# THE STABLE TRACE FORMULA FOR SHIMURA VARIETIES OF ABELIAN TYPE

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Abstract. We express the Frobenius–Hecke traces on the compactly supported cohomology of a Shimura variety of abelian type in terms of elliptic parts of stable Arthur–Selberg trace formulas for the endoscopic groups. This confirms predictions of Langlands and Kottwitz at primes where the level is hyperspecial.

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*Date*: October 11, 2021.

## Introduction

0.1. **The main results.** Shimura varieties have provided a testing ground for many conjectures in the Langlands Program, and have been indispensable in the (partial) solutions of some of these conjectures. Motivated by the work of Eichler, Shimura, Kuga, Sato, and Ihara, Langlands formulated the problem of expressing the Hasse–Weil zeta function of a Shimura variety in terms of automorphic $L$-functions. This question is itself a special case of Langlands' conjecture that all motivic $L$-functions are automorphic.

In a series of papers [Lan73, Lan76, Lan77, Lan79a, Lan79b], Langlands developed the idea of systematically using trace formulas to attack this problem. In his initial investigations, he encountered the phenomenon of $L$-indistinguishability, which motivated the theory of endoscopy. Based on the latter, Langlands predicted that one should be able to compare a Lefschetz-type trace formula for the Shimura variety with trace formulas arising in the theory of automorphic representations after stabilizing both types of the formulas. This prediction was formulated as a precise conjecture in Kottwitz's paper [Kot90].

The main result of the present paper is a verification of this conjecture for Shimura varieties of abelian type: We prove an identity between a Grothendieck–Lefschetz–Verdier trace formula on the Shimura variety and elliptic parts of stable Arthur–Selberg trace formulas for the endoscopic groups.

To state our main result more precisely, we fix some notation. Let $(G, X)$ be a Shimura datum with reflex field $E$. Fix a prime $\ell$, and let $\xi$ be an algebraic representation of $G$ over $\overline{\mathbb{Q}}_\ell$. Let

$$\mathbf{H}^i_c(\mathrm{Sh}, \xi) := \varinjlim_K \mathbf{H}^i_c(\mathrm{Sh}_K(G, X)_{\overline{E}}, \mathscr{L}_\xi),$$

where $K$ runs through all sufficiently small compact open subgroups of $G(\mathbb{A}_f)$, and for each $K$ we denote by $\mathrm{Sh}_K(G, X)$ the Shimura variety at level $K$, and by $\mathscr{L}_\xi$ the automorphic $\ell$-adic sheaf attached to $\xi$. (We need a technical assumption on $\xi$ so that $\mathscr{L}_\xi$ is well defined, but we omit this here. In the introduction the reader can assume $\xi$ is trivial and $\mathscr{L}_\xi = \overline{\mathbb{Q}}_\ell$.) Then $\mathbf{H}^i_c(\mathrm{Sh}, \xi)$ admits commuting actions by $\mathrm{Gal}(\overline{E}/E)$ and $G(\mathbb{A}_f)$.

Let $p \neq \ell$ be a prime, and let $\Phi \in \mathrm{Gal}(\overline{E}/E)$ be a geometric Frobenius element at a place $\mathfrak{p}$ of $E$ above $p$. Let $f$ be an element of the Hecke algebra of $G(\mathbb{A}_f)$. We always assume that $f$ is of the form $1_{K_p} f^p$, where $f^p$ is in the Hecke algebra of $G(\mathbb{A}^p_f)$ and $1_{K_p}$ is the characteristic function of a hyperspecial subgroup $K_p \subset G(\mathbb{Q}_p)$. (When $f$ is fixed, this condition is satisfied for almost all primes $p$.) For $m$ an integer we define

$$T(m, f) := \sum_i (-1)^i \mathrm{tr}\,(f \times \Phi^m \mid \mathbf{H}^i_c(\mathrm{Sh}, \xi)).$$

Note that if $f$ is the characteristic function of some compact open subgroup $K \subset G(\mathbb{A}_f)$ and $\xi$ is trivial, then $T(m, f)$ is directly related to the Euler factor at $\mathfrak{p}$ of the Hasse–Weil zeta function of $\mathrm{Sh}_K(G, X)$, when $p$ is sufficiently large.

**Theorem 1** (see Theorem 8.3.11). *Assume that $(G, X)$ is of abelian type. For all sufficiently large $m$ we have*

$$T(m, f) = \sum_{\mathfrak{e}} \iota(\mathfrak{e}) ST^{H_1}_{\mathrm{ell},\chi_{H_1}}(f^{H_1}), \tag{0.1.1}$$

*where $\mathfrak{e}$ runs through the elliptic endoscopic data for $G$ up to isomorphism, $\iota(\mathfrak{e}) \in \mathbb{Q}$ is a constant depending only on $\mathfrak{e}$, and $ST^{H_1}_{\mathrm{ell},\chi_{H_1}}$ is an elliptic stable distribution associated with $\mathfrak{e}$ (defined on a $z$-extension $H_1$ of the endoscopic group in $\mathfrak{e}$).*

If the derived subgroup $G_{\mathrm{der}}$ of $G$ is simply connected and the center $Z_G$ of $G$ is cuspidal (i.e., having equal $\mathbb{Q}$-rank and $\mathbb{R}$-rank), then $ST^{H_1}_{\mathrm{ell},\chi_{H_1}}$ is the elliptic part of the stable trace formula for $\mathfrak{e}$. Without these assumptions, the definition involves $z$-extensions and fixed central characters. The functions $f^{H_1}$ will be explained after we state Theorem 3 below. The requirement that $m$ is sufficiently large is needed to ensure that the local terms in the Grothendieck–Lefschetz–Verdier trace formula can be calculated naively; this is a special case of Deligne's conjecture, which has been proved in general by Fujiwara [Fuj97] and Varshavsky [Var05]. For applications this restriction turns out to be harmless. Note that by contrast, knowing (0.1.1) only for all sufficiently divisible $m$ would be insufficient for most applications.

Kottwitz [Kot90, §3, §7] conjectured the equality in Theorem 1 for general Shimura varieties, and proved it in the case of PEL type A or C in [Kot92b, §19] and [Kot90, Thm. 7.2]. By results of Matsushima [Mat67] and Franke [Fra98], the $G(\mathbb{A}_f)$-action on $\mathbf{H}^i_c(\mathrm{Sh}, \xi)$ can be understood in terms of automorphic representations of $G$. It is expected that the equality in Theorem 1 should lead to a description of $\mathbf{H}^i_c(\mathrm{Sh}, \xi)$, or a variant when the Shimura varieties are non-compact, as a $\mathrm{Gal}(\overline{E}/E) \times G(\mathbb{A}_f)$-module. This description should involve the global Langlands correspondence between automorphic representations and Galois representations, as well as Arthur's conjectures on automorphic multiplicities. This would lead to an expression of the Hasse–Weil zeta function in terms of automorphic $L$-functions. See [Kot90, Part II] for an explanation of this circle of ideas. In the non-compact case one expects that replacing $\mathbf{H}^i_c(\mathrm{Sh}, \xi)$ by the intersection cohomology of the Baily–Borel compactification will lead to a description similar to the compact case. We do not prove this variant of Theorem 1 for intersection cohomology in the present paper, but Theorem 1 and the point counting formula in Theorem 2 below are expected to play a crucial role in the proof of such a result; see for instance [Mor10, Mor11, Zhu18].

The proof of Theorem 1 consists of two steps. The first step is to prove a "point counting formula", expressing $T(m, f)$ in terms of orbital integrals and twisted orbital integrals on $G$ in a way resembling the geometric side of the Arthur–Selberg trace formula. The second step is stabilization, which relates the (twisted) orbital integrals on $G$ with the terms constituting $ST^{H_1}_{\mathrm{ell},\chi_{H_1}}(f^{H_1})$.

When $G_{\mathrm{der}}$ is simply connected and $Z_G$ is cuspidal, the point counting formula was already conjectured by Kottwitz [Kot90, §3]. Let $q = p^r$ be the cardinality of the residue field of $\mathfrak{p}$. For $m$ sufficiently large, the conjecture states that

$$T(m, f) = \sum_{\substack{(\gamma_0,\gamma,\delta) \in \mathfrak{KP}_{\mathrm{cla}}(q^m)/\sim, \\ \alpha(\gamma_0,\gamma,\delta)=0}} c_1(\gamma_0, \gamma, \delta) c_2(\gamma_0) O_\gamma(f^p) TO_\delta(\phi_{mr}) \mathrm{tr}\, \xi(\gamma_0). \tag{0.1.2}$$

Here $\mathfrak{KP}_{\mathrm{cla}}(q^m)$ consists of triples $(\gamma_0, \gamma, \delta) \in G(\mathbb{Q}) \times G(\mathbb{A}_f^p) \times G(\mathbb{Q}_{q^m})$ such that $\gamma_0$ is $\mathbb{R}$-elliptic, stably conjugate to $\gamma$, and stably conjugate to the degree $mr$ norm of $\delta$. There is also a technical assumption on $\delta$ which we omit here. (The notation $\mathfrak{KP}_{\mathrm{cla}}$ stands for "classical Kottwitz parameters".) The equivalence relation $\sim$ is given by stable conjugacy on the first factor, conjugacy on the second factor, and $\sigma$-conjugacy on the third factor. Kottwitz defines a Galois cohomological invariant $\alpha(\gamma_0, \gamma, \delta)$ for each $(\gamma_0, \gamma, \delta) \in \mathfrak{KP}_{\mathrm{cla}}(q^m)$, and in (0.1.2) the summation is subject to the condition $\alpha(\gamma_0, \gamma, \delta) = 0$. In each summand, we have an orbital integral $O_\gamma(f^p)$ on $G(\mathbb{A}_f^p)$, a twisted orbital integral $TO_\delta(\phi_{mr})$ on $G(\mathbb{Q}_{q^m})$ (where $\phi_{mr}$ is an explicit function on $G(\mathbb{Q}_{q^m})$), the character $\mathrm{tr}\,\xi$ of $\xi$ evaluated at $\gamma_0$, a volume term $c_1(\gamma_0, \gamma, \delta)$, and a term $c_2(\gamma_0)$ defined via Galois cohomology.

In the conjectural formula (0.1.2), the assumption that $G_{\mathrm{der}}$ is simply connected is quite serious. Without it, Kottwitz's construction of the invariant $\alpha(\gamma_0, \gamma, \delta)$ for $(\gamma_0, \gamma, \delta) \in \mathfrak{KP}_{\mathrm{cla}}(q^m)$ no longer works, and also the volume term $c_1(\gamma_0, \gamma, \delta)$ is not well defined. These problems are caused by the possible disconnectedness of $G_{\gamma_0}$ for a semi-simple $\gamma_0 \in G(\mathbb{Q})$. In the following theorem, the point counting formula we prove is a generalization of (0.1.2) without any assumptions on $G_{\mathrm{der}}$ and $Z_G$.

**Theorem 2** (see Theorem 6.3.6)**.** *If $(G, X)$ is of abelian type, then for all sufficiently large $m$ we have*

$$(0.1.3) \quad T(m, f) = \sum_{\gamma_0 \in \Sigma} \sum_{\substack{\mathfrak{c} \in \mathfrak{KP}(\gamma_0, q^m) \\ \alpha(\mathfrak{c}) = 0}} \left| (G_{\gamma_0}/G_{\gamma_0}^0)(\mathbb{Q}) \right|^{-1} c_1(\mathfrak{c}) c_2(\gamma_0) O_{\mathfrak{c}}(f^p) TO_{\mathfrak{c}}(\phi_{mr}) \mathrm{tr}\,\xi(\gamma_0),$$

*where the terms $c_1(\mathfrak{c}), c_2(\gamma_0), O_{\mathfrak{c}}(f^p), TO_{\mathfrak{c}}(\phi_{mr})$ are defined analogously as the terms in (0.1.2).*

The most significant new feature of (0.1.3) is that the summation index set $\mathfrak{KP}_{\mathrm{cla}}(q^m)/\sim$ in (0.1.2) has been replaced by a more refined set

$$\coprod_{\gamma_0 \in \Sigma} \mathfrak{KP}(\gamma_0, q^m)$$

which admits a map to the former. Here $\Sigma$ is a certain subset of the set of $\mathbb{R}$-elliptic elements of $G(\mathbb{Q})$, and for each $\gamma_0 \in \Sigma$ the definition of $\mathfrak{KP}(\gamma_0, q^m)$ is Galois cohomological in nature. (We also allow $Z_G$ to be non-cuspidal, in which case $\Sigma$ depends on the choice of a compact open subgroup $K^p \subset G(\mathbb{A}_f^p)$ such that $f^p$ is $K^p$-bi-invariant.) For each $\mathfrak{c} \in \mathfrak{KP}(\gamma_0, q^m)$, we define an invariant $\alpha(\mathfrak{c})$ lying in an abelian group that depends only on $G_{\gamma_0}^0$ and $G$. This definition specializes to Kottwitz's invariant $\alpha(\gamma_0, \gamma, \delta)$ when $G_{\mathrm{der}}$ is simply connected. In (0.1.3) the condition $\alpha(\mathfrak{c}) = 0$ is imposed, similarly as in (0.1.2).

Once Theorem 2 is proved, in order to prove Theorem 1 we need to stabilize the right hand side of (0.1.3). We prove this stabilization in general as in the next theorem, without assuming that $(G, X)$ is of abelian type.

**Theorem 3** (see Theorem 8.3.10)**.** *The right hand side of (0.1.3) is equal to*

$$\sum_{\mathfrak{e}} \iota(\mathfrak{e}) ST^{H_1}_{\mathrm{ell}, \chi_{H_1}}(f^{H_1}).$$

Theorem 1 is immediate from Theorems 2 and 3. The proof of Theorem 3 follows the outline of [Kot90, §7]. Namely, after applying a Fourier transform on the finite abelian group of which $\alpha(\mathfrak{c})$ is a character, we can turn the right hand side of (0.1.3) into the sum of $\kappa$-orbital integrals (twisted at $p$) over adelic conjugacy classes. To rewrite the sum in terms of stable distributions on endoscopic groups, the key input is the transfer of orbital integrals via the Kottwitz–Langlands–Shelstad transfer and the fundamental lemma. More precisely, $f^{H_1}$ away from $\{p, \infty\}$ is obtained from $f^p$ via the usual untwisted transfer, whereas $f^{H_1}$ at $p$ is a twisted transfer of $\phi_{mr}$, and $f^{H_1}$ at $\infty$ is constructed explicitly as a finite linear combination of certain stably cuspidal functions.

We carry out the stabilization without the simplifying hypotheses in [Kot90, §7] that $G_{\mathrm{der}}$ is simply connected and that $Z_G$ is cuspidal, by working systematically with $z$-extensions and fixed central characters. Here a useful fact is that once a $z$-extension $G_1$ of $G$ is fixed, it induces $z$-extensions $H_1$ of endoscopic groups $H$ for $G$. To transfer functions with fixed central characters (thus the functions are not compactly supported in general), the main point is that the transfer factors enjoy an equivariance property with respect to the translation by central elements. It is also worth mentioning the improvement that, unlike [Kot90, Thm. 7.2], Theorem 1 has no $(G, H)$-regularity condition imposed in the stable distributions. The reason is that there is no contribution coming from the non-$(G, H)$-regular semisimple terms, as shown by Morel (§8.2.5 and Lemma 8.2.8 below).

0.2. **Applications.** Theorem 1, or its proof (Theorem 5), has already been used to obtain the following results.

- With Kret, one of us (Shin) has constructed the automorphic to Galois direction of the Langlands correspondence for $\mathrm{GSp}_{2n}$ and (a form of) $\mathrm{GSO}_{2n}$ over totally real fields, under a technical local hypothesis [KS16, KS20]. This involves constructing Galois representations into GSpin groups, cf. [FC90, p. 268].
- One of us (Zhu) has given a description of the Hasse–Weil zeta function and the Hecke–Galois action on the intersection cohomology of the Baily–Borel compactification of Shimura varieties for some global forms of $\mathrm{SO}(N, 2)$ in terms of automorphic representations [Zhu18]. This completes the Langlands–Kottwitz program for these varieties at almost all primes.
- Youcis [You] has extended Scholze's version [Sch13] of the Langlands–Kottwitz method for Shimura varieties with bad reduction from the case of PEL type to the case of abelian type.
- Mack-Crane [MC21] has obtained a trace formula for Igusa varieties of Hodge type which is analogous to Theorem 2 in the case of Hodge type. This generalizes [Shi09]. A generalization to the case of abelian type, as well as a stabilization analogous to Theorem 1 is expected, cf. [Shi10].)

We stress that the Shimura data appearing in concrete applications, as in the first two items, are typically of abelian type but *not* of Hodge type. The same is true with the three applications below. As we will explain in §0.3 below, the proof of Theorem 1 in the case of Hodge type is substantially easier, but this does not suffice for many applications.

In general, Theorem 1 is the key to determining the fundamental virtual $G(\mathbb{A}_f)\times \mathrm{Gal}(\overline{E}/E)$-module $[\mathbf{H}_c(\mathrm{Sh},\xi)] := \sum_i (-1)^i\, \mathbf{H}^i_c(\mathrm{Sh},\xi)$ in terms of automorphic representations. A notable corollary is then to express the Hasse–Weil zeta function of $\mathrm{Sh}_K$ as an alternating product of automorphic $L$-functions for sufficiently small compact open subgroups $K\subset G(\mathbb{A}_f)$, possibly up to Euler factors at finitely many primes. We intend to work out the details in a sequel and obtain unconditional results in various special cases, which will provide important ingredients for some remarkable arithmetic results:

- The Beilinson–Bloch–Kato conjecture for Rankin–Selberg motives using Shimura varieties of unitary groups [LTX+19],
- Higher dimensional Gross–Zagier formula (i.e., arithmetic inner product formula) using Shimura varieties of unitary groups [LL20].
- Euler systems arising from Shimura varieties of $\mathrm{SO}(2n-1,2)$ [Cor18].

Related to the applications in [LTX+19, LL20], it is worth pointing out that when one passes between a Shimura datum of abelian type and an "isogenous" Shimura datum of Hodge type, the reflex field is often not preserved. Thus even if one is just interested in *constructing* representations of $\mathrm{Gal}(\overline{E}/E)$ using the cohomology of a Shimura variety of a unitary group with reflex field $E$ (which is of abelian type but not of Hodge type), one cannot pass to a Shimura variety of Hodge type without having to enlarge $E$ in general.

To understand the structure of $[\mathbf{H}_c(\mathrm{Sh},\xi)]$ in the general case of abelian type, there are two main obstacles to proving an unconditional theorem. Let us briefly address them, leaving the details to §9.2 below.

When $G$ is anisotropic modulo center over $\mathbb{Q}$, or equivalently when the finite-level Shimura varieties $\mathrm{Sh}_K(G,X)$ are projective, the first problem is to show that

$$ST^{H_1}_{\mathrm{ell},\chi_{H_1}}(f^{H_1}) = ST^{H_1}_{\chi_{H_1}}(f^{H_1})$$

in the summand of Theorem 3, namely that the non-elliptic terms in the stable distribution cancel each other out. The resolution of this problem is within reach at least in various special cases that are of interest for applications, and this will be treated in the sequel. The second problem is that the endoscopic classification for automorphic representations is not available for general reductive groups.

If $G$ is isotropic modulo center, the second problem remains the same. In place of the first problem, however, it is desirable to promote Theorem 1 by proving an equality where the compactly supported cohomology and $ST^{H_1}_{\mathrm{ell},\chi_{H_1}}$ are replaced with the intersection cohomology of the Baily–Borel compactification and $ST^{H_1}_{\chi_{H_1}}$, respectively. As mentioned above, such an upgrade is obtained for $\mathrm{SO}(N,2)$ in [Zhu18]. Some results were previously known for Shimura varieties of PEL type A and C [LR92, Mor08, Mor10, Mor11].

Another application of our work would be the analogues of Theorems 1 and 2 for Shimura varieties of parahoric level at $p$, in light of recent advances on the Haines–Kottwitz test function conjecture [HR21, HR20] and the Langlands–Rapoport conjecture in the parahoric case [Zho20, van20]. The latter takes as an input the hyperspecial case through the earlier work [Kis17]; a strengthening should be possible by appealing to our improvement (Theorem 5) instead.

## 0.3. **Variants of the Langlands–Rapoport Conjecture.** 

We now discuss the proof of Theorem 2. To simplify the exposition we assume that $\xi$ is trivial (so that $\mathscr{L}_\xi = \overline{\mathbb{Q}}_\ell$), and that $f^p = 1_{K^p}$ for a sufficiently small compact open subgroup $K^p \subset G(\mathbb{A}_f^p)$. We continue to assume that $K_p$ is hyperspecial. If $\mathrm{Sh}_{K_pK^p}$ is proper over $E$, then one expects that there exists a proper smooth integral model, $\mathscr{S}_{K_pK^p}$, of $\mathrm{Sh}_{K_pK^p}$ over $\mathcal{O}_{E,(\mathfrak{p})}$. In this case we have

$$T(m,f) = \#\mathscr{S}_{K_pK^p}(\mathbb{F}_{q^m}). \tag{0.3.1}$$

If $\mathrm{Sh}_{K_pK^p}$ is not proper, one still conjectures that there exists a canonical smooth integral model $\mathscr{S}_{K_pK^p}$ satisfying (0.3.1) (among other conditions). Hence in all cases we seek for a formula for $\#\mathscr{S}_{K_pK^p}(\mathbb{F}_{q^m})$, thus the name "point counting formula".

For Shimura varieties of Hodge type, it is possible to establish a point counting formula by generalizing the considerations of Kottwitz [Kot92b] in the PEL-type setting, with the aid of the results from [Kis17]. In this approach one attaches group-theoretic invariants to isogeny classes over a fixed finite field $\mathbb{F}_{q^m}$; see [Lee18]. It does not seem to be possible to deduce Theorem 2 for general Shimura varieties of abelian type from such results in the case of Hodge type. In the current paper, we take the point of view of Langlands–Rapoport [LR87], which relates $\overline{\mathbb{F}}_q$-isogeny classes and certain Galois gerbs. Although the statements we prove in the case of Hodge type require more effort, they have the merit that one can then infer similar statements in the case of abelian type, and hence deduce the point counting formula.

Write $\mathscr{S}_{K_p}$ for $\varprojlim_{K^p} \mathscr{S}_{K_pK^p}$. The Langlands–Rapoport Conjecture states that there is a $G(\mathbb{A}_f^p) \times \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-equivariant bijection

$$\mathscr{S}_{K_p}(\overline{\mathbb{F}}_q) \xrightarrow{\sim} \coprod_\phi \varprojlim_{K^p} I_\phi(\mathbb{Q})\backslash X(\phi)/K^p.$$

Here $\phi$ runs through conjugacy classes of admissible morphisms from a pro-(Galois gerb) $\mathfrak{Q}$ over $\mathbb{Q}$, called the *quasi-motivic gerb*, to the neutral gerb associated with $G$. For each admissible morphism $\phi$, we have a reductive group $I_\phi$ over $\mathbb{Q}$, and a set $X(\phi)$ equipped with commuting actions by $I_\phi(\mathbb{A}_f)$, $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, and $G(\mathbb{A}_f^p)$.

Currently, the Langlands–Rapoport Conjecture is open even for the Siegel modular varieties. (For some quaternionic Shimura varieties the conjecture has been proved by Reimann [Rei97].) In [Kis17], a weaker version of the conjecture is proved for the canonical integral models of Shimura varieties of abelian type, which are constructed in [Kis10] for $p > 2$ and in [KMP16] for $p = 2$, and are shown to satisfy (0.3.1) in [LS18]. (The assumption that $p > 2$ in [Kis17] can be dropped; see the proof of Theorem 6.2.4.) In this weaker version, the set

$$\varprojlim_{K^p} I_\phi(\mathbb{Q})\backslash X(\phi)/K^p$$

is replaced by

$$\varprojlim_{K^p} I_\phi(\mathbb{Q})_{\tau(\phi)}\backslash X(\phi)/K^p,$$

where $\tau(\phi)$ is an unspecified element of $I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$, and $I_\phi(\mathbb{Q})_{\tau(\phi)}$ is the image of $I_\phi(\mathbb{Q})$ under

$$I_\phi(\mathbb{Q}) \hookrightarrow I_\phi(\mathbb{A}_f) \xrightarrow{\mathrm{Int}(\tau(\phi))} I_\phi(\mathbb{A}_f).$$

It turns out that in order to deduce (0.1.3) from such a weaker statement, one must have better control of the elements $\tau(\phi)$. We formulate the desiderata in what we call the "Langlands–Rapoport–$\tau$ Conjecture".

We introduce some definitions in order to state the conjecture. For each admissible morphism $\phi$, we have the *algebraic part* $\phi^\Delta$ of $\phi$, which is a $\overline{\mathbb{Q}}$-homomorphism from a pro-torus $\mathfrak{Q}^\Delta$ to $G_{\overline{\mathbb{Q}}}$. The double quotient set

$$\mathcal{H}(\phi) := I_\phi(\mathbb{A}_f)\backslash I_\phi^{\mathrm{ad}}(\mathbb{A}_f)/I_\phi^{\mathrm{ad}}(\mathbb{Q})$$

is an abelian group, and up to canonical isomorphism it depends only on the $G(\overline{\mathbb{Q}})$-conjugacy class of $\phi^\Delta$. For each maximal torus $T$ in $I_\phi$, write $\mathcal{H}(\phi)_T$ for the cokernel of the localization map

$$\ker\left(\mathbf{H}^1(\mathbb{Q},T)\to\mathbf{H}^1(\mathbb{R},T)\oplus\mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q},G)\right)\longrightarrow\mathbf{H}^1(\mathbb{A}_f,T).$$

There is a natural homomorphism

$$\mathcal{H}(\phi)\longrightarrow\mathcal{H}(\phi)_T,$$

see Definition 2.6.19.

**Conjecture 1** ("Langlands–Rapoport–$\tau$"; see Conjecture 2.7.3)**.** *There is a bijection*

$$\mathscr{S}_{K_p}(\overline{\mathbb{F}}_q)\xrightarrow{\sim}\coprod_\phi\varprojlim_{K^p}I_\phi(\mathbb{Q})_{\tau(\phi)}\backslash X(\phi)/K^p,$$

*which is $G(\mathbb{A}_f^p)\times\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-equivariant, with respect to elements $\tau(\phi)\in I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$ satisfying the following conditions.*

(i) *The image of $\tau(\phi)$ in $\mathcal{H}(\phi)$ depends only on the $G(\overline{\mathbb{Q}})$-conjugacy class of $\phi^\Delta$.*
(ii) *For each maximal torus $T$ in $I_\phi$, the image of $\tau(\phi)$ in $\mathcal{H}(\phi)$ lies in the kernel of $\mathcal{H}(\phi)\to\mathcal{H}(\phi)_T$.*

Note that the original Langlands–Rapoport Conjecture implies Conjecture 1, as we can take all $\tau(\phi)$ to be 1. Also Conjecture 1 is stronger than the version of Langlands–Rapoport proved in [Kis17], as two non-trivial conditions on $\tau(\phi)$ are imposed.

**Theorem 4** (see Theorem 2.7.4)**.** *Conjecture 1 implies (0.1.3).*

The proof of Theorem 4 is group theoretic in nature (and works without the abelian type assumption). Some of the key ingredients come from [Kot84a] and [LR87, §5]. If one assumes the original Langlands–Rapoport Conjecture, that $G_{\mathrm{der}}$ is simply connected, and that every admissible morphism factors through the pseudo-motivic gerb, then the proof of (0.1.3) is essentially given in *loc. cit.*, as explained in [Mil92]. However, our proof of Theorem 4 does not logically follow from [LR87] or [Mil92], as we have the following new features:

- We need to show that the possibly non-trivial elements $\tau(\phi)$ do not affect the desired "point counting" on $\coprod_\phi\varprojlim_{K^p}I_\phi(\mathbb{Q})_{\tau(\phi)}\backslash X(\phi)/K^p$, as long as they satisfy the two conditions in Conjecture 1.
- We use the corrected construction of the quasi-motivic gerb $\mathfrak{Q}$ given by Reimann [Rei97], and do not assume that $G_{\mathrm{der}}$ is simply connected. As a result several definitions and arguments in [LR87, §5] need to be modified. (In the general case of abelian type, it is not enough to work with the pseudo-motivic gerb as is done in [Mil92].)

- We need to work with the more refined set $\coprod_{\gamma_0 \in \Sigma} \mathfrak{KP}(\gamma_0, q^m)$ as opposed to $\mathfrak{KP}_{\mathrm{cla}}(q^m)/\sim$, for the reasons explained below Theorem 2.

The logical relations between the various conjectures are depicted in the following diagram. All the implications are valid without the abelian type assumption.

$$\begin{tikzcd}
& \boxed{\text{weak LR Conj. as in [Kis17]}} \\
\boxed{\text{LR Conj.}} \arrow[r, Rightarrow] & \boxed{\text{LR–}\tau\text{ Conj. (Conj. 1)}} \arrow[u, Rightarrow] \arrow[d, Rightarrow, "\text{Thm. 4}"] \\
& \boxed{\text{point counting formula (0.1.3)}} \arrow[d, Rightarrow, "\text{Thm. 3}"] \\
& \boxed{\text{stable trace formula (0.1.1)}}
\end{tikzcd}$$

**Theorem 5** (see Theorem 6.3.5)**.** *If $(G, X)$ is of abelian type, then Conjecture 1 holds with respect to the canonical integral models.*

Theorem 2 follows from Theorems 4 and 5. In the rest of the introduction we discuss the proof of Theorem 5.

0.4. **The conjecture in the case of Hodge type.** Questions about Shimura varieties of abelian type can often be reduced to the same questions for Shimura varieties of Hodge type, plus some additional information on connected components. For instance, this is what is done in [Kis17] for the weak form of the Langlands–Rapoport conjecture. In Conjecture 1, we have imposed the minimal set of conditions that allow one to deduce the point counting formula (0.1.3) from the conjecture (see Theorem 4), regardless of the type of the Shimura datum. However, one may strengthen the conjecture by requiring certain compatibility conditions with connected components. It is this stronger version of Conjecture 1 which we prove in the case of Hodge type. We then use this to prove Conjecture 1 in the general case of abelian type.

We now discuss some key ideas in the proof of Conjecture 1 for a $(G, X)$ of Hodge type, and indicate the kind of strengthening we obtain. We postpone to §0.6 the explanation of how our results in the case of Hodge type imply Conjecture 1 in the general case of abelian type.

By the theory of integral models in [Kis10], after fixing a suitable embedding of $(G, X)$ into a Siegel Shimura datum, for each $x \in \mathscr{S}_{K_p}(\overline{\mathbb{F}}_q)$ we obtain an abelian variety $\mathcal{A}_x$ (up to prime-to-$p$ isogeny) over the residue field of $x$, together with tensors over the $\mathbb{A}_f^p$-Tate module and over the (integral) Dieudonné module of $\mathcal{A}_x$. Recall that these tensors arise by specializing Hodge cycles on abelian varieties over points in the generic fiber of $\mathscr{S}_{K_p}$. The set $\mathscr{S}_{K_p}(\overline{\mathbb{F}}_q)$ is partitioned into *isogeny classes*, where two points $x, x'$ are called isogenous if there exists a quasi-isogeny $\mathcal{A}_x \to \mathcal{A}_{x'}$ preserving the tensors.

Let $\mathcal{G}$ be the reductive group scheme over $\mathbb{Z}_p$ corresponding to $K_p$. For $x \in \mathscr{S}_{K_p}(\overline{\mathbb{F}}_q)$, the relative Frobenius on the $\mathbb{A}_f^p$-Tate module and the absolute Frobenius on the Dieudonné module of $\mathcal{A}_x$ give rise to an element of

$$G^{\sim} := \left( \varinjlim_n G(\mathbb{A}_f^p) \right) \times \left\{ \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})\text{-}\sigma\text{-conjugacy classes in } G(\mathbb{Q}_p^{\mathrm{ur}}) \right\},$$

where the direct limit is over positive integers $n$ ordered by divisibility and with respect to the transition maps $\gamma \mapsto \gamma^{n'/n}$ for $n|n'$.

As we have already indicated, in the current case of Hodge type, we would like to keep track of some information about connected components. For technical reasons we work with a set $\pi^*(G, X)$ that is equipped with a map from (but is not equal to) the set of connected components of $\mathscr{S}_{K_p, \overline{\mathbb{F}}_q}$. We then have a natural map

$$\mathscr{S}_{K_p}(\overline{\mathbb{F}}_q) \longrightarrow G^{\sim} \times \pi^*(G, X).$$

Analogously, for each admissible morphism $\phi$, we have a natural map

$$X(\phi) \longrightarrow G^{\sim} \times \pi^*(G, X).$$

**Definition 1.** By an *amicable pair*, we mean a pair $(\phi, \mathscr{I})$ consisting of an admissible morphism $\phi$ and an isogeny class $\mathscr{I} \subset \mathscr{S}_{K_p}(\overline{\mathbb{F}}_q)$ such that the images of $X(\phi)$ and $\mathscr{I}$ in $G^{\sim} \times \pi^*(G, X)$ have non-empty intersection.

**Proposition 1** (see Theorem 5.13.9)**.** *Let $\mathbb{I}$ be the set of isogeny classes, and let $\mathbb{J}$ be the set of admissible morphisms up to conjugacy.*[1] *There exists a bijection $\mathscr{B} : \mathbb{J} \xrightarrow{\sim} \mathbb{I}$ such that for each $\phi \in \mathbb{J}$ the pair $(\phi, \mathscr{B}(\phi))$ is amicable, and such that $\mathscr{B}$ is equivariant with respect to Galois cohomological twistings.*

We explain the last requirement on $\mathscr{B}$. The Galois cohomological twisting on $\mathbb{I}$ is defined by twisting a $\mathbb{Q}$-isogeny class of abelian varieties (with the additional tensors) in its $\overline{\mathbb{Q}}$-isogeny class, and the Galois cohomological twisting on $\mathbb{J}$ is defined by replacing an admissible morphism $\phi$ with another admissible morphism $\phi'$ such that $\phi^{\Delta} = \phi'^{,\Delta}$. For an arbitrary amicable pair $(\phi, \mathscr{I})$, the set of Galois cohomology classes that can be used to twist $\phi$ is canonically identified with the corresponding set for $\mathscr{I}$. Since $(\phi, \mathscr{B}(\phi))$ is required to be amicable, it makes sense to require that $\mathscr{B}$ is equivariant with respect to the two twisting operations.

We now explain the proof of Proposition 1. We make use of the following diagram:

(0.4.1)
$$\begin{tikzcd}
& \{\text{special point data}\} \arrow[d, dashed] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
& \{\text{amicable pairs}\} \arrow[dl] \arrow[dr] & \\
\mathbb{J} & & \mathbb{I}
\end{tikzcd}$$

Here a *special point datum* refers to a triple $(T, i, h)$, where $(T, h)$ is a Shimura datum on a torus, and $i$ is an embedding of Shimura data $(T, h) \to (G, X)$ such that $i(T)$ is a maximal torus in $G$. Given $(T, i, h)$, we obtain a special point $\widetilde{x}_{(T,i,h)}$ in the generic fiber of $\mathscr{S}_{K_p}$, which specializes to a point $x_{(T,i,h)} \in \mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$. Define $\mathscr{I}(T, i, h)$ to be the isogeny class of $x_{(T,i,h)}$. Similarly, $(T, i, h)$ gives rise to an

[1] In the rest of the introduction we shall deliberately conflate an admissible morphism with its conjugacy class to simplify the exposition.

admissible morphism $\phi(T,i,h)$. We thus have maps from the set of special point data to $\mathbb{I}$ and $\mathbb{J}$. The following proposition asserts that we can fill in the dashed arrow in the above diagram.

**Proposition 2** (see Corollary 5.11.9)**.** *Let $(T,i,h)$ be a special point datum. Then the pair $(\phi(T,i,h),\mathscr{I}(T,i,h))$ is amicable.*

The key arithmetic input to the proof of Proposition 2 is the construction of certain *integral special points* and the computation of their images in $G^{\sim}\times\pi^*(G,X)$. We first sketch the construction. Attached to the special point $\widetilde{x}_{(T,i,h)}$ we have a CM abelian variety $\widetilde{A}$ defined over some number field, together with a canonical $\mathcal{G}$-representation on the dual $p$-adic Tate module

$$\Lambda := T_p(\widetilde{A})^\vee.$$

Let $\mathcal{T}^\circ$ be the connected Néron model of $T_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$. Then $T_{\mathbb{Q}_p}$ acts on $\Lambda[1/p]$ via $i : T_{\mathbb{Q}_p} \to G_{\mathbb{Q}_p}$, and we fix a $\mathcal{T}^\circ$-stable $\mathbb{Z}_p$-lattice $\Lambda'$ in $\Lambda[1/p]$. The lattice $\Lambda'$ corresponds to a $\mathbb{Q}_p$-isogeny $\iota : \widetilde{A} \to \widetilde{A}'$ between CM abelian varieties. We can choose a finite extension $F/\mathbb{Q}_p$ such that both $\widetilde{A}$ and $\widetilde{A}'$ are defined over $F$ and have good reduction, and such that $\iota$ is also defined over $F$. Let $M$ and $M'$ be the base changes to $\mathbb{Z}_p^{\mathrm{ur}}$ of the contravariant Dieudonné modules of the reductions of $\widetilde{A}$ and $\widetilde{A}'$, respectively. The reduction of $\iota$ induces a Frobenius-equivariant isomorphism $\iota^* : M'[1/p] \xrightarrow{\sim} M[1/p]$.

Using some integral $p$-adic Hodge theory to be discussed in §0.5 below, we construct a $\mathbb{Z}_p^{\mathrm{ur}}$-linear isomorphism

$$\eta : M' \xrightarrow{\sim} \Lambda' \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{ur}}, \tag{0.4.2}$$

which is canonical up to automorphisms of the right hand side induced by elements of $\mathcal{T}^\circ(\mathbb{Z}_p^{\mathrm{ur}})$. Let $M''$ be the image of the composite map

$$\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{ur}} \hookrightarrow \Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p^{\mathrm{ur}} \cong \Lambda' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p^{\mathrm{ur}} \xrightarrow{\eta^{-1}} M'[1/p] \xrightarrow{\iota^*} M[1/p].$$

Then $M''$ is a $\mathbb{Z}_p^{\mathrm{ur}}$-lattice in $M[1/p]$. Moreover, we have a $\mathcal{G}_{\mathbb{Z}_p^{\mathrm{ur}}}$-representation on $M$ (canonical up to $\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$-conjugation), and $M'' \subset M[1/p]$ is a translate of $M$ by an element $g \in G(\mathbb{Q}_p^{\mathrm{ur}})$. [2]

We would like to define a point of $\mathscr{I}(T,i,h)$ corresponding to $M''$, but this is not possible in general, as $g \in G(\mathbb{Q}_p^{\mathrm{ur}})$ may not satisfy the defining condition of an affine Deligne–Lusztig set, so that $g \cdot M$ may not be the Dieudonné module of an abelian variety. To remedy this, for each isogeny class $\mathscr{I}$ we introduce a canonical enlargement $\mathscr{I}^* \supset \mathscr{I}$, and extend the map $\mathscr{I} \to G^{\sim} \times \pi^*(G,X)$ to $\mathscr{I}^*$. In the current situation, every $G(\mathbb{Q}_p^{\mathrm{ur}})$-translate of $M$ defines an element of $\mathscr{I}(T,i,h)^*$, and in particular we view $M''$ as an element of $\mathscr{I}(T,i,h)^*$, called an *integral special point*.

We then need to compute the image of $M'' \in \mathscr{I}(T,i,h)^*$ in $G^{\sim}\times\pi^*(G,X)$. This is based on the following result, whose proof will be discussed in §0.5 below.

**Proposition 3.** *Write $\Gamma_{p,0}$ for the inertia subgroup of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$*. Then we have*

[2] By contrast, since $\Lambda'$ is not required to be a $G(\mathbb{Q}_p)$-translate of $\Lambda$, there is no reason to expect that the image of $M'$ under $\iota^*$ is a $G(\mathbb{Q}_p^{\mathrm{ur}})$-translate of $M$.

(i) *The Frobenius on* $M'[1/p]$ *corresponds via* $\eta$ *to* $\delta\sigma$, *with* $\delta \in T(\mathbb{Q}_p^{\mathrm{ur}}) \subset \mathrm{GL}(\Lambda' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p^{\mathrm{ur}})$ *such that the image of* $\delta$ *in* $X_*(T)_{\Gamma_{p,0}}$ *under the Kottwitz homomorphism is equal to the natural image of* $\mu_h$, *the Hodge cocharacter of* $h : \mathbb{S} \to T_{\mathbb{R}}$.

(ii) $M''$ *is a* $G_{\mathrm{der}}(\mathbb{Q}_p^{\mathrm{ur}})$-*translate of* $M$ *inside* $M[1/p]$.

Using part (i) of the proposition and the Shimura–Taniyama reciprocity law, we obtain an explicit description of the image of $M''$ in $G^{\sim}$. Using part (ii), we prove that the image of $M''$ in $\pi^*(G, X)$ is equal to that of $x_{(T,i,h)} \in \mathscr{I}(T, i, h)$.

For Galois gerbs there is parallel construction of integral special points. By comparing the images of integral special points in $G^{\sim} \times \pi^*(G, X)$ in the two contexts, we obtain Proposition 2.

Given Proposition 2, Proposition 1 is proved in two stages. In the first stage, we construct subsets $\mathbb{J}' \subset \mathbb{J}$ and $\mathbb{I}' \subset \mathbb{I}$ and a bijection $\mathscr{B} : \mathbb{J}' \xrightarrow{\sim} \mathbb{I}'$ such that for each $\phi \in \mathbb{J}'$ there exists a special point datum which induces both $\phi$ and $\mathscr{B}(\phi)$. By Proposition 2 we know that $(\phi, \mathscr{B}(\phi))$ is amicable for such $\phi$. In the second stage, we use Galois cohomological twisting on both sides to extend $\mathscr{B}$ to a bijection $\mathbb{J} \xrightarrow{\sim} \mathbb{I}$. For this, we need to show that if $(\phi, \mathscr{I})$ is an amicable pair then, after we twist $\phi$ and $\mathscr{I}$ by a common Galois cohomology class, we again obtain an amicable pair. To show this we again utilize integral special points. More precisely, we use the fact that for each amicable pair $(\phi, \mathscr{I})$ and each maximal torus $T \subset I_\phi$, there exist two special point data of the form $(T, i, h)$ and $(T, i', h)$ such that

$$\mathscr{I} = \mathscr{I}(T, i, h) \quad \text{and} \quad \phi = \phi(T, i', h). \tag{0.4.3}$$

(Here the second equality is up to conjugacy.) This fact follows from the special point lifting theorem in [Kis17] and a similar result for Galois gerbs, and it allows us to understand arbitrary Galois cohomological twistings by studying the twisting of integral special points.

Note that Proposition 1 does not yet give a bijection $\mathbb{J} \xrightarrow{\sim} \mathbb{I}$ compatible with the diagram (0.4.1). It remains an interesting open problem to show the existence of such a compatible bijection (which is necessarily unique). We expect that the solution would lead to better understanding of the Langlands–Rapoport Conjecture.

Having shown Proposition 1, we proceed to prove Conjecture 1 in the case of Hodge type as follows: Fix a bijection $\mathscr{B}$ as in Proposition 1. For each $\phi \in \mathbb{J}$, using that $(\phi, \mathscr{B}(\phi))$ is amicable, we can find an element $\tau(\phi) \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$ and a $G(\mathbb{A}_f^p) \times \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-equivariant bijection

$$f_\phi : \mathscr{B}(\phi) \xrightarrow{\sim} \varprojlim_{K^p} I_\phi(\mathbb{Q})_{\tau(\phi)} \backslash X(\phi)/K^p$$

which commutes with the natural maps from the two sides to $G^{\sim} \times \pi^*(G, X)$. We also require that the map $X(\phi) \to \mathscr{B}(\phi)$ induced by $f_\phi^{-1}$ satisfies some natural equivariance conditions. Here neither $\tau(\phi)$ nor $f_\phi$ is unique, but essentially our requirements on $f_\phi$ restrict the ambiguity of $\tau(\phi)$ such that the image of $\tau(\phi)$ in $\mathcal{H}(\phi)$ depends only on the pair $(\phi, \mathscr{B}(\phi))$ and thus only on $\phi$ if we keep $\mathscr{B}$ fixed. We then need to show that these canonical elements of $\mathcal{H}(\phi)$ for all $\phi$ satisfy the two conditions in Conjecture 1.

Condition (i) follows from the fact that $\mathscr{B}$ is compatible with Galois cohomological twistings. It is proved as a byproduct of the second stage of the proof of Proposition 2.

Condition (ii) follows from the fact that $(\phi, \mathscr{B}(\phi))$ is amicable for all $\phi$, and that for each amicable pair $(\phi, \mathscr{I})$ and each maximal torus $T$ in $I_\phi$ we can arrange (0.4.3).

We now discuss the strengthening of Conjecture 1 that we obtain in the case of Hodge type. We have already mentioned that the requirements on $f_\phi$ restrict the ambiguity of $\tau(\phi) \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$ such that it has a well-defined image in $\mathcal{H}(\phi)$. In fact we can do better. Let $Z_\phi^\dagger$ be the intersection of the center of $I_\phi$ with $G_{\mathrm{der}}$, which is a $\mathbb{Q}$-subgroup of $I_\phi$. We define $\mathfrak{H}(\phi)$ to be the quotient of

$$\operatorname{coker}\left( \ker\left( \mathbf{H}^1(\mathbb{Q}, Z_\phi^\dagger) \;\to\; \mathbf{H}^1(\mathbb{R}, Z_\phi^\dagger) \,\oplus\, \mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}, G_{\mathrm{der}})\right) \;\longrightarrow\; \mathbf{H}^1(\mathbb{A}_f, Z_\phi^\dagger)\right)$$

by the image of a certain map $\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p) \to \mathbf{H}^1(\mathbb{A}_f, Z_\phi^\dagger)$; see §5.10.5. Using especially the fact that $f_\phi$ is compatible with the maps to $\pi^*(G, X)$, we know that the image of $\tau(\phi)$ in $\mathfrak{H}(\phi)$ is also well defined, i.e., it depends only on the pair $(\phi, \mathscr{B}(\phi))$. We prove a strengthened version of condition (i) in Conjecture 1, where $\mathcal{H}(\phi)$ is replaced by $\mathfrak{H}(\phi) \oplus \mathcal{H}(\phi)$. Similarly, for each maximal torus $T$ in $I_\phi$, writing $T^\dagger := T \cap G_{\mathrm{der}}$ we define $\mathfrak{H}(\phi)_T$ to be the quotient of

$$\operatorname{coker}\left( \ker\left( \mathbf{H}^1(\mathbb{Q}, T^\dagger) \;\to\; \mathbf{H}^1(\mathbb{R}, T^\dagger) \,\oplus\, \mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}, G_{\mathrm{der}})\right) \;\longrightarrow\; \mathbf{H}^1(\mathbb{A}_f, T^\dagger)\right)$$

by the image of a certain map $\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p) \to \mathbf{H}^1(\mathbb{A}_f, T^\dagger)$; see §5.12.1. In analogy with the natural map $\mathcal{H}(\phi) \to \mathcal{H}(\phi)_T$ we have a natural map $\mathfrak{H}(\phi) \to \mathfrak{H}(\phi)_T$. We prove a strengthened version of condition (ii) in Conjecture 1 where the kernel of $\mathcal{H}(\phi) \to \mathcal{H}(\phi)_T$ is replaced by the kernel of $\mathfrak{H}(\phi) \oplus \mathcal{H}(\phi) \to \mathfrak{H}(\phi)_T \oplus \mathcal{H}(\phi)_T$.

0.5. **Integral $p$-adic Hodge theory.** We now explain the ingredients from $p$-adic Hodge theory that go into the construction and study of the integral special points in §0.4.

Let $\mathcal{P}$ be a parahoric group scheme over $\mathbb{Z}_p$. Write $\mathrm{Rep}\mathcal{P}$ for the category of $\mathcal{P}$-representations on finite free $\mathbb{Z}_p$-modules. Let $F/\mathbb{Q}_p$ be a finite extension with residue field $k$. Write $W = W(k)$ and $F_0 = W(k)[1/p]$. Consider a crystalline representation $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathcal{P}(\mathbb{Z}_p)$. Then for each $\Lambda \in \mathrm{Rep}\mathcal{P}$, we can view $\Lambda$ as a $\mathrm{Gal}(\overline{F}/F)$-stable lattice in the crystalline representation $\Lambda[1/p]$. Using the functor $\mathfrak{M}$ in [Kis06] we obtain from $\Lambda$ a pair

$$(M_{\mathrm{cris}}(\Lambda), \varphi),$$

where $M_{\mathrm{cris}}(\Lambda) := \mathfrak{M}(\Lambda) \otimes_{W[\![u]\!]} W$ is a finite free $W$-module and $\varphi$ is a $\sigma$-linear automorphism of $M_{\mathrm{cris}}(\Lambda)[1/p]$, called the Frobenius. This yields a $\otimes$-functor $\Upsilon_\rho$ from $\mathrm{Rep}\mathcal{P}$ to the category of pairs as above. Using a purity result proved recently by Anschütz [Ans18][3] we show that the $\otimes$-functor $\Lambda \mapsto M_{\mathrm{cris}}(\Lambda)$ (where we forget the Frobenius) is isomorphic to $\Lambda \mapsto \Lambda \otimes_{\mathbb{Z}_p} W$. We denote by $Y(\Upsilon_\rho)^\circ$ the $\mathcal{P}(W)$-torsor of isomorphisms between the two $\otimes$-functors.

*Remark* 1. For our purposes it is important to know that the formation of $M_{\mathrm{cris}}(\Lambda)$ for a crystalline $\mathrm{Gal}(\overline{F}/F)$-lattice $\Lambda$ is compatible with replacing $F$ by an arbitrary finite extension of $F$. This has been proved by T. Liu [Liu18].

[3]The special cases for connected reductive group schemes and parahoric group schemes with tamely ramified generic fibers were previously shown in [Kis10] and [KP18] respectively.

If we fix an element of $Y(\Upsilon_\rho)^\circ$, then the Frobenius structure on $\Upsilon_\rho$ is given by an element $\delta \in \mathcal{P}(F_0)$. We prove the following result about $\delta$.

**Proposition 4** (see Proposition 4.4.7)**.** *The image of $\delta$ under the Kottwitz homomorphism $\mathcal{P}(\mathbb{Q}_p^{\mathrm{ur}}) \to \pi_1(\mathcal{P}_{\mathbb{Q}_p})_{\Gamma_{p,0}}$ is equal to the image of the Hodge–Tate cocharacter for $\rho$.*

Before discussing the proof of Proposition 4, we explain how the above local theory is applied in the global situation of §0.4, namely for the construction of (0.4.2) and the proof of Proposition 3. We use the notation as in the paragraph preceding (0.4.2). Up to enlarging $F$, the natural action of $\mathrm{Gal}(\overline{F}/F)$ on $\Lambda'$ is induced by a crystalline representation $\rho_T : \mathrm{Gal}(\overline{F}/F) \to \mathcal{T}^\circ(\mathbb{Z}_p)$. We apply the local theory to $\rho_T$ and the group scheme $\mathcal{T}^\circ$. On choosing an element of the $\mathcal{T}^\circ(W)$-torsor $Y(\Upsilon_{\rho_T})^\circ$, we obtain an isomorphism $M_{\mathrm{cris}}(\Lambda') \xrightarrow{\sim} \Lambda' \otimes_{\mathbb{Z}_p} W$. On the other hand, we have a canonical *integral comparison isomorphism* $M_{\mathrm{cris}}(\Lambda') \otimes_W \mathbb{Z}_p^{\mathrm{ur}} \xrightarrow{\sim} M'$. See §5.2.2 for references. We define $\eta$ to be the composition of the two isomorphisms. Now part (i) of Proposition 3 follows from Proposition 4. For part (ii) of Proposition 3, we use that the $\mathrm{Gal}(\overline{F}/F)$-action on $\Lambda$ is induced by a crystalline representation $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$, and apply the local theory to $\rho$ and the group scheme $\mathcal{G}$. There is no direct map between the $\mathcal{T}^\circ(W)$-torsor $Y(\Upsilon_{\rho_T})^\circ$ and the $\mathcal{G}(W)$-torsor $Y(\Upsilon_\rho)^\circ$ since there is (in general) no $\mathbb{Z}_p$-homomorphism $\mathcal{T}^\circ \to \mathcal{G}$. Nevertheless, we have natural $\mathbb{Z}_p$-homomorphisms $\mathcal{G} \to \mathcal{G}^{\mathrm{ab}}$ and $\mathcal{T}^\circ \to \mathcal{G}^{\mathrm{ab}}$, and $\rho$ and $\rho_T$ induce the same crystalline representation $\rho^{\mathrm{ab}} : \mathrm{Gal}(\overline{F}/F) \to \mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p)$. We can apply the local theory for the third time, to $\rho^{\mathrm{ab}}$ and the group scheme $\mathcal{G}^{\mathrm{ab}}$. Comparing each of $Y(\Upsilon_\rho)^\circ$ and $Y(\Upsilon_{\rho_T})^\circ$ with the $\mathcal{G}^{\mathrm{ab}}(W)$-torsor $Y(\Upsilon_{\rho^{\mathrm{ab}}})$, we obtain part (ii). See Proposition 4.5.2 and Remark 4.5.3 for details of this argument.

We now explain the proof of Proposition 4. As usual we write $\mathcal{O}_F$ as $W[\![u]\!]/E$, where $E$ is an Eisenstein polynomial in $W[u]$. Write $\mathfrak{S}$ for $W[\![u]\!]$. We construct a homomorphism

$$\mathcal{P}(\operatorname{Frac}\mathfrak{S}) \longrightarrow \pi_1(P)_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/K_0)}, \tag{0.5.1}$$

which can be viewed as an $E$-adic variant of the Kottwitz homomorphism. The proof of Proposition 4 has the following two steps.

(i) We show that $\delta$ comes from an element $\delta_{\mathfrak{S}} \in \mathcal{P}(\mathfrak{S}[1/E])$ under the specialization $u \mapsto 0$, and that the image of $\delta_{\mathfrak{S}}$ under (0.5.1) is equal to the image of the Hodge–Tate cocharacter for $\rho$.

(ii) We show that if an element $g \in \mathcal{P}(\mathfrak{S}[1/E]) \subset \mathcal{P}(\operatorname{Frac}\mathfrak{S})$ specializes to $g_0 \in \mathcal{P}(K_0)$ under $u \mapsto 0$, then the image of $g$ under (0.5.1) is equal to the image of $g_0$ under the $p$-adic (i.e., classical) Kottwitz homomorphism.

In step (i), we use properties of the functor $\mathfrak{M}$ in [Kis06]. In both steps we make use of the following result about "Kottwitz homomorphisms in families", which may be of independent interest.

**Proposition 5** (see §1.3)**.** *Let $F$ be a field of characteristic $0$, $R$ an $F$-algebra, and $v$ a discrete valuation on $R$. Then for each reductive group $P$ over $F$ there is a natural map*

$$\kappa_P^v : P(R) \to \pi_1(P)_{\mathrm{Gal}(\overline{F}/F)},$$

*generalizing the Kottwitz homomorphism, which is given by $v : R^\times \to \mathbb{Z}$ when $P = \mathbb{G}_m$. Moreover we have*

(i) *Suppose that* $\operatorname{Spec} R$ *has trivial Picard group. For discrete valuations* $v_1, \cdots, v_n$ *on* $R$ *and integers* $a_1, \cdots, a_n$ *such that* $\sum_i a_v v_i$ *vanishes on* $R^\times$*, we have*

$$\sum_i a_i \kappa_P^{v_i}(P(R)) = 0$$

*for all reductive groups* $P$ *over* $F$*.*

(ii) *Suppose that* $R = F$*, and let* $v$ *be a discrete valuation on* $F$ *with valuation ring* $\mathcal{O}_F$*. For any reductive group* $P$ *over* $F$ *and any smooth affine group scheme* $\mathcal{P}$ *over* $\mathcal{O}_F$ *with connected fibers extending* $P$*, we have*

$$\kappa_P^v(\mathcal{P}(\mathcal{O}_F)) = 0.$$

0.6. **From Hodge type to abelian type.** We now explain how our proof of Conjecture 1 in the case of Hodge type, together with the strengthening discussed at the end of §0.4, implies the conjecture in the case of abelian type. For this we follow the reduction method in [Kis17], which takes as input a construction of the elements $\tau(\phi)$ in the case of Hodge type, and outputs a construction of them in the case of abelian type. Thus we only need to transport the properties of $\tau(\phi)$ proved in the case of Hodge type to the case of abelian type. Specifically, for a Shimura datum $(G_2, X_2)$ of abelian type that is of interest, we pick an auxiliary Shimura datum $(G, X)$ of Hodge type satisfying some standard compatibility conditions with $(G_2, X_2)$. We show that the strengthened version of Conjecture 1, which we have proved for $(G, X)$, implies the original Conjecture 1 for $(G_2, X_2)$, provided that $(G, X)$ satisfies the following technical hypothesis:

- The $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module $X_*(G^{\mathrm{ab}})$ is generated by the image of a Hodge cocharacter.

That for any $(G_2, X_2)$ we can indeed find such $(G, X)$ follows from a construction of Deligne.

For a fixed $(G_2, X_2)$, the auxiliary $(G, X)$ that we find does not, in general, have connected center, which violates an assumption in the reduction method in [Kis17]. For this reason, we need a generalization of [Kis17, Lem. 1.2.18] from reductive group schemes to (certain) parahoric group schemes; see Corollary 4.4.16. The proof again uses the local theory exposed in §0.5, as well as Proposition 5.

**Organization of the paper.** In §1, after some group theoretic preparation, we state the conjectural point counting formula as in (0.1.3) for a general Shimura datum (with hyperspecial level at $p$; see Conjecture 1.8.8). Of note is §1.3, in which we generalize the Kottwitz homomorphism to families, as discussed in Proposition 5 above.

In §2, we first recall the formalism of Galois gerbs and the Langlands–Rapoport Conjecture, and then state the Langlands–Rapoport–$\tau$ Conjecture (see Conjecture 2.7.3). For the formulation of the conjecture we need a general twisting construction for admissible morphisms. We study this in §2.6.

In §3, we show that the Langlands–Rapoport–$\tau$ Conjecture implies the desired point counting formula. The key step is the assignment of a Kottwitz parameter (i.e., a summation index in the point counting formula) to a gg LR pair (an object within the realm of Galois gerbs and the Langlands–Rapoport Conjecture). We study this construction in §3.5 in the presence of general twisting elements $\tau(\phi)$. In §3.6, we study the special case when the elements $\tau(\phi)$ are controlled as in the Langlands–Rapoport–$\tau$ Conjecture. Roughly speaking, we show that the set of Kottwitz parameters that can arise from this construction is unaffected by the presence of these twisting elements.

In §4, we develop the input from integral $p$-adic Hodge theory. We define *integral $F$-isocrystals with $G$-structure* in §4.2, and attach them to $G$-valued crystalline representations in §4.4 (for $G$ a $\mathbb{Z}_p$-group scheme satisfying "property KL"). Most of the content of this section is of a local nature, with the exception of the later parts of §4.3, where global abelian Galois representations related to the Shimura–Taniyama reciprocity law are considered.

In §5 and §6, we prove the Langlands–Rapoport–$\tau$ Conjecture for Shimura varieties of abelian type. In §5.12 and §5.13, we prove intermediate results for Shimura varieties of Hodge type. These results constitute the strengthened version of Langlands–Rapoport–$\tau$ mentioned above. As we have already explained, the crucial innovations needed for proving these results are the construction and study of integral special points. These are carried out in §5.7 and §5.10, in the geometric context and the Galois gerb context respectively. In §6.3, we combine the results proved in the case of Hodge type with the reduction method in [Kis17] and a construction of Deligne to prove the Langlands–Rapoport–$\tau$ Conjecture in the case of abelian type.

We devote §§7–8 to stabilization. In §7, we have preparatory discussions on central character data in §7.1, endoscopic data and $z$-extensions in §7.2, Galois cohomology invariants in §7.3, and the Langlands–Shelstad–Kottwitz transfer in §7.4. By implementing central characters and $z$-extensions, we make it unnecessary to assume any undesirable technical hypothesis such as cuspidality of $Z_G$ or simple connectedness of $G_{\mathrm{der}}$.

In §8, we present the stabilization in three steps following Kottwitz. We rewrite the point counting formula in terms of adelic $\kappa$-orbital integrals (§8.1), transfer $\kappa$-orbital integrals to stable orbital integrals on $z$-extensions of endoscopic groups (§8.2), and then finish by reorganizing the terms into the sum of stable distributions intrinsic to the endoscopic groups and their $z$-extensions (§8.3).

Finally, in §9, after some recollection of the general stable trace formula, we indicate what extra information and steps are needed, in addition to our main results, for understanding the cohomology of Shimura varieties unconditionally.

**Acknowledgments.** We thank T. Haines and R. Zhou for answering our questions on parahoric group schemes. We thank M. Rapoport for intriguing conversations on the Langlands–Rapoport Conjecture and useful suggestions. We thank M. Harris for his long-term interest in this project and encouragement.

M. K. is partially supported by NSF grant DMS-1902158. S. W. S. is partially supported by NSF grant DMS-1802039/2101688, NSF RTG grant DMS-1646385, and a Miller Professorship. Y. Z. is partially supported by NSF grant DMS-1802292, and by a startup grant at University of Maryland.

**Conventions.**

- When $G$ is a group acting on a set $X$ on the left, we denote the action map $G \times X \to X$ by $(\rho, x) \mapsto {}^{\rho}x$, or $(\rho, x) \mapsto \rho(x)$. We shall *not* use the notation $x^{\rho}$.
- If $g, h$ are elements of a group, we define $\mathrm{Int}(g)h$ to be $ghg^{-1}$.
- Given a field $F$, we denote by $\overline{F}$ a fixed algebraic closure. Throughout we fix field embeddings $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_v$ for all places $v$ of $\mathbb{Q}$. The Galois groups will

sometimes be abbreviated as follows:

$$\Gamma = \Gamma_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \quad \Gamma_v = \Gamma_{\mathbb{Q}_v} = \mathrm{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v).$$

Using the field embeddings fixed above, we view $\Gamma_v$ as a subgroup of $\Gamma$. When $v$ is non-archimedean, we write $\Gamma_{v,0}$ for the inertia subgroup of $\Gamma_v$.

- If $L$ is a subfield of $\overline{\mathbb{Q}}$ and $v$ is a place of $\mathbb{Q}$, we denote by $L_v$ the completion of $L$ inside $\overline{\mathbb{Q}}_v$, with respect to the fixed embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_v$.
- For a prime $p$, we denote by $\mathbb{Q}_p^{\mathrm{ur}}$ the maximal unramified extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$. We write $\mathbb{Q}_{p^n}$ for the degree $n$ unramified extension of $\mathbb{Q}_p$ in $\mathbb{Q}_p^{\mathrm{ur}}$, and write $\mathbb{Z}_{p^n}$ for its ring of integers. We write $\mathbb{Z}_p^{\mathrm{ur}}$ for the strict henselization of $\mathbb{Z}_p$, i.e., $\mathbb{Z}_p^{\mathrm{ur}} = \bigcup_n \mathbb{Z}_{p^n}$.
- For a prime $p$, we denote by $\breve{\mathbb{Q}}_p$ the completion of $\mathbb{Q}_p^{\mathrm{ur}}$, and denote by $\breve{\mathbb{Z}}_p$ its ring of integers. We fix an embedding $\overline{\mathbb{Q}}_p \hookrightarrow \overline{\breve{\mathbb{Q}}}_p$, and thereby identify $\Gamma_{\breve{\mathbb{Q}}_p} = \mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/\breve{\mathbb{Q}}_p)$ with the inertia subgroup $\Gamma_{p,0}$ of $\Gamma_p$. We denote by $\sigma$ the arithmetic $p$-Frobenius in $\mathrm{Aut}(\breve{\mathbb{Q}}_p)$.
- We denote by $\mathbb{A}, \mathbb{A}_f, \mathbb{A}_f^p$ respectively the adeles over $\mathbb{Q}$, the finite adeles over $\mathbb{Q}$, and the finite adeles away from $p$. We also denote by $\mathbb{A}_f^*$ the product $\mathbb{A}_f^p \times \mathbb{Q}_p^{\mathrm{ur}}$, when $p$ is clear in the context.
- For a perfect field $k$, we write $W(k)$ for the ring of Witt vectors over $k$. When $k = \mathbb{F}_{p^n}$ we identify $W(k)$ with $\mathbb{Z}_{p^n}$.
- By a *reductive group* over a field, we always mean a connected reductive group.
- For a connected reductive group $I$ over a field, we write $I_{\mathrm{der}}, I_{\mathrm{sc}}, I^{\mathrm{ad}}$ for the derived subgroup, the simply connected cover of the derived subgroup, and the adjoint group respectively. We define $I^{\mathrm{ab}}$ to be $I/I_{\mathrm{der}}$, the maximal torus quotient of $I$.
- For a reductive group $I$ over $\mathbb{R}$, we write $I(\mathbb{R})^+$ for the identity connected component of the real Lie group $I(\mathbb{R})$, and write $I(\mathbb{R})_+$ for the preimage of $I^{\mathrm{ad}}(\mathbb{R})^+$ under $I(\mathbb{R}) \to I^{\mathrm{ad}}(\mathbb{R})$. If $I$ is defined over $\mathbb{Q}$, we write $I(\mathbb{Q})^+$ for $I(\mathbb{Q}) \cap I(\mathbb{R})^+$, and write $I(\mathbb{Q})_+$ for $I(\mathbb{Q}) \cap I(\mathbb{R})_+$.
- All group cohomology classes or cocycles for profinite groups (e.g. Galois groups) are understood in the continuous sense. We shall denote a 1-cochain by $(g_\rho)_\rho$, or $\rho \mapsto g_\rho$, or simply $g_\rho$. A 1-cocycle $g_\rho$ satisfies $g_{\rho\sigma} = g_\rho {}^\rho g_\sigma$.
- Let $I$ be a linear algebraic group over a field $F$ of characteristic zero. Let $F'/F$ be a Galois extension. We denote by $Z^1(F'/F, I(F'))$ the set of continuous 1-cocycles $\mathrm{Gal}(F'/F) \to I(F')$. Denote by $\mathbf{H}^1(F'/F, I(F'))$ the corresponding cohomology set. When $F' = \overline{F}$ is an algebraic closure of $F$, we write $Z^1(F, I)$ and $\mathbf{H}^1(F, I)$. When $I$ is reductive, for $\tau \in I^{\mathrm{ad}}(F)$, its *image* in $\mathbf{H}^1(F, Z_I)$ is understood to be the class of the cocycle $\rho \mapsto \widetilde{\tau}^{-1} {}^\rho\widetilde{\tau}$, where $\widetilde{\tau} \in I(\overline{F})$ is an (arbitrary) lift of $\tau$.
- When $I$ is a connected reductive group over $F$, we denote by $I(F)_{\mathrm{ss}}$ the set of semi-simple elements of $I(F)$.
- We denote by $\mathbb{S}$ the Deligne torus $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$.

# Part 1. Axiomatic point counting

## 1. The point counting formula

### 1.1. Abelianized Galois cohomology.

**1.1.1.** Let $F$ be a field of characteristic zero, $\overline{F}$ an algebraic closure, and $\Gamma = \mathrm{Gal}(\overline{F}/F)$. Let $\Gamma$-Mod (resp. $\Gamma$-$\mathrm{Mod}_f$) be the abelian category of discrete $\mathbb{Z}[\Gamma]$-modules that are finitely generated (resp. finite free) over $\mathbb{Z}$. Let $\mathrm{Mult}(F)$ be the abelian category of algebraic groups of multiplicative type over $F$, and $\mathrm{Tori}(F) \subset \mathrm{Mult}(F)$ the full subcategory of tori. Each of the above four categories is naturally an exact category, and we have the corresponding bounded and unbounded derived categories; see for instance [Kel96].

We have an exact anti-equivalence of abelian categories

$$X^* : \mathrm{Mult}(F) \longrightarrow \Gamma\text{-Mod},$$

sending each multiplicative group to its group of characters. We set

$$X_* = \mathrm{Hom}_{\mathbb{Z}}(\cdot, \mathbb{Z}) \circ X^* : \mathrm{Mult}(F) \longrightarrow \Gamma\text{-Mod}_f,$$

sending each multiplicative group to its group of cocharacters.

**Proposition 1.1.2.** *We have a commutative diagram (that is, commuting up to natural isomorphisms) of equivalences of triangulated categories*

$$\begin{tikzcd}
\mathcal{D}^b(\mathrm{Tori}(F)) \arrow[r, "\sim"] \arrow[d, "\sim"] & \mathcal{D}^b(\Gamma\text{-Mod}_f) \arrow[d, "\sim"] \\
\mathcal{D}^b(\mathrm{Mult}(F)) \arrow[r, "\sim"] & \mathcal{D}^b(\Gamma\text{-Mod})
\end{tikzcd}$$

*where the top functor is induced by $X_*$, and the bottom functor realizes the derived functor $\mathrm{R}X_*$ of $X_* : \mathrm{Mult}(F) \to \Gamma$-Mod.*

*Proof.* The vertical functors are induced by the inclusions $\Gamma$-$\mathrm{Mod}_f \subset \Gamma$-Mod and $\mathrm{Tori}(F) \subset \mathrm{Mult}(F)$. Note that every $M \in \Gamma$-$\mathrm{Mod}_f$ is a $\mathbb{Z}[\Gamma/U]$-module for some open normal subgroup $U \subset \Gamma$. Using this one checks that the fully exact subcategory $\Gamma$-$\mathrm{Mod}_f \subset \Gamma$-Mod satisfies the dual versions of conditions C1 and C2 in [Kel96, §12]. By [Kel96, Theorem 12.1], the canonical functor $\mathcal{D}^-(\Gamma\text{-Mod}_f) \to \mathcal{D}^-(\Gamma\text{-Mod})$ is an equivalence. This implies that the vertical functor on the right is an equivalence, in view of the observation that the acyclicity of a complex in $\Gamma$-$\mathrm{Mod}_f$ at a given degree (in the sense of [Kel96, §11]) is equivalent to the vanishing of the cohomology at the same degree computed in $\Gamma$-Mod. Now that the vertical functor on the left is an equivalence follows by applying the exact anti-equivalence $X^*$, or more precisely its quasi-inverse given by $\underline{\mathrm{Hom}}_{\mathrm{Spec}\, F}(\_, \mathbb{G}_m)$.

Since $X_* : \mathrm{Tori}(F) \to \Gamma$-$\mathrm{Mod}_f$ is an exact equivalence, the functor on the top is an equivalence. Now we can fill in the equivalence in the bottom row. The exactness also implies that $X_* : \mathrm{Tori}(F) \to \Gamma$-$\mathrm{Mod}_f$ preserves acyclicity of bounded complexes, which implies that the bottom row realizes the derived functor of $X_*$, by [Har66, §I, Thm. 5.1]. $\square$

**Definition 1.1.3.** Let $G$ be a connected reductive group over $F$. Let $Z_G, Z_{G_{\mathrm{sc}}}$ be the centers of $G, G_{\mathrm{sc}}$ respectively. Let $\mathscr{Z}_G$ be the complex $Z_{G_{\mathrm{sc}}} \to Z_G$ in $\mathrm{Mult}(F)$, at degrees $-1, 0$.

**Proposition 1.1.4.** *There is a canonical isomorphism* $\mathrm{R}X_*(\mathscr{Z}_G) \cong \pi_1(G)$ *inside* $\mathcal{D}^b(\Gamma\text{-Mod})$.

*Proof.* Let $T$ be a maximal torus in $G$, and let $\widetilde{T}$ be its inverse image in $G_{\mathrm{sc}}$. Recall that $\pi_1(G)$ is defined to be the $\Gamma$-module $X_*(T)/X_*(\widetilde{T})$. If $S$ is another maximal torus in $G$, with preimage $\widetilde{S}$ in $G_{\mathrm{sc}}$, and $g$ is any element of $G(\overline{F})$ such that $\mathrm{Int}(g)(T_{\overline{F}}) = S_{\overline{F}}$, then $g$ induces an isomorphism of $\Gamma$-modules $\iota_{T,S} : X_*(T)/X_*(\widetilde{T}) \xrightarrow{\sim} X_*(S)/X_*(\widetilde{S})$, which is independent of $g$. Thus $\pi_1(G)$ does not depend on $T$.

