Weil}}(v_k, v'_l)\right) = \mathrm{tr}_{D/\mathbb{Q}}\left(\langle \alpha(\epsilon_k), \alpha(\epsilon_l)\rangle_1^* \cdot \widetilde{e}^{L_2,\mathrm{Weil}}(v_k, v'_l)\right)$$

$$= \mathrm{tr}_{D/\mathbb{Q}}\left(\langle \epsilon_k, \epsilon_l\rangle_1^* \cdot \widetilde{e}^{L_2,\mathrm{Weil}}(v_k, v'_l)\right) = \widehat{e}^{L,\mathrm{Weil}}(v, v').$$

□

3.3. **Level structure.** We wish to view Lemma 3.11 as presenting the analogue of a classical reductive dual pair. This is more than an analogy when there is a level structure, as treated in [Shi, §6] from which we import notation. To begin with, assume that S is a $\mathbb{Q}$-scheme. A nontrivial morphism of group schemes $\psi : \mathbb{A}^\infty \to \mathbb{G}_m$ over S (where $\mathbb{A}^\infty$ is a constant group scheme) plays the role of additive character. Let $\langle\cdot,\cdot\rangle_{2,\psi}$ denote the pairing $W_2 \otimes \mathbb{A}^\infty \times W_2 \otimes \mathbb{A}^\infty \to \mathbb{G}_m$ obtained from $\langle\cdot,\cdot\rangle_2$ and ψ in the obvious manner. Similarly $\langle\cdot,\cdot\rangle_\psi : V \otimes \mathbb{A}^\infty \times V \otimes \mathbb{A}^\infty \to \mathbb{G}_m$ is defined. Let

$$\alpha_2 : (W_2 \otimes \mathbb{A}^\infty)_S \simeq VA_2$$

be a $D \otimes_{\mathbb{Q}} \mathbb{A}^\infty$-linear isomorphism (of ind-group schemes over S) compatible with $(D \otimes_{\mathbb{Q}} \mathbb{A}^\infty, *)$-skew Hermitian pairings $\langle\cdot,\cdot\rangle_{2,\psi}$ and $\widehat{e}^{L_2}$. As we have $VA \simeq W_1 \otimes_D VA_2$ by construction, the isomorphism α_2 induces

$$\alpha : VA \simeq (W_1 \otimes_D W_2) \otimes_{\mathbb{Q}} \mathbb{A}^\infty$$

matching $\widehat{e}^L$ and $\langle\cdot,\cdot\rangle_\psi$. Then α_2 and α induce

$$\begin{array}{ccccc} \underline{\mathrm{Aut}}(VA_2, \widetilde{e}^{L_2,\mathrm{Weil}}, \iota_2) & \simeq & \underline{\mathrm{Aut}}_D(W_2 \otimes \mathbb{A}^\infty, \langle\cdot,\cdot\rangle_2)_S & = & G_2 \times_{F_0} S \\ \underline{\mathrm{Aut}}(VA, \widehat{e}^L, \iota) & \simeq & \mathrm{Sp}_{F_0}(W \otimes \mathbb{A}^\infty, \langle\cdot,\cdot\rangle)_S & = & G \times_{F_0} S. \end{array}$$

Using the above isomorphisms along with Lemma 3.11, one precisely recovers the classical reductive dual pair of §3.1. Now when S is an $\mathbb{F}_p$-scheme, the level structure in the second case of [Shi, §6.5] can be likewise adapted to produce a "dual pair", which is classical outside p but looks different at the p-components.

3.4. **The p-adic case.** Only in §3.4 (and in §4.3) we change the notation from §3.2. Let F_0 be a finite extension of $\mathbb{Q}_p$. Let D be a division $\mathbb{Q}_p$-algebra with involution $*$ such that $F_0 \subset D^{*=1}$ and $\dim_{\mathbb{Q}_p} D < \infty$. Let W_1 be a finite right D-module, and let $\langle\cdot,\cdot\rangle_1 : W_1 \times W_1 \to D$ be a $(D, *)$-Hermitian pairing. Let (A_2, L_2, ι_2) be as in §3.2, except that we take $\iota_2 : D \hookrightarrow \mathrm{End}^0(A_2) \otimes_{\mathbb{Q}} \mathbb{Q}_p$. The contents of the previous subsection carries over with obvious changes. For instance, the analogue of Lemma 3.11 produces a dual pair

$$\underline{\mathrm{Aut}}_D(W_1, \langle\cdot,\cdot\rangle_1) \times \underline{\mathrm{Aut}}(V_pA_2, \widetilde{e}^{L_2,\mathrm{Weil}}, \iota_2) \to \underline{\mathrm{Sp}}(V_pA, \widehat{e}^L_p, \iota) \hookrightarrow \underline{\mathrm{Sp}}(V_pA, \widehat{e}^L_p)$$

which is injective on $\underline{\mathrm{Aut}}_D(W_1, \langle\cdot,\cdot\rangle_1) \times \{1\}$ and $\{1\} \times \underline{\mathrm{Aut}}(V_pA_2, \widetilde{e}^{L_2,\mathrm{Weil}}, \iota_2)$.

When $S = \mathrm{Spec}\,\overline{\mathbb{F}}_p$ (and $p \neq 2$ for safety), we can formulate everything also only in terms of p-divisible groups using [Shi, §6.4]. As this is not part of the classical theory, we provide some examples in this unfamiliar territory. Let $D_{1/2}$ be a quaternion division algebra with center $\mathbb{Q}_p$ and $\Sigma_{1/2}$ a simple p-divisible group over $\overline{\mathbb{F}}_p$ with slope 1/2. It is well known that $\mathrm{End}^0(\Sigma_{1/2}) \simeq D_{1/2}$.