Now the natural map $\mathscr{Z}_G \to (\widetilde{T} \to T)$ is a quasi-isomorphism, as the cone of this map is easily seen to be acyclic (here we regard $\widetilde{T} \to T$ as being in degrees $-1$ and 0). Thus

$$\mathrm{R}X_*(\mathscr{Z}_G) \cong \mathrm{R}X_*(\widetilde{T} \to T) = X_*(T)/X_*(\widetilde{T}) = \pi_1(G).$$

$\square$

**1.1.5.** We now review the theory of abelianized Galois cohomology, developed by Borovoi [Bor98] and Labesse [Lab99].

For the rest of this subsection $F$ will be a local or global field of characteristic zero. We introduce a symbol ? as follows. When $F$ is local, ? denotes $F$. When $F$ is global, ? denotes one of $F$, $\mathbb{A}_F/F$, or $\mathbb{A}_F^S$, where $S$ is a finite set of places of $F$, and $\mathbb{A}_F^S$ is the ring of adeles away from $S$. Let $\Gamma = \mathrm{Gal}(\overline{F}/F)$, and define the discrete $\Gamma$-module

$$\mathbf{D}_? = \begin{cases} \overline{F}^\times, & \text{if } ? = F, \\ (\bar{\mathbb{A}}_F^S)^\times, & \text{if } ? = \mathbb{A}_F^S, \\ (\bar{\mathbb{A}}_F)^\times/\overline{F}^\times, & \text{if } ? = \mathbb{A}_F/F. \end{cases}$$

Here $\bar{\mathbb{A}}_F^S$ denotes $\overline{F} \otimes_F \mathbb{A}_F^S$.

For any bounded complex $C^\bullet$ in $\mathrm{Mult}(F)$, we define the abelian groups

$$\mathbf{H}^i(?, C^\bullet) := \mathbf{H}^i(\Gamma, \mathrm{R}X_*(C^\bullet) \overset{\mathrm{L}}{\otimes}_{\mathbb{Z}} \mathbf{D}_?), \quad i \in \mathbb{Z},$$

cf. [Lab99, p. 22, p. 26]. Here the term on the right denotes the continuous group cohomology of the profinite group $\Gamma$

**1.1.6.** Let $G$ be a connected reductive group over $F$. We define

$$\mathbf{H}^i_{\mathrm{ab}}(?, G) := \mathbf{H}^i(?, \mathscr{Z}_G),$$

cf. [Lab99, §1.6]. When ? is not $\mathbb{A}_F/F$, we have the usual Galois/adelic cohomology

$$\mathbf{H}^i(?, G), \quad i = 0, 1,$$

defined to be the continuous cohomology of $\Gamma$ acting on $G(\overline{F})$ or $G(\bar{\mathbb{A}}_F^S)$ according as ? is $F$ or $\mathbb{A}_F^S$. This is a group for $i = 0$ and a pointed set for $i = 1$. We have natural "abelianization" maps

$$\mathrm{ab}^i_? : \mathbf{H}^i(?, G) \to \mathbf{H}^i_{\mathrm{ab}}(?, G), \quad i = 0, 1,$$

which is a group homomorphism for $i = 0$ and a map of pointed sets for $i = 1$. By [Lab99, Prop. 1.6.7], the map $\mathrm{ab}^1_F : \mathbf{H}^1(F, G) \to \mathbf{H}^1_{\mathrm{ab}}(F, G)$ is surjective, and is bijective when $F$ is local non-archimedean. In particular, when $F$ is local non-archimedean, $\mathbf{H}^1(F, G)$ has a canonical structure of an abelian group.

When $F$ is global, we have

$$\mathbf{H}^1(\mathbb{A}_F^S, G) \cong \prod_{\substack{\text{places } v \text{ of } F,\\ v \notin S}}{}' \mathbf{H}^1(F_v, G), \tag{1.1.6.1}$$

where the restricted product is with respect to the trivial elements; see for instance [PR94, p. 298, Cor. 1]. Analogously we have

$$\mathbf{H}^1_{\mathrm{ab}}(\mathbb{A}_F^S, G) \cong \bigoplus_{\substack{\text{places } v \text{ of } F,\\ v \notin S}} \mathbf{H}^1_{\mathrm{ab}}(F_v, G). \tag{1.1.6.2}$$

The decompositions (1.1.6.1) and (1.1.6.2) are compatible with $\mathrm{ab}^1_{\mathbb{A}_F^S}$ and $\mathrm{ab}^1_{F_v}$. If $S$ contains all archimedean places of $F$, then $\mathrm{ab}^1_{\mathbb{A}_F^S} : \mathbf{H}^1(\mathbb{A}_F^S, G) \to \mathbf{H}^1_{\mathrm{ab}}(\mathbb{A}_F^S, G)$ is bijective, giving $\mathbf{H}^1(\mathbb{A}_F^S, G)$ a canonical structure of an abelian group.

**1.1.7.** Let $f : I \to G$ be an $F$-homomorphism between connected reductive groups over $F$. Let $\mathscr{Z}_{I\to G}$ be the mapping cone of the map of complexes $\mathscr{Z}_I \to \mathscr{Z}_G$ induced by $f$. We set (cf. [Lab99, §1.8])

$$\mathbf{H}^i_{\mathrm{ab}}(?, I \to G) = \mathbf{H}^i(?, \mathscr{Z}_{I\to G}).$$

Assume ? is not $\mathbb{A}_F/F$. We follow [Lab99] in writing:

$$\mathfrak{D}(I, G; ?) := \ker\left(\mathbf{H}^1(?, I) \to \mathbf{H}^1(?, G)\right),$$
$$\mathfrak{E}(I, G; ?) := \ker\left(\mathbf{H}^1_{\mathrm{ab}}(?, I) \to \mathbf{H}^1_{\mathrm{ab}}(?, G)\right).$$

Thus $\mathfrak{D}(I, G; ?)$ is a pointed set, and $\mathfrak{E}(I, G; ?)$ is an abelian group. We have a map of pointed sets

$$\mathfrak{D}(I, G; ?) \longrightarrow \mathfrak{E}(I, G; ?)$$

induced by $\mathrm{ab}^1_?$. This map is bijective in the following two cases:

- $F$ is local non-archimedean and $? = F$.
- $F$ is global, $? = \mathbb{A}_F^S$, and $S$ contains all the archimedean places of $F$.

**1.1.8.** Let $\Gamma = \mathrm{Gal}(\overline{F}/F)$, and $M \in \Gamma$-Mod. When $F$ is global or local non-archimedean, we set $\mathcal{A}_F(M) = M_{\Gamma,\mathrm{tors}}$, the torsion subgroup of the coinvariants $M_\Gamma$. When $F$ is local archimedean, we set $\mathcal{A}_F(M) = \widehat{\mathbf{H}}^{-1}(\Gamma, M)$. Note that in both cases, there is a canonical embedding $\mathcal{A}_F(M) \hookrightarrow M_{\Gamma,\mathrm{tors}}$ for each $M \in \Gamma$-Mod.

Now suppose $F$ is global. For each place $v$ of $F$ we fix an embedding $\overline{F} \to \overline{F}_v$, and thereby view $\Gamma_v = \mathrm{Gal}(\overline{F}_v/F_v)$ as a subgroup of $\Gamma = \mathrm{Gal}(\overline{F}/F)$. For $M \in \Gamma$-Mod, set

$$\mathcal{B}_F(M) := \bigoplus_{\text{places } v \text{ of } F} \mathcal{A}_{F_v}(M),$$

and define the map $\mathscr{P}(M) : \mathcal{B}_F(M) \to \mathcal{A}_F(M)$ to be the direct sum over $v$ of the maps

$$\mathcal{A}_{F_v}(M) \hookrightarrow M_{\Gamma_v,\mathrm{tors}} \xrightarrow{\dagger} M_{\Gamma,\mathrm{tors}} = \mathcal{A}_F(M),$$

where † is induced by the identity on $M$. Then $\mathscr{P}$ is a natural transformation $\mathcal{B}_F \to \mathcal{A}_F$ between functors from $\Gamma$-Mod to finite abelian groups.

More generally, if $S$ is a finite set of places of $F$, we set

$$\mathcal{B}_F^S(M) = \bigoplus_{\substack{\text{places } v \text{ of } F, \\ v \notin S}} \mathcal{A}_{F_v}(M).$$

**Proposition 1.1.9.** *Let $F$ be local or global, and let $I \to G$ be a homomorphism of connected reductive groups over $F$. Assume that the induced map $\pi_1(I) \to \pi_1(G)$ is surjective, and denote its kernel by $K$.*

*(i) Let $? = F$ when $F$ is local, and let $? = \mathbb{A}_F/F$ when $F$ is global. Then the exact sequence*

$$\mathbf{H}^0_{\mathrm{ab}}(?, I \to G) \to \mathbf{H}^1_{\mathrm{ab}}(?, I) \to \mathbf{H}^1_{\mathrm{ab}}(?, G)$$

*is canonically isomorphic to the natural sequence*

$$\mathcal{A}_F(K) \to \mathcal{A}_F(\pi_1(I)) \to \mathcal{A}_F(\pi_1(G)).$$

*If $F$ is global and $? = \mathbb{A}_F^S$ for a finite set $S$ of places of $F$, then the above statement still holds with $\mathcal{A}_F$ replaced by $\mathcal{B}_F^S$.*

*(ii) When $F$ is global, the commutative diagram with exact rows*

$$\begin{array}{ccccc} \mathbf{H}^0_{\mathrm{ab}}(\mathbb{A}_F, I \to G) & \longrightarrow & \mathbf{H}^1_{\mathrm{ab}}(\mathbb{A}_F, I) & \longrightarrow & \mathbf{H}^1_{\mathrm{ab}}(\mathbb{A}_F, G) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{H}^0_{\mathrm{ab}}(\mathbb{A}_F/F, I \to G) & \longrightarrow & \mathbf{H}^1_{\mathrm{ab}}(\mathbb{A}_F/F, I) & \longrightarrow & \mathbf{H}^1_{\mathrm{ab}}(\mathbb{A}_F/F, G) \end{array}$$

*is canonically identified with the natural commutative diagram*

$$\begin{array}{ccccc} \mathcal{B}_F(K) & \longrightarrow & \mathcal{B}_F(\pi_1(I)) & \longrightarrow & \mathcal{B}_F(\pi_1(G)) \\ \downarrow{\scriptstyle \mathscr{P}(K)} & & \downarrow{\scriptstyle \mathscr{P}(\pi_1(I))} & & \downarrow{\scriptstyle \mathscr{P}(\pi_1(G))} \\ \mathcal{A}_F(K) & \longrightarrow & \mathcal{A}_F(\pi_1(I)) & \longrightarrow & \mathcal{A}_F(\pi_1(G)). \end{array}$$

*Proof.* This follows from the results of [Bor98, §4], Proposition 1.1.4 applied to $I$ and $G$, and the analogous fact that $\mathrm{R}X_*(\mathscr{Z}_{I\to G})$ is represented by the complex $K[1]$ (which uses the surjectivity of $\pi_1(I) \to \pi_1(G)$). □

## 1.2. Inner twistings and local triviality conditions.

**Definition 1.2.1.** Let $F$ be a field, and let $\overline{F}$ be an algebraic closure. Let $H, H_1$ be algebraic groups over $F$.

(i) By an *inner twisting* from $H$ to $H_1$, we mean an $\overline{F}$-group isomorphism $\psi : H_{\overline{F}} \xrightarrow{\sim} H_{1,\overline{F}}$ such that for each $\rho \in \Gamma$ the automorphism $({}^{\rho}\psi)^{-1}\psi$ of $H_{\overline{F}}$ is inner.

(ii) Two inner twistings from $H$ to $H_1$ are called *equivalent*, if they differ by an inner automorphism of $H_{\overline{F}}$.

**Definition 1.2.2.** Let $F$ be a field, and let $H$ be an algebraic group over $F$. By an *inner form of* $H$, we mean a pair $(H_1, [\psi])$, where $H_1$ is an algebraic group over $F$ and $[\psi]$ is an equivalence class of inner twistings from $H$ to $H_1$. By an *isomorphism* between two inner forms $(H_1, [\psi])$ and $(H_1', [\psi'])$ of $H$, we mean an $F$-group isomorphism $H_1 \xrightarrow{\sim} H_1'$ under which $[\psi]$ is identified with $[\psi']$. By abuse of notation, we often denote an inner form $(H_1, [\psi])$ simply by $H_1$, if no confusion can arise.

*Remark* 1.2.3. In the setting of Definition 1.2.2, there is a bijection from the set of isomorphism classes of inner forms of $H$ to $\mathbf{H}^1(F, H^{\mathrm{ad}})$, sending $(H_1, [\psi])$ to the class of the cocycle $\rho \mapsto \psi^{-1} \circ {}^\rho\psi$. Note that the natural map $\mathbf{H}^1(F, H^{\mathrm{ad}}) \to \mathbf{H}^1(\mathrm{Gal}(\overline{F}/F), \mathrm{Aut}(H_{\overline{F}}))$ is in general not injective. The image of this map classifies the $F$-isomorphism classes of algebraic groups $H_1$ over $F$ which can be extended to an inner form $(H_1, [\psi])$ of $H$.

**Definition 1.2.4.** Let $I$ and $G$ be connected reductive groups over a field $F$. By an *inner transfer datum from $I$ to $G$*, we mean a pair $(f, \mathcal{W})$, where $f$ is an injective $\overline{F}$-homomorphism $I_{\overline{F}} \to G_{\overline{F}}$, and $\mathcal{W}$ is a non-empty subset of $G(\overline{F})$, satisfying the following conditions:

(i) For each $g \in \mathcal{W}$, there is an $F$-subgroup $\mathcal{I}_g \subset G$, such that $\mathrm{Int}(g)(\mathrm{im}\, f) = (\mathcal{I}_g)_{\overline{F}}$, and such that the $\overline{F}$-isomorphism $\psi_g := \mathrm{Int}(g) \circ f : I_{\overline{F}} \xrightarrow{\sim} (\mathcal{I}_g)_{\overline{F}}$ is an inner twisting between the $F$-groups $I$ and $\mathcal{I}_g$.

(ii) For all $g_1, g_2 \in \mathcal{W}$, the $\overline{F}$-isomorphism $\psi_{g_1,g_2} := \mathrm{Int}(g_2 g_1^{-1}) : (\mathcal{I}_{g_1})_{\overline{F}} \xrightarrow{\sim} (\mathcal{I}_{g_2})_{\overline{F}}$ is an inner twisting between the $F$-groups $\mathcal{I}_{g_1}$ and $\mathcal{I}_{g_2}$.

**1.2.5.** Let $F$ be a local or global field of characteristic zero, and let the symbol ? be as in §1.1.5. Let $I, G$ be connected reductive groups over $F$, and let $(f, \mathcal{W})$ be an inner transfer datum from $I$ to $G$ (Definition 1.2.4). Choose an element $g \in \mathcal{W}$, and let $\mathcal{I}_g, \psi_g$ be as in Definition 1.2.4. Since $\psi_g$ is an inner twisting, it induces an isomorphism $\mathscr{Z}_I \to \mathscr{Z}_{\mathcal{I}_g}$ between complexes in $\mathrm{Mult}(F)$, and in particular an isomorphism $\psi_{g,*} : \mathbf{H}^i_{\mathrm{ab}}(?, I) \xrightarrow{\sim} \mathbf{H}^i_{\mathrm{ab}}(?, \mathcal{I}_g), \forall i \in \mathbb{Z}$. Since inner automorphisms of $G_{\overline{F}}$ act as the identity on $\mathscr{Z}_G$, the composite homomorphism

$$\mathbf{H}^i_{\mathrm{ab}}(?, I) \xrightarrow{\psi_{g,*}} \mathbf{H}^i_{\mathrm{ab}}(?, \mathcal{I}_g) \to \mathbf{H}^i_{\mathrm{ab}}(?, G)$$

is independent of the choice of $g$, and we say that it is induced by $(f, \mathcal{W})$.

Now assume that $F$ is global, and let $S$ be a (finite or infinite) set of places of $F$ containing all the archimedean places. We let

$$\text{Ш}^S(F, I) := \ker(\mathbf{H}^1(F, I) \to \prod_{v \in S} \mathbf{H}^1(F_v, I)),$$
$$\text{Ш}^S_{\mathrm{ab}}(F, I) := \ker(\mathbf{H}^1_{\mathrm{ab}}(F, I) \to \prod_{v \in S} \mathbf{H}^1_{\mathrm{ab}}(F_v, I)).$$

By [Bor98, Thm. 5.12 (i)], we have a canonical isomorphism

$$\text{Ш}^S(F, I) \cong \text{Ш}^S_{\mathrm{ab}}(F, I) \tag{1.2.5.1}$$

induced by $\mathrm{ab}^1_F$. In the sequel we shall often make this identification implicitly.

The homomorphism $\mathbf{H}^1_{\mathrm{ab}}(F, I) \to \mathbf{H}^1_{\mathrm{ab}}(F, G)$ induced by $(f, \mathcal{W})$ restricts to a homomorphism

$$\text{Ш}^S_{\mathrm{ab}}(F, I) \longrightarrow \text{Ш}^S_{\mathrm{ab}}(F, G).$$

We denote the kernel of this homomorphism by

$$\text{Ш}^S_G(F, I).$$

Via (1.2.5.1) we also view $\text{Ш}^S_G(F, I)$ as a subset of $\mathbf{H}^1(F, I)$.

More generally, for any $\mathbb{Q}$-subgroup $I' \subset I$, we denote by $\text{Ш}^S_G(F, I')$ the kernel of the composite

$$\text{Ш}^S_{\mathrm{ab}}(F, I') \to \text{Ш}^S_{\mathrm{ab}}(F, I) \to \text{Ш}^S_{\mathrm{ab}}(F, G).$$

If $S$ is the set of all places we omit $S$ from the notation, and write $Ш(F,I)$, $Ш_G(F,I)$, etc.

**Lemma 1.2.6.** *Let $G$ be a connected reductive group over $\mathbb{Q}$. Let $\mathcal{I}$ be a connected reductive subgroup of $G$ defined over $\mathbb{Q}$. Let $\widetilde{\mathcal{I}}$ be the inverse image of $\mathcal{I}$ in $G_{\mathrm{sc}}$. Assume that $\mathcal{I}$ contains a maximal torus in $G$ defined over $\mathbb{Q}$, and assume that $\widetilde{\mathcal{I}}$ is connected reductive. Then the natural map $Ш^\infty_{G_{\mathrm{sc}}}(\mathbb{Q},\widetilde{\mathcal{I}}) \to Ш^\infty_G(\mathbb{Q},\mathcal{I})$ is surjective. (Here $Ш^\infty_{G_{\mathrm{sc}}}(\mathbb{Q},\widetilde{\mathcal{I}})$ is defined using the trivial inner transfer datum $(f,\mathcal{W})$, where $f$ is the inclusion and $\mathcal{W}$ contains $1$. Similarly, $Ш^\infty_G(\mathbb{Q},\mathcal{I})$ is defined using the trivial inner transfer datum.)*

*Proof.* In the current situation $Ш^\infty_G(\mathbb{Q},\mathcal{I})$ is nothing but the kernel of the natural map $\mathbf{H}^1(\mathbb{Q},\mathcal{I}) \to \mathbf{H}^1(\mathbb{R},\mathcal{I}) \oplus \mathbf{H}^1(\mathbb{Q},G)$, whose second component is induced by the inclusion $\mathcal{I} \hookrightarrow G$. The similar remark holds for $Ш^\infty_{G_{\mathrm{sc}}}(\mathbb{Q},\widetilde{\mathcal{I}})$. Now let $\beta \in Ш^\infty_G(\mathbb{Q},\mathcal{I})$. Then there exists $g \in G(\overline{\mathbb{Q}})$ such that $\beta$ is represented by the cocycle

$$\Gamma \ni \tau \mapsto g^{-1}{}^\tau g \in \mathcal{I}(\overline{\mathbb{Q}}).$$

Note that we may replace $g$ by any other element of $G(\mathbb{Q})g\mathcal{I}(\overline{\mathbb{Q}})$, without changing the class $\beta$. Since $\beta$ is trivial at $\infty$, we have $g \in G(\mathbb{R})\mathcal{I}(\mathbb{C})$. By real approximation, we can left-multiply $g$ by an element of $G(\mathbb{Q})$ to arrange that

$$g \in G(\mathbb{R})^+\mathcal{I}(\mathbb{C}).$$

Let $\pi$ denote the projection $G(\mathbb{R}) \to G^{\mathrm{ab}}(\mathbb{R})$. Since $\pi(G(\mathbb{R})^+) \subset G^{\mathrm{ab}}(\mathbb{R})^+ = \pi(T(\mathbb{R})^+)$, we have $G(\mathbb{R})^+ \subset G_{\mathrm{der}}(\mathbb{R})T(\mathbb{R})^+ \subset G_{\mathrm{der}}(\mathbb{R})\mathcal{I}(\mathbb{R})$. Thus we have

$$g \in G_{\mathrm{der}}(\mathbb{R})\mathcal{I}(\mathbb{C}).$$

Again by real approximation, we may further left-multiply $g$ by an element of $G_{\mathrm{der}}(\mathbb{Q})$ to arrange that

$$g \in G_{\mathrm{der}}(\mathbb{R})^+\mathcal{I}(\mathbb{C}).$$

Now since $G(\overline{\mathbb{Q}}) = G_{\mathrm{der}}(\overline{\mathbb{Q}})T(\overline{\mathbb{Q}})$, we may right-multiply $g$ by an element of $T(\overline{\mathbb{Q}})$ to arrange that

$$g \in G_{\mathrm{der}}(\overline{\mathbb{Q}}) \cap G_{\mathrm{der}}(\mathbb{R})^+\mathcal{I}(\mathbb{C}).$$

Now we pick a lift $\widetilde{g} \in G_{\mathrm{sc}}(\overline{\mathbb{Q}})$ of $g \in G_{\mathrm{der}}(\overline{\mathbb{Q}})$. Since $G_{\mathrm{sc}}(\mathbb{R})$ (which is connected by Cartan's theorem) maps onto $G_{\mathrm{der}}(\mathbb{R})^+$, we have $\widetilde{g} \in G_{\mathrm{sc}}(\mathbb{R})\widetilde{\mathcal{I}}(\mathbb{C})$. The cocycle

$$\Gamma \ni \tau \mapsto \widetilde{g}^{-1}{}^\tau(\widetilde{g})$$

is then valued in $\widetilde{\mathcal{I}}(\overline{\mathbb{Q}})$ (since $g^{-1}{}^\tau g \in \mathcal{I}(\overline{\mathbb{Q}})$), and represents a class in $Ш^\infty_{G_{\mathrm{sc}}}(\mathbb{Q},\widetilde{\mathcal{I}})$ lifting $\beta$. □

*Remark* 1.2.7. In the setting of the above lemma, we in fact have $Ш^\infty_{G_{\mathrm{sc}}}(\mathbb{Q},\widetilde{\mathcal{I}}) = Ш^\infty_{\mathrm{ab}}(\mathbb{Q},\widetilde{\mathcal{I}})$, because $\mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q},G_{\mathrm{sc}}) = 0$.

**1.2.8.** Let $I,G$ be connected reductive groups over $\mathbb{Q}$, and let $(f,\mathcal{W})$ be an inner transfer datum from $I$ to $G$. Assume that $I$ and $G$ have the same absolute rank. For each $g \in \mathcal{W}$, we know that $\mathcal{I}_g$ contains a maximal torus in $G$ defined over $\mathbb{Q}$, and in particular $\mathcal{I}_g$ contains $Z_G$. Let $Z$ be a $\mathbb{Q}$-subgroup of $Z_G$. Note that $f^{-1}(Z)$ is a $\mathbb{Q}$-subgroup of $I$, and $f$ induces a $\mathbb{Q}$-isomorphism $f^{-1}(Z) \xrightarrow{\sim} Z$. Let $\bar{I} := I/f^{-1}(Z)$ and $\bar{G} := G/Z$. Then $(f,\mathcal{W})$ induces an inner transfer datum $(\bar{f},\overline{\mathcal{W}})$ from $\bar{I}$ to $\bar{G}$. We use $(f,\mathcal{W})$ to define $Ш^\infty_G(\mathbb{Q},I)$, and use $(\bar{f},\overline{\mathcal{W}})$ to define $Ш^\infty_{\bar{G}}(\mathbb{Q},\bar{I})$

**Corollary 1.2.9.** *In the setting of §1.2.8, the natural map $Ш^{\infty}_G(\mathbb{Q}, I) \to Ш^{\infty}_{\bar{G}}(\mathbb{Q}, \bar{I})$ is surjective.*

*Proof.* By picking an arbitrary element $g \in \mathcal{W}$ and replacing $I$ by $\mathcal{I}_g$, we reduce to the following situation:

- $I$ is a $\mathbb{Q}$-subgroup of $G$ containing a maximal torus in $G$,
- $f$ is the inclusion,
- $\mathcal{W}$ contains 1.

Since $G_{\mathrm{sc}} = (\bar{G})_{\mathrm{sc}}$, and since the inverse image of $I$ in $G_{\mathrm{sc}}$ is equal to the inverse image of $\bar{I}$ in $(\bar{G})_{\mathrm{sc}}$, the corollary immediately follows from Lemma 1.2.6. □

**Lemma 1.2.10.** *Let $F$ be a field of characteristic zero. Let $I \to S$ be a surjective homomorphism from a connected reductive group $I$ to a torus $S$ over $F$. Let $I'$ be the kernel.*

*(i) We have $Z_I \cap I' = Z_{I'}$, and we have short exact sequences*

$$1 \to Z_{I'} \to I' \to I^{\mathrm{ad}} \to 1$$

*and*

$$1 \to Z_{I'} \to Z_I \to S \to 1.$$

*(ii) The maps*

$$f_1 : I(F) \to I^{\mathrm{ad}}(F) \xrightarrow{\delta^1} \mathbf{H}^1(F, Z_{I'})$$

*and*

$$f_2 : I(F) \to S(F) \xrightarrow{\delta^2} \mathbf{H}^1(F, Z_{I'})$$

*differ by a sign. Here the maps $\delta^1, \delta^2$ are the boundary maps induced by the short exact sequences in (i). In particular, every element of the image of $f_1$ or $f_2$ has trivial image in $\mathbf{H}^1(F, I')$ and trivial image in $\mathbf{H}^1(F, Z_I)$.*

*Proof.* Part (i) follows from the fact that $I'$ contains $I_{\mathrm{der}}$. For part (ii), let $i \in I(F)$ and write $i = i_0 i_1 = i_1 i_0$ with $i_0 \in I'(\overline{F})$ and $i_1 \in Z_I(\overline{F})$. Then $f_2(i)$ is represented by the cocycle $(i_1^{-1} {}^{\rho} i_1)_{\rho}$ in $Z^1(F, Z_{I'})$. Since ${}^{\rho} i = i$, this cocycle equals $(i_0 {}^{\rho} i_0^{-1})_{\rho}$, which represents $-f_1(i)$. □

**Corollary 1.2.11.** *In Lemma 1.2.10, take $F = \mathbb{Q}$, and take $I \to S$ to be the natural map $I \to I^{\mathrm{ab}}$, so that $I' = I_{\mathrm{der}}$. Then the image of $I(\mathbb{Q})_+$ in $\mathbf{H}^1(\mathbb{Q}, Z_{I_{\mathrm{der}}})$, under either of the maps $f_1$ or $f_2$, is contained in $Ш^{\infty}_{I_{\mathrm{der}}}(\mathbb{Q}, Z_{I_{\mathrm{der}}})$.*

*Proof.* Using the notation of Lemma 1.2.10, the image of $I^{\mathrm{ad}}(\mathbb{R})^+$ under the boundary map $\delta^1 : I^{\mathrm{ad}}(\mathbb{R}) \to \mathbf{H}^1(\mathbb{R}, Z_{I_{\mathrm{der}}})$ is trivial, because the image of $I_{\mathrm{der}}(\mathbb{R}) \to I^{\mathrm{ad}}(\mathbb{R})$ contains $I^{\mathrm{ad}}(\mathbb{R})^+$. The corollary then follows from Lemma 1.2.10. □

## 1.3. **The Kottwitz homomorphism.**

**1.3.1.** Let $F$ be a field of characteristic 0, $\overline{F}$ an algebraic closure, and $\Gamma_F = \mathrm{Gal}(\overline{F}/F)$. Let $G$ be a reductive group over $F$. Recall that, when $F$ is complete, discretely valued, with algebraically closed residue field, the *Kottwitz homomorphism* is a homomorphism $G(F) \to \pi_1(G)_{\Gamma_F}$ which is functorial in $G$, and for $G = \mathbb{G}_m$ is the valuation map $F^{\times} \to \mathbb{Z}$. The original construction in [Kot97, §7] relies on Steinberg's theorem for $F$.

Here we generalize the construction of the Kottwitz homomorphism. We shall obtain a homomorphism $\kappa_G^{R,v} : G(R) \to \pi_1(G)_{\Gamma_F}$, where $R$ is any $F$-algebra (with

$F$ arbitrary) equipped with a discrete valuation $v$. This will allow us to show that the Kottwitz homomorphism is constant in certain families.

**1.3.2.** Let $\mathsf{S}$ be the big fpqc site of Spec $F$. Let $\mathcal{A}b\mathcal{S}hv(\mathsf{S})$ be the category of abelian sheaves on $\mathsf{S}$. We shall view Mult$(F)$ and Tori$(F)$ (see §1.1.1) as full subcategories of $\mathcal{A}b\mathcal{S}hv(\mathsf{S})$. Let $\mathcal{D}(\mathsf{S})$ be the derived category of $\mathcal{A}b\mathcal{S}hv(\mathsf{S})$, and let $\mathcal{D}^{[-1,0]}(\mathsf{S})$ be the full subcategory of $\mathcal{D}(\mathsf{S})$ consisting of those $L \in \mathcal{D}^b(\mathsf{S})$ such that $\mathbf{H}^i(L) = 0$ unless $i \in \{-1, 0\}$.

We shall need the formalism of Picard stacks[4] on $\mathsf{S}$, as in [AG73, Exposé XVIII, §1.4]. Following *loc. cit.*, let $\mathcal{C}h^\flat(\mathsf{S})$ be the category whose objects are the small Picard stacks on $\mathsf{S}$, and whose morphisms are the isomorphism classes of additive functors between Picard stacks. By [AG73, Exposé XVIII, Prop. 1.4.15], we have an equivalence of categories

$$\mathrm{ch} : \mathcal{D}^{[-1,0]}(\mathsf{S}) \to \mathcal{C}h^\flat(\mathsf{S}).$$

For a complex $C^\bullet = (C^{-1} \to C^0)$ in Mult$(F)$ (at degrees $-1, 0$), $\mathrm{ch}(C^\bullet)$ is given by the quotient stack $[C^{-1}\backslash C^0]$.

**1.3.3.** As in Proposition 1.1.2, we have an equivalence of categories

$$\mathrm{R}X_* : \mathcal{D}^b(\mathrm{Mult}(F)) \to \mathcal{D}^b(\Gamma_F\text{-Mod}).$$

We fix once and for all a quasi-inverse $\mathcal{Y}$ of $\mathrm{R}X_*$, and natural isomorphisms $\epsilon : \mathrm{R}X_* \circ \mathcal{Y} \to \mathrm{id}$ and $\eta : \mathrm{id} \to \mathcal{Y} \circ \mathrm{R}X_*$.

Let $G$ be a reductive group over $F$. Let $\mathscr{Z}_G$ be as in Definition 1.1.3. By Proposition 1.1.4, we have a canonical isomorphism $\mathrm{R}X_*(\mathscr{Z}_G) \cong \pi_1(G)$ in $\mathcal{D}^b(\Gamma_F\text{-Mod})$. Let $\mathrm{pr} : \pi_1(G) \to \pi_1(G)_{\Gamma_F}$ be the canonical map, viewed as a morphism in $\mathcal{D}^b(\Gamma_F\text{-Mod})$. Let

$$\mathscr{Z}_G^F := \mathcal{Y}(\pi_1(G)_{\Gamma_F}) \in \mathcal{D}^b(\mathrm{Mult}(F)),$$

and let

$$\kappa_0 : \mathscr{Z}_G \longrightarrow \mathscr{Z}_G^F$$

be given by the composite

$$\mathscr{Z}_G \xrightarrow{\eta} \mathcal{Y}(\mathrm{R}X_*(\mathscr{Z}_G)) \xrightarrow{\mathcal{Y}(\mathrm{pr})} \mathcal{Y}(\pi_1(G)_{\Gamma_F}) = \mathscr{Z}_G^F.$$

Thus we have a canonical isomorphism

$$\epsilon : \mathrm{R}X_*(\mathscr{Z}_G^F) \xrightarrow{\sim} \pi_1(G)_{\Gamma_F}.$$

**Lemma 1.3.4.** *In $\mathcal{D}^b(\mathrm{Mult}(F))$, $\mathscr{Z}_G^F$ is isomorphic to a complex of the form*

$$T^{-1} \longrightarrow T^0,$$

*where $T^{-1}$ and $T^0$ are split tori over $F$ and located at degrees $-1$ and $0$. In particular, the image of $\mathscr{Z}_G^F$ in $\mathcal{D}(\mathsf{S})$ lies in $\mathcal{D}^{[-1,0]}(\mathsf{S})$.*

*Proof.* This follows from the fact that $\pi_1(G)_{\Gamma_F}$ is isomorphic to a complex $L^{-1} \to L^0$ in $\mathcal{D}^b(\Gamma_F\text{-Mod})$, where $L^{-1}$ and $L^0$ are finite free $\mathbb{Z}$-modules with the trivial $\Gamma$-action. □

[4]We omit the adjective "strictly commutative", as that will always be understood.

**1.3.5.** We write $\mathscr{K}_G^F$ for $\mathrm{ch}(\mathscr{Z}_G^F) \in \mathcal{C}h^\flat(\mathsf{S})$, and call it the *Kottwitz stack* of $G$ over $F$. The inclusions induce a canonical equivalence between quotient stacks $[Z_{G_{\mathrm{sc}}}\backslash Z_G] \to [G_{\mathrm{sc}}\backslash G]$. Thus we obtain a functor between stacks on $\mathsf{S}$:

$$\kappa_G^{\mathrm{can}} : G \longrightarrow [G_{\mathrm{sc}}\backslash G] \xrightarrow{\sim} [Z_{G_{\mathrm{sc}}}\backslash Z_G] \cong \mathrm{ch}(\mathscr{Z}_G) \xrightarrow{\mathrm{ch}(\kappa_0)} \mathrm{ch}(\mathscr{Z}_G^F) = \mathscr{K}_G^F,$$

which is canonical up to isomorphism.

For any small Picard category $P$ (strictly commutative, as always), we denote by $\pi_0(P)$ the set of isomorphism classes of $P$, which is naturally an abelian group. Then $\kappa_G^{\mathrm{can}}$ induces a morphism

$$\pi_0(\kappa_G^{\mathrm{can}}) : G \longrightarrow \pi_0(\mathscr{K}_G^F(\cdot))$$

of presheaves in groups[5] on $\mathsf{S}$.

**1.3.6.** Now choose $T^\bullet = (T^{-1} \to T^0)$ as in Lemma 1.3.4, and choose an isomorphism $f : T^\bullet \xrightarrow{\sim} \mathscr{Z}_G^F$ in $\mathcal{D}^b(\mathrm{Mult}(F))$. If $R$ is any $F$-algebra with $\mathrm{Pic}(\mathrm{Spec}\, R) = \{1\}$, then we have canonical isomorphisms of abelian groups

$$\begin{aligned}(1.3.6.1)\qquad \pi_0\big(\mathrm{ch}(T^\bullet)(R)\big) &\cong T^0(R)/T^{-1}(R) \\ &\cong (X_*(T^0)/X_*(T^{-1})) \otimes_{\mathbb{Z}} R^\times \cong \mathbf{H}^0(\mathrm{R}X_*(T^\bullet)) \otimes_{\mathbb{Z}} R^\times,\end{aligned}$$

since $T^{-1}$ is a split torus. In this case, consider the composite isomorphism:

$$\begin{aligned}\gamma_R : \pi_0(\mathscr{K}_G^F(R)) \xrightarrow{f^{-1}} \pi_0\big(\mathrm{ch}(T^\bullet)(R)\big) &\cong \mathbf{H}^0(\mathrm{R}X_*(T^\bullet)) \otimes R^\times \\ &\xrightarrow{f} \mathbf{H}^0(\mathrm{R}X_*(\mathscr{Z}_G^F)) \otimes R^\times \xrightarrow{\epsilon} \pi_1(G)_{\Gamma_F} \otimes R^\times.\end{aligned}$$

Then $\gamma_R$ is independent of the choice of $(T^\bullet, f)$, by the functoriality of (1.3.6.1) in $T^\bullet$.

**1.3.7.** If $R$ is a commutative ring, by a *discrete valuation on* $R$, we mean a function $v : R \to \mathbb{Z} \cup \{\infty\}$ satisfying $v(0) = \infty$, $v(1) = 0$, $v(ab) = v(a) + v(b)$, and $v(a+b) \geq \min(v(a), v(b))$, for all $a, b \in R$. (Here $\infty > n, \forall n \in \mathbb{Z}$, and we do not require $v(a) = \infty \Rightarrow a = 0$.)

Now consider an $F$-algebra $R$ satisfying $\mathrm{Pic}(\mathrm{Spec}\, R) = \{1\}$, and a discrete valuation $v$ on $R$. Composing the canonical map $\pi_0(\kappa_G^{\mathrm{can}})$ in §1.3.5 with the canonical map $\gamma_R$ in §1.3.6, we obtain the canonical map

$$\kappa_G^R : G(R) \xrightarrow{\pi_0(\kappa_G^{\mathrm{can}})} \pi_0(\mathscr{K}_G^F(R)) \xrightarrow{\gamma_R} \pi_1(G)_{\Gamma_F} \otimes R^\times,$$

which is a group homomorphism. On composing the above with $v : R^\times \to \mathbb{Z}$, we obtain the group homomorphism

$$(1.3.7.1)\qquad \kappa_G^{R,v} : G(R) \xrightarrow{\kappa_G^R} \pi_1(G)_{\Gamma_F} \otimes R^\times \xrightarrow{v} \pi_1(G)_{\Gamma_F}.$$

We now extend the definition of $\kappa_G^{R,v}$, dropping the hypothesis $\mathrm{Pic}(\mathrm{Spec}\, R) = \{1\}$.

**Definition 1.3.8.** Let $R$ be an arbitrary $F$-algebra, and let $v$ be a discrete valuation on $R$. The elements $r \in R$ with $v(r) = \infty$ form a prime ideal $\mathfrak{p}$, and $v$ factors as $R \to \mathrm{Frac}\,(R/\mathfrak{p}) \xrightarrow{\bar{v}} \mathbb{Z} \cup \{\infty\}$. We define $\kappa_G^{R,v}$ to be the composition

$$G(R) \longrightarrow G(\mathrm{Frac}\,(R/\mathfrak{p})) \xrightarrow{\kappa_G^{\mathrm{Frac}\,(R/\mathfrak{p}),\bar{v}}} \pi_1(G)_{\Gamma_F}.$$

[5] We caution the reader that the right hand side is not a sheaf.

We call $\kappa_G^{R,v}$ the *Kottwitz homomorphism.* We often simply write $\kappa_G^v$ for $\kappa_G^{R,v}$.

**1.3.9.** In Definition 1.3.8, if $R$ satisfies $\mathrm{Pic}(\mathrm{Spec}\, R) = \{1\}$, one checks that the definition of $\kappa_G^{R,v}$ agrees with the previous definition (1.3.7.1). Moreover, the generally defined $\kappa_G^{R,v}$ (without the hypothesis $\mathrm{Pic}(\mathrm{Spec}\, R) = \{1\}$) is functorial in the pair $(R, v)$ in the following sense. Let $R'$ be another $F$-algebra equipped with a discrete valuation $v'$. Suppose there is an $F$-algebra map $h : R \to R'$ such that $v$ is the pull-back of $v'$ along $h$. Then $\kappa_G^{R,v}$ equals the composition $G(R) \xrightarrow{h} G(R') \xrightarrow{\kappa_G^{R',v'}} \pi_1(G)_{\Gamma_F}$. For fixed $(R, v)$, the homomorphism $\kappa_G^{R,v}$ is also functorial in the reductive group $G$ over $F$, i.e., it is a natural transformation between the functors $G \mapsto G(R)$ and $G \mapsto \pi_1(G)_{\Gamma_F}$. Using this, one easily checks that $\kappa_G^{F,v}$ agrees with Kottwitz's original construction in [Kot97, §7], in the special case when $(F, v)$ is a complete discretely valued field with algebraically closed residue field.

**Proposition 1.3.10.** *Let $R$ be an $F$-algebra, and let $v_1, \dots, v_n$ be a collection of discrete valuations on $R$. Let $a_1, \dots, a_n \in \mathbb{Z}$. Suppose that*

*(i) $\sum_{i=1}^n a_i v_i(u) = 0$ for all $u \in R^\times$, and*
*(ii) the group $\mathrm{Pic}(\mathrm{Spec}\, R)$ is trivial.*

*Then $\sum_{i=1}^n a_i \kappa_G^{v_i}(h) = 0$ for all $h \in G(R)$.*

*Proof.* By condition (ii), each $\kappa_G^{v_i}$ factors as in (1.3.7.1). Thus the map $\sum_i a_i \kappa_G^{v_i}$ factors through the map $\mathrm{id} \otimes \sum a_i v_i : \pi_1(G)_{\Gamma_F} \otimes R^\times \to \pi_1(G)_{\Gamma_F}$, which is zero by (i). $\square$

**Proposition 1.3.11.** *Let $R \supset F$ be a domain, and $v_1, \dots, v_n$ a collection of discrete valuations on $R$. Let $a_1, \dots, a_n \in \mathbb{Z}$. Suppose that $R = R^\circ[1/f_j]_{j=1}^m$, where $R^\circ \subset R$ is a subring and $f_j \in R^\circ$ are non-zero prime elements, satisfying the following conditions.*

*(i) The ring $R^\circ$ is a noetherian locally factorial domain.*
*(ii) For $i = 1, \dots, n$, we have $v_i(R^\circ) \subset \mathbb{Z}_{\geq 0} \cup \{\infty\}$.*
*(iii) $\sum_{i=1}^n a_i v_i(f_j) = 0$ for each $j = 1, \dots, m$.*

*Then $\sum_{i=1}^n a_i \kappa_G^{v_i}(h) = 0$ for all $h \in G(R)$.*

*Proof.* Let $R^{\circ\prime}$ be the ring obtained from $R^\circ$ by inverting all elements $f$ such that $v_i(f) = 0$ for all $i$. The conditions of the proposition continue to hold if we replace $R^\circ$ (resp. $R$) by $R^{\circ\prime}$ (resp. $R^{\circ\prime}[1/f_j]_{j=1}^m$), and omit from the list of $f_j$ those elements such that $v_i(f_j) = 0$ for all $i$ (as they become units in $R^{\circ\prime}$). By the functoriality of the Kottwitz map as discussed in §1.3.9, we reduce to the case where $R^\circ = R^{\circ\prime}$. Then $R^\circ$ is semi-local, as each proper ideal is contained in one of the prime ideals $\mathfrak{p}_i = \{x \in R^\circ \mid v_i(x) > 0\}, i = 1, \dots, n$.

Since $R^\circ$ is noetherian and locally factorial, the restriction map $\mathrm{Pic}(\mathrm{Spec}\, R^\circ) \to \mathrm{Pic}(\mathrm{Spec}\, R)$ is surjective (see [Gro67, Cor. 21.6.11]). Since $R^\circ$ is semi-local, we have $\mathrm{Pic}(\mathrm{Spec}\, R^\circ) = \mathrm{Pic}(\mathrm{Spec}\, R) = \{1\}$.

Now since the $f_j$ are prime in $R^\circ$, any unit in $R$ has the form $u = w f_1^{e_1} \dots f_m^{e_m}$ where $w \in R^{\circ\times}, e_j \in \mathbb{Z}$. Since $v_i(w) = 0$ for all $i$, we have

$$\sum_{i=1}^n a_i v_i(u) = \sum_{j=1}^m e_j \sum_{i=1}^n a_i v_i(f_j) = 0.$$

The proposition now follows from Proposition 1.3.10. $\square$

**Corollary 1.3.12.** *Suppose that $F$ is equipped with a discrete valuation $v_F : F \to \mathbb{Z} \cup \{\infty\}$ with ring of integers $\mathcal{O}_F$, and that $\mathcal{G}$ is a smooth affine group scheme over $\mathcal{O}_F$ extending $G$. Assume that $\mathcal{G}$ has connected fibers. Then $\kappa_G^{v_F} : G(F) \to \pi_1(G)_{\Gamma_F}$ maps $\mathcal{G}(\mathcal{O}_F) \subset G(F)$ to $\{0\}$.*

*Proof.* Let $R^\circ$ be the affine ring of $\mathcal{G}$. Let $\pi_F \in \mathcal{O}_F$ be a uniformizer. Our conditions imply that $\pi_F$ is a prime element of the noetherian domain $R^\circ$, and hence $R^\circ_{(\pi_F)}$ is a DVR. Let $v_0$ be the pull-back to $R^\circ$ of the canonical discrete valuation on $R^\circ_{(\pi_F)}$. Let $g \in \mathcal{G}(\mathcal{O}_F)$. We also consider the valuation $v_g$ given by $R^\circ \xrightarrow{g} \mathcal{O}_F \xrightarrow{v_F} \mathbb{Z} \cup \{\infty\}$. In the following we show that $\kappa_G^{v_F}(g) = 0$.

Let $R = R^\circ[1/\pi_F]$. Note that $v_0(\pi_F) = v_g(\pi_F) = 1$. In particular, $v_0$ and $v_g$ extend to $R$. Since $R^\circ$ and $v_0, v_g$ satisfy conditions (i) (ii) in Proposition 1.3.11, and since $v_0(\pi_F) - v_g(\pi_F) = 0$, we may apply that proposition to conclude that $\kappa_G^{v_0}(h) - \kappa_G^{v_g}(h) = 0$ for all $h \in G(R)$. Applying this to $h = g_u$, where $g_u$ is the universal point in $\mathcal{G}(R^\circ) \subset G(R)$, we get

$$\kappa_G^{v_0}(g_u) = \kappa_G^{v_g}(g_u) = \kappa_G^{v_F}(g),$$

where the second equality follows from functoriality (§1.3.9). This shows that $\kappa_G^{v_F}(g)$ does not depend on $g \in G(R)$. Hence it must be 0, its value on the identity. $\square$

*Remark* 1.3.13. In Corollary 1.3.12, if $\mathcal{G}$ is a parahoric group scheme ([HR08]), and if the discretely valued field $(F, v_F)$ is strictly henselian, then the conclusion follows from [HR08, Prop. 3].

**1.3.14.** Keep the setting and notation of Corollary 1.3.12. Assume that $\mathcal{G}$ is reductive, and $F$ is complete. Let $\pi_F \in F$ be a uniformizer. Let $\mathcal{S} \subset \mathcal{G}$ be a maximal split torus. Then we have the Cartan–Iwahori–Matsumoto decomposition

$$G(F) = \bigcup_{\mu \in X_*(\mathcal{S})} G(\mathcal{O}_F)\mu(\pi_F)G(\mathcal{O}_F).$$

(The union is not disjoint.) The decomposition in this generality is proved in [AHH19, Thm. 1.3, Rmk. 3.5].

**Corollary 1.3.15.** *Keep the setting of §1.3.14. If $g \in G(F)$ belongs to the double coset indexed by $\mu \in X_*(\mathcal{S})$, then $\kappa_G^{v_F}(g) = [\mu]$, where $[\mu]$ is the image of $\mu$ under the natural map $X_*(\mathcal{S}) = \pi_1(\mathcal{S}_F) \to \pi_1(G) \to \pi_1(G)_{\Gamma_F}$.*

*Proof.* By Corollary 1.3.12, it suffices to show that $\kappa_G^{v_F}(\mu(\pi_F)) = [\mu]$. But this follows from the functoriality of the Kottwitz map in the group $G$. $\square$

1.4. **Decent elements and twisting.** Throughout this subsection, we fix a prime $p$, and denote by $\sigma$ the arithmetic $p$-Frobenius in $\mathrm{Aut}(\breve{\mathbb{Q}}_p)$.

**1.4.1.** Let $G$ be a connected reductive group over $\mathbb{Q}_p$. For $b \in G(\breve{\mathbb{Q}}_p)$, we write $\nu_b$ for the Newton cocharacter of $b$, which is a fractional cocharacter of $G_{\breve{\mathbb{Q}}_p}$; see [Kot85, §4] and [RZ96, §1.7]. Following [RZ96, Def. 1.8], we say that $b$ is *decent*, if there exists $n \in \mathbb{Z}_{\geq 1}$ such that $n\nu_b$ is a cocharacter of $G_{\breve{\mathbb{Q}}_p}$, and such that

$$b\sigma(b) \cdots \sigma^{n-1}(b) = (n\nu_b)(p). \tag{1.4.1.1}$$

In this case, we also say that $b$ is *$n$-decent*.

For any $n$-decent $b \in G(\breve{\mathbb{Q}}_p)$, it is shown in [RZ96, Cor. 1.9] that $b \in G(\mathbb{Q}_{p^n})$, and that $\nu_b$ is defined over $\mathbb{Q}_{p^n}$. In particular, if $b$ is $n$-decent, then it is also $n'$-decent for $n|n'$. Clearly the condition that an element of $G(\mathbb{Q}_{p^n})$ is $n$-decent is invariant under $\sigma$-conjugation by $G(\mathbb{Q}_{p^n})$. Conversely, if $b, b'$ are $n$-decent and if $b' = gb\sigma(g)^{-1}$ for some $g \in G(\breve{\mathbb{Q}}_p)$, then necessarily $g \in G(\mathbb{Q}_{p^n})$; see [RZ96, Cor. 1.10].

**1.4.2.** We denote by $\mathrm{B}(G)$ the set of $\sigma$-conjugacy classes in $G(\breve{\mathbb{Q}}_p)$. For $b \in G(\breve{\mathbb{Q}}_p)$, we denote its class in $\mathrm{B}(G)$ by $[b]$. We recall Kottwitz's classification of elements of $\mathrm{B}(G)$. Let $\mathcal{N}(G)$ denote the set of $\sigma$-stable $G(\breve{\mathbb{Q}}_p)$-conjugacy classes of fractional cocharacters of $G_{\breve{\mathbb{Q}}_p}$. The association $b \mapsto \nu_b$ descends to the *Newton map* $\bar{\nu} : \mathrm{B}(G) \to \mathcal{N}(G)$. As in §1.3, we have the Kottwitz homomorphism $w_G : G(\breve{\mathbb{Q}}_p) \to \pi_1(G)_{\Gamma_{p,0}}$ associated with the $p$-adic valuation on $\breve{\mathbb{Q}}_p$. By [Kot97, §7], $w_G$ is surjective, and descends to a map $\kappa_G : \mathrm{B}(G) \to \pi_1(G)_{\Gamma_p}$, called the *Kottwitz map*. By [Kot97, §4.13], the map

$$(\bar{\nu}, \kappa_G) : \mathrm{B}(G) \longrightarrow \mathcal{N}(G) \times \pi_1(G)_{\Gamma_p}$$

is injective.

It is proved by Kottwitz [Kot85, §4.3] (cf. [RZ96, §1.11]) that every $\sigma$-conjugacy class in $G(\breve{\mathbb{Q}}_p)$ is represented by a decent element. Thus $\mathrm{B}(G)$ is in natural bijection with the set of $G(\mathbb{Q}_p^{\mathrm{ur}})$-orbits in the set of decent elements of $G(\mathbb{Q}_p^{\mathrm{ur}})$, where $G(\mathbb{Q}_p^{\mathrm{ur}})$ acts by $\sigma$-conjugation.

**1.4.3.** Let $b \in G(\breve{\mathbb{Q}}_p)$. The functor sending any $\mathbb{Q}_p$-algebra $R$ to the group

$$J_b(R) := \left\{ g \in G(R \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p) \mid gb = b\sigma(g) \right\}$$

is represented by a reductive group $J_b$ over $\mathbb{Q}_p$. We shall also write $J_b^G$ for $J_b$, to make the presence of $G$ explicit. If $b$ is decent (so $b \in G(\mathbb{Q}_p^{\mathrm{ur}})$), then by [RZ96, Cor. 1.14], there is a canonical $\mathbb{Q}_p^{\mathrm{ur}}$-isomorphism from $J_{b,\mathbb{Q}_p^{\mathrm{ur}}}$ to the centralizer $G_{\mathbb{Q}_p^{\mathrm{ur}},\nu_b}$ of $\nu_b$ in $G_{\mathbb{Q}_p^{\mathrm{ur}}}$. (In this case, for any $\mathbb{Q}_p^{\mathrm{ur}}$-algebra $R$ we have $J_b(R) \subset G(R \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{ur}})$, and the embedding $J_{b,\mathbb{Q}_p^{\mathrm{ur}}} \to G_{\mathbb{Q}_p^{\mathrm{ur}}}$ is induced by the natural map $R \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{ur}} \to R$.) In this case the action of $\sigma$ on $G_{\mathbb{Q}_p^{\mathrm{ur}},\nu_b}(\mathbb{Q}_p^{\mathrm{ur}})$ with respect to the $\mathbb{Q}_p$-form $J_b$ is given by $g \mapsto b\sigma(g)b^{-1}$, where $\sigma(\cdot)$ is defined with respect to the $\mathbb{Q}_p$-form $G$.

If $b$ is decent and if

$$\mathcal{W} := \left\{ c \in G(\mathbb{Q}_p^{\mathrm{ur}}) \mid c\nu_b c^{-1} \text{ is defined over } \mathbb{Q}_p \right\} \neq \emptyset, \tag{1.4.3.1}$$

then the canonical embedding $J_{b,\mathbb{Q}_p^{\mathrm{ur}}} \to G_{\mathbb{Q}_p^{\mathrm{ur}}}$ and $\mathcal{W}$ form an inner transfer datum from $J_b$ to $G$ (Definition 1.2.4). We thus obtain a canonical map

$$\mathbf{H}^1(\mathbb{Q}_p, J_b) \cong \mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}_p, J_b) \to \mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}_p, G). \tag{1.4.3.2}$$

Note that for $b$ decent, (1.4.3.1) holds in the following two cases:

(i) $G$ is quasi-split over $\mathbb{Q}_p$.
(ii) $b$ is basic in $G$, i.e., $\nu_b$ is central.

Indeed, the $G(\mathbb{Q}_p^{\mathrm{ur}})$-conjugacy class of $\nu_b$ is always stable under $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$. In case (i), this conjugacy class must contain a fractional cocharacter defined over $\mathbb{Q}_p$. In case (ii), $\nu_b$ itself is already defined over $\mathbb{Q}_p$.

We remark that in case (ii), the canonical inner transfer datum from $J_b$ to $G$ equips $J_b$ with the structure of an inner form of $G$. The isomorphism class of this inner form of $G$ (see Definition 1.2.2) depends only on $[b] \in \mathrm{B}(G)$, not on the decent representative $b$.

**1.4.4.** Let $G$ be a connected reductive group over $\mathbb{Q}_p$. Let $b \in G(\breve{\mathbb{Q}}_p)$ be a decent element, and fix an element $\beta \in \mathbf{H}^1(\mathbb{Q}_p, J_b)$. By Steinberg's theorem, $\beta$ is represented by a cocycle $(a_\rho)_\rho \in Z^1(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p, J_b(\mathbb{Q}_p^{\mathrm{ur}}))$. Under the canonical isomorphism $J_{b,\mathbb{Q}_p^{\mathrm{ur}}} \xrightarrow{\sim} G_{\mathbb{Q}_p^{\mathrm{ur}},\nu_b}$, we view $a_\sigma \in J_b(\mathbb{Q}_p^{\mathrm{ur}})$ as an element of $G(\mathbb{Q}_p^{\mathrm{ur}})$, and define $b' := a_\sigma b \in G(\mathbb{Q}_p^{\mathrm{ur}})$. It is easy to see that the $\sigma$-conjugacy class of $b'$ in $G(\mathbb{Q}_p^{\mathrm{ur}})$ depends only on $b, \beta$, not on $(a_\rho)_\rho$. We shall say that (the $\sigma$-conjugacy class of) $b'$ is the twist of $b$ by $\beta$.

**Proposition 1.4.5.** *In the setting of §1.4.4, we have $\nu_{b'} = \nu_b$, and $b'$ is decent. Moreover, if (1.4.3.1) holds for $b$ (e.g., if $G$ is quasi-split), then $\kappa_G([b']) - \kappa_G([b])$ is equal to the image of $\beta$ under*

$$\mathbf{H}^1(\mathbb{Q}_p, J_b) \to \mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}_p, G) \xrightarrow{\sim} \pi_1(G)_{\Gamma_p,\mathrm{tors}},$$

*where the first map is (1.4.3.2), and the second isomorphism is as in Proposition 1.1.9.*

*Proof.* Choose $n \in \mathbb{Z}_{\geq 1}$ to be divisible enough such that

$$(a_\rho)_\rho \in Z^1(\mathbb{Q}_{p^n}/\mathbb{Q}_p, J_b(\mathbb{Q}_{p^n})),$$

and such that $b$ is $n$-decent. Using that $(a_\rho)_\rho$ is a cocycle, one shows by induction that

$$a_{\sigma^i} = b'\sigma(b') \cdots \sigma^{i-1}(b')\sigma^{i-1}(b^{-1}) \cdots \sigma(b^{-1})b^{-1},$$

for each $i \in \mathbb{Z}_{\geq 1}$. Since $a_{\sigma^n} = 1$, we have

$$b'\sigma(b') \cdots \sigma^{n-1}(b') = b\sigma(b) \cdots \sigma^{n-1}(b).$$

Since $b$ is $n$-decent, the right hand side is equal to $p^{n\nu_b}$. By the characterization of $\nu_{b'}$ (see [Kot85, §4]), we conclude that $\nu_{b'} = \nu_b$, and that $b'$ is $n$-decent.

We now prove the statement about $\kappa_G([b']) - \kappa_G([b])$. Since (1.4.3.1) holds, we can replace $b$ by a $\sigma$-conjugate in $G(\mathbb{Q}_p^{\mathrm{ur}})$ and assume that $\nu_b$ is defined over $\mathbb{Q}_p$. Then we can replace $G$ by $G_{\nu_b}$ and reduce to the case where $\nu_b$ is central. To finish the proof we only need to show that the image of $\beta$ under the composite isomorphism $\mathbf{H}^1(\mathbb{Q}_p, J_b) \cong \mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}_p, J_b) \cong \pi_1(J_b)_{\Gamma_p,\mathrm{tors}}$ is equal to $\kappa_{J_b}(a_\sigma)$. This follows from [Kot85, Rmk. 5.7] applied to $J_b$ (cf. [RV14, Rmk. 2.2 (iv)]). □

## 1.5. **Shimura varieties and their cohomology.**

**Definition 1.5.1.** Let $H$ be a locally profinite group admitting a countable neighborhood basis of the identity. Let $B$ be a locally noetherian scheme. Let $S$ be a $B$-scheme equipped with a right $H$-action via $B$-automorphisms. We say that the action is *admissible* if there exists a class $\mathscr{K}$ of compact open subgroups of $H$ satisfying the following conditions.

(i) The class $\mathscr{K}$ contains all sufficiently small compact open subgroups of $H$ (i.e., all open subgroups of a fixed compact open subgroup).
(ii) For $K \in \mathscr{K}$, the categorical quotient $S/K$ in the category of $B$-schemes exists, and is smooth, separated, and of finite type over $B$.

(iii) For $K_1, K_2 \in \mathscr{K}$ of $H$ such that $K_1 \subset K_2$, the natural map $S/K_1 \to S/K_2$ is finite étale.
(iv) The maps $S \to S/K$ identify $S$ with $\varprojlim_{K\in\mathscr{K}} S/K$.

**1.5.2.** Let $H$, $B$, $S$ and $\mathscr{K}$ be as in Definition 1.5.1. For $K \in \mathscr{K}$, we write $S_K$ for $S/K$. Let $\ell$ be a prime number invertible on $B$. The construction in [HT01, §III.3] can be generalized to define $\ell$-adic sheaves on $S_K$ and the Hecke action on the cohomology. We explain this in the following.

Let $K, U \in \mathscr{K}$, with $U$ normal in $K$. The group $K/U$ acts on $S_U$ via $S_K$-automorphisms. Since $S_K = S_U/K$ and since the map $S_U \to S_K$ is finite étale, we know that $S_U \to S_K$ is a Galois étale cover, and moreover $\mathrm{Gal}(S_U/S_K)$ is identified with the maximal quotient of $K/U$ that acts faithfully on $S_U$, cf. [Gro03, Exposé V, Prop. 3.1].

For each $K \in \mathscr{K}$, we define the profinite group

$$\mathrm{Gal}(S/S_K) := \varprojlim_{U \trianglelefteq K \text{ open}} \mathrm{Gal}(S_U/S_K). \tag{1.5.2.1}$$

Since there exists neighborhood basis of 1 in $H$ consisting of countably many open normal subgroups $U_i$ of $K$ with

$$\cdots \subset U_i \subset U_{i-1} \subset \cdots U_1 \subset K,$$

we have a presentation

$$\mathrm{Gal}(S/S_K) \cong \varprojlim_i \mathrm{Gal}(S_{U_i}/S_K). \tag{1.5.2.2}$$

Thus we are in a special case of the general setting at the beginning of [HT01, §III.2], with our $S_K$ playing the role of $X$, and our $\mathrm{Gal}(S/S_K)$ playing the role of $\Gamma$. By the construction in *loc. cit.*, every continuous $\mathrm{Gal}(S/S_K)$-representation $\rho$ on a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space gives rise to a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathscr{L}_\rho$ on $S_K$.

Note that for each $K \in \mathscr{K}$, the natural homomorphism $K \to \mathrm{Gal}(S/S_K)$ is surjective, which can be seen from (1.5.2.2) and the similar presentation $K \cong \varprojlim_i K/U_i$ (using that the indexing set is countable). Now let $\xi$ be a continuous representation of $H$ on a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space $W$. We make the following assumption on $\xi$:

- For all sufficiently small $K \in \mathscr{K}$, the restriction $\xi|_K$ factors through $\mathrm{Gal}(S/S_K)$.

Given such a $\xi$, we may and shall shrink $\mathscr{K}$ and assume that the above condition holds for all $K \in \mathscr{K}$. Then for each $K \in \mathscr{K}$ we apply the previous construction to the representation of $\mathrm{Gal}(S/S_K)$ on $W$ induced by $\xi$, and obtain a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on $S_K$, denoted by $\mathscr{L}_{\xi,K}$.

Consider $K_1, K_2 \in \mathscr{K}$ and $g \in H$ such that $g^{-1}K_1 g \subset K_2$. The action of $g$ on $S$ induces a map $g : S_{K_1} \to S_{K_2}$. As on p. 96 of [HT01], the actions of $g$ on $S$ and on $W$ together define a morphism

$$\overrightarrow{g}^* : g^*\mathscr{L}_{\xi,K_2} \longrightarrow \mathscr{L}_{\xi,K_1}$$

between $\overline{\mathbb{Q}}_\ell$-sheaves on $S_{K_1}$. For any geometric point $x$ of $B$, this induces a map

$$[g]_{K_2,K_1} : \mathbf{H}^i_c(S_{K_2,x}, \mathscr{L}_{\xi,K_2}) \longrightarrow \mathbf{H}^i_c(S_{K_1,x}, \mathscr{L}_{\xi,K_1}).$$

Define

$$\mathbf{H}^i_c(S_x, \xi) := \varinjlim_{K\in\mathscr{K}} \mathbf{H}^i_c(S_{K,x}, \mathscr{L}_{\xi,K}),$$

where the transition maps are given by $[1]_{K_2,K_1}$ for $K_1 \subset K_2$. The maps $[g]_{K_2,K_1}$ for varying $g, K_1, K_2$ give rise to a left $H$-action on $\mathbf{H}^i_c(S_x, \xi)$, called the *Hecke action*. As on p. 97 of [HT01], $\mathbf{H}^i_c(S_x, \xi)$ is an admissible $H$-module over $\overline{\mathbb{Q}}_\ell$. Indeed, using the Hochschild–Serre spectral sequence and the fact that the cohomology of a finite group acting on a $\overline{\mathbb{Q}}_\ell$-vector space vanishes in positive degrees, we see that for each $K \in \mathscr{K}$ the natural map $\mathbf{H}^i_c(S_{K,x}, \mathscr{L}_{\xi,K}) \to \mathbf{H}^i_c(S_x, \xi)$ is injective, and its image is the subspace of $K$-invariants. Since $\mathbf{H}^i_c(S_{K,x}, \mathscr{L}_{\xi,K})$ is finite-dimensional, we know that $\mathbf{H}^i_c(S_x, \xi)$ is an admissible $H$-module.

**1.5.3.** Let $(G, X)$ be a Shimura datum with reflex field $E = E(G, X) \subset \mathbb{C}$. By the theory of canonical models due to Shimura [Shi63, Shi64, Shi65, Shi66, Shi67a, Shi67b, Shi70a, Shi70b], Deligne [Del71, Del79], Milne [Mil83] (cf. [Mil90a]), and Borovoi [Bor84], we have a canonical $E$-scheme $\mathrm{Sh} = \mathrm{Sh}(G, X)$ equipped with a right $G(\mathbb{A}_f)$-action that is admissible in the sense of Definition 1.5.1. The class $\mathscr{K}$ as in Definition 1.5.1 can be taken to be the class of neat compact open subgroups of $G(\mathbb{A}_f)$ (as defined in [Pin90, §0.6]). For each $K \in \mathscr{K}$, we denote $\mathrm{Sh}/K$ by $\mathrm{Sh}_K = \mathrm{Sh}_K(G, X)$. This is a smooth, quasi-projective $E$-scheme, whose analytification over $\mathbb{C}$ is identified with the hermitian locally symmetric variety $G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$.

If $G = T$ is a torus, then $X$ consists of a single $\mathbb{R}$-homomorphism $h : \mathbb{S} \to T_{\mathbb{R}}$, and $\mathrm{Sh}_K(\overline{E})$ is identified with the finite set $\mathrm{Sh}_K(\mathbb{C}) = T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K$. The action of $\mathrm{Gal}(\overline{E}/E)$ on this finite set is given by the *reciprocity law*, which we now recall in order to fix the sign convention. Let $\mu = \mu_h$[6], which is a cocharacter of $T$ defined over $E$. Consider the composite homomorphism of $\mathbb{Q}$-algebraic groups

$$r(\mu)^{\mathrm{alg}} : \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m \xrightarrow{\mathrm{Res}_{E/\mathbb{Q}} \mu} \mathrm{Res}_{E/\mathbb{Q}} T \xrightarrow{\mathrm{N}_{E/\mathbb{Q}}} T.$$

This induces a group homomorphism

$$\pi_0(E^\times\backslash \mathbb{A}_E^\times) \longrightarrow \pi_0(T(\mathbb{Q})\backslash T(\mathbb{A})).$$

Now the left hand side is identified with $\mathrm{Gal}(E^{\mathrm{ab}}/E)$ under the global Artin map (normalized geometrically, i.e., uniformizers correspond to geometric Frobenius elements at the finite places), while the right hand side admits a natural map to $T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K$ (cf. [Del79, §2.2.3]). We thus obtain a group homomorphism

$$r : \mathrm{Gal}(E^{\mathrm{ab}}/E) \longrightarrow T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K.$$

For $\sigma \in \mathrm{Gal}(\overline{E}/E)$ and $x \in \mathrm{Sh}_K(\overline{E}) \cong T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K$, we have the *reciprocity law*

$$\sigma(x) = r(\sigma) \cdot x.$$

This uniquely determines the $E$-scheme structure of $\mathrm{Sh}_K$. Note that the above reciprocity law differs from [Del79] in the sign of $\mu$. Thus the $E$-scheme which we call the canonical model for $(T, \{h\})$ would be called the canonical model for $(T, \{h^{-1}\})$ according to *loc. cit.* Our sign convention is the same as that used by Pink [Pin90, Pin92a] and Morel [Mor10].

For general Shimura data, the canonical models are uniquely characterized by functoriality and the case of tori. According to our sign convention, the Siegel modular varieties classifying polarized abelian varieties are canonical models for the Siegel Shimura data as specified in [Kis10, §2.1.5]. (This is proved in [Del71,

[6]The convention used here is the same as in [Del79]. If $T_{\mathbb{R}} = \mathbb{G}_{m,\mathbb{R}}$ and $h$ is given by $h(z) = z^p \bar{z}^q$, then $\mu_h$ is given by $\mu_h(z) = z^{-p}$.

Thm. 4.21].) The discrepancy between this fact and Deligne's sign conventions in [Del79] was observed in [Mil90b].

**Definition 1.5.4.** Let $T$ be a $\mathbb{Q}$-torus. We denote by $T_a$ be the maximal $\mathbb{Q}$-anisotropic subtorus of $T$ (see [Spr98, Prop. 13.2.4]). We denote by $T_{ac}$ the smallest $\mathbb{Q}$-subgroup of $T_a$ whose base change to $\mathbb{R}$ contains the maximal $\mathbb{R}$-split subtorus of $T_{a,\mathbb{R}}$. (Clearly $T_{ac}$ exists and is a torus.) We call $T_{ac}$ the *anti-cuspidal part* of $T$. We say that $T$ is *cuspidal*, if $T$ has equal $\mathbb{Q}$-split rank and $\mathbb{R}$-split rank (cf. [Mor10, Def. 3.1.1]).

**Lemma 1.5.5.** *Let $T$ be a $\mathbb{Q}$-torus. The following statements are equivalent.*

- *(i) $T$ is cuspidal.*
- *(ii) $T$ is isogenous over $\mathbb{Q}$ to the product of a split $\mathbb{Q}$-torus and a $\mathbb{Q}$-torus that is anisotropic over $\mathbb{R}$.*
- *(iii) $T_a$ is $\mathbb{R}$-anisotropic.*
- *(iv) $T_{ac}$ is trivial.*
- *(v) $T(\mathbb{Q})$ is discrete in $T(\mathbb{A}_f)$.*
- *(vi) All arithmetic subgroups of $T(\mathbb{Q})$ are finite.*
- *(vii) $T$ satisfies the following* Serre condition *(cf.* [Kis17, §3.5.6]*). Fix a complex conjugation $\iota \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For all $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and all $\mu \in X_*(T)$, we have*
$$(\tau - 1)(\iota + 1)\mu = (\iota + 1)(\tau - 1)\mu = 0.$$

*In general, $T_{ac}$ is the smallest $\mathbb{Q}$-subgroup $S$ of $T$ such that $T/S$ is cuspidal.*

*Proof.* The equivalence of (i), (ii), (iii), and (iv) follows from [Spr98, Prop. 13.2.4]. The equivalence of (i), (v), and (vi) is shown in [Gro99, Prop. 1.4]. We now show that (vii) is equivalent to the other conditions. Note that (vii) is invariant under isogeny over $\mathbb{Q}$, and is satisfied when $T$ is is either split over $\mathbb{Q}$ or anisotropic over $\mathbb{R}$. Hence (ii) implies (vii). Conversely, if (vii) holds, then every $\mu \in X_*(T)^{\iota=1}$ is fixed by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, since $(\iota+1)\mu = 2\mu$ is fixed by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This implies (i).

The last assertion in the lemma is clear since (i) and (iii) are equivalent. □

**1.5.6.** Let $(G, X)$ be a Shimura datum with reflex field $E$. Write $Z$ for $Z_G$, and write $Z_{ac}$ for $(Z^0)_{ac}$. For each compact open subgroup $K \subset G(\mathbb{A}_f)$, we write $Z(\mathbb{Q})_K$ for $Z(\mathbb{Q}) \cap K$, and write $Z_K$ for $Z(\mathbb{A}_f) \cap K$. Here both intersections are inside $G(\mathbb{A}_f)$.

**Lemma 1.5.7.** *Let $K \subset G(\mathbb{A}_f)$ be a neat compact open subgroup. Then $Z(\mathbb{Q})_K$ is contained in $Z_{ac}(\mathbb{Q})$.*

*Proof.* By Lemma 1.5.5, $Z^0/Z_{ac}$, all congruence subgroups of $(Z^0/Z_{ac})(\mathbb{Q})$ are finite. Thus the same is true for all congruence subgroups of $(Z/Z_{ac})(\mathbb{Q})$. It follows that the image of $Z(\mathbb{Q})_K$ in $(Z/Z_{ac})(\mathbb{Q})$ is finite. But this image is also neat inside $(Z/Z_{ac})(\mathbb{A}_f)$, so it is trivial. □

**1.5.8.** Fix a prime number $\ell$. Let $\xi$ be an irreducible algebraic representation of $G$ over $\overline{\mathbb{Q}}_\ell$. We set $G^c := G/Z_{ac}$, and assume that $\xi$ factors through $G^c$.[7] In the following we construct $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathrm{Sh}_K$ associated with $\xi$, for all sufficiently small $K$, by applying the general formalism in §1.5.2. This construction is well

[7] In [Mil90a, §III], $G^c$ is defined to be $G/Z_s$, where $Z_s$ is the maximal $\mathbb{Q}$-subtorus of $Z^0$ that is $\mathbb{Q}$-anisotropic and $\mathbb{R}$-split. Note that it is assumed in [Mil90a, §II, (2.1.4)] that $Z^0$ splits over a CM field. Under this assumption, $Z_s$ is equal to $Z_{ac}$. In general, the two can be different.

known. See for instance [Kot92b, §6], [Pin92a, §5], [HT01, §3.2], and [LS18, §3], which give this construction at different levels of generality. Note that in all but the last reference, the Shimura varieties being considered satisfy $G = G^c$.

Let $K$ and $U$ be neat compact open subgroups of $G(\mathbb{A}_f)$, with $U$ normal in $K$. Since each neat congruence subgroup $\Lambda$ of $G(\mathbb{Q})$ acts on $X$ with kernel $\Lambda \cap Z(\mathbb{Q})$, we have

$$\mathrm{Gal}(\mathrm{Sh}_U / \mathrm{Sh}_K) = K/(Z(\mathbb{Q})_K U). \tag{1.5.8.1}$$

Write $Z(\mathbb{Q})^-$ for the closure of $Z(\mathbb{Q})$ in $Z(\mathbb{A}_f)$, and write $Z(\mathbb{Q})^-_K$ for the intersection $Z(\mathbb{Q})^- \cap K$ inside $G(\mathbb{A}_f)$. Note that $Z(\mathbb{Q})^-_K$ is also the closure of $Z(\mathbb{Q})_K$ inside $K$, since $K$ is open and closed in $G(\mathbb{A}_f)$. Define $\mathrm{Gal}(\mathrm{Sh} / \mathrm{Sh}_K)$ as in (1.5.2.1). By (1.5.8.1), we have

$$\mathrm{Gal}(\mathrm{Sh} / \mathrm{Sh}_K) \cong K/Z(\mathbb{Q})^-_K, \tag{1.5.8.2}$$

(cf. [Del79, §2.1.9]). By Lemma 1.5.7, the natural map $K \to G^c(\mathbb{A}_f)$ factors through $\mathrm{Gal}(\mathrm{Sh} / \mathrm{Sh}_K)$.[8]

Via the projection $G(\mathbb{A}_f) \to G(\mathbb{Q}_\ell)$, we obtain a continuous representation of $G(\mathbb{A}_f)$ on a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space induced by $\xi$. This continuous representation satisfies the assumption in §1.5.2, namely its restriction to $K$ factors through $\mathrm{Gal}(\mathrm{Sh} / \mathrm{Sh}_K)$ for all sufficiently small (in fact, all neat) $K$. By the construction in §1.5.2, we obtain a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathscr{L}_{\xi,K}$ on $\mathrm{Sh}_K$ for all neat $K$, and obtain the admissible $G(\mathbb{A}_f)$-module

$$\mathbf{H}^i_c(\mathrm{Sh}_{\overline{E}}, \xi) := \varinjlim_K \mathbf{H}^i_c(\mathrm{Sh}_{K,\overline{E}}, \mathscr{L}_{\xi,K}).$$

We have a natural continuous $\mathrm{Gal}(\overline{E}/E)$-action on $\mathbf{H}^i_c(\mathrm{Sh}_{\overline{E}}, \xi)$ that commutes with the $G(\mathbb{A}_f)$-action. Our main interest is to understand the virtual $G(\mathbb{A}_f) \times \mathrm{Gal}(\overline{E}/E)$-module

$$\sum_i (-1)^i \, \mathbf{H}^i_c(\mathrm{Sh}_{\overline{E}}, \xi).$$

## 1.6. **Kottwitz parameters.**

**1.6.1.** Let $(G, X)$ be a Shimura datum, and let $p$ be a prime number. In this subsection we define *Kottwitz parameters* with respect to $(G, X)$ and $p$, generalizing the considerations in [Kot90, §2] where $G_{\mathrm{der}}$ is assumed to be simply connected.

Let $E \subset \mathbb{C}$ be the reflex field of $(G, X)$. From the fixed embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, we obtain a prime $\mathfrak{p}$ of $E$ over $p$. Let $p^r$ be the cardinality of the residue field of $\mathfrak{p}$. Fix a positive multiple $n$ of $r$.

The Hodge cocharacters attached to $h \in X$ all have the same image in $\pi_1(G)$, which we denote by $[\mu]_X \in \pi_1(G)$.

We say that an element $\gamma_0 \in G(\mathbb{Q})$ is *semi-simple and* $\mathbb{R}$*-elliptic*, if $\epsilon \in T(\mathbb{R})$ for some elliptic maximal torus $T$ in $G_{\mathbb{R}}$. Since $G$ is part of a Shimura datum, $G^{\mathrm{ad}}_{\mathbb{R}}$ admits a Cartan involution. Therefore $G_{\mathbb{R}}$ contains elliptic maximal tori, and our definition of $\mathbb{R}$-elliptic elements agrees with the more general definition in the literature, cf. [Kot86, §9.1].

[8] In fact, the induced map $\mathrm{Gal}(\mathrm{Sh} / \mathrm{Sh}_K) \cong K/Z(\mathbb{Q})^-_K \to G^c(\mathbb{A}_f)$ is never injective, if $Z_{ac}$ is non-trivial. This follows from the fact that $Z_{ac}(\mathbb{Q})(K \cap Z_{ac}(\mathbb{A}_f))$ has finite index in $Z_{ac}(\mathbb{A}_f)$ ([Bor63, Thm. 5.1]), and the fact that $Z_{ac}(\mathbb{Q})^-$ has infinite index in $Z_{ac}(\mathbb{A}_f)$ ([PR94, Prop. 7.13(2)]). In [Mil90a, §III, Rmk. 6.1] it is incorrectly stated that $\mathrm{Gal}(\mathrm{Sh} / \mathrm{Sh}_K)$ is isomorphic to the image of $K$ in $G^c(\mathbb{A}_f)$, cf. §3 of the updated online version of [LS18] and its erratum.

**Definition 1.6.2.** A *classical Kottwitz parameter of degree* $n$ (with respect to $(G, X)$ and $p$) is a triple

$$(\gamma_0, \gamma = (\gamma_l)_{l \neq p}, \delta) \in G(\mathbb{Q}) \times G(\mathbb{A}_f^p) \times G(\mathbb{Q}_{p^n}),$$

satisfying the following conditions.

**CKP1:** The element $\gamma_0 \in G(\mathbb{Q})$ is semi-simple and $\mathbb{R}$-elliptic.
**CKP2:** For each prime $l \neq p$, $\gamma_l$ is stably conjugate to $\gamma_0$ as elements of $G(\mathbb{Q}_l)$.
**CKP3:** The image of $\gamma_0$ in $G(\mathbb{Q}_p)$ is a degree $n$ norm of $\delta$ (see [Kot82, §5]).
**CKP4:** The image of $\delta$ under the Kottwitz map $\kappa_G : \mathrm{B}(G_{\mathbb{Q}_p}) \to \pi_1(G)_{\Gamma_p}$ is equal to the image of $-[\mu]_X$.

We denote by $\mathfrak{KP}_{\mathrm{cla}}(p^n)$ the set of classical Kottwitz parameters of degree $n$.

**1.6.3.** We define an equivalence relation $\sim$ on $G(\mathbb{Q}) \times G(\mathbb{A}_f^p) \times G(\mathbb{Q}_{p^n})$ by declaring $(\gamma_0, \gamma, \delta) \sim (\gamma_0{}', \gamma', \delta')$ if the following conditions are satisfied:

- The elements $\gamma_0$ and $\gamma_0{}'$ are stably conjugate in $G(\mathbb{Q})$.
- The elements $\gamma$ and $\gamma'$ are conjugate in $G(\mathbb{A}_f^p)$.
- The elements $\delta$ and $\delta'$ are $\sigma$-conjugate in $G(\mathbb{Q}_{p^n})$.

The subset $\mathfrak{KP}_{\mathrm{cla}}(p^n) \subset G(\mathbb{Q}) \times G(\mathbb{A}_f^p) \times G(\mathbb{Q}_{p^n})$ is stable under $\sim$.

**Definition 1.6.4.** A *Kottwitz parameter* (with respect to $(G, X)$ and $p$) is a tuple $(\gamma_0, a, [b])$, consisting of:

- a semi-simple and $\mathbb{R}$-elliptic element $\gamma_0 \in G(\mathbb{Q})$,
- an element $a \in \mathfrak{D}(G_{\gamma_0}^0, G; \mathbb{A}_f^p)$,
- a $\sigma$-conjugacy class $[b]$ in $G_{\gamma_0}^0(\breve{\mathbb{Q}}_p)$ (i.e., $[b] \in \mathrm{B}(G^0_{\gamma_0, \mathbb{Q}_p})$),

satisfying the following condition.

**KP0:** Let $[b]_G$ be the image of $[b]$ in $\mathrm{B}(G_{\mathbb{Q}_p})$. Then the element $\kappa_G([b]_G) \in \pi_1(G)_{\Gamma_p}$ is equal to the image of $-[\mu]_X$.

We denote by $\mathfrak{KP}$ the set of all Kottwitz parameters. For $\mathfrak{c} = (\gamma_0, a, [b]) \in \mathfrak{KP}$, we write $I_0(\mathfrak{c})$ for $G_{\gamma_0}^0$. When $\mathfrak{c}$ is fixed in the context we also simply write $I_0$ for $I_0(\mathfrak{c})$.

**Definition 1.6.5.** We say that $(\gamma_0, a, [b]) \in \mathfrak{KP}$ is $p^n$-*admissible*, if it satisfies the following condition.

**KP1:** Let $b \in I_0(\breve{\mathbb{Q}}_p)$ be a representative of $[b]$. There exists $c \in G(\breve{\mathbb{Q}}_p)$ such that, letting $\delta := c^{-1} b \sigma(c) \in G(\breve{\mathbb{Q}}_p)$, we have

$$c^{-1} \gamma_0 c = \delta \cdot \sigma(\delta) \cdots \sigma^{n-1}(\delta). \tag{1.6.5.1}$$

(Clearly this condition is independent of the representative $b$ of $[b]$.) We denote by $\mathfrak{KP}_{\mathrm{a}}(p^n)$ the set of $p^n$-admissible Kottwitz parameters.

**1.6.6.** Next we deduce some consequences of the condition **KP1**. We shall work in a local setting as follows. Let $\gamma_0 \in G(\mathbb{Q}_p)_{\mathrm{ss}}$ and let $I_0 := (G_{\mathbb{Q}_p})_{\gamma_0}^0$. Let $[b] \in \mathrm{B}(I_0)$, and assume that **KP1** holds for $[b]$ and $\gamma_0$.

**Lemma 1.6.7.** *Keep the setting of §1.6.6. Choose $b$ and $c$ as in **KP1** with respect to $[b]$ and $\gamma_0$. Let $\delta = c^{-1} b \sigma(c)$. Then we have $\delta \in G(\mathbb{Q}_{p^n})$. The $\sigma$-conjugacy class of $\delta$ in $G(\mathbb{Q}_{p^n})$ depends only on $[b] \in \mathrm{B}(I_0)$, not on the choices of $b$ and $c$. Moreover, $\gamma_0 \in G(\mathbb{Q}_p)$ is a degree $n$ norm of $\delta \in G(\mathbb{Q}_{p^n})$.*

*Proof.* By (1.6.5.1) we have $(\delta \rtimes \sigma)^n = c^{-1}\gamma_0 c \rtimes \sigma^n$. Since $b$ and $1 \rtimes \sigma$ both commute with $\gamma_0$, we know that $\delta \rtimes \sigma = c^{-1}(b \rtimes \sigma)c$ commutes with $c^{-1}\gamma_0 c$. Hence $\delta \rtimes \sigma$ commutes with $\sigma^n$, which means that $\delta \in G(\mathbb{Q}_{p^n})$.

To prove the second statement, we first note that when $b$ is fixed, any choice of $c$ has to satisfy the equation

$$\gamma_0 c = b\sigma(b) \cdots \sigma^{n-1}(b)\sigma^n(c).$$

Hence two choices of $c$ give rise to the same coset $cG(\mathbb{Q}_{p^n})$. It follows that the $\sigma$-conjugacy class of $\delta$ in $G(\mathbb{Q}_{p^n})$ is independent of the choice of $c$. Now suppose we choose another representative $b' \in I_0(\breve{\mathbb{Q}}_p)$ of $[b] \in \mathrm{B}(I_0)$. Then $b' = db\sigma(d)^{-1}$ for some $d \in I_0(\breve{\mathbb{Q}}_p)$. Letting $c' := dc$, we have $c'^{-1}b'\sigma(c') = \delta$, and

$$c'^{-1}\gamma_0 c' = c^{-1}\gamma_0 c = \delta\sigma(\delta) \cdots \sigma^{n-1}(\delta).$$

Hence $b'$ still determines the same $\delta \in G(\mathbb{Q}_{p^n})$ up to $\sigma$-conjugacy.

Finally, we show that $\gamma_0$ is a degree $n$ norm of $\delta$. If $G_{\mathrm{der}}$ is simply connected, then the statement follows from (1.6.5.1) and the definition of norm. In general, take a $z$-extension $1 \to Z \to H \to G_{\mathbb{Q}_p} \to 1$ over $\mathbb{Q}_p$. Let $\widetilde{c} \in H(\breve{\mathbb{Q}}_p)$ be a lift of $c \in G(\breve{\mathbb{Q}}_p)$, and let $\widetilde{\delta} \in H(\mathbb{Q}_{p^n})$ be a lift of $\delta \in G(\mathbb{Q}_{p^n})$. Define $\widetilde{\gamma}_0 \in H(\breve{\mathbb{Q}}_p)$ by the equation

$$\widetilde{c}^{-1}\widetilde{\gamma}_0\widetilde{c} = \widetilde{\delta}\sigma(\widetilde{\delta}) \cdots \sigma^{n-1}(\widetilde{\delta}). \tag{1.6.7.1}$$

Then $\widetilde{\gamma}_0$ is a lift of $\gamma_0$. We claim that $\widetilde{\gamma}_0 \in H(\mathbb{Q}_p)$. Once the claim is proved, we know that $\widetilde{\gamma}_0$ is a degree $n$ norm of $\widetilde{\delta}$ by (1.6.7.1), and it follows that $\gamma_0$ is a degree $n$ norm of $\delta$ (see [Kot82, §5]).

It remains to prove the claim. Let $\widetilde{b} = \widetilde{c}\widetilde{\delta}\sigma(\widetilde{c})^{-1} \in H(\breve{\mathbb{Q}}_p)$. By (1.6.7.1) we have $(\widetilde{\delta} \rtimes \sigma)^n = \widetilde{c}^{-1}\widetilde{\gamma}_0\widetilde{c} \rtimes \sigma^n$. Since $\widetilde{\delta} \rtimes \sigma$ commutes with $\sigma^n$ (i.e., $\widetilde{\delta} \in H(\mathbb{Q}_{p^n})$), it must commute with $\widetilde{c}^{-1}\widetilde{\gamma}_0\widetilde{c}$. Hence $\widetilde{b} \rtimes \sigma = \widetilde{c}(\widetilde{\delta} \rtimes \sigma)\widetilde{c}^{-1}$ commutes with $\widetilde{\gamma}_0$. On the other hand since $\widetilde{b}$ is a lift of $b \in I_0$, we know that $\widetilde{b}$ commutes with $\widetilde{\gamma}_0$ by [Kot82, Lem. 3.1 (1)]. Therefore $\sigma$ commutes with $\widetilde{\gamma}_0$, which proves the claim. $\square$

**Lemma 1.6.8.** *Let $\mathcal{H}$ be a smooth affine group scheme over $\mathbb{Z}_p$ with connected fibers. Then every element of $\mathcal{H}(\breve{\mathbb{Q}}_p)/\mathcal{H}(\breve{\mathbb{Z}}_p)$ is fixed by some power of $\sigma$. The natural map $\mathcal{H}(\mathbb{Q}_p^{\mathrm{ur}})/\mathcal{H}(\mathbb{Z}_p^{\mathrm{ur}}) \to \mathcal{H}(\breve{\mathbb{Q}}_p)/\mathcal{H}(\breve{\mathbb{Z}}_p)$ is a bijection.*

*Proof.* By [Bro13, Lem. 3.2], there exists a closed embedding of $\mathbb{Z}_p$-groups $\mathcal{H} \to \mathrm{GL}_n$. Hence the first statement in the lemma reduces to the case where $\mathcal{H} = \mathrm{GL}_n$, and it follows from the fact that $\mathrm{GL}_n(\mathbb{Q}_p^{\mathrm{ur}})$ is dense in $\mathrm{GL}_n(\breve{\mathbb{Q}}_p)$ under the $p$-adic topology. The second statement (for general $\mathcal{H}$) follows from the first statement and Greenberg's theorem [Gre63, Prop. 3] asserting the surjectivity of the map $\mathcal{H}(\breve{\mathbb{Z}}_p) \to \mathcal{H}(\breve{\mathbb{Z}}_p), g \mapsto g \cdot \sigma^n(g)^{-1}$ for arbitrary $n \in \mathbb{Z}_{\geq 1}$. $\square$

**Lemma 1.6.9.** *Keep the setting of §1.6.6. Let $b \in I_0(\breve{\mathbb{Q}}_p)$ be a decent representative of $[b] \in \mathrm{B}(I_0)$ (see §1.4.2). Then there exists $t \in \mathbb{Z}_{\geq 1}$ such that $b$ is $t$-decent, and such that*

$$\gamma_0^t = p^{nt\nu_b}k, \tag{1.6.9.1}$$

*where $k$ lies in a bounded subgroup of $G(\breve{\mathbb{Q}}_p)$ (in the sense of* [Tit79, §2.2]*).*

*Proof.* Assume $b$ is $t_0$-decent. Let $c$ be as in **KP1**. Let $\mathcal{G}$ be a parahoric model of $G_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$, and write $\bar{c}$ for the image of $c$ in $\in G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p)$. By (1.6.5.1), we have $\gamma_0 \cdot \bar{c} = (b \rtimes \sigma)^n \cdot \bar{c}$. Since $\gamma_0$ commutes with $b$ and $1 \rtimes \sigma$, for any multiple $t$ of $t_0$ we have

$$\gamma_0^t \cdot \bar{c} = (b \rtimes \sigma)^{nt} \cdot \bar{c} = (p^{nt\nu_b} \rtimes \sigma^{nt}) \cdot \bar{c}.$$

By Lemma 1.6.8, when $t$ is sufficiently divisible we have $\sigma^{nt}\bar{c} = \bar{c}$. Then $k := p^{-nt\nu_b}\gamma_0^t$ lies in $c\mathcal{G}(\breve{\mathbb{Z}}_p)c^{-1}$, which is a bounded subgroup of $G(\breve{\mathbb{Q}}_p)$. □

One can view (1.6.9.1) as a "polar decomposition" of $\gamma_0^t$. Such a decomposition satisfies a very strong sense of uniqueness, as specified in the following lemma.

**Lemma 1.6.10.** *Let $F$ be a complete discretely valued field. Let $H$ be a linear algebraic group over $F$, and let $\epsilon$ be a semi-simple element of $H(\overline{F})$. Then there exists at most one cocharacter $\nu$ of $H_\epsilon$ defined over $\overline{F}$ such that for some uniformizer $\pi \in F^\times$, $\epsilon^{-1}\pi^\nu$ lies in a $H(\overline{F})$-conjugate of a bounded subgroup of $H(F)$.*

*Proof.* Let $\rho : H \to \mathrm{GL}_N$ be a faithful representation of $H$ over $F$. If $k$ is an element of a bounded subgroup of $H(F)$, then $\rho(k)$ lies in a $\mathrm{GL}_N(F)$-conjugate of $\mathrm{GL}_N(\mathcal{O}_F)$ (cf. the proof of [Kis10, Lem. 2.3.1]), and hence all eigenvalues of $\rho(k)$ have valuation zero. (Here and below, by eigenvalues we always mean eigenvalues in $\overline{F}$.) To prove the lemma it suffices to prove that for each semi-simple $\epsilon \in \mathrm{GL}_N(\overline{F})$, there exists at most one cocharacter $\nu$ of $\mathrm{GL}_{N,\epsilon,\overline{F}}$ such that for some uniformizer $\pi \in F$ all eigenvalues of $\epsilon^{-1}\pi^\nu$ have valuation zero.

Without loss of generality we may assume that $\epsilon = \mathrm{diag}(\lambda_1 I_{N_1}, \cdots, \lambda_r I_{N_r})$ with distinct $\lambda_1, \cdots, \lambda_r \in \overline{F}^\times$. Then $\mathrm{GL}_{N,\epsilon}$ is naturally identified with $\mathrm{GL}_{N_1} \times \cdots \times \mathrm{GL}_{N_r}$. Note that if a semi-simple element $k \in \mathrm{GL}_{N,\epsilon}(\overline{F})$ is such that all its eigenvalues have valuation zero, then the projection of $k$ in each $\mathrm{GL}_{N_i}(\overline{F})$ satisfies the analogous condition. We have thus reduced to the case where $\epsilon$ is central in $\mathrm{GL}_N(\overline{F})$. In this case, if $\nu$ is a cocharacter of $\mathrm{GL}_{N,\overline{F}}$ such that for some uniformizer $\pi$ all eigenvalues of $\epsilon^{-1}\pi^\nu$ have valuation zero, then $\nu$ must be given by $z \mapsto \mathrm{diag}(z^m, \cdots, z^m)$ where $m$ is the valuation of the unique eigenvalue of $\epsilon$. This proves the uniqueness of $\nu$. □

**Corollary 1.6.11.** *Keep the setting of §1.6.6. Then $[b]$ is basic in $\mathrm{B}(I_0)$. If $[b'] \in \mathrm{B}(I_0)$ is another class satisfying **KP1** with respect to $\gamma_0$, then $\nu_b = \nu_{b'}$.*

*Proof.* Let $b$ be a decent representative of $[b]$. By Lemma 1.6.9 and Lemma 1.6.10 (the latter applied to $F = \breve{\mathbb{Q}}_p$, $\pi = p$, $H = G_{\breve{\mathbb{Q}}_p}$, $\epsilon = \gamma_0$), any element of $G(\breve{\mathbb{Q}}_p)$ that centralizes $\gamma_0$ has to centralize $\nu_b$. Therefore $\nu_b$ factors through the center of $I_0$, and $[b]$ is basic in $\mathrm{B}(I_0)$. The second statement also follows from these two lemmas. □

**Corollary 1.6.12.** *Let $\mathfrak{c} = (\gamma_0, a, [b]) \in \mathfrak{KP}(p^n)$. Then $[b]$ is basic in $\mathrm{B}(I_0(\mathfrak{c})_{\mathbb{Q}_p})$. If $(\gamma_0, a', [b'])$ is another element of $\mathfrak{KP}(p^n)$, then $\nu_b = \nu_{b'}$.*

*Proof.* This follows from Corollary 1.6.11. □

**1.6.13.** Our next goal is to define the notion of an isomorphism between Kottwitz parameters. We first consider a general construction. Let $\gamma_0 \in G(\mathbb{Q})_{\mathrm{ss}}$, and let $I_0 := G^0_{\gamma_0}$. Let $u \in G(\overline{\mathbb{Q}})$ be an element satisfying

$$\gamma_0' := u\gamma_0 u^{-1} \in G(\mathbb{Q})$$

and

$$u^{-1\rho}u \in I_0, \quad \forall \rho \in \Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

Let $I_0' := G^0_{\gamma_0'}$. We have a bijection

$$u_* : \mathfrak{D}(I_0, G; \mathbb{A}_f^p) \xrightarrow{\sim} \mathfrak{D}(I_0', G; \mathbb{A}_f^p), \tag{1.6.13.1}$$

sending the class of a cocycle $(a_\rho)_\rho \in Z^1(\Gamma, I_0(\bar{\mathbb{A}}_f^p))$ to the class of $(ua_\rho{}^\rho u^{-1})_\rho$. Next note that the cocycle $(u^{-1\rho}u)_\rho \in Z^1(\mathbb{Q}_p, I_0)$ becomes trivial in $\mathbf{H}^1(\breve{\mathbb{Q}}_p, I_0)$ by Steinberg's theorem. Hence there exists $d \in I_0(\overline{\breve{\mathbb{Q}}}_p)$ such that

$$u^{-1\rho}u = d^{-1\rho}d, \quad \forall \rho \in \Gamma_{p,0}.$$

Then $u_0 := ud^{-1}$ lies in $G(\breve{\mathbb{Q}}_p)$. We have

$$u_0\gamma_0 u_0^{-1} = \gamma_0',$$

and

$$u_0^{-1\sigma}u_0 \in I_0(\breve{\mathbb{Q}}_p).$$

By the previous two properties of $u_0$, we have a bijection

$$\begin{aligned} u_* : \mathrm{B}(I_{0,\mathbb{Q}_p}) &\xrightarrow{\sim} \mathrm{B}(I'_{0,\mathbb{Q}_p}) \\ [b] &\longmapsto [u_0 b\sigma(u_0)^{-1}], \end{aligned} \tag{1.6.13.2}$$

which depends only on $u$, not on the choice of $d$.

**Definition 1.6.14.** Let $\mathfrak{c} = (\gamma_0, a, [b]), \mathfrak{c}' = (\gamma_0', a', [b']) \in \mathfrak{KP}$. By an *isomorphism from* $\mathfrak{c}$ *to* $\mathfrak{c}'$, we mean an element $u \in G(\overline{\mathbb{Q}})$, satisfying the following conditions.

(i) We have $\mathrm{Int}(u)\gamma_0 = \gamma_0'$, and $u^{-1\rho}u \in I_0(\mathfrak{c})$ for all $\rho \in \Gamma$.

(ii) The bijection $u_* : \mathfrak{D}(I_0(\mathfrak{c}), G; \mathbb{A}_f^p) \xrightarrow{\sim} \mathfrak{D}(I_0(\mathfrak{c}'), G; \mathbb{A}_f^p)$ as in (1.6.13.1) sends $a$ to $a'$.

(iii) The bijection $u_* : \mathrm{B}(I_0(\mathfrak{c})_{\mathbb{Q}_p}) \xrightarrow{\sim} \mathrm{B}(I_0(\mathfrak{c}')_{\mathbb{Q}_p})$ as in (1.6.13.2) sends $[b]$ to $[b']$.

In such a situation we write $u : \mathfrak{c} \xrightarrow{\sim} \mathfrak{c}'$.

**1.6.15.** If $u : \mathfrak{c} \xrightarrow{\sim} \mathfrak{c}'$ and $v : \mathfrak{c}' \xrightarrow{\sim} \mathfrak{c}''$ are two isomorphisms between Kottwitz parameters, then $vu \in G(\overline{\mathbb{Q}})$ is an isomorphism $\mathfrak{c} \xrightarrow{\sim} \mathfrak{c}''$, and $u^{-1} \in G(\overline{\mathbb{Q}})$ is the isomorphism $\mathfrak{c}' \xrightarrow{\sim} \mathfrak{c}$ inverse to $u$. Moreover, one checks that $p^n$-admissibility of Kottwitz parameters (Definition 1.6.5) is preserved under isomorphisms.

We denote by $\mathfrak{KP}/\cong$ the set of isomorphism classes of Kottwitz parameters, and by $\mathfrak{KP}_{\mathrm{a}}(p^n)/\cong$ the set of isomorphism classes of $p^n$-admissible Kottwitz parameters.

**1.6.16.** We define a natural map

$$\mathfrak{KP}_{\mathrm{a}}(p^n) \longrightarrow \mathfrak{KP}_{\mathrm{cla}}(p^n)/\sim \tag{1.6.16.1}$$

as follows. Let $\mathfrak{c} = (\gamma_0, a, [b]) \in \mathfrak{KP}_{\mathrm{a}}(p^n)$. The element $a \in \mathfrak{D}(I_0(\mathfrak{c}), G; \mathbb{A}_f^p)$ determines a conjugacy class in $G(\mathbb{A}_f^p)$ which is stably conjugate to $\gamma_0$. Take $\gamma$ to be an arbitrary element of this conjugacy class. By Lemma 1.6.7, $[b]$ determines a $\sigma$-conjugacy class in $G(\mathbb{Q}_{p^n})$, of which $\gamma_0$ is a degree $n$ norm. Take $\delta$ to be an arbitrary element of this $\sigma$-conjugacy class. Then $(\gamma_0, \gamma, \delta)$ is an element of $\mathfrak{KP}_{\mathrm{cla}}(p^n)$, and its equivalence class depends only on $\mathfrak{c}$. We define the map (1.6.16.1) by sending $\mathfrak{c}$ to the equivalence class of $(\gamma_0, \gamma, \delta)$.

One checks that the map (1.6.16.1) factors through $\mathfrak{KP}_{\mathrm{a}}(p^n)/\cong$. Moreover, when $G_{\mathrm{der}}$ is simply connected, the induced map $\mathfrak{KP}_{\mathrm{a}}(p^n)/\cong \longrightarrow \mathfrak{KP}_{\mathrm{cla}}(p^n)/\sim$ is a bijection. We will not need this fact in the paper, but we outline here how to recover $[b] \in \mathrm{B}(I_0(\mathfrak{c})_{\mathbb{Q}_p})$ from $\delta$, which is perhaps the only non-obvious part of the argument. Since $G_{\mathrm{der}}$ is simply connected, $I_0(\mathfrak{c}) = G_{\gamma_0}$. Since $\gamma_0$ is a degree $n$ norm of $\delta$ and since $\mathbf{H}^1(\breve{\mathbb{Q}}_p, I_0)$ is trivial by Steinberg's theorem, there exists $c \in G(\breve{\mathbb{Q}}_p)$ such that (1.6.5.1) holds. Define $b := c\delta\sigma(c) \in G(\breve{\mathbb{Q}}_p)$. Since $(\delta \rtimes \sigma)^n = c^{-1}\gamma_0 c \rtimes \sigma^n$ and since $\delta \rtimes \sigma$ commutes with $\sigma^n$, we know that $\delta \rtimes \sigma$ commutes with $c^{-1}\gamma_0 c$, or equivalently that $b \rtimes \sigma$ commutes with $\gamma_0$. Since $\gamma_0$ is $\sigma$-invariant, we know that $b$ commutes with $\gamma_0$, i.e., $b \in I_0(\mathfrak{c})(\breve{\mathbb{Q}}_p)$. In this way we have recovered $[b] \in \mathrm{B}(I_0(\mathfrak{c})_{\mathbb{Q}_p})$ from $\delta$.

## 1.7. **The Kottwitz invariant.**

**1.7.1.** In this subsection we define an invariant attached to each Kottwitz parameter. We first construct the abelian group where the invariant lies in. We start with a general setting.

Let $G$ be a reductive group over $\mathbb{Q}$, and let $I$ be a reductive subgroup of $G$. We have an infinite commutative diagram with exact rows and columns

$$\begin{tikzcd}
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & \mathbf{H}^i_{\mathrm{ab}}(\mathbb{Q}, I) \arrow[r] \arrow[d] & \mathbf{H}^i_{\mathrm{ab}}(\mathbb{Q}, G) \arrow[r] \arrow[d] & \mathbf{H}^i_{\mathrm{ab}}(\mathbb{Q}, I \to G) \arrow[r] \arrow[d] & \mathbf{H}^{i+1}_{\mathrm{ab}}(\mathbb{Q}, I) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathbf{H}^i_{\mathrm{ab}}(\mathbb{A}, I) \arrow[r] \arrow[d] & \mathbf{H}^i_{\mathrm{ab}}(\mathbb{A}, G) \arrow[r] \arrow[d] \arrow[dr, dashed] & \mathbf{H}^i_{\mathrm{ab}}(\mathbb{A}, I \to G) \arrow[r] \arrow[d] & \mathbf{H}^{i+1}_{\mathrm{ab}}(\mathbb{A}, I) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathbf{H}^i_{\mathrm{ab}}(\mathbb{A}/\mathbb{Q}, I) \arrow[r] \arrow[d] & \mathbf{H}^i_{\mathrm{ab}}(\mathbb{A}/\mathbb{Q}, G) \arrow[r] \arrow[d] & \mathbf{H}^i_{\mathrm{ab}}(\mathbb{A}/\mathbb{Q}, I \to G) \arrow[r] \arrow[d] & \mathbf{H}^{i+1}_{\mathrm{ab}}(\mathbb{A}/\mathbb{Q}, I) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathbf{H}^{i+1}_{\mathrm{ab}}(\mathbb{Q}, I) \arrow[r] \arrow[d] & \mathbf{H}^{i+1}_{\mathrm{ab}}(\mathbb{Q}, G) \arrow[r] \arrow[d] & \mathbf{H}^{i+1}_{\mathrm{ab}}(\mathbb{Q}, I \to G) \arrow[r] \arrow[d] & \mathbf{H}^{i+2}_{\mathrm{ab}}(\mathbb{Q}, I) \arrow[r] \arrow[d] & \cdots \\
& \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

We define $\mathfrak{E}(I, G; \mathbb{A}/\mathbb{Q})$ to be the cokernel of the map

$$\mathbf{H}^0_{\mathrm{ab}}(\mathbb{A}, G) \longrightarrow \mathbf{H}^0_{\mathrm{ab}}(\mathbb{A}/\mathbb{Q}, I \to G)$$

given by the dashed arrow in the above diagram for $i = 0$. We have a natural map $\mathfrak{E}(I, G; \mathbb{A}) \to \mathfrak{E}(I, G; \mathbb{A}/\mathbb{Q})$ defined by first lifting an element of $\mathfrak{E}(I, G; \mathbb{A})$ to $\mathbf{H}^0_{\mathrm{ab}}(\mathbb{A}, I \to G)$, and then mapping the lift to $\mathbf{H}^0_{\mathrm{ab}}(\mathbb{A}/\mathbb{Q}, I \to G)$ and then to $\mathfrak{E}(I, G; \mathbb{A}/\mathbb{Q})$. We know that the sequence

$$\mathfrak{E}(I, G; \mathbb{Q}) \to \mathfrak{E}(I, G; \mathbb{A}) \to \mathfrak{E}(I, G; \mathbb{A}/\mathbb{Q}) \tag{1.7.1.1}$$

is exact; see [Lab99, Prop. 1.8.4].

We write $K$ for the kernel of $\pi_1(I) \to \pi_1(G)$. Recall that the functor $\mathcal{A}_{\mathbb{Q}}$ is introduced in §1.1.8. As usual we write $\Gamma$ for $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

**Lemma 1.7.2.** *The natural inclusions*

$$\ker\Big(\mathcal{A}_{\mathbb{Q}}(K) \to \mathcal{A}_{\mathbb{Q}}(\pi_1(I))\Big) \hookrightarrow \ker\Big(K_{\Gamma,\mathrm{tors}} \to \pi_1(I)_{\Gamma,\mathrm{tors}}\Big) \hookrightarrow \ker\Big(K_{\Gamma} \to \pi_1(I)_{\Gamma}\Big)$$

*are equalities.*

*Proof.* As in the proof of [Mil92, Prop. B.4], there exists a finite Galois extension $E/\mathbb{Q}$ such that the actions of $\Gamma$ on $K$, $\pi_1(I)$, and $\pi_1(G)$ all factor through $\mathrm{Gal}(E/\mathbb{Q})$ and such that

$$\mathcal{A}_{\mathbb{Q}}(K) = \widehat{\mathbf{H}}^{-1}(\mathrm{Gal}(E/\mathbb{Q}), K),$$
$$\mathcal{A}_{\mathbb{Q}}(\pi_1(I)) = \widehat{\mathbf{H}}^{-1}(\mathrm{Gal}(E/\mathbb{Q}), \pi_1(I)).$$

Thus the first group in the lemma is equal to the kernel of

$$\widehat{\mathbf{H}}^{-1}(\mathrm{Gal}(E/\mathbb{Q}), K) \longrightarrow \widehat{\mathbf{H}}^{-1}(\mathrm{Gal}(E/\mathbb{Q}), \pi_1(I)).$$

From this, it is clear that the first and third groups in the lemma are both canonically identified with the cokernel of

$$\widehat{\mathbf{H}}^{-2}(\mathrm{Gal}(E/\mathbb{Q}), \pi_1(I)) \longrightarrow \widehat{\mathbf{H}}^{-2}(\mathrm{Gal}(E/\mathbb{Q}), \pi_1(G)).$$

The lemma follows. □

**Proposition 1.7.3.** *Assume that $I$ contains a maximal torus in $G$. Then the abelian group $\mathfrak{E}(I, G; \mathbb{A}/\mathbb{Q})$ is canonically identified with*

$$\frac{K_{\Gamma,\mathrm{tors}}}{\bigoplus_v \ker(K_{\Gamma_v,\mathrm{tors}} \to \pi_1(I)_{\Gamma_v})}.$$

*Here $v$ runs through all the places of $\mathbb{Q}$, and the quotient is with respect to the natural maps $K_{\Gamma_v,\mathrm{tors}} \to K_{\Gamma,\mathrm{tors}}$. In particular, $\mathfrak{E}(I, G; \mathbb{A}/\mathbb{Q})$ is finite.*

*Proof.* Under our assumption, the map $\pi_1(I) \to \pi_1(G)$ is surjective. Applying Proposition 1.1.9 (ii), we know that $\mathfrak{E}(I, G; \mathbb{A}/\mathbb{Q})$ is canonically identified with the cokernel of

$$\mathscr{P}(K) : \ker\left(\mathcal{B}_{\mathbb{Q}}(K) \to \mathcal{B}_{\mathbb{Q}}(\pi_1(I))\right) \longrightarrow \mathcal{A}_{\mathbb{Q}}(K).$$

By Lemma 1.7.2, we have

$$\ker\left(\mathcal{B}_{\mathbb{Q}}(K) \to \mathcal{B}_{\mathbb{Q}}(\pi_1(I))\right) = \bigoplus_v \ker(K_{\Gamma_v,\mathrm{tors}} \to \pi_1(I)_{\Gamma_v}).$$

The corollary follows. □

For any Hausdorff locally compact abelian group $H$, we denote by $H^D$ the Pontryagin dual. The following result is well known to experts, cf. [Lab99, p. 43, Remarque].

**Corollary 1.7.4.** *Under the assumption in Proposition 1.7.3, the abelian group $\mathfrak{E}(I, G; \mathbb{A}/\mathbb{Q})$ is canonically identified with $\mathfrak{K}(I/\mathbb{Q})^D$, where $\mathfrak{K}(I/\mathbb{Q})$ is defined in* [Kot86, §4.6].

*Proof.* We shall freely use the definitions and results in [Kot86]. Recall that $\mathfrak{K}(I/\mathbb{Q})$ is defined as the subgroup of $\pi_0([Z(\widehat{I})/Z(\widehat{G})]^\Gamma)$ consisting of those elements whose images in $\mathbf{H}^1(F, Z(\widehat{G}))$ are locally trivial.

Since $X^*(Z(\widehat{I})) \cong \pi_1(I)$ and $X^*(Z(\widehat{G})) \cong \pi_1(G)$, we have $X^*(Z(\widehat{I})/Z(\widehat{G})) \cong K$. Hence

$$X^*\left(\pi_0([Z(\widehat{I})/Z(\widehat{G})]^\Gamma)\right) \cong K_{\Gamma,\mathrm{tors}}.$$

Also, for each place $v$, we have

$$X^*\left(\pi_0(Z(\widehat{I})^{\Gamma_v})\right) \cong \pi_1(I)_{\Gamma_v,\mathrm{tors}}$$

and

$$X^*\left(\pi_0([Z(\widehat{I})/Z(\widehat{G})]^{\Gamma_v})\right) \cong K_{\Gamma_v,\mathrm{tors}}.$$

By the exact sequence

$$\pi_0(Z(\widehat{I})^{\Gamma_v}) \to \pi_0([Z(\widehat{I})/Z(\widehat{G})]^{\Gamma_v}) \to \mathbf{H}^1(F_v, Z(\widehat{G})),$$

we can identify $\mathfrak{K}(I/\mathbb{Q})$ with the set of $x \in (K_{\Gamma,\mathrm{tors}})^D$ such that for each place $v$ the composite map $x_v : K_{\Gamma_v,\mathrm{tors}} \to K_{\Gamma,\mathrm{tors}} \xrightarrow{x} \mathbb{C}^\times$ equals the composite map $K_{\Gamma_v,\mathrm{tors}} \to \pi_1(I)_{\Gamma_v,\mathrm{tors}} \xrightarrow{y_v} \mathbb{C}^\times$, for some $y_v \in (\pi_1(I)_{\Gamma_v,\mathrm{tors}})^D$. By the exact sequence

$$(\pi_1(I)_{\Gamma_v,\mathrm{tors}})^D \to (K_{\Gamma_v,\mathrm{tors}})^D \to \left(\ker(K_{\Gamma_v,\mathrm{tors}} \to \pi_1(I)_{\Gamma_v,\mathrm{tors}})\right)^D,$$

the last condition on $x_v$ is equivalent to requiring that $x$ kills the image of

$$\ker(K_{\Gamma_v,\mathrm{tors}} \to \pi_1(I)_{\Gamma_v,\mathrm{tors}})$$

in $K_{\Gamma,\mathrm{tors}}$. Comparing this description of $\mathfrak{K}(I/\mathbb{Q})$ and Proposition 1.7.3, we see that $\mathfrak{K}(I/F) \cong \mathfrak{E}(I,G;\mathbb{A}_F/F)^D$. □

**1.7.5.** We now keep the setting of §1.6.1. Let $\mathfrak{c} = (\gamma_0, a, [b]) \in \mathfrak{KP}$, and write $I_0$ for $I_0(\mathfrak{c}) = G^0_{\gamma_0}$. We now construct an element

$$\alpha(\mathfrak{c}) \in \mathfrak{E}(I_0, G; \mathbb{A}/\mathbb{Q}),$$

called the *Kottwitz invariant* of $\mathfrak{c}$. This generalizes the construction in [Kot90, §2].

We write $\beta^{p,\infty}(\mathfrak{c})$ for the element $a \in \mathfrak{D}(I_0, G; \mathbb{A}_f^p)$. As discussed in §1.1.6, the abelianization map induces an isomorphism $\mathfrak{D}(I_0, G; \mathbb{A}_f^p) \cong \mathfrak{E}(I_0, G; \mathbb{A}_f^p) \subset \mathbf{H}^1_{\mathrm{ab}}(\mathbb{A}_f^p, I_0)$. By Proposition 1.1.9 (i), we have a canonical isomorphism

$$\mathbf{H}^1_{\mathrm{ab}}(\mathbb{A}_f^p, I_0) \cong \bigoplus_{v \neq p,\infty} \pi_1(I_0)_{\Gamma_v,\mathrm{tors}}.$$

Hence we also view $\beta^{p,\infty}(\mathfrak{c})$ as an element of

$$\bigoplus_{v \neq p,\infty} \pi_1(I_0)_{\Gamma_v,\mathrm{tors}}.$$

For each place $v \notin \{p,\infty\}$, we write $\beta_v(\mathfrak{c}) \in \pi_1(I_0)_{\Gamma_v,\mathrm{tors}}$ for the component of $\beta^{p,\infty}(\mathfrak{c})$ at $v$. We pick a lift $\widetilde{\beta}_v(\mathfrak{c}) \in \pi_1(I_0)$ of $\beta_v(\mathfrak{c})$ that maps to zero in $\pi_1(G)$. Such a lift exists, since $\beta_v(\mathfrak{c})$ maps to zero in $\pi_1(G)_{\Gamma_v}$ and since the map $\pi_1(I_0) \to \pi_1(G)$ is surjective. Since $\beta_v(\mathfrak{c}) = 0$ for almost all $v$, we may and shall assume that $\widetilde{\beta}_v(\mathfrak{c}) = 0$ for almost all $v$.

Let $\beta_p(\mathfrak{c}) := \kappa_{I_0}([b]) \in \pi_1(I_0)_{\Gamma_p}$. We pick a lift $\widetilde{\beta}_p(\mathfrak{c}) \in \pi_1(I_0)$ of $\beta_p(\mathfrak{c})$ that maps to $-[\mu]_X \in \pi_1(G)$. Such a lift exists by condition **KP0** in Definition 1.6.4 and by the surjectivity of the map $\pi_1(I_0) \to \pi_1(G)$.

Now we take an elliptic maximal torus $T$ in $G_{\mathbb{R}}$ such that $\gamma_0 \in T(\mathbb{R})$. Then $T \subset I_{0,\mathbb{R}}$. Since $T$ is elliptic, there exists $h \in X$ that factors through $T$. Let $\beta_\infty(\mathfrak{c})$ be the image of $\mu_h \in X_*(T)$ in $\pi_1(I_0)_{\Gamma_\infty}$. By [Kot90, Lem. 5.1], we know that the image of $\mu_h$ in $X_*(T)_{\Gamma_\infty}$ is independent of the choice of $h$. Moreover, since all elliptic maximal tori in $I_{0,\mathbb{R}}$ are conjugate under $I_0(\mathbb{R})$, the element $\beta_\infty(\mathfrak{c})$ is independent of the choice of $(T,h)$ as above. (For more details see [Kot90, p. 167].) We pick a lift $\widetilde{\beta}_\infty(\mathfrak{c}) \in \pi_1(I_0)$ of $\beta_\infty(\mathfrak{c})$ that maps to $[\mu]_X \in \pi_1(G)$. Such a lift exists, since the image of $\beta_\infty(\mathfrak{c})$ under $\pi_1(I_0)_{\Gamma_\infty} \to \pi_1(G)_{\Gamma_\infty}$ equals the image of $[\mu]_X$, and since the map $\pi_1(I_0) \to \pi_1(G)$ is surjective.

Write $K$ for $\ker(\pi_1(I_0) \to \pi_1(G))$. By the above construction, we have an element $\widetilde{\beta}_v(\mathfrak{c}) \in K$ for each place $v \notin \{p,\infty\}$, as well as an element $\widetilde{\beta}_p(\mathfrak{c}) + \widetilde{\beta}_\infty(\mathfrak{c}) \in K$. We define

$$\widetilde{\beta}(\mathfrak{c}) := \sum_v \widetilde{\beta}_v(\mathfrak{c}) \in K.$$

Here the summation is over all places $v$ of $\mathbb{Q}$, and only finitely many terms are non-zero.

Note that $K_{\Gamma_\infty}$ is torsion. Indeed, if we take an $\mathbb{R}$-elliptic maximal torus $T$ in $I_{0,\mathbb{R}}$ and let $\widetilde{T}$ and $\widetilde{S}$ be the inverse images of $T$ in $G_{\mathrm{sc},\mathbb{R}}$ and $I_{0,\mathrm{sc},\mathbb{R}}$ respectively, then $K \cong X_*(\widetilde{T})/X_*(\widetilde{S})$. Since $\widetilde{T}$ is anisotropic over $\mathbb{R}$, we know that $X_*(\widetilde{T})_{\Gamma_\infty}$ is torsion. It follows that $K_{\Gamma_\infty}$ is torsion. In particular, $K_\Gamma$ is torsion. Hence $\widetilde{\beta}(\mathfrak{c})$ gives rise to an element $\alpha(\mathfrak{c}) \in \mathfrak{E}(I_0, G; \mathbb{A}/\mathbb{Q})$, by Proposition 1.7.3. Note that the ambiguity in $\widetilde{\beta}(\mathfrak{c})$ caused by the choices of $\widetilde{\beta}_v(\mathfrak{c})$ always comes from $\bigoplus_v \ker(K \to \pi_1(I_0)_{\Gamma_v})$. Hence $\alpha(\mathfrak{c})$ is well defined.

**1.7.6.** It is convenient to have the definition of local components of Kottwitz invariants when $\gamma_0$ is not $\mathbb{Q}$-rational for the stabilization of the trace formula. Suppose $\gamma_0$ is a semi-simple element of $G(\mathbb{A}_f^p)$. We have $I_0 = G^0_{\gamma_0}$ over $\mathbb{A}_f^p$, and we define the pointed set $\mathfrak{D}(I_0, G; \mathbb{A}_f^p)$ to be the restricted product $\prod'_v \mathfrak{D}(I_0, G; \mathbb{Q}_v)$ with respect to the trivial elements. (The cohomology of $\Gamma$ acting on $I_0(\bar{\mathbb{A}}_f^p)$ no longer makes sense.) If we are given an element $a \in \mathfrak{D}(I_0, G; \mathbb{A}_f^p)$, we will also write $\beta^{p,\infty}(\gamma_0, a)$ for $a$, generalizing the notation $\beta^{p,\infty}(\mathfrak{c})$ in §1.7.5. Similarly, given $\gamma_0 \in G(\mathbb{Q}_p)_{\mathrm{ss}}$ and given $[b] \in \mathrm{B}((G_{\mathbb{Q}_p})^0_{\gamma_0})$ satisfying **KP0** in Definition 1.6.4, we have $\beta_p(\gamma_0, [b])$ and $\widetilde{\beta}_p(\gamma_0, [b])$ (the latter involving an extra choice), generalizing $\beta_p(\mathfrak{c})$ and $\widetilde{\beta}_p(\mathfrak{c})$ in §1.7.5. Finally, starting from a semi-simple $\mathbb{R}$-elliptic element $\gamma_0 \in G(\mathbb{R})$, we can define $\beta_\infty(\gamma_0)$ and $\widetilde{\beta}_\infty(\gamma_0)$ (the latter involving an extra choice) in the same way as in §1.7.5, generalizing $\beta_\infty(\mathfrak{c})$ and $\widetilde{\beta}_\infty(\mathfrak{c})$.

**1.7.7.** Let us check that our definition of Kottwitz invariants coincides with Kottwitz's definition in [Kot90, §2], when $G_{\mathrm{der}}$ is simply connected. This verification allows us to freely import results from [Kot90] during the stabilization process.

Under the identification $\pi_1(G) \cong X^*(Z(\widehat{G}))$, we view $[\mu]_X \in \pi_1(G)$ as a character on $Z(\widehat{G})$. Under the current assumption on $G_{\mathrm{der}}$, recall that Kottwitz attaches an invariant $\alpha(\mathfrak{k}) \in \mathfrak{K}(I_0/\mathbb{Q})^D$ to a classical Kottwitz parameter $\mathfrak{k} = (\gamma_0, \gamma, \delta)$ of degree $n$. The outline is as follows. Let $I_0 := G_{\gamma_0}$, which is connected by the assumption on $G_{\mathrm{der}}$. Kottwitz first defines $\alpha_v(\mathfrak{k}) \in \pi_1(I_0)_{\Gamma_v} \cong X^*(Z(\widehat{I}_0)^{\Gamma_v})$ at every place $v$. Then the character $\alpha_v(\mathfrak{k})$ on $Z(\widehat{I}_0)^{\Gamma_v}$ can be extended to a character $\beta'_v(\mathfrak{k})$ on $Z(\widehat{I}_0)^{\Gamma_v} Z(\widehat{G})$, uniquely by the requirement that $\beta'_v(\mathfrak{k})$ is either trivial or equal to $-[\mu]$ or $[\mu]$ on $Z(\widehat{G})$, according as $v \notin \{p,\infty\}$ or $v = p$ or $v = \infty$, respectively. [9] Thereby one obtains a character $\beta(\mathfrak{k}) := \prod_v \beta'_v(\mathfrak{k})$ on $\bigcap_v Z(\widehat{I}_0)^{\Gamma_v} Z(\widehat{G})$. Since $\beta(\mathfrak{k})$ is trivial on $Z(\widehat{G})$ by construction, it gives rise to a character $\alpha(\mathfrak{k})$ on

[9] We write $\beta'$ for Kottwitz's $\beta$ to avoid conflict with our own notation.

$\left(\bigcap_v Z(\widehat{I_0})^{\Gamma_v} Z(\widehat{G})\right)/Z(\widehat{G})$. The last group is canonically isomorphic to $\mathfrak{K}(I_0/\mathbb{Q})$ if $\gamma_0$ is elliptic in $G(\mathbb{Q})$, which is true since $\gamma_0$ is $\mathbb{R}$-elliptic. We note that the canonical map from $K = \ker(\pi_1(I_0) \to \pi_1(G))$ to the character group of $\cap_v Z(\widehat{I_0})^{\Gamma_v} Z(\widehat{G})$ is compatible with the canonical map from $K$ to $\mathfrak{E}(I_0, G, \mathbb{A}/\mathbb{Q})$ via the isomorphisms

$$\mathfrak{E}(I_0, G; \mathbb{A}/\mathbb{Q}) \cong \mathfrak{K}(I_0/\mathbb{Q})^D \cong \left(\frac{\cap_v Z(\widehat{I_0})^{\Gamma_v} Z(\widehat{G})}{Z(\widehat{G})}\right)^D$$

as can be seen from the proof of Corollary 1.7.4.

When $G_{\mathrm{der}}$ is simply connected, we remarked in §1.6.16 that we have a bijection $\mathfrak{KP}_{\mathrm{a}}(p^n)/\cong \ \to \mathfrak{KP}_{\mathrm{cla}}(p^n)/\sim$. Suppose $\mathfrak{c} = (\gamma_0, a, [b]) \in \mathfrak{KP}_{\mathrm{a}}(p^n)$ corresponds to $\mathfrak{k} = (\gamma_0, \gamma, \delta) \in \mathfrak{KP}_{\mathrm{cla}}(p^n)$. We defined $\beta_v(\mathfrak{c})$ and $\widetilde{\beta}_v(\mathfrak{c})$ in §1.7.5, for each place $v$. For $v \in \{p, \infty\}$, note that $\widetilde{\beta}_v(\mathfrak{c})$ is a character on $Z(\widehat{I_0})$, extending the character $\beta_v(\mathfrak{c})$ on $Z(\widehat{I_0})^{\Gamma_v}$. Inspecting the definition we see that

$$\beta_v(\mathfrak{c}) = \alpha_v(\mathfrak{k}), \quad \forall v,$$

and

$$\widetilde{\beta}_v(\mathfrak{c})\Big|_{Z(\widehat{I_0})^{\Gamma_v} Z(\widehat{G})} = \beta'_v(\mathfrak{k}), \quad \forall v.$$

Therefore the product $\prod_v \widetilde{\beta}_v(\mathfrak{c})$ gives the element $\alpha(\mathfrak{c}) \in \mathfrak{K}(I_0/\mathbb{Q})^D$. Comparing with the definition of $\alpha(\mathfrak{c})$ in §1.7.5, we conclude that

$$\alpha(\mathfrak{c}) = \alpha(\mathfrak{k}).$$

From now on we return to the general setting, i.e., we do not assume that $G_{\mathrm{der}}$ is simply connected.

**Proposition 1.7.8.** *Let $\mathfrak{c} = (\gamma_0, a, [b])$, $\mathfrak{c}' = (\gamma_0, a', [b]) \in \mathfrak{KP}$. Write $I_0$ for the group $I_0(\mathfrak{c}) = I_0(\mathfrak{c}')$. The difference $\alpha(\mathfrak{c}) - \alpha(\mathfrak{c}') \in \mathfrak{E}(I_0, G; \mathbb{A}/\mathbb{Q})$ is equal to the image of $a - a' \in \mathfrak{D}(I_0, G; \mathbb{A}_f^p)$ under the composite map*

$$\mathfrak{D}(I_0, G; \mathbb{A}_f^p) \cong \mathfrak{E}(I_0, G; \mathbb{A}_f^p) \hookrightarrow \mathfrak{E}(I_0, G; \mathbb{A}) \to \mathfrak{E}(I_0, G; \mathbb{A}/\mathbb{Q}). \tag{1.7.8.1}$$

*Proof.* Write $K$ for $\ker(\pi_1(I_0) \to \pi_1(G))$. By Proposition 1.1.9 (ii), the diagram

$$\begin{array}{ccc} \mathbf{H}^0_{\mathrm{ab}}(\mathbb{A}, I_0 \to G) & \longrightarrow & \mathbf{H}^1(\mathbb{A}, I_0) \\ \downarrow & & \\ \mathbf{H}^0_{\mathrm{ab}}(\mathbb{A}/\mathbb{Q}, I_0 \to G) & & \end{array}$$

is canonically isomorphic to the diagram

(1.7.8.2)

$$\begin{array}{ccc} \widehat{\mathbf{H}}^{-1}(\Gamma_\infty, K) \oplus \bigoplus_{v \neq \infty} K_{\Gamma_v, \mathrm{tors}} & \longrightarrow & \widehat{\mathbf{H}}^{-1}(\Gamma_\infty, \pi_1(I_0)) \oplus \bigoplus_{v \neq \infty} \pi_1(I_0)_{\Gamma_v, \mathrm{tors}} \\ \downarrow & & \\ K_{\Gamma, \mathrm{tors}} & & \end{array}$$

For each place $v \notin \{p, \infty\}$, we choose $\widetilde{\beta}_v(\mathfrak{c})$ and $\widetilde{\beta}_v(\mathfrak{c}')$ in $K$ as in §1.7.5. The $\Gamma_v$-action on $\pi_1(G)$ factors through some finite quotient $\Gamma'_v$ of $\Gamma_v$, and we have an exact sequence $\mathbf{H}_1(\Gamma'_v, \pi_1(G)) \to K_{\Gamma_v} \to \pi_1(I_0)_{\Gamma_v}$. Since the image of $\widetilde{\beta}_v(\mathfrak{c})$ in $\pi_1(I_0)_{\Gamma_v}$ is the torsion element $\beta_v(\mathfrak{c})$, and since $\mathbf{H}_1(\Gamma'_v, \pi_1(G))$ is torsion, the image of $\widetilde{\beta}_v(\mathfrak{c})$ in $K_{\Gamma_v}$ is torsion. We denote this image by $\bar{\beta}_v(\mathfrak{c}) \in K_{\Gamma_v,\mathrm{tors}}$. Similarly we define $\bar{\beta}_v(\mathfrak{c}') \in K_{\Gamma_v,\mathrm{tors}}$. Let

$$\Delta := (\bar{\beta}_v(\mathfrak{c}) - \bar{\beta}_v(\mathfrak{c}'))_{v\neq p,\infty} \in \bigoplus_{v\neq p,\infty} K_{\Gamma_v,\mathrm{tors}}.$$

Then $\Delta$ is sent to $a-a'$ by the horizontal map in (1.7.8.2). By the above discussion, the image of $a-a'$ under (1.7.8.1) is equal to the image of $\Delta$ under the composite map

$$\bigoplus_{v\neq p,\infty} K_{\Gamma_v,\mathrm{tors}} \to K_{\Gamma,\mathrm{tors}} \to \mathfrak{E}(I_0, G; \mathbb{A}/\mathbb{Q}),$$

where the last map is the quotient map as in Proposition 1.7.3. On the other hand, by the construction of the Kottwitz invariant, the image of $\Delta$ in $\mathfrak{E}(I_0, G; \mathbb{A}/\mathbb{Q})$ is $\alpha(\mathfrak{c}) - \alpha(\mathfrak{c}')$. Hence the image of $a - a'$ under (1.7.8.1) is $\alpha(\mathfrak{c}) - \alpha(\mathfrak{c}')$. □

**1.7.9.** Let $u : \mathfrak{c} \xrightarrow{\sim} \mathfrak{c}'$ be an isomorphism between Kottwitz parameters. Then the isomorphism $\mathrm{Int}(u) : I_0(\mathfrak{c})_{\overline{\mathbb{Q}}} \xrightarrow{\sim} I_0(\mathfrak{c}')_{\overline{\mathbb{Q}}}$ is an inner twisting, and in particular it induces an isomorphism of abelian groups

$$\mathfrak{E}(I_0(\mathfrak{c}), G; \mathbb{A}/\mathbb{Q}) \xrightarrow{\sim} \mathfrak{E}(I_0(\mathfrak{c}'), G; \mathbb{A}/\mathbb{Q}) \tag{1.7.9.1}$$

The following result justifies that the Kottwitz invariant is indeed an "invariant".

**Proposition 1.7.10.** *The isomorphism (1.7.9.1) takes $\alpha(\mathfrak{c})$ to $\alpha(\mathfrak{c}')$.*

*Proof.* We write $I_0$ and $I_0'$ for $I_0(\mathfrak{c})$ and $I_0(\mathfrak{c}')$. Let $K = \ker(\pi_1(I_0) \to \pi_1(G))$ and $K' = \ker(\pi_1(I_0') \to \pi_1(G))$. The inner twisting $\mathrm{Int}(u) : I_{0,\overline{\mathbb{Q}}} \xrightarrow{\sim} I'_{0,\overline{\mathbb{Q}}}$ induces a $\Gamma$-equivariant isomorphism $\pi_1(I_0) \xrightarrow{\sim} \pi_1(I_0')$, which we denote by $f$. Then $f$ restricts to an isomorphism $K \xrightarrow{\sim} K'$. Moreover, if we identify the two sides of (1.7.9.1) with quotients of $K_{\Gamma,\mathrm{tors}}$ and $K'_{\Gamma,\mathrm{tors}}$ respectively as in Proposition 1.7.3, then (1.7.9.1) is induced by $f : K \xrightarrow{\sim} K'$.

Let $\omega \in \mathfrak{D}(I_0, G; \mathbb{Q})$ be the class of the cocycle $(u^{-1}{}^{\rho}u)_{\rho\in\Gamma}$. For each place $v$ of $\mathbb{Q}$, we denote by $\beta_v(\omega)$ the image of $\omega$ under the composite map

$$\mathbf{H}^1(\mathbb{Q}, I_0) \to \mathbf{H}^1(\mathbb{Q}_v, I_0) \xrightarrow{\mathrm{ab}^1_{\mathbb{Q}_v}} \mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}_v, I_0).$$

Using the isomorphism $\mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}_v, I_0) \cong \mathcal{A}_{\mathbb{Q}_v}(\pi_1(I_0))\pi_1(I_0)_{\Gamma_v,\mathrm{tors}}$ in Proposition 1.1.9, we also view $\beta_v(\omega)$ as an element of $\pi_1(I_0)_{\Gamma_v,\mathrm{tors}}$. We claim that for each place $v$, the image of $\beta_v(\mathfrak{c}) + \beta_v(\omega)$ under the isomorphism $\pi_1(I_0)_{\Gamma_v} \xrightarrow{\sim} \pi_1(I_0')_{\Gamma_v}$ induced by $f$ equals $\beta_v(\mathfrak{c}')$.

Our claim for $v \notin \{p, \infty\}$ follows from the following commutative diagram, which is a special case of [Bor98, Lem. 3.15.1]

$$\begin{tikzcd}
\mathfrak{D}(I_0, G; \mathbb{Q}_v) \arrow[rr, "u_*"] \arrow[d, "\mathrm{ab}^1"', "\cong"] & & \mathfrak{D}(I_0', G; \mathbb{Q}_v) \arrow[d, "\mathrm{ab}^1"', "\cong"] \\
\mathfrak{E}(I_0, G; \mathbb{Q}_v) \arrow[r, "c"] & \mathfrak{E}(I_0, G; \mathbb{Q}_v) \arrow[r, "d", "\sim"'] & \mathfrak{E}(I_0', G; \mathbb{Q}_v)
\end{tikzcd}$$

Here $u_*$ is the component at $v$ of the bijection (1.6.13.1), $c$ is the translation map $x \mapsto x - \beta_v(\omega)$, and $d$ is the group isomorphism induced by the inner twisting $\mathrm{Int}(u) : I_{0,\overline{\mathbb{Q}}} \xrightarrow{\sim} I'_{0,\overline{\mathbb{Q}}}$.

Similarly, the bijection $u_* : \mathrm{B}(I_{0,\mathbb{Q}_p}) \xrightarrow{\sim} \mathrm{B}(I'_{0,\mathbb{Q}_p})$ as in (1.6.13.2) fits in the commutative diagram

$$\begin{tikzcd} \mathrm{B}(I_{0,\mathbb{Q}_p}) \arrow[rr, "u_*"] \arrow[d, "\kappa_{I_0}"'] & & \mathrm{B}(I'_{0,\mathbb{Q}_p}) \arrow[d, "\kappa_{I'_0}"'] \\ \pi_1(I_0)_{\Gamma_p} \arrow[r, "c"] & \pi_1(I_0)_{\Gamma_p} \arrow[r, "f", "\sim"'] & \pi_1(I'_0)_{\Gamma_p} \end{tikzcd}$$

where $c$ is the translation map $x \mapsto x - \beta_p(\omega)$. To see this, we use the fact that if we choose $u_0$ as in §1.6.13, then the image of $u_0^{-1\sigma}u_0 \in I_0(\breve{\mathbb{Q}}_p)$ in $\pi_1(I_0)_{\Gamma_p}$ under $\kappa_{I_0}$ equals $\beta_p(\omega)$; see [Kot85, Rmk. 5.7] and [RV14, Rmk. 2.2 (iv)]. Our claim for $v = p$ follows from the above commutative diagram.

Now we prove the claim for $v = \infty$. As in §1.7.5, we choose $(T, h)$ to define $\beta_\infty(\mathfrak{c})$ and choose $(T', h')$ to define $\beta_\infty(\mathfrak{c}')$. Without loss of generality we may assume that $T' = \mathrm{Int}(k)(T)$ and $h = \mathrm{Int}(k) \circ h$ for some $k \in G(\mathbb{R})$. Since elliptic maximal tori transfer between inner forms, there exists $j \in I_0(\mathbb{C})$ such that the map $\mathrm{Int}(uj) : T_{\mathbb{C}} \to I'_{0,\mathbb{C}}$ is defined over $\mathbb{R}$ and has image $T'$. We have a commutative diagram

$$\begin{tikzcd} X_*(T) \arrow[r, "\mathrm{Int}(uj)", "\sim"'] \arrow[d] & X_*(T') \arrow[d] \\ \pi_1(I_0) \arrow[r, "f", "\sim"'] & \pi_1(I'_0) \end{tikzcd},$$

where the vertical maps are the natural quotient maps. Let $n := (uj)^{-1}k \in G(\mathbb{C})$. Then $n$ is in the normalizer of $T$ in $G$, and $\mathrm{Int}(uj)^{-1}(\mu_{h'}) = \mathrm{Int}(n)(\mu_h)$. Let $\Delta_\infty$ be the image of $\mathrm{Int}(n)(\mu_h) - \mu_h$ under $X_*(T) \to \pi_1(I_0)_{\Gamma_\infty}$. Then by the above discussion we know that the image of $\beta_\infty(\mathfrak{c}) + \Delta_\infty$ under $f : \pi_1(I_0)_{\Gamma_\infty} \xrightarrow{\sim} \pi_1(I'_0)_{\Gamma_\infty}$ equals $\beta_\infty(\mathfrak{c}')$. On the other hand, by [Kot90, Lem. 5.1], we have $\Delta_\infty = \beta_\infty(\omega)$. Our claim for $v = \infty$ follows.

We have proved the claim. Now for each place $v$, since $\beta_v(\omega)$ maps to zero in $\pi_1(G)_{\Gamma_v}$, there exists $\widetilde{\beta}_v(\omega) \in K$ lifting $\beta_v(\omega)$. By the claim, we may choose the lifts $\widetilde{\beta}_v(\mathfrak{c})$ and $\widetilde{\beta}_v(\mathfrak{c}')$ as in §1.7.5, in such a way that $\widetilde{\beta}_v(\mathfrak{c}) + \widetilde{\beta}_v(\omega)$ maps to $\widetilde{\beta}_v(\mathfrak{c}')$ under $f : \pi_1(I_0) \xrightarrow{\sim} \pi_1(I'_0)$. To complete the proof, it remains to show that the element

$$\Omega := \sum_v \widetilde{\beta}_v(\omega) \in K$$

is sent to zero under $K \to K_\Gamma = K_{\Gamma,\mathrm{tors}} \to \mathfrak{E}(I_0, G; \mathbb{A}/\mathbb{Q})$.

In fact, we show that there is a way to choose $\widetilde{\beta}_v(\omega)$ such that the image of $\Omega$ in $K_\Gamma$ is already zero. Pick an element $\Omega' \in \mathbf{H}^0_{\mathrm{ab}}(\mathbb{Q}, I_0 \to G)$ whose image along the surjection $\mathbf{H}^0_{\mathrm{ab}}(\mathbb{Q}, I_0 \to G) \to \mathfrak{E}(I_0, G; \mathbb{Q}) \cong \mathfrak{D}(I_0, G; \mathbb{Q})$ is $\omega$. Write $(\Omega'_v)_v$ for the image of $\Omega'$ under the composite map (see Proposition 1.1.9)

$$\mathbf{H}^0_{\mathrm{ab}}(\mathbb{Q}, I_0 \to G) \to \mathbf{H}^0_{\mathrm{ab}}(\mathbb{A}, I_0 \to G) \cong \bigoplus_v \mathcal{A}_{\mathbb{Q}_v}(K) \subset \bigoplus_v K_{\Gamma_v,\mathrm{tors}}. \tag{1.7.10.1}$$

Then for each $v$, $\Omega'_v \in K_{\Gamma_v,\mathrm{tors}}$ is a lift of $\beta_v(\omega) \in \pi_1(I_0)_{\Gamma_v}$. Thus we may and shall choose the lift $\widetilde{\beta}_v(\omega) \in K$ of $\beta_v(\omega)$ such that $\widetilde{\beta}_v(\omega)$ is a lift of $\Omega'_v$. In this case, to show that $\Omega$ is sent to zero in $K_\Gamma$, it suffices to note that the composition of (1.7.10.1) with the natural map $\bigoplus_v K_{\Gamma_v,\mathrm{tors}} \to K_{\Gamma,\mathrm{tors}}$ is zero. Indeed, by Proposition 1.1.9, this composition is identified with the composition

$$\mathbf{H}^0_{\mathrm{ab}}(\mathbb{Q}, I_0 \to G) \to \mathbf{H}^0_{\mathrm{ab}}(\mathbb{A}, I_0 \to G) \to \mathbf{H}^0_{\mathrm{ab}}(\mathbb{A}/\mathbb{Q}, I_0 \to G),$$

which is zero as desired. □

**1.7.11.** Let $\mathfrak{c} = (\gamma_0, a, [b]) \in \mathfrak{KP}$, and write $I_0$ for $I_0(\mathfrak{c})$. Assume that $[b] \in \mathrm{B}(I_{0,\mathbb{Q}_p})$ is basic. This is the case, for example, when $\mathfrak{c}$ is $p^n$-admissible, by Corollary 1.6.12. Recall that the notion of inner forms and isomorphisms between inner forms are given in Definition 1.2.2. The image of $a$ in $\mathbf{H}^1(\mathbb{A}_f^p, I_0^{\mathrm{ad}})$ determines an inner form $I_v$ of $I_0$ over $\mathbb{Q}_v$ up to isomorphism for each place $v \neq p, \infty$. The basic element $[b] \in \mathrm{B}(I_{0,\mathbb{Q}_p})$ determines an inner form $I_p$ of $I_0$ over $\mathbb{Q}_p$ up to isomorphism, namely $I_p := J_b^{I_0}$ for a decent representative $b$ of $[b]$ (see §1.4.3). Finally, let $(T, h)$ be as in §1.7.5. Then $\mathrm{Int}(h(i))$ induces a Cartan involution on $(I_0/Z_G)_{\mathbb{R}}$, from which we obtain an inner form $I_\infty$ of $I_0$ over $\mathbb{R}$ that is anisotropic modulo $Z_{G,\mathbb{R}}$. The isomorphism class of this inner form $I_\infty$ depends only on $\mathfrak{c}$.

**Proposition 1.7.12.** *In the situation of §1.7.11, assume that the Kottwitz invariant $\alpha(\mathfrak{c})$ is zero. Then there exists an inner form $I = I(\mathfrak{c})$ of $I_0$ over $\mathbb{Q}$, unique up to isomorphism between inner forms, such that its localization over $\mathbb{Q}_v$ is isomorphic to $I_v$ as inner forms of $I_{0,\mathbb{Q}_v}$ for each place $v$ .*

*Proof.* The uniqueness follows from the Hasse principle for $I_0^{\mathrm{ad}}$. To prove the existence, for each place $v$ we denote by $\eta_v$ the cohomology class in $\mathbf{H}^1(\mathbb{Q}_v, I_0^{\mathrm{ad}})$ corresponding to the inner form $I_v$ (cf. Remark 1.2.3). By [Kot86, Prop. 2.6] (cf. [Bor98, Thm. 5.16]) we have an exact sequence of pointed sets

$$\mathbf{H}^1(\mathbb{Q}, I_0^{\mathrm{ad}}) \to \bigoplus_v \mathbf{H}^1(\mathbb{Q}_v, I_0^{\mathrm{ad}}) \xrightarrow{m} \pi_1(I_0^{\mathrm{ad}})_{\Gamma,\mathrm{tors}}.$$

Here $m$ is defined as follows. For each place $v$, let $m_v$ be the composite

$$\mathbf{H}^1(\mathbb{Q}_v, I_0^{\mathrm{ad}}) \xrightarrow{\mathrm{ab}^1} \mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}_v, I_0^{\mathrm{ad}}) \cong \mathcal{A}_{\mathbb{Q}_v}(\pi_1(I_0^{\mathrm{ad}})) \hookrightarrow \pi_1(I_0^{\mathrm{ad}})_{\Gamma_v,\mathrm{tors}},$$

(note that these maps are all isomorphisms for $v$ finite) and let $i_v$ be the natural map $\pi_1(I_0^{\mathrm{ad}})_{\Gamma_v,\mathrm{tors}} \to \pi_1(I_0^{\mathrm{ad}})_{\Gamma,\mathrm{tors}}$. Then $m := \sum_v i_v \circ m_v$.