Example 3.12. Let $S = \mathrm{Spec}\,\overline{\mathbb{F}}_p$, $(W_1, \langle\cdot,\cdot\rangle_1)$ be an orthogonal $\mathbb{Q}_p$-vector space of dimension g_1 and $\Sigma_2 = (\Sigma_{1/2})^{g_2}$. Then the resulting dual pair may be identified with

$$O(W_1, \langle\cdot,\cdot\rangle_1) \times \mathrm{Sp}_{g_2}(D_{1/2}) \to \mathrm{Sp}_{g_1g_2}(D_{1/2}).$$

Example 3.13. Let $S = \operatorname{Spec} \overline{\mathbb{F}}_p$ and E be a quadratic extension of $\mathbb{Q}_p$ with $1 \neq * \in \mathrm{Gal}(E/\mathbb{Q}_p)$. Let $(W_1, \langle \cdot, \cdot \rangle_1)$ be a Hermitian E-vector space of dimension g_1. Let $\Sigma_2 = (\Sigma_{1/2})^{g_2}$ be equipped with $\iota_2 : E \hookrightarrow \mathrm{End}^0(\Sigma_2)$ and a $\mathbb{Q}_p$-linear nondegenerate pairing $\langle \cdot, \cdot \rangle_0 : \Sigma_2 \times \Sigma_2 \to \mu_{p^\infty}$ such that $\langle ex, y \rangle_2 = \langle x, e^* y \rangle_2$ for all $e \in E$ and $x, y \in \Sigma_2$. Let $\langle \cdot, \cdot \rangle$ be the pairing on $V_p\Sigma_2$ obtained from $\langle \cdot, \cdot \rangle_0$ as in [Shi, §6.3]. This produces a dual pair

$$U(W_1, \langle \cdot, \cdot \rangle_1) \times U(V_p\Sigma_2, \langle \cdot, \cdot \rangle, \iota_2) \to \mathrm{Sp}_{g_1 g_2}(D_{1/2}),$$

where $U(V_p\Sigma_2, \langle \cdot, \cdot \rangle, \iota_2)$ is an inner form of the quasi-split unitary group in g_2 variables over $\mathbb{Q}_p$.

4. Reductive dual pairs of type II

4.1. **Classical reductive dual pairs of type II.** We only recall the number field case as the p-adic case is completely analogous. Let

- F_0 be a finite extension field of $\mathbb{Q}$,
- W be a finite dimensional F_0-vector space of even dimension,
- $\langle \cdot, \cdot \rangle : W \times W \to F_0$ be a non-degenerate alternating F_0-linear pairing.

Recall ([How79, §5], [MVW87, 1.19-1.20]) that a type II dual reductive pair arises from the following data.

- a division algebra D whose center E contains F_0,
- a right D-module W_1 and a left D-module W_2,

such that $W_1 \otimes_D W_2$ embeds into W as a maximal isotropic subspace for $\langle \cdot, \cdot \rangle$. For $i = 1, 2$, define $G_i := GL_D(W_i)$. (Namely $G_i(R) = GL_{D \otimes_{F_0} R}(W_i \otimes_{F_0} R)$ for any F_0-algebra R.) The two groups G_1 and G_2 form a reductive dual pair of type II in $\mathrm{Sp}_{F_0}(W, \langle \cdot, \cdot, \rangle)$.

4.2. **Setup for a geometric analogue.** Consider the data at the start of §3.2. We further assume that

- $*$ is an involution of the first kind,
- $W_1 = \oplus_{i=1}^{g_1} \epsilon_i \cdot D$ is a finite right D-module,
- an orthogonal pairing $\langle \cdot, \cdot \rangle_1$ is chosen for W_1 such that Hypothesis 3.4 is satisfied,
- there is a complete polarization $VA_2 \simeq V_2' \times V_2''$ with respect to $\widehat{e}^{L_2}$ (so that V_2' and V_2'' are isotropic) such that V_2' and V_2'' are stable under the action of $\iota_2(D)$.

Construct (A, L, ι) exactly as in §3.2. Then we have $VA \simeq W_1 \otimes_D VA_2$, where $\widehat{e}^{L,\mathrm{Weil}}$ is matched with $\mathrm{tr}_{D/\mathbb{Q}}(\langle \cdot, \cdot \rangle_1 \otimes \widetilde{e}^{L_2,\mathrm{Weil}})$. As we explained in that subsection, there is a map (3.5). On the other hand, there is a map $\underline{\mathrm{Aut}}(V_2', \iota_2) \hookrightarrow \underline{\mathrm{Aut}}(V_2', \iota_2) \times \underline{\mathrm{Aut}}(V_2'', \iota_2)$ given by $\alpha \mapsto (\alpha, \alpha^\vee)$, where $\alpha^\vee$ comes from the duality between V_2' and V_2'' via $\widehat{e}^{L_2}$. Composing with $\underline{\mathrm{Aut}}(V_2', \iota_2) \times \underline{\mathrm{Aut}}(V_2'', \iota_2) \hookrightarrow \underline{\mathrm{Aut}}(VA_2, \iota_2)$, we obtain $\underline{\mathrm{Aut}}(V_2', \iota_2) \hookrightarrow \underline{\mathrm{Aut}}(VA_2, \iota_2)$. Moreover, the image lies in $\underline{\mathrm{Aut}}(VA_2, \widehat{e}^{L_2}, \iota_2)$. Together with (3.5), we have

$$\underline{\mathrm{Aut}}_D(W_1) \times \underline{\mathrm{Aut}}(V_2', \iota_2) \to \underline{\mathrm{Aut}}_D(W_1) \times \underline{\mathrm{Aut}}(VA_2, \widehat{e}^{L_2}, \iota_2) \to \underline{\mathrm{Aut}}(VA, \iota). \tag{4.1}$$

Lemma 4.1. *The map* (4.1) *induces a map of group functors on* (Sch/S)

$$\underline{\mathrm{Aut}}_D(W_1) \times \underline{\mathrm{Aut}}(V_2', \iota_2) \to \underline{\mathrm{Sp}}(VA, \widehat{e}^L, \iota) \hookrightarrow \underline{\mathrm{Sp}}(VA, \widehat{e}^L)$$

injective on $\underline{\mathrm{Aut}}_D(W_1) \times \{1\}$ *and* $\{1\} \times \underline{\mathrm{Aut}}(V_2', \iota_2)$.

Proof. The essentially same argument as in the proof of Lemma 3.11 works here. □

4.3. **The p-adic case.** Let F_0, F and D be as in §3.4. Just like we adapted §3.2 to §3.4, we can rework §4.2 in the p-adic case. When $S = \operatorname{Spec} \overline{\mathbb{F}}_p$ ($p \neq 2$), things can be reformulated in terms of p-divisible groups. We would like to elaborate on this point, as the special case of ordinary p-divisible groups will be investigated further in §§5.4-5.5. (Similarly the situation of type I pairs can be recast in terms of p-divisible groups.)

- $W_1 = \oplus_{i=1}^{g_1} \epsilon_i \cdot D$ is a finite right D-module,
- (Σ_2', ι_2') is a p-divisible group over $\overline{\mathbb{F}}_p$ with a $\mathbb{Q}_p$-algebra map $\iota_2' : D \hookrightarrow \mathrm{End}^0(\Sigma_2')$.
- $\Sigma' = (\Sigma_2')^{g_1}$, $\Sigma := (\Sigma')^\vee \times \Sigma'$.
- $\iota' : D \hookrightarrow \mathrm{End}^0(\Sigma')$ is induced by the diagonal action via ι_2'.
- $\iota : D \hookrightarrow \mathrm{End}^0(\Sigma)$ is the composition of ι' with $\mathrm{End}^0(\Sigma') \hookrightarrow \mathrm{End}^0(\Sigma)$ given by $\beta \mapsto (\beta, \beta^\vee)$.