We only need to prove that

$$\sum_v i_v \circ m_v(\eta_v) = 0.$$

For each $v$, we claim that $m_v(\eta_v)$ equals the image of $\beta_v(\mathfrak{c})$ under $\pi_1(I_0)_{\Gamma_v} \to \pi_1(I_0^{\mathrm{ad}})_{\Gamma_v} = \pi_1(I_0^{\mathrm{ad}})_{\Gamma_v,\mathrm{tors}}$. Indeed, this statement is non-trivial only for $v \in \{p, \infty\}$. For $v = p$, there is a canonical bijection between $\mathbf{H}^1(\mathbb{Q}_p, I_0^{\mathrm{ad}})$ and the set of basic elements of $\mathrm{B}(I^{\mathrm{ad}}_{0,\mathbb{Q}_p})$. If we identify $\mathbf{H}^1(\mathbb{Q}_p, I_0^{\mathrm{ad}})$ with $\pi_1(I_0^{\mathrm{ad}})_{\Gamma_p} = \pi_1(I_0^{\mathrm{ad}})_{\Gamma_p,\mathrm{tors}}$, then this bijection is a section of the Kottwitz map $\mathrm{B}(I^{\mathrm{ad}}_{0,\mathbb{Q}_p}) \to \pi_1(I_0^{\mathrm{ad}})_{\Gamma_p}$. Moreover, this bijection sends $\eta_p$ to the image of $[b]$ in $\mathrm{B}(I^{\mathrm{ad}}_{0,\mathbb{Q}_p})$. For more details see the end of [RV14, §2.1]. The claim for $v = p$ follows. For $v = \infty$, let $(T, h)$ be as in §1.7.5, and let $\bar{T} := T/Z_{G,\mathbb{R}}$. Let $\bar{h}$ (resp. $\bar{\mu}_h$) be the composition of $h : \mathbb{S} \to T$ (resp. $\mu_h : \mathbb{G}_m \to T_{\mathbb{C}}$) with $T \to \bar{T}$. Since $\bar{T}$ is anisotropic, we have $\widehat{\mathbf{H}}^{-1}(\Gamma_\infty, X_*(\bar{T})) = X_*(\bar{T})_{\Gamma_\infty}$, and the Tate–Nakayama isomorphism $\widehat{\mathbf{H}}^{-1}(\Gamma_\infty, X_*(\bar{T})) \xrightarrow{\sim} \mathbf{H}^1(\mathbb{R}, \bar{T})$ is induced by the map

$$\begin{aligned} X_*(\bar{T}) &\longrightarrow Z^1(\mathbb{R}, \bar{T}) \\ \lambda &\longmapsto (1 \mapsto 1, \tau \mapsto \lambda(-1)), \end{aligned}$$

where $\tau$ denotes the complex conjugation. By definition, $\eta_\infty$ is represented by the cocycle $(1 \mapsto 1, \tau \mapsto \bar{h}(i))$, whereas $\beta_\infty(\mathfrak{c})$ is represented by $\mu_h \in X_*(T)$. Thus to verify our claim it suffices to check that $\bar{h}(i) = \bar{\mu}_h(-1)$. This follows from the equality $h(i) = \mu_h(-1)w_h(i)$, where $w_h$ is the weight cocharacter of $h$ and factors through $Z_{G,\mathbb{R}}$.

Write $K$ for $\ker(\pi_1(I_0) \to \pi_1(G))$. By the above claim, $\sum_v i_v \circ m_v(\eta_v)$ is equal to the image of $\alpha(\mathfrak{c})$ under the composite map

$$\mathfrak{E}(I_0, G; \mathbb{A}/\mathbb{Q}) \cong \frac{K_{\Gamma,\mathrm{tors}}}{\bigoplus_v \ker(K_{\Gamma_v,\mathrm{tors}} \to \pi_1(I_0)_{\Gamma_v})} \to \pi_1(I_0)_{\Gamma,\mathrm{tors}} \to \pi_1(I_0^{\mathrm{ad}})_{\Gamma,\mathrm{tors}},$$

where the first isomorphism is as in Proposition 1.7.3, and the second map is induced by the inclusion $K \hookrightarrow \pi_1(I_0)$. Since $\alpha(\mathfrak{c}) = 0$, we have $\sum_v i_v \circ m_v(\eta_v) = 0$, as desired. $\square$

## 1.8. **Stating the point counting formula.**

**1.8.1.** Let $(G, X)$ be a Shimura datum, and let $p$ be a prime number. We assume that $G$ is unramified over $\mathbb{Q}_p$, and fix a reductive model $\mathcal{G}$ of $G_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$. In the sequel we shall call such a quadruple $(G, X, p, \mathcal{G})$ an *unramified Shimura datum*. Let $E$ be the reflex field of $(G, X)$, and let $\mathfrak{p}$ and $q = p^r$ be as in §1.6.1. In the current case $E_\mathfrak{p}$ is unramified over $\mathbb{Q}_p$, so we identify $E_\mathfrak{p}$ with $\mathbb{Q}_{p^r}$. We write $K_p$ for the hyperspecial subgroup $\mathcal{G}(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$.

We fix notations for Hecke algebras. For each compact open subgroup $K^p \subset G(\mathbb{A}_f^p)$, let $\mathcal{H}(G(\mathbb{A}_f^p)/\!\!/K^p)$ be the Hecke algebra of $\mathbb{C}$-valued smooth compactly supported $K^p$-bi-invariant distributions on $G(\mathbb{A}_f^p)$. Let $\mathcal{H}(G(\mathbb{A}_f^p)/\!\!/K^p)_\mathbb{Q}$ be the $\mathbb{Q}$-subalgebra of $\mathcal{H}(G(\mathbb{A}_f^p)/\!\!/K^p)$ consisting of distributions that are rational on characteristic functions of compact open subgroups of $G(\mathbb{A}_f^p)$. Elements of $\mathcal{H}(G(\mathbb{A}_f^p)/\!\!/K^p)$ can be represented as $f^p dg^p$, where $f^p$ is a $\mathbb{C}$-valued smooth compactly supported $K^p$-bi-invariant function on $G(\mathbb{A}_f^p)$, and $dg^p$ is a Haar measure on $G(\mathbb{A}_f^p)$ assigning rational volumes to compact open subgroups. Elements of $\mathcal{H}(G(\mathbb{A}_f^p)/\!\!/K^p)_\mathbb{Q}$ can similarly be represented as $f^p dg^p$

Fix an irreducible algebraic representation $\xi$ of $G$ over $\overline{\mathbb{Q}}$ that factors through $G^c$. Fix a prime number $\ell \neq p$, and view $\xi$ as a representation over $\overline{\mathbb{Q}}_\ell$. As explained in §1.5.8, we have the $\mathrm{Gal}(\overline{E}/E) \times G(\mathbb{A}_f)$-module

$$\mathbf{H}_c^i(\mathrm{Sh}_{\overline{E}}, \xi),$$

which is admissible as an $G(\mathbb{A}_f)$-module. For each compact open subgroup $K^p \subset G(\mathbb{A}_f^p)$, we have the induced action of $\mathcal{H}(G(\mathbb{A}_f^p) /\!\!/ K^p)_\mathbb{Q}$ on the admissible $G(\mathbb{A}_f^p)$-module $\mathbf{H}_c^i(\mathrm{Sh}_{\overline{E}}, \xi)^{K_p}$. Fix a decomposition subgroup $D_\mathfrak{p} \subset \mathrm{Gal}(\overline{E}/E)$ at $\mathfrak{p}$, and fix an element $\Phi_\mathfrak{p} \in D_\mathfrak{p}$ that lifts the geometric $q$-Frobenius.

For $m \in \mathbb{Z}_{\geq 1}$ and $f^p dg^p \in \mathcal{H}(G(\mathbb{A}_f^p)/\!\!/K^p)_\mathbb{Q}$, we define

$$T(\Phi_\mathfrak{p}^m, f^p dg^p) := \sum_i (-1)^i \mathrm{tr}\left(\Phi_\mathfrak{p}^m \times (f^p dg^p) \mid \mathbf{H}_c^i(\mathrm{Sh}_{\overline{E}}, \xi)^{K_p}\right) \in \overline{\mathbb{Q}}_\ell.$$

Our goal in the rest of this subsection is to state a conjectural formula for the above quantity. In what follows we keep $f^p dg^p$ fixed.

**1.8.2.** Let $m \in \mathbb{Z}_{\geq 1}$ and let $n = mr$. Fix $\mathfrak{c} = (\gamma_0, a, [b]) \in \mathfrak{KP}_{\mathrm{a}}(p^n)$, satisfying $\alpha(\mathfrak{c}) = 0$. As in §1.6.16, $\mathfrak{c}$ gives rise to a classical Kottwitz parameter $(\gamma_0, \gamma, \delta) \in \mathfrak{KP}_{\mathrm{cla}}(p^n)$ of degree $n$, well defined up to equivalence. Let $I(\mathfrak{c})$ be the global inner form of $I_0(\mathfrak{c})$ as in Proposition 1.7.12. Let $R := \mathrm{Res}_{\mathbb{Q}_{p^n}/\mathbb{Q}_p} G$, and we view $\delta$ as an element of $R(\mathbb{Q}_p)$. Let $\theta$ be the $\mathbb{Q}_p$-automorphism of $R$ corresponding to the arithmetic $p$-Frobenius $\sigma \in \mathrm{Gal}(\mathbb{Q}_{p^n}/\mathbb{Q}_p)$. Let $R_{\delta \rtimes \theta}$ denote the fixed subgroup of $R$ under the automorphism $\mathrm{Int}(\delta) \circ \theta$.

Note that the $\mathbb{A}_f^p$-group $G_\gamma^0$ is isomorphic to $I(\mathfrak{c})_{\mathbb{A}_f^p}$, and the $\mathbb{Q}_p$-group $R^0_{\delta \rtimes \theta}$ is isomorphic to $I(\mathfrak{c})_{\mathbb{Q}_p}$. Moreover, these isomorphisms are canonical up to inner automorphisms defined over $\mathbb{A}_f^p$ and $\mathbb{Q}_p$ respectively. Choose Haar measures $di^p$ on $I(\mathbb{A}_f^p)$ and $di_p$ on $I(\mathbb{Q}_p)$. They can be transported to $G_\gamma^0(\mathbb{A}_f^p)$ and $R^0_{\delta \rtimes \sigma}(\mathbb{Q}_p)$ respectively in an unambiguous way. We denote the resulting Haar measures on $G_\gamma^0(\mathbb{A}_f^p)$ and $R^0_{\delta \rtimes \sigma}(\mathbb{Q}_p)$ still by $di^p$ and $di_p$.

Since $r|n$, and since $G$ is quasi-split over $\mathbb{Q}_p$ (as it is unramified), the Hodge cocharacters $\mu_h$ of $h \in X$ determine a $G(\mathbb{Q}_{p^n})$-conjugacy class of cocharacters of $G_{\mathbb{Q}_{p^n}}$, cf. [Kot84a, §1.3]. The *negative* of this conjugacy class of cocharacters (i.e., with all members replaced by their inverses) further determines a $\mathcal{G}(\mathbb{Z}_{p^n})$-double coset in $G(\mathbb{Q}_{p^n})$ via the Cartan decomposition, and we denote the characteristic function of this double coset by $\phi_n : G(\mathbb{Q}_{p^n}) \to \{0, 1\}$ (cf. [Kot84a, §2.1]). We define

$$O(\mathfrak{c}, m, f^p dg^p, di_p di^p) := O_\gamma(f^p dg^p) TO_\delta(\phi_n) \in \mathbb{C},$$

where $O_\gamma(f^p dg^p)$ is the orbital integral

$$\int_{G_\gamma^0(\mathbb{A}_f^p) \backslash G(\mathbb{A}_f^p)} f^p(g^{-1}\gamma g) \frac{dg^p}{di^p},$$

and $TO_\delta(\phi_n)$ is the twisted orbital integral

$$\int_{R^0_{\delta \rtimes \theta}(\mathbb{Q}_p) \backslash R(\mathbb{Q}_p)} \phi_n(r^{-1}\delta\theta(r)) \frac{dr_p}{di_p},$$

with $dr_p$ the Haar measure on $R(\mathbb{Q}_p) = G(\mathbb{Q}_{p^n})$ giving volume 1 to $\mathcal{G}(\mathbb{Z}_{p^n})$.

*Remark* 1.8.3. As the notation suggests, the dependence of $O(\mathfrak{c}, m, f^p dg^p, di_p di^p)$ on the two Haar measures $di_p$ and $di^p$ is only via the product measure $di_p di^p$ on $I(\mathfrak{c})(\mathbb{A}_f)$. Moreover, if $di_p di^p$ is rational on compact open subgroups, then $O(\mathfrak{c}, m, f^p dg^p, di_p di^p)$ lies in $\mathbb{Q}$. To see this, we may assume that $f^p = 1_{K^p a K^p}$ for a compact open subgroup $K^p \subset G(\mathbb{A}_f^p)$ and some $a \in G(\mathbb{A}_f^p)$, and that each of $dg^p, di_p, di^p$ is rational on compact open subgroups. We then know that $O_\gamma(f^p dg^p)$ lies in $\mathbb{Q}$ by adapting [Kot05, (3.4.1)] from the local setting to the adelic setting.[10] Similarly, we have $TO_\delta(\phi_n) \in \mathbb{Q}$ by a formula similar to [Kot05, (3.4.1)], cf. the proof of [ZZ20, Lem. 4.2.3].

[10] Our definition of the orbital integral equals $[G_\gamma(\mathbb{A}_f^p) : G_\gamma^0(\mathbb{A}_f^p)]$ times the adelic integral analogous to [Kot05, (3.4.1)].

*Remark* 1.8.4. Up to normalizations of the Haar measures on $G^0_\gamma(\mathbb{A}^p_f)$ and $R^0_{\delta\rtimes\theta}(\mathbb{Q}_p)$, the dependence of $O(\mathfrak{c}, m, f^p dg^p, di_p di^p)$ on $\mathfrak{c}$ is only via $(\gamma, \delta)$. However for later purposes it is important to normalize these Haar measures by choosing a Haar measure on $I(\mathfrak{c})(\mathbb{A}_f)$. Note that the classical Kottwitz parameter $(\gamma_0, \gamma, \delta)$ alone does not determine the global inner form $I(\mathfrak{c})$ of $G^0_{\gamma_0}$ (unless $G_{\mathrm{der}}$ is simply connected), so this way of normalization only makes sense with the presence of $\mathfrak{c}$.

**Lemma 1.8.5.** *Let $I$ be a connected reductive group over $\mathbb{Q}$, and let $Z$ be a $\mathbb{Q}$-subgroup of $Z_I$. Assume that $I/Z$ is anisotropic over $\mathbb{R}$. Then for any open subgroup $U \subset Z(\mathbb{A}_f)$, $I(\mathbb{Q})U$ is a closed subgroup of $I(\mathbb{A}_f)$.*

*Proof.* We write $\bar{I}$ for $I/Z$. Since $\bar{I}$ is anisotropic over $\mathbb{R}$, and since $\bar{I}(\mathbb{Q})$ is discrete in $\bar{I}(\mathbb{A})$, we know that $\bar{I}(\mathbb{Q})$ is a discrete (and hence closed) subgroup of $\bar{I}(\mathbb{A}_f)$. Let $f$ denote the map $I(\mathbb{A}_f) \to \bar{I}(\mathbb{A}_f)$. Let $V := f^{-1}(\bar{I}(\mathbb{Q}))$. Then $Z(\mathbb{A}_f) = f^{-1}(\{1\})$ is open in $V$, and $V$ is closed in $I(\mathbb{A}_f)$. Since $U$ is open in $Z(\mathbb{A}_f)$, it is open in $V$. Hence $I(\mathbb{Q})U$ is an open subgroup of $V$, and therefore closed in $V$. We have seen that $V$ is closed in $I(\mathbb{A}_f)$, so $I(\mathbb{Q})U$ is closed in $I(\mathbb{A}_f)$. □

**1.8.6.** Keep the setting of §1.8.2. As in §1.5.6, for each compact open subgroup $K^p \subset G(\mathbb{A}_f)$ we write $Z_{K_pK^p}$ for $Z_G(\mathbb{A}_f) \cap K_pK^p$, and write $Z(\mathbb{Q})_{K_pK^p}$ for $Z_G(\mathbb{Q}) \cap K_pK^p$. Since $Z := Z_G$ and $I := I(\mathfrak{c})$ satisfy the assumptions in Lemma 1.8.5, we know that $I(\mathfrak{c})(\mathbb{Q})Z_{K_pK^p}$ is a closed subgroup of $I(\mathfrak{c})(\mathbb{A}_f)$. Recall from [Bor63, Thm. 5.1] that $I(\mathfrak{c})(\mathbb{Q})\backslash I(\mathfrak{c})(\mathbb{A}_f)/U$ is finite for every compact open subgroup $U \subset I(\mathbb{A}_f)$. It follows that $I(\mathfrak{c})(\mathbb{Q})Z_{K_pK^p}\backslash I(\mathfrak{c})(\mathbb{A}_f)$ is compact Hausdorff. We equip $I(\mathfrak{c})(\mathbb{A}_f)$ with the Haar measure $di_p di^p$, and equip $I(\mathfrak{c})(\mathbb{Q})Z_{K_pK^p}$ with the Haar measure that gives volume 1 to its open subgroup $I(\mathfrak{c})(\mathbb{Q})Z_{K_pK^p}$. Then $I(\mathfrak{c})(\mathbb{Q})Z_{K_pK^p}\backslash I(\mathfrak{c})(\mathbb{A}_f)$ has finite volume under the quotient measure, and we denote this volume by

$$c_1(\mathfrak{c}, K^p, di_p di^p).$$

We also define

$$c_2(\mathfrak{c}) = c_2(\gamma_0) := \left|\text{Ш}_G(\mathbb{Q}, G^0_{\gamma_0})\right|.$$

Note that the product

$$c_1(\mathfrak{c}, K^p, di_p di^p) O(\mathfrak{c}, m, f^p dg^p, di_p di^p)$$

is independent of $di_p di^p$. Combined with Remark 1.8.3, this implies that the above product lies in $\mathbb{Q}$. In the sequel, we shall denote this product simply by

$$c_1(\mathfrak{c}, K^p) O(\mathfrak{c}, m, f^p dg^p) \in \mathbb{Q}.$$

**1.8.7.** Let $\Sigma_{\mathbb{R}\text{-ell}}(G)$ be the set of stable conjugacy classes of semi-simple, $\mathbb{R}$-elliptic elements of $G(\mathbb{Q})$. (This is well defined, since $\mathbb{R}$-elliptic maximal tori transfer between inner forms of reductive groups over $\mathbb{R}$.) We fix a compact open subgroup $K^p \subset G(\mathbb{A}^p_f)$ such that $K_pK^p$ is neat and such that $f^p$ is $K^p$-bi-invariant. We fix a subset $\Sigma_{K^p}$ of $G(\mathbb{Q})$ such that each $Z(\mathbb{Q})_{K_pK^p}$-translation-orbit in $\Sigma_{\mathbb{R}\text{-ell}}(G)$ is represented by exactly one element of $\Sigma_{K^p}$.

For each $\gamma_0 \in \Sigma_{K^p}$, we write $\mathfrak{KP}(\gamma_0)$ for the set of $\mathfrak{c} \in \mathfrak{KP}$ whose first component is $\gamma_0$.

For any reductive group $H$ over $\mathbb{Q}$ and any $\epsilon \in H(\mathbb{Q})_{\mathrm{ss}}$, we know that $(H_\epsilon/H^0_\epsilon)(\overline{\mathbb{Q}})$ is isomorphic to a subgroup of the abelian group $\pi_1(H_{\mathrm{der}})$ by [Ste75, Cor. 2.16 (a)].

It follows that $H_\epsilon/H_\epsilon^0$ is a finite commutative algebraic group over $\mathbb{Q}$. We define

$$\iota_H(\epsilon) := [H_\epsilon(\mathbb{Q}) : H_\epsilon^0(\mathbb{Q})],$$
$$\bar\iota_H(\epsilon) := \left|(H_\epsilon/H_\epsilon^0)(\mathbb{Q})\right|.$$

**Conjecture 1.8.8.** *For all sufficiently large integers $m$ (in a way depending on $f^p dg^p$), we have*

$$(1.8.8.1)\quad T(\Phi_{\mathfrak{p}}^m, f^p dg^p) = \sum_{\gamma_0\in\Sigma_{K^p}} \bar\iota_G(\gamma_0)^{-1} c_2(\gamma_0) \mathrm{tr}\,\xi(\gamma_0) \sum_{\substack{\mathfrak{c}\in\mathfrak{KP}(\gamma_0)\cap\mathfrak{KP}_{\mathrm{a}}(p^n)\\ \alpha(\mathfrak{c})=0}} c_1(\mathfrak{c}, K^p) O(\mathfrak{c}, m, f^p dg^p).$$

*Remark* 1.8.9. On the right hand side of (1.8.8.1), each summand indexed by $\gamma_0$ is of the form $\mathrm{tr}\,\xi(\gamma_0)$ (which lies in $\overline{\mathbb{Q}}$) times a rational number, by the discussion at the end of §1.8.6. Hence the right hand side of (1.8.8.1) in fact lies in the smallest number field containing $\{\mathrm{tr}\,\xi(\gamma_0) \mid \gamma_0 \in \Sigma_{K^p}\}$.

*Remark* 1.8.10. When $G_{\mathrm{der}}$ is simply connected and $Z_G$ is cuspidal, the right hand side of (1.8.8.1) recovers the formula conjectured by Kottwitz in [Kot90, §3]. We have formulated the conjecture only for $m$ sufficiently large, in anticipation of the fact that the local terms in the Grothendieck–Lefschetz–Verdier formula are equal to the naive local terms (i.e., Deligne's conjecture) only for $m$ sufficiently large. For applications, it is important (and usually sufficient) to know that (1.8.8.1) holds for all sufficiently large $m$, not just all sufficiently divisible $m$.

## 2. Variants of the Langlands–Rapoport Conjecture

2.1. **The formalism of Galois gerbs.** The Langlands–Rapoport Conjecture, in its original form in [LR87], is formulated using Galois gerbs. In this subsection we recall the basic definitions in the formalism of Galois gerbs. We mainly follow [Kis17, §3.1], while we make some corrections (see especially Remark 2.1.2) and provide some complementary explanations.

In the following, let $k'/k$ be a Galois extension of fields of characteristic zero.

**Definition 2.1.1.** By a *$k'/k$-Galois gerb*, we mean a pair $(G, \mathfrak{G})$, where $G$ is a connected linear algebraic group over $k'$, and $\mathfrak{G}$ is an extension of topological groups

$$1 \to (G(k'), \text{discrete topology}) \to \mathfrak{G} \to \mathrm{Gal}(k'/k) \to 1$$

satisfying the following conditions.

(i) For each $g \in \mathfrak{G}$, there is a $k'$-group isomorphism $g^{\mathrm{alg}} : \tau^*G \to G$, where $\tau$ is the image of $g$ in $\mathrm{Gal}(k'/k)$, such that the conjugation action of $g$ on $G(k')$ is given by $G(k') \xrightarrow{\tau} (\tau^*G)(k') \xrightarrow{g^{\mathrm{alg}}} G(k')$.

(ii) There exists a continuous group theoretic section of $\mathfrak{G} \to \mathrm{Gal}(k'/k)$ defined on an open subgroup of $\mathrm{Gal}(k'/k)$.

We often write $\mathfrak{G}$ for a $k'/k$-Galois gerb $(G, \mathfrak{G})$, and write $\mathfrak{G}^\Delta$ for $G$, called the *kernel* of $\mathfrak{G}$. A *morphism* between two $k'/k$-Galois gerbs $\mathfrak{G}_1$ and $\mathfrak{G}_2$ is a pair $(\phi^\Delta, \phi)$ consisting of a $k'$-homomorphism $\phi^\Delta : \mathfrak{G}_1^\Delta \to \mathfrak{G}_2^\Delta$ and a continuous homomorphism $\phi : \mathfrak{G}_1 \to \mathfrak{G}_2$ satisfying the following conditions.

(a) $\phi$ commutes with the maps $\mathfrak{G}_i \to \mathrm{Gal}(k'/k)$.

(b) The restriction of $\phi$ to $\mathfrak{G}_1^\Delta(k')$ is given by $\phi^\Delta$.

We write $\mathcal{G}rb(k'/k)$ for the category of $k'/k$-Galois gerbs.

*Remark* 2.1.2. The assumption in Definition 2.1.1 that $G$ is connected is missing in [Kis17, §3.1], and should be added. This assumption implies that $G(k')$ is Zariski dense in $G$ (see [Bor91, Cor. 18.3]). In particular, each $g \in \mathfrak{G}$ uniquely determines the isomorphism $g^{\mathrm{alg}}$, which is vital for various constructions. Another consequence is that for a morphism $(\phi^\Delta, \phi) : \mathfrak{G}_1 \to \mathfrak{G}_2$ between $k'/k$-Galois gerbs, $\phi^\Delta$ is uniquely determined by $\phi$. For this reason we can view $\phi$ alone as a morphism $\mathfrak{G}_1 \to \mathfrak{G}_2$.

*Remark* 2.1.3. Let $\mathfrak{G} \in \mathcal{G}rb(k'/k)$. Then all continuous group theoretic sections of $\mathfrak{G} \to \mathrm{Gal}(k'/k)$ defined on open subgroups of $\mathrm{Gal}(k'/k)$ belong to the same germ. Moreover, this germ can be extended to a continuous set theoretic section of $\mathfrak{G} \to \mathrm{Gal}(k'/k)$ defined on the whole $\mathrm{Gal}(k'/k)$.

**Definition 2.1.4.** Let $G$ be a connected linear algebraic group over $k$. Let $\mathfrak{G}_G$ be the split extension $G(k') \rtimes \mathrm{Gal}(k'/k)$, where $\mathrm{Gal}(k'/k)$ acts naturally on $G(k')$. Then $(G_{k'}, \mathfrak{G}_G) \in \mathcal{G}rb(k'/k)$, and it is called the *neutral $k'/k$-Galois gerb* associated with $G$.

**Definition 2.1.5.** Let $\mathfrak{G} \in \mathcal{G}rb(k'/k)$. For each $g \in \mathfrak{G}^\Delta(k')$, conjugation by $g$ induces an automorphism $\mathrm{Int}(g)$ of $\mathfrak{G}$. Let $\phi, \psi : \mathfrak{H} \to \mathfrak{G}$ be two morphisms in $\mathcal{G}rb(k'/k)$. We say that $\phi$ and $\psi$ are *conjugate* (or $\mathfrak{G}^\Delta(k')$*-conjugate*, for clarity), if there exists $g \in \mathfrak{G}^\Delta(k')$ such that $\phi = \mathrm{Int}(g) \circ \psi$.

**2.1.6.** Let $\phi : \mathfrak{H} \to \mathfrak{G}$ be a morphism in $\mathcal{G}rb(k'/k)$. A $k$-algebraic group $I_\phi$ is defined in [Kis17, §3.1.1, Lem. 3.1.2][11]. We have a canonical identification between $I_{\phi,k'}$ and the centralizer $\mathfrak{G}^\Delta_{\phi^\Delta}$ of $\mathrm{im}(\phi^\Delta)$ in $\mathfrak{G}^\Delta$. Under this identification, $I_\phi(k)$ is the group of $g \in \mathfrak{G}^\Delta(k')$ such that $\mathrm{Int}(g) \circ \phi = \phi$. Moreover, if we choose a continuous set theoretic section $\mathrm{Gal}(k'/k) \to \mathfrak{H}, \tau \mapsto q_\tau$ of $\mathfrak{H} \to \mathrm{Gal}(k'/k)$, then the action of $\tau \in \mathrm{Gal}(k'/k)$ on $I_\phi(k') \cong \mathfrak{G}^\Delta_{\phi^\Delta}(k')$ with respect to the $k$-form $I_\phi$ is induced by conjugation by $\phi(q_\tau)$ inside $\mathfrak{G}$.

In fact, the axioms for $k'/k$-Galois-gerbs guarantee that the above description of the $\mathrm{Gal}(k'/k)$-action on $\mathfrak{G}^\Delta_{\phi^\Delta}(k')$ can be naturally upgraded to a $k'/k$-Galois descent datum that gives the $k$-form $I_\phi$ of the $k'$-group $\mathfrak{G}^\Delta_{\phi^\Delta}$. We refer the reader to the proof of [Kis17, Lem. 3.1.2] for more details. Here we only remark that the cocycle condition for the descent datum amounts to the fact that for all $\tau, \rho \in \mathrm{Gal}(k'/k)$, $\phi(q_\tau q_\rho q_{\tau\rho}^{-1})$ lies in $(\mathrm{im}\, \phi^\Delta)(k')$, and hence lies in the center of $\mathfrak{G}^\Delta_{\phi^\Delta}$.

If $\zeta : \mathfrak{K} \to \mathfrak{H}$ and $\phi : \mathfrak{H} \to \mathfrak{G}$ are two morphisms in $\mathcal{G}rb(k'/k)$, then the inclusion $\mathfrak{G}^\Delta_{\phi^\Delta} \hookrightarrow \mathfrak{G}^\Delta_{(\phi\circ\zeta)^\Delta}$ induces an injective $k$-homomorphism

$$I_\phi \hookrightarrow I_{\phi\circ\zeta}. \tag{2.1.6.1}$$

If $\phi : \mathfrak{H} \to \mathfrak{G}_G$ is a morphism in $\mathcal{G}rb(k'/k)$ with $G$ a connected linear algebraic group over $k$, then $I_\phi$ contains $Z_G$ as a $k$-subgroup.

*Remark* 2.1.7. Let $\mathfrak{G} \in \mathcal{G}rb(k'/k)$, and let $\phi$ be the identity $\mathfrak{G} \to \mathfrak{G}$. Then $I_\phi$ is a canonical $k$-form of $\mathfrak{G}^\Delta_{\phi^\Delta} = Z_{\mathfrak{G}^\Delta}$. In particular, if $\mathfrak{G}^\Delta$ is a torus, then we have a canonical $k$-form of $\mathfrak{G}^\Delta$.

[11]The assumption in [Kis17, Lem. 3.1.2] that the target of $\phi$ is a neutral gerb is not needed.

**2.1.8.** Let $\phi : \mathfrak{H} \to \mathfrak{G}_G$ be a morphism in $\mathcal{G}rb(k'/k)$, where the target is a neutral gerb associated with a reductive group $G$ over $k$. We assume that $\mathfrak{H}^\Delta$ is a torus. In this situation we define $k$-groups $I_\phi^\dagger$ and $\widetilde{I}_\phi$ that are closely related to $I_\phi$.

Let $M := G_{k',\phi^\Delta}$. Then $M$ is a $k'$-subgroup of $G_{k'}$ whose base change to an algebraic closure of $k'$ becomes a Levi subgroup. Let $M^\dagger := M \cap G_{\mathrm{der},k'}$, and let $\widetilde{M}$ be the inverse image of $M^\dagger$ in $G_{\mathrm{sc},k'}$. Then $M$, $M^\dagger$, and $\widetilde{M}$ are reductive groups over $k'$, and the natural maps $\widetilde{M} \to M^\dagger \to M$ induce isomorphisms between the respective adjoint groups.

The usual conjugation action of $G(k')$ on $G_{\mathrm{sc}}(k')$ together with the natural action of $\mathrm{Gal}(k'/k)$ on $G_{\mathrm{sc}}(k')$ gives rise to an action of $\mathfrak{G}_G = G(k') \rtimes \mathrm{Gal}(k'/k)$ on $G_{\mathrm{sc}}(k')$, which we denote by $\mathrm{Int}_G^{G_{\mathrm{sc}}}$. Choose a continuous set theoretic section $\mathrm{Gal}(k'/k) \to \mathfrak{H}, \tau \mapsto q_\tau$ of $\mathfrak{H} \to \mathrm{Gal}(k'/k)$. We define $\widetilde{I}_\phi$ to be the $k$-form of $\widetilde{M}$ corresponding to the following $\mathrm{Gal}(k'/k)$-action on $\widetilde{M}(k')$: Each $\tau \in \mathrm{Gal}(k'/k)$ acts by $\mathrm{Int}_G^{G_{\mathrm{sc}}}(\phi(q_\tau))$. More precisely, just as the definition of $I_\phi$ via Galois descent discussed in §2.1.6, this Galois action can be naturally upgraded to a $k'/k$-Galois descent datum on $\widetilde{M}$. The cocycle condition in the current context amounts to the requirement that for all $\tau, \rho \in \mathrm{Gal}(k'/k)$, $\mathrm{Int}_G^{G_{\mathrm{sc}}}(\phi(q_\tau q_\rho q_{\tau\rho}^{-1}))$ acts trivially on $\widetilde{M}$. This is indeed true, because $\phi(q_\tau q_\rho q_{\tau\rho}^{-1})$ lies in $Z_M$, and any element of $Z_M$ acts trivially on $\widetilde{M}$ via $\mathrm{Int}_G^{G_{\mathrm{sc}}}$. Using the same principle, one sees that the Galois descent datum does not depend on the choice of $\tau \mapsto q_\tau$. Thus we obtain the $k$-group $\widetilde{I}_\phi$ canonically.

In the same way we define a $k$-form $I_\phi^\dagger$ of $M^\dagger$. The natural $k'$-homomorphism $\widetilde{M} \to M^\dagger \hookrightarrow M$ induce $k$-homomorphisms $\widetilde{I}_\phi \to I_\phi^\dagger \hookrightarrow I_\phi$ between reductive groups. Note that the composite $k'$-homomorphism $I_{\phi,k'} \hookrightarrow G_{k'} \to G_{k'}^{\mathrm{ab}}$ is defined over $k$, and its kernel is naturally identified with $I_\phi^\dagger$.

**2.1.9.** Let $l'/l$ be another Galois extension of fields of characteristic zero, equipped with compatible embeddings $k \hookrightarrow l$ and $k' \hookrightarrow l'$. In this situation we have the *pull-back* functor

$$\mathrm{PB} : \mathcal{G}rb(k'/k) \longrightarrow \mathcal{G}rb(l'/l). \tag{2.1.9.1}$$

We explain its definition.

We first define PB of an object. Let $\mathfrak{G} \in \mathcal{G}rb(k'/k)$, with kernel $G$. We have a short exact sequence

$$1 \to G(k') \to \mathfrak{G}^0_{l'/l} \to \mathrm{Gal}(l'/l) \to 1,$$

where $\mathfrak{G}^0_{l'/l}$ is the fiber product $\mathfrak{G} \times_{\mathrm{Gal}(k'/k)} \mathrm{Gal}(l'/l)$ in the category of topological groups. The above short exact sequence is in fact an extension of topological groups, which follows easily from Remark 2.1.3.

Given any $g \in \mathfrak{G}^0_{l'/l}$ with image $\tau \in \mathrm{Gal}(l'/l)$, the conjugation action of $g$ on $G(k')$ is induced by a $k'$-isomorphism $(\tau|_{k'})^* G \xrightarrow{\sim} G$ uniquely determined by $g$ (i.e., the isomorphism $h^{\mathrm{alg}}$, where $h$ is the image of $g$ in $\mathfrak{G}$). The last $k'$-isomorphism induces an $l'$-isomorphism $u_g : \tau^*(G_{l'}) \xrightarrow{\sim} G_{l'}$, and in particular an automorphism of $G(l')$ given by

$$G(l') = G_{l'}(l') \xrightarrow{\tau} (\tau^* G_{l'})(l') \xrightarrow{u_g} G_{l'}(l') = G(l').$$

In this way we obtain an action of $\mathfrak{G}^0_{l'/l}$ on $G(l')$ via group automorphisms.

Let $\mathfrak{G}_{l'/l}$ be the quotient group $(G(l') \rtimes \mathfrak{G}^0_{l'/l})/\left\{x \rtimes x^{-1} \mid x \in G(k')\right\}$. We shall denote elements of $\mathfrak{G}_{l'/l}$ by $[x,(u,\tau)]$, for $x \in G(l'), (u,\tau) \in \mathfrak{G}^0_{l'/l}$. We have a short exact sequence

$$1 \to G(l') \xrightarrow{x \mapsto [x,(1,1)]} \mathfrak{G}_{l'/l} \xrightarrow{[x,(u,\tau)] \mapsto \tau} \mathrm{Gal}(l'/l) \to 1. \tag{2.1.9.2}$$

We equip $\mathfrak{G}_{l'/l}$ with the quotient topology of the product topology on $G(l') \rtimes \mathfrak{G}^0_{l'/l}$. (As always, $G(l')$ has the discrete topology.) Then $\mathfrak{G}_{l'/l}$ is a topological group, and (2.1.9.2) is an extension of topological groups. One checks that $(G_{l'}, \mathfrak{G}_{l'/l}) \in \mathcal{G}rb(l'/l)$. We define $\mathrm{PB}(\mathfrak{G})$ to be $(G_{l'}, \mathfrak{G}_{l'/l})$.

We now define PB of a morphism. Given any morphism $\phi : \mathfrak{G} \to \mathfrak{H}$ in $\mathcal{G}rb(k'/k)$, we define $\mathrm{PB}(\phi)$ to be $(\phi^\Delta, \phi_{l'/l}) : \mathrm{PB}(\mathfrak{G}) \to \mathrm{PB}(\mathfrak{H})$, where $\phi_{l'/l}$ is given by

$$\begin{aligned} \phi_{l'/l} : \mathfrak{G}_{l'/l} &\longrightarrow \mathfrak{H}_{l'/l} \\ [x,(u,\tau)] &\longmapsto [\phi^\Delta(x), (\phi(u),\tau)], \quad x \in G(l'), (u,\tau) \in \mathfrak{G}^0_{l'/l}. \end{aligned}$$

This concludes the definition of the functor PB.

Note that for $\mathfrak{G} \in \mathcal{G}rb(k'/k)$, there is a canonical group homomorphism

$$\varsigma^{\mathfrak{G}}_{\mathrm{can}} : \mathrm{Gal}(l'/lk') \longrightarrow \mathfrak{G}_{l'/l}, \quad \tau \longmapsto [1,(1,\tau)],$$

which is a section of $\mathfrak{G}_{l'/l} \to \mathrm{Gal}(l'/l)$.

**Lemma 2.1.10.** *Keep the setting of §2.1.9, and assume in addition that $l = k$. Let $\mathfrak{G}, \mathfrak{H} \in \mathcal{G}rb(k'/k)$. A morphism $\psi : \mathrm{PB}(\mathfrak{G}) \to \mathrm{PB}(\mathfrak{H})$ is of the form $\mathrm{PB}(\phi)$ for some morphism $\phi : \mathfrak{G} \to \mathfrak{H}$ if and only if $\psi \circ \varsigma^{\mathfrak{G}}_{\mathrm{can}} = \varsigma^{\mathfrak{H}}_{\mathrm{can}}$. Moreover, when this is the case, $\phi$ is unique.*

*Proof.* The "only if" part is trivial. We show the "if" part.

From the hypothesis on $\psi$, it follows that $\psi^\Delta$ is the base change to $l'$ of a $k'$-homomorphism $\phi^\Delta : \mathfrak{G}^\Delta \to \mathfrak{H}^\Delta$, and that $\psi[1,(u,\tau)]$ is of the form $[1,(\phi(u),\tau)]$ for some function $\phi : \mathfrak{G} \to \mathfrak{H}$. We now check that $(\phi^\Delta, \phi)$ is a morphism in $\mathcal{G}rb(k'/k)$. In fact, only the continuity of $\phi$ is non-obvious. For this, we observe that the map $\mathfrak{H}^0_{l'/l} \to \mathfrak{H}^\Delta(l') \rtimes \mathfrak{H}^0_{l'/l}, (u,\tau) \mapsto (1,u,\tau)$ is continuous and open, since $\mathfrak{H}^\Delta(l')$ is discrete. Hence the induced injective map $\mathfrak{H}^0_{l'/l} \to \mathfrak{H}_{l'/l}$ is also continuous and open, i.e., a homeomorphism onto its image. It follows that the map $\mathfrak{G}^0_{l'/l} \to \mathfrak{H}^0_{l'/l}, (u,\tau) \to (\phi(u),\tau)$ is continuous. The continuity of $\phi$ then follows from the openness of the map $\mathrm{Gal}(l'/l) \to \mathrm{Gal}(k'/k)$.

Given that $(\phi^\Delta, \phi)$ is a morphism, it is clear that $\psi = \mathrm{PB}(\phi)$.

Finally, we show that if $\psi = \mathrm{PB}(\phi)$ then $\phi$ is unique. Suppose $\phi_1$ also satisfies the condition. Then

$$\psi[1,(u,\tau)] = [1,(\phi(u),\tau)] = [1,(\phi_1(u),\tau)],$$

and in particular $\phi(u) = \phi_1(u)$, for all $(u,\tau) \in \mathfrak{G}^0_{l'/l}$. Since the projection $\mathfrak{G}^0_{l'/l} \to \mathfrak{G}, (u,\tau) \mapsto u$ is surjective, we have $\phi = \phi_1$. □

**Definition 2.1.11.** Let pro-$\mathcal{G}rb(k'/k)$ be the category whose objects are projective systems $(\mathfrak{G}_i)_{i\in I}$ in $\mathcal{G}rb(k'/k)$ indexed by directed sets $(I, \leq)$ and whose morphisms are given by

$$\mathrm{Hom}_{\text{pro-}\mathcal{G}rb(k'/k)}((\mathfrak{G}_i)_{i\in I}, (\mathfrak{H}_j)_{j\in J}) := \varprojlim_{j\in J} \varinjlim_{i\in I} \mathrm{Hom}_{\mathcal{G}rb(k'/k)}(\mathfrak{G}_i, \mathfrak{H}_j).$$

Objects of pro-$\mathcal{G}rb(k'/k)$ are called *pro-$k'/k$-Galois gerbs*. We view $\mathcal{G}rb(k'/k)$ naturally as a full subcategory of pro-$\mathcal{G}rb(k'/k)$. When we are in the situation of §2.1.9, the pull-back functor (2.1.9.1) naturally extends to a functor pro-$\mathcal{G}rb(k'/k) \to$ pro-$\mathcal{G}rb(l'/l)$, which we still call *pull-back*.

**2.1.12.** Let $\mathfrak{G} = (\mathfrak{G}_i)_{i\in I} \in$ pro-$\mathcal{G}rb(k'/k)$. Then we can take the projective limit $\mathfrak{G}^\Delta := \varprojlim_i \mathfrak{G}_i^\Delta$ in the category of affine $k'$-group schemes, and take the projective limit $\mathfrak{G}^{\mathrm{top}} := \varprojlim_i \mathfrak{G}_i$ in the category of topological groups. If $\phi : \mathfrak{G} \to \mathfrak{H}$ is a morphism in pro-$\mathcal{G}rb(k'/k)$, then $\phi$ naturally induces a homomorphism of affine $k'$-group schemes $\phi^\Delta : \mathfrak{G}^\Delta \to \mathfrak{H}^\Delta$, and a continuous homomorphism $\phi^{\mathrm{top}} : \mathfrak{G}^{\mathrm{top}} \to \mathfrak{H}^{\mathrm{top}}$. In the sequel, if $d$ is an element of $\mathfrak{G}^{\mathrm{top}}$, we shall simply write $d \in \mathfrak{G}$. Also, we shall simply write $\phi(d) \in \mathfrak{H}$ for $\phi^{\mathrm{top}}(d) \in \mathfrak{H}^{\mathrm{top}}$.

The following definition generalizes Definition 2.1.5.

**Definition 2.1.13.** Let $\mathfrak{H} = (\mathfrak{H}_i)_{i\in I} \in$ pro-$\mathcal{G}rb(k'/k), \mathfrak{G} \in \mathcal{G}rb(k'/k)$, and let $\phi, \psi : \mathfrak{H} \to \mathfrak{G}$ be two morphisms in pro-$\mathcal{G}rb(k'/k)$. We say that $\phi$ and $\psi$ are *conjugate* (or $\mathfrak{G}^\Delta(k')$*-conjugate*) if there exists $g \in \mathfrak{G}^\Delta(k')$ such that $\psi = \mathrm{Int}(g) \circ \phi$ as morphisms in pro-$\mathcal{G}rb(k'/k)$.

**2.1.14.** Let $\mathfrak{H} = (\mathfrak{H}_i)_{i\in I} \in$ pro-$\mathcal{G}rb(k'/k), \mathfrak{G} \in \mathcal{G}rb(k'/k)$, and let $\phi : \mathfrak{H} \to \mathfrak{G}$ be a morphism in pro-$\mathcal{G}rb(k'/k)$. We now define $I_\phi$, generalizing the definition in §2.1.6.

Choose $i_0 \in I$ such that $\phi$ is induced by a morphism $\phi_{i_0} : \mathfrak{H}_{i_0} \to \mathfrak{G}$. For each $i \in I$ with $i \geq i_0$, let $\phi_i$ be the composition $\mathfrak{H}_i \to \mathfrak{H}_{i_0} \xrightarrow{\phi_{i_0}} \mathfrak{G}$. For $j \geq i \geq i_0$, there is a natural $k$-homomorphism $I_{\phi_i} \to I_{\phi_j}$ as in (2.1.6.1), whose base change to $k'$ is identified with the inclusion map $\mathfrak{G}^\Delta_{\phi_i^\Delta} \to \mathfrak{G}^\Delta_{\phi_j^\Delta}$. For sufficiently large $i$, the decreasing subgroups $\mathrm{im}(\phi_i^\Delta)$ of $\mathfrak{G}^\Delta$ stabilize, since $\mathfrak{G}^\Delta$ is noetherian. Hence $\mathfrak{G}^\Delta_{\phi_i^\Delta}$ and $I_{\phi_i}$ also stabilize. We can thus define

$$\mathrm{im}(\phi^\Delta) := \mathrm{im}(\phi_i^\Delta),$$
$$\mathfrak{G}^\Delta_{\phi^\Delta} := \mathfrak{G}^\Delta_{\phi_i^\Delta},$$
$$I_\phi := I_{\phi_i},$$

for $i \in I$ sufficiently large. Clearly these definitions are independent of the initial choices of $i_0$ and $\phi_{i_0}$.

By construction, $I_{\phi,k'}$ is canonically identified with $\mathfrak{G}^\Delta_{\phi^\Delta}$. It is also easy to see that $I_\phi(k)$ precisely consists of those $g \in \mathfrak{G}^\Delta(k')$ such that $\mathrm{Int}(g) \circ \phi = \phi$ (as morphisms in pro-$\mathcal{G}rb(k'/k)$).

If $\mathfrak{G} = \mathfrak{G}_G$ for some reductive group $G$ over $k$, and if $\mathfrak{H}_i^\Delta$ are tori for all $i \in I$, then we also extend the definitions of $I_\phi^\dagger$ and $\widetilde{I}_\phi$ in §2.1.8 to the present case, in the obvious way. In this case, each of $I_\phi, I_\phi^\dagger, \widetilde{I}_\phi$ is a reductive group. The group $I_\phi$ has the same absolute rank as $G$, and contains $Z_G$ as a $\mathbb{Q}$-subgroup.

**2.1.15.** Let $\mathfrak{H}, \mathfrak{G}, \phi$ be as at the beginning of §2.1.14. Given a continuous 1-cocycle $a = (a_\rho) \in Z^1(k'/k, I_\phi(k'))$, there is a morphism $a\phi : \mathfrak{H} \to \mathfrak{G}$ defined as follows. Choose $i \in I$ such that $\phi$ is induced by some $\phi_i : \mathfrak{H}_i \to \mathfrak{G}$ and such that $I_\phi = I_{\phi_i}$. Denote by $\pi$ the structural map $\mathfrak{H}_i \to \mathrm{Gal}(k'/k)$. We define

$$a\phi_i : \mathfrak{H}_i \longrightarrow \mathfrak{G}, \quad g \longmapsto a_{\pi(g)}\phi_i(g).$$

Here, we view $a_{\pi(g)} \in I_\phi(k')$ as an element of $\mathfrak{G}^\Delta(k') \subset \mathfrak{G}$ via the canonical embedding $I_{\phi,k'} \hookrightarrow \mathfrak{G}^\Delta$. Then $a\phi_i$ is a morphism in $\mathcal{G}rb(k'/k)$, and we define $a\phi$ to be the morphism induced by $a\phi_i$. This definition is independent of choices.

**Lemma 2.1.16.** *In the setting of §2.1.15, the map $a \mapsto a\phi$ is a bijection from $Z^1(k'/k, I_\phi(k'))$ to the set of morphisms $\phi' : \mathfrak{H} \to \mathfrak{G}$ such that $\phi'^{,\Delta} = \phi^\Delta$. Moreover, for $a, a' \in Z^1(k'/k, I_\phi(k'))$, we have $a\phi$ is conjugate to $a'\phi$ if and only if $a$ is cohomologous to $a'$.*

*Proof.* Since $\mathfrak{G}^\Delta$ is finitely presented over $k'$, the natural map

$$\varinjlim_i \mathrm{Hom}_{k'}(\mathfrak{H}_i^\Delta, G_{k'}) \longrightarrow \mathrm{Hom}_{k'}(\mathfrak{H}^\Delta, G_{k'})$$

is a bijection. Here $\mathrm{Hom}_{k'}$ denotes the set of homomorphisms of $k'$-group schemes. The lemma then reduces to the case where $\mathfrak{H}$ is in $\mathcal{G}rb(k'/k)$. In this case the proof is exactly the same as the proof of part (2) of [Kis17, Lem. 3.1.2]. □

**Definition 2.1.17.** For $a$ and $\phi$ as in §2.1.15, we call $a\phi$ the *twist of $\phi$ by $a$*. Similarly, for a class $\beta \in \mathbf{H}^1(k'/k, I_\phi(k'))$, we define the *twist of $\phi$ by $\beta$*, denoted by $\phi^\beta$, to be the conjugacy class of $a\phi$ where $a$ is any cocycle representing $\beta$. This is well defined by Lemma 2.1.16.[12] We shall sometimes also write $\phi^\beta$ for an unspecified member of this conjugacy class.

**2.1.18.** We explain how to obtain Galois gerbs from Reimann's explicit cocycle construction in [Rei97, App. B]. We first sketch the idea behind the construction informally. Let $\mathfrak{G} \in \mathcal{G}rb(k'/k)$. Suppose that there is a continuous set theoretic section $\varsigma : \mathrm{Gal}(k'/k) \to \mathfrak{G}$ of $\mathfrak{G} \to \mathrm{Gal}(k'/k)$ such that $\varsigma(\rho)\varsigma(\tau)\varsigma(\rho\tau)^{-1}$ lies in $Z_{\mathfrak{G}^\Delta}(k')$ for all $\rho, \tau \in \mathrm{Gal}(k'/k)$. Then the isomorphisms $(\varsigma(\tau))^{\mathrm{alg}} : \tau^*\mathfrak{G}^\Delta \xrightarrow{\sim} \mathfrak{G}^\Delta$ for $\tau \in \mathrm{Gal}(k'/k)$ form a $k'/k$-Galois descent datum. Let $G$ be the corresponding $k$-form of $\mathfrak{G}^\Delta$. Then the isomorphism class of $\mathfrak{G}$ can be recovered from $G$ and the map $\mathrm{Gal}(k'/k) \times \mathrm{Gal}(k'/k) \to Z_G(k'), (\rho, \tau) \mapsto \varsigma(\rho)\varsigma(\tau)\varsigma(\rho\tau)^{-1}$, which is in fact a continuous 2-cocycle.

We now give the formal construction. First we define a category $\mathcal{R}(k'/k)$. The objects are pairs $(G, z)$, where $G$ is a connected linear algebraic group over $k$, and $z = (z_{\rho,\tau})$ is a continuous 2-cocycle $\mathrm{Gal}(k'/k) \times \mathrm{Gal}(k'/k) \to Z_G(k')$ satisfying $z_{1,1} = 1$. A morphism $(G', z') \to (G, z)$ is a pair $(\phi^\Delta, f)$, where $\phi^\Delta : G'_{k'} \to G_{k'}$ is a homomorphism of $k'$-groups, and $f = (f_\rho)$ is a continuous 1-cochain $\mathrm{Gal}(k'/k) \to G(k')$ satisfying

$$z_{\rho,\tau} f_\rho {}^\rho f_\tau f_{\rho\tau}^{-1} = \phi^\Delta(z'_{\rho,\tau}),$$
$$\mathrm{Int}(f_\rho) \circ \rho^*(\phi^\Delta) = \phi^\Delta,$$

for all $\rho, \tau \in \mathrm{Gal}(k'/k)$. (In particular, $f_1 = 1$.) The composition of morphisms is given by

$$(\phi^\Delta, f) \circ (\psi^\Delta, h) := (\phi^\Delta \circ \psi^\Delta, (\phi^\Delta(h_\rho) f_\rho)_\rho).$$

We define a fully faithful functor

$$\mathcal{E} : \mathcal{R}(k'/k) \longrightarrow \mathcal{G}rb(k'/k). \tag{2.1.18.1}$$

[12]Since we define $\phi^\beta$ to be the whole conjugacy class, its members are not necessarily of the form $a\phi$ for any $a \in Z^1(k'/k, I_\phi(k'))$.

Let $(G, z) \in \mathcal{R}(k'/k)$. To define $\mathcal{E}(G, z)$, we let $\mathfrak{G} := G(k') \times \mathrm{Gal}(k'/k)$, equipped with the product topology. Define a binary operation on $\mathfrak{G}$ by

$$(g, \rho) \cdot (h, \tau) := (g\rho(h)z_{\rho,\tau}, \rho\tau).$$

Then $(\mathfrak{G}, \cdot)$ is a topological group, and the maps $G(k') \to \mathfrak{G}, g \mapsto (g, 1)$ and $\mathfrak{G} \to \mathrm{Gal}(k'/k), (g, \rho) \mapsto \rho$ make $\mathfrak{G}$ a topological group extension of $\mathrm{Gal}(k'/k)$ by $G(k')$. Then $(G_{k'}, \mathfrak{G}) \in \mathcal{G}rb(k'/k)$. (To check condition (ii) in Definition 2.1.1, use that $z_{\rho,\tau} = 1$ for all $\rho, \tau$ sufficiently close to 1.) We define $\mathcal{E}(G, z)$ to be $(G_{k'}, \mathfrak{G})$.

For a morphism $(\phi^{\Delta}, f) : (G', z') \to (G, z)$ in $\mathcal{R}(k'/k)$, we define

$$\begin{aligned} \mathcal{E}(\phi^{\Delta}, f) : \mathcal{E}(G', z') = G'(k') \times \mathrm{Gal}(k'/k) &\longrightarrow \mathcal{E}(G, z) = G(k') \times \mathrm{Gal}(k'/k) \\ (g', \rho) &\longmapsto (\phi^{\Delta}(g')f_{\rho}, \rho). \end{aligned}$$

This completes the definition of the functor $\mathcal{E}$. We omit the proof that $\mathcal{E}$ is fully faithful, since this fact will not be used.

Analogous to Definition 2.1.11, we consider the category pro-$\mathcal{R}(k'/k)$ of pro-objects in $\mathcal{R}(k'/k)$ indexed by directed sets. The functor (2.1.18.1) naturally extends to a functor

$$\mathcal{E} : \text{pro-}\mathcal{R}(k'/k) \longrightarrow \text{pro-}\mathcal{G}rb(k'/k) \tag{2.1.18.2}$$

which is also fully faithful.

In [Rei97, App. B], various *affine groupoids* are defined, which are needed for the correct formulation of the Langlands–Rapoport Conjecture. There is a functor from pro-$\mathcal{R}(k'/k)$ to the category of affine $k'/k$-groupoids, and Reimann obtains the desired groupoids by constructing explicit objects in pro-$\mathcal{R}(k'/k)$ (for suitable $k'/k$). In the present paper, we shall not need affine groupoids, but we shall import Reimann's explicit constructions and obtain pro-Galois gerbs via the functor (2.1.18.2). This is the same as the point of view taken in [Kis17].

## 2.2. **The Dieudonné gerb and the quasi-motivic gerb.**

**2.2.1.** Fix a prime $p$. We recall the definition of the *Dieudonné gerb* in terms of the functor (2.1.18.2), cf. [Rei97, pp. 109–110]. For each $n \in \mathbb{Z}_{\geq 1}$, let $\kappa_n : \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p) \times \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p) \to \mathbb{Z}$ be the unique continuous function satisfying

$$\kappa_n(\sigma^i, \sigma^j) = \lfloor i/n \rfloor + \lfloor j/n \rfloor - \lfloor (i+j)/n \rfloor, \quad \forall i, j \in \mathbb{Z}.$$

Here $\sigma$ denotes the arithmetic $p$-Frobenius as usual. Let $\mathcal{D}_n$ be the object in $\mathcal{R}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$ given by $(\mathbb{G}_m, (p^{\kappa_n(\rho,\tau)})_{\rho,\tau})$. For $n, n' \in \mathbb{Z}_{\geq 1}$ with $n|n'$, let $\lambda_{n,n'} : \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p) \to \mathbb{Z}$ be the unique continuous function satisfying

$$\lambda_{n,n'}(\sigma^i) = \lfloor i/n' \rfloor n'/n - \lfloor i/n \rfloor, \quad \forall i \in \mathbb{Z}.$$

Let $\delta_{n',n} : \mathcal{D}_{n'} \to \mathcal{D}_n$ be the morphism in $\mathcal{R}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$ given by

$$(x \mapsto x^{n'/n}, (p^{\lambda_{n,n'}(\rho)})_{\rho}).$$

Then $(\mathcal{D}_n)_{n \in \mathbb{Z}_{\geq 1}}$ equipped with the transition morphisms $\delta_{n',n}$ is an object in pro-$\mathcal{R}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$.

Applying the functor (2.1.18.2) to $(\mathcal{D}_n)_n$, we obtain an object $\mathfrak{D} = (\mathfrak{D}_n)_n = (\mathcal{E}(\mathcal{D}_n))_n$ in pro-$\mathcal{G}rb(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$. This is called the *Dieudonné gerb*.

We denote by $\mathbb{D}$ the pro-torus $\varprojlim_{n\in\mathbb{Z}_{\geq 1}} \mathbb{G}_m$ over $\operatorname{Spec}\mathbb{Z}$, where for $n|n'$ the transition map from the $n'$-th $\mathbb{G}_m$ to the $n$-th $\mathbb{G}_m$ is $z\mapsto z^{n'/n}$. We have

$$\mathfrak{D}^{\Delta}=\mathbb{D}_{\mathbb{Q}_p^{\mathrm{ur}}}.$$

Let $n\in\mathbb{Z}_{\geq 1}$. By construction, the set underlying $\mathfrak{D}_n$ is $\mathbb{G}_m(\mathbb{Q}_p^{\mathrm{ur}})\times\operatorname{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$. Using this we define a canonical element

$$d_{\sigma,n}:=(p^{-\lfloor 1/n\rfloor},\sigma)\in\mathfrak{D}_n,$$

and a canonical map

$$\varsigma_n:\operatorname{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_{p^n})\longrightarrow\mathfrak{D}_n,\quad \rho\longmapsto(1,\rho),$$

which is a continuous section of $\mathfrak{D}_n\to\operatorname{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$. Clearly $\kappa_n(\sigma^i,\sigma^j)=0$ when $i,j$ are divisible by $n$, so $\kappa_n$ vanishes on $\operatorname{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_{p^n})\times\operatorname{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_{p^n})$ by continuity. It follows that $\varsigma_n$ is a group homomorphism. For future use, we compute:

$$d_{\sigma,n}^n=(p^{-\lfloor 1/n\rfloor}\prod_{i=1}^{n-1}p^{\kappa_n(\sigma^i,\sigma)},\sigma^n)=(p^{-1},\sigma^n)=p^{-1}\varsigma_n(\sigma^n),\quad\forall n. \tag{2.2.1.1}$$

$$\mathcal{E}(\delta_{n,n'})(d_{\sigma,n'})=(p^{-\lfloor 1/n'\rfloor\lambda_{n,n'}(\sigma)},\sigma)=(p^{-\lfloor 1/n\rfloor},\sigma)=d_{\sigma,n},\quad\forall n|n'. \tag{2.2.1.2}$$

(2.2.1.3)

$$\mathcal{E}(\delta_{n,n'})(\varsigma_{n'}(\rho))=(p^{\lambda_{n,n'}(\rho)},\rho)=(1,\rho)=\varsigma_n(\rho),\quad\forall n|n',\forall\rho\in\operatorname{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_{p^{n'}}).$$

By (2.2.1.2), the system $(d_{\sigma,n})_n$ defines an element $d_\sigma\in\mathfrak{D}^{\mathrm{top}}$.

**Definition 2.2.2.** Let $\mathfrak{G}_p\in$ pro-$\mathcal{G}rb(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ be the pull back of $\mathfrak{D}$. For each $n\in\mathbb{Z}_{\geq 1}$, let $\mathfrak{G}_{p,n}\in$ pro-$\mathcal{G}rb(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ be the pull back of $\mathfrak{D}_n$.[13] Thus $\mathfrak{G}_p=(\mathfrak{G}_{p,n})_n$.

**Definition 2.2.3.** Let $G$ be a connected linear algebraic group over $\mathbb{Q}_p$. Let $\mathfrak{G}_G\in\mathcal{G}rb(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $\mathfrak{G}_G^{\mathrm{ur}}\in\mathcal{G}rb(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$ be the associated neutral gerbs. A morphism $\theta:\mathfrak{G}_p\to\mathfrak{G}_G$ in pro-$\mathcal{G}rb(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is called *unramified*, if it is the pull-back of a morphism $\theta^{\mathrm{ur}}:\mathfrak{D}\to\mathfrak{G}_G^{\mathrm{ur}}$ in pro-$\mathcal{G}rb(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$. By the obvious generalization of Lemma 2.1.10, $\theta^{\mathrm{ur}}$ is uniquely determined by $\theta$. For general $\theta$, we write $\mathcal{UR}(\theta)$ for the set of $g\in G(\overline{\mathbb{Q}}_p)$ such that $\operatorname{Int}(g^{-1})\circ\theta$ is unramified.

**Lemma 2.2.4.** *Keep the notation of Definition 2.2.3. The following statements hold.*

(i) *For any morphism $\theta:\mathfrak{G}_p\to\mathfrak{G}_G$, the set $\mathcal{UR}(\theta)$ is a $G(\mathbb{Q}_p^{\mathrm{ur}})$-torsor, where $G(\mathbb{Q}_p^{\mathrm{ur}})$ multiplies on the right.*

(ii) *Let $\phi:\mathfrak{D}\to\mathfrak{G}_G^{\mathrm{ur}}$ be a morphism. For sufficiently divisible $n$, we have $\phi(d_\sigma)^n=\phi_n^{\Delta}(p^{-1})\rtimes\sigma^n\in\mathfrak{G}_G^{\mathrm{ur}}$, where $\phi_n$ is a morphism $\mathfrak{D}_n\to\mathfrak{G}_G^{\mathrm{ur}}$ inducing $\phi$.*

*Proof.* By the discussion in §2.1.9, the fact that $\mathfrak{G}_p$ is the pull-back of $\mathfrak{D}$ gives rise to a canonical homomorphism $\varsigma:\Gamma_{p,0}=\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{ur}})\to\mathfrak{G}_p^{\mathrm{top}}$, which is a section of $\mathfrak{G}_p^{\mathrm{top}}\to\Gamma_p$. By (the obvious generalization of) Lemma 2.1.10, a morphism $\theta:\mathfrak{G}_p\to\mathfrak{G}_G$ is unramified if and only if $\theta(\varsigma(\tau))=1\rtimes\tau$ for all $\tau\in\Gamma_{p,0}$.

For part (i), we write $\theta(\varsigma(\tau))=a_\tau\rtimes\tau\in\mathfrak{G}_G$, for $\tau\in\Gamma_{p,0}$. Then $(a_\tau)_\tau\in Z^1(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{ur}},G(\overline{\mathbb{Q}}_p))$. By the previous paragraph, an element $g\in G(\overline{\mathbb{Q}}_p)$ lies in

[13] In [Kis17, §3.1.6], our $\mathfrak{G}_{p,n}$ is denoted by $\widetilde{\mathfrak{G}}_p^{\mathbb{Q}_{p^n}}$.

$\mathcal{UR}(\theta)$ if and only if $g^{-1}a_\tau\tau(g) = 1$ for all $\tau \in \Gamma_{p,0}$. By Steinberg's theorem, such a $g$ exists since $G$ is connected. It is then clear that $\mathcal{UR}(\theta)$ is a $G(\mathbb{Q}_p^{\mathrm{ur}})$-torsor.

We now show part (ii). First pick $n$ such that $\phi$ is induced by a morphism $\phi_n : \mathfrak{D}_n \to \mathfrak{G}_G^{\mathrm{ur}}$. Since $\phi_n$ is continuous, there exists an open subgroup $U$ of $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_{p^n})$ such that $\phi_n(\varsigma_n(\rho)) = 1 \rtimes \rho$ for all $\rho \in U$. We may assume that $U = \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_{p^{n'}})$ for some $n'$ divisible by $n$. Using (2.2.1.3), we may replace $n$ by $n'$ and assume that $n = n'$. We then have

$$\phi(d_\sigma)^n = \phi_n(d_{\sigma,n}^n) = \phi_n(p^{-1}\varsigma_n(\sigma^n)) = \phi_n^\Delta(p^{-1}) \rtimes \sigma^n,$$

where the second equality is by (2.2.1.1). □

**Definition 2.2.5.** Let $G$ be a connected linear algebraic group over $\mathbb{Q}_p$. For any unramified morphism $\theta : \mathfrak{G}_p \to \mathfrak{G}_G$, we define $b_\theta \in G(\mathbb{Q}_p^{\mathrm{ur}})$ by the formula $\theta^{\mathrm{ur}}(d_\sigma) = b_\theta \rtimes \sigma$.

**Proposition 2.2.6.** *Let $G$ be a reductive group over $\mathbb{Q}_p$, and let $\theta : \mathfrak{G}_p \to \mathfrak{G}_G$ be an unramified morphism. The following statements hold.*

*(i) Viewing $\theta^{\mathrm{ur},\Delta} : \mathbb{D}_{\mathbb{Q}_p^{\mathrm{ur}}} \to G_{\mathbb{Q}_p^{\mathrm{ur}}}$ as a fractional cocharacter of $G_{\mathbb{Q}_p^{\mathrm{ur}}}$, we have $\theta^{\mathrm{ur},\Delta} = -\nu_{b_\theta}$. Moreover $b_\theta$ is decent (see §1.4.1).*

*(ii) There are natural $\mathbb{Q}_p$-isomorphisms $J_{b_\theta} \cong I_{\theta^{\mathrm{ur}}} \cong I_\theta$.*

*(iii) Let $\beta \in \mathbf{H}^1(\mathbb{Q}_p, I_\theta)$. Then there is a member $\theta'$ of the conjugacy class $\theta^\beta$ (see Definition 2.1.17) satisfying the following conditions:*

*(a) The morphism $\theta' : \mathfrak{G}_p \to \mathfrak{G}_G$ is unramified.*

*(b) By part (ii), we view $\beta$ as an element of $\mathbf{H}^1(\mathbb{Q}_p, J_{b_\theta})$. Then the $\sigma$-conjugacy class of $b_{\theta'}$ in $G(\mathbb{Q}_p^{\mathrm{ur}})$ is given by the twist of $b_\theta$ by $\beta$, as in §1.4.4.*

*Proof.* **(i)** Choose $n \in \mathbb{Z}_{\geq 1}$ such that $\theta^{\mathrm{ur}}$ is induced by a morphism $\theta_n : \mathfrak{D}_n \to \mathfrak{G}_G^{\mathrm{ur}}$. Let $\nu_n := \theta_n^\Delta$. Then $\nu_n$ is a cocharacter of $G_{\mathbb{Q}_p^{\mathrm{ur}}}$, and $\theta^{\mathrm{ur},\Delta} = n^{-1}\nu_n$. Up to enlarging $n$, we may assume that $\nu_n$ is defined over $\mathbb{Q}_{p^n}$.

By Lemma 2.2.4 (iii), up to enlarging $n$ we have

$$b_\theta\sigma(b_\theta)\cdots\sigma^{n-1}(b_\theta) = \nu_n(p^{-1}) \in G(\mathbb{Q}_p^{\mathrm{ur}}).$$

We conclude that $\nu_{b_\theta} = -n^{-1}\nu_n = -\theta^{\mathrm{ur},\Delta}$, and that $b_\theta$ is $n$-decent.

**(ii)** As in [RZ96, Cor. 1.14], the fact that $b_\theta$ is decent implies that

$$J_{b_\theta}(R) = \left\{ g \in G_{\mathbb{Q}_p^{\mathrm{ur}},\nu_{b_\theta}}(R \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{ur}}) \mid gb_\theta = b_\theta\sigma(g) \right\},$$

for any $\mathbb{Q}_p$-algebra $R$. In view of $G_{\mathbb{Q}_p^{\mathrm{ur}},\nu_{b_\theta}} = G_{\mathbb{Q}_p^{\mathrm{ur}},\theta^{\mathrm{ur},\Delta}}$, the above description of $J_{b_\theta}$ agrees with the explicit description of $I_{\theta^{\mathrm{ur}}}$ as in §2.1.6. The natural isomorphism $I_{\theta^{\mathrm{ur}}} \cong I_\theta$ arises from the fact that they are the same $\mathbb{Q}_p$-form of $G_{\mathbb{Q}_p^{\mathrm{ur}},\theta^{\mathrm{ur},\Delta}}$.

**(iii)** By Steinberg's theorem, $\beta$ is represented by a cocycle

$$a = (a_\rho) \in Z^1(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p, J_b(\mathbb{Q}_p^{\mathrm{ur}})).$$

Viewing $a$ as in $Z^1(\mathbb{Q}_p, I_\theta)$, we define $\theta' := a\theta$, which is in the conjugacy class $\theta^\beta$. Then $\theta'$ is unramified, and we have $b_{\theta'} = a_\sigma b$, which implies condition (b). □

**2.2.7.** Let $\mathcal{G}$ be a reductive group scheme over $\mathbb{Z}_p$, with generic fiber $G$. Let $r \in \mathbb{Z}_{\geq 1}$, and let $\mu$ be a cocharacter of $\mathcal{G}_{\mathbb{Z}_{p^r}}$.

For each morphism $\theta : \mathfrak{G}_p \to \mathfrak{G}_G$ in pro-$\mathcal{G}rb(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, we recall the definition of a set $X_\mu(\theta)$ and a bijection $\Phi : X_\mu(\theta) \to X_\mu(\theta)$ as in [Kis17, §3.3.3]. For each $g \in \mathcal{UR}(\theta)$, we write $b_g$ for $b_{\mathrm{Int}(g^{-1})\circ\theta} \in G(\mathbb{Q}_p^{\mathrm{ur}})$. Define

$$Y_\mu(\theta) := \left\{ g \in \mathcal{UR}(\theta) \mid b_g \in \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}) p^\mu \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}) \subset G(\mathbb{Q}_p^{\mathrm{ur}}) \right\}.$$

The group $\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$ acts on $Y_\mu(\theta)$ via right multiplication, and we define

$$X_\mu(\theta) := Y_\mu(\theta)/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}).$$

By the Cartan decomposition, the subset $\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}) p^\mu \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}) \subset G(\mathbb{Q}_p^{\mathrm{ur}})$ depends on $\mu$ only via the $\mathcal{G}(\mathbb{Z}_{p^r})$-conjugacy class of $\mu$, and so the same holds for $Y_\mu(\theta)$ and $X_\mu(\theta)$.