Let $\alpha \in \mathrm{Aut}_D(W_1)$. For $1 \le i \le g_1$, $\alpha(\epsilon_i) = \sum_{j=1}^{g_1} \epsilon_j \cdot d_{ji}$ for some $d_{ji} \in D$. Make α act on Σ' in the isogeny category of p-divisible groups by the same formula as (3.4). This induces $\underline{\mathrm{Aut}}_D(W_1) \hookrightarrow \underline{\mathrm{Aut}}^0(\Sigma', \iota') \simeq \underline{\mathrm{Aut}}(V_p\Sigma', \iota')$. Together with the diagonal action $\underline{\mathrm{Aut}}(V_p\Sigma'_2, \iota'_2) \hookrightarrow \underline{\mathrm{Aut}}(V_p\Sigma', \iota')$, this induces a morphism of group functors

$$\underline{\mathrm{Aut}}_D(W_1) \times \underline{\mathrm{Aut}}(V_p\Sigma'_2, \iota'_2) \to \underline{\mathrm{Aut}}(V_p\Sigma', \iota'). \tag{4.2}$$

This is essentially the desired local type II pair. However Σ' is (in general) neither self-dual nor equipped with a symplectic pairing, without which there is no construction of the Weil representation. Thus we interpret the latter automorphism group in terms of (Σ, ι).

Consider any F_0-linear (via ι) symplectic pairing $\langle\cdot,\cdot\rangle : V_p\Sigma \times V_p\Sigma \to F_0 \otimes_{\mathbb{Q}_p} V_p\mu_{p^\infty}$ for which $V_p(\Sigma')^\vee$ and $V_p\Sigma'$ are totally isotropic. Then $\langle\cdot,\cdot\rangle$ induces $\delta : \underline{\mathrm{Aut}}(V_p\Sigma') \simeq \underline{\mathrm{Aut}}(V_p(\Sigma')^\vee)$ so that the two elements corresponding via δ are an adjoint pair with respect to $\langle\cdot,\cdot\rangle$. Then (δ, id) defines an embedding $\underline{\mathrm{Aut}}(V_p\Sigma', \iota') \hookrightarrow \underline{\mathrm{Aut}}(V_p\Sigma, \langle\cdot,\cdot\rangle, \iota)$. Composing with (4.2), obtain

$$\underline{\mathrm{Aut}}_D(W_1) \times \underline{\mathrm{Aut}}(V_p\Sigma'_2, \iota'_2) \to \underline{\mathrm{Aut}}(V_p\Sigma, \langle\cdot,\cdot\rangle, \iota) \hookrightarrow \underline{\mathrm{Aut}}(V_p\Sigma, \langle\cdot,\cdot\rangle) \tag{4.3}$$

injective on $\underline{\mathrm{Aut}}_D(W_1) \times \{1\}$ and $\{1\} \times \underline{\mathrm{Aut}}(V_p\Sigma'_2, \iota'_2)$.

Example 4.2. Let $S = \mathrm{Spec}\,\overline{\mathbb{F}}_p$. Let $[D : F_0] = d$, $\dim_D W_1 = n_1 \ge 1$. The height of Σ'_2 is dn_2 for some $n_2 \ge 1$. Suppose that Σ'_2 (thus also Σ') is étale, in which case we may view $V_p\Sigma'_2$ simply as a left D-module W_2, and $V_p\Sigma'$ as a left D-module $W_1 \otimes_D W_2$. Then (4.2) may be identified with

$$GL_D(W_1) \times GL_D(W_2) \to GL_{F_0}(W_1 \otimes_D W_2).$$

Let us explicitly describe the restriction of the Weil representation via the above map.

For simplicity, assume $D = F_0 = \mathbb{Q}_p$ for now. We can find $\langle\cdot,\cdot\rangle$ starting from a perfect pairing $\langle\cdot,\cdot\rangle_0 : \Sigma \times \Sigma \to \mu_{p^\infty}$. The Schrodinger model is

$$\omega = C_c^\infty(W_1 \otimes_{\mathbb{Q}_p} W_2, \overline{\mathbb{F}}_p)$$

equipped with the action of an ind $\overline{\mathbb{F}}_p$-group scheme P (which is analogous to the Siegel parabolic subgroup) as in [Shi, Cor 7.7]. Put $G_1 := GL_{\mathbb{Q}_p}(W_1)$, $G_2 := GL_{\mathbb{Q}_p}(W_2)$. We need not recall the definition of P here. It is enough to record that by restricting via $G_1 \times G_2 \to GL_{\mathbb{Q}_p}(W_1 \otimes_{\mathbb{Q}_p} W_2) \hookrightarrow P$, we obtain a $G_1 \times G_2$-representation $\omega|_{G_1 \times G_2}$ satisfying

$$((g_1, g_2) \cdot \phi)(w_1 \otimes w_2) = \phi(g_1^{-1}w_1 \otimes g_2^{-1}w_2), \quad \phi \in C_c^\infty(W_1 \otimes_{\mathbb{Q}_p} W_2, \overline{\mathbb{F}}_p).$$

By identifying $G_1 = GL_{\mathbb{Q}_p}(W_1^\vee)$ via $g_1 \mapsto (g_1^{-1})^\vee$ and using the canonical isomorphism $W_1 \otimes_{\mathbb{Q}_p} W_2 \simeq \mathrm{Hom}_{\mathbb{Q}_p}(W_1^\vee, W_2)$, we can also view $\omega|_{G_1 \times G_2}$ as a representation given by

$$((g_1, g_2) \cdot \phi)(f) = \phi(g_2^{-1} f g_1), \quad \phi \in C_c^\infty(\mathrm{Hom}_{\mathbb{Q}_p}(W_1^\vee, W_2), \overline{\mathbb{F}}_p),\ f \in \mathrm{Hom}_{\mathbb{Q}_p}(W_1^\vee, W_2). \tag{4.4}$$

More generally, when the assumption that $D = F_0 = \mathbb{Q}_p$ is dropped, we have $G_1 = GL_D(W_1^\vee)$, $G_2 = GL_D(W_2)$ and $\omega|_{G_1 \times G_2}$ is a representation on $C_c^\infty(\mathrm{Hom}_D(W_1^\vee, W_2), \overline{\mathbb{F}}_p)$ given by the same formula as (4.4).