Consider the map

$$\begin{aligned} \widetilde{\Phi} : \mathcal{UR}(\theta) &\longrightarrow \mathcal{UR}(\theta) \\ g &\longmapsto g b_g \sigma(b_g) \cdots \sigma^{r-1}(b_g). \end{aligned}$$

For $g \in \mathcal{UR}(\theta)$ we have $b_{\widetilde{\Phi}(g)} = \sigma^r(b_g)$. It follows that $\widetilde{\Phi}$ restricts to a map $Y_\mu(\theta) \to Y_\mu(\theta)$. If we fix an element $g_0 \in \mathcal{UR}(\theta)$ and use it to identify the $G(\mathbb{Q}_p^{\mathrm{ur}})$-torsor $\mathcal{UR}(\theta)$ with $G(\mathbb{Q}_p^{\mathrm{ur}})$, then the map $\widetilde{\Phi}$ becomes the map $G(\mathbb{Q}_p^{\mathrm{ur}}) \to G(\mathbb{Q}_p^{\mathrm{ur}}), g \mapsto b_{g_0}\sigma(b_{g_0}) \cdots \sigma^{r-1}(b_{g_0})\sigma^r(g)$. This shows that $\widetilde{\Phi}$ restricts to a $\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$-equivariant bijection $Y_\mu(\theta) \to Y_\mu(\theta)$. Hence $\widetilde{\Phi}$ induces a bijection

$$\Phi : X_\mu(\theta) \xrightarrow{\sim} X_\mu(\theta),$$

which we call the *$p^r$-Frobenius.*

The isomorphism class of the $\Phi^{\mathbb{Z}}$-set $X_\mu(\theta)$ depends on $\theta$ only through the conjugacy class of $\theta$. Moreover, after fixing an element $g_0 \in \mathcal{UR}(\theta)$, from the previous paragraph we know that the map $G(\overline{\mathbb{Q}}_p) \to G(\overline{\mathbb{Q}}_p), g \mapsto g_0^{-1} g$ induces a bijection from $X_\mu(\theta)$ to the *affine Deligne–Lusztig set*

$$\begin{aligned} X_\mu(b_{g_0}) &:= \left\{ g \in G(\mathbb{Q}_p^{\mathrm{ur}})/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}) \mid g^{-1} b_{g_0} \sigma(g) \in \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}) p^\mu \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}) \right\} \\ &\xrightarrow{\sim} \left\{ g \in G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p) \mid g^{-1} b_{g_0} \sigma(g) \in \mathcal{G}(\breve{\mathbb{Z}}_p) p^\mu \mathcal{G}(\breve{\mathbb{Z}}_p) \right\}, \end{aligned} \tag{2.2.7.1}$$

on which $\Phi$ acts by $g \mapsto b_{g_0}\sigma(b_{g_0}) \cdots \sigma^{r-1}(b_{g_0})\sigma^r(g)$. The second line is the usual definition of an affine Deligne–Lusztig set found in the literature, and we have a natural map (2.2.7.1) induced by the inclusion $G(\mathbb{Q}_p^{\mathrm{ur}}) \hookrightarrow G(\breve{\mathbb{Q}}_p)$. That this map is a bijection follows from Lemma 1.6.8 and the functoriality of the Cartan decomposition.

**2.2.8.** We keep fixing a prime $p$. For each finite prime $v \neq p$, let

$$\mathfrak{G}_v := \Gamma_v \in \mathcal{G}rb(\overline{\mathbb{Q}}_v/\mathbb{Q}_v).$$

Let

$$\mathfrak{G}_\infty \in \mathcal{G}rb(\mathbb{C}/\mathbb{R})$$

be the Weil group of $\mathbb{R}$, and let

$$\mathfrak{G}_p \in \text{pro-}\mathcal{G}rb(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$$

be as in Definition 2.2.2.

Consider the pro-torus $\mathrm{Res}_{\overline{\mathbb{Q}}/\mathbb{Q}} \mathbb{G}_m := \varprojlim_L \mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_m$ over $\mathbb{Q}$, where $L$ runs through the set of finite Galois extensions of $\mathbb{Q}$ contained in $\overline{\mathbb{Q}}$, ordered by inclusion, and the transition maps are the norm maps. The neutral gerbs $\mathfrak{G}_{\mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_m} \in \mathcal{G}rb_{\mathbb{Q}}$

for different $L$ form a projective system, i.e., an object in pro-$\mathcal{G}rb(\overline{\mathbb{Q}}/\mathbb{Q})$. We denote this object by $\mathfrak{G}_{\mathrm{Res}_{\overline{\mathbb{Q}}/\mathbb{Q}}\mathbb{G}_m}$.

For each object $\mathfrak{G}$ (resp. morphism $\phi$) in pro-$\mathcal{G}rb(\overline{\mathbb{Q}}/\mathbb{Q})$, we denote its pull-back in pro-$\mathcal{G}rb(\overline{\mathbb{Q}}_v/\mathbb{Q}_v)$ by $\mathfrak{G}(v)$ (resp. $\phi(v)$), for each place $v$ of $\mathbb{Q}$. Here the pull-back functor is defined with respect to our fixed embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_v$. Reimann [Rei97, §B.2] has constructed a *quasi-motivic Galois gerb*, which is an object

$$\mathfrak{Q} \in \text{pro-}\mathcal{G}rb(\overline{\mathbb{Q}}/\mathbb{Q}),$$

equipped with a morphism $\zeta_v : \mathfrak{G}_v \to \mathfrak{Q}(v)$ in pro-$\mathcal{G}rb(\overline{\mathbb{Q}}_v/\mathbb{Q}_v)$ for each place $v$ of $\mathbb{Q}$, and a morphism $\psi : \mathfrak{Q} \to \mathfrak{G}_{\mathrm{Res}_{\overline{\mathbb{Q}}/\mathbb{Q}}\mathbb{G}_m}$ in pro-$\mathcal{G}rb(\overline{\mathbb{Q}}/\mathbb{Q})$. More precisely, Reimann constructs in the proof of [Rei97, Thm. B.2.8] versions of $\mathfrak{Q}$, $\zeta_v$, and $\psi$ in the categories pro-$\mathcal{R}(\overline{\mathbb{Q}}/\mathbb{Q})$ and pro-$\mathcal{R}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v)$. We transport his constructions via the functors (2.1.18.2). The unique characterization of the tuple $(\mathfrak{Q}, (\zeta_v)_v, \psi)$ is delicate to state. We omit this and refer the reader to *loc. cit.* and [Kis17, Thm. 3.1.9].

By construction, $\mathfrak{Q}$ is given by a projective system $(\mathfrak{Q}^L)_L$ in $\mathcal{G}rb_{\mathbb{Q}}$, indexed by the set of finite Galois extensions $L/\mathbb{Q}$ contained in $\overline{\mathbb{Q}}$, ordered by inclusion. For each $L$, we have $\mathfrak{Q}^{L,\Delta} = Q^L_{\overline{\mathbb{Q}}}$, where $Q^L$ is a $\mathbb{Q}$-torus explicitly described in [Rei97, §B.2]. (Here $Q^L$ agrees with the canonical $\mathbb{Q}$-form of $\mathfrak{Q}^{L,\Delta}$ as in Remark 2.1.7.) If $L \subset L' \subset \overline{\mathbb{Q}}$, then the transition morphism $\mathfrak{Q}^{L'} \to \mathfrak{Q}^L$ is surjective, and its kernel $Q^{L'}_{\overline{\mathbb{Q}}} \to Q^L_{\overline{\mathbb{Q}}}$ is defined over $\mathbb{Q}$. We write $Q$ for the pro-torus $(\varprojlim_L Q^L)$ over $\mathbb{Q}$. Since the projective system $(\mathfrak{Q}^L)_L$ is indexed by a countable set and since the transition morphisms are surjective, we conclude that the projections $\mathfrak{Q}^{\mathrm{top}} \to \mathfrak{Q}^L$ are surjective. (See §2.1.12 for the notation $\mathfrak{Q}^{\mathrm{top}}$.)

For $v \in \{p, \infty\}$, we denote the group scheme homomorphism $\zeta_v^\Delta : \mathfrak{G}_v^\Delta \to \mathfrak{Q}(v)^\Delta$ (see §2.1.12) by $\nu(v)$. By construction, $\nu(v)$ is defined over $\mathbb{Q}_v$. Thus we have $\nu(p) : \mathbb{D}_{\mathbb{Q}_p} \to Q_{\mathbb{Q}_p}$ and $\nu(\infty) : \mathbb{G}_{m,\mathbb{R}} \to Q_{\mathbb{R}}$.

**2.2.9.** Let $T$ be a torus over $\mathbb{Q}$, and let $\mu \in X_*(T)$. Let $L/\mathbb{Q}$ be a finite Galois extension contained in $\overline{\mathbb{Q}}$ such that $\mu$ is defined over $L$. Then $\mu$ induces a $\mathbb{Q}$-homomorphism

$$\mu_* : \mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m \xrightarrow{\mathrm{Res}_{L/\mathbb{Q}}\mu} \mathrm{Res}_{L/\mathbb{Q}} T \xrightarrow{\mathrm{N}_{L/\mathbb{Q}}} T.$$

We obtain a morphism $\Psi_{T,\mu} : \mathfrak{Q} \to \mathfrak{G}_T$ in pro-$\mathcal{G}rb(\overline{\mathbb{Q}}/\mathbb{Q})$ via the composition

$$\mathfrak{Q} \xrightarrow{\psi} \mathfrak{G}_{\mathrm{Res}_{\overline{\mathbb{Q}}/\mathbb{Q}}\mathbb{G}_m} \to \mathfrak{G}_{\mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m} \xrightarrow{\mu_*} \mathfrak{G}_T.$$

This is independent of the choice of $L$.

**Lemma 2.2.10.** *Let $\theta = \Psi_{T,\mu}(p) \circ \zeta_p : \mathfrak{G}_p \to \mathfrak{G}_T(p)$. Choose $g \in \mathcal{UR}(\theta)$, and let $[b] \in \mathrm{B}(T_{\mathbb{Q}_p})$ be the $\sigma$-conjugacy class of $b_{\mathrm{Int}(g^{-1})\circ\theta} \in T(\mathbb{Q}_p^{\mathrm{ur}})$ in $T(\breve{\mathbb{Q}}_p)$ (which is well defined, by Lemma 2.2.4 (i)). Then $\kappa_T([b]) \in X_*(T)_{\Gamma_p}$ is equal to the image of $-\mu \in X_*(T)$.*

*Proof.* The proof reduces to the "universal case", where $T = \mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m$, and the map $\mu_* : \mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m \to T$ is the identity. In this case, $X_*(T)_{\Gamma_p}$ is torsion free, and therefore the homomorphism $\iota : X_*(T)_{\Gamma_p} \to X_*(T) \otimes \mathbb{Q}$ induced by taking averages of $\Gamma_p$-orbits in $X_*(T)$ is injective. For each $[b'] \in \mathrm{B}(T_{\mathbb{Q}_p})$, we have $\iota(\kappa_T([b'])) = \nu_{b'}$ by [Kot85, §2.8]. Hence to prove the lemma it suffices to prove that $\iota(\mu) = -\nu_b$. By

Proposition 2.2.6 (i), we have $-\nu_b = (\mathrm{Int}(g^{-1}) \circ \theta)^{\mathrm{ur},\Delta} = \theta^{\Delta}$. By [Kis17, (3.1.11)], $\theta^{\Delta}$ is indeed equal to $\iota(\mu)$, as desired.[14] $\square$

## 2.3. **Strictly monoidal categories.**

**2.3.1.** Let $G, H$ be two strictly monoidal categories (which we always assume to be small). By a *strictly monoidal functor* $G \to H$, we mean a functor that strictly respects the monoidal structures. By a *monoidal isomorphism* between two strictly monoidal functors $\phi, \psi : G \to H$, we mean an isomorphism of functors $\mathcal{A} : \phi \xrightarrow{\sim} \psi$ such that for any two objects $g_1, g_2 \in G$ the following diagram commutes:

$$\begin{tikzcd} \phi(g_1 \otimes g_2) \arrow[r, "\mathcal{A}(g_1 \otimes g_2)"] \arrow[d, equal] & \psi(g_1 \otimes g_2) \arrow[d, equal] \\ \phi(g_1) \otimes \phi(g_2) \arrow[r, "\mathcal{A}(g_1) \otimes \mathcal{A}(g_2)"] & \psi(g_1) \otimes \psi(g_2) \end{tikzcd}$$

Every group can be naturally viewed as a strictly monoidal category, where the only morphisms are the identities. For two groups $G$ and $H$, the set of strictly monoidal functors $G \to H$ is the same as the set of group homomorphisms $G \to H$. More generally, each crossed module $(\widetilde{H} \to H)$ also determines a strictly monoidal category denoted by $H/\widetilde{H}$; see [Kis17, §3.2.1]. When $(\widetilde{H} \to H)$ is a crossed module, each element $h \in H$ induces via conjugation a strictly monoidal functor $\mathrm{Int}(h) : H/\widetilde{H} \to H/\widetilde{H}$.

**2.3.2.** Now let $G$ be a group and $(\widetilde{H} \xrightarrow{\varrho} H)$ be a crossed module. We denote the structural action of $H$ on $\widetilde{H}$ by Int. Consider two strictly monoidal functors $\phi, \psi : G \to H/\widetilde{H}$. If $\mathcal{A} : \phi \xrightarrow{\sim} \psi$ is an isomorphism of functors, then for each $q \in G$ the isomorphism $\mathcal{A}(q) : \phi(q) \xrightarrow{\sim} \psi(q)$ corresponds to an element $\mathcal{A}(q) \in \widetilde{H}$. Thus we may view $\mathcal{A}$ as a function $G \to \widetilde{H}$. In this way, there is a one-to-one correspondence between monoidal isomorphisms $\mathcal{A} : \phi \xrightarrow{\sim} \psi$ and functions $\mathcal{A} : G \to \widetilde{H}$ satisfying

$$\mathcal{A}(qr) = \mathcal{A}(q) \cdot \mathrm{Int}(\phi(q))(\mathcal{A}(r)),$$
$$\varrho(\mathcal{A}(q)) \cdot \phi(q) = \psi(q), \quad \forall q, r \in G.$$

When $G$ is equipped with a topology, we shall call a monoidal isomorphism $\mathcal{A} : \phi \xrightarrow{\sim} \psi$ *continuous*, if the corresponding function $\mathcal{A} : G \to \widetilde{H}$ is continuous with respect to the given topology on $G$ and the discrete topology on $\widetilde{H}$.

**2.3.3.** Let $k$ be a field of characteristic zero, and let $\bar{k}$ be a fixed algebraic closure. Let $G$ be a reductive group over $k$, and let $\mathfrak{G}_G$ be the associated neutral gerb in $\mathcal{G}rb(\bar{k}/k)$. As in [Kis17, §3.2.2], we have a crossed module $G_{\mathrm{sc}}(\bar{k}) \to \mathfrak{G}_G$, and we denote the corresponding strictly monoidal category $\mathfrak{G}_G/G_{\mathrm{sc}}(\bar{k})$ by $\mathfrak{G}_{G/G_{\mathrm{sc}}}$. We have a canonical strictly monoidal functor $\mathfrak{G}_G \to \mathfrak{G}_{G/G_{\mathrm{sc}}}$.

**Lemma 2.3.4.** *Keep the setting and notation of §2.3.3. Let $\mathfrak{H} = (\mathfrak{H}_i)_{i \in I}$ be an object in* pro-$\mathcal{G}rb(\bar{k}/k)$ *such that the projections $\mathfrak{H}^{\mathrm{top}} \to \mathfrak{H}_i$ are surjective and such that $\mathfrak{H}_i^{\Delta}$ are tori for all $i \in I$. Let $\phi : \mathfrak{H} \to \mathfrak{G}_G$ be a morphism in* pro-$\mathcal{G}rb(\bar{k}/k)$*. Let*

[14] A similar argument is made in the proof of [Kis17, Lem. 3.4.2] in order to determine $\kappa_T([b])$ for $T = \mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_m$. There $T$ is unramified over $\mathbb{Q}_p$, so the cited result [RR96, Thm. 4.2 (ii)] is valid. Our present argument does not need this unramifiedness assumption.

*$a \in Z^1(k, I_\phi)$ and let $\phi' = a\phi$ (see §2.1.15). Let $\phi_{\widetilde{\mathrm{ab}}}$ (resp. $\phi'_{\widetilde{\mathrm{ab}}}$) be the composite strictly monoidal functor*

$$\mathfrak{H}^{\mathrm{top}} \xrightarrow{\phi^{\mathrm{top}}\ (resp.\ \phi^{\prime,\mathrm{top}})} \mathfrak{G}_G \longrightarrow \mathfrak{G}_{G/G_{\mathrm{sc}}}.$$

*Then there exist $g \in G(\bar{k})$ and a continuous monoidal isomorphism $\phi_{\widetilde{\mathrm{ab}}} \xrightarrow{\sim} \mathrm{Int}(g) \circ \phi'_{\widetilde{\mathrm{ab}}}$, if and only if the class of $a$ in $\mathbf{H}^1(k, I_\phi)$ lies in the image of $\mathbf{H}^1(k, \widetilde{I}_\phi) \to \mathbf{H}^1(k, I_\phi)$. Here $\widetilde{I}_\phi$ is defined in §2.1.8 and §2.1.14.*

*Proof.* By the assumption that $\mathfrak{H}^{\mathrm{top}} \to \mathfrak{H}_i$ are surjective, the lemma reduces to the case where $\mathfrak{H} \in \mathcal{G}rb(\bar{k}/k)$, which we now assume. Write $M$ for $G_{\bar{k},\phi^\Delta}$. Let $\varrho$ denote the natural map $G_{\mathrm{sc}} \to G$, and let $\widetilde{M} := \varrho^{-1}(M) \subset G_{\mathrm{sc},\bar{k}}$. Write $\pi$ for the structural map $\mathfrak{H} \to \mathrm{Gal}(\bar{k}/k)$. For each $q \in \mathfrak{H}$, we write $\phi(q) = g_q \rtimes \pi(q)$, with $g_q \in G(\bar{k})$.

Assume there exist $g \in G(\bar{k})$ and a continuous monoidal isomorphism $\mathcal{A} : \phi_{\widetilde{\mathrm{ab}}} \xrightarrow{\sim} \mathrm{Int}(g) \circ \phi'_{\widetilde{\mathrm{ab}}}$. As discussed in §2.3.2, $\mathcal{A}$ can be viewed as a continuous function $\mathfrak{H} \to G_{\mathrm{sc}}(\bar{k})$ satisfying

$$\mathcal{A}(qr) = \mathcal{A}(q) \cdot \mathrm{Int}(\phi(q))(\mathcal{A}(r)), \tag{2.3.4.1}$$

$$\varrho(\mathcal{A}(q)) \cdot \phi(q) = \mathrm{Int}(g)[a_{\pi(q)}\phi(q)], \quad \forall q, r \in \mathfrak{H}. \tag{2.3.4.2}$$

Decompose $g$ as $g = g'z$, with $g' \in G_{\mathrm{der}}(\bar{k}), z \in Z_G(\bar{k})$. Fix a lift $\widetilde{g} \in G_{\mathrm{sc}}(\bar{k})$ of $g'$, and define a continuous map $\mathcal{B} : \mathfrak{H} \to G_{\mathrm{sc}}(\bar{k})$ by

$$\mathcal{B}(q) := \widetilde{g}^{-1}\mathcal{A}(q) \cdot \mathrm{Int}(\phi(q))(\widetilde{g}) \in G_{\mathrm{sc}}(\bar{k}), \quad \forall q \in \mathfrak{H}.$$

Then by (2.3.4.1), we have

$$\mathcal{B}(qr) = \mathcal{B}(q) \cdot \mathrm{Int}(\phi(q))(\mathcal{B}(r)), \quad \forall q, r \in \mathfrak{H}. \tag{2.3.4.3}$$

By (2.3.4.2), we have

$$\begin{aligned} \varrho(\mathcal{B}(q)) &= (g')^{-1} \cdot [\varrho(\mathcal{A}(q)) \cdot \phi(q)] \cdot g'\phi(q)^{-1} \\ &= z a_{\pi(q)}\phi(q) g^{-1} \cdot g'\phi(q)^{-1} = z a_{\pi(q)} g_q \pi(q) z^{-1} \pi(q)^{-1} g_q^{-1}, \end{aligned}$$

i.e.,

$$\varrho(\mathcal{B}(q)) = z^{\pi(q)} z^{-1} a_{\pi(q)}, \quad \forall q \in \mathfrak{H}. \tag{2.3.4.4}$$

From (2.3.4.4), we see that

$$\mathcal{B}(q) \in \widetilde{M}(\bar{k}),\ \forall q \in \mathfrak{H}; \tag{2.3.4.5}$$

$$\mathcal{B}(q) \in Z_{G_{\mathrm{sc}}}(\bar{k}),\ \forall q \in \mathfrak{H}^\Delta(\bar{k}). \tag{2.3.4.6}$$

Using (2.3.4.3) and (2.3.4.5), we see that the map $\mathcal{B}|_{\mathfrak{H}^\Delta(\bar{k})} : \mathfrak{H}^\Delta(\bar{k}) \to Z_{G_{\mathrm{sc}}}(\bar{k})$ is a group homomorphism. Since $\mathfrak{H}^\Delta(\bar{k})$ is a divisible abelian group and $Z_{G_{\mathrm{sc}}}(\bar{k})$ is finite, we have $\mathcal{B}|_{\mathfrak{H}^\Delta(\bar{k})} \equiv 1$. Combining the last fact with (2.3.4.3), we see that $\mathcal{B}(q)$ depends only on $\pi(q)$, i.e., $\mathcal{B} = B \circ \pi$ for a continuous map $B : \mathrm{Gal}(\bar{k}/k) \to \widetilde{M}(\bar{k})$. Now by the definition of the $k$-form $\widetilde{I}_\phi$ of $\widetilde{M}$, the relation (2.3.4.3) precisely means that $B \in Z^1(k, \widetilde{I}_\phi)$. Note that the inclusion of $\bar{k}$-groups $Z_{G,\bar{k}} \hookrightarrow M$ induces an inclusion of $k$-groups $Z_G \to I_\phi$. Hence (2.3.4.4) implies that the class of $a$ in $\mathbf{H}^1(k, I_\phi)$ equals the image of the class of $B$ in $\mathbf{H}^1(k, \widetilde{I}_\phi)$.

Conversely, assume the class of $a$ in $\mathbf{H}^1(k, I_\phi)$ lies in the image of $\mathbf{H}^1(k, \widetilde{I}_\phi)$. Then there exist $g \in I_\phi(\bar{k})$ and $B \in Z^1(k, \widetilde{I}_\phi)$ such that

$$a_\tau = g \cdot \varrho(B(\tau)) \cdot \mathrm{Int}(g_\tau)({}^\tau g^{-1}), \quad \forall \tau \in \mathrm{Gal}(\bar{k}/k).$$

Decompose $g$ as $g = \varrho(\widetilde{g})z^{-1}$, with $\widetilde{g} \in \widetilde{M}(\bar{k})$ and $z \in Z_G(\bar{k})$. Then we can absorb $\widetilde{g}$ into $B$ by replacing each $B(\tau)$ with $\widetilde{g}B(\tau)\,\mathrm{Int}(g_\tau)({}^\tau\widetilde{g}^{-1})$. Hence we may assume that $g = z^{-1}$. Let $\mathcal{B} := B \circ \pi : \mathfrak{H} \to \widetilde{M}(\bar{k})$. Then $\mathcal{B}$ is continuous and satisfies (2.3.4.3) and (2.3.4.4). Note that the relation (2.3.4.4) can also be written as

$$\varrho(\mathcal{B}(q)) \cdot \phi(q) = \mathrm{Int}(z)[a_{\pi(q)}\phi(q)] \tag{2.3.4.7}$$

(since ${}^{\pi(q)}z = \mathrm{Int}(\phi(q))(z)$). By (2.3.4.3) and (2.3.4.7), $\mathcal{B}$ is a continuous monoidal isomorphism $\phi_{\widetilde{\mathrm{ab}}} \xrightarrow{\sim} \mathrm{Int}(z) \circ \phi'_{\widetilde{\mathrm{ab}}}$. □

*Remark* 2.3.5. By the discussion in §2.2.8, the assumptions on $\mathfrak{H}$ in Lemma 2.3.4 are satisfied by the quasi-motivic Galois gerb $\mathfrak{Q} \in$ pro-$\mathcal{G}rb(\overline{\mathbb{Q}}/\mathbb{Q})$.

## 2.4. **Admissible morphisms for an unramified Shimura datum.**

**2.4.1.** Let $(G, X, p, \mathcal{G})$ be an unramified Shimura datum as in §1.8.1. Let $E$ be the reflex field, and let $\mathfrak{p}$ and $q = p^r$ be as in §1.6.1. We will use the following notation throughout the paper. For each field extension $F/E$ such that $G_F$ is quasi-split, the Hodge cocharacters $\mu_h$ attached to $h \in X$ determine a $G(F)$-conjugacy class of cocharacters of $G_F$. We denote this conjugacy class by $\mathbb{\mu}_X(F)$. Now inside $\mathbb{\mu}_X(\mathbb{Q}_{p^r})$, there is a canonical $\mathcal{G}(\mathbb{Z}_{p^r})$-conjugacy class consisting of those cocharacters in $\mathbb{\mu}_X(\mathbb{Q}_{p^r})$ that extend to cocharacters of $\mathcal{G}_{\mathbb{Z}_{p^r}}$. We denote this $\mathcal{G}(\mathbb{Z}_{p^r})$-conjugacy class by $\mathbb{\mu}_X^{\mathcal{G}}$.

A choice of $x \in X$ gives rise to a morphism

$$\xi_\infty : \mathfrak{G}_\infty \longrightarrow \mathfrak{G}_G(\infty)$$

in $\mathcal{G}rb(\mathbb{C}/\mathbb{R})$, whose conjugacy class depends only on $X$. See [Kis17, §3.3.5] for the explicit construction. For a finite prime $v$ unequal to $p$, let

$$\xi_v : \mathfrak{G}_v = \Gamma_v \longrightarrow \mathfrak{G}_G(v) = G(\overline{\mathbb{Q}}_v) \rtimes \Gamma_v$$

be the natural section. Then $\xi_v$ is a morphism in $\mathcal{G}rb(\overline{\mathbb{Q}}_v/\mathbb{Q}_v)$.

Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be a morphism in pro-$\mathcal{G}rb(\overline{\mathbb{Q}}/\mathbb{Q})$. (Here $\mathfrak{Q}$ is defined with respect to the fixed $p$.) For each place $v$ of $\mathbb{Q}$ except $p$, we define

$$X_v(\phi) := \left\{ g \in G(\overline{\mathbb{Q}}_v) \mid \mathrm{Int}(g) \circ \xi_v = \phi(v) \circ \zeta_v \right\}.$$

We define

$$X_p(\phi) := X_{-\mu}(\phi(p) \circ \zeta_p)$$

for $\mu \in \mathbb{\mu}_X^{\mathcal{G}}$. Here the right hand side is as in §2.2.7, and it is independent of the choice of $\mu$. We have the $p^r$-Frobenius $\Phi : X_p(\phi) \xrightarrow{\sim} X_p(\phi)$.

**Definition 2.4.2** (cf. [Kis17, §3.3.6])**.** Keep the setting of §2.4.1. A morphism $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ in pro-$\mathcal{G}rb(\overline{\mathbb{Q}}/\mathbb{Q})$ is called *admissible*, if the following conditions are satisfied.

(i) Let $\mu \in \mathbb{\mu}_X(\overline{\mathbb{Q}})$. Let $\psi_{\mu_{\widetilde{\mathrm{ab}}}} : \mathfrak{Q} \longrightarrow \mathfrak{G}_{G/G_{\mathrm{sc}}}$ be the associated strictly monoidal functor, defined in [Kis17, §3.3.1]. Let $\phi_{\widetilde{\mathrm{ab}}}$ be the composite strictly monoidal functor $\mathfrak{Q} \xrightarrow{\phi} \mathfrak{G}_G \to \mathfrak{G}_{G/G_{\mathrm{sc}}}$. We require that there exist $g \in G(\overline{\mathbb{Q}})$ and a continuous monoidal isomorphism $\mathcal{A} : \mathrm{Int}(g) \circ \psi_{\mu_{\widetilde{\mathrm{ab}}}} \xrightarrow{\sim} \phi_{\widetilde{\mathrm{ab}}}$.

(ii) For each place $v$ of $\mathbb{Q}$, $X_v(\phi) \neq \emptyset$.

*Remark* 2.4.3. Condition (i) in Definition 2.4.2 is a correction of condition (1) in [Kis17, §3.3.6] in that we add the requirement that $\mathcal{A}$ should be continuous. The results on admissible morphisms in [Kis17, §3.4], especially the statement and proof of [Kis17, Prop. 3.4.11], are only correct with the present definition.

*Remark* 2.4.4. Given a morphism $\mathfrak{Q} \to \mathfrak{G}_G$ in pro-$\mathcal{G}rb(\overline{\mathbb{Q}}/\mathbb{Q})$, whether it is admissible depends only on its conjugacy class. We may thus speak of admissible conjugacy classes of morphisms $\mathfrak{Q} \to \mathfrak{G}_G$.

*Remark* 2.4.5. As is pointed out by Reimann [Rei97, App. B], the original construction of the quasi-motivic gerb $\mathfrak{Q}$ in [LR87] is incorrect. As such, the definitions and results in [LR87, §5] about "admissible morphisms" do not directly apply to the objects defined in Definition 2.4.2. Nevertheless, most of the content of *loc. cit.* can be modified to suit the corrected definition of $\mathfrak{Q}$. In the sequel, we shall cite *loc. cit.* only for those technical results that are essentially independent of the actual construction of $\mathfrak{Q}$.

**2.4.6.** Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be an arbitrary morphism in pro-$\mathcal{G}rb(\overline{\mathbb{Q}}/\mathbb{Q})$. By condition (ii) in [Rei97, Def. B2.7], there exists a continuous cocycle $\zeta_\phi^{p,\infty} : \Gamma \to G(\bar{\mathbb{A}}_f^p)$ that induces the morphisms $\phi(l) \circ \zeta_l$ for all finite primes $l \neq p$, in the following sense. We have a canonical map $\bar{\mathbb{A}}_f^p = \mathbb{A}_f^p \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_l$ given by the projection $\mathbb{A}_f^p \to \mathbb{Q}_l$ and the fixed embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_l$. Denote the composition

$$\Gamma_l \hookrightarrow \Gamma \xrightarrow{\zeta_\phi^{p,\infty}} G(\bar{\mathbb{A}}_f^p) \to G(\overline{\mathbb{Q}}_l)$$

by $\zeta_{\phi,l}$. Then for each $\tau \in \mathfrak{G}_l = \Gamma_l$ we have

$$(\phi(l) \circ \zeta_l)(\tau) = \zeta_{\phi,l}(\tau) \rtimes \tau \in G(\overline{\mathbb{Q}}_l) \rtimes \Gamma_l.$$

If we choose an arbitrary $\mathbb{Z}$-structure on $G$, then for almost all primes $l \neq p$ the set $X_l(\phi)$ contains integral points in $G(\overline{\mathbb{Q}}_l)$ (and is *a fortiori* non-empty). Indeed, for almost all $l$, the chosen $\mathbb{Z}$-structure on $G$ has connected smooth reduction at $l$, and $\zeta_{\phi,l}$ is induced by a continuous unramified cocycle $\mathrm{Gal}(\mathbb{Q}_l^{\mathrm{ur}}/\mathbb{Q}_l) \to G(\mathbb{Z}_l^{\mathrm{ur}})$. It is a standard result (see for instance [PR94, p. 294, Thm. 6.8']) that any such cocycle is a coboundary, and this precisely means that $X_l(\phi)$ contains integral points.

**2.4.7.** Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be an admissible morphism. On choosing a $\mathbb{Z}$-structure on $G$, we form the restricted product

$$X^p(\phi) := \prod_{l \notin \{p,\infty\}}' X_l(\phi)$$

with respect to the subsets of integral elements of the $X_l(\phi)$ (cf. §2.4.6). Clearly $X^p(\phi)$ is independent of the choice of $\mathbb{Z}$-structure, and is a $G(\mathbb{A}_f^p)$-torsor under right multiplication. (It is non-empty since $\phi$ is admissible.) Equivalently, with the notation in §2.4.6, $X^p(\phi)$ is the right $G(\mathbb{A}_f^p)$-torsor consisting of $x \in G(\bar{\mathbb{A}}_f^p)$ such that

$$x^{-1} \cdot \zeta_\phi^{p,\infty}(\tau) \cdot {}^\tau x = 1, \quad \forall \tau \in \Gamma.$$

We now define

$$X(\phi) := X_p(\phi) \times X^p(\phi),$$

which is equipped with the action of $\Phi^{\mathbb{Z}} \times G(\mathbb{A}_f^p)$. We still call $\Phi$ the *q-Frobenius* on $X(\phi)$.

By definition, $X(\phi)$ is a subset of $G(\overline{\mathbb{Q}}_p)/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}) \times G(\bar{\mathbb{A}}_f^p)$. Under the canonical embedding $I_{\phi,\overline{\mathbb{Q}}} \hookrightarrow G_{\overline{\mathbb{Q}}}$, we let the group $I_\phi(\mathbb{A}_f)$ act on $G(\overline{\mathbb{Q}}_p)/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}) \times G(\bar{\mathbb{A}}_f^p)$ via left multiplication. This induces a left action of $I_\phi(\mathbb{A}_f)$ on $X(\phi)$. For each $\tau \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$, we set

$$S_\tau(\phi) := \varprojlim_{K^p} I_\phi(\mathbb{Q})_\tau \backslash X(\phi)/K^p,$$

where $K^p$ runs through the compact open subgroups of $G(\mathbb{A}_f^p)$, and $I_\phi(\mathbb{Q})_\tau$ denotes the image of

$$I_\phi(\mathbb{Q}) \hookrightarrow I_\phi(\mathbb{A}_f) \xrightarrow{\mathrm{Int}(\tau)} I_\phi(\mathbb{A}_f).$$

Then $S_\tau(\phi)$ inherits the action of $\Phi^{\mathbb{Z}} \times G(\mathbb{A}_f^p)$. When $\tau = 1$, we write $S(\phi)$ for $S_\tau(\phi)$.

## 2.5. **Integral models and the Langlands–Rapoport Conjecture.**

**2.5.1.** Keep the setting and notation of §2.4.1. Let $K_p = \mathcal{G}(\mathbb{Z}_p)$, and let $\mathrm{Sh}_{K_p} = \mathrm{Sh}_{K_p}(G, X)$ be the inverse limit

$$\varprojlim_{K^p} \mathrm{Sh}_{K_pK^p}(G, X),$$

where $K^p$ runs through compact open subgroups of $G(\mathbb{A}_f^p)$. This inverse limit exists as an $E$-scheme, since the transition maps are finite. The right $G(\mathbb{A}_f^p)$-action on $\mathrm{Sh}_{K_p}$ induced by the $G(\mathbb{A}_f)$-action on $\mathrm{Sh}(G, X)$ is admissible in the sense of Definition 1.5.1.

**Definition 2.5.2.** By a *smooth integral model* of $\mathrm{Sh}_{K_p}$, we mean a scheme $\mathscr{S}_{K_p}$ over $\mathcal{O}_{E,(\mathfrak{p})}$ extending $\mathrm{Sh}_{K_p}$, equipped with an admissible right $G(\mathbb{A}_f^p)$-action extending the $G(\mathbb{A}_f^p)$-action on $\mathrm{Sh}_{K_p}$.[15] When $\mathscr{S}_{K_p}$ is given, we write $\mathscr{S}_{K_pK^p}$ for $\mathscr{S}_{K_p}/K^p$ for all sufficiently small compact open subgroups $K^p \subset G(\mathbb{A}_f^p)$.

The following theorem is proved in [Kis10] for $p > 2$, and in [KMP16] for $p = 2$.

**Theorem 2.5.3** ([Kis10, KMP16])**.** *If $(G, X)$ is of abelian type, then there exists a smooth integral model of* $\mathrm{Sh}_{K_p}$*. This model is uniquely characterized by the extension property as detailed in* [Kis10, §2.3.7].

*Remark* 2.5.4. In [Kis10] and [KMP16], it is not explicitly verified that the $G(\mathbb{A}_f^p)$-action on the integral model satisfies the separatedness in condition (ii) and condition (iii) in Definition 1.5.1. The former follows from the facts that the Siegel modular schemes at finite levels are separated over $\mathbb{Z}_{(p)}$, that normalization maps and closed immersions are separated, and that taking finite free quotients preserve separatedness. For the latter, see [LS18, §3] for an explanation.

**2.5.5.** Now fix a prime $\ell \neq p$ and fix an irreducible $\overline{\mathbb{Q}}_\ell$-representation $\xi$ of $G$ that factors through $G^c = G/Z_{ac}$, as in §1.5.8. Suppose a smooth integral model $\mathscr{S}_{K_p}$ is given. As explained in §1.5.2, for sufficiently small compact open subgroups $U^p \subset K^p \subset G(\mathbb{A}_f^p)$, the map $\mathscr{S}_{K_pU^p} \to \mathscr{S}_{K_pK^p}$ is finite étale Galois and the Galois

[15] The adjective "smooth" refers to the smoothness requirement in condition (ii) in Definition 1.5.1 for the admissible $G(\mathbb{A}_f^p)$-action on $\mathscr{S}_{K_p}$. The scheme $\mathscr{S}_{K_p}$ is typically not locally of finite presentation over $\mathcal{O}_{E,(\mathfrak{p})}$.

group is identified with the maximal quotient of $K^p/U^p$ that acts faithfully on $\mathscr{S}_{K_pU^p}$. Since $\mathrm{Sh}_{K_pU^p}$ is dense in $\mathscr{S}_{K_pU^p}$, we see that the last group is identified with $\mathrm{Gal}(\mathrm{Sh}_{K_pU^p}/\mathrm{Sh}_{K_pK^p})$. Thus by (1.5.8.1), we have

$$\mathrm{Gal}(\mathscr{S}_{K_pU^p}/\mathscr{S}_{K_pK^p}) \cong K^p/U^p Z(\mathbb{Q})^{-,(p)}_{K_pK^p},$$

where $Z(\mathbb{Q})^{-,(p)}_{K_pK^p}$ is the image of $Z(\mathbb{Q})^{-}_{K_pK^p}$ under the projection $G(\mathbb{A}_f) \to G(\mathbb{A}_f^p)$. Hence $\mathrm{Gal}(\mathscr{S}_{K_p}/\mathscr{S}_{K_pK^p})$ as defined in (1.5.2.1) is the quotient of $K^p$ by the closure of $Z(\mathbb{Q})^{-,(p)}_{K_pK^p}$ in $G(\mathbb{A}_f^p)$. (Actually $Z(\mathbb{Q})^{-,(p)}_{K_pK^p}$ is already closed in $G(\mathbb{A}_f^p)$, because $Z(\mathbb{Q})^{-}_{K_pK^p}$ is compact.) By Lemma 1.5.7, (the closure of) $Z(\mathbb{Q})^{-,(p)}_{K_pK^p}$ is contained in $Z_{ac}(\mathbb{A}_f^p)$ when $K^p$ is sufficiently small. We now view $\xi$ as a continuous representation of $G(\mathbb{A}_f^p)$ via the projection $G(\mathbb{A}_f^p) \to G(\mathbb{Q}_\ell)$. Then for all sufficiently small $K^p$, the restriction $\xi|_{K^p}$ factors through $\mathrm{Gal}(\mathscr{S}_{K_p}/\mathscr{S}_{K_pK^p})$ by the above discussion. Thus as in §1.5.2, for each sufficiently small $K^p$ we obtain a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathscr{L}_{\xi,K^p}$ on $\mathscr{S}_{K_pK^p}$, and for each geometric point $x$ of $\mathrm{Spec}\,\mathcal{O}_{E,(\mathfrak{p})}$ we have the admissible $G(\mathbb{A}_f^p)$-module

$$\mathbf{H}^i_c(\mathscr{S}_{K_p,x},\xi) := \varinjlim_{K^p} \mathbf{H}^i_c(\mathscr{S}_{K_pK^p,x},\mathscr{L}_{\xi,K^p}).$$

When $x = \mathrm{Spec}\,\overline{E}$, the above is identified with $\mathbf{H}^i_c(\mathrm{Sh}_{\overline{E}},\xi)^{K_p}$. Moreover, we have a canonical adjunction morphism

$$\mathbf{H}^i_c(\mathscr{S}_{K_p,\overline{\mathbb{F}}_q},\xi) \longrightarrow \mathbf{H}^i_c(\mathscr{S}_{K_p,\overline{E}},\xi) \cong \mathbf{H}^i_c(\mathrm{Sh}_{\overline{E}},\xi)^{K_p}, \tag{2.5.5.1}$$

which is $\mathrm{Gal}(\overline{E_\mathfrak{p}}/E_\mathfrak{p}) \times G(\mathbb{A}_f^p)$-equivariant. Here $\mathrm{Gal}(\overline{E_\mathfrak{p}}/E_\mathfrak{p})$ acts on the left via the quotient $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, and acts on the right via the embedding into $\mathrm{Gal}(\overline{E}/E)$.

**Definition 2.5.6.** We say that $\mathscr{S}_{K_p}$ has *well-behaved* $\mathbf{H}^*_c$, if (2.5.5.1) is an isomorphism for all choices of $\ell \neq p$, $\xi$, and $i$.

**Theorem 2.5.7** ([LS18, Cor. 4.6])**.** *The canonical smooth integral model in Theorem 2.5.3 has well-behaved* $\mathbf{H}^*_c$.

Recall that for each admissible morphism $\phi : \mathfrak{Q} \to \mathfrak{G}_G$, we have defined in §2.4.7 a set $S(\phi)$ equipped with an action of $G(\mathbb{A}_f^p)$ and a $q$-Frobenius $\Phi$, where $q$ is the residue cardinality of $\mathfrak{p}$.

**Conjecture 2.5.8** (Langlands–Rapoport)**.** *There exists a smooth integral model* $\mathscr{S}_{K_p}$ *of* $\mathrm{Sh}_{K_p}$ *over* $\mathcal{O}_{E,(\mathfrak{p})}$ *which has well-behaved* $\mathbf{H}^*_c$ *and for which there is a bijection*

$$\mathscr{S}_{K_p}(\overline{\mathbb{F}}_q) \xrightarrow{\sim} \coprod_{\phi} S(\phi)$$

*compatible with the actions of* $G(\mathbb{A}_f^p)$ *and the* $q$*-Frobenius* $\Phi$*. Here* $\phi$ *runs through a set of representatives for the conjugacy classes of admissible morphisms* $\mathfrak{Q} \to \mathfrak{G}_G$.

In the rest of this section, we formulate a variant of the above conjecture, which we call "the Langlands–Rapoport–$\tau$ Conjecture".

2.6. **Preparations for the Langlands–Rapoport–$\tau$ Conjecture.** In this subsection we develop the prerequisites for our formulation of the Langlands–Rapoport–$\tau$ Conjecture.

**2.6.1.** Using that the projections $\mathfrak{Q}^{\mathrm{top}} \to \mathfrak{Q}^L$ are surjective (see §2.2.8), for each $\tau \in \Gamma$ we can choose a lift $q_\tau \in \mathfrak{Q}^{\mathrm{top}}$ of $\tau$. We fix such a choice in the sequel. We first study a "well-positioned" condition for morphisms from $\mathfrak{Q}$ to neutral gerbs. Let $G$ be an arbitrary reductive group over $\mathbb{Q}$. Recall from §2.2.8 that $\mathfrak{Q}^\Delta$ has the canonical $\mathbb{Q}$-structure $Q$.

**Definition 2.6.2.** A morphism $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ in pro-$\mathcal{G}rb(\overline{\mathbb{Q}}/\mathbb{Q})$ is called *günstig gelegen* (to be abbreviated as *gg*)[16], if $\phi^\Delta : \mathfrak{Q}^\Delta \to G_{\overline{\mathbb{Q}}}$ is defined over $\mathbb{Q}$.

**Lemma 2.6.3.** *Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be a morphism in* pro-$\mathcal{G}rb(\overline{\mathbb{Q}}/\mathbb{Q})$. *For each $\tau \in \Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, write $\phi(q_\tau) = g_\tau \rtimes \tau$, with $g_\tau \in G(\overline{\mathbb{Q}})$. Then $\phi$ is gg if and only if $g_\tau$ lies in $G_{\overline{\mathbb{Q}},\phi^\Delta}(\overline{\mathbb{Q}})$ for each $\tau \in \Gamma$. If $\phi$ is gg, then the canonical $\overline{\mathbb{Q}}$-isomorphism $I_{\phi,\overline{\mathbb{Q}}} \xrightarrow{\sim} G_{\phi^\Delta,\overline{\mathbb{Q}}}$ is an inner twistings between the underlying $\mathbb{Q}$-groups.*

*Proof.* Assume that $\phi$ is induced by a morphism $\phi_0 : \mathfrak{Q}^L \to \mathfrak{G}_G$ in $\mathcal{G}rb(\overline{\mathbb{Q}}/\mathbb{Q})$. Let $y \in Q^L(\overline{\mathbb{Q}}) = \mathfrak{Q}^{L,\Delta}(\overline{\mathbb{Q}})$ and $\tau \in \Gamma$ be arbitrary. We denote the image of $q_\tau \in \mathfrak{Q}^{\mathrm{top}}$ in $\mathfrak{Q}^L$ still by $q_\tau$. Then

$$\phi_0^\Delta({}^\tau y) = \phi_0(q_\tau y q_\tau^{-1}) = (g_\tau \rtimes \tau)\phi_0^\Delta(y)(g_\tau \rtimes \tau)^{-1} = g_\tau {}^\tau[\phi_0^\Delta(y)]g_\tau^{-1}. \tag{2.6.3.1}$$

Now $\phi^\Delta$ is defined over $\mathbb{Q}$ if and only if $\phi_0^\Delta$ is defined over $\mathbb{Q}$ (since the transition maps in the pro-torus $Q = \varprojlim_L Q^L$ are all surjective). By (2.6.3.1), this is equivalent to the condition that each $g_\tau$ centralizes $\mathrm{im}(\phi_0^\Delta)$. Since $\mathrm{im}(\phi_0^\Delta) = \mathrm{im}(\phi^\Delta)$, the last condition is equivalent to the condition that each $g_\tau$ lies in $G_{\overline{\mathbb{Q}},\phi^\Delta}(\overline{\mathbb{Q}})$.

Now if $\phi$ is gg, then for each $\tau \in \Gamma$ the $\overline{\mathbb{Q}}$-isomorphism $I_{\phi,\overline{\mathbb{Q}}} \xrightarrow{\sim} G_{\phi^\Delta,\overline{\mathbb{Q}}}$ differs from its $\tau$-twist by composition with $\mathrm{Int}(g_\tau)$. Since $g_\tau \in G_{\phi^\Delta}(\overline{\mathbb{Q}})$, this means that the $\overline{\mathbb{Q}}$-isomorphism is an inner twisting. $\square$

**Definition 2.6.4.** Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be a morphism in pro-$\mathcal{G}rb(\overline{\mathbb{Q}}/\mathbb{Q})$. By a *$G$-rational maximal torus in $I_\phi$*, we mean a maximal torus $T \subset I_\phi$ (defined over $\mathbb{Q}$) such that the composite embedding $T_{\overline{\mathbb{Q}}} \hookrightarrow I_{\phi,\overline{\mathbb{Q}}} \cong G_{\overline{\mathbb{Q}},\phi^\Delta} \hookrightarrow G_{\overline{\mathbb{Q}}}$ is defined over $\mathbb{Q}$.

*Remark* 2.6.5. In Definition 2.6.4, $T$ is necessarily a maximal torus in $G$. This is because $\mathfrak{Q}^\Delta$ is a pro-torus, and as a result $I_\phi$ is a reductive group having the same absolute rank as $G$, cf. §2.1.14.

**Lemma 2.6.6.** *Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be a morphism in* pro-$\mathcal{G}rb(\overline{\mathbb{Q}}/\mathbb{Q})$ *such that $I_\phi$ contains a $G$-rational maximal torus $T$. Then $\phi$ is gg. Moreover, let $f$ denote the $\mathbb{Q}$-embedding underlying $T_{\overline{\mathbb{Q}}} \hookrightarrow I_{\phi,\overline{\mathbb{Q}}} \cong G_{\overline{\mathbb{Q}},\phi^\Delta} \hookrightarrow G_{\overline{\mathbb{Q}}}$. Then $\phi$ factors as $\mathfrak{Q} \xrightarrow{\phi_T} \mathfrak{G}_T \xrightarrow{f} \mathfrak{G}_G$.*

*Proof.* For each $\tau \in \Gamma$, define $g_\tau \in G(\overline{\mathbb{Q}})$ as in Lemma 2.6.3. For $t \in T(\overline{\mathbb{Q}})$, we have

$$f({}^\tau t) = g_\tau {}^\tau[f(t)]g_\tau^{-1},$$

by the definition of the $\mathbb{Q}$-structure of $I_\phi$; see §2.1.6 and §2.1.8. Since $f$ is defined over $\mathbb{Q}$, we have $f({}^\tau t) = {}^\tau[f(t)]$. Hence $g_\tau$ commutes with $f(T(\overline{\mathbb{Q}}))$. Since $f(T)$ is a maximal torus in $G$ (see Remark 2.6.5), we have $g_\tau \in f(T)(\overline{\mathbb{Q}})$. We conclude that $\phi$ is gg by Lemma 2.6.3. Moreover, since $Z_{I_\phi} \subset T$ and since $\phi^\Delta$ factors through the

[16] This terminology comes from Langlands–Rapoport [LR87, §5]. However, the definition of günstig gelegen morphisms given by Langlands–Rapoport uses the elements $\delta_n$, which do not directly make sense with the corrected definition of $\mathfrak{Q}$.

center of $G_{\phi^\Delta}$, we know that $\phi^\Delta$ factors through $f(T) \subset G$. We have already seen that each $g_\tau$ lies in $f(T)(\overline{\mathbb{Q}})$. It follows that $\phi$ factors through $f : \mathfrak{G}_T \to \mathfrak{G}_G$. $\square$

**Lemma 2.6.7.** *Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be an admissible morphism. Then the $\mathbb{R}$-group $I_{\phi(\infty)\circ\zeta_\infty}$ is an inner form of $G_\mathbb{R}$. Moreover, the $\mathbb{R}$-groups $(I_\phi/Z_G)_\mathbb{R} = I_{\phi(\infty)}/Z_{G,\mathbb{R}}$ and $I_{\phi(\infty)\circ\zeta_\infty}/Z_{G,\mathbb{R}}$ are both anisotropic.*

*Proof.* As in (2.1.6.1) we have an $\mathbb{R}$-embedding $I_{\phi(\infty)} \hookrightarrow I_{\phi(\infty)\circ\zeta_\infty}$. Since $\phi$ is admissible, $\phi(\infty) \circ \zeta_\infty$ is conjugate to $\xi_\infty$. Hence there is an $\mathbb{R}$-isomorphism $I_{\phi(\infty)\circ\zeta_\infty} \xrightarrow{\sim} I_{\xi_\infty}$ induced by $\mathrm{Int}(g)$ for some $g \in G(\mathbb{C})$. As discussed in [Kis17, §3.3.5], $I_{\xi_\infty}$ is the inner form of $G_\mathbb{R}$ with anisotropic adjoint group. The lemma follows. $\square$

**2.6.8.** We now return to the setting of §2.4.1. Thus we have an unramified Shimura datum $(G, X, p, \mathcal{G})$ and the notion of admissible morphisms $\mathfrak{Q} \to \mathfrak{G}_G$.

**Proposition 2.6.9.** *Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be an admissible morphism. For each maximal torus $T \subset I_\phi$ defined over $\mathbb{Q}$, there exists $g \in G(\overline{\mathbb{Q}})$ such that* $\mathrm{Int}(g)(T)$ *is a $G$-rational maximal torus in $I_{\mathrm{Int}(g)\circ\phi}$.*

*Proof.* This is essentially proved by Langlands–Rapoport, when they prove [LR87, Lem. 5.23]. We sketch the argument, as the precise statement of the proposition is not explicit in [LR87].[17]

Let $\psi : G_{\overline{\mathbb{Q}}} \xrightarrow{\sim} G^*_{\overline{\mathbb{Q}}}$ be an inner twisting from $G$ to a fixed quasi-split reductive group $G^*$ over $\mathbb{Q}$. Then the $G^*(\overline{\mathbb{Q}})$-conjugacy class of the composite embedding $\iota : T_{\overline{\mathbb{Q}}} \hookrightarrow G_{\overline{\mathbb{Q}}} \xrightarrow{\psi} G^*_{\overline{\mathbb{Q}}}$ is stable under $\Gamma$. By [Kot82, Cor. 2.2], we can modify $\psi$ by an inner automorphism to arrange that $\iota$ is defined over $\mathbb{Q}$.

Let $T^* = \iota(T)$. Then $T^*$ is a maximal torus in $G^*$ defined over $\mathbb{Q}$, and we have a $\mathbb{Q}$-isomorphism $\iota : T \xrightarrow{\sim} T^*$. We now check that $T^*$ transfers to $G$ locally at all places $v$, or equivalently, that some $G(\overline{\mathbb{Q}}_v)$-conjugate of $T$ is a $\mathbb{Q}_v$-torus in $G_{\mathbb{Q}_v}$. For $v = p$, this follows from the assumption that $G$ is unramified, and hence quasi-split, over $\mathbb{Q}_p$. For $v = \infty$, this follows from the fact that $T^*/Z_{G^*} \cong T/Z_G$ is anisotropic over $\mathbb{R}$ (Lemma 2.6.7). For $v \notin \{\infty, p\}$, pick $u_v \in G(\overline{\mathbb{Q}}_v)$ such that $\mathrm{Int}(u_v^{-1}) \circ \phi(v) \circ \zeta_v = \xi_v$, which exists since $\phi$ is admissible. Then the canonical embedding $I_{\mathrm{Int}(u_v)\circ\phi(v),\overline{\mathbb{Q}}_v} \hookrightarrow G_{\overline{\mathbb{Q}}_v}$ is defined over $\mathbb{Q}_v$, and hence $\mathrm{Int}(u_v)(T_{\mathbb{Q}_v})$ is a $\mathbb{Q}_v$-maximal torus in $G_{\mathbb{Q}_v}$, as desired.

Since $T^*$ transfers to $G$ locally and is elliptic over $\mathbb{R}$, it transfers globally to $G$ by [LR87, Lem. 5.6]. This means there exists $g \in G(\overline{\mathbb{Q}})$ such that $\mathrm{Int}(g)(T)$ is a $\mathbb{Q}$-maximal torus in $G$ and such that the isomorphism $\mathrm{Int}(g) \circ \psi^{-1} : T^* \to \mathrm{Int}(g)(T)$ is defined over $\mathbb{Q}$. It follows that $\mathrm{Int}(g)(T)$ is a $G$-rational maximal torus in $I_{\mathrm{Int}(g)\circ\phi}$. $\square$

**Corollary 2.6.10.** *Every admissible morphism $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ is conjugate to a gg morphism.*

*Proof.* By Proposition 2.6.9, $\phi$ is conjugate to a morphism $\phi' : \mathfrak{Q} \to \mathfrak{G}_G$ such that $I_{\phi'}$ contains a $G$-rational maximal torus. By Lemma 2.6.6, $\phi'$ is gg. $\square$

[17] The only information about $\mathfrak{Q}$ used in this argument is the fact that $\mathfrak{Q}^\Delta$ is a pro-torus with surjective transition maps. Hence the validity of this argument is unaffected by Reimann's correction of the definition of $\mathfrak{Q}$.

**2.6.11.** Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be an admissible morphism. By Corollary 2.6.10, the set $\mathcal{W} := \{g \in G(\overline{\mathbb{Q}}) \mid \operatorname{Int} g \circ \phi \text{ is gg}\}$ is non-empty. Using Lemma 2.6.3, one checks that the canonical embedding $I_{\phi,\overline{\mathbb{Q}}} \to G_{\overline{\mathbb{Q}}}$ and $\mathcal{W}$ together form an inner transfer datum from $I_\phi$ to $G$ (Definition 1.2.4). We thus obtain a canonical map

$$\mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}, I_\phi) \longrightarrow \mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}, G), \tag{2.6.11.1}$$

and we define $Ш^\infty_G(\mathbb{Q}, I_\phi) \subset \mathbf{H}^1(\mathbb{Q}, I_\phi)$ as in §1.2.5.

**Proposition 2.6.12.** *Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be an admissible morphism, and $\beta \in \mathbf{H}^1(\mathbb{Q}, I_\phi)$. Then $\phi^\beta$ (see Definition 2.1.17) is admissible if and only if $\beta$ belongs to $Ш^\infty_G(\mathbb{Q}, I_\phi)$.*

*Proof.* The "if" part is proved in [Kis17, Lem. 4.5.6], under the assumption that $Z^0_G$ is cuspidal. Below we give complete proofs of both directions of the implication, taking into account the correction of the definition of admissible morphisms mentioned in Remark 2.4.3. Since the admissibility condition is invariant under conjugacy, we use Corollary 2.6.10 to reduce to the case where $\phi$ is gg. We now assume that $\phi$ is gg and fix a cocycle $a \in Z^1(\mathbb{Q}, I_\phi)$ representing $\beta$.

**Step 1.** We show that $\beta$ has zero image under the composite

$$\mathbf{H}^1(\mathbb{Q}, I_\phi) \xrightarrow{\mathrm{ab}^1_{\mathbb{Q}}} \mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}, I_\phi) \xrightarrow{(2.6.11.1)} \mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}, G) \tag{2.6.12.1}$$

if and only if there exist $h \in G(\overline{\mathbb{Q}})$ and a continuous monoidal isomorphism $\operatorname{Int}(h) \circ \phi_{\widetilde{\mathrm{ab}}} \xrightarrow{\sim} (a\phi)_{\widetilde{\mathrm{ab}}}$. (See Definition 2.4.2 for the notation.) By Lemma 2.3.4 and Remark 2.3.5, the latter condition is equivalent to asking that $\beta$ comes from $\mathbf{H}^1(\mathbb{Q}, \widetilde{I}_\phi)$. We have a natural exact sequence of pointed sets

$$\mathbf{H}^1(\mathbb{Q}, \widetilde{I}_\phi) \to \mathbf{H}^1(\mathbb{Q}, I_\phi) \to \mathbf{H}^1(\mathbb{Q}, \widetilde{I}_\phi \to I_\phi),$$

where $\mathbf{H}^1(\mathbb{Q}, \widetilde{I}_\phi \to I_\phi)$ is the Galois cohomology of the crossed module $(\widetilde{I}_\phi \to I_\phi)$ of $\mathbb{Q}$-groups; see [Bor98, §3]. (The crossed module structure is the one inherited from the crossed module $G_{\mathrm{sc}} \to G$.) The natural map $(Z_{G_{\mathrm{sc}}} \to Z_G) \to (\widetilde{I}_\phi \to I_\phi)$ is a quasi-isomorphism of crossed modules, and therefore $\mathbf{H}^1(\mathbb{Q}, \widetilde{I}_\phi \to I_\phi)$ is naturally isomorphic to $\mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}, G)$. The composition $\mathbf{H}^1(\mathbb{Q}, I_\phi) \to \mathbf{H}^1(\mathbb{Q}, \widetilde{I}_\phi \to I_\phi) \cong \mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}, G)$ is equal to (2.6.12.1). This proves the desired statement.

**Step 2.** We show that $\beta$ has trivial image in $\mathbf{H}^1(\mathbb{R}, I_\phi)$ if and only if $\phi(\infty) \circ \zeta_\infty$ is conjugate to $(a\phi)(\infty) \circ \zeta_\infty$. By Lemma 2.1.16, the latter condition is equivalent to the vanishing of the image of $\beta$ under

$$\mathbf{H}^1(\mathbb{Q}, I_\phi) \to \mathbf{H}^1(\mathbb{R}, I_\phi) = \mathbf{H}^1(\mathbb{R}, I_{\phi(\infty)}) \xrightarrow{\dagger} \mathbf{H}^1(\mathbb{R}, I_{\phi(\infty)\circ\zeta_\infty}),$$

where $\dagger$ is induced by the $\mathbb{R}$-inclusion $I_{\phi(\infty)} \hookrightarrow I_{\phi(\infty)\circ\zeta_\infty}$. Thus the desired statement boils down to $\dagger$ having trivial kernel, which follows from Lemma 2.6.7 and [Kis17, Lem. 4.4.5].

**Step 3.** We show that if $\beta$ has zero image under (2.6.12.1), then $\phi(l) \circ \zeta_l$ is conjugate to $(a\phi)(l) \circ \zeta_l$ for all finite primes $l \neq p$. For this, it suffices to show that $\beta$ has trivial image under the composite map

$$\mathbf{H}^1(\mathbb{Q}, I_\phi) \to \mathbf{H}^1(\mathbb{Q}_l, I_\phi) = \mathbf{H}^1(\mathbb{Q}_l, I_{\phi(l)}) \to \mathbf{H}^1(\mathbb{Q}_l, I_{\phi(l)\circ\zeta_l}). \tag{2.6.12.2}$$

Since $\phi$ is admissible, $\phi(l) \circ \zeta_l$ is conjugate to $\xi_l$. It follows that the canonical $\overline{\mathbb{Q}}_l$-embedding $I_{\phi(l)\circ\zeta_l,\overline{\mathbb{Q}}_l} \to G_{\overline{\mathbb{Q}}_l}$ is an inner twisting between $\mathbb{Q}_l$-groups. This induces a canonical isomorphism

$$\mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}_l, I_{\phi(l)\circ\zeta_l}) \xrightarrow{\sim} \mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}_l, G). \tag{2.6.12.3}$$

Now we have a commutative diagram

$$\begin{tikzcd}
\mathbf{H}^1(\mathbb{Q}, I_\phi) \arrow[r, "(2.6.12.2)"] \arrow[d, "(2.6.12.1)"] & \mathbf{H}^1(\mathbb{Q}_l, I_{\phi(l)\circ\zeta_l}) \arrow[r, "\mathrm{ab}^1_{\mathbb{Q}_l}", "\cong"'] & \mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}_l, I_{\phi(l)\circ\zeta_l}) \arrow[d, "\cong"', "(2.6.12.3)"] \\
\mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}, G) \arrow[rr, "\text{localization}"] & & \mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}_l, G)
\end{tikzcd}$$

which implies the desired statement.

**Step 4.** We show that if $\beta$ has zero image under (2.6.12.1), then $X_p(a\phi) \neq \emptyset$. We fix $\mu \in \mathbb{\mu}_X^{\mathcal{G}}$ as in §2.4.1. Let $\theta := \phi(p) \circ \zeta_p$, and let $\theta' := (a\phi)(p) \circ \zeta_p$. By definition $X_p(\phi) = X_{-\mu}(\theta)$ and $X_p(a\phi) = X_{-\mu}(\theta')$.

Fix an arbitrary $g_0 \in \mathcal{UR}(\theta)$, and write $\theta_0$ for $\mathrm{Int}(g_0^{-1})\theta$. Thus $\theta_0$ is unramified. Now $\mathrm{Int}(g_0^{-1})$ induces a $\mathbb{Q}_p$-isomorphism $I_\theta \xrightarrow{\sim} I_{\theta_0}$. Let $\beta_0$ denote the image of $\beta$ under the composite

$$\mathbf{H}^1(\mathbb{Q}, I_\phi) \to \mathbf{H}^1(\mathbb{Q}_p, I_\phi) \to \mathbf{H}^1(\mathbb{Q}_p, I_\theta) \xrightarrow{\mathrm{Int}(g_0^{-1})} \mathbf{H}^1(\mathbb{Q}_p, I_{\theta_0}).$$

Then $\theta'$ belongs to the conjugacy class of $\theta_0^{\beta_0}$.

By Proposition 2.2.6 (iii), the conjugacy class $\theta_0^{\beta_0}$ contains an unramified member $\theta_0'$ such that $b' := b_{\theta_0'}$ is obtained from $b := b_{\theta_0}$ by twisting by $\beta_0$. Since $G$ is quasi-split over $\mathbb{Q}_p$, we can apply Proposition 1.4.5 to conclude that $\nu_b = \nu_{b'}$, and that $\kappa_G(b') - \kappa_G(b)$ is the image of $\beta_0$ in $\pi_1(G)_{\Gamma_p,\mathrm{tors}}$. Our assumption on $\beta$ implies that the last image is zero. Hence we have $[b] = [b']$ in $\mathrm{B}(G_{\mathbb{Q}_p})$, by Kottwitz's classification (see §1.4.2).

Since $\theta_0$ (resp. $\theta_0'$) is an unramified member in the conjugacy class of $\theta$ (resp. $\theta'$), by the discussion in §2.2.7, we have $X_{-\mu}(\theta) \cong X_{-\mu}(b)$, and $X_{-\mu}(\theta') \cong X_{-\mu}(b')$. Since $[b] = [b']$, we have $X_{-\mu}(b) \cong X_{-\mu}(b')$. Thus the non-emptiness of $X_{-\mu}(\theta)$ implies the non-emptiness of $X_{-\mu}(\theta')$.

The proof of the proposition is completed by combining the above four steps. (The "only if" part follows from Steps 1 and 2 alone.) □

**2.6.13.** Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be an admissible morphism. We define

$$\mathcal{H}(\phi) := I_\phi(\mathbb{A}_f)\backslash I_\phi^{\mathrm{ad}}(\mathbb{A}_f)/I_\phi^{\mathrm{ad}}(\mathbb{Q}),$$
$$\mathfrak{E}^p(\phi) := I_\phi(\mathbb{A}_f^p)\backslash I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p).$$

We have a natural map

$$\mathfrak{E}^p(\phi) \longrightarrow \mathcal{H}(\phi),$$

and it is surjective by weak approximation (see [PR94, Thm. 7.8]) applied to $I_\phi^{\mathrm{ad}}$.

The boundary map arising from the short exact sequence $1 \to Z_{I_\phi} \to I_\phi \to I_\phi^{\mathrm{ad}} \to 1$ induces an isomorphism of pointed sets $\mathfrak{E}^p(\phi) \cong \mathfrak{D}(Z_{I_\phi}, I_\phi; \mathbb{A}_f^p)$. Since $\mathfrak{D}(Z_{I_\phi}, I_\phi; \mathbb{A}_f^p) \cong \mathfrak{E}(Z_{I_\phi}, I_\phi; \mathbb{A}_f^p)$ is an abelian group, we have a canonical abelian group structure on $\mathfrak{E}^p(\phi)$.

**Lemma 2.6.14.** *The surjection $\mathfrak{E}^p(\phi) \to \mathcal{H}(\phi)$ induces an abelian group structure on $\mathcal{H}(\phi)$. Moreover, we have a commutative diagram*

$$\begin{tikzcd} \mathfrak{E}^p(\phi) \arrow[r] \arrow[d] & \mathfrak{E}(Z_{I_\phi}, I_\phi; \mathbb{A}_f^p) \arrow[d] \\ \mathcal{H}(\phi) \arrow[r] & \mathfrak{E}(Z_{I_\phi}, I_\phi; \mathbb{A}_f)/\mathfrak{D}(Z_{I_{\phi,\mathrm{sc}}}, I_{\phi,\mathrm{sc}}; \mathbb{Q}) \end{tikzcd} \tag{2.6.14.1}$$

*where the rows are isomorphisms. Here the right vertical arrow is induced by the inclusion $\mathfrak{E}(Z_{I_\phi}, I_\phi; \mathbb{A}_f^p) \to \mathfrak{E}(Z_{I_\phi}, I_\phi; \mathbb{A}_f)$, and the bottom arrow is induced by the boundary map $\delta : I_\phi^{\mathrm{ad}}(\mathbb{A}_f) \to \mathfrak{D}(Z_{I_\phi}, I_\phi; \mathbb{A}_f) \cong \mathfrak{E}(Z_{I_\phi}, I_\phi; \mathbb{A}_f)$ arising from the short exact sequence $1 \to Z_{I_\phi} \to I_\phi \to I_\phi^{\mathrm{ad}} \to 1$.*

*Proof.* First note that $\mathfrak{D}(Z_{I_{\phi,\mathrm{sc}}}, I_{\phi,\mathrm{sc}}; \mathbb{Q})$ is a subgroup of $\mathbf{H}^1(\mathbb{Q}, Z_{I_{\phi,\mathrm{sc}}})$, so the quotient on the lower right corner of the diagram is defined. Now the boundary map $\delta$ induces a bijection $I_\phi(\mathbb{A}_f)\backslash I_\phi^{\mathrm{ad}}(\mathbb{A}_f) \xrightarrow{\sim} \mathfrak{E}(Z_{I_\phi}, I_\phi; \mathbb{A}_f)$, and maps $I_\phi^{\mathrm{ad}}(\mathbb{Q}) \subset I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$ onto the image of $\mathfrak{D}(Z_{I_{\phi,\mathrm{sc}}}, I_{\phi,\mathrm{sc}}; \mathbb{Q}) \to \mathfrak{E}(Z_{I_\phi}, I_\phi; \mathbb{A}_f)$ (since we have a surjective boundary map $I_\phi^{\mathrm{ad}}(\mathbb{Q}) \to \mathfrak{D}(Z_{I_{\phi,\mathrm{sc}}}, I_{\phi,\mathrm{sc}}; \mathbb{Q})$ associated with the short exact sequence $1 \to Z_{I_{\phi,\mathrm{sc}}} \to I_{\phi,\mathrm{sc}} \to I_\phi^{\mathrm{ad}} \to 1$). The lemma follows. $\square$

**Lemma 2.6.15.** *The subset $\mathfrak{D}(Z_{I_{\phi,\mathrm{sc}}}, I_{\phi,\mathrm{sc}}; \mathbb{Q})$ of $\mathbf{H}^1(\mathbb{Q}, Z_{I_{\phi,\mathrm{sc}}})$ (which is a subgroup) is equal to the kernel of the composite map of pointed sets*

$$\mathbf{H}^1(\mathbb{Q}, Z_{I_{\phi,\mathrm{sc}}}) \to \mathbf{H}^1(\mathbb{R}, Z_{I_{\phi,\mathrm{sc}}}) \to \mathbf{H}^1(\mathbb{R}, I_{\phi,\mathrm{sc}}).$$

*Proof.* By the Kneser–Harder–Chernousov Theorem (see [Bor98, Thm. 5.0.3]), the localization map $\mathbf{H}^1(\mathbb{Q}, I_{\mathrm{sc}}) \to \mathbf{H}^1(\mathbb{R}, I_{\mathrm{sc}})$ is a bijection. The lemma follows. $\square$

**2.6.16.** Let $\mathcal{AM} = \mathcal{AM}(G, X, p, \mathcal{G})$ be the set of all admissible morphisms $\mathfrak{Q} \to \mathfrak{G}_G$. On this set we define an equivalence relation $\approx$ by declaring $\phi_1 \approx \phi_2$ if and only if $\phi_1^\Delta$ is $G(\overline{\mathbb{Q}})$-conjugate to $\phi_2^\Delta$. Clearly $\approx$ is weaker than the equivalence relation defined by conjugacy among admissible morphisms. By Lemma 2.1.16, we know that $\phi_1 \approx \phi_2$ if and only if there exists a (necessarily unique) $\beta \in \mathbf{H}^1(\mathbb{Q}, I_\phi)$ such that $\phi_2$ belongs to the conjugacy class $\phi_1^\beta$. Moreover, when this is the case, we know that $\beta$ lies in $Ш_G^\infty(\mathbb{Q}, I_{\phi_1})$ by Proposition 2.6.12.

As a consequence of Lemma 2.6.14 and Lemma 2.6.15, we know that for $\phi \in \mathcal{AM}$, the abelian group $\mathcal{H}(\phi)$ depends only on the groups $Z_{I_\phi}, Z_{I_{\phi,\mathrm{sc}}}, I_{\phi,\mathrm{sc},\mathbb{R}}$ and the maps between them. The same is true for the abelian group $\mathfrak{E}^p(\phi) \cong \mathfrak{E}(Z_{I_\phi}, I_\phi; \mathbb{A}_f^p)$. Now if $\phi_1, \phi_2 \in \mathcal{AM}$ are such that $\phi_1 \approx \phi_2$, then for any $g \in G(\overline{\mathbb{Q}})$ such that $\mathrm{Int}(g) \circ \phi_1^\Delta = \phi_2^\Delta$, the $\overline{\mathbb{Q}}$-isomorphism $\mathrm{Int}(g) : G_{\overline{\mathbb{Q}}, \phi_1^\Delta} \xrightarrow{\sim} G_{\overline{\mathbb{Q}}, \phi_2^\Delta}$ induces an inner twisting $\mathrm{Comp}_g : I_{\phi_1, \overline{\mathbb{Q}}} \xrightarrow{\sim} I_{\phi_2, \overline{\mathbb{Q}}}$ between $\mathbb{Q}$-groups. Clearly the equivalence class (Definition 1.2.1) of the inner twisting $\mathrm{Comp}_g$ is independent of the choice of $g$. It follows that $Z_{I_{\phi_1}}, Z_{I_{\phi_1,\mathrm{sc}}}$ are canonically identified with $Z_{I_{\phi_2}}, Z_{I_{\phi_2,\mathrm{sc}}}$ respectively. If we let $\mathrm{Comp}_{g,\mathrm{sc}} : I_{\phi_1,\mathrm{sc},\overline{\mathbb{Q}}} \xrightarrow{\sim} I_{\phi_2,\mathrm{sc},\overline{\mathbb{Q}}}$ be the inner twisting induced by $\mathrm{Comp}_g$, then the equivalence class of $\mathrm{Comp}_{g,\mathrm{sc}}$ is also independent of $g$. Moreover, $\mathrm{Comp}_{g,\mathrm{sc}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ is the composition of an $\mathbb{R}$-isomorphism with an inner automorphism defined over $\mathbb{C}$. This is because if we let $\beta$ be the element of $Ш_G^\infty(\mathbb{Q}, I_{\phi_1})$ such that $\phi_2 \in \phi_1^\beta$, then the class of $\mathrm{Comp}_{g,\mathrm{sc}}$ in $\mathbf{H}^1(\mathbb{Q}, I_{\phi_1}^{\mathrm{ad}})$ is the image of $\beta$, and hence has trivial image in $\mathbf{H}^1(\mathbb{R}, I_{\phi_1}^{\mathrm{ad}})$. Thus $I_{\phi_1,\mathrm{sc},\mathbb{R}}$ is canonically identified with $I_{\phi_2,\mathrm{sc},\mathbb{R}}$ up to inner automorphisms defined over $\mathbb{R}$. From this analysis, we see that there are canonical abelian group isomorphisms

$$\mathrm{Comp}_{\phi_1,\phi_2} : \mathcal{H}(\phi_1) \xrightarrow{\sim} \mathcal{H}(\phi_2)$$
$$\mathrm{Comp}^{\mathfrak{E}^p}_{\phi_1,\phi_2} : \mathfrak{E}^p(\phi_1) \xrightarrow{\sim} \mathfrak{E}^p(\phi_2)$$

which depend only on $\phi_1$ and $\phi_2$. These maps commute with the natural surjections $\mathfrak{E}^p(\phi_i) \to \mathcal{H}(\phi_i)$. For any three $\phi_1, \phi_2, \phi_3 \in \mathcal{AM}$ such that $\phi_1 \approx \phi_2 \approx \phi_3$, we have the following cocycle relations

$$\mathrm{Comp}_{\phi_2,\phi_3} \circ \mathrm{Comp}_{\phi_1,\phi_2} = \mathrm{Comp}_{\phi_1,\phi_3}, \quad \mathrm{Comp}_{\phi_1,\phi_1} = \mathrm{id}_{\mathcal{H}(\phi_1)};$$
$$\mathrm{Comp}^{\mathfrak{E}^p}_{\phi_2,\phi_3} \circ \mathrm{Comp}^{\mathfrak{E}^p}_{\phi_1,\phi_2} = \mathrm{Comp}^{\mathfrak{E}^p}_{\phi_1,\phi_3}, \quad \mathrm{Comp}^{\mathfrak{E}^p}_{\phi_1,\phi_1} = \mathrm{id}_{\mathfrak{E}^p(\phi_1)}.$$

We now view $\mathcal{AM} = \mathcal{AM}(G, X, p, \mathcal{G})$ as a discrete topological space, and define sheaves of abelian groups $\mathfrak{E}^p = \mathfrak{E}^p_{(G,X,p,\mathcal{G})}$ and $\mathcal{H} = \mathcal{H}_{(G,X,p,\mathcal{G})}$ on $\mathcal{AM}$ whose stalks at each $\phi \in \mathcal{AM}$ are $\mathfrak{E}^p(\phi)$ and $\mathcal{H}(\phi)$ respectively. We have a surjective homomorphism $\mathfrak{E}^p \to \mathcal{H}$. The above discussion implies that $\mathfrak{E}^p$ and $\mathcal{H}$ are the pull-backs of unique (up to unique isomorphism) sheaves of abelian groups $\mathfrak{E}^p_{\approx}$ and $\mathcal{H}_{\approx}$ on the quotient space $\mathcal{AM}/\approx$. The surjective homomorphism $\mathfrak{E}^p \to \mathcal{H}$ is the pull-back of a unique surjective homomorphism $\mathfrak{E}^p_{\approx} \to \mathcal{H}_{\approx}$.

The quotient map $\mathcal{AM} \to \mathcal{AM}/\approx$ factors through $\mathcal{AM}/\mathrm{conj}$, the set of conjugacy classes of admissible morphisms. We let $\mathcal{H}_{\mathrm{conj}}$ (resp. $\mathfrak{E}^p_{\mathrm{conj}}$) be the pull-back of $\mathcal{H}_{\approx}$ (resp. $\mathfrak{E}^p_{\approx}$) to $\mathcal{AM}/\mathrm{conj}$.

**Definition 2.6.17.** For $\mathcal{F} \in \{\mathfrak{E}^p, \mathcal{H}\}$, we denote by $\Gamma(\mathcal{F})$ the group of global sections of the sheaf $\mathcal{F}$ on $\mathcal{AM}$. For $\underline{\tau} \in \Gamma(\mathcal{F})$, we write $\underline{\tau}(\phi) \in \mathcal{F}(\phi)$ for the germ of $\underline{\tau}$ at each $\phi \in \mathcal{AM}$. We denote by $\Gamma(\mathcal{F})_0$ (resp. $\Gamma(\mathcal{F})_1$) the subgroup of $\Gamma(\mathcal{F})$ consisting of those global sections that descend to global sections of $\mathcal{F}_{\approx}$ over $\mathcal{AM}/\approx$ (resp. global sections of $\mathcal{F}_{\mathrm{conj}}$ over $\mathcal{AM}/\mathrm{conj}$). Thus $\Gamma(\mathcal{F})_0 \subset \Gamma(\mathcal{F})_1 \subset \Gamma(\mathcal{F})$.

**2.6.18.** Let $\phi \in \mathcal{AM}$. The boundary map $I_\phi(\mathbb{A}_f)\backslash I^{\mathrm{ad}}_\phi(\mathbb{A}_f) \to \mathbf{H}^1(\mathbb{A}_f, Z_{I_\phi})$ arising from the short exact sequence $1 \to Z_{I_\phi} \to I_\phi \to I^{\mathrm{ad}}_\phi \to 1$ induces a map

$$\mathcal{H}(\phi) = I_\phi(\mathbb{A}_f)\backslash I^{\mathrm{ad}}_\phi(\mathbb{A}_f)/I^{\mathrm{ad}}_\phi(\mathbb{Q}) \longrightarrow \mathbf{H}^1(\mathbb{A}_f, Z_{I_\phi})/\text{Ш}^\infty_{I_\phi}(\mathbb{Q}, Z_{I_\phi}). \tag{2.6.18.1}$$

Indeed, since $I^{\mathrm{ad}}_\phi(\mathbb{R})$ is compact (by Lemma 2.6.7), it is connected (see [Bor91, §24.6]). Hence the map $I_\phi(\mathbb{R}) \to I^{\mathrm{ad}}_\phi(\mathbb{R})$ is onto, and the boundary map $I^{\mathrm{ad}}_\phi(\mathbb{R}) \to \mathbf{H}^1(\mathbb{R}, Z_{I_\phi})$ is trivial. In particular, the image of $I^{\mathrm{ad}}_\phi(\mathbb{Q})$ in $\mathbf{H}^1(\mathbb{Q}, Z_{I_\phi})$ lies in $\text{Ш}^\infty_{I_\phi}(\mathbb{Q}, Z_{I_\phi})$, and it follows that (2.6.18.1) is well defined. We have a commutative diagram:

$$\begin{array}{ccc} \mathfrak{E}^p(\phi) & \xrightarrow{\cong} & \mathfrak{E}(Z_{I_\phi}, I_\phi; \mathbb{A}^p_f) \\ \downarrow & & \downarrow \\ \mathcal{H}(\phi) & \xrightarrow{(2.6.18.1)} & \mathbf{H}^1(\mathbb{A}_f, Z_{I_\phi})/\text{Ш}^\infty_{I_\phi}(\mathbb{Q}, Z_{I_\phi}) \end{array} \tag{2.6.18.2}$$

where the left vertical arrow is the natural surjection and the right vertical arrow is induced by the inclusion $\mathbf{H}^1(\mathbb{A}^p_f, Z_{I_\phi}) \to \mathbf{H}^1(\mathbb{A}_f, Z_{I_\phi})$.

**Definition 2.6.19.** Let $\underline{\tau} \in \Gamma(\mathcal{H})$, and $\underline{\sigma} \in \Gamma(\mathfrak{E}^p)$.

(i) We say that $\underline{\tau}$ is *tori-rational*, if for each $\phi \in \mathcal{AM}$ and for each maximal torus $T \subset I_\phi$, the image of $\underline{\tau}(\phi)$ is trivial under the composite map

$$\mathcal{H}(\phi) \xrightarrow{(2.6.18.1)} \mathbf{H}^1(\mathbb{A}_f, Z_{I_\phi})/\text{Ш}^\infty_{I_\phi}(\mathbb{Q}, Z_{I_\phi}) \to \mathbf{H}^1(\mathbb{A}_f, T)/\text{Ш}^\infty_G(\mathbb{Q}, T). \tag{2.6.19.1}$$

(ii) We say that $\underline{\sigma}$ is *tori-rational*, if for each $\phi \in \mathcal{AM}$ and for each maximal torus $T \subset I_\phi$, the image of $\underline{\sigma}(\phi)$ is trivial under the composite map

$$\mathfrak{E}^p(\phi) \cong \mathfrak{E}(Z_{I_\phi}, I_\phi; \mathbb{A}_f^p) \subset \mathbf{H}^1(\mathbb{A}_f^p, Z_{I_\phi}) \to \mathbf{H}^1(\mathbb{A}_f^p, T) \to \mathbf{H}^1(\mathbb{A}_f^p, T)/\text{Ш}^{\infty,p}_G(\mathbb{Q}, T).$$

In the above, $\text{Ш}^\infty_G(\mathbb{Q}, T)$ denotes the kernel of the composite map $\text{Ш}^\infty(\mathbb{Q}, T) \to \text{Ш}^\infty(\mathbb{Q}, I_\phi) \to \text{Ш}^\infty(\mathbb{Q}, G)$, and similarly for $\text{Ш}^{\infty,p}_G(\mathbb{Q}, T)$; see §1.2.5 and §2.6.11.

The next lemma relates the two notions of tori-rationality for elements of $\Gamma(\mathcal{H})$ and of $\Gamma(\mathfrak{E}^p)$.

**Lemma 2.6.20.** *Let $\underline{\tau} \in \Gamma(\mathcal{H})$. The following statements are equivalent.*

*(i) $\underline{\tau}$ is tori-rational.*
*(ii) The section $\underline{\tau}$ has a lift $\underline{\sigma} \in \Gamma(\mathfrak{E}^p)$ along the natural surjection $\Gamma(\mathfrak{E}^p) \to \Gamma(\mathcal{H})$ such that $\underline{\sigma}$ is tori-rational.*
*(iii) Every $\underline{\sigma} \in \Gamma(\mathfrak{E}^p)$ lifting $\underline{\tau}$ is tori-rational.*

*Proof.* The implication (ii) $\Rightarrow$ (i) follows from the commutative diagram (2.6.18.2). Obviously (iii) $\Rightarrow$ (ii). It remains to show (i) $\Rightarrow$ (iii).

Let $\underline{\sigma} \in \Gamma(\mathfrak{E}^p)$ be a lift of $\underline{\tau}$. For each $\phi \in \mathcal{AM}$, we have a natural surjection $\mathfrak{E}(Z_{I_\phi}, I_\phi; \mathbb{A}_f) \to \mathcal{H}(\phi)$ as in Lemma 2.6.14. Fix an element $\epsilon_\phi \in \mathfrak{E}(Z_{I_\phi}, I_\phi; \mathbb{A}_f)$ lifting $\underline{\tau}(\phi)$. Then the image of $\underline{\tau}(\phi)$ under (2.6.18.1) is represented by $\epsilon_\phi$.

By the commutative diagram (2.6.18.2), the image of $\underline{\sigma}(\phi)$ under $\mathbf{H}^1(\mathbb{A}_f^p, Z_{I_\phi}) \to \mathbf{H}^1(\mathbb{A}_f, Z_{I_\phi})$ equals the sum of $\epsilon_\phi$ and the image of some $\upsilon_\phi \in \text{Ш}^\infty_{I_\phi}(\mathbb{Q}, Z_{I_\phi})$. For each maximal torus $T \subset I_\phi$, by tori-rationality of $\underline{\tau}$ there exists an element $\beta_{\phi,T} \in \text{Ш}^\infty_G(\mathbb{Q}, T)$ whose image in $\mathbf{H}^1(\mathbb{A}_f, T)$ equals that of $\epsilon_\phi$. Let $\beta'_{\phi,T} \in \text{Ш}^\infty_G(\mathbb{Q}, T)$ be the sum of $\beta_{\phi,T}$ and the image of $\upsilon_\phi$ in $\text{Ш}^\infty_{I_\phi}(\mathbb{Q}, T) \subset \text{Ш}^\infty_G(\mathbb{Q}, T)$. Then the image of $\underline{\sigma}(\phi)$ under

$$\mathfrak{E}(Z_{I_\phi}, I_\phi; \mathbb{A}_f^p) \to \mathfrak{E}(Z_{I_\phi}, I_\phi; \mathbb{A}_f) \to \mathbf{H}^1(\mathbb{A}_f, T)$$

equals that of $\beta'_{\phi,T}$. It follows that $\beta'_{\phi,T}$ lies in $\text{Ш}^{\infty,p}_G(\mathbb{Q}, T)$, and that the image of $\underline{\sigma}(\phi)$ in $\mathbf{H}^1(\mathbb{A}_f^p, T)/\text{Ш}^{\infty,p}_G(\mathbb{Q}, T)$ is trivial, as desired. $\square$

## 2.7. **The Langlands–Rapoport–$\tau$ Conjecture.**

**2.7.1.** Let $(G, X, p, \mathcal{G})$ be an unramified Shimura datum. Let $\underline{\tau} \in \Gamma(\mathcal{H})_1$ (Definition 2.6.17). For each admissible morphism $\phi : \mathfrak{Q}_f \to \mathfrak{G}_G$, we set

$$S_{\underline{\tau}}(\phi) := S_{\widetilde{\underline{\tau}(\phi)}}(\phi),$$

where $\widetilde{\underline{\tau}}(\phi) \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$ is any lift of $\underline{\tau}(\phi) \in \mathcal{H}(\phi)$, and $S_{\widetilde{\underline{\tau}(\phi)}}(\phi)$ is defined as in §2.4.7. The isomorphism class of the $\Phi^{\mathbb{Z}} \times G(\mathbb{A}_f^p)$-set $S_{\underline{\tau}}(\phi)$ is independent of the choice of $\widetilde{\underline{\tau}}(\phi)$. Moreover, from the assumption that $\underline{\tau} \in \Gamma(\mathcal{H})_1$, it follows that the isomorphism class of the $\Phi^{\mathbb{Z}} \times G(\mathbb{A}_f^p)$-set $S_{\underline{\tau}}(\phi)$ depends on $\phi$ only via its conjugacy class.

We write $\mathsf{LR}(G, X, p, \mathcal{G}, \underline{\tau})$ for the modification of Conjecture 2.5.8 where each $S(\phi)$ is replaced by $S_{\underline{\tau}}(\phi)$.

Combined with Theorem 2.5.3 and Theorem 2.5.7, the main result of [Kis17] can be stated as follows.

**Theorem 2.7.2.** *Let $(G, X, p, \mathcal{G})$ be an unramified Shimura datum such that $(G, X)$ is of abelian type. Assume $p > 2$. Then there exists $\underline{\tau} \in \Gamma(\mathcal{H})_1$ such that the statement $\mathsf{LR}(G, X, p, \mathcal{G}, \underline{\tau})$ holds.*

We refer the reader to Theorem 6.2.4 below for a more precise version of the above theorem. There we will also explain that the assumption $p > 2$ can be removed. In the following conjecture, we impose better control of $\underline{\tau}$ than the condition that $\underline{\tau}$ belongs to $\Gamma(\mathcal{H})_1$.

**Conjecture 2.7.3** (Langlands–Rapoport–$\tau$)**.** *For each unramified Shimura datum $(G, X, p, \mathcal{G})$, there exists a tori-rational element $\underline{\tau} \in \Gamma(\mathcal{H})_0$ such that the statement $\mathsf{LR}(G, X, p, \mathcal{G}, \underline{\tau})$ holds.*

**Theorem 2.7.4.** *Conjecture 2.7.3 implies Conjecture 1.8.8.*

We devote the next section to the proof of Theorem 2.7.4. Note that Conjecture 2.7.3 is weaker than Conjecture 2.5.8, as the latter asserts that $\underline{\tau}$ can be taken to be trivial. In view of Theorem 2.7.4, Conjecture 2.7.3 is a viable substitute for Conjecture 2.5.8 for applications to computing zeta functions and $\ell$-adic cohomology.

Theorem 2.7.2 is weaker than Conjecture 2.7.3 in that no extra control of $\underline{\tau} \in \Gamma(\mathcal{H})_1$ is provided. In Part 2 we shall improve on Theorem 2.7.2 and prove Conjecture 2.7.3 in the case of abelian type.

## 3. Langlands–Rapoport–$\tau$ implies point counting

Throughout §3, we fix an unramified Shimura datum $(G, X, p, \mathcal{G})$, and keep the notation $E, \mathfrak{p}, q = p^r$ as in §2.4.1. Our goal is to prove Theorem 2.7.4.

### 3.1. Semi-admissible and admissible Langlands–Rapoport pairs.

**Definition 3.1.1.** By a *Langlands–Rapoport pair* (*LR pair*), we mean a pair $(\phi, \epsilon)$, where $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ is a morphism in pro-$\mathcal{G}rb_{\mathbb{Q}}$, and $\epsilon$ is an element of $I_\phi(\mathbb{Q})$. We call such a pair $(\phi, \epsilon)$ *semi-admissible*, if $\phi$ is admissible. We denote by $\mathcal{LRP}$ the set of all LR pairs, and by $\mathcal{LRP}_{\mathrm{sa}}$ the subset of semi-admissible LR pairs.

*Remark* 3.1.2. If $(\phi, \epsilon) \in \mathcal{LRP}_{\mathrm{sa}}$, then $\epsilon$ is semi-simple. This is because $(I_\phi/Z_G)(\mathbb{R})$ is anisotropic by Lemma 2.6.7.

**3.1.3.** The group $G(\overline{\mathbb{Q}})$ acts on the set $\mathcal{LRP}$ by *conjugation* in the following sense. If $(\phi, \epsilon) \in \mathcal{LRP}$ and $g \in G(\overline{\mathbb{Q}})$, then $\mathrm{Int}(g)(\phi, \epsilon) := (\mathrm{Int}(g) \circ \phi, \mathrm{Int}(g)\epsilon)$ is also an element of $\mathcal{LRP}$. We write

$$\langle \mathcal{LRP} \rangle := \mathcal{LRP}/G(\overline{\mathbb{Q}})\text{-conjugacy}.$$

For $(\phi, \epsilon) \in \mathcal{LRP}$, we denote by

$$\langle \phi, \epsilon \rangle \in \langle \mathcal{LRP} \rangle$$

the $G(\overline{\mathbb{Q}})$-conjugacy class of $(\phi, \epsilon)$. The subset $\mathcal{LRP}_{\mathrm{sa}} \subset \mathcal{LRP}$ is stable under $G(\overline{\mathbb{Q}})$-conjugacy, and we write

$$\langle \mathcal{LRP}_{\mathrm{sa}} \rangle := \mathcal{LRP}_{\mathrm{sa}}/G(\overline{\mathbb{Q}})\text{-conjugacy}.$$

**3.1.4.** Let $(\phi, \epsilon) \in \mathcal{LRP}_{\rm sa}$. Let $\theta = \phi(p) \circ \zeta_p : \mathfrak{G}_p \to \mathfrak{G}_G(p)$, and choose $g \in \mathcal{UR}(\theta)$ (see Definition 2.2.3 and Lemma 2.2.4). Let $b_g := b_{\mathrm{Int}(g^{-1})\circ\theta} \in G(\mathbb{Q}_p^{\rm ur})$ (see Definition 2.2.5), and let $\epsilon_g := \mathrm{Int}(g^{-1})(\epsilon) \in G(\overline{\mathbb{Q}})$. Since $\epsilon$ is semi-simple (Remark 3.1.2), so is $\epsilon_g$.

We have $\epsilon_g \in I_{\mathrm{Int}(g)^{-1}\circ\theta}(\mathbb{Q}_p)$, and hence $\epsilon_g \in J_{b_g}(\mathbb{Q}_p)$ by Proposition 2.2.6. Also, by the same proposition, $b_g$ is decent. It then also follows that $\epsilon_g \in G(\mathbb{Q}_p^{\rm ur})$, as $J_{b_g}(\mathbb{Q}_p) \subset G(\mathbb{Q}_p^{\rm ur})$ (see §1.4.3). We let

$$\mathrm{cls}_p(\phi, \epsilon) := \{(b_g, \epsilon_g) \mid g \in \mathcal{UR}(\theta)\}.$$

Let $G(\mathbb{Q}_p^{\rm ur})$ act on $\mathrm{cls}_p(\phi, \epsilon)$ on the left by $h \cdot (b, \epsilon') := (hb\sigma(h)^{-1}, h\epsilon' h^{-1})$. Since $\mathcal{UR}(\theta)$ is a $G(\mathbb{Q}_p^{\rm ur})$-torsor, the $G(\mathbb{Q}_p^{\rm ur})$-action on $\mathrm{cls}_p(\phi, \epsilon)$ is transitive.

**3.1.5.** Fix a positive integer $m$, and let $n = mr$. (Recall that $q = p^r$ is the residue cardinality of $\mathfrak{p}$.) We will define *$q^m$-admissible LR pairs*, which will serve to describe $\mathbb{F}_{q^m}$-points of the Shimura variety.

Let $(\phi, \epsilon) \in \mathcal{LRP}_{\rm sa}$, and let $(b, \epsilon') \in \mathrm{cls}_p(\phi, \epsilon)$. Recall from §2.2.7 and §2.4.1 that the set $X_p(\phi)$ is identified with the set $X_{-\mu}(b)$, where $\mu \in \mathfrak{u}_X^{\mathcal{G}}$. The action of $\Phi$ on $X_p(\phi)$ corresponds to the left multiplication by $\Phi_b := (b \rtimes \sigma)^r$ on $X_{-\mu}(b)$. The action of $\epsilon$ on $X_p(\phi)$ corresponds to the left multiplication by $\epsilon'$ on $X_{-\mu_X}(b)$. Let

$$X_p(\phi, \epsilon, q^m) := \{x \in X_p(\phi) \mid \epsilon x = \Phi^m x\}.$$

Then we have an identification

$$X_p(\phi, \epsilon, q^m) \cong X_{-\mu}(b, \epsilon', q^m) := \{x \in X_{-\mu}(b) \mid \epsilon' x = \Phi_b^m x\}.$$

This motivates the following definition.

**Definition 3.1.6.** We say that an element $(\phi, \epsilon) \in \mathcal{LRP}_{\rm sa}$ is *$q^m$-admissible*, if for one (and hence every) element $(b, \epsilon')$ of $\mathrm{cls}_p(\phi, \epsilon)$, we have

$$\left\{x \in G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p) \mid \epsilon' x = \Phi_b^m x\right\} \neq \emptyset.$$

We denote by $\mathcal{LRP}_{\rm a}(q^m)$ the set of $q^m$-admissible elements of $\mathcal{LRP}_{\rm sa}$. This subset is stable under $G(\overline{\mathbb{Q}})$-conjugacy, and we write

$$\langle \mathcal{LRP}_{\rm a}(q^m)\rangle := \mathcal{LRP}_{\rm a}(q^m)/G(\overline{\mathbb{Q}})\text{-conjugacy}.$$

**Lemma 3.1.7.** *Let $(\phi, \epsilon) \in \mathcal{LRP}_{\rm a}(q^m)$, and let $(b, \epsilon') \in \mathrm{cls}_p(\phi, \epsilon)$. Then there exists $t \in \mathbb{Z}_{\geq 1}$ satisfying the following conditions.*

*(i) The fractional cocharacter $t\nu_b$ is a cocharacter of $G$ defined over $\mathbb{Q}_{p^t}$.*

*(ii) We have $\epsilon'^t = p^{nt\nu_b}k$, for $k$ lying in some conjugate of $\mathcal{G}(\breve{\mathbb{Z}}_p)$ in $G(\breve{\mathbb{Q}}_p)$.*

*Proof.* We have seen in §3.1.4 that $b$ is decent. Take $t$ such that $b$ is $t$-decent. Then condition (i) is already satisfied. Also by assumption there exists $x \in G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p)$ such that

$$\epsilon' x = \Phi_b^m x.$$

Since $\epsilon'$ commutes with $\Phi_b = (b \rtimes \sigma)^r$, we have

$$\epsilon'^t x = \Phi_b^{mt} x.$$

By Lemma 1.6.8, we can replace $t$ by a multiple, and assume that $\sigma^t x = x$. Let $s = tn = tmr$. Then

$$\epsilon'^t x = \Phi_b^{mt} x = (b \rtimes \sigma)^s x = b^\sigma b^{\sigma^2} b \cdots {}^{\sigma^{s-1}} bx.$$

Therefore

$$k := (b^{\sigma} b^{\sigma^2} b \cdots {}^{\sigma^{s-1}} b)^{-1} \epsilon'^{t} \in x \mathcal{G}(\breve{\mathbb{Z}}_p) x^{-1}. \tag{3.1.7.1}$$

Finally, since $b$ is $t$-decent we have $k = p^{-nt\nu_b} {\epsilon'}^{t}$. This proves condition (ii). $\square$

Recall from §2.2.8 that we have a homomorphism of $\mathbb{Q}_p$-group schemes $\nu(p) : \mathbb{D}_{\mathbb{Q}_p} \to Q_{\mathbb{Q}_p}$.

**Proposition 3.1.8.** *Let $(\phi_1, \epsilon_1), (\phi_2, \epsilon_2) \in \mathcal{LRP}_{\mathrm{a}}(q^m)$. Suppose $\epsilon_1 = \epsilon_2$ as elements of $G(\overline{\mathbb{Q}})$. Then $\phi_1^{\Delta} \circ \nu(p) = \phi_2^{\Delta} \circ \nu(p)$ as homomorphisms $\mathbb{D}_{\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$.*

*Proof.* As in §3.1.4, for $i = 1, 2$, we choose $g_i \in \mathcal{UR}(\phi_i(p) \circ \zeta_p)$, and let $(b_i, \epsilon_i')$ be the element of $\mathrm{cls}_p(\phi_i, \epsilon_i)$ associated with $g_i$. By Lemma 3.1.7, we can find $t \in \mathbb{Z}_{\geq 1}$ such that

$${\epsilon_i'}^{t} = p^{nt\nu_{b_i}} k_i,$$

where $k_i$ lies in some $G(\breve{\mathbb{Q}}_p)$-conjugate of $\mathcal{G}(\breve{\mathbb{Z}}_p)$, for $i = 1, 2$. Since $\epsilon_i'$ commutes with $b_i \rtimes \sigma$, we know that ${\epsilon_i'}^{t}$ commutes with $nt\nu_{b_i}$. Also $k_i$ lies in a bounded subgroup of $G(\breve{\mathbb{Q}}_p)$. Applying Lemma 1.6.10 to $F = \breve{\mathbb{Q}}_p$, we see that $nt\nu_{b_i}$ is the unique cocharacter $\nu$ of $G$ over $\overline{\breve{\mathbb{Q}}}_p$ commuting with ${\epsilon_i'}^{t}$ such that ${\epsilon_i'}^{-t} p^{\nu}$ lies in a $G(\overline{\breve{\mathbb{Q}}}_p)$-conjugate of a bounded subgroup of $G(\breve{\mathbb{Q}}_p)$. Let $g = g_2^{-1} g_1 \in G(\overline{\mathbb{Q}}_p)$. Then we have $\epsilon_2' = \mathrm{Int}(g)\epsilon_1'$. By the above-mentioned uniqueness of $nt\nu_{b_i}$ with respect to ${\epsilon_i'}^{t}$, we have

$$\nu_{b_2} = \mathrm{Int}(g) \circ \nu_{b_1}. \tag{3.1.8.1}$$

By Proposition 2.2.6 (i) we have

$$-\nu_{b_i} = \left(\mathrm{Int}(g_i^{-1}) \circ \phi_i(p) \circ \zeta_p\right)^{\Delta} = \mathrm{Int}(g_i^{-1}) \circ \phi_i^{\Delta} \circ \nu(p). \tag{3.1.8.2}$$

Comparing (3.1.8.1) with (3.1.8.2), we have $\phi_1^{\Delta} \circ \nu(p) = \phi_2^{\Delta} \circ \nu(p)$ as desired. $\square$

Recall from §2.2.8 that we have a homomorphism of $\mathbb{R}$-group schemes $\nu_{\infty} : \mathbb{G}_m \to Q_{\mathbb{R}}$.

**Lemma 3.1.9.** *Let $\phi_1, \phi_2 : \mathfrak{Q} \to \mathfrak{G}_G$ be two admissible morphisms. Then $\phi_1^{\Delta} \circ \nu(\infty) = \phi_2^{\Delta} \circ \nu(\infty)$.*

*Proof.* By condition (ii) in Definition 2.4.2 with $v = \infty$, we know that $\phi_i(\infty) \circ \zeta_{\infty}$ is conjugate to $\xi_{\infty}$. Hence $(\phi_i(\infty) \circ \zeta_{\infty})^{\Delta} = \phi_i^{\Delta} \circ \nu(\infty)$ is conjugate to $\xi_{\infty}^{\Delta}$. By the definition of $\xi_{\infty}$ in [Kis17, §3.3.5], $\xi_{\infty}^{\Delta}$ is equal to the weight cocharacter for the Shimura datum $(G, X)$, which is central in $G$ (see [Del79, §2.1.1]). The lemma follows. $\square$

**Corollary 3.1.10.** *Let $(\phi_1, \epsilon_1), (\phi_2, \epsilon_2) \in \mathcal{LRP}_{\mathrm{a}}(q^m)$. For $i = 1, 2$, assume that $\phi_i$ is gg (Definition 2.6.2), and that $\epsilon_i$ lies in $G(\mathbb{Q})$. Then for all $g \in G(\overline{\mathbb{Q}})$ such that $\mathrm{Int}(g)(\epsilon_1) = \epsilon_2$, we have $\mathrm{Int}(g) \circ \phi_1^{\Delta} = \phi_2^{\Delta}$.*

*Proof.* Write $H := \left\{g \in G(\overline{\mathbb{Q}}) \mid \mathrm{Int}(g)\epsilon_1 = \epsilon_2\right\}$. Let $g \in H$. For all $\tau \in \Gamma$, we have ${}^{\tau}g \in H$, since $\epsilon_1, \epsilon_2 \in G(\mathbb{Q})$. Applying Proposition 3.1.8 and Lemma 3.1.9 to $(\phi_2, \epsilon_2)$ and $\mathrm{Int}({}^{\tau}g)(\phi_1, \epsilon_1)$, we get

$$\phi_2^{\Delta} \circ \nu(v) = \mathrm{Int}({}^{\tau}g) \circ \phi_1^{\Delta} \circ \nu(v), \quad \text{for } v = p, \infty. \tag{3.1.10.1}$$

Now we apply $\tau^{-1}$ to both sides of (3.1.10.1). As $\phi_1^{\Delta}$ and $\phi_2^{\Delta}$ are defined over $\mathbb{Q}$ (by the gg assumption), we get

$$\phi_2^{\Delta} \circ {}^{\tau^{-1}}\nu(v) = \mathrm{Int}(g) \circ \phi_1^{\Delta} \circ {}^{\tau^{-1}}\nu(v), \quad \text{for } v = p, \infty. \tag{3.1.10.2}$$

By construction, for each finite Galois extension $L/\mathbb{Q}$ contained in $\overline{\mathbb{Q}}$, the $\mathbb{Q}$-torus $Q^L$ is split over $L$, and the $\mathbb{Q}$-vector space $X_*(Q^L) \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by the $\mathrm{Gal}(L/\mathbb{Q})$-conjugates of the fractional cocharacters

$$\mathbb{D}_{\overline{\mathbb{Q}}_p} \xrightarrow{\nu(p)} Q_{\overline{\mathbb{Q}}_p} \to Q^L_{\overline{\mathbb{Q}}_p}$$

and

$$\mathbb{G}_{m,\mathbb{C}} \xrightarrow{\nu(\infty)} Q_{\mathbb{C}} \to Q^L_{\mathbb{C}}.$$

See [Kis17, §3.1] for details. Hence (3.1.10.2) for all $\tau$ implies that $\phi_2^{\Delta} = \mathrm{Int}(g) \circ \phi_1^{\Delta}$. □

3.2. **The günstig gelegen condition.** As in §2.6.1, for each $\tau \in \Gamma$ we choose a lift $q_\tau \in \mathfrak{Q}^{\mathrm{top}}$. For $(\phi, \epsilon) \in \mathcal{LRP}$, we write $I^0_{\phi,\epsilon}$ for $(I_\phi)^0_\epsilon$.

**Definition 3.2.1.** We say that an LR pair $(\phi, \epsilon) \in \mathcal{LRP}$ is *günstig gelegen* (*to be abbreviated as gg*)[18], if the following conditions are satisfied:

- The embedding $I_{\phi,\overline{\mathbb{Q}}} \hookrightarrow G_{\overline{\mathbb{Q}}}$ maps $\epsilon \in I_\phi(\mathbb{Q})$ into $G(\mathbb{Q})$. Moreover, $\epsilon$ is $\mathbb{R}$-elliptic in $G$, cf. §1.6.1.
- For each $\tau \in \Gamma$, write $\phi(q_\tau) = g_\tau \rtimes \tau$, with $g_\tau \in G(\overline{\mathbb{Q}})$. Then $g_\tau$ lies in $(G_{\overline{\mathbb{Q}},\phi^\Delta})^0_\epsilon(\overline{\mathbb{Q}}) = I^0_{\phi,\epsilon}(\overline{\mathbb{Q}})$.

We denote by $\mathcal{LRP}^{\mathrm{gg}}$ the set of gg LR pairs.

*Remark* 3.2.2. Let $(\phi, \epsilon) \in \mathcal{LRP}$, and let $\phi(q_\tau) = g_\tau \rtimes \tau$, for $\tau \in \Gamma$. If one changes the choice of $q_\tau$, then $g_\tau$ is left multiplied by a $\overline{\mathbb{Q}}$-point of $\mathrm{im}(\phi^\Delta)$, which is a central torus in $G_{\overline{\mathbb{Q}},\phi^\Delta}$. Hence the second condition in Definition 3.2.1 is independent of the choice of $q_\tau$.

**3.2.3.** Let $(\phi, \epsilon) \in \mathcal{LRP}^{\mathrm{gg}}$. By Lemma 2.6.3, $\phi$ is gg, and the canonical $\overline{\mathbb{Q}}$-isomorphism $I_{\phi,\overline{\mathbb{Q}}} \xrightarrow{\sim} G_{\phi^\Delta,\overline{\mathbb{Q}}}$ is an inner twisting between the $\mathbb{Q}$-groups $I_\phi$ and $G_{\phi^\Delta}$. Moreover, in the current case this inner twisting restricts to an inner twisting between the $\mathbb{Q}$-groups $I_{\phi,\epsilon}$ and $(G_{\phi^\Delta})_\epsilon$, and an inner twisting between the $\mathbb{Q}$-groups $I^0_{\phi,\epsilon}$ and $(G_{\phi^\Delta})^0_\epsilon$.

In fact, the inner twisting between $I^0_{\phi,\epsilon}$ and $(G_{\phi^\Delta})^0_\epsilon$ can be interpreted as follows. Let $I_0 = G^0_\epsilon \subset G$. Since $(\phi, \epsilon)$ is gg, $\phi$ factors as $\mathfrak{Q} \to \mathfrak{G}_{I_0} \to \mathfrak{G}_G$. We write $\phi_{I_0}$ for $\phi$ when we view it as a morphism $\mathfrak{Q} \to \mathfrak{G}_{I_0}$. Then $\phi_{I_0}$ is itself gg. Hence by Lemma 2.6.3 applied to $\phi_{I_0}$, we obtain an inner twisting between $I_{\phi_{I_0}}$ and $(I_0)_{(\phi_{I_0})^\Delta} = (I_0)_{\phi^\Delta}$. It is easy to see that as $\mathbb{Q}$-groups we have $I_{\phi_{I_0}} = I^0_{\phi,\epsilon}$ and $(I_0)_{\phi^\Delta} = (G_{\phi^\Delta})^0_\epsilon$.

**Definition 3.2.4.** We write $[\mathcal{LRP}^{\mathrm{gg}}]$ for the quotient set of $\mathcal{LRP}^{\mathrm{gg}}$ divided by the equivalence relation of $G(\overline{\mathbb{Q}})$-conjugacy.[19] For $(\phi, \epsilon) \in \mathcal{LRP}^{\mathrm{gg}}$, we denote its image in $[\mathcal{LRP}^{\mathrm{gg}}]$ by $[\phi, \epsilon]$. We denote the natural injection $[\mathcal{LRP}^{\mathrm{gg}}] \to \langle \mathcal{LRP} \rangle$ by $\mathfrak{v}$ (standing for *vergessen*).

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[18] As in Definition 2.6.2, this terminology comes from [LR87, §5], but our definition is modified to suit the corrected definition of $\mathfrak{Q}$.

[19] We caution the reader that the subset $\mathcal{LRP}^{\mathrm{gg}} \subset \mathcal{LRP}$ is not stable under $G(\overline{\mathbb{Q}})$-conjugacy.

**Lemma 3.2.5.** *Let $(\phi, \epsilon) \in \mathcal{LRP}$, and let $a = (a_\tau)_\tau \in Z^1(\mathbb{Q}, I_{\phi,\epsilon})$. Then $(a\phi, \epsilon) \in \mathcal{LRP}$ (i.e., the element $\epsilon \in G(\overline{\mathbb{Q}})$ lies in the image of $I_{a\phi}(\mathbb{Q}) \hookrightarrow I_{a\phi}(\overline{\mathbb{Q}}) \hookrightarrow G(\overline{\mathbb{Q}})$). Here $a\phi$ is the twist of $\phi$ by the cocycle $a$ as in §2.1.15. If in addition we have $(\phi, \epsilon) \in \mathcal{LRP}^{\mathrm{gg}}$ and $a \in Z^1(\mathbb{Q}, I^0_{\phi,\epsilon})$, then $(a\phi, \epsilon) \in \mathcal{LRP}^{\mathrm{gg}}$.*

*Proof.* First recall that $(a\phi)^\Delta = \phi^\Delta$, and $I_{\phi,\overline{\mathbb{Q}}}$ and $I_{a\phi,\overline{\mathbb{Q}}}$ are equal as $\overline{\mathbb{Q}}$-subgroups of $G_{\overline{\mathbb{Q}}}$. Write $\phi(q_\tau) = g_\tau \rtimes \tau$ for each $\tau \in \Gamma$. Then we have $(a\phi)(q_\tau) = a_\tau g_\tau \rtimes \tau$. Since $\epsilon \in I_\phi(\mathbb{Q})$, we have $g_\tau {}^\tau\epsilon g_\tau^{-1} = \epsilon$. Since $\epsilon$ commutes with $a_\tau$, we have

$$a_\tau g_\tau {}^\tau\epsilon g_\tau^{-1} a_\tau^{-1} = \epsilon,$$

which means that $\epsilon \in I_{a\phi}(\mathbb{Q})$. Thus we have shown that $(a\phi, \epsilon) \in \mathcal{LRP}$.

To show the second statement, we need to check that $a_\tau g_\tau \in I^0_{a\phi,\epsilon}(\overline{\mathbb{Q}})$ for all $\tau$. But both $g_\tau$ and $a_\tau$ lie in $I^0_{\phi,\epsilon}(\overline{\mathbb{Q}})$, and we have $I^0_{a\phi,\epsilon}(\overline{\mathbb{Q}}) = I^0_{\phi,\epsilon}(\overline{\mathbb{Q}})$. The desired statement follows. □

**Lemma 3.2.6.** *Let $(\phi, \epsilon) \in \mathcal{LRP}$. Let $a, b \in Z^1(\mathbb{Q}, I_{\phi,\epsilon})$. Then $a, b$ are cohomologous in $I_{\phi,\epsilon}$ if and only if $\langle a\phi, \epsilon\rangle = \langle b\phi, \epsilon\rangle$.*

*Proof.* Write $\phi(q_\tau) = g_\tau \rtimes \tau$, for each $\tau \in \Gamma$. If $a, b$ are cohomologous in $I_{\phi,\epsilon}$, then there exists $u \in I_{\phi,\epsilon}(\overline{\mathbb{Q}})$ such that

$$a_\tau = u^{-1} b_\tau g_\tau {}^\tau u g_\tau^{-1}, \quad \forall \tau \in \Gamma. \tag{3.2.6.1}$$

Here ${}^\tau u$ denotes the action of $\tau$ on $u$ viewed as in $G(\overline{\mathbb{Q}})$. Then

$$(a\phi)(q_\tau) = a_\tau g_\tau \rtimes \tau = u^{-1} b_\tau g_\tau {}^\tau u \rtimes \tau = \mathrm{Int}(u^{-1}) \circ (b\phi)(q_\tau).$$

Hence we have $\mathrm{Int}(u^{-1}) \circ (b\phi) = a\phi$ as they already agree on the kernel. Also $\mathrm{Int}(u^{-1})\epsilon = \epsilon$. Therefore $(a\phi, \epsilon) = \mathrm{Int}(u^{-1})(b\phi, \epsilon)$, and so $\langle a\phi, \epsilon\rangle = \langle b\phi, \epsilon\rangle$.

Conversely, assume that $\langle a\phi, \epsilon\rangle = \langle b\phi, \epsilon\rangle$. Then there exists $u \in G(\overline{\mathbb{Q}})$ such that $(a\phi, \epsilon) = \mathrm{Int}(u^{-1})(b\phi, \epsilon)$. Since $(a\phi)^\Delta = (b\phi)^\Delta$, we have $u \in I_{\phi,\epsilon}(\overline{\mathbb{Q}})$. Now the relation $\mathrm{Int}(u^{-1}) \circ (b\phi) = a\phi$ is equivalent to (3.2.6.1), which shows that $a$ and $b$ are cohomologous. □

**3.2.7.** Let $(\psi, \delta) \in \mathcal{LRP}$ and $(\phi, \epsilon) \in \mathcal{LRP}^{\mathrm{gg}}$. In view of Lemma 3.2.5 and Lemma 3.2.6, we have well-defined maps

$$Z^1(\mathbb{Q}, I_{\psi,\delta}) \to \mathcal{LRP}, \quad a \mapsto (a\psi, \delta)$$

and

$$Z^1(\mathbb{Q}, I^0_{\phi,\epsilon}) \to \mathcal{LRP}^{\mathrm{gg}}, \quad a \mapsto (a\phi, \epsilon),$$

which induce maps

$$\iota_{\psi,\delta} : \mathbf{H}^1(\mathbb{Q}, I_{\psi,\delta}) \longrightarrow \langle \mathcal{LRP} \rangle$$

and

$$\eta_{\phi,\epsilon} : \mathbf{H}^1(\mathbb{Q}, I^0_{\phi,\epsilon}) \longrightarrow [\mathcal{LRP}^{\mathrm{gg}}]$$

respectively. Moreover, by Lemma 3.2.6 the map $\iota_{\psi,\delta}$ is injective.

**Lemma 3.2.8.** *Let $(\phi, \epsilon) \in \mathcal{LRP}^{\mathrm{gg}}$. The subset $\mathrm{im}(\eta_{\phi,\epsilon})$ of $[\mathcal{LRP}^{\mathrm{gg}}]$ depends only on $[\phi, \epsilon] \in [\mathcal{LRP}^{\mathrm{gg}}]$.*

*Proof.* Suppose we have $(\phi', \epsilon') \in \mathcal{LRP}^{\mathrm{gg}}$ such that $[\phi, \epsilon] = [\phi', \epsilon']$. Then $(\phi', \epsilon') = \mathrm{Int}(g)(\phi, \epsilon)$ for some $g \in G(\overline{\mathbb{Q}})$. We have an isomorphism $\mathrm{Int}(g) : I^0_{\phi,\epsilon} \to I^0_{\phi',\epsilon'}$ defined over $\mathbb{Q}$. This induces a bijection

$$g_* : \mathbf{H}^1(\mathbb{Q}, I^0_{\phi,\epsilon}) \to \mathbf{H}^1(\mathbb{Q}, I^0_{\phi',\epsilon'}),$$

and we have $\eta_{\phi,\epsilon} = \eta_{\phi',\epsilon'} \circ g_*$. $\square$

**Definition 3.2.9.** For $x = [\phi, \epsilon] \in [\mathcal{LRP}^{\mathrm{gg}}]$, we define the subset

$$\mathcal{C}_x := \mathrm{im}(\eta_{\phi,\epsilon}) \subset [\mathcal{LRP}^{\mathrm{gg}}].$$

This is well defined by Lemma 3.2.8.

**Lemma 3.2.10.** *Let $x, y \in [\mathcal{LRP}^{\mathrm{gg}}]$. Then $y \in \mathcal{C}_x$ if and only if $x \in \mathcal{C}_y$. In particular, subsets of the form $\mathcal{C}_x$ form a partition of $[\mathcal{LRP}^{\mathrm{gg}}]$.*

*Proof.* Let $x = [\phi, \epsilon] \in [\mathcal{LRP}^{\mathrm{gg}}]$. Assume that $y \in \mathcal{C}_x$. Then there exists $a = (a_\tau)_\tau \in Z^1(\mathbb{Q}, I^0_{\phi,\epsilon})$ such that $y = [a\phi, \epsilon]$. We identify $I_\phi(\overline{\mathbb{Q}})$ with $I_{a\phi}(\overline{\mathbb{Q}})$. For each $\tau \in \Gamma$, let $\widehat{\tau}(\cdot)$ (resp. $\check{\tau}(\cdot)$) denote the action of $\tau$ on $I_\phi(\overline{\mathbb{Q}})$ with respect to the $\mathbb{Q}$-structure $I_\phi$ (resp. $I_{a\phi}$). We have

$$\check{\tau}(\cdot) = a_\tau \widehat{\tau}(\cdot) a_\tau^{-1}.$$

For each $\tau$, let $b_\tau := a_\tau^{-1} \in I_{a\phi}(\overline{\mathbb{Q}})$. Then for $\sigma, \tau \in \Gamma$ we have

$$b_{\sigma\tau} = (a_\sigma \widehat{\sigma}(a_\tau))^{-1} = (\check{\sigma}(a_\tau) a_\sigma)^{-1} = b_\sigma \check{\sigma}(b_\tau),$$

showing that $(b_\tau)_\tau \in Z^1(\mathbb{Q}, I_{a\phi})$. Clearly we have $\phi = b(a\phi)$, and so $x \in \mathcal{C}_y$. $\square$

**Lemma 3.2.11.** *Let $(\phi_1, \epsilon_1), (\phi_2, \epsilon_2) \in \mathcal{LRP}^{\mathrm{gg}}$ be elements having the same image in $[\mathcal{LRP}^{\mathrm{gg}}]$, and let $g \in G(\overline{\mathbb{Q}})$ be such that* $\mathrm{Int}(g)(\phi_1, \epsilon_1) = (\phi_2, \epsilon_2)$ *(which exists by the first assumption). Then we have $g\, {}^\tau g^{-1} \in G^0_{\epsilon_2}(\overline{\mathbb{Q}})$ for all $\tau \in \Gamma$. In particular, $\epsilon_1, \epsilon_2 \in G(\mathbb{Q})$ are stably conjugate.*

*Proof.* For $i = 1, 2$, we write $\phi_i(q_\tau) = g_{i,\tau} \rtimes \tau$. Then

$$g_{2,\tau} = g g_{1,\tau} {}^\tau g^{-1} = g g_{1,\tau} g^{-1} g\, {}^\tau g^{-1},$$

and hence

$$g\, {}^\tau g^{-1} = (g g_{1,\tau} g^{-1})^{-1} g_{2,\tau}.$$

Since $(\phi_1, \epsilon_1), (\phi_2, \epsilon_2) \in \mathcal{LRP}^{\mathrm{gg}}$, we have $g_{1,\tau} \in G^0_{\epsilon_1}(\overline{\mathbb{Q}})$ and $g_{2,\tau} \in G^0_{\epsilon_2}(\overline{\mathbb{Q}})$. It follows that $g g_{1,\tau} g^{-1} \in G^0_{\epsilon_2}(\overline{\mathbb{Q}})$. Hence $g\, {}^\tau g^{-1} \in G^0_{\epsilon_2}(\overline{\mathbb{Q}})$, as desired. $\square$

**Definition 3.2.12.** As in §1.8.7, let $\Sigma_{\mathbb{R}\text{-ell}}(G)$ be the set of stable conjugacy classes of semi-simple, $\mathbb{R}$-elliptic elements of $G(\mathbb{Q})$. We define the *stable conjugacy class map*

$$\mathfrak{scc} : [\mathcal{LRP}^{\mathrm{gg}}] \longrightarrow \Sigma_{\mathbb{R}\text{-ell}}(G),$$

sending $[\phi, \epsilon]$ to the stable conjugacy class of $\epsilon$. This is well defined by Lemma 3.2.11.

**Definition 3.2.13.** Fix $m$ as in §3.1.5. We set

$$\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}} := \mathcal{LRP}^{\mathrm{gg}} \cap \mathcal{LRP}_{\mathrm{sa}}, \quad \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m) := \mathcal{LRP}^{\mathrm{gg}} \cap \mathcal{LRP}_{\mathrm{a}}(q^m),$$

$$[\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}] := \mathfrak{v}^{-1}(\langle \mathcal{LRP}_{\mathrm{sa}} \rangle), \quad [\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)] := \mathfrak{v}^{-1}(\langle \mathcal{LRP}_{\mathrm{a}}(q^m) \rangle).$$

**3.2.14.** Let $(\phi, \epsilon) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)$. Then $\phi$ is gg (see §3.2.3), and $\epsilon \in G(\mathbb{Q})$. Hence by Corollary 3.1.10 the $\overline{\mathbb{Q}}$-inclusion $I_{\phi,\overline{\mathbb{Q}}} \hookrightarrow G_{\overline{\mathbb{Q}}}$ induces a $\overline{\mathbb{Q}}$-isomorphism $I_{\phi,\epsilon,\overline{\mathbb{Q}}} \xrightarrow{\sim} G_{\epsilon,\overline{\mathbb{Q}}}$. By this fact and by the discussion in §3.2.3, we obtain canonical inner twistings

$$(I^0_{\phi,\epsilon})_{\overline{\mathbb{Q}}} \xrightarrow{\sim} (G^0_\epsilon)_{\overline{\mathbb{Q}}},$$
$$(I_{\phi,\epsilon})_{\overline{\mathbb{Q}}} \xrightarrow{\sim} (G_\epsilon)_{\overline{\mathbb{Q}}}.$$

**Lemma 3.2.15.** *Let $(\phi, \epsilon) \in \mathcal{LRP}$. Assume that $I_\phi$ contains a $G$-rational maximal torus $T$ (see Definition 2.6.4) such that $\epsilon \in T(\mathbb{Q})$ and such that $T/Z_G$ is anisotropic over $\mathbb{R}$. Then $(\phi, \epsilon) \in \mathcal{LRP}^{\mathrm{gg}}$.*

*Proof.* The assumptions imply that $\epsilon$ is a semi-simple $\mathbb{R}$-elliptic element of $G(\mathbb{Q})$. By the proof of Lemma 2.6.6, if we write $\phi(q_\tau) = g_\tau \rtimes \tau$, then $g_\tau$ lies in $T(\overline{\mathbb{Q}})$ for each $\tau \in \Gamma$. Note that $T \subset I^0_{\phi,\epsilon}$. Hence $g_\tau$ lies in $I^0_{\phi,\epsilon}(\mathbb{Q})$ as desired. □

**Lemma 3.2.16.** *Let $(\phi, \epsilon) \in \mathcal{LRP}_{\mathrm{sa}}$. Then there exists $g \in G(\overline{\mathbb{Q}})$ such that $(\phi', \epsilon') := \mathrm{Int}(g)(\phi, \epsilon) \in \mathcal{LRP}_{\mathrm{sa}}$ satisfies the following condition: The group $I_{\phi'}$ contains a $G$-rational maximal torus $T'$ such that $\epsilon' \in T'(\mathbb{Q})$ and such that $T'/Z_G$ is anisotropic over $\mathbb{R}$. Moreover, in this case we have $(\phi', \epsilon') \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}$.*

*Proof.* Choose a maximal torus $T \subset I_\phi$ defined over $\mathbb{Q}$ such that $\epsilon \in T(\mathbb{Q})$. (This is possible since $\epsilon$ is semi-simple; see Remark 3.1.2.) Since $\phi$ is admissible, Proposition 2.6.9 implies that there exists $g \in G(\overline{\mathbb{Q}})$ such that $\mathrm{Int}(g)(T)$ is a $G$-rational maximal torus in $I_{\mathrm{Int}(g)\circ\phi}$. We write $(\phi', \epsilon')$ for $\mathrm{Int}(g)(\phi, \epsilon)$, and write $T'$ for $\mathrm{Int}(g)(T)$. Then $T'$ is a $G$-rational maximal torus in $I_{\phi'}$, and $\epsilon' \in T'(\mathbb{Q})$. Since $\phi'$ is admissible, we know that $T'/Z_G$ is anisotropic over $\mathbb{R}$ by Lemma 2.6.7. Finally, we have $(\phi', \epsilon') \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}$ by Lemma 3.2.15. □

**Corollary 3.2.17.** *The injection $\mathfrak{v} : [\mathcal{LRP}^{\mathrm{gg}}] \to \langle \mathcal{LRP} \rangle$ restricts to bijections $[\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}] \xrightarrow{\sim} \langle \mathcal{LRP}_{\mathrm{sa}} \rangle$ and $[\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)] \xrightarrow{\sim} \langle \mathcal{LRP}_{\mathrm{a}}(q^m) \rangle$.*

*Proof.* By Lemma 3.2.16, every $G(\overline{\mathbb{Q}})$-conjugacy class in $\mathcal{LRP}_{\mathrm{sa}}$ contains an element of $\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}$. Hence $\mathfrak{v}$ restricts to a bijection $[\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}] \xrightarrow{\sim} \langle \mathcal{LRP}_{\mathrm{sa}} \rangle$. Since $[\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)]$ is by definition $\mathfrak{v}^{-1}(\langle \mathcal{LRP}_{\mathrm{a}}(q^m) \rangle)$, we see that $\mathfrak{v}$ also restricts to a bijection $[\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)] \xrightarrow{\sim} \langle \mathcal{LRP}_{\mathrm{a}}(q^m) \rangle$. □

**3.2.18.** Let $(\phi, \epsilon) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}$. As $I^0_{\phi,\epsilon}$ is a $\mathbb{Q}$-subgroup of $I_\phi$, we use the canonical inner transfer datum from $I_\phi$ to $G$ as in §2.6.11 to define $\text{Ш}^\infty_G(\mathbb{Q}, I^0_{\phi,\epsilon})$; see §1.2.5.

**Proposition 3.2.19.** *Let $(\phi, \epsilon) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}$. Then $\eta^{-1}_{\phi,\epsilon}([\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}]) = \text{Ш}^\infty_G(\mathbb{Q}, I^0_{\phi,\epsilon})$.*

*Proof.* By Proposition 2.6.12, an element $\beta \in \mathbf{H}^1(\mathbb{Q}, I^0_{\phi,\epsilon})$ lies in $\eta^{-1}_{\phi,\epsilon}([\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}])$ if and only if the image of $\beta$ in $\mathbf{H}^1(\mathbb{Q}, I_\phi)$ lies in $\text{Ш}^\infty_G(\mathbb{Q}, I_\phi)$. Thus we only need to check that the map $\mathbf{H}^1(\mathbb{R}, I^0_{\phi,\epsilon}) \to \mathbf{H}^1(\mathbb{R}, I_\phi)$ has trivial kernel. Note that $(I^0_{\phi,\epsilon})_\mathbb{R}$ and $(I_\phi)_\mathbb{R}$ are both reductive groups over $\mathbb{R}$, and their centers both contain $Z_{G,\mathbb{R}}$. The desired statement then follows from Lemma 2.6.7 and [Kis17, Lem. 4.4.5]. □

**Proposition 3.2.20.** *Let $x \in [\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)]$, and let $e = \mathfrak{scc}(x) \in \Sigma_{\mathbb{R}\text{-ell}}(G)$. Then*

$$\mathcal{C}_x \cap [\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)] = \mathfrak{scc}^{-1}(e) \cap [\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)].$$

*Proof.* We only need to show the containment $\mathfrak{scc}^{-1}(e) \cap [\mathcal{LRP}_{\mathrm{a}}^{\mathrm{gg}}(q^m)] \subset \mathcal{C}_x$. Let $y \in \mathfrak{scc}^{-1}(e) \cap [\mathcal{LRP}_{\mathrm{a}}^{\mathrm{gg}}(q^m)]$ be arbitrary. Write $x = [\phi_0, \epsilon_0], y = [\phi_1, \epsilon_1]$, for some $(\phi_i, \epsilon_i) \in \mathcal{LRP}_{\mathrm{a}}^{\mathrm{gg}}(q^m)$, $i = 0, 1$. Since $\mathfrak{scc}(x) = \mathfrak{scc}(y)$, there exists $g \in G(\overline{\mathbb{Q}})$ such that

$$\mathrm{Int}(g)\epsilon_1 = \epsilon_0, \tag{3.2.20.1}$$

$$g^\tau g^{-1} \in G^0_{\epsilon_0}(\overline{\mathbb{Q}}), \quad \forall \tau \in \Gamma. \tag{3.2.20.2}$$

Let $\phi' = \mathrm{Int}(g) \circ \phi_1$. By (3.2.20.1), we have $(\phi', \epsilon_0) \in \mathcal{LRP}$, and $(\phi', \epsilon_0)$ is $G(\overline{\mathbb{Q}})$-conjugate to $(\phi_1, \epsilon_1)$. By Corollary 3.1.10 applied to the relation (3.2.20.1), we have $\phi'^{\Delta} = \phi_0^{\Delta}$. By Lemma 2.1.16 we have $\phi' = a\phi_0$ for some $a \in Z^1(\mathbb{Q}, I_{\phi_0})$. For $i = 0, 1$, we write $\phi_i(q_\tau) = g_\tau^{(i)} \rtimes \tau$. Then

$$\phi'(q_\tau) = a_\tau g_\tau^{(0)} \rtimes \tau. \tag{3.2.20.3}$$

By the definition of $\phi'$ we have

$$\phi'(q_\tau) = \mathrm{Int}(g) \circ \phi_1(q_\tau) = g(g_\tau^{(1)} \rtimes \tau)g^{-1} = g g_\tau^{(1)\tau} g^{-1} \rtimes \tau. \tag{3.2.20.4}$$

Comparing (3.2.20.3) and (3.2.20.4), we have

$$a_\tau = g g_\tau^{(1)\tau} g^{-1} (g_\tau^{(0)})^{-1} = \mathrm{Int}(g)(g_\tau^{(1)}) \cdot (g^\tau g^{-1}) \cdot (g_\tau^{(0)})^{-1}. \tag{3.2.20.5}$$

Since $(\phi_i, \epsilon_i)$ is gg, we have $g_\tau^{(i)} \in G^0_{\epsilon_i}(\overline{\mathbb{Q}})$. Hence $\mathrm{Int}(g)(g_\tau^{(1)})$ and $(g_\tau^{(0)})^{-1}$ both lie in $G^0_{\epsilon_0}(\overline{\mathbb{Q}})$. Thus by (3.2.20.2) and (3.2.20.5), we have $a_\tau \in G^0_{\epsilon_0}(\overline{\mathbb{Q}})$. By the discussion in §3.2.14, we have $(G^0_{\epsilon_0})_{\overline{\mathbb{Q}}} = (I^0_{\phi_0,\epsilon_0})_{\overline{\mathbb{Q}}}$, since $(\phi_0, \epsilon_0) \in \mathcal{LRP}_{\mathrm{a}}^{\mathrm{gg}}(q^m)$. Hence $a = (a_\tau)$ is a cocycle in $Z^1(\mathbb{Q}, I^0_{\phi_0,\epsilon_0})$. It follows from Lemma 3.2.5 that the pair $(\phi', \epsilon_0) = (a\phi_0, \epsilon_0)$ is gg. Since this pair is $G(\overline{\mathbb{Q}})$-conjugate to $(\phi_1, \epsilon_1)$, we have $y = [\phi', \epsilon_0] = [a\phi_0, \epsilon_0] \in \mathrm{im}(\eta_{\phi_0,\epsilon_0}) = \mathcal{C}_x$. $\square$

3.3. **Admissible morphisms and maximal tori.** We first explain a result which considerably strengthens Corollary 2.6.10.

**Definition 3.3.1.** By a *special point datum* for $(G, X)$, we mean a triple $(T, i, h)$, where $T$ is a torus over $\mathbb{Q}$, $i : T \to G$ is an injective $\mathbb{Q}$-homomorphism whose image is a maximal torus in $G$, and $h : \mathbb{S} \to T_{\mathbb{R}}$ is an $\mathbb{R}$-homomorphism such that $i \circ h \in X$. We denote the set of special point data by $\mathcal{SPD}(G, X)$.

**Definition 3.3.2.** Let $(T, i, h) \in \mathcal{SPD}(G, X)$. Let $\mu_h \in X_*(T)$ be the Hodge cocharacter associated with $h$. We denote by $\phi(T, i, h)$ the composite morphism

$$\mathfrak{Q} \xrightarrow{\Psi_{T,\mu_h}} \mathfrak{G}_T \xrightarrow{i} \mathfrak{G}_G$$

in pro-$\mathcal{G}rb(\overline{\mathbb{Q}}/\mathbb{Q})$. (See §2.2.9 for $\Psi_{T,\mu_h}$.)

**Theorem 3.3.3** ([Kis17, Lem. 3.5.8, Thm. 3.5.11])**.** *A morphism $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ is admissible if and only if there exists $(T, i, h) \in \mathcal{SPD}(G, X)$ such that $\phi$ is conjugate to $\phi(T, i, h)$.*

*Remark* 3.3.4. Let $(T, i, h) \in \mathcal{SPD}(G, X)$, and let $\phi = \phi(T, i, h)$. Then $i(T_{\overline{\mathbb{Q}}})$ is contained in $I_{\phi,\overline{\mathbb{Q}}}$ (when they are both viewed as subgroups of $G_{\overline{\mathbb{Q}}}$), and the inclusion $T_{\overline{\mathbb{Q}}} \hookrightarrow I_{\phi,\overline{\mathbb{Q}}}$ is defined over $\mathbb{Q}$. In other words, $T$ is naturally a $G$-rational maximal torus in $I_\phi$. By Lemma 2.6.6, $\phi$ is gg. With this observation, we see that Theorem 3.3.3 strengthens Corollary 2.6.10.

**3.3.5.** Our next goal is to prove a more precise version of the "only if" direction in Theorem 3.3.3. We introduce some general notation. For a $\mathbb{Q}_p$-torus $T$ and a cocharacter $\lambda \in X_*(T)$, we write $\bar{\lambda}^T$, or simply $\bar{\lambda}$, for

$$[L : \mathbb{Q}_p]^{-1} \sum_{\tau \in \mathrm{Gal}(L/\mathbb{Q}_p)} \tau(\lambda) \in X_*(T) \otimes \mathbb{Q},$$

where $L/\mathbb{Q}_p$ is any finite Galois extension over which $\lambda$ is defined. For an unramified reductive group $M$ over $\mathbb{Q}_p$, we write

$$w_M : M(\breve{\mathbb{Q}}_p) \longrightarrow \pi_1(M)_{\Gamma_{p,0}} = \pi_1(M)$$

for the Kottwitz homomorphism associated with the $p$-adic valuation on $\breve{\mathbb{Q}}_p$, as in §1.3 and §1.4.2. (Here $\Gamma_{p,0}$ acts trivially on $\pi_1(M)$ since $M$ is unramified over $\mathbb{Q}_p$.)

We fix $\mu \in \mathbb{\mu}_X^{\mathcal{G}}$ as in §2.4.1. For each $b \in G(\breve{\mathbb{Q}}_p)$, we define $X_{-\mu}(b)$ as in §2.2.7.

**Lemma 3.3.6.** *Let $T \subset G_{\mathbb{Q}_p}$ be a maximal torus over $\mathbb{Q}_p$. Let $b \in T(\breve{\mathbb{Q}}_p) \subset G(\breve{\mathbb{Q}}_p)$ be such that $X_{-\mu}(b)$ is non-empty. Then there exists $\mu_T \in X_*(T)$ which is $G(\overline{\mathbb{Q}}_p)$-conjugate to $\mu$ and such that $\nu_b$ is equal to $-\overline{\mu_T}^T$ as elements of $X_*(T) \otimes \mathbb{Q}$.*

*Proof.* This is proved by Langlands–Rapoport, when they prove [LR87, Lem. 5.11]. We recall the argument with the suitable changes in notation.

First note that $\nu_b$ is a fractional cocharacter of $T$ defined over $\mathbb{Q}_p$ (as this holds for arbitrary $b \in T(\breve{\mathbb{Q}}_p)$). Let $M$ be the centralizer in $G_{\mathbb{Q}_p}$ of the maximal $\mathbb{Q}_p$-split subtorus of $T$. Then $M$ is a $\mathbb{Q}_p$-Levi subgroup of $G_{\mathbb{Q}_p}$ containing $T$, and $\nu_b$ factors through the center of $M$. Up to conjugating $T$ and $b$ by an element of $G(\mathbb{Q}_p)$, we may assume that $M$ contains a maximal torus $T'$ that is the centralizer of (the generic fiber of) a maximal $\mathbb{Z}_p$-split torus in $\mathcal{G}$. In particular, there is a reductive model $\mathcal{M}$ of $M$ over $\mathbb{Z}_p$ such that the embedding $M \hookrightarrow G_{\mathbb{Q}_p}$ extends to $\mathcal{M} \hookrightarrow \mathcal{G}$. Without loss of generality, we may also assume that $\mu$ is a cocharacter of $T'$ defined over $E_{\mathfrak{p}} = \mathbb{Q}_{p^r}$.

Let $\Omega$ (resp. $\Omega_M$) denote the absolute Weyl group of $G$ (resp. $M$). The Cartan decompositions give rise to maps

$$c_{\mathcal{G}} : G(\breve{\mathbb{Q}}_p) \longrightarrow \mathcal{G}(\breve{\mathbb{Z}}_p)\backslash G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p) \xrightarrow{\sim} \Omega\backslash X_*(T'),$$
$$c_{\mathcal{M}} : M(\breve{\mathbb{Q}}_p) \longrightarrow \mathcal{M}(\breve{\mathbb{Z}}_p)\backslash M(\breve{\mathbb{Q}}_p)/\mathcal{M}(\breve{\mathbb{Z}}_p) \xrightarrow{\sim} \Omega_M\backslash X_*(T').$$

These maps lift the Kottwitz homomorphisms $w_G$ and $w_M$ respectively, cf. Corollary 1.3.15.

Now by the assumption that $X_{-\mu}(b) \neq \emptyset$, there exists $x \in G(\breve{\mathbb{Q}}_p)$ such that

$$c_{\mathcal{G}}(x^{-1} b \sigma(x)) = \Omega \cdot (-\mu).$$

From this, it is shown on p. 178 of [LR87] that there exists $m \in M(\breve{\mathbb{Q}}_p)$ such that

$$c_{\mathcal{M}}(m^{-1} b \sigma(m)) = \Omega_M \cdot (-\mu). \tag{3.3.6.1}$$

(The argument uses the Iwasawa decomposition and the fact that $\mu$ is minuscule, cf. also the proof of [Kis17, Lem. 2.2.2].)

We take the desired $\mu_T$ to be any element of $X_*(T)$ that is conjugate to $\mu \in X_*(T')$ by $M(\overline{\mathbb{Q}}_p)$. It remains to check that $\nu_b = -\overline{\mu_T}^T$. Note that $\overline{\mu_T}^T$ factors through the maximal $\mathbb{Q}_p$-split subtorus of $T$, and is therefore central in $M$. We have seen that $\nu_b$ is also central in $M$. Hence in order to check $\nu_b = -\overline{\mu_T}^T$, it suffices to check that $\nu_b$ and $-\overline{\mu_T}^T$ have equal image in $\pi_1(M)_{\mathbb{Q}} := \pi_1(M) \otimes_{\mathbb{Z}} \mathbb{Q}$. Without loss of generality, we may replace $b$ by a $\sigma$-conjugate in $T(\breve{\mathbb{Q}}_p)$, and assume that $b$ is decent in $T(\breve{\mathbb{Q}}_p)$ (see §1.4.1). Then for sufficiently divisible $n$ we have

$$b\sigma(b)\cdots\sigma^{n-1}(b) = p^{n\nu_b}.$$

For each $\lambda \in X_*(T') \otimes \mathbb{Q}$, we denote its image in $\pi_1(M)_{\mathbb{Q}}$ by $[\lambda]$. We compute

$$\begin{aligned}[n\nu_b] = w_M(p^{n\nu_b}) &= w_M(b\sigma(b)\cdots\sigma^{n-1}(b)) \\ &= w_M(m) - w_M(\sigma^n(m)) - [\mu + \sigma(\mu) + \cdots + \sigma^{n-1}(\mu)],\end{aligned}$$

where the last equality follows from (3.3.6.1). (Note that the action of $\sigma$ on $X_*(T')$ is indeed well defined as $T'$ is unramified.) We can choose $n$ divisible enough such that the coset $m\mathcal{M}(\breve{\mathbb{Z}}_p) \in M(\breve{\mathbb{Q}}_p)/\mathcal{M}(\breve{\mathbb{Z}}_p)$ is fixed by $\sigma^n$ (see Lemma 1.6.8), and such that $T'$ splits over $\mathbb{Q}_{p^n}$. Then the above relation becomes

$$[\nu_b] = -[\bar{\mu}^{T'}].$$

Finally, $[\bar{\mu}^{T'}]$ is equal to the image of $\overline{\mu_T}^T$ in $\pi_1(M)_{\mathbb{Q}}$. This is because $\mu$ and $\mu_T$ have the same image in $\pi_1(M)$, and the Galois actions on both $X_*(T)$ and $X_*(T')$ are compatible with that on $\pi_1(M)$. Thus we conclude that $\nu_b$ and $-\overline{\mu_T}^T$ have the same image in $\pi_1(M)_{\mathbb{Q}}$, as desired. □

**Lemma 3.3.7.** *Let $T \subset G$ be a maximal torus over $\mathbb{Q}$ such that $T_{\mathbb{R}}$ is elliptic in $G_{\mathbb{R}}$. Let $i$ denote the inclusion $T \hookrightarrow G$. Let $\mu_T \in X_*(T)$ be such that $i \circ \mu_T$ lies in $\mathfrak{p}_X(\overline{\mathbb{Q}})$. Then there exist $u \in G(\overline{\mathbb{Q}})$ and an $\mathbb{R}$-homomorphism $h : \mathbb{S} \to T_{\mathbb{R}}$, satisfying the following conditions.*

(i) *$i' := \mathrm{Int}(u) \circ i : T_{\overline{\mathbb{Q}}} \to G_{\overline{\mathbb{Q}}}$ is defined over $\mathbb{Q}$.*
(ii) *$\mu_T = \mu_h$.*
(iii) *$i' \circ h \in X$.*

*Proof.* This is proved by Langlands–Rapoport, when they prove [LR87, Lem. 5.12]. In fact, in that lemma $\mu_T$ is assumed to be of the form $\omega\mu_{h_0}$, where $\omega$ is an element of the absolute Weyl group of $(G, T)$, and $h_0$ is an $\mathbb{R}$-homomorphism $\mathbb{S} \to T_{\mathbb{R}}$ such that $i \circ h_0 \in X$. We explain why our hypothesis implies that setting. Since $T_{\mathbb{R}}$ is elliptic in $G_{\mathbb{R}}$, there indeed exists an $\mathbb{R}$-homomorphism $h_0 : \mathbb{S} \to T_{\mathbb{R}}$ such that $i \circ h_0 \in X$. Then $i \circ \mu_{h_0}$ and $i \circ \mu_T$ are conjugate by $G(\mathbb{C})$, and it follows that $\mu = \omega\mu_{h_0}$ for some $\omega$ in the absolute Weyl group. □

**3.3.8.** Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be an admissible morphism. Let $T \subset I_\phi$ be a maximal torus over $\mathbb{Q}$. Since $(\phi(p) \circ \zeta_p)^\Delta$ is a central fractional cocharacter of $I_{\phi,\overline{\mathbb{Q}}_p}$, it can be viewed as an element of $X_*(T) \otimes \mathbb{Q}$. We say that a cocharacter $\mu_T \in X_*(T)$ is *$\phi$-admissible*, if the composition $\mathbb{G}_{m,\overline{\mathbb{Q}}} \xrightarrow{\mu_T} T_{\overline{\mathbb{Q}}} \hookrightarrow I_{\phi,\overline{\mathbb{Q}}} \hookrightarrow G_{\overline{\mathbb{Q}}}$ is a cocharacter in $\mathfrak{p}_X(\overline{\mathbb{Q}})$, and if $\overline{\mu_T}^{T_{\mathbb{Q}_p}} = (\phi(p) \circ \zeta_p)^\Delta$ as elements of $X_*(T) \otimes \mathbb{Q}$.