5. A remark on mod p theta correspondence

The last section begins with some recollection of classical conjectures and results (§§5.1-5.2), which could be omitted but were included for the reader's convenience. Then we take the liberty to speculate on the possibility of a theta correspondence for p-adic groups when the coefficient field is $\overline{\mathbb{F}}_p$. There are one negative and one positive results. A somewhat surprising fact, though it is easy to prove, is that the naive analogue of Howe's conjecture on the bijective correspondence is almost vacuous already for the (GL_1, GL_1)-pair (Proposition 5.9). On the other hand, we verify that a weak analogue of Howe's unramified theta correspondence still works for type II pairs (Theorem 5.14).

Often G_i will denote the group of F-points rather than the underlying algebraic group by abuse of notation. Now F_0 and F are finite extensions of $\mathbb{Q}_p$ (rather than $\mathbb{Q}$).

5.1. **Review of Howe's conjecture.** Let (G_1, G_2) be a (classical) reductive dual pair of type 1 or 2 as in §3.1 or §4.1 which comes with an inclusion

$$G_1 \times G_2 \hookrightarrow \mathrm{Sp}_{F_0}(W, \langle\cdot,\cdot\rangle).$$

Write $\widetilde{G}_1$ (resp. $\widetilde{G}_2$) for the preimage of $G_1 \times \{1\}$ (resp. $\{1\} \times G_2$) via the classical double covering $\widetilde{\mathrm{Sp}}_{F_0}(W, \langle\cdot,\cdot\rangle) \to \mathrm{Sp}_{F_0}(W, \langle\cdot,\cdot\rangle)$. The subgroup of $\widetilde{\mathrm{Sp}}_{F_0}(W, \langle\cdot,\cdot\rangle)$ generated by $\widetilde{G}_1$ and $\widetilde{G}_2$ is denoted $\widetilde{G}_1\widetilde{G}_2$. It is known ([MVW87, Ch2,III.1]) that

Lemma 5.1. *Let $\widetilde{g}_1, \widetilde{g}_2 \in \widetilde{\mathrm{Sp}}_{F_0}(W, \langle\cdot,\cdot\rangle)$ with images g_1, g_2 in $\mathrm{Sp}_{F_0}(W, \langle\cdot,\cdot\rangle)$, respectively. If $g_1g_2 = g_2g_1$ then $\widetilde{g}_1\widetilde{g}_2 = \widetilde{g}_2\widetilde{g}_1$.*

In particular, $\widetilde{G}_1$ and $\widetilde{G}_2$ are centralizing each other in $\widetilde{\mathrm{Sp}}_{F_0}(W, \langle\cdot,\cdot\rangle)$. Thus the pullback of the Weil representation via

$$\widetilde{G}_1 \times \widetilde{G}_2 \stackrel{\text{product}}{\to} \widetilde{G}_1\widetilde{G}_2 \hookrightarrow \widetilde{\mathrm{Sp}}_{F_0}(W, \langle\cdot,\cdot\rangle)$$

makes a good sense and is going to be written as $\omega|_{\widetilde{G}_1\times\widetilde{G}_2}$. Define

$$R(\widetilde{G}_1\widetilde{G}_2) = \{\pi_1 \otimes \pi_2 \in \mathrm{Irr}(\widetilde{G}_1 \times \widetilde{G}_2) \,|\, \mathrm{Hom}_{\widetilde{G}_1\times\widetilde{G}_2}(\omega|_{\widetilde{G}_1\times\widetilde{G}_2}, \pi_1 \otimes \pi_2) \neq 0\}.$$

For $i = 1, 2$, let $R(\widetilde{G}_i)$ denote the image of $R(\widetilde{G}_1\widetilde{G}_2)$ in $\mathrm{Irr}(\widetilde{G}_i)$ under the projection $\pi_1 \otimes \pi_2 \mapsto \pi_i$. The following theorem was conjectured by Howe in [How79, §6] (also see [MVW87, III.2]). Parts (i) and (ii) were established by Waldspurger ([Wal90]) and Minguez ([Min08a]), respectively.

Theorem 5.2. *Suppose that either*

(i) F has residue field characteristic different from 2 or
(ii) (G_1, G_2) is a type 2 pair.

Then $R(\widetilde{G}_1\widetilde{G}_2)$ is a graph of bijection in $R(\widetilde{G}_1) \times R(\widetilde{G}_2)$.

Remark 5.3. Note that we only consider F as a finite extension of $\mathbb{Q}_p$. The function field case of the conjecture is known to be true except type I pairs with $\mathrm{char} F = 2$.

5.2. **Howe's conjecture in the unramified case.** This subsection follows [How79, §7], where the reader can find more detail. Assume that F is an unramified extension of F_0, that G_1, G_2 are unramified groups, that D splits over F, and that the residue field characteristic of F is odd. Then there exists an $\mathcal{O}_{F_0}$-lattice $\Lambda \subset W$ which is self-dual with respect to $\langle\cdot,\cdot\rangle$. Set $J := \mathrm{Sp}_{\mathcal{O}_{F_0}}(\Lambda, \langle\cdot,\cdot\rangle)$ and $\widetilde{J}$ to be the preimage of J in $\widetilde{\mathrm{Sp}}_{F_0}(W, \langle\cdot,\cdot\rangle)$. Then $\widetilde{J} \simeq J \times \{1, -1\}$. Let $K_i \subset G_i$ be a hyperspecial maximal compact subgroup for $i = 1, 2$. By conjugation one can assume that $K_1, K_2 \subset J$. Thereby the inclusions $K_i \hookrightarrow \widetilde{G}_i$ are obtained. Following Howe, define for $i = 1, 2$,

$$R(\widetilde{G}_i, K_i) := \{\pi_i \in \mathrm{Irr}(\widetilde{G}_i) :\ \pi_i^{K_i} \neq 0\}$$

$$R(\widetilde{G}_1\widetilde{G}_2, K_1K_2) := \{(\pi_1, \pi_2) \in R(\widetilde{G}_1\widetilde{G}_2) :\ \pi_i^{K_i} \neq 0,\ i = 1, 2\}.$$

Let $\mathscr{H}(\widetilde{G}_i//K_i)$ denote the Hecke algebra of K_i-bi-invariant $\mathbb{C}$-valued functions on $\widetilde{G}_i$. Each element of $\mathscr{H}(\widetilde{G}_i//K_i)$ defines an endomorphism of $(\omega|_{\widetilde{G}_1\times\widetilde{G}_2})^{K_1K_2}$.

Theorem 5.4. *(*[How79, Thm 7.1]*, cf.* [MVW87, §5]*)*

(i) $R(\widetilde{G}_1\widetilde{G}_2, K_1K_2)$ is a graph of bijection in $R(\widetilde{G}_1, K_1) \times R(\widetilde{G}_2, K_2)$.
(ii) Suppose $(\pi_1, \pi_2) \in R(\widetilde{G}_1\widetilde{G}_2)$. Then $\pi_1 \in R(\widetilde{G}_1, K_1)$ if and only if $\pi_2 \in R(\widetilde{G}_2, K_2)$.
(iii) The images of $\mathscr{H}(\widetilde{G}_i, K_i)$ in $\mathrm{End}_{\mathbb{C}}((\omega|_{\widetilde{G}_1\times\widetilde{G}_2})^{K_1K_2})$ are the same for $i = 1, 2$.

5.3. **Commuting elements in the metaplectic group.** In order to formulate a theta correspondence in our geometric context, it is fundamental to have the analogue of Lemma 5.1 for $\underline{\mathrm{Mp}}(V_pA, \widehat{e}_p^L)$ and $\underline{\mathrm{Sp}}(V_pA, \widehat{e}_p^L)$. We will state this as a hypothesis. Recall from [Shi, §5.1, §5.5] that for (A, L) (as in this article) there is a natural sequence of group functors on (Sch/S)

$$1 \to \mathbb{G}_m \to \underline{\mathrm{Mp}}(V_pA, \widehat{e}_p^L) \to \underline{\mathrm{Sp}}(V_pA, \widehat{e}_p^L) \to 1,$$

whose T-valued points yield an exact sequence of groups for each S-scheme T. For a p-divisible group Σ with a perfect alternating pairing $\langle \cdot, \cdot \rangle : \Sigma \times \Sigma \to \mu_{p^\infty}$ over S there is a similar sequence

$$1 \to \mathbb{G}_m \to \underline{\mathrm{Mp}}(V_p\Sigma, \langle \cdot, \cdot \rangle) \to \underline{\mathrm{Sp}}(V_p\Sigma, \langle \cdot, \cdot \rangle) \to 1.$$

Hypothesis 5.5. Let $\widetilde{g}_1, \widetilde{g}_2 \in \underline{\mathrm{Mp}}(V_pA, \widehat{e}_p^L)(T)$ for an S-scheme T. If the images of $\widetilde{g}_1$ and $\widetilde{g}_2$ in $\underline{\mathrm{Sp}}(V_pA, \widehat{e}_p^L)(T)$ commute then $\widetilde{g}_1\widetilde{g}_2 = \widetilde{g}_2\widetilde{g}_1$. The same holds with $(V_p\Sigma, \langle \cdot, \cdot \rangle)$ in place of $(V_pA, \widehat{e}_p^L)$.

The same analogue for $\underline{\mathrm{Mp}}(VA, \widehat{e}^L)$ should follow from the hypothesis at every prime p. In this subsection we focus on one prime p at a time. Although it is tempting to conjecture that the hypothesis is always true, we do not have good evidence other than analogy with the case of classical dual pairs. Nevertheless in some simple cases we can show:

Lemma 5.6. *Hypothesis 5.5 holds true*

- *(i) for $\underline{\mathrm{Mp}}(V_pA, \widehat{e}_p^L)(T)$ if p is invertible in S and there is a level structure $\eta : V_p \simeq V_pA$ for a symplectic $\mathbb{Q}$-vector space V_p with $\langle \cdot, \cdot \rangle_\psi : V_p \times V_p \to \mathbb{G}_m$ (cf. §3.3,* [Shi, §6.2]*), or*
- *(ii) for $\underline{\mathrm{Mp}}(V_p\Sigma, \langle \cdot, \cdot \rangle)$ if S is an $\mathbb{F}_p$-scheme and Σ is an ordinary p-divisible group over $\mathbb{F}_p$, or*
- *(iii) for $\underline{\mathrm{Mp}}(V_pA, \widehat{e}_p^L)(T)$ if S and $(\Sigma, \langle \cdot, \cdot \rangle)$ are as in (ii) and there is a level structure $\zeta : V_p\Sigma \simeq V_pA$ (cf.* [Shi, §6.3]*).*

Proof. We may assume that T is connected. In case (i), $\underline{\mathrm{Mp}}(V_pA, \widehat{e}_p^L)$ may be identified with the classical metaplectic group $\mathrm{Mp}(V_p, \langle \cdot, \cdot \rangle_\psi)$ (as a constant group scheme), for which the assertion is well known ([MVW87, Ch 2, Lem II.5]). In case (ii), the proof is obvious as $\underline{\mathrm{Mp}}(V_p\Sigma, \langle \cdot, \cdot \rangle) \simeq \mathbb{G}_m \times \underline{\mathrm{Sp}}(V_p\Sigma, \langle \cdot, \cdot \rangle)$ ([Shi, Cor 7.8]). It is easy to deduce (iii) from (ii). □

In the notation of Lemmas 3.11 and 4.1, let $\widetilde{G}_1$ (resp. $\widetilde{G}_2$) denote the pullback of $\underline{\mathrm{Mp}}(V_pA, \widehat{e}^L)$ along $\underline{\mathrm{Aut}}_D(W_1, \langle \cdot, \cdot \rangle_1) \times \{1\} \hookrightarrow \underline{\mathrm{Sp}}(V_pA, \widehat{e}^L)$ (resp. $\{1\} \times \underline{\mathrm{Aut}}(V_pA_2, \widetilde{e}^{L_2,\mathrm{Weil}}, \iota_2) \hookrightarrow \underline{\mathrm{Sp}}(V_pA, \widehat{e}^L)$) for type I pairs and along $\underline{\mathrm{Aut}}_D(W_1) \times \{1\} \to \underline{\mathrm{Sp}}(V_pA, \widehat{e}^L)$ (resp. $\{1\} \times \underline{\mathrm{Aut}}(V_2', \iota_2) \to \underline{\mathrm{Sp}}(V_pA, \widehat{e}^L)$) for type II pairs. Thus $\widetilde{G}_1$ and $\widetilde{G}_2$ are subgroup functors of $\underline{\mathrm{Mp}}(V_pA, \widehat{e}^L)$. We have a similar definition of $\widetilde{G}_1$ and $\widetilde{G}_2$ in the setting of p-divisible groups, cf. (4.3).