**Theorem 3.3.9.** *Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be an admissible morphism, and let $T \subset I_\phi$ be a maximal torus over $\mathbb{Q}$. The following statements hold.*

(i) *There exists $\mu_T \in X_*(T)$ that is $\phi$-admissible in the sense of §3.3.8.*
(ii) *Let $\mu_T \in X_*(T)$ be as in (i). Then there exists a special point datum of the form $(T, i, h) \in \mathcal{SPD}(G, X)$, satisfying the following conditions:*
  (a) *We have $\mu_T = \mu_h$.*

(b) *There exists* $g \in G(\overline{\mathbb{Q}})$ *such that* $\mathrm{Int}(g) \circ \phi = \phi(T, i, h)$, *and such that the embedding* $i : T \to G$ *equals the composition*

$$T_{\overline{\mathbb{Q}}} \hookrightarrow I_{\phi,\overline{\mathbb{Q}}} \xrightarrow{\mathrm{Int}(g)} I_{\phi(T,i,h),\overline{\mathbb{Q}}} \hookrightarrow G_{\overline{\mathbb{Q}}}.$$

*(Here the first two maps are defined over* $\mathbb{Q}$, *and the second map is an isomorphism.)*

*Proof.* This essentially follows from the proof of [LR87, Satz 5.3]. We reproduce the argument for the convenience of the reader, and we remove the assumption in *loc. cit.* that $G_{\mathrm{der}}$ is simply connected.

**(i)** By Proposition 2.6.9, we may assume that $T$ is a $G$-rational maximal torus in $I_\phi$. By Lemma 2.6.6, $\phi$ factors through $\mathfrak{G}_T \subset \mathfrak{G}_G$. Hence the $G(\mathbb{Q}_p^{\mathrm{ur}})$-torsor $\mathcal{UR}(\phi(p) \circ \zeta_p)$ contains elements of $T(\overline{\mathbb{Q}}_p)$. We choose $t \in \mathcal{UR}(\phi(p) \circ \zeta_p) \cap T(\overline{\mathbb{Q}}_p)$, and let $b = b_{\mathrm{Int}(t^{-1}) \circ \phi(p) \circ \zeta_p} \in G(\mathbb{Q}_p^{\mathrm{ur}})$ (see Definition 2.2.5). Then $b \in T(\mathbb{Q}_p^{\mathrm{ur}})$. Since $\phi$ is admissible, we have $X_{-\mu}(b) \neq \emptyset$ for $\mu \in \mathfrak{u}_X^{\mathcal{G}}$. Hence by Lemma 3.3.6 we find $\mu_T \in X_*(T)$ such that $\overline{\mu_T}^{T_{\mathbb{Q}_p}} = -\nu_b$. Finally, by Proposition 2.2.6 (i), we have $-\nu_b = (\mathrm{Int}(t)^{-1} \circ \phi(p) \circ \zeta_p)^\Delta$, which equals $(\phi(p) \circ \zeta_p)^\Delta$. Hence $\mu_T$ is $\phi$-admissible.

**(ii)** Again by Proposition 2.6.9, we may assume that $T$ is a $G$-rational maximal torus in $I_\phi$. We denote by $i_0$ the inclusion $T_{\overline{\mathbb{Q}}} \hookrightarrow I_{\phi,\overline{\mathbb{Q}}} \hookrightarrow G_{\overline{\mathbb{Q}}}$, which is defined over $\mathbb{Q}$. By Lemma 2.6.7, $i_0(T_{\mathbb{R}})$ is an elliptic maximal torus in $G_{\mathbb{R}}$. Applying Lemma 3.3.7, we find $u \in G(\overline{\mathbb{Q}})$ and an $\mathbb{R}$-homomorphism $h : \mathbb{S} \to T_{\mathbb{R}}$ such that $\mathrm{Int}(u) \circ i_0 : T \to G$ is still defined over $\mathbb{Q}$, such that $\mu_T = \mu_h$, and such that $\mathrm{Int}(u) \circ i_0 \circ h \in X$.

We may replace $\phi$ and $i_0$ by $\mathrm{Int}(u) \circ \phi$ and $\mathrm{Int}(u) \circ i_0$ respectively, and assume that $u = 1$. This does not change the property that $T$ is $G$-rational, and we now have $(T, i_0, h) \in \mathcal{SPD}(G, X)$ such that $\mu_T = \mu_h$.

By Lemma 2.6.6, we may factorize $\phi$ uniquely as $\phi = i_0 \circ \phi_T$, where $\phi_T$ is a morphism $\mathfrak{Q} \to \mathfrak{G}_T$. Analogously, $\phi(T, i_0, h)$ factors as $i_0 \circ \Psi_{T,\mu_T}$ by its definition. We claim that $\phi_T^\Delta = \Psi_{T,\mu_T}^\Delta$, or equivalently $\phi^\Delta = \phi(T, i_0, h)^\Delta$. In fact, the $\phi$-admissibility of $\mu_T$ implies that

$$(\phi_T(p) \circ \zeta_p)^\Delta = (\Psi_{T,\mu_T}(p) \circ \zeta_p)^\Delta \tag{3.3.9.1}$$

(see [Kis17, (3.1.11)]). Also, since $\phi$ and $\phi(T, i_0, h)$ are both admissible (the former by assumption and the latter by the "if" direction in Theorem 3.3.3), we have

$$(\phi_T(\infty) \circ \zeta_\infty)^\Delta = (\Psi_{T,\mu_T}(\infty) \circ \zeta_\infty)^\Delta \tag{3.3.9.2}$$

by Lemma 3.1.9. Since $\phi_T^\Delta$ and $\Psi_{T,\mu_T}^\Delta$ are both homomorphisms $\mathfrak{Q}^\Delta \to T_{\overline{\mathbb{Q}}}$ defined over $\mathbb{Q}$ (for instance by Lemma 2.6.6), the above relations (3.3.9.1) and (3.3.9.2) imply that $\phi_T^\Delta = \Psi_{T,\mu_T}^\Delta$ as desired (cf. the discussion on the tori $Q^L$ in the proof of Corollary 3.1.10).

By the claim above and by Lemma 2.1.16, we have $\phi_T = a\Psi_{T,\mu_T}$, for some element $a \in Z^1(\mathbb{Q}, I_{\Psi_{T,\mu_T}}) = Z^1(\mathbb{Q}, T)$. By the admissibility of $\phi$ and $\phi(T, i_0, h)$ and by Proposition 2.6.12, the image of $a$ under

$$Z^1(\mathbb{Q}, T) \to \mathbf{H}^1(\mathbb{Q}, T) \to \mathbf{H}^1(\mathbb{Q}, I_{\phi(T,i_0,h)})$$

lies in $Ш_G^\infty(\mathbb{Q}, I_{\phi(T,i_0,h)})$. By Lemma 2.6.7 and [Kis17, Lem. 4.4.5], the map $\mathbf{H}^1(\mathbb{R}, T) \to \mathbf{H}^1(\mathbb{R}, I_{\phi(T,i_0,h)})$ has trivial kernel. Thus the class of $a$ in $\mathbf{H}^1(\mathbb{Q}, T)$

lies in the kernel of the map $Ш^\infty(\mathbb{Q}, T) \to Ш^\infty(\mathbb{Q}, G)$ induced by $i_0 : T \to G$. It follows that there exists $g \in G(\overline{\mathbb{Q}})$, satisfying:

$$i_0(a_\tau) = g^{-1}\,{}^\tau g, \quad \forall \tau \in \Gamma; \tag{3.3.9.3}$$

$$g \in G(\mathbb{R}) \cdot i_0(T(\mathbb{C})) \subset G(\mathbb{C}). \tag{3.3.9.4}$$

For each $\tau \in \Gamma$, we write $\phi(T, i_0, h)(q_\tau) = g_\tau \rtimes \tau$, and $\phi(q_\tau) = g'_\tau \rtimes \tau$. (Here $q_\tau \in \mathfrak{Q}$ is a lift of $\tau$, as always). Because $i_0(a_\tau)$ and $g_\tau$ both lie in $i_0(T(\overline{\mathbb{Q}}))$ and commute with each other, we have $g'_\tau = i_0(a_\tau)g_\tau = g_\tau i_0(a_\tau) = g_\tau g^{-1}\,{}^\tau g$. Then we have

$$(\mathrm{Int}(g) \circ \phi)(q_\tau) = g g'_\tau\,{}^\tau g^{-1} \rtimes \tau = g g_\tau g^{-1}. \tag{3.3.9.5}$$

Now let $i := \mathrm{Int}(g) \circ i_0$. By (3.3.9.3), $i$ is a $\mathbb{Q}$-embedding $T \to G$. By (3.3.9.4), $i \circ h \in X$. In particular, $(T, i, h) \in \mathcal{SPD}(G, X)$. By (3.3.9.5), we have $(\mathrm{Int}(g) \circ \phi)(q_\tau) = \phi(T, i, h)(q_\tau)$. Since we also have $(\mathrm{Int}(g) \circ \phi)^\Delta = \phi(T, i, h)^\Delta$ (because $\phi_T^\Delta = \Psi_{T,\mu_T}^\Delta$), we have $\mathrm{Int}(g) \circ \phi = \phi(T, i, h)$. Thus $(T, i, h)$ and $g$ are the desired elements. $\square$

3.4. **Admissible stable conjugacy classes.** The goal of this subsection is to construct certain elements of the image of $[\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)] \subset [\mathcal{LRP}^{\mathrm{gg}}]$ under the map $\mathfrak{scc} : [\mathcal{LRP}^{\mathrm{gg}}] \to \Sigma_{\mathbb{R}\text{-ell}}(G)$ in Definition 3.2.12. For $M$ a reductive group over $\mathbb{Q}_p$, we write $w_M : M(\breve{\mathbb{Q}}_p) \to \pi_1(M)_{\Gamma_{p,0}}$ for the Kottwitz homomorphism, as in §1.4.2. For each $k \in \mathbb{Z}_{\geq 1}$, we denote by $\mathrm{B}^{(k)}(M)$ the set of $\sigma^k$-conjugacy classes in $M(\breve{\mathbb{Q}}_p)$.

**Lemma 3.4.1.** *Let $M$ be an unramified reductive group over $\mathbb{Q}_p$. Let $x \in M(\breve{\mathbb{Q}}_p)$ be in the kernel of $w_M$. For all $k \in \mathbb{Z}_{\geq 1}$, the class of $x$ in $\mathrm{B}^{(k)}(M)$ is in the image of the natural map $\mathrm{B}^{(k)}(M_{\mathrm{sc}}) \to \mathrm{B}^{(k)}(M)$.*

*Proof.* Write $\tau := \sigma^k$. We first assume that $M_{\mathrm{der}} = M_{\mathrm{sc}}$. Consider $M^{\mathrm{ab}} = M/M_{\mathrm{der}}$, which is an unramified torus over $\mathbb{Q}_p$. The image $\bar{x}$ of $x$ in $M^{\mathrm{ab}}(\breve{\mathbb{Q}}_p)$ is in the kernel of $w_{M^{\mathrm{ab}}}$, and this kernel is the unique parahoric subgroup of $M^{\mathrm{ab}}(\breve{\mathbb{Q}}_p)$ (which is hyperspecial in this case). By Greenberg's theorem [Gre63, Prop. 3], we have $\bar{x} = \bar{c} \cdot {}^\tau\bar{c}^{-1}$ for some $\bar{c} \in M^{\mathrm{ab}}(\breve{\mathbb{Q}}_p)$. Let $c \in M(\breve{\mathbb{Q}}_p)$ be a lift of $\bar{c}$, which exists because $\mathbf{H}^1(\breve{\mathbb{Q}}_p, M_{\mathrm{der}})$ is trivial by Steinberg's theorem. Then $c^{-1}x\,{}^\tau c \in M_{\mathrm{der}}(\breve{\mathbb{Q}}_p)$, which means that the class of $x$ in $\mathrm{B}^{(k)}(M)$ comes from $\mathrm{B}^{(k)}(M_{\mathrm{der}}) = \mathrm{B}^{(k)}(M_{\mathrm{sc}})$.

In the general case, as in [Kot84a, §3] we take a $z$-extension

$$1 \longrightarrow Z \longrightarrow H \longrightarrow M \longrightarrow 1$$

over $\mathbb{Q}_p$, where $H$ is an unramified reductive group with simply connected derived subgroup, and $Z$ is an unramified induced torus contained in the center of $H$. Then we have a commutative diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & Z(\breve{\mathbb{Q}}_p) \arrow[r] \arrow[d, "w_Z"] & H(\breve{\mathbb{Q}}_p) \arrow[r] \arrow[d, "w_H"] & M(\breve{\mathbb{Q}}_p) \arrow[r] \arrow[d, "w_M"] & 1 \\
1 \arrow[r] & \pi_1(Z) \arrow[r] & \pi_1(H) \arrow[r] & \pi_1(M) \arrow[r] & 1
\end{tikzcd}$$

By [Kot97, §7], the vertical arrows in the above diagram are surjective. Since $w_M(x) = 0$, there exists $h \in \ker(w_H) \subset H(\breve{\mathbb{Q}}_p)$ that maps to $x \in M(\breve{\mathbb{Q}}_p)$. Applying the first part of the proof to $H$, we know that the class of $h$ in $\mathrm{B}^{(k)}(H)$ comes from $\mathrm{B}^{(k)}(H_{\mathrm{sc}})$. But the composite map $H_{\mathrm{sc}} \to H \to M$ factors through $M_{\mathrm{sc}}$. The lemma follows. $\square$

**Lemma 3.4.2.** *Let $M$ be an unramified reductive group over $\mathbb{Q}_p$. Let $T \subset M$ be an elliptic maximal torus. Let $x \in T(\mathbb{Q}_p)$ be such that $w_M(x) = 0 \in \pi_1(M)$. Then some integer power of $x$ lies in a compact subgroup of $T(\mathbb{Q}_p)$.*

*Proof.* Let $\mathcal{T}^\circ$ be the connected Néron model of $T$ over $\mathbb{Z}_p$. Then the kernel of $w_T : T(\breve{\mathbb{Q}}_p) \to X_*(T)_{\Gamma_{p,0}}$ is $\mathcal{T}^\circ(\breve{\mathbb{Z}}_p)$; see [Rap05, Rmk. 2.2 (iii)]. If $w_T(x^k) = 0$ for some integer $k$, then $x^k$ lies in $\mathcal{T}^\circ(\breve{\mathbb{Z}}_p) \cap T(\mathbb{Q}_p) = \mathcal{T}^\circ(\mathbb{Z}_p)$, which is a compact subgroup of $T(\mathbb{Q}_p)$. Hence it remains to show that $w_T(x^k) = 0$ for some $k$. We know that $w_T$ maps $T(\mathbb{Q}_p) \subset T(\breve{\mathbb{Q}}_p)$ into the group of $\sigma$-invariants $(X_*(T)_{\Gamma_{p,0}})^\sigma$. It remains to show that the natural map

$$(X_*(T)_{\Gamma_{p,0}})^\sigma \hookrightarrow X_*(T)_{\Gamma_{p,0}} \to \pi_1(M)_{\Gamma_{p,0}} = \pi_1(M)$$

has torsion kernel. For this, let $A$ be the maximal $\mathbb{Q}_p$-split subtorus of $Z_M$. Since $T$ is elliptic in $M$, $A$ is also the maximal $\mathbb{Q}_p$-split subtorus of $T$. In particular $X_*(A)$ is identified with $X_*(T)^{\Gamma_p}$. The inclusion map $X_*(A) \cong X_*(T)^{\Gamma_p} \hookrightarrow X_*(T)$ induces an isomorphism

$$X_*(A) \otimes \mathbb{Q} \xrightarrow{\sim} (X_*(T)_{\Gamma_{p,0}})^\sigma \otimes \mathbb{Q}.$$

(The inverse map is induced by taking average over $\Gamma_{p,0}$-orbits in $X_*(T)$.) Thus we reduce to showing that the natural map $X_*(A) \otimes \mathbb{Q} \to \pi_1(M) \otimes \mathbb{Q}$ is injective, but this is clear. □

**Definition 3.4.3.** For $\epsilon \in G(\mathbb{Q}_p)_{\mathrm{ss}}$, we let $M(\epsilon)$ be the Levi subgroup of $G_{\mathbb{Q}_p}$ that is the centralizer of the maximal $\mathbb{Q}_p$-split subtorus of the center of $G_\epsilon^0$. (Equivalently, $M(\epsilon)$ is the smallest Levi subgroup of $G_{\mathbb{Q}_p}$ containing $G^0_{\mathbb{Q}_p,\epsilon}$.)

**Definition 3.4.4.** Let $\epsilon \in G(\mathbb{Q}_p)_{\mathrm{ss}}$, and let $M := M(\epsilon)$. Let $n = mr$ as in §3.1.5. We say that $\epsilon$ is *$p^n$-admissible*, if there exists a cocharacter $\mu_M$ of $M_{\mathbb{Q}_{p^n}}$ satisfying the following conditions.

- $\mu_M \in \mathbb{\mu}_X(\mathbb{Q}_{p^n})$. (Here $\mathbb{\mu}_X(\mathbb{Q}_{p^n})$ is well defined since $E_{\mathfrak{p}} = \mathbb{Q}_{p^r} \subset \mathbb{Q}_{p^n}$.)
- We have

$$w_M(\epsilon) = -\sum_{i=0}^{n-1} \sigma^i[\mu_M]^M \in \pi_1(M), \tag{3.4.4.1}$$

  where $[\mu_M]^M$ denotes the image of $\mu_M$ in $\pi_1(M)$.

The following lemma generalizes [LR87, Lem. 5.17] to the case where $G_{\mathrm{der}}$ is not necessarily simply connected.

**Lemma 3.4.5.** *The set of $p^n$-admissible elements in $G(\mathbb{Q}_p)_{\mathrm{ss}}$ is invariant under stable conjugacy over $\mathbb{Q}_p$.*

*Proof.* Evidently this set is invariant under $G(\mathbb{Q}_p)$-conjugacy. Now let $\epsilon, \epsilon' \in G(\mathbb{Q}_p)_{\mathrm{ss}}$ be stably conjugate and suppose that $\epsilon$ is $p^n$-admissible. We show that $\epsilon'$ is $p^n$-admissible. Let $M = M(\epsilon)$. Since $M$ is a Levi subgroup of $G_{\mathbb{Q}_p}$, the inclusion $M \subset G_{\mathbb{Q}_p}$ induces an injection $\mathbf{H}^1(\mathbb{Q}_p, M) \to \mathbf{H}^1(\mathbb{Q}_p, G)$, cf. [Hai09, §4.1]. Also note that $G_\epsilon^0 = M_\epsilon^0$. Hence we have a natural bijection $\mathfrak{D}(M_\epsilon^0, M; \mathbb{Q}_p) \xrightarrow{\sim} \mathfrak{D}(G_\epsilon^0, G; \mathbb{Q}_p)$. It follows that we have a natural surjection from the set of $M(\mathbb{Q}_p)$-conjugacy classes in the stable conjugacy class of $\epsilon$ in $M$ onto the set of $G(\mathbb{Q}_p)$-conjugacy classes in the stable conjugacy class of $\epsilon$ in $G_{\mathbb{Q}_p}$.[20] Conjugating $\epsilon'$ by an

[20]Recall that these two sets are mapped onto by $\mathfrak{D}(M_\epsilon^0, M; \mathbb{Q}_p)$ and $\mathfrak{D}(G_\epsilon^0, G; \mathbb{Q}_p)$ respectively.

element of $G(\mathbb{Q}_p)$ if necessary, we may assume that $\epsilon'$ lies in $M(\mathbb{Q}_p)$, and that $\epsilon'$ is stably conjugate to $\epsilon$ inside $M$. Under these assumptions, we have $M = M(\epsilon')$. To check that $\epsilon'$ is $p^n$-admissible, it suffices to show that $w_M(\epsilon) = w_M(\epsilon')$. For this, note that since $\epsilon$ and $\epsilon'$ are stably conjugate in $M$, they are conjugate in $M(\breve{\mathbb{Q}}_p)$ by Steinberg's theorem. The desired statement follows from the fact that $w_M$ is a group homomorphism from $M(\breve{\mathbb{Q}}_p)$ to an abelian group. □

By Lemma 3.4.5, we have a well-defined notion of $p^n$*-admissibility* for stable conjugacy classes in $G(\mathbb{Q}_p)_{\mathrm{ss}}$. The following result is a generalization of one direction in [LR87, Satz 5.21].

**Theorem 3.4.6.** *Let $\epsilon \in G(\mathbb{Q})_{\mathrm{ss}}$ represent a stable conjugacy class in $\Sigma_{\mathbb{R}\text{-ell}}(G)$ whose localization over $\mathbb{Q}_p$ is $p^n$-admissible. Then the stable conjugacy class of $\epsilon$ lies in the image of $[\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)]$ under the map $\mathfrak{scc} : [\mathcal{LRP}^{\mathrm{gg}}] \to \Sigma_{\mathbb{R}\text{-ell}}(G)$. (Here $n = mr$ and $p^n = q^m$.)*

*Proof.* By assumption there exists an $\mathbb{R}$-elliptic maximal torus in $(G^0_\epsilon)_{\mathbb{R}}$. By a theorem of Kneser [Kne65], there exists a $\mathbb{Q}_p$-elliptic maximal torus in $(G^0_\epsilon)_{\mathbb{Q}_p}$. It then follows from [LR87, Lem. 5.10] that there exists a maximal torus $T$ in $G^0_\epsilon$ such that $T_{\mathbb{Q}_v}$ is elliptic in $(G^0_\epsilon)_{\mathbb{Q}_v}$ for $v = \infty$ and $p$.

Let $M = M(\epsilon)$ (which is defined over $\mathbb{Q}_p$) be as in Definition 3.4.3, and let $\mu_M$ be as in Definition 3.4.4. Then $T_{\mathbb{Q}_p}$ is a maximal torus in $M$. Let $\mu_T \in X_*(T)$ be a conjugate of $\mu_M$ under $M(\overline{\mathbb{Q}}_p)$. By Lemma 3.3.7, we find $u \in G(\overline{\mathbb{Q}})$ such that $\mathrm{Int}(u) : T \to G$ is defined over $\mathbb{Q}$, and such that $\mathrm{Int}(u) \circ \mu_T = \mu_h$ for some $h \in X$. Note that $\epsilon$ is stably conjugate to $\mathrm{Int}(u)(\epsilon)$, because $u^{-1}{}^\tau u \in T(\overline{\mathbb{Q}}) \subset G^0_\epsilon(\overline{\mathbb{Q}})$ for all $\tau \in \Gamma$. Hence we may replace $\epsilon, T, M$ by $\mathrm{Int}(u)(\epsilon), \mathrm{Int}(u)(T), \mathrm{Int}(u)(M)$ respectively, and assume that $\mu_T = \mu_h$ for some $h \in X$ (which necessarily factors through $T_{\mathbb{R}}$).

We denote the inclusion $T \hookrightarrow G$ by $i$. Then we have $(T, i, h) \in \mathcal{SPD}(G, X)$. Let $\phi = \phi(T, i, h)$ (Definition 3.3.2). Then $\epsilon \in T(\mathbb{Q}) \subset I_\phi(\mathbb{Q})$, and we have $(\phi, \epsilon) \in \mathcal{LRP}_{\mathrm{sa}}$ by Theorem 3.3.3. Moreover, by Remark 3.3.4 we have $(\phi, \epsilon) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}$. It remains to show that $(\phi, \epsilon) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)$.

Since $M$ is an unramified reductive group over $\mathbb{Q}_p$, it contains an unramified *and* elliptic maximal torus $T'$ (see [LR87, p. 171] or [DeB06, §2.4]). Let $\mu' \in X_*(T')$ be a conjugate of $\mu_T$ and $\mu_M$ under $M(\overline{\mathbb{Q}}_p)$. Let $b' = \mu'(p^{-1}) \in T'(\mathbb{Q}_p^{\mathrm{ur}})$. Then one immediately checks the following properties:

(i) The element $b'$ is decent in $T'(\breve{\mathbb{Q}}_p)$.
(ii) We have $w_{T'}(b') = -\mu' \in X_*(T')$. In particular, we have

$$w_M(b') = -[\mu_M]^M \in \pi_1(M). \tag{3.4.6.1}$$

Since $\phi = \phi(T, i, h) = i \circ \Psi_{T,\mu_T}$, the $G(\mathbb{Q}_p^{\mathrm{ur}})$-torsor $\mathcal{UR}(\phi(p) \circ \zeta_p)$ contains the $T(\mathbb{Q}_p^{\mathrm{ur}})$-torsor $\mathcal{UR}(\Psi_{T,\mu_T}(p) \circ \zeta_p)$. Let $(b_0, \epsilon_0) \in \mathrm{cls}_p(\phi, \epsilon)$ be the element associated with some $g \in \mathcal{UR}(\Psi_{T,\mu_T}(p) \circ \zeta_p)$ (see §3.1.4). Then $\epsilon_0 = \epsilon$, and $b_0 \in T(\mathbb{Q}_p^{\mathrm{ur}})$. Moreover, by Lemma 2.2.10, the element $\kappa_T(b_0) \in X_*(T)_{\Gamma_p}$ equals the image of $-\mu_T$.

Since $T_{\mathbb{Q}_p}$ and $T'$ are both elliptic in $M$, we know that $b_0 \in T(\breve{\mathbb{Q}}_p)$ and $b' \in T'(\breve{\mathbb{Q}}_p)$ are both basic in $M(\breve{\mathbb{Q}}_p)$. By what we have seen about $w_{T'}(b')$ and $\kappa_T(b_0)$, we have $\kappa_M(b') = \kappa_M(b_0) \in \pi_1(M)_{\Gamma_p}$. It follows that $b'$ and $b_0$ represent the same (basic) class in $\mathrm{B}(M)$ (see [Kot85, Prop. 5.6]). Since $b_0$ and $b'$ are decent, there exists $s \in M(\mathbb{Q}_p^{\mathrm{ur}})$ such that $b' = sb_0\sigma(s)^{-1}$ (see §1.4.1). We then have $(b', s\epsilon s^{-1}) \in \mathrm{cls}_p(\phi, \epsilon)$.

We write $\epsilon'$ for $s\epsilon s^{-1}$, which is an element of $M(\mathbb{Q}_p^{\mathrm{ur}})$. It remains to show that $\epsilon'^{-1}(b' \rtimes \sigma)^n$ has a fixed point in $G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p)$. For each $k \in \mathbb{Z}_{\geq 1}$, let

$$U_k := (\epsilon')^{-k} \cdot b' \cdot \sigma(b') \cdots \sigma^{kn-1}(b') \in M(\mathbb{Q}_p^{\mathrm{ur}}),$$

so that

$$(\epsilon')^{-k}(b' \rtimes \sigma)^{kn} = U_k \rtimes \sigma^{kn}. \tag{3.4.6.2}$$

Since $\epsilon'$ is $M(\mathbb{Q}_p^{\mathrm{ur}})$-conjugate to $\epsilon$, we have $w_M(\epsilon') = w_M(\epsilon)$. Hence $w_M(U_k) = 0$ by (3.4.4.1) and (3.4.6.1). By Lemma 3.4.1, the class of $U_1$ in $\mathrm{B}^{(n)}(M)$ comes from $\mathrm{B}^{(n)}(M_{\mathrm{sc}})$. We claim that the class of $U_1$ in $\mathrm{B}^{(n)}(M)$ is basic with trivial Newton point. Given the claim, we use the fact that the only basic class in $\mathrm{B}^{(n)}(M_{\mathrm{sc}})$ is the trivial class ([Kot85, Prop. 5.4]) to deduce that $U_1 = c\sigma^n(c^{-1})$ for some $c \in M(\breve{\mathbb{Q}}_p)$. It follows that $\epsilon'^{-1}(b' \rtimes \sigma)^n = U_1 \rtimes \sigma^n$ has a fixed point in $G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p)$, namely $c\mathcal{G}(\breve{\mathbb{Z}}_p)$. The proof of the theorem will then be finished.

To prove the claim, it suffices to find $t \in \mathbb{Z}_{\geq 1}$ and $e \in M(\breve{\mathbb{Q}}_p)$ such that

$$U_1 \cdot \sigma^n(U_1) \cdot \sigma^{2n}(U_1) \cdots \sigma^{(t-1)n}(U_1) = e \cdot \sigma^{tn}(e^{-1}). \tag{3.4.6.3}$$

Since $\epsilon'$ commutes with $b' \rtimes \sigma$, it easily follows from (3.4.6.2) that the left hand side of (3.4.6.3) is equal to $U_t$. Fix an arbitrary reductive model $\mathcal{M}$ of $M$ over $\mathbb{Z}_p$. By Greenberg's theorem [Gre63, Prop. 3] (cf. [Kot84a, Lem. 1.4.9]), in order to find $t, e$ such that $U_t = e \cdot \sigma^{tn}(e^{-1})$ it suffices to find $t$ such that $U_t \in \mathcal{M}(\breve{\mathbb{Z}}_p)$.

Now for each $k \in \mathbb{Z}_{\geq 1}$ we have

$$U_k = (\epsilon')^{-k} p^{\lambda_k}, \tag{3.4.6.4}$$

where

$$\lambda_k := -\sum_{j=0}^{kn-1} \sigma^j(\mu').$$

Since $T'$ is unramified, for sufficiently divisible $k$ we have $\sigma^{kn}(\mu') = \mu'$. In this case $\lambda_k$ is defined over $\mathbb{Q}_p$, and in particular it is central in $M$ by the ellipticity of $T'$. Hence for sufficiently divisible $k$ we have

$$s^{-1}U_k s = \epsilon^{-k} p^{\lambda_k} \in T(\mathbb{Q}_p).$$

Since $w_M(s^{-1}U_k s) = w_M(U_k) = 0$, and since $T_{\mathbb{Q}_p}$ is elliptic in $M$, we apply Lemma 3.4.2 to conclude that some power of $s^{-1}U_k s$ lies in a compact subgroup of $T(\mathbb{Q}_p)$. In particular, for any given neighborhood $\mathcal{N}$ of 1 in $T(\mathbb{Q}_p)$, all sufficiently divisible powers of $s^{-1}U_k s$ lie in $\mathcal{N}$. Now observe that when $\lambda_k$ is defined over $\mathbb{Q}_p$, we have $\lambda_{kl} = l \cdot \lambda_k$ for all $l \in \mathbb{Z}_{\geq 1}$, and therefore

$$s^{-1}U_{kl}s = (s^{-1}U_k s)^l, \quad \forall l \in \mathbb{Z}_{\geq 1}.$$

We conclude that for $\mathcal{N}$ as above, we have $s^{-1}U_k s \in \mathcal{N}$ for all sufficiently divisible $k$. If we take $\mathcal{N}$ to be $(s^{-1}\mathcal{M}(\breve{\mathbb{Z}}_p)s) \cap T(\mathbb{Q}_p)$, then we see that $U_k \in \mathcal{M}(\breve{\mathbb{Z}}_p)$ for all sufficiently divisible $k$, as desired. □

For a quasi-split reductive group $M$ over $\mathbb{Q}_p$, recall that the *degree $n$ norm* is a map from the set of $\sigma$-conjugacy classes in $M(\mathbb{Q}_{p^n})$ to the set of stable conjugacy classes in $M(\mathbb{Q}_p)$; see [Kot82, §5].

**Lemma 3.4.7.** *Let $\epsilon \in G(\mathbb{Q}_p)_{\mathrm{ss}}$, and let $M = M(\epsilon)$. Assume that there exists a cocharacter $\mu_M$ of $M_{\mathbb{Q}_{p^n}}$ satisfying the first condition in Definition 3.4.4 and such that $\epsilon$ is a degree $n$ norm with respect to $M$ of some $\delta \in M(\mathbb{Q}_{p^n})$ satisfying $w_M(\delta) = -[\mu_M]^M$. Then $\epsilon$ is $p^n$-admissible.*

*Proof.* It suffices to check (3.4.4.1), for the given $\mu_M$. First assume that $M_{\mathrm{der}}$ is simply connected. Then we have $\pi_1(M) = \pi_1(M^{\mathrm{ab}})$. Let $\bar{\epsilon}$ and $\bar{\delta}$ be the images of $\epsilon$ and $\delta$ in $M^{\mathrm{ab}}(\mathbb{Q}_p)$ and $M^{\mathrm{ab}}(\mathbb{Q}_{p^n})$ respectively. Then we have $\bar{\epsilon} = \bar{\delta}\sigma(\bar{\delta}) \cdots \sigma^{n-1}(\bar{\delta})$. It follows that

$$w_M(\epsilon) = w_{M^{\mathrm{ab}}}(\bar{\epsilon}) = \sum_{i=0}^{n-1} \sigma^i(w_{M^{\mathrm{ab}}}(\bar{\delta})) = \sum_{i=0}^{n-1} \sigma^i(w_M(\delta)), \tag{3.4.7.1}$$

which gives the desired (3.4.4.1).

In the general case, we take an unramified $z$-extension $1 \to Z \to H \to M \to 1$ as in the proof of Lemma 3.4.1. Let $\widetilde{\delta} \in H(\mathbb{Q}_{p^n})$ be a lift of $\delta$, and let $\widetilde{\epsilon} \in H(\mathbb{Q}_p)$ be a degree $n$ norm of $\widetilde{\delta}$. By the identity (3.4.7.1) applied to $H, \widetilde{\epsilon}, \widetilde{\delta}$, we have

$$w_H(\widetilde{\epsilon}) = \sum_{i=0}^{n-1} \sigma^i(w_H(\widetilde{\delta})). \tag{3.4.7.2}$$

Now the image of $\widetilde{\epsilon}$ in $M(\mathbb{Q}_p)$ is stably conjugate to $\epsilon$ over $\mathbb{Q}_p$, and therefore $M(\breve{\mathbb{Q}}_p)$-conjugate to $\epsilon$. Hence the image of $w_H(\widetilde{\epsilon}) \in \pi_1(H)$ under $\pi_1(H) \to \pi_1(M)$ equals $w_M(\epsilon)$. Obviously the image of $w_H(\widetilde{\delta}) \in \pi_1(H)$ under $\pi_1(H) \to \pi_1(M)$ equals $w_M(\delta)$. Hence from (3.4.7.2) we get

$$w_M(\epsilon) = \sum_{i=0}^{n-1} \sigma^i(w_M(\delta)),$$

which gives the desired (3.4.4.1). □

In the next proposition, let the function $\phi_n : G(\mathbb{Q}_{p^n}) \to \{0, 1\}$ be as in §1.8.2. Thus $\phi_n$ is the characteristic function of the double coset $\mathcal{G}(\mathbb{Z}_{p^n})p^{-\mu}\mathcal{G}(\mathbb{Z}_{p^n})$ for arbitrary $\mu \in \mathfrak{y}_X^{\mathcal{G}}$, in the notation of §2.4.1.

**Proposition 3.4.8.** *Let $\epsilon \in G(\mathbb{Q})_{\mathrm{ss}}$ represent a stable conjugacy class in $\Sigma_{\mathbb{R}\text{-ell}}(G)$. Assume that $\epsilon \in G(\mathbb{Q}_p)$ is a degree $n$ norm of some $\delta \in G(\mathbb{Q}_{p^n})$ whose $\sigma$-conjugacy class in $G(\mathbb{Q}_{p^n})$ intersects non-trivially with the support of $\phi_n$. Then the stable conjugacy class of $\epsilon$ is in the image of $[\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)]$ under $\mathfrak{scc}$.*

*Proof.* Let $\epsilon'$ be a $G(\mathbb{Q}_p)$-conjugate of $\epsilon$ such that $M(\epsilon')$ contains the generic fiber of a maximal $\mathbb{Z}_p$-split torus $\mathcal{A}$ in $\mathcal{G}$. By Theorem 3.4.6, we only need to check that $\epsilon'$ is $p^n$-admissible.

Write $M$ for $M(\epsilon')$. Since $M \supset \mathcal{A}_{\mathbb{Q}_p}$, we have a reductive model $\mathcal{M}$ of $M$ over $\mathbb{Z}_p$ such that the embedding $M \hookrightarrow G_{\mathbb{Q}_p}$ extends to $\mathcal{M} \hookrightarrow \mathcal{G}$. Let $\mathcal{S}$ be a maximal $\mathbb{Z}_{p^n}$-split torus in $\mathcal{G}_{\mathbb{Z}_{p^n}}$ such that $\mathcal{S}$ contains $\mathcal{A}_{\mathbb{Z}_{p^n}}$. In particular, $S := \mathcal{S}_{\mathbb{Q}_{p^n}}$ is a maximal $\mathbb{Q}_{p^n}$-split torus in $G_{\mathbb{Q}_{p^n}}$, and it is contained in $M_{\mathbb{Q}_{p^n}}$. There exists $\mu' \in X_*(S)$ that is $\mathcal{G}(\mathbb{Z}_{p^n})$-conjugate to some $\mu \in \mathfrak{y}_X^{\mathcal{G}}$. We fix such a $\mu'$. Note that $\phi_n$ is the characteristic function of $\mathcal{G}(\mathbb{Z}_{p^n})p^{-\mu'}\mathcal{G}(\mathbb{Z}_{p^n})$.

Denote by $s_M(\phi_n)$ the function in the unramified Hecke algebra of $M(\mathbb{Q}_{p^n})$ with respect to $\mathcal{M}(\mathbb{Z}_{p^n})$, obtained from $\phi_n$ via the partial Satake transform (a.k.a. normalized constant term). Now [Hai09, Lem. 4.2.1] implies that there exists $\delta_M \in$

$M(\mathbb{Q}_{p^n})$ which is $\sigma$-conjugate to $\delta$ in $G(\mathbb{Q}_{p^n})$ and whose degree $n$ norm with respect to $M$ is the stable conjugacy class of $\epsilon$ in $M(\mathbb{Q}_p)$. By the descent formula [Hai09, (4.4.4)], the $\sigma$-conjugacy class of $\delta_M$ in $M(\mathbb{Q}_{p^n})$ intersects non-trivially with the support of $s_M(\phi_n)$. On the other hand, one deduces from the computation of Satake transform on p. 297 of [Kot84a] that $s_M(\phi_n)$ is a non-negative linear combination of the characteristic functions of the double cosets $\mathcal{M}(\mathbb{Z}_{p^n})p^{-u\mu'}\mathcal{M}(\mathbb{Z}_{p^n})$, where $u$ runs over the $\mathbb{Q}_{p^n}$-rational Weyl group of $M$ in $G$ (acting on $S$). Hence we may assume that $\delta_M$ lies in $\mathcal{M}(\mathbb{Z}_{p^n})p^{-u\mu'}\mathcal{M}(\mathbb{Z}_{p^n})$, for some $u$ as above. Let $\mu_M := u\mu'$. Then $\mu_M$ satisfies the first condition in Definition 3.4.4, and $w_M(\delta_M) = -[\mu_M]^M$. Since the stable conjugacy class of $\epsilon'$ in $M(\mathbb{Q}_p)$ is the degree $n$ norm of $\delta_M$ with respect to $M$, we apply Lemma 3.4.7 to conclude that $\epsilon'$ is $p^n$-admissible. □

## 3.5. **Constructing Kottwitz parameters.**

**3.5.1.** Let $(\phi, \epsilon) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}$, and let $\tau \in I^{\mathrm{ad}}_\phi(\mathbb{A}^p_f)$ be an arbitrary element. Our first goal is to assign to the triple $(\phi, \epsilon, \tau)$ a Kottwitz parameter (see Definition 1.6.4)

$$\mathbf{t}(\phi, \epsilon, \tau) \in \mathfrak{KP}.$$

Let $\gamma_0 := \epsilon$, and $I_0 := G^0_{\gamma_0}$. The Kottwitz parameter $\mathbf{t}(\phi, \epsilon, \tau)$ shall be of the form $(\gamma_0, a, [b])$. Note that by the first condition in Definition 3.2.1, $\gamma_0$ is indeed a semi-simple and $\mathbb{R}$-elliptic element of $G(\mathbb{Q})$, meeting the requirement in Definition 1.6.4.

We construct $a \in \mathfrak{D}(I_0, G; \mathbb{A}^p_f)$. Fix a lift $\widetilde{\tau} \in I_\phi(\bar{\mathbb{A}}^p_f)$ of $\tau$, and let $\zeta^{p,\infty}_\phi : \Gamma \to G(\bar{\mathbb{A}}^p_f)$ be the cocycle as in §2.4.6. Since $(\phi, \epsilon)$ is gg, $\zeta^{p,\infty}_\phi$ takes values in $I_0(\bar{\mathbb{A}}^p_f)$. For each $\rho \in \Gamma$, let $t_\rho := \widetilde{\tau}^{-1} \cdot {}^\rho\widetilde{\tau} \in Z_{I_\phi}(\bar{\mathbb{A}}^p_f)$. (Here ${}^\rho\widetilde{\tau}$ is defined with respect to the $\mathbb{Q}$-structure of $I_\phi$.) Then $(t_\rho)_\rho$ is a continuous cocycle $\Gamma \to Z_{I_\phi}(\bar{\mathbb{A}}^p_f)$. Consider the map

$$\begin{aligned} A : \Gamma &\longrightarrow G(\bar{\mathbb{A}}^p_f) \\ \rho &\longmapsto t_\rho \zeta^{p,\infty}_\phi(\rho). \end{aligned}$$

Using the fact that the natural map $Z_{I_\phi} \to G$ is defined over $\mathbb{Q}$ and factors through $I_0$, we know that $A$ is a continuous cocycle $\Gamma \to I_0(\bar{\mathbb{A}}^p_f)$. We claim that $A$ has trivial image in $\mathbf{H}^1(\Gamma, G(\bar{\mathbb{A}}^p_f))$. In fact, if we denote by $\widetilde{\tau}_G$ the image of $\widetilde{\tau}$ under the canonical embedding $I_\phi(\bar{\mathbb{A}}_f) \to G(\bar{\mathbb{A}}_f)$, then

$$t_\rho = \widetilde{\tau}_G^{-1} \cdot \zeta^{p,\infty}_\phi(\rho) \cdot {}^\rho\widetilde{\tau}_G \cdot \zeta^{p,\infty}_\phi(\rho)^{-1}.$$

(Here ${}^\rho\widetilde{\tau}_G$ is defined with respect to the $\mathbb{Q}$-structure of $G$.) Hence

$$A(\rho) = \widetilde{\tau}_G^{-1} \cdot \zeta^{p,\infty}_\phi(\rho) \cdot {}^\rho\widetilde{\tau}_G, \quad \forall \rho \in \Gamma. \tag{3.5.1.1}$$

Since $\phi$ is admissible, the cocycle $\zeta^{p,\infty}_\phi$ has trivial image in $\mathbf{H}^1(\Gamma, G(\bar{\mathbb{A}}^p_f))$. It follows from (3.5.1.1) that $A$ also has trivial image in $\mathbf{H}^1(\Gamma, G(\bar{\mathbb{A}}^p_f))$. Now from (3.5.1.1) it is clear that the class of $A$ in $\mathbf{H}^1(\mathbb{A}^p_f, I_0)$ is independent of the choice of $\widetilde{\tau}$. We define the desired element $a \in \mathfrak{D}(I_0, G; \mathbb{A}^p_f)$ to be the class of $A$.

Next we construct $[b] \in \mathrm{B}(I_{0,\mathbb{Q}_p})$. By §3.2.3, we may view $\phi$ as a morphism $\phi_{I_0} : \mathfrak{Q} \to \mathfrak{G}_{I_0}$. We choose $g \in \mathcal{UR}(\phi_{I_0}(p) \circ \zeta_p)$ and let $b = b_{\mathrm{Int}(g^{-1}) \circ \phi_{I_0}(p) \circ \zeta_p} \in I_0(\mathbb{Q}^{\mathrm{ur}}_p)$; see Definition 2.2.5. By Lemma 2.2.4 (i), the class $[b]$ of $b$ in $\mathrm{B}(I_{0,\mathbb{Q}_p})$ is independent of choices. We now check condition **KP0** in Definition 1.6.4 for $[b]$. By the admissibility of $\phi$, we have $X_p(\phi) \neq \emptyset$. Comparing §2.2.7 and §2.4.1, we have

$X_p(\phi) \cong X_{-\mu}(b)$ for $\mu \in \mathfrak{M}_X^{\mathcal{G}}$, and so $X_{-\mu}(b) \neq \emptyset$. It immediately follows that $b$ and $-\mu$ have the same image in $\pi_1(G)_{\Gamma_p}$ (cf. Corollary 1.3.15). Thus **KP0** is satisfied by $[b]$. This finishes the construction of $\mathbf{t}(\phi, \epsilon, \tau) \in \mathfrak{KP}$.

**Proposition 3.5.2.** *Let $(\phi, \epsilon) \in \mathcal{LRP}_{\mathrm{sa}}^{\mathrm{gg}}$, and let $\tau \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p)$. Let*

$$\begin{aligned}(\gamma_0, a, [b]) &:= \mathbf{t}(\phi, \epsilon, 1),\\ (\gamma_0, a', [b']) &:= \mathbf{t}(\phi, \epsilon, \tau).\end{aligned}$$

*Then $[b'] = [b]$. The difference*

$$a' - a \in \mathfrak{D}(I_0, G; \mathbb{A}_f^p) \cong \mathfrak{E}(I_0, G; \mathbb{A}_f^p)$$

*is equal to the image of $\tau$ under the composite map*

$$I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p) \to \mathfrak{D}(Z_{I_\phi}, I_\phi; \mathbb{A}_f^p) \xrightarrow{\sim} \mathfrak{E}(Z_{I_\phi}, I_\phi; \mathbb{A}_f^p) \to \mathfrak{E}(I_0, G; \mathbb{A}_f^p). \tag{3.5.2.1}$$

*In particular, the dependence of $\mathbf{t}(\phi, \epsilon, \tau)$ on $\tau$ is only through the image of $\tau$ in $\mathfrak{E}(I_0, G; \mathbb{A}_f^p)$.*

*Proof.* The first statement follows from the definition of $[b]$ and $[b']$. For the second statement, using the notation in §3.5.1, we know that $a$ is represented by the cocycle $\rho \mapsto \zeta_\phi^{p,\infty}(\rho)$ whereas $a'$ is represented by the cocycle $\rho \mapsto t_\rho \zeta_\phi^{p,\infty}(\rho)$, where $t_\rho$ is determined by a choice of $\widetilde{\tau}$ lifting $\tau$. The difference $a' - a$ in $\mathbf{H}^1_{\mathrm{ab}}(\mathbb{A}_f^p, I_0)$ is then given by the image of the class of $(t_\rho)_\rho$ under $\mathbf{H}^1(\mathbb{A}_f^p, Z_{I_\phi}) = \mathbf{H}^1_{\mathrm{ab}}(\mathbb{A}_f^p, Z_{I_\phi}) \to \mathbf{H}^1_{\mathrm{ab}}(\mathbb{A}_f^p, I_0)$. The desired statement follows. □

**Proposition 3.5.3.** *Let $(\phi, \epsilon), (\phi', \epsilon') \in \mathcal{LRP}_{\mathrm{sa}}^{\mathrm{gg}}$ such that $[\phi, \epsilon] = [\phi', \epsilon']$. Let $u \in G(\overline{\mathbb{Q}})$ be an element such that $\mathrm{Int}(u)(\phi, \epsilon) = (\phi', \epsilon')$. Let $\tau \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p)$, and let $\tau' \in I_{\phi'}^{\mathrm{ad}}(\mathbb{A}_f^p)$ be the image of $\tau$ under the $\mathbb{Q}$-isomorphism $I_\phi^{\mathrm{ad}} \xrightarrow{\sim} I_{\phi'}^{\mathrm{ad}}$ induced by the $\mathbb{Q}$-isomorphism $\mathrm{Int}(u) : I_\phi \xrightarrow{\sim} I_{\phi'}$. Then $u$ is an isomorphism $\mathbf{t}(\phi, \epsilon, \tau) \xrightarrow{\sim} \mathbf{t}(\phi', \epsilon', \tau')$ in the sense of Definition 1.6.14.*

*Proof.* We write $\mathbf{t}(\phi, \epsilon, \tau) = (\gamma_0, a, [b])$ and $\mathbf{t}(\phi', \epsilon', \tau') = (\gamma_0', a', [b'])$. By definition, $\gamma_0 = \epsilon$ and $\gamma_0' = \epsilon'$. We check that $u$ satisfies the three conditions in Definition 1.6.14. Condition (i) follows from Lemma 3.2.11.

To check condition (ii), we define $t_\rho$ as in §3.5.1, with respect to $(\phi, \epsilon, \tau)$. Then the counterpart $t'_\rho$ with respect to $(\phi', \epsilon', \tau')$ can be chosen to be $ut_\rho u^{-1}$. Now $a$ is represented by the cocycle $A : \rho \mapsto t_\rho \zeta_\phi^{p,\infty}(\rho)$, and $a'$ is represented by the cocycle $A' : \rho \mapsto t'_\rho \zeta_{\phi'}^{p,\infty}(\rho) = ut_\rho u^{-1} \zeta_{\phi'}^{p,\infty}(\rho)$. Note that $\zeta_{\phi'}^{p,\infty}(\rho) = u \cdot \zeta_\phi^{p,\infty}(\rho) \cdot {}^\rho u^{-1}$, since $\phi' = \mathrm{Int}(u) \circ \phi$. Hence

$$A'(\rho) = uA(\rho)^\rho u^{-1}, \quad \forall \rho \in \Gamma.$$

This proved condition (ii).

To check condition (iii), we write $I_0$ for $G_\epsilon^0$ and write $I_0'$ for $G_{\epsilon'}^0$. We choose $d \in I_0(\breve{\overline{\mathbb{Q}}}_p)$ such that $u_0 := ud^{-1}$ lies in $G(\breve{\mathbb{Q}}_p)$, as in §1.6.13. Choose $g \in \mathcal{UR}(\phi_{I_0}(p) \circ \zeta_p) \subset I_0(\overline{\mathbb{Q}}_p)$, and $g' \in \mathcal{UR}(\phi'_{I_0'}(p) \circ \zeta_p) \subset I_0'(\overline{\mathbb{Q}}_p)$. We may assume that $b = b_{\mathrm{Int}(g^{-1}) \circ \phi_{I_0}(p) \circ \zeta_p}$ and $b' = b_{\mathrm{Int}(g'^{-1}) \circ \phi'_{I_0'}(p) \circ \zeta_p}$. Then we have

$$g^{-1} u^{-1} g' \in G(\mathbb{Q}_p^{\mathrm{ur}}), \tag{3.5.3.1}$$

since this element conjugates the unramified morphism $\mathrm{Int}(g'^{-1}) \circ \phi'(p) \circ \zeta_p : \mathfrak{G}_p \to \mathfrak{G}_G(p)$ to the unramified morphism $\mathrm{Int}(g^{-1}) \circ \phi(p) \circ \zeta_p : \mathfrak{G}_p \to \mathfrak{G}_G(p)$. For the same reason we have

$$b = (g^{-1}u^{-1}g')b'\sigma(g^{-1}u^{-1}g')^{-1}. \tag{3.5.3.2}$$

The bijection $u_* : \mathrm{B}(I_{0,\mathbb{Q}_p}) \to \mathrm{B}(I'_{0,\mathbb{Q}_p})$ as in (1.6.13.2) sends $[b]$ to $[u_0 b\sigma(u_0)^{-1}]$, and by (3.5.3.2) the latter element equals $[wb'\sigma(w)^{-1}]$, where

$$w = u_0 g^{-1} u^{-1} g' \in G(\overline{\breve{\mathbb{Q}}}_p).$$

To finish the proof it suffices to show that $w \in I'_0(\breve{\mathbb{Q}}_p)$. Since $d^{-1}, g^{-1} \in I_0(\overline{\breve{\mathbb{Q}}}_p)$, we have $u_0 g^{-1} u^{-1} = \mathrm{Int}(u)(d^{-1}g^{-1}) \in I'_0(\overline{\breve{\mathbb{Q}}}_p)$. Also $g' \in I'_0(\overline{\mathbb{Q}}_p)$, so $w \in I'_0(\overline{\breve{\mathbb{Q}}}_p)$. By the fact that $u_0 \in G(\breve{\mathbb{Q}}_p)$ and by (3.5.3.1), we have $w \in G(\breve{\mathbb{Q}}_p)$. Hence $w \in I'_0(\breve{\mathbb{Q}}_p)$, as desired. $\square$

**3.5.4.** Let $(\phi, \epsilon) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}$. Let $I_0 := G^0_\epsilon$. Recall that the group $\text{Ш}^\infty_G(\mathbb{Q}, I^0_{\phi,\epsilon})$ is defined in §3.2.18. Using the canonical inner twisting between $I^0_{\phi,\epsilon}$ and $(G_{\phi^\Delta})^0_\epsilon \subset I_0$ as in §3.2.3, we have a natural homomorphism $\text{Ш}^\infty_G(\mathbb{Q}, I^0_{\phi,\epsilon}) \to \mathfrak{E}(I_0, G; \mathbb{Q})$. Let $e = (e_\rho)_\rho \in Z^1(\mathbb{Q}, I^0_{\phi,\epsilon})$ be a cocycle representing a class in $\text{Ш}^\infty_G(\mathbb{Q}, I^0_{\phi,\epsilon})$. By Proposition 3.2.19, we obtain $(e\phi, \epsilon) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}$.

**Proposition 3.5.5.** *Let $(\epsilon, a, [b]) := \mathbf{t}(\phi, \epsilon, 1)$, and $(\epsilon, a', [b']) := \mathbf{t}(e\phi, \epsilon, 1)$. Then the difference $a' - a \in \mathfrak{E}(I_0, G, \mathbb{A}^p_f)$ is equal to the natural image of $e$. The elements $[b'], [b] \in \mathrm{B}(I_{0,\mathbb{Q}_p})$ have conjugate Newton cocharacters. Moreover, if $\nu_b$ is central in $I_0$, then $\kappa_{I_0}(b') - \kappa_{I_0}(b) \in \pi_1(I_0)_{\Gamma_p}$ is equal to the image of $e$ in $\mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}_p, I_0)$, which is identified with $\pi_1(I_0)_{\Gamma_p,\mathrm{tors}}$ as in Proposition 1.1.9.*

*Proof.* By construction, $a$ and $a'$ are represented by the cocycles $\zeta^{p,\infty}_\phi$ and $\zeta^{p,\infty}_{e\phi}$ respectively. Clearly

$$\zeta^{p,\infty}_{e\phi}(\rho) = e_\rho \zeta^{p,\infty}_\phi(\rho), \quad \forall \rho \in \Gamma \tag{3.5.5.1}$$

The statement about $a' - a$ follows from this and [Bor98, Lem. 3.15.1].

We now prove the statement about $[b]$ and $[b']$. As in §3.2.3, we may view $\phi$ as a morphism $\mathfrak{Q} \to \mathfrak{G}_{I_0}$, which we denote by $\phi_{I_0}$. We may also view $e$ as a cocycle in $Z^1(\mathbb{Q}, I_{\phi_{I_0}})$. Then $e\phi$ is induced by $e\phi_{I_0} : \mathfrak{Q} \to \mathfrak{G}_{I_0}$. By construction, $b$ (resp. $b'$) is associated with the choice of an element $g$ in $\mathcal{UR}(\phi_{I_0}(p) \circ \zeta_p)$ (resp. $g'$ in $\mathcal{UR}((e\phi_{I_0})(p) \circ \zeta_p)$). Let $\theta := \mathrm{Int}(g)^{-1} \circ \phi_{I_0}(p) \circ \zeta_p$ and let $\theta' := \mathrm{Int}(g')^{-1} \circ (e\phi_{I_0})(p) \circ \zeta_p$, and we view both as unramified morphisms $\mathfrak{G}_p \to \mathfrak{G}_{I_0}(p)$. Thus $b = b_\theta$ and $b' = b_{\theta'}$. Now $\mathrm{Int}(g^{-1})$ induces a $\mathbb{Q}_p$-map $I_{\phi_{I_0},\mathbb{Q}_p} \to I_\theta$. We let $e_\theta$ denote the image of $e$ under

$$Z^1(\mathbb{Q}, I_{\phi_{I_0}}) \longrightarrow Z^1(\mathbb{Q}_p, I_{\phi_{I_0}}) \xrightarrow{\mathrm{Int}(g^{-1})} Z^1(\mathbb{Q}_p, I_\theta).$$

By Proposition 2.2.6 (ii), we have a canonical isomorphism $I_\theta \cong J^{I_0}_b$. (See §1.4.3 for the notation $J^{I_0}_b$.) We write $e_b$ for $e_\theta$ when we view it as an element of $Z^1(\mathbb{Q}_p, J^{I_0}_b)$. Now $\theta'$ is in the conjugacy class of $e_\theta\theta$ (as morphisms $\mathfrak{G}_p \to \mathfrak{G}_{I_0}(p)$). By Proposition 2.2.6 (iii), the $\sigma$-conjugacy class of $b'$ in $I_0(\mathbb{Q}^{\mathrm{ur}}_p)$ is the twist of $b$ by $e_b$. In particular, $[b]$ and $[b']$ have conjugate Newton cocharacters by Proposition 1.4.5.

If $\nu_b$ is central, then we can apply Proposition 1.4.5 (to the group $I_{0,\mathbb{Q}_p}$), and conclude that $\kappa_{I_0}(b') - \kappa_{I_0}(b)$ is equal to the image of $e_b$ under

$$\mathbf{H}^1(\mathbb{Q}_p, J_b^{I_0}) \xrightarrow{\sim} \mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}_p, J_b^{I_0}) \xrightarrow{\sim} \mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}_p, I_0) \xrightarrow{\sim} \pi_1(I_0)_{\Gamma_p,\mathrm{tors}}.$$

Here the second isomorphism is induced by the canonical inner twisting $\iota$ between the $\mathbb{Q}_p$-groups $J_b^{I_0}$ and $I_{0,\mathbb{Q}_p}$, as in case (ii) of §1.4.3. We are left to check that the above image of $e_b$ in $\pi_1(I_0)_{\Gamma_p,\mathrm{tors}}$ is equal to the image of $e$ in $\pi_1(I_0)_{\Gamma_p,\mathrm{tors}}$ as in the proposition. This boils down to the commutativity of the following diagram up to inner automorphisms:

$$\begin{tikzcd}
(I_{\phi_{I_0}})_{\overline{\mathbb{Q}}_p} \arrow[r] \arrow[d, "\mathrm{Int}(g^{-1})", "\cong"'] & I_{0,\overline{\mathbb{Q}}_p} \\
I_{\theta,\overline{\mathbb{Q}}_p} \arrow[r, "\cong"] & (J_b^{I_0})_{\overline{\mathbb{Q}}_p} \arrow[u, "\iota"', "\cong"]
\end{tikzcd}$$

Here the top arrow is given by the canonical inner twisting between $I_{\phi_{I_0}}$ and $(I_0)_{\phi^\Delta} \subset I_0$. The bottom arrow is the canonical isomorphism in Proposition 2.2.6 (ii). It suffices to check that the composition $I_{\theta,\overline{\mathbb{Q}}_p} \xrightarrow{\sim} (J_b^{I_0})_{\overline{\mathbb{Q}}_p} \xrightarrow{\iota} I_{0,\overline{\mathbb{Q}}_p}$ is equal to the canonical embedding $I_{\theta,\overline{\mathbb{Q}}_p} \to I_{0,\overline{\mathbb{Q}}_p}$ attached to $\theta : \mathfrak{G}_p \to \mathfrak{G}_{I_0}(p)$. This is straightforward by the proof of Proposition 2.2.6 (ii). $\square$

**Proposition 3.5.6.** *Let $(\phi, \epsilon) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}$, and let $\mathfrak{c} = \mathfrak{t}(\phi, \epsilon, 1) \in \mathfrak{KP}$. Then the Kottwitz invariant $\alpha(\mathfrak{c})$ is trivial.*

*Proof.* By Proposition 3.5.3 and Proposition 1.7.10, we may replace $(\phi, \epsilon)$ by any other element $(\phi', \epsilon') \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}$ such that $[\phi, \epsilon] = [\phi', \epsilon']$, in the course of the proof. Choose a maximal torus $T \subset I_\phi$ defined over $\mathbb{Q}$ such that $\epsilon \in T(\mathbb{Q})$. (This is possible since $\epsilon$ is semi-simple.) By Theorem 3.3.9, there exists $(T, i, h) \in \mathcal{SPD}(G, X)$ and $g \in G(\overline{\mathbb{Q}})$ such that $\mathrm{Int}(g) \circ \phi = \phi(T, i, h)$, and such that $\mathrm{Int}(g)(\epsilon) = i(\epsilon) \in G(\mathbb{Q})$. Using Lemma 3.2.15 one checks that $(\mathrm{Int}(g) \circ \phi, \mathrm{Int}(g)(\epsilon)) \in \mathcal{LRP}^{\mathrm{gg}}$. Hence up to replacing $(\phi, \epsilon)$ by $(\mathrm{Int}(g) \circ \phi, \mathrm{Int}(g)(\epsilon))$, we may assume that $\phi = \phi(T, i, h)$, that $\epsilon \in T(\mathbb{Q})$, and that the embedding $T_{\overline{\mathbb{Q}}} \hookrightarrow I_{\phi,\overline{\mathbb{Q}}} \hookrightarrow G_{\overline{\mathbb{Q}}}$ coincides with $i$.

Write $\mathfrak{c} = (\gamma_0 = \epsilon, a, [b])$. By definition (see §2.2.9 and Definition 3.3.2), $\phi$ factors as $\mathfrak{Q} \to \mathfrak{G}_{\mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_m} \to \mathfrak{G}_G$, where $L/\mathbb{Q}$ be a finite Galois extension contained in $\overline{\mathbb{Q}}$ splitting $T$, and $\mathfrak{G}_{\mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_m} \to \mathfrak{G}_G$ is induced by a $\mathbb{Q}$-homomorphism $f : \mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_m \to G$. In particular, $\zeta_\phi^{p,\infty}$ is induced by cocycle $\Gamma \to (\mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_m)(\bar{\mathbb{A}}_f^p)$ and $f$. By Shapiro's lemma and Hilbert 90, the class $a$ is trivial.

Recall that $b = b_\theta$, where $\theta = \mathrm{Int}(g^{-1}) \circ \phi(p) \circ \zeta_p$ for some $g \in \mathcal{UR}(\phi(p) \circ \zeta_p) \cap I_0(\overline{\mathbb{Q}}_p)$. Since $\phi$ factors through $i : \mathfrak{G}_T \to \mathfrak{G}_G$, we may take $g$ to be inside $i(T)(\overline{\mathbb{Q}}_p)$. Then $\theta$ factors through $i : \mathfrak{G}_T(p) \to \mathfrak{G}_G(p)$, and $b \in i(T)(\mathbb{Q}_p^{\mathrm{ur}})$. Write $b = i(b_T)$. By Lemma 2.2.10, the element $\kappa_T(b_T) \in X_*(T)_{\Gamma_p}$ is equal to the image of $-\mu_h \in X_*(T)$. Therefore, keeping the notation in §1.7.5, we may choose $\widetilde{\beta}_p(\mathfrak{c})$ to be the image of $-\mu_h \in X_*(T)$ in $\pi_1(I_0(\mathfrak{c}))$ (with respect to $i : T \to I_0(\mathfrak{c})$). Also, we may choose $\widetilde{\beta}_\infty(\mathfrak{c})$ to be the image of $\mu_h \in X_*(T)$ in $\pi_1(I_0(\mathfrak{c}))$. We have seen that $a$ is trivial, so we may choose $\widetilde{\beta}_l(\mathfrak{c})$ to be zero for all $l \notin \{p, \infty\}$. Then we have $\alpha(\mathfrak{c}) = 0$. $\square$

**Corollary 3.5.7.** *Let* $(\phi, \epsilon) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}$ *and let* $\tau \in I^{\mathrm{ad}}_{\phi}(\mathbb{A}^p_f)$. *Let* $\mathfrak{c} = \mathbf{t}(\phi, \epsilon, \tau) \in \mathfrak{KP}$. *Then* $\alpha(\mathfrak{c})$ *is equal to the image of* $\tau$ *under*

$$I^{\mathrm{ad}}_{\phi}(\mathbb{A}^p_f) \xrightarrow{(3.5.2.1)} \mathfrak{E}(I_0(\mathfrak{c}), G; \mathbb{A}^p_f) \hookrightarrow \mathfrak{E}(I_0(\mathfrak{c}), G; \mathbb{A}) \to \mathfrak{E}(I_0(\mathfrak{c}), G; \mathbb{A}/\mathbb{Q}).$$

*Proof.* This follows from Proposition 3.5.6, Proposition 3.5.2, and Proposition 1.7.8. $\square$

As in §3.1.5, let $n$ be a positive multiple of $r$.

**Proposition 3.5.8.** *Let* $(\phi, \epsilon) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}$ *and* $\tau \in I^{\mathrm{ad}}_{\phi}(\mathbb{A}^p_f)$. *Then* $(\phi, \epsilon)$ *is* $p^n$-*admissible if and only if* $\mathbf{t}(\phi, \epsilon, \tau)$ *is* $p^n$-*admissible.*

*Proof.* Write $\mathbf{t}(\phi, \epsilon, \tau) = (\epsilon, a, [b])$ and write $I_0$ for $G^0_{\epsilon}$. By construction, $[b] \in \mathrm{B}(I_{0,\mathbb{Q}_p})$ has a representative $b \in I_0(\mathbb{Q}^{\mathrm{ur}}_p)$ such that $b = b_{\theta}$, where $\theta = \mathrm{Int}(g^{-1}) \circ \phi(p) \circ \zeta_p$ for some $g \in \mathcal{UR}(\phi(p) \circ \zeta_p) \cap I_0(\overline{\mathbb{Q}}_p)$. Since $g$ commutes with $\epsilon$, we see that $(b, \epsilon) \in \mathrm{cls}_p(\phi, \epsilon)$. By [Kot84a, Lem. 1.4.9], $\mathbf{t}(\phi, \epsilon, \tau)$ is $p^n$-admissible if and only if $\epsilon^{-1}(b \rtimes \sigma)^n$ has a fixed point in $G(\breve{\mathbb{Q}}_p)/G(\breve{\mathbb{Z}}_p)$. The latter condition is precisely the definition that $(\phi, \epsilon)$ is $p^n$-admissible. $\square$

**Proposition 3.5.9.** *Let* $\mathfrak{c} = (\gamma_0, a, [b]) \in \mathfrak{KP}_{\mathrm{a}}(p^n)$ *with* $\alpha(\mathfrak{c}) = 0$. *Assume that there exists an element of* $\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(p^n)$ *of the form* $(\phi_0, \gamma_0)$. *Then there exists an element of* $\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(p^n)$ *of the form* $(\phi, \gamma_0)$ *such that* $\mathfrak{c} = \mathbf{t}(\phi, \gamma_0, 1)$.

*Proof.* Let $I_0 := G^0_{\gamma_0}$. Write $\mathbf{t}(\phi_0, \gamma_0, 1) = \mathfrak{c}_0 = (\gamma_0, a_0, [b_0]) \in \mathfrak{KP}$. By Proposition 3.5.8, we have $\mathfrak{c}_0 \in \mathfrak{KP}_{\mathrm{a}}(p^n)$. Hence by Corollary 1.6.12 we know that $\nu_{b_0}$ is central in $I_0$ and that $\nu_b = \nu_{b_0}$. In particular, $\kappa_{I_0}(b)$ and $\kappa_{I_0}(b_0)$ both lie in $\pi_1(I_0)_{\Gamma_p,\mathrm{tors}}$. Consider the element

$$\begin{array}{rcccccc} e_0 & := & (a - a_0) & \oplus & (\kappa_{I_0}(b) - \kappa_{I_0}(b_0)) & & \\ & \in & \mathbf{H}^1_{\mathrm{ab}}(\mathbb{A}^p_f, I_0) & \oplus & \pi_1(I_0)_{\Gamma_p,\mathrm{tors}} & \cong & \mathbf{H}^1_{\mathrm{ab}}(\mathbb{A}_f, I_0). \end{array}$$

By **KP0** in Definition 1.6.4, the element $\kappa_{I_0}(b) - \kappa_{I_0}(b_0)$ goes to zero in $\pi_1(G)_{\Gamma_p}$. Therefore $e_0 \in \mathfrak{E}(I_0, G; \mathbb{A}_f)$.

Note that the image of $e_0$ in $\mathfrak{E}(I_0, G; \mathbb{A}/\mathbb{Q})$ is just $\alpha(\mathfrak{c}) - \alpha(\mathfrak{c}_0)$. We have $\alpha(\mathfrak{c}) = 0$ by hypothesis, and $\alpha(\mathfrak{c}_0) = 0$ by Proposition 3.5.6. By the exact sequence (1.7.1.1), there exists a lift $e_1 \in Ш^{\infty}_G(\mathbb{Q}, I_0)$ of $e_0$. Let $I := I^0_{\phi_0,\gamma_0}$. Then we have a canonical inner twisting $I_{\overline{\mathbb{Q}}} \xrightarrow{\sim} I_{0,\overline{\mathbb{Q}}}$ as in §3.2.14, and hence a canonical isomorphism $Ш^{\infty}_G(\mathbb{Q}, I) \cong Ш^{\infty}_G(\mathbb{Q}, I_0)$. Let $e_2 \in Ш^{\infty}_G(\mathbb{Q}, I)$ be the element corresponding to $e_1 \in Ш^{\infty}_G(\mathbb{Q}, I_0)$.

By Proposition 3.2.19, we obtain an element $(e_2\phi_0, \gamma_0) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}$. We claim that $\mathbf{t}(e_2\phi_0, \gamma_0) = \mathfrak{c}$. Write $\mathbf{t}(e_2\phi_0, \gamma_0) = (\gamma_0, a', [b'])$. By Proposition 3.5.5, we have $a' - a_0 = a - a_0$, and so $a' = a$. As $\nu_{b_0}$ is central in $I_0$, we can Proposition 3.5.5 to conclude that $\nu_{b'} = \nu_{b_0}$, and that $\kappa_{I_0}(b') - \kappa_{I_0}(b_0) = \kappa_{I_0}(b) - \kappa_{I_0}(b_0)$. Thus we have $[b] = [b']$ by the classification of $\mathrm{B}(I_{0,\mathbb{Q}_p})$ (see §1.4.2). Having checked that $\mathbf{t}(e_2\phi_0, \gamma_0) = \mathfrak{c}$, we deduce that $(e_2\phi_0, \gamma_0) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(p^n)$ by Proposition 3.5.8. $\square$

3.6. **The effect of a controlled twist.** Recall that Conjecture 2.7.3 predicts the existence of a tori-rational element $\underline{\tau} \in \Gamma(\mathcal{H})_0$ such that $\mathsf{LR}(G, X, p, \mathcal{G}, \underline{\tau})$ holds. In this subsection we fix such a $\underline{\tau}$. By Lemma 2.6.20 there exists a tori-rational $\underline{\sigma} \in \Gamma(\mathfrak{E}^p)_0$ lifting $\underline{\tau}$. By the last assertion in Proposition 3.5.2, we have a well-defined Kottwitz parameter $\mathbf{t}(\phi, \epsilon, \underline{\sigma}(\phi))$ assigned to each $(\phi, \epsilon) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}$. We shall write $\mathbf{t}_{\underline{\sigma}}(\phi, \epsilon)$ for $\mathbf{t}(\phi, \epsilon, \underline{\sigma}(\phi))$. We establish analogues of the results in §3.5, for the function $\mathbf{t}_{\underline{\sigma}} : \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}} \to \mathfrak{KP}$.

As in §3.1.5, let $n$ be a positive multiple of $r$.