Corollary 5.7. *Suppose that Hypothesis 5.5 is true for (A, L) or $(\Sigma, \langle \cdot, \cdot \rangle)$. (For instance suppose that one of the conditions of Lemma 5.6 holds.) Then $\widetilde{G}_1$ and $\widetilde{G}_2$ defined above commute.*

Proof. This is immediate since the images of $\widetilde{G}_1$ and $\widetilde{G}_2$ in $\underline{\mathrm{Sp}}(V_pA, \widehat{e}_p^L)$ commute by construction. □

Remark 5.8. Deligne informed us that Hypothesis 5.5 is false for an arbitrary metaplectic group. More precisely, let G be a connected reductive group, say over $\mathbb{Q}_p$. Let $\widetilde{G} \to G(\mathbb{Q}_p)$ be an arbitrary metaplectic extension of $G(\mathbb{Q}_p)$ by $\mathbb{G}_m$ or μ_n for some $n \geq 2$. Then there are counterexamples where $\widetilde{g}_1, \widetilde{g}_2$ commute in $\widetilde{G}$ but their images in $G(\mathbb{Q}_p)$ do note commute. In this regard, the double covering of a symplectic group may be somewhat special, and we are hoping that its generalization $\underline{\mathrm{Mp}}(V_pA, \widehat{e}_p^L)$ also has this special property.

5.4. **Quotients of the mod p Weil representation of a p-adic metaplectic group.** Here we explore a mod p analogue of Howe's conjecture for p-adic groups. A mod l analogue of Howe's conjecture for a prime $l \neq p$ was considered in [Min08b]. Interestingly Minguez observed that there is a counterexample to the naive analogue of Theorem 5.2 in the mod l setting when l is not a so-called banal prime.

In the mod p case the problem is even more serious. The naive analogue of Theorem 5.2 hopelessly fails already in the case of the most elementary type II pair $(GL_1(\mathbb{Q}_p), GL_1(\mathbb{Q}_p))$. To study this case, put $k := \overline{\mathbb{F}}_p$ and recall that recall that $\omega = C_c^\infty(\mathbb{Q}_p, k)$ is a representation of $GL_1(\mathbb{Q}_p) \times GL_1(\mathbb{Q}_p)$ such that $((g_1, g_2) \cdot \phi)(g) = \phi(g_2^{-1} g g_1)$ (Example (4.2)). For our purpose it suffices to consider the first copy of $GL_1(\mathbb{Q}_p)$, and we will view ω as a $GL_1(\mathbb{Q}_p)$-representation as such.

Proposition 5.9. *Let V be a nonzero finite dimensional smooth representation of $GL_1(\mathbb{Q}_p)$.*

(i) Any $k[GL_1(\mathbb{Q}_p)]$-linear map $C_c^\infty(\mathbb{Q}_p^\times, k) \to V$ is zero for each V.

(ii) Any nonzero $k[GL_1(\mathbb{Q}_p)]$-linear map $\omega = C_c^\infty(\mathbb{Q}_p, k) \to V$ has 1-dimensional image which is the trivial representation of $GL_1(\mathbb{Q}_p)$.

Proof. (i) Let $\xi : C_c^\infty(\mathbb{Q}_p^\times, k) \to V$ be a $k[GL_1(\mathbb{Q}_p)]$-linear map. It suffices to prove that there exists $N \geq 1$ such that $\xi(\mathrm{char}_{a(1+p^n\mathbb{Z}_p)}) = 0$ for all $a \in \mathbb{Q}_p^\times$ and all $n \geq N$. As V is smooth, there exists $m \geq 1$ such that $1 + p^m\mathbb{Z}_p$ acts trivially on V. Take any $n \geq m$. Let $b_0, ..., b_{p-1}$ be any coset representatives for $(1 + p^n\mathbb{Z}_p)/(1 + p^{n+1}\mathbb{Z}_p)$. Then

$$\xi(\mathrm{char}_{a(1+p^n\mathbb{Z}_p)}) = \sum_{i=0}^{p-1} \xi(\mathrm{char}_{ab_i(1+p^{n+1}\mathbb{Z}_p)}) = \sum_{i=0}^{p-1} b_i^{-1} \cdot \xi(\mathrm{char}_{a(1+p^{n+1}\mathbb{Z}_p)}) = \sum_{i=0}^{p-1} \xi(\mathrm{char}_{a(1+p^{n+1}\mathbb{Z}_p)}) = 0.$$

(ii) Let $\mathbf{1}$ denotes the trivial representation of $GL_1(\mathbb{Q}_p)$. The map $\omega \to \mathbf{1}$ by $\phi \mapsto \phi(0)$ induces an exact sequence of $k[GL_1(\mathbb{Q}_p)]$-modules

$$0 \to C_c^\infty(\mathbb{Q}_p^\times, k) \to \omega \to \mathbf{1} \to 0.$$

This and the assertion (i) imply (ii). □

Remark 5.10. If $k = \mathbb{C}$ then any continuous character $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ is realized as a quotient of $C_c^\infty(\mathbb{Q}_p^\times, \mathbb{C})$ (with multiplicity one). If μ is a $\mathbb{C}$-valued Haar measure on $\mathbb{Q}_p^\times$ then $\phi \mapsto \int_{\mathbb{Q}_p^\times} \phi(x)\chi^{-1}(x)d\mu$ exhibits a nonzero map in $\mathrm{Hom}_{\mathbb{Q}_p^\times}(C_c^\infty(\mathbb{Q}_p^\times, \mathbb{C}), \chi)$. See [MVW87, Ch 3, Lem 2.3] for the study of quotients of $C_c^\infty(GL_n(\mathbb{Q}_p), \mathbb{C})$, where a $\mathbb{C}$-valued Haar measure on $GL_n(\mathbb{Q}_p)$ is indispensable. The case $k = \overline{\mathbb{F}}_p$ is special due to the lack of an $\overline{\mathbb{F}}_p$-valued Haar measure. Indeed, the argument of Proposition 5.9.(i) essentially proves the nonexistence of an $\overline{\mathbb{F}}_p$-valued Haar measure on any pro-p subgroup of $\mathbb{Q}_p^\times$.

The above proposition tells us that in the mod p setting, there are no interesting naive analogues of $R(\widetilde{G}_1\widetilde{G}_2)$ and $R(\widetilde{G}_1\widetilde{G}_2, K_1K_2)$. We do not know how to overcome this difficulty and make a plausible conjecture in a similar spirit as Howe's conjecture: Even if we allow π_1 and π_2 to be *reducible* representations of finite length, the mod p Weil representation still has few quotients of the form $\pi_1 \otimes \pi_2$. It is not immediately clear whether replacing Hom with Ext would help.

5.5. **A weak analogue of Howe's conjecture in the unramified case.** Despite the negative result of §5.4, we would like to ask (cf. Remark 5.15)

Question 5.11. Is a suitable analogue of Theorem 5.4 true in the mod p case?

The aim of this subsection is to verify a weak analogue of Theorem 5.4.(iii) for ordinary type II pairs. More precisely we will show that the Weil representation admits a finite filtration whose quotients have the property described in Theorem 5.4.(iii). (In fact we prove slightly more; see Theorem 5.14 below.) Our argument was inspired by [MVW87] and [Min08a].

Assume that $n_1 \geq n_2 \geq 1$. Let us set up some notation.

- F is a finite extension of $\mathbb{Q}_p$ with valuation $v_F : F^\times \to \mathbb{Z}$; its ring of integers is $\mathcal{O}_F$.
- $G_i := GL_{n_i}(F)$, $K_i := GL_{n_i}(\mathcal{O}_F)$ for $i = 1, 2$.
- $H_r := GL_r(F)$, $U_r := GL_r(\mathcal{O}_F)$ (where $0 \leq r \leq n_2$), agreeing $H_0 = U_0 = \{1\}$.
- $M_{n_i-r,r} := H_{n_i-r} \times H_r$, $K_{n_i-r,r} := U_{n_i-r} \times U_r$.
- T_i is the diagonal maximal torus of G_i for $i = 1, 2$.
- S_r is the diagonal maximal torus of H_r.
- B_1 (resp. B_2) is the Borel subgroup of upper (resp. lower) triangular matrices in G_1 (resp. G_2).
- B_r^+ (resp. B_r^-) is the Borel subgroup of upper (resp. lower) triangular matrices in H_r.
- Φ_i is the set of B_i-positive roots of T_i in G_i for $i = 1, 2$.
- Φ_r^+ (resp. Φ_r^-) is the set of B_r^+-positive (resp. B_r^--positive) roots of T_r in H_r.
- All Hecke algebras and spaces of functions (e.g. $C^\infty(G_1)$) will have coefficients in $k = \overline{\mathbb{F}}_p$. The reference to k will be omitted most of the time.

Let $\omega|_{G_1\times G_2}$ denote the restricted Weil representation at the end of Example 4.2 (with $D = F$). By choosing a basis of $W_1^\vee$ and W_2 in that example, we can identify $\omega|_{G_1\times G_2} = C_c^\infty(M_{n_1,n_2}(F), k)$ with $G_1 \times G_2$-action described by

$$((g_1, g_2)\phi)(x) = \phi(g_2^{-1}xg_1), \quad \forall x \in M_{n_1,n_2}(F).$$

For $0 \le r \le n_2 + 1$, let Ω_r denote the k-subspace of $\omega|_{G_1\times G_2}$ consisting of functions supported on $n_1 \times n_2$-matrices of rank $\ge r$. Then there is a $G_1 \times G_2$-stable filtration

$$\{0\} = \Omega_{n_2+1} \subsetneq \Omega_{n_2} \subsetneq \Omega_{n_2-1} \subsetneq \cdots \subsetneq \Omega_1 \subsetneq \Omega_0 = \omega|_{G_1\times G_2}.$$

Then the space of $K_1 \times K_2$-invariant vectors on quotients are equipped with Hecke actions

$$\mathscr{H}(G_1, K_1) \times \mathscr{H}(G_2, K_2) \to \mathrm{End}_k((\Omega_r/\Omega_{r+1})^{K_1\times K_2}). \tag{5.1}$$

For $0 \le r \le n$, let μ_r denote the $H_r \times H_r$-representation on $C_c^\infty(H_r, k)$, where the action is given by

$$((h_1, h_2) \cdot \phi)(h) = \phi(h_2^{-1}hh_1).$$

Observe that $\mu_r^{U_r\times U_r} = C_c^\infty(U_r\backslash H_r/U_r, k)$ comes equipped with a natural action of $\mathscr{H}(H_r, U_r)\times\mathscr{H}(H_r, U_r)$. Let $\#$ denote the order 2 automorphism of $\mathscr{H}(H_r, U_r)$ induced by $g \mapsto g^{-1}$ on H_r.

Lemma 5.12. *For every $f \in \mathscr{H}(H_r, U_r)$, the actions of $(f, 1)$ and $(1, f^\#)$ on $\mu_r^{U_r\times U_r}$ are the same.*

Proof. Take $V_1 = V_2 = \mathbf{1}$ in [Her, Prop 6.2]. Note that our action of $\mathscr{H}_k(H_r, U_r) \times \{1\}$ is a left action, which differs from Herzig's right action by $f \mapsto f^\#$. ☐

Lemma 5.13. *For every $0 \le r \le n_2$, there is an isomorphism of $G_1 \times G_2$-representations*

$$\Omega_r/\Omega_{r+1} \simeq \mathrm{Ind}_{P^+_{n_2-r,r}\times P^-_{n_2-r,r}}^{G_1\times G_2}(\mathbf{1} \otimes \mu_r).$$

Proof. It is enough to observe that Lemme 1.3 and the paragraph above Définition 2.1 of [Min08a] carry over to the case of $\overline{\mathbb{F}}_p$-coefficients. Note that our G_1 (resp. G_2) is his G'_m (resp. G_n). ☐

There are partial Satake transforms (as $\mathscr{S}_G^M$ of [Her, §2.3])

$$\mathscr{S}_i : \mathscr{H}(G_i, K_i) \hookrightarrow \mathscr{H}(M_{n_i-r,r}, K_{n_i-r,r}), \quad i = 1, 2$$

defined with respect to B_i. (In other words, require the P of [Her, §2.3] to contain B_i.) Define

$$S_r^- := \{t \in S_r : v_F(\alpha(t)) \le 0,\ \forall \alpha \in \Phi_r^+\} \tag{5.2}$$

and also S_r^+, using Φ_r^- in place of Φ_r^+. Let $\mathscr{H}_{S_r}^-$ denote the subalgebra of $\mathscr{H}_{S_r}(\mathbf{1}) = C^\infty((GL_1(F)/GL_1(\mathcal{O}_F))^r)$ consisting of functions whose supports are contained in S_r^-. Similarly define $\mathscr{H}_{S_r}^+$, $\mathscr{H}_{T_1}^-$, and $\mathscr{H}_{T_2}^+$ replacing Φ_r^+ in (5.2) respectively with Φ_r^-, the set of $B_1 \cap M_{n_1-r,r}$-positive roots, and the set of $B_2 \cap M_{n_2-r,r}$-positive roots. Consider the diagram

$$\begin{array}{ccc} \mathscr{H}(M_{n_i-r,r}, K_{n_i-r,r}) & \overset{\exists!\,\mathscr{T}_i}{\dashrightarrow} & \mathscr{H}(H_r, U_r) \\ \downarrow \wr & & \downarrow \wr \\ \mathscr{H}_{T_i}^* & \longrightarrow & \mathscr{H}_{S_r}^* \end{array} \tag{5.3}$$

where $* = -$ if $i = 1$ and $* = +$ if $i = 2$. The vertical maps are the Satake isomorphisms of [Her11] with respect to $B_i \cap M_{n_i-r,r}$ and B_r^*. The bottom horizontal arrow is induced by the inclusion

$$S_r = GL_1(F)^r \hookrightarrow T_i = GL_1(F)^{n_i}, \quad (x_1, ..., x_r) \mapsto (x_1, ..., x_r, 1, ..., 1).$$

Then there exists a unique map $\mathscr{T}_i$ which makes the diagram commute. Now consider

$$\begin{array}{ccccccc} \mathscr{H}(G_1, K_1) & \xrightarrow{\mathscr{S}_1} & \mathscr{H}(M_{n_1-r,r}, K_{n_1-r,r}) & \xrightarrow{\mathscr{T}_1} & \mathscr{H}(H_r, U_r) & \xrightarrow{\sim} & \mathscr{H}_{S_r}^- \\ & & & & \downarrow \# & & \downarrow \# \\ \mathscr{H}(G_2, K_2) & \xrightarrow{\mathscr{S}_2} & \mathscr{H}(M_{n_2-r,r}, K_{n_2-r,r}) & \xrightarrow{\mathscr{T}_2} & \mathscr{H}(H_r, U_r) & \xrightarrow{\sim} & \mathscr{H}_{S_r}^+ \end{array} \tag{5.4}$$

The third arrow in each row is the same as in the right vertical arrow of (5.3). The rightmost vertical map of (5.4) is induced by $t \mapsto t^{-1}$ on S_r and denoted # by abuse of notation. Obviously the right rectangle commutes.

Theorem 5.14. *(i) Let $f_i \in \mathscr{H}(G_i, K_i)$, $i = 1, 2$. If f_1 and f_2 have the same image in $\mathscr{H}^+_{S_r}$ via (5.4) then $(f_1, 1)$ and $(1, f_2)$ have the same image via (5.1).*

(ii) The composite maps $\mathscr{H}(G_1, K_1) \to \mathscr{H}^-_{S_r}$ and $\mathscr{H}(G_2, K_2) \to \mathscr{H}^+_{S_r}$ are surjective.

(iii) $\mathscr{H}(G_1, K_1) \times \{1\}$ and $\{1\} \times \mathscr{H}(G_2, K_2)$ have the same image in $\mathrm{End}_k((\Omega_r/\Omega_{r+1})^{K_1 \times K_2})$.

Proof. (i) Recall that $\Omega_r/\Omega_{r+1} \simeq \mathrm{Ind}^{G_1 \times G_2}_{P^+_{n_2-r,r} \times P^-_{n_2-r,r}} (\mathbf{1} \otimes \mu_r)$ from Lemma 5.13. By sending ϕ of the induced representation to $\phi(1)$, we obtain an isomorphism of k-vector spaces (cf. (2.12) of [Her])

$$(\Omega_r/\Omega_{r+1})^{K_1 \times K_2} \xrightarrow{\sim} (\mathbf{1} \otimes \mu_r)^{K_{n_1-r,r} \times K_{n_2-r,r}}.$$

By [Her, Lem 2.13], the above map is equivariant for the action of $\prod_{i=1}^2 \mathscr{H}(G_i, K_i)$, if the latter acts on the right hand side through $\prod_{i=1}^2 \mathscr{H}(M_{n_i-r,r}, K_{n_i-r,r})$ via $(\mathscr{S}_1, \mathscr{S}_2)$. Since the action of $\prod_{i=1}^2 M_{n_i-r,r}$ on $\mathbf{1} \otimes \mu_r$ factors through its projections onto $H_r \times H_r$, we see that $\prod_{i=1}^2 \mathscr{H}(M_{n_i-r,r}, K_{n_i-r,r})$ acts through $\mathscr{H}(H_r, U_r) \times \mathscr{H}(H_r, U_r)$ via $(\mathscr{T}_1, \mathscr{T}_2)$. At this point, observe that the assumption of (i) implies that $\mathscr{T}_1(\mathscr{S}_1(f_1))$ maps to $\mathscr{T}_2(\mathscr{S}_2(f_2))$ via #. Lemma 5.12 shows that $\mathscr{T}_1(\mathscr{S}_1(f_1))$ and $\mathscr{T}_2(\mathscr{S}_2(f_2))$ have the same action on $(\mathbf{1} \otimes \mu_r)^{K_{n_1-r,r} \times K_{n_2-r,r}}$.

(ii) The image of $\mathscr{S}_1$, mapped to $\mathscr{H}^-_{T_1}$ via (5.3) consists exactly of

$$\phi \in C^\infty_c(T_1), \quad \mathrm{supp}\,\phi \subset \{t \in T_1 : v_F(\alpha(t)) \le 0,\ \forall \alpha \in \Phi_1\}.$$

Clearly the set of such ϕ maps onto $\mathscr{H}^-_{S_r}$ via (5.3), noting that Φ_1 restricts to Φ^+_r via $S_r \hookrightarrow T_1$. Therefore $\mathscr{H}(G_1, K_1) \to \mathscr{H}^-_{S_r}$ is onto. The same argument shows that $\mathscr{H}(G_2, K_2) \to \mathscr{H}^+_{S_r}$ is onto as well.

(iii) This is immediate from (i) and (ii). □

Remark 5.15. A variant of Question 5.11 can be asked, replacing the hyperspecial subgroups by either the maximal pro-p subgroups of Iwahori subgroups or the congruence subgroups consisting of elements which are 1 modulo p. For instance, one can ask whether one gets a correspondence of "Serre weights".

Remark 5.16. Let us conclude with a speculative remark. Our belief is that the conjectural mod p theta correspondence for p-adic groups should be compatible with the conjectural (global) theta correspondence for mod p automorphic forms. However it is not easy to make sense of this, as no satisfactory representation-theoretic approach seems available to study mod p automorphic forms in general. Some anomalies are discussed in [Ser96], for instance.

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