**Proposition 3.6.1.** *Let $(\phi, \epsilon)$ and $(\phi', \epsilon')$ be elements of $\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}$ such that $[\phi, \epsilon] = [\phi', \epsilon']$. Let $u \in G(\overline{\mathbb{Q}})$ be an element such that $\mathrm{Int}(u)(\phi, \epsilon) = (\phi', \epsilon')$. Then $u$ is an isomorphism $\mathbf{t}_{\underline{\sigma}}(\phi, \epsilon) \xrightarrow{\sim} \mathbf{t}_{\underline{\sigma}}(\phi', \epsilon')$. In particular, the isomorphism class of $\mathbf{t}_{\underline{\sigma}}(\phi, \epsilon)$ depends on $(\phi, \epsilon)$ only through $[\phi, \epsilon] \in [\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}]$.*

*Proof.* Since $\underline{\sigma} \in \Gamma(\mathfrak{E}^p)_0$, we may find a lift $\tau \in I^{\mathrm{ad}}_{\phi}(\mathbb{A}^p_f)$ of $\underline{\sigma}(\phi)$ and a lift $\tau' \in I^{\mathrm{ad}}_{\phi'}(\mathbb{A}^p_f)$ such that $\tau$ maps to $\tau'$ under the isomorphism $I^{\mathrm{ad}}_{\phi} \xrightarrow{\sim} I^{\mathrm{ad}}_{\phi'}$ induced by $\mathrm{Int}(u) : I_\phi \xrightarrow{\sim} I_{\phi'}$. The proposition then follows from Proposition 3.5.3. □

**Proposition 3.6.2.** *Keep the setting and notation of §3.5.4, and assume in addition that $(\phi, \epsilon) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(p^n)$. Let $(\epsilon, a, [b]) := \mathbf{t}_{\underline{\sigma}}(\phi, \epsilon)$, and $(\epsilon, a', [b']) := \mathbf{t}_{\underline{\sigma}}(e\phi, \epsilon)$. The following statements hold.*

*(i) The difference $a' - a \in \mathfrak{E}(I_0, G; \mathbb{A}^p_f)$ is equal to the natural image of $e$.*

*(ii) The elements $[b'], [b] \in \mathrm{B}(I_{0,\mathbb{Q}_p})$ are basic and have equal Newton cocharacter. The difference $\kappa_{I_0}(b') - \kappa_{I_0}(b) \in \pi_1(I_0)_{\Gamma_p}$ is equal to the image of $e$ in $\pi_1(I_0)_{\Gamma_p,\mathrm{tors}}$ as in Proposition 3.5.5.*

*(iii) Let $e' \in Z^1(\mathbb{Q}, I^0_{\phi,\epsilon})$ be another cocycle representing the same cohomology class as $e$. Then we have $\mathbf{t}_{\underline{\sigma}}(e\phi, \epsilon) = \mathbf{t}_{\underline{\sigma}}(e'\phi, \epsilon)$. (These two Kottwitz parameters are equal, not just isomorphic.)*

*Proof.* For (i), in view of Proposition 3.5.2 and the statement about $a' - a$ in Proposition 3.5.5, it suffices to check that $\underline{\sigma}(\phi)$ and $\underline{\sigma}(e\phi)$ have the same image in $\mathfrak{E}(I_0, G; \mathbb{A}^p_f)$. But this follows easily from the fact that $\underline{\sigma} \in \Gamma(\mathfrak{E}^p)_0$.

By Proposition 3.5.2, the components $[b]$ and $[b']$ are unaffected by $\underline{\sigma}$. Hence to prove (ii) we may assume that $\underline{\sigma}$ is trivial. By Proposition 3.5.8 and Corollary 1.6.12, we know that $[b]$ is basic in $\mathrm{B}(I_0)$. The remaining statements in (ii) follow from Proposition 3.5.5.

Part (iii) follows from the previous two parts, and Kottwitz's classification of $\mathrm{B}(I_{0,\mathbb{Q}_p})$ (see §1.4.2). □

For each $\mathfrak{c} \in \mathfrak{KP}$, recall that we have defined the Kottwitz invariant $\alpha(\mathfrak{c})$ in §1.7.5.

**Proposition 3.6.3.** *For each $(\phi, \epsilon) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}$, we have $\alpha(\mathbf{t}_{\underline{\sigma}}(\phi, \epsilon)) = 0$.*

*Proof.* Choose a maximal torus $T \subset I_\phi$ defined over $\mathbb{Q}$ such that $\epsilon \in T(\mathbb{Q})$. (This is possible since $\epsilon$ is semi-simple.) By Corollary 3.5.7 and the exact sequence (1.7.1.1), it suffices to show that the image of $\underline{\sigma}(\phi)$ in $\mathbf{H}^1(\mathbb{A}^p_f, T)$ comes from $Ш^{\infty,p}_G(\mathbb{Q}, T)$. This follows from the fact that $\underline{\sigma}$ is tori-rational. □

Let $n$ be a positive multiple of $r$.

**Lemma 3.6.4.** *Let $(\phi_1, \epsilon_1) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(p^n)$. Let $g \in G(\overline{\mathbb{Q}})$ be an element that stably conjugates $\epsilon_1$ to some $\epsilon_2 \in G(\mathbb{Q})$, i.e., $\mathrm{Int}(g)(\epsilon_1) = \epsilon_2$, and $g\, {}^\tau g^{-1} \in G^0_{\epsilon_2}(\overline{\mathbb{Q}})$ for all $\tau \in \Gamma$. Define $\phi_2 := \mathrm{Int}(g) \circ \phi_1$. Then $(\phi_2, \epsilon_2) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(p^n)$.*

*Proof.* We only need to show that $(\phi_2, \epsilon_2) \in \mathcal{LRP}^{\mathrm{gg}}$. Since $\epsilon_2 \in G(\mathbb{Q})$ is stably conjugate to $\epsilon_1$, it is $\mathbb{R}$-elliptic in $G$ as $\epsilon_1$ is (cf. §1.8.7). Now write $\phi_i(q_\tau) = g_{i,\tau} \rtimes \tau$, for $i = 1, 2$. It remains to show that $g_{2,\tau} \in I^0_{\phi_2,\epsilon_2}(\overline{\mathbb{Q}})$, for each $\tau \in \Gamma$. As in the proof of Lemma 3.2.11, we have

$$g\ {}^{\tau}g^{-1} = [g g_{1,\tau} g^{-1}]^{-1} g_{2,\tau}.$$

The left hand side lies in $G^0_{\epsilon_2}(\overline{\mathbb{Q}})$ by hypothesis, and the term $g g_{1,\tau} g^{-1}$ lies in $G^0_{\epsilon_2}(\overline{\mathbb{Q}})$ since $g_{1,\tau} \in G^0_{\epsilon_1}(\overline{\mathbb{Q}})$. Hence we have $g_{2,\tau} \in G^0_{\epsilon_2}(\overline{\mathbb{Q}})$. By §3.2.14, the last group is in fact equal to $I^0_{\phi_2,\epsilon_2}(\overline{\mathbb{Q}})$. □

**Proposition 3.6.5.** *Let $\mathfrak{c} = (\gamma_0, a, [b]) \in \mathfrak{KP}_{\mathrm{a}}(p^n)$. Let $(\gamma_0, \gamma, \delta) \in \mathfrak{KP}_{\mathrm{cla}}(p^n)$ be the classical Kottwitz parameter of degree $n$ (up to equivalence) assigned to $\mathfrak{c}$ as in §1.6.16. Assume that the $\sigma$-conjugacy class of $\delta$ in $G(\mathbb{Q}_{p^n})$ intersects non-trivially with the support of $\phi_n$. (Here $\phi_n$ is as in §1.8.2.) Assume that $\alpha(\mathfrak{c}) = 0$. Then there exists an element of $\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)$ of the form $(\phi, \gamma_0)$ such that $\mathbf{t}_{\underline{\sigma}}(\phi, \gamma_0) = \mathfrak{c}$.*

*Proof.* Let $I_0 := G^0_{\gamma_0}$. Since $\gamma_0 \in G(\mathbb{Q}_p)$ is a degree $n$ norm of $\delta \in G(\mathbb{Q}_{p^n})$, by Proposition 3.4.8 and Lemma 3.6.4 there exists an element of $\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(p^n)$ of the form $(\phi_0, \gamma_0)$. By Proposition 3.5.9, we find $(\phi_1, \gamma_0) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(p^n)$ such that $\mathbf{t}(\phi_1, \gamma_0, 1) = \mathfrak{c}$.

Consider the Kottwitz parameter

$$(\gamma_0, a_1, [b_1]) := \mathbf{t}_{\underline{\sigma}}(\phi_1, \gamma_0). \tag{3.6.5.1}$$

By Proposition 3.5.2, we know that $[b_1] = [b]$, and that $a_1 - a$ is equal to the image of $\underline{\sigma}(\phi_1)$ in $\mathfrak{E}(I_0, G; \mathbb{A}^p_f)$. Fix a maximal torus $T \subset I_{\phi_1}$ such that $\gamma_0 \in T(\mathbb{Q})$. Then by tori-rationality of $\underline{\sigma}$, the image of $\underline{\sigma}(\phi_1)$ in $\mathbf{H}^1(\mathbb{A}^p_f, T)$ is equal to the image of some $\beta \in Ш^{\infty,p}_G(\mathbb{Q}, T)$.

Under $T \hookrightarrow I^0_{\phi_1,\gamma_0}$ the class $-\beta$ determines a class in $Ш^{\infty,p}_G(\mathbb{Q}, I^0_{\phi_1,\gamma_0})$. Fix a cocycle $e \in Z^1(\mathbb{Q}, I^0_{\phi_1,\gamma_0})$ representing the latter class. By Proposition 3.2.19, we obtain $(e\phi_1, \gamma_0) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}$. Write $\phi$ for $e\phi_1$, and let

$$(\gamma_0, a', [b']) := \mathbf{t}_{\underline{\sigma}}(\phi, \gamma_0). \tag{3.6.5.2}$$

Comparing (3.6.5.1) and (3.6.5.2), we see from Proposition 3.6.2 that $a' - a_1$ equals the image of $-\beta$ in $\mathfrak{E}(I_0, G; \mathbb{A}^p_f)$, and that $[b'] = [b_1]$ (since $-\beta$ is trivial at $p$). Thus we have $[a'] = [a]$ and $[b'] = [b]$. Hence $\mathbf{t}_{\underline{\sigma}}(\phi, \gamma_0) = \mathfrak{c}$. □

## 3.7. **Proof of Theorem 2.7.4.**

**3.7.1.** Throughout we fix a tori-rational element $\underline{\tau} \in \Gamma(\mathcal{H})_0$ such that the statement $\mathsf{LR}(G, X, p, \mathcal{G}, \underline{\tau})$ holds. Namely, we have a smooth integral model $\mathscr{S}_{K_p}$ of $\mathrm{Sh}_{K_p}$ over $\mathcal{O}_{E,(\mathfrak{p})}$ which has well-behaved $\mathbf{H}^*_c$, and we have a bijection

$$\mathscr{S}_{K_p}(\overline{\mathbb{F}}_q) \xrightarrow{\sim} \coprod_{\phi} S_{\underline{\tau}}(\phi) \tag{3.7.1.1}$$

compatible with the actions of $G(\mathbb{A}^p_f)$ and the $q$-Frobenius $\Phi$.

Our goal is to prove (1.8.8.1) for all sufficiently large $m$. First observe that in the proof we may arbitrarily replace $K^p$ by an open subgroup. In particular, we may and shall assume that $K^p$ is neat (see [Lan13, Def. 1.4.1.8, Rmk. 1.4.1.9]), and that $\mathscr{S}_{K_pU^p}$ is defined for each open subgroup $U^p \subset K^p$. It follows from the neatness of $K^p$ that $K = K_pK^p$ is neat in the sense of Pink [Pin90, §0.6]. In the sequel we write $Z(\mathbb{Q})_K$ for $Z_G(\mathbb{Q}) \cap K$ and write $Z_K$ for $Z_G(\mathbb{A}_f) \cap K$, as in §1.8.6.

By linearity, we may assume that $f^p = 1_{K^p g^{-1} K^p}$ for some $g \in G(\mathbb{A}_f^p)$, and that $dg^p$ assigns volume 1 to $K^p$.

By Lemma 2.6.20 we fix a tori-rational $\underline{\sigma} \in \Gamma(\mathfrak{E}^p)_0$ lifting $\underline{\tau}$. For each admissible morphism $\phi$, we fix a lift $\tau_\phi \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p)$ of $\underline{\sigma}(\phi) \in \mathfrak{E}^p(\phi) = I_\phi(\mathbb{A}_f^p)\backslash I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p)$. We let $X^p(\phi)'$ be the $I_\phi(\mathbb{A}_f^p)$-set whose underlying set is $X^p(\phi)$, but the $I_\phi(\mathbb{A}_f^p)$-action is given by the natural action pre-composed with conjugation by $\tau_\phi$. Thus $S_{\underline{\tau}}(\phi)$ is isomorphic to

$$\varprojlim_{U^p} I_\phi(\mathbb{Q})\backslash(X_p(\phi) \times X^p(\phi)'/U^p).$$

**Lemma 3.7.2.** *For each admissible morphism $\phi$, the following statements hold.*

*(i) Let $\epsilon'$ be an element of $I_\phi(\mathbb{A}_f)$ which is conjugate under $I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$ to some $\epsilon \in I_\phi(\mathbb{Q})$. If there exist $y^p \in X^p(\phi)/K_g^p$ and $y_p \in X_p(\phi)$ such that*

$$y^p g \equiv \epsilon' y^p g \mod K^p, \quad \text{and} \quad \epsilon' y_p = y_p,$$

*then $\epsilon' \in Z(\mathbb{Q})_K$.*

*(ii) We have $I_\phi(\mathbb{Q})_{\mathrm{der}} \cap Z(\mathbb{Q})_K = \{1\}$.*

*Proof.* **(i)** The proof follows the same idea as [Mil92, Lem. 5.5]. We view $\epsilon$ as an element of $G(\overline{\mathbb{Q}})$ and let $\bar{\epsilon}$ be its image in $G^{\mathrm{ad}}(\overline{\mathbb{Q}})$. By Lemma 2.6.7, $\epsilon$ is semi-simple, and $\bar{\epsilon}$ lies in the $\mathbb{R}$-points of a compact form of $G^{\mathrm{ad}}_{\mathbb{R}}$. By the existence of $y^p$, the image of $\epsilon$ under $G(\overline{\mathbb{Q}}) \to G(\bar{\mathbb{A}}_f^p)$ has a conjugate $u$ that lies in $K^p \subset G(\mathbb{A}_f^p) \subset G(\bar{\mathbb{A}}_f^p)$. By the existence of $y_p$, the image of $\epsilon$ under $G(\overline{\mathbb{Q}}) \to G(\overline{\mathbb{Q}}_p)$ has a conjugate $v$ that lies in $\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$. It follows that $\bar{\epsilon}$ is torsion. Now let $\bar{u}$ be the image of $u$ in $G^{\mathrm{ad}}(\mathbb{A}_f^p)$. Then $\bar{\epsilon}$ is conjugate to $\bar{u}$ inside $G^{\mathrm{ad}}(\bar{\mathbb{A}}_f^p)$, and so $\bar{u}$ is torsion. But $\bar{u}$ lies in the image of $K^p$ under $G(\mathbb{A}_f^p) \to G^{\mathrm{ad}}(\mathbb{A}_f^p)$, which is neat. Hence at least one local component of $\bar{u}$ is trivial. It follows that $\bar{\epsilon} = 1$. We have thus shown that $\epsilon \in Z(\overline{\mathbb{Q}})$, and in particular $\epsilon = \epsilon'$.

Since the natural embedding $Z \to I_\phi$ is defined over $\mathbb{Q}$ and since $\epsilon \in I_\phi(\mathbb{Q})$, we have $\epsilon \in Z(\mathbb{Q})$. Now using the existence of $y^p$ and $y_p$ it is easy to see that $\epsilon \in Z(\mathbb{Q})_K$.

**(ii)** Clearly $I_\phi(\mathbb{Q})_{\mathrm{der}} \cap Z(\mathbb{Q})_K$ is contained in the $\mathbb{Q}$-points of the center of $I_{\phi,\mathrm{der}}$, and is hence finite. Since $K$ is neat, $Z(\mathbb{Q})_K$ is torsion free, and so $I_\phi(\mathbb{Q})_{\mathrm{der}} \cap Z(\mathbb{Q})_K = \{1\}$. $\square$

**Lemma 3.7.3.** *We keep the assumptions on $f^p$ and $dg^p$ in §3.7.1. When $m$ is sufficiently large (in a way depending on $K^p$ and $f^p$), we have*

$$T(\Phi_{\mathfrak{p}}^m, f^p dg^p) = \sum_\phi \sum_\epsilon \#\mathscr{O}(\phi, \epsilon, m, g) \cdot \mathrm{tr}\, \xi(\epsilon), \tag{3.7.3.1}$$

*where*

- *the first summation is over a set of representatives for the conjugacy classes of admissible morphisms $\phi$.*
- *for each $\phi$ the second summation is over a subset of $I_\phi(\mathbb{Q})$ such that each conjugacy class in $I_\phi(\mathbb{Q})/Z(\mathbb{Q})_K$ is represented exactly once.*
- *the set $\mathscr{O}(\phi, \epsilon, m, g)$ is defined as the quotient of*

$$X_p(\phi, \epsilon, q^m) \times \left\{y^p \in X^p(\phi)'/K_g^p \mid y^p \equiv \epsilon y^p g \mod K^p\right\}$$

*by the diagonal left action of $I_{\phi,\epsilon}(\mathbb{Q})$. Here $X_p(\phi, \epsilon, q^m)$ is defined in §3.1.5.*

*Proof.* Since (2.5.5.1) is an isomorphism by assumption, $T(\Phi_{\mathfrak{p}}^m, f^p dg^p)$ is equal to

$$\sum_i (-1)^i \mathrm{tr}\left(\Phi_{\mathfrak{p}}^m \times (f^p dg^p) \mid \mathbf{H}_c^i(\mathscr{S}_{K_p,\overline{\mathbb{F}}_q}, \xi)\right). \tag{3.7.3.2}$$

Let $K_g^p := K^p \cap gKg^{-1}$. We have two maps $\pi_g, \pi_1 : \mathscr{S}_{K_p K_g^p} \to \mathscr{S}_K$, induced by the actions of $g \in G(\mathbb{A}_f^p)$ and $1 \in G(\mathbb{A}_f^p)$ on $\mathscr{S}_{K_p}$, respectively. By our specific choices of $f^p$ and $dg^p$, the endomorphism $\Phi_{\mathfrak{p}}^m \times (f^p dg^p)$ of $\mathbf{H}_c^i(\mathscr{S}_{K,\overline{\mathbb{F}}_q}, \mathscr{L}_{\xi,K^p})$is induced by the correspondence

$$\begin{tikzcd}[column sep=small] & \mathscr{S}_{K_pK_g^p,\mathbb{F}_q} \arrow[dl, "\pi_g"'] \arrow[dr, "F^{mr}\circ\pi_1"] & \\ \mathscr{S}_{K,\mathbb{F}_q} & & \mathscr{S}_{K,\mathbb{F}_q} \end{tikzcd} \tag{3.7.3.3}$$

and the cohomological correspondence

$$\pi_g^* \mathscr{L}_{\xi,K^p} \xrightarrow{\overrightarrow{g}^*} \mathscr{L}_{\xi,K_g^p} \xrightarrow{(\overrightarrow{1}^*)^{-1}} \pi_1^* \mathscr{L}_{\xi,K^p} \cong \pi_1^*(F^{mr})^* \mathscr{L}_{\xi,K^p}. \tag{3.7.3.4}$$

(See §1.5.2 for $\overrightarrow{g}^*$ and $\overrightarrow{1}^*$.) Here $F$ is the absolute $p$-Frobenius endomorphism, and the last isomorphism in (3.7.3.4) is induced by the canonical isomorphism between any étale sheaf and its pull-back under $F$.

We now apply the Grothendieck–Lefschetz–Verdier trace formula together with Deligne's conjecture to compute (3.7.3.2). The latter has been proved by Fujiwara [Fuj97] and Varshavsky [Var05] (cf. also [Pin92b]), and states that the local terms in the trace formula can be replaced by the naive local terms, under the assumption that $m$ is sufficiently large (while fixing $K^p$ and $g$).

Let $\mathcal{FIX}$ be the set of $\overline{\mathbb{F}}_q$-valued fixed points of the correspondence (3.7.3.3). Using the bijection (3.7.1.1), we obtain a description of $\mathcal{FIX}$ as follows. For each admissible morphism $\phi$, by Lemma 3.7.2 we know that the data

$$\begin{cases} Y = X_p(\phi) \times X^p(\phi)'/K_g^p, \\ X = X_p(\phi) \times X^p(\phi)'/K^p, \\ I = I_\phi(\mathbb{Q}), \\ C = Z(\mathbb{Q})_K, \\ a : Y \to X, \ (y_p, y^p) \mapsto (y_p, y^p g \mod K^p), \\ b : Y \to X, \ (y_p, y^p) \mapsto (\Phi^m y_p, y^p \mod K^p). \end{cases} \tag{3.7.3.5}$$

satisfy the hypotheses of [Mil92, Lem. 5.3]. By *loc. cit.*, (3.7.1.1) induces a bijection

$$\mathcal{FIX} \cong \coprod_\phi \coprod_\epsilon \mathscr{O}(\phi, \epsilon, m, g), \tag{3.7.3.6}$$

where $\phi$ and $\epsilon$ run through the same ranges as in (3.7.3.1).

It remains to calculate the naive local term at each point in $\mathcal{FIX}$. We need to show that if $x \in \mathcal{FIX}$ corresponds to a point in $\mathscr{O}(\phi, \epsilon, m, g)$ under (3.7.3.6), then the naive local term at $x$ is equal to $\mathrm{tr}\,\xi(\epsilon)$. Note that $x$ only determines the conjugacy class of $\epsilon$ in $I_\phi(\mathbb{Q})/Z(\mathbb{Q})_K$. By our assumption that $\xi$ factors through $G^c$ and by Lemma 1.5.7, we know that $\mathrm{tr}\,\xi$ is invariant under $Z(\mathbb{Q})_K$ since $K$ is neat. Hence $\mathrm{tr}\,\xi$ defines a class function on $G(\overline{\mathbb{Q}})$ that is translation-invariant under $Z(\mathbb{Q})_K$, and our desideratum makes sense.

Now let $\widetilde{x}$ be a lift of $x$ in $\mathscr{S}_{K_p}(\overline{\mathbb{F}}_q)$. Since $x \in \mathcal{FIX}$, there exists $k_{\widetilde{x}} \in K^p$ such that

$$\Phi^m \widetilde{x} = \widetilde{x} g k_{\widetilde{x}}.$$

Let $k_\ell \in G(\mathbb{Q}_\ell)$ be the component of $k_{\widetilde{x}}$ at $\ell$, and let $g_\ell \in G(\mathbb{Q}_\ell)$ be the component of $g \in G(\mathbb{A}_f^p)$ at $\ell$. By the same argument as [Kot92b, p. 433], the naive local term at $x$ is given by

$$\operatorname{tr} \xi(k_\ell^{-1} g_\ell^{-1}).$$

If $x$ corresponds to an element of $\mathscr{O}(\phi, \epsilon, m, g)$ under (3.7.3.6), then $k_\ell^{-1} g_\ell^{-1}$ is conjugate to $\epsilon$ in $G(\overline{\mathbb{Q}}_\ell)$. Thus the naive local term at $x$ is $\operatorname{tr} \xi(\epsilon)$, as desired. $\square$

**Lemma 3.7.4.** *Let $(\phi, \epsilon) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}$, and let $\mathfrak{c} = \mathbf{t}_{\underline{\sigma}}(\phi, \epsilon)$. The set $\mathcal{O}(\phi, \epsilon, m, g)$ defined in Lemma 3.7.3 is empty unless $(\phi, \epsilon) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)$. In the latter case, we have $\mathfrak{c} \in \mathfrak{KP}_{\mathrm{a}}(p^n)$ by Proposition 3.5.8, and we define $c_1(\mathfrak{c}, K^p) O(\mathfrak{c}, m, f^p dg^p)$ as in §1.8.2 and §1.8.6. We have*

$$\#\mathcal{O}(\phi, \epsilon, m, g) = \iota_{I_\phi}(\epsilon)^{-1} c_1(\mathfrak{c}, K^p) O(\mathfrak{c}, m, f^p dg^p).$$

*Proof.* We write $\mathfrak{c} = (\gamma_0, \gamma, [b])$, where $\gamma_0 = \epsilon$, and let $(\gamma_0, \gamma, \delta) \in \mathfrak{KP}_{\mathrm{cla}}(p^n)$ be the classical Kottwitz parameter associated with $\mathfrak{c}$ (which is well defined up to equivalence). By construction, $(b, \epsilon) \in \mathrm{cls}_p(\phi, \epsilon)$, so $X_p(\phi, \epsilon, q^m)$ is identified with $X_{-\mu_X}(b, \epsilon, q^m)$; see §3.1.5. If $\mathcal{O}(\phi, \epsilon, m, g) \neq \emptyset$, then $X_p(\phi, \epsilon, q^m) \neq \emptyset$, and it follows that $(\phi, \epsilon)$ is $q^m$-admissible.

Now assume that $(\phi, \epsilon)$ is $q^m$-admissible. The computation of $\#\mathcal{O}(\phi, \epsilon, m, g)$ is essentially the same as the computation by Kottwitz in [Kot84a, §1.4, §1.5]. We explain how to make the transition from our setting to the setting of *loc. cit.* We write $Y^p$ for the set $\{y^p \in X^p(\phi)'/K^p_g \mid y^p \equiv \epsilon y^p g \mod K^p\}$, and write $Y_p$ for the set $X_p(\phi, \epsilon, q^m)$. Thus $\mathcal{O}(\phi, \epsilon, m, g) = I_{\phi,\epsilon}(\mathbb{Q}) \backslash (Y_p \times Y^p)$. Note that $\#\mathcal{O}(\phi, \epsilon, m, g)$ is equal to the cardinality of $I^0_{\phi,\epsilon}(\mathbb{Q}) \backslash (Y_p \times Y^p)$ multiplied by $\iota_{I_\phi}(\epsilon)^{-1}$. This is because, if an element $u \in I_{\phi,\epsilon}(\mathbb{Q})$ has a fixed point in $Y_p \times Y^p$, then by Lemma 3.7.2 (applied to $g = 1$), we must have $u \in Z(\mathbb{Q})_K \subset I^0_{\phi,\epsilon}(\mathbb{Q})$. Thus it remains to show that

$$\#\Big(I^0_{\phi,\epsilon}(\mathbb{Q}) \backslash (Y_p \times Y^p)\Big) = c_1(\mathfrak{c}) O(\mathfrak{c}, m, f^p dg^p). \tag{3.7.4.1}$$

As explained in §2.4.7, we identify $X^p(\phi)' = X^p(\phi)$ with a right $G(\mathbb{A}_f^p)$-coset inside $G(\bar{\mathbb{A}}_f^p)$. Fix $y_0^p \in X^p(\phi)' \subset G(\bar{\mathbb{A}}_f^p)$. Inspecting definitions, we see that $\gamma \in G(\mathbb{A}_f^p)$ is conjugate to $(\mathrm{Int}(y_0^p)^{-1} \circ \mathrm{Int}(\tau_\phi))(\epsilon) \in G(\mathbb{A}_f^p)$ inside $G(\mathbb{A}_f^p)$. We may assume that they are equal. Now if we use the "base point" $y_0^p$ to identify $X^p(\phi)'$ with $G(\mathbb{A}_f^p)$, then we have a bijection

$$Y^p \xrightarrow{\sim} W^p := \left\{ y^p \in G(\mathbb{A}_f^p)/K_g^p \mid y^p \equiv \gamma y^p g \mod K^p \right\}.$$

Under this bijection, the action of $I^0_{\phi,\epsilon}(\mathbb{Q})$ on $Y^p$ corresponds to the action of $I^0_{\phi,\epsilon}(\mathbb{Q})$ on $W^p$ given by the composition of the $\mathbb{A}_f^p$-isomorphism $\mathrm{Int}(y_0^p)^{-1} \circ \mathrm{Int}(\tau_\phi) : (I^0_{\phi,\epsilon})_{\mathbb{A}_f^p} \hookrightarrow (G^0_\gamma)_{\mathbb{A}_f^p}$ (see §3.2.14) followed by left multiplication of $G^0_\gamma(\mathbb{A}_f^p)$ on $W^p$.

Let $\mu \in \mathfrak{p}_X^{\mathcal{G}}$. We have already seen that $Y_p$ can be identified with $X_{-\mu}(b, \epsilon, q^m)$ as in §3.1.5. More precisely, write $I_0$ for $G^0_\epsilon$, and suppose that $b = b_{\mathrm{Int}(g^{-1}) \circ \phi_{I_0}(p) \circ \zeta_p}$

for $g \in \mathcal{UR}(\phi(p) \circ \zeta_p) \cap I_0(\overline{\mathbb{Q}}_p)$. (Here we follow the notation in §3.5.1.) Then $y_p \mapsto g^{-1} y_p$ induces a bijection

$$Y_p \xrightarrow{\sim} X_{-\mu}(b, \epsilon, q^m),$$

see §2.2.7 and §3.1.5. Moreover, under this bijection the action of $I^0_{\phi,\epsilon}(\mathbb{Q})$ on $Y_p$ corresponds to the action of $I^0_{\phi,\epsilon}(\mathbb{Q})$ on $X_{-\mu}(b, \epsilon, q^m)$ given by the composition of the $\mathbb{Q}_p$-isomorphism $\mathrm{Int}(g^{-1}) : (I^0_{\phi,\epsilon})_{\mathbb{Q}_p} \xrightarrow{\sim} J_b^{I_0}$ followed by left multiplication of $J_b^{I_0}(\mathbb{Q}_p)$ on $X_{-\mu}(b, \epsilon, q^m)$. Here, to see that $\mathrm{Int}(g^{-1}) : (I^0_{\phi,\epsilon})_{\mathbb{Q}_p} \xrightarrow{\sim} J_b^{I_0}$ is an isomorphism, use the identification $I^0_{\phi,\epsilon} \cong I_{\phi_{I_0}}$ as in §3.2.3 and §3.2.14, and use the fact that $b$ is basic in $I_0$ by Corollary 1.6.12.

If we fix $c \in G(\breve{\mathbb{Q}}_p)$ such that $\delta = c^{-1} b \sigma(c)$ and such that (1.6.5.1) holds, then the map $x \mapsto c^{-1}x$ induces a bijection

$$X_{-\mu}(b, \epsilon, q^m) \xrightarrow{\sim} W_p := \left\{ y_p \in G(\mathbb{Q}_{p^n})/\mathcal{G}(\mathbb{Z}_{p^n}) \mid y_p^{-1} \delta \sigma(y_p) \in \mathcal{G}(\mathbb{Z}_{p^n}) p^{-\mu} \mathcal{G}(\mathbb{Z}_{p^n}) \right\}.$$

Under this bijection, the action of $J_b^{I_0}(\mathbb{Q}_p)$ on the left corresponds to the following action on $W_p$: We have an injective $\mathbb{Q}_p$-homomorphism $J_b^{I_0} \to R^0_{\delta \rtimes \sigma}$ induced by $\mathrm{Int}(c^{-1})$. (See §1.8.2 for $R^0_{\delta \rtimes \theta}$.) This homomorphism is in fact an isomorphism, because both groups are connected, and their dimensions are equal to that of $I_0$. We thus identify $J_b^{I_0}(\mathbb{Q}_p)$ with $R^0_{\delta \rtimes \theta}(\mathbb{Q}_p)$, and let the latter group act on $W_p$ by left multiplication.

In conclusion, we have bijections

(3.7.4.2)

$$I^0_{\phi,\epsilon}(\mathbb{Q}) \backslash (Y_p \times Y^p) \xrightarrow{\sim} I^0_{\phi,\epsilon}(\mathbb{Q}) \backslash (X_{-\mu}(b, \epsilon, q^m) \times W^p) \xrightarrow{\sim} I^0_{\phi,\epsilon}(\mathbb{Q}) \backslash (W_p \times W^p),$$

where $I^0_{\phi,\epsilon}(\mathbb{Q})$ acts on $X_{-\mu}(b, \epsilon, q^m)$, $W^p$, and $W_p$ in the way described above. By abuse of notation, we still write $I^0_{\phi,\epsilon}(\mathbb{Q})$ for the image of $I^0_{\phi,\epsilon}(\mathbb{Q})$ inside $J_b^{I_0}(\mathbb{Q}_p) \times G^0_\gamma(\mathbb{A}_f^p) \cong R^0_{\delta \rtimes \theta}(\mathbb{Q}_p) \times G^0_\gamma(\mathbb{A}_f^p)$, under the embeddings described above. We also identify the last two product groups with $I(\mathfrak{c})(\mathbb{A}_f)$, canonically up to conjugation by $I(\mathfrak{c})^{\mathrm{ad}}(\mathbb{A}_f)$. We assume for a moment that $I^0_{\phi,\epsilon}(\mathbb{Q}) Z_K$ is closed and has finite co-volume inside $I(\mathfrak{c})(\mathbb{A}_f)$. Then the computation in [Kot84a, §1.5][21] shows that the cardinality of the third set in (3.7.4.2) is given by

$$\mathrm{vol}(I^0_{\phi,\epsilon}(\mathbb{Q}) Z_K \backslash I(\mathfrak{c})(\mathbb{A}_f)) \cdot O(\mathfrak{c}, m, f^p dg^p).$$

Here, $I^0_{\phi,\epsilon}(\mathbb{Q}) Z_K$ is equipped with the Haar measure giving volume 1 to its open subgroup $Z_K = Z_G(\mathbb{A}_f) \cap K$.

To complete the proof, we need to verify our assumption on $I^0_{\phi,\epsilon}(\mathbb{Q}) Z_K$, and we need to identify $\mathrm{vol}(I^0_{\phi,\epsilon}(\mathbb{Q}) Z_K \backslash I_{\mathfrak{c}}(\mathbb{A}_f))$ with $c_1(\mathfrak{c})$. For both purposes, it suffices to prove the following claim: The image of $I^0_{\phi,\epsilon}(\mathbb{Q})$ inside $I(\mathfrak{c})(\mathbb{A}_f)$ is $I(\mathfrak{c})^{\mathrm{ad}}(\mathbb{A}_f)$-conjugate to $I(\mathfrak{c})(\mathbb{Q})$. (Note that the Haar measure on $I(\mathfrak{c})(\mathbb{A}_f)$ is invariant under conjugation by $I(\mathfrak{c})^{\mathrm{ad}}(\mathbb{A}_f)$.)

To prove the claim, note that we have a canonical inner twisting between the $\mathbb{Q}$-groups $I^0_{\phi,\epsilon}$ and $I_0$, as in §3.2.14. We have described an $\mathbb{A}_f^p$-isomorphism $I^0_{\phi,\epsilon} \xrightarrow{\sim} G^0_\gamma$, and a $\mathbb{Q}_p$-isomorphism $I^0_{\phi,\epsilon} \xrightarrow{\sim} J_b^{I_0}$. One checks that these isomorphisms are isomorphisms between inner forms of $I_0$ (in the sense of Definition 1.2.2). Also,

[21]In *loc. cit.*, it is assumed that $G_{\mathrm{der}}$ is simply connected, so that $G_\gamma$ and $R_{\delta \rtimes \sigma}$ (which is denoted by $G^\sigma_\delta$) are connected. To transport the computation to the current situation, one simply replaces all appearances of $G_\gamma$ and $R_{\delta \rtimes \sigma}$ by their identity components.

$I^0_{\phi,\epsilon}/Z_G$ is anisotropic over $\mathbb{R}$, and up to isomorphism there is at most one inner form of $I_0$ over $\mathbb{R}$ which is anisotropic modulo $Z_G$. The claim then follows from the unique characterization of $I(\mathfrak{c})$ as an inner form of $I_0$ over $\mathbb{Q}$ (up to isomorphism), in Lemma 1.7.12. $\square$

**3.7.5.** We have a natural action

$$\begin{aligned} Z(\mathbb{Q})_K \times [\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)] &\longrightarrow [\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)] \\ (z, [\phi, \epsilon]) &\longmapsto [\phi, z\epsilon]. \end{aligned}$$

We fix a set of representatives for the $Z(\mathbb{Q})_K$-orbits in $[\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)]$ and by abuse of notation denote this subset of $[\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)]$ by

$$[\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)]/Z(\mathbb{Q})_K.$$

By the first statement in Lemma 3.7.4 and by Corollary 3.2.17, we may replace the two summations in (3.7.3.1) by the summation over $[\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)]/Z(\mathbb{Q})_K$. Applying Lemma 3.7.4 to the summands, we obtain

$$\begin{aligned} (3.7.5.1)\quad & T(\Phi^m_{\mathfrak{p}}, f^p dg^p) \\ &= \sum_{[\phi,\epsilon]\in[\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)]/Z(\mathbb{Q})_K} \iota_{I_\phi}(\epsilon)^{-1} c_1(\mathbf{t}_{\underline{\sigma}}(\phi,\epsilon))\ O\big(\mathbf{t}_{\underline{\sigma}}(\phi,\epsilon), m, f^p dg^p\big) \operatorname{tr}\xi(\epsilon), \end{aligned}$$

for all sufficiently large $m$. Here, for $(\phi,\epsilon) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)$, we know that the isomorphism class of $\mathbf{t}_{\underline{\sigma}}(\phi,\epsilon)$ depends only on $[\phi,\epsilon]$, by Proposition 3.6.1. Moreover, the value of

$$c_1(\mathbf{t}_{\underline{\sigma}}(\phi,\epsilon)) \cdot O\big(\mathbf{t}_{\underline{\sigma}}(\phi,\epsilon), m, f^p dg^p\big)$$

depends on $\mathbf{t}_{\underline{\sigma}}(\phi,\epsilon)$ only via its isomorphism class, which can be checked using the definitions.

Let $\Sigma_{K^p}$ be as in §1.8.7, and for each $\gamma_0 \in \Sigma_{K^p}$ we write $\mathfrak{KP}(\gamma_0, q^m)_\alpha$ for the set of $\mathfrak{c} \in \mathfrak{KP}(\gamma_0) \cap \mathfrak{KP}_{\mathrm{a}}(q^m)$ satisfying $\alpha(\mathfrak{c}) = 0$. (See §1.8.7 for the notation $\mathfrak{KP}(\gamma_0)$.) By Proposition 1.7.10, $\mathfrak{KP}(\gamma_0, q^m)_\alpha$ is stable under isomorphisms between Kottwitz parameters. Using Proposition 3.6.3, we rewrite (3.7.5.1) as

$$(3.7.5.2)\quad T(\Phi^m_{\mathfrak{p}}, f^p dg^p) = \sum_{\gamma_0\in\Sigma_{K^p}} \operatorname{tr}\xi(\gamma_0) \sum_{\mathfrak{c}\in\mathfrak{KP}(\gamma_0,q^m)_\alpha/\cong} c_1(\mathfrak{c}) O(\mathfrak{c}, m, f^p dg^p) \mathscr{A}(\mathfrak{c}),$$

where

$$(3.7.5.3)\qquad \mathscr{A}(\mathfrak{c}) := \sum_{\substack{[\phi,\epsilon]\in[\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)] \\ \mathbf{t}_{\underline{\sigma}}(\phi,\epsilon)\cong\mathfrak{c}}} \iota_{I_\phi}(\epsilon)^{-1}.$$

**Lemma 3.7.6.** *Let $\gamma_0 \in \Sigma_{K^p}$ and let $\mathfrak{c} \in \mathfrak{KP}(\gamma_0, q^m)_\alpha$. Assume that $O(\mathfrak{c}, m, f^p dg^p)$ is non-zero. Let $k$ be the number of elements of $\mathfrak{KP}(\gamma_0, q^m)_\alpha$ that are isomorphic to $\mathfrak{c}$. Then we have*

$$\mathscr{A}(\mathfrak{c}) = k \cdot \bar{\iota}_G(\gamma_0)^{-1} \cdot c_2(\gamma_0).$$

*Here the notation is as in §1.8.6 and §1.8.7.*

*Proof.* First note that $k$ is finite, since it is at most $\left|G_{\gamma_0}/G^0_{\gamma_0}\right|$. From the non-vanishing $O(\mathfrak{c}, m, f^p dg^p)$ (or rather just the non-vanishing of the twisted orbital integral at $p$), it follows that $\mathfrak{c}$ satisfies the assumptions of Proposition 3.6.5 (for $n = mr$). By that proposition, there exists an element of $\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)$ of the form $(\phi_0, \gamma_0)$ such that $\mathbf{t}_{\underline{\sigma}}(\phi_0, \gamma_0) = \mathfrak{c}$. By Proposition 3.2.20, all $[\phi, \epsilon]$ appearing in the summation in (3.7.5.3) necessarily lie inside $\mathcal{C}_{[\phi_0,\gamma_0]}$. On the other hand, for each $[\phi,\epsilon]\in\mathcal{C}_{[\phi_0,\gamma_0]}\cap[\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}]$, if $\mathbf{t}_{\underline{\sigma}}(\phi,\epsilon)\cong\mathfrak{c}$, then $[\phi,\epsilon]$ is automatically $q^m$-admissible, by Proposition 3.5.8. Hence we can rewrite (3.7.5.3) as

$$\mathscr{A}(\mathfrak{c})=\sum_{\substack{[\phi,\epsilon]\in\mathcal{C}_{[\phi_0,\gamma_0]}\cap[\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}]\\ \mathbf{t}_{\underline{\sigma}}(\phi,\epsilon)\cong\mathfrak{c}}}\iota_{I_\phi}(\epsilon)^{-1}. \tag{3.7.6.1}$$

Let $\widetilde{\text{Ш}}^\infty_G(\mathbb{Q},I^0_{\phi_0,\gamma_0})$ denote the inverse image of $\text{Ш}^\infty_G(\mathbb{Q},I^0_{\phi_0,\gamma_0})$ in $Z^1(\mathbb{Q},I^0_{\phi_0,\gamma_0})$. By Definition 3.2.9 and Proposition 3.2.19, we have a surjection

$$\begin{aligned}\widetilde{\text{Ш}}^\infty_G(\mathbb{Q},I^0_{\phi_0,\gamma_0})&\longrightarrow\mathcal{C}_{[\phi_0,\gamma_0]}\cap[\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}]\\ e&\longmapsto[e\phi_0,\gamma_0].\end{aligned}$$

which factors through a surjection

$$\eta_{\phi_0,\gamma_0}:\text{Ш}^\infty_G(\mathbb{Q},I^0_{\phi_0,\gamma_0})\longrightarrow\mathcal{C}_{[\phi_0,\gamma_0]}\cap[\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}].$$

Define $\widetilde{D}$ to be the subset of $\widetilde{\text{Ш}}^\infty_G(\mathbb{Q},I^0_{\phi_0,\gamma_0})$ consisting of elements $e$ such that

$$\mathbf{t}_{\underline{\sigma}}(e\phi_0,\gamma_0)\cong\mathfrak{c}.$$

Now let $\mathfrak{c}_1,\mathfrak{c}_2,\cdots,\mathfrak{c}_k$ be all the distinct elements of $\mathfrak{KP}(\gamma_0,q^m)_\alpha$ that are isomorphic to $\mathfrak{c}$. For each $e\in\widetilde{D}$, we have $\mathbf{t}_{\underline{\sigma}}(e\phi_0,\gamma_0)=\mathfrak{c}_i$ for a unique $i\in\{1,\cdots,k\}$. We thus obtain a partition

$$\widetilde{D}=\coprod_{i=1}^k\widetilde{D}_i.$$

By Proposition 3.6.2 (iii), for each $1\le i\le k$, the set $\widetilde{D}_i$ is the inverse image of a subset $D_i$ of $\text{Ш}^\infty_G(\mathbb{Q},I^0_{\phi_0,\gamma_0})$. We can thus rewrite (3.7.6.1) as

$$\mathscr{A}(\mathfrak{c})=\sum_{i=1}^k\sum_{\beta\in D_i}\frac{1}{\iota_{I_{e\phi_0}}(\gamma_0)\cdot\#\eta^{-1}_{\phi_0,\gamma_0}(\eta_{\phi_0,\gamma_0}(\beta))} \tag{3.7.6.2}$$

where $e$ is a cocycle representing $\beta$. (We will soon see that the fibers of $\eta_{\phi_0,\gamma_0}$ are indeed finite.)

By §3.2.14, the $\overline{\mathbb{Q}}$-embedding $(I^0_{\phi_0,\gamma_0})_{\overline{\mathbb{Q}}}\to(G^0_{\gamma_0})_{\overline{\mathbb{Q}}}$ is an isomorphism and is an inner twisting between the $\mathbb{Q}$-groups $I^0_{\phi_0,\gamma_0}$ and $G^0_{\gamma_0}$. Using this observation and Proposition 3.6.2, it is easy to see that for each $1\le i\le k$, the set $D_i$ is either empty, or a coset of $\text{Ш}(\mathbb{Q},I^0_{\phi_0,\gamma_0})$ inside $\text{Ш}^\infty_G(\mathbb{Q},I^0_{\phi_0,\gamma_0})$. We claim that it is never empty. Let $u_i\in G(\overline{\mathbb{Q}})$ be an isomorphism $\mathfrak{c}\xrightarrow{\sim}\mathfrak{c}_i$. Then $u_i\in G_{\gamma_0}(\overline{\mathbb{Q}})$, and $u_i{}^\tau u_i^{-1}\in G^0_{\gamma_0}(\overline{\mathbb{Q}}),\forall\tau\in\Gamma$. By Lemma 3.6.4, we have $(\mathrm{Int}(u_i)\circ\phi_0,\gamma_0)\in\mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)$. By Proposition 3.6.1, we have $\mathbf{t}_{\underline{\sigma}}(\mathrm{Int}(u_i)\circ\phi_0,\gamma_0)=\mathfrak{c}_i$. It remains to check that $\mathrm{Int}(u_i)\circ\phi_0$ is of the form $e\phi_0$ for some $e\in Z^1(\mathbb{Q},I^0_{\phi_0,\gamma_0})$. For this, we fix a lift $q_\tau\in\mathfrak{Q}$ of each $\tau\in\Gamma$, and write $\phi_0(q_\tau)=g_\tau\rtimes\tau$. Define $e_\tau:=u_ig_\tau{}^\tau u_i^{-1}g_\tau^{-1}$. Then $e=(e_\tau)_\tau$ is a cocycle in $Z^1(\mathbb{Q},I_{\phi_0})$, and $\mathrm{Int}(u_i)\circ\phi_0=e\phi_0$. It remains to check that $e_\tau\in I^0_{\phi_0,\gamma_0}(\overline{\mathbb{Q}})$ for each $\tau$. Let $\pi_0:=(G_{\gamma_0}/G^0_{\gamma_0})(\overline{\mathbb{Q}})$, which is an abelian group as explained in §1.8.7. We have seen that $u_i$ and ${}^\tau u_i$ map to the same element of $\pi_0$. Since $(\phi_0,\gamma_0)$ is gg, $g_\tau$ maps to the identity in $\pi_0$. Hence $e_\tau$ maps to the identity in $\pi_0$, i.e., $e_\tau\in G^0_{\gamma_0}(\overline{\mathbb{Q}})=I^0_{\phi_0,\gamma_0}(\overline{\mathbb{Q}})$, as desired.

We have proved the claim. Hence for each $1\le i\le k$, we have

$$|D_i|=\left|\text{Ш}(\mathbb{Q},I^0_{\phi_0,\gamma_0})\right|.$$

Since $I^0_{\phi_0,\gamma_0}$ is an inner form of $G^0_{\gamma_0}$ over $\mathbb{Q}$, this number is equal to $c_2(\gamma_0) = \left|\text{Ш}(\mathbb{Q}, G^0_{\gamma_0})\right|$.

To complete the proof of the lemma, it suffices to check that each summand in (3.7.6.2) is equal to $\bar{\iota}_G(\gamma_0)^{-1}$. Recall from §3.2.7 that the composition of $\eta_{\phi_0,\gamma_0}$ with the natural injection $\mathfrak{v} : [\mathcal{LRP}^{\mathrm{gg}}] \to \langle\mathcal{LRP}\rangle$ factors as

$$\mathbf{H}^1(\mathbb{Q}, I^0_{\phi_0,\gamma_0}) \to \mathbf{H}^1(\mathbb{Q}, I_{\phi_0,\gamma_0}) \xrightarrow{\iota_{\phi_0,\gamma_0}} \langle\mathcal{LRP}\rangle.$$

The map $\iota_{\phi_0,\gamma_0}$ is injective. Hence for each $\beta \in \mathbf{H}^1(\mathbb{Q}, I^0_{\phi_0,\gamma_0})$, the set

$$\eta^{-1}_{\phi_0,\gamma_0}(\eta_{\phi_0,\gamma_0}(\beta))$$

is equal to the fiber of the map $\mathbf{H}^1(\mathbb{Q}, I^0_{\phi_0,\gamma_0}) \to \mathbf{H}^1(\mathbb{Q}, I_{\phi_0,\gamma_0})$ containing $\beta$. This fiber can be identified with the kernel of $\mathbf{H}^1(\mathbb{Q}, I^0_{e\phi_0,\gamma_0}) \to \mathbf{H}^1(\mathbb{Q}, I_{e\phi_0,\gamma_0})$, where $e$ is a cocycle representing $\beta$. By [Lab04, Cor. III.1.3], the cardinality of this kernel is

$$\bar{\iota}_{I_{e\phi_0}}(\gamma_0) \cdot (\iota_{I_{e\phi_0}}(\gamma_0))^{-1}.$$

From this, we see that if a summand in (3.7.6.2) is indexed by $\beta$, then this summand is equal to $\bar{\iota}_{I_{e\phi_0}}(\gamma_0)^{-1}$ for any cocycle $e$ representing $\beta$. Since $\beta \in D_i$ for some $i$ and since $\mathfrak{c}_i$ is $q^m$-admissible (as $\mathfrak{c}$ is), we have $(e\phi_0, \gamma_0) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{a}}(q^m)$ by Proposition 3.5.8. Thus by §3.2.14, we have canonical inner twistings $(I_{e\phi_0,\gamma_0})_{\overline{\mathbb{Q}}} \xrightarrow{\sim} (G_{\gamma_0})_{\overline{\mathbb{Q}}}$ and $(I^0_{e\phi_0,\gamma_0})_{\overline{\mathbb{Q}}} \xrightarrow{\sim} (G^0_{\gamma_0})_{\overline{\mathbb{Q}}}$. In particular, we have an inner twisting between the commutative groups $I_{e\phi_0,\gamma_0}/I^0_{e\phi_0,\gamma_0}$ and $G_{\gamma_0}/G^0_{\gamma_0}$ (see §1.8.7), which must be an isomorphism over $\mathbb{Q}$. It follows that $\bar{\iota}_{I_{e\phi_0}}(\gamma_0) = \bar{\iota}_G(\gamma_0)$. The proof is complete. □

The proof of (1.8.8.1) is completed by combining (3.7.5.2) and Lemma 3.7.6.

# Part 2. Shimura varieties of abelian type

## 4. Results on crystalline representations

Throughout this section we fix a prime number $p$.

### 4.1. Generalities on fiber functors.

**4.1.1.** Let $G$ be a flat, finite-type, affine group scheme over $\mathbb{Z}_p$, and $\mathrm{Rep}_{\mathbb{Z}_p} G$ the category of representations of $G$ on finite free $\mathbb{Z}_p$-modules. For any commutative ring $S$, we write $\mathrm{Mod}_S^{\mathrm{fp}}$ for the category of finite projective $S$-modules. Now let $R$ be a faithfully flat $\mathbb{Z}_p$-algebra. By a *fiber functor over* $R$, we mean a faithful exact $\otimes$-functor $\omega : \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}_R^{\mathrm{fp}}$. We denote by $\mathbb{1}_R : \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}_R^{\mathrm{fp}}$ the functor which takes $L$ to $L \otimes R$. Then $\mathbb{1}_R$ is a fiber functor, called *the standard fiber functor*.

If $\omega : \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}_R^{\mathrm{fp}}$ is a $\otimes$-functor and $S$ is an $R$-algebra, we write $\omega_S$ for the composition of $\otimes$-functors

$$\mathrm{Rep}_{\mathbb{Z}_p} G \xrightarrow{\omega} \mathrm{Mod}_R^{\mathrm{fp}} \xrightarrow{\cdot \otimes_R S} \mathrm{Mod}_S^{\mathrm{fp}},$$

called the *base change* or *pull-back* of $\omega$ over $S$.

For two $\otimes$-functors $\omega_1, \omega_2 : \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}_R^{\mathrm{fp}}$, we write $\underline{\mathrm{Isom}}^{\otimes}(\omega_1, \omega_2)$ for the $R$-scheme of $\otimes$-isomorphisms from $\omega_1$ to $\omega_2$; see [Del90, §1.11]. Thus for any $R$-algebra $S$, $\underline{\mathrm{Isom}}^{\otimes}(\omega_1, \omega_2)(S)$ is the set of $\otimes$-isomorphisms $\omega_{1,S} \xrightarrow{\sim} \omega_{2,S}$. If $\omega_1 = \omega_2$, we write $\underline{\mathrm{Aut}}^{\otimes}(\omega_1)$ for $\underline{\mathrm{Isom}}^{\otimes}(\omega_1, \omega_1)$.

The reconstruction theorem, which is well known over a field, is valid in our current setting, since $\mathbb{Z}_p$ is a Dedekind domain and $G$ is affine flat. This theorem says that the natural morphism $G \to \underline{\mathrm{Aut}}^{\otimes}(\mathbb{1}_{\mathbb{Z}_p})$ is an isomorphism of $\mathbb{Z}_p$-group schemes; see for instance [Mil12, X, Thm. 1.2, Rmk. 1.6], or [Wed04, Thm. 5.17].

Let $\mathrm{Rep}_{\mathbb{Q}_p} G$ be the category of $G$-representations on finite-dimensional $\mathbb{Q}_p$-vector spaces. By [Ser68, §1.5], every representation in $\mathrm{Rep}_{\mathbb{Q}_p} G$ contains a $G$-stable $\mathbb{Z}_p$-lattice (since $\mathbb{Z}_p$ is noetherian and $G$ is affine flat). Using this fact, for each fiber functor $\omega : \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}_R^{\mathrm{fp}}$ we can define a $\otimes$-functor

$$\omega[1/p] : \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Mod}_{R[1/p]}^{\mathrm{fp}}$$

as follows: If $V$ is in $\mathrm{Rep}_{\mathbb{Q}_p} G$, then we write $V = \varinjlim_i V_i$ as a direct limit of $G$-stable $\mathbb{Z}_p$-lattices. We set $\omega[1/p](V) = \varinjlim_i \omega(V_i)$, which is naturally isomorphic to $\omega(V_i) \otimes_R R[1/p]$ for any $i$. Note that $\omega[1/p]$ is again a fiber functor, i.e., a faithful exact $\otimes$-functor.

Given the above construction, it is easy to see that the category of fiber functors $\mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}_R^{\mathrm{fp}}$ is equivalent to the category of fiber-wise faithful exact functors between fibered categories $\mathbf{Rep}\ G \to \mathbf{Bun}_{\mathrm{Spec}\, R}$ (fibered over the small Zariski site of $\mathrm{Spec}\, \mathbb{Z}_p$), as considered in [Bro13]. Thus we shall freely import results from *loc. cit.* for the fiber functors in our sense. Also cf. the last remark in the introduction of *loc. cit.*

Let $\omega : \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}_R^{\mathrm{fp}}$ be a fiber functor. Then $\underline{\mathrm{Isom}}^{\otimes}(\mathbb{1}_R, \omega)$ has a right $G_R$-action via the natural homomorphism $G_R \to \underline{\mathrm{Aut}}^{\otimes}(\mathbb{1}_R)$ (which we have seen is an isomorphism). By [Bro13, Thm. 4.8], $\underline{\mathrm{Isom}}^{\otimes}(\mathbb{1}_R, \omega)$ is in fact a right $G_R$-torsor over $R$ (for the fppf topology). This result could be viewed as a generalization of the reconstruction theorem recalled above. In the sequel, we denote the $G_R$-torsor $\underline{\mathrm{Isom}}^{\otimes}(\mathbb{1}_R, \omega)$ by $P_\omega$.

**4.1.2.** Let $G$ and $R$ be as in §4.1.1. We introduce a short-hand notation. Let $\omega : \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}_R^{\mathrm{fp}}$ be a fiber functor, and let $S$ be a $R[1/p]$-algebra. Then the base change $\omega_S : \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}_S^{\mathrm{fp}}$ of $\omega$ over $S$ factors as

$$\mathrm{Rep}_{\mathbb{Z}_p} G \xrightarrow{\cdot \otimes \mathbb{Q}_p} \mathrm{Rep}_{\mathbb{Q}_p} G \xrightarrow{\omega[1/p]} \mathrm{Mod}_{R[1/p]}^{\mathrm{fp}} \xrightarrow{\cdot \otimes_{R[1/p]} S} \mathrm{Mod}_S^{\mathrm{fp}}.$$

We denote the composite functor

$$(\cdot \otimes_{R[1/p]} S) \circ \omega[1/p] : \mathrm{Rep}_{\mathbb{Q}_p} G \longrightarrow \mathrm{Mod}_S^{\mathrm{fp}}$$

again by $\omega_S$.

**Definition 4.1.3.** For any finite free module $M$ over any commutative ring $S$, we denote by $M^{\otimes}$ the direct sum of all the $S$-modules which can be formed from $M$ using the operations of taking duals, tensor products, symmetric powers, and exterior powers. Elements of $M^{\otimes}$ are called *tensors over* $M$.

**4.1.4.** Let $G$ and $R$ be as in §4.1.1. Let $\omega : \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}_R^{\mathrm{fp}}$ be a fiber functor. Then $\omega$ is compatible with the operations considered in Definition 4.1.3, by [Bro13, Thm. 4.8, Rmk. 4.2]. (The compatibility with taking duals follows from rigidity; see [DM82, Prop. 1.9].) In this case, for each $L$ in $\mathrm{Rep}_{\mathbb{Z}_p} G$ and each $G$-invariant element $s \in L^{\otimes}$, we may think of $s$ as a morphism $\mathbf{1} \to L^{\otimes}$ (where $\mathbf{1}$ is the unit object) and apply $\omega$ to it. We then get a morphism $\omega(s) : \mathbf{1} \to \omega(L^{\otimes}) \cong \omega(L)^{\otimes}$, or equivalently an element $\omega(s) \in \omega(L)^{\otimes}$. Here, it is understood that $\omega$ has been extended to infinite direct sums of objects, when we apply it to $L^{\otimes}$.

**Definition 4.1.5.** Let $G$ be a flat, finite-type, affine group scheme over $\mathbb{Z}_p$. We say that $G$ is *definable by tensors*, if there exists $L$ in $\mathrm{Rep}_{\mathbb{Z}_p} G$, and a subset $(s_\alpha)_{\alpha \in \boldsymbol{\alpha}} \subset L^{\otimes}$, such that the $\mathbb{Z}_p$-homomorphism $G \to \mathrm{GL}(L)$ is a closed embedding whose image is the point-wise stabilizer of $(s_\alpha)_{\alpha \in \boldsymbol{\alpha}}$. When this is the case we call $(L, (s_\alpha)_{\alpha \in \boldsymbol{\alpha}})$ a *defining datum for* $G$.

*Remark* 4.1.6. If $G$ is a flat, finite-type, affine group scheme over $\mathbb{Z}_p$ such that $G_{\mathbb{Q}_p}$ is reductive, then $G$ is definable by tensors, by combining [Bro13, Lem. 3.2] and [Kis10, Prop. 1.3.2].

**4.1.7.** Let $G$ be a flat, finite-type, affine group scheme over $\mathbb{Z}_p$ that is definable by tensors. Fix a defining datum $(L, (s_\alpha)_{\alpha \in \boldsymbol{\alpha}})$ for $G$. Let $R$ be a faithfully flat $\mathbb{Z}_p$-algebra. It will be useful to give a description of more explicit data giving rise to a fiber functor over $R$.

Let $D$ be a finite free $R$-module equipped with a collection of tensors $(s_{\alpha,0})_{\alpha \in \boldsymbol{\alpha}} \subset D^{\otimes}$ indexed by the set $\boldsymbol{\alpha}$, and suppose that there exists an $R$-module isomorphism $L \otimes_{\mathbb{Z}_p} R \xrightarrow{\sim} D$ taking each $s_\alpha$ to $s_{\alpha,0}$. (Obviously such $D$ and $(s_{\alpha,0})_{\alpha \in \boldsymbol{\alpha}}$ always exist.)

**Lemma 4.1.8.** *Keep the setting of §4.1.7. There exists a fiber functor*

$$\omega : \mathrm{Rep}_{\mathbb{Z}_p} G \longrightarrow \mathrm{Mod}_R^{\mathrm{fp}}$$

*equipped with an isomorphism* $\iota : \omega(L) \xrightarrow{\sim} D$ *such that* $\iota$ *maps* $\omega(s_\alpha) \in \omega(L)^{\otimes}$ *to* $s_{\alpha,0}$ *for each* $\alpha \in \boldsymbol{\alpha}$. *The pair* $(\omega, \iota)$ *is unique up to unique isomorphism in the following sense. Given two such pairs* $(\omega, \iota)$, $(\omega', \iota')$, *there is a unique* $\otimes$*-isomorphism* $\omega \xrightarrow{\sim} \omega'$ *which takes* $\iota$ *to* $\iota'$.

*Proof.* To show the existence of $\omega$, we set $\omega(Q) = Q \otimes_{\mathbb{Z}_p} R$ for $Q$ in $\mathrm{Rep}_{\mathbb{Z}_p} G$, and we take $\iota$ to be any isomorphism $L \otimes_{\mathbb{Z}_p} R \xrightarrow{\sim} D$ taking $s_\alpha$ to $s_{\alpha,0}$. (Such an isomorphism is assumed to exist in §4.1.7.)

For the uniqueness, consider two such pairs $(\omega, \iota)$, $(\omega', \iota')$. We may assume that $(\omega, \iota)$ is as constructed above, so in particular $\omega = \mathbb{1}_R$. Consider the $R$-scheme $\underline{\mathrm{Isom}}_{(s_\alpha)}(\omega(L), \omega(L'))$ whose points valued in any $R$-algebra $S$ classify $S$-linear isomorphisms $\omega(L) \otimes_R S \xrightarrow{\sim} \omega'(L) \otimes_R S$ taking $\omega(s_\alpha)$ to $\omega'(s_\alpha)$ for each $\alpha$. Using the existence of $\iota$ and $\iota'$, one sees that $\underline{\mathrm{Isom}}_{(s_\alpha)}(\omega(L), \omega(L'))$ is a trivial $G_R$-torsor. (Here $G_R$ acts on $\omega(L) = L \otimes_{\mathbb{Z}_p} R$.) There is a natural $G_R$-equivariant map

$$P_{\omega'} = \underline{\mathrm{Isom}}^{\otimes}(\mathbb{1}_R, \omega') \longrightarrow \underline{\mathrm{Isom}}_{(s_\alpha)}(\omega(L), \omega(L')),$$

which is necessarily an isomorphism since $P_{\omega'}$ is a $G_R$-torsor. It follows that $P_{\omega'}$ is a trivial $G_R$-torsor, and there exists an isomorphism of $\otimes$-functors $\omega \xrightarrow{\sim} \omega'$ which is unique up to multiplication by elements of $G(R)$. In particular, there is a unique choice of such an isomorphism which takes $\iota$ to $\iota'$. □

*Remark* 4.1.9. The proof of Lemma 4.1.8 also shows that if $(\omega, \iota)$ is a pair as in that lemma, then $\omega$ is $\otimes$-isomorphic to $\mathbb{1}_R$.

**Lemma 4.1.10.** *Keep the setting of §4.1.7, and let $(\omega, \iota)$ be a pair as in Lemma 4.1.8. Let $\omega'$ be a fiber functor $\mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Mod}^{\mathrm{fp}}_{R[1/p]}$, and let $\iota'$ be an isomorphism $\omega'(L[1/p]) \xrightarrow{\sim} D[1/p]$ taking $\omega'(s_\alpha) \in \omega'(L[1/p])^{\otimes}$ to $s_{\alpha,0} \in D[1/p]^{\otimes}$ for each $\alpha \in \boldsymbol{\alpha}$. Then there is a unique $\otimes$-isomorphism $\omega[1/p] \xrightarrow{\sim} \omega'$ which takes $\iota$ to $\iota'$.*

*Proof.* By Remark 4.1.9, we may assume that $\omega = \mathbb{1}_R$ without loss of generality. The proof of the lemma is then completely analogous to the proof of the uniqueness in Lemma 4.1.8. □

**Lemma 4.1.11.** *Let $G$ be a smooth affine group scheme over $\mathbb{Z}_p$ with connected fibers. (We do not need to assume that $G$ is definable by tensors.) Let $R$ be the ring of integers in either a finite unramified extension of $\mathbb{Q}_p$ or $\mathbb{Q}_p^{\mathrm{ur}}$. Let $\omega : \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}^{\mathrm{fp}}_R$ be a fiber functor. Then $P_\omega$ is a trivial $G_R$-torsor. In particular, the set $P_\omega(R)$ is a $G(R)$-torsor.*

*Proof.* Our assumptions, together with Lang's theorem, imply that any $G_R$-torsor over $R$, such as $P_\omega$, is necessarily trivial. □

**4.1.12.** We continue to consider $G$ and $R$ as in §4.1.1. For each fiber functor $\omega : \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}^{\mathrm{fp}}_R$ we set

$$Y(\omega) := P_\omega(R[1/p]) = \{\otimes\text{-isomorphisms } \mathbb{1}_{R[1/p]} \xrightarrow{\sim} \omega_{R[1/p]}\}.$$

This is either empty or a right $G(R[1/p])$-torsor (i.e., for $\eta \in Y(\omega)$ and $g \in G(R[1/p])$, we define $\eta \cdot g := \eta \circ g$). When $P_\omega(R)$ is non-empty and thus a (right) $G(R)$-torsor, there is a canonical isomorphism

$$Y(\omega) \cong P_\omega(R) \times^{G(R)} G(R[1/p]).$$

In this case we write $Y(\omega)^\circ$ for $P_\omega(R)$ when we view it as a subset of $Y(\omega)$. Its elements are called *integral points* of $Y(\omega)$.

4.2. **Integral $F$-isocrystals with $G$-structure.**

**4.2.1.** Let $K_0$ be either a finite unramified extension of $\mathbb{Q}_p$, or $\mathbb{Q}_p^{\mathrm{ur}}$. Let $\sigma \in \mathrm{Aut}(K_0)$ be the arithmetic $p$-Frobenius. We will apply the considerations in §4.1 to $F$-isocrystals over $K_0$ with additional structures.

Recall that an *$F$-isocrystal over $K_0$* is a finite-dimensional $K_0$-vector space $V$, equipped with an isomorphism $\varphi_V : \sigma^* V \xrightarrow{\sim} V$ called the *Frobenius*. We denote by $\mathrm{Isoc}_{K_0}$ the category of $F$-isocrystals over $K_0$.

Let $\mathrm{Isoc}^{\circ}_{K_0}$ be the category of pairs $(M, \varphi_M)$, where $M$ is a finite free $\mathcal{O}_{K_0}$-module, and $(M[1/p], \varphi_M)$ is an $F$-isocrystal. We shall often write $\varphi$ for $\varphi_M$ when no confusion can arise. Morphisms in this category are by definition morphisms of $\mathcal{O}_{K_0}$-modules that respect the Frobenii (after inverting $p$). We call an object of $\mathrm{Isoc}^{\circ}_{K_0}$ an *integral $F$-isocrystal over $K_0$*. Note that $\mathrm{Isoc}^{\circ}_{K_0}$ contains the category of the usual $F$-crystals, but has the advantage of containing duals of objects. We write

$$v : \mathrm{Isoc}^{\circ}_{K_0} \longrightarrow \mathrm{Mod}^{\mathrm{fp}}_{\mathcal{O}_{K_0}}$$

for the forgetful functor, taking each $(M, \varphi_M)$ to $M$. (Here $v$ stands for "vergessen", as in [RZ96, §1].)

Let $G$ be a flat, finite-type, affine group scheme over $\mathbb{Z}_p$. By an *integral $F$-isocrystal with $G$-structure over $K_0$*, we mean a faithful, exact, $\otimes$-functor

$$\Upsilon : \mathrm{Rep}_{\mathbb{Z}_p} G \longrightarrow \mathrm{Isoc}^{\circ}_{K_0}.$$

Equivalently, $\Upsilon$ is a $\otimes$-functor such that the composition $v \circ \Upsilon : \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}^{\mathrm{fp}}_{\mathcal{O}_{K_0}}$ is a fiber functor. Note that we do not require $v \circ \Upsilon$ to be equal to $\mathbb{1}_{K_0}$.

Similarly, by an *$F$-isocrystal with $G$-structure over $K_0$*, we mean a faithful, exact, $\otimes$-functor

$$\Upsilon : \mathrm{Rep}_{\mathbb{Q}_p} G \longrightarrow \mathrm{Isoc}_{K_0}.$$

If $\Upsilon$ is an integral $F$-isocrystal with $G$-structure, then the similar construction as in §4.1.1 yields a natural $F$-isocrystal with $G$-structure $\Upsilon[1/p]$.

We denote the categories of integral $F$-isocrystals (resp. $F$-isocrystals) with $G$-structure over $K_0$ by $G$-$\mathrm{Isoc}^{\circ}_{K_0}$(resp. $G$-$\mathrm{Isoc}_{K_0}$). In these categories, morphisms are by definition $\otimes$-isomorphisms between $\otimes$-functors $\mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Isoc}^{\circ}_{K_0}$ (resp. between $\otimes$-functors.

**4.2.2.** Let $G$ be a flat, finite-type, affine group scheme over $\mathbb{Z}_p$ that is definable by tensors, and fix a defining datum $(L, (s_\alpha)_{\alpha \in \boldsymbol{\alpha}})$ for $G$ (see §4.1.7). Let $(D, \varphi_D)$ be an object in $\mathrm{Isoc}^{\circ}_{K_0}$, equipped with a collection of $\varphi_D$-invariant tensors $(s_{\alpha,0})_{\alpha \in \boldsymbol{\alpha}} \subset D^{\otimes}$, such that there exists an $\mathcal{O}_{K_0}$-module isomorphism $L \otimes \mathcal{O}_{K_0} \xrightarrow{\sim} D$ taking each $s_\alpha$ to $s_{\alpha,0}$. (Obviously such a tuple $(D, \varphi_D, (s_{\alpha,0})_{\alpha \in \boldsymbol{\alpha}})$ always exists.)

**Lemma 4.2.3.** *Keep the setting of §4.2.2. There exists an integral $F$-isocrystal with $G$-structure $\Upsilon$ equipped with an isomorphism $\iota : \Upsilon(L) \xrightarrow{\sim} (D, \varphi_D)$ in $\mathrm{Isoc}^{\circ}_{K_0}$ such that $\iota$ maps $\Upsilon(s_\alpha) \in \Upsilon(L)^{\otimes}$ to $s_{\alpha,0}$ for each $\alpha \in \boldsymbol{\alpha}$. The pair $(\Upsilon, \iota)$ is unique up to unique isomorphism in the following sense. Given two such pairs $(\Upsilon, \iota)$, $(\Upsilon', \iota')$, there is a unique isomorphism $\Upsilon \xrightarrow{\sim} \Upsilon'$ in the category $G$-$\mathrm{Isoc}^{\circ}_{K_0}$ which takes $\iota$ to $\iota'$.*

*Proof.* Fix an $\mathcal{O}_{K_0}$-module isomorphism $\iota : L \otimes \mathcal{O}_{K_0} \simeq D$ taking each $s_\alpha$ to $s_{\alpha,0}$. The composite map

$$L \otimes K_0 \xrightarrow[\sigma^*\iota]{\sim} \sigma^* D[1/p] \xrightarrow{\varphi_D} D[1/p] \xrightarrow[\iota^{-1}]{\sim} L \otimes K_0$$

fixes $(s_\alpha)$, and hence has the form $\delta\sigma$ for some $\delta \in G(K_0)$. For $Q$ in $\mathrm{Rep}_{\mathbb{Z}_p} G$ we set $\Upsilon(Q) = Q \otimes \mathcal{O}_{K_0}$, and set $\varphi_Q := \delta\sigma : \sigma^* Q[1/p] \xrightarrow{\sim} Q$. This shows the existence of $(\Upsilon, \iota)$.

To show uniqueness, let $(\Upsilon, \iota)$, $(\Upsilon', \iota')$, be two pairs as in the lemma. Let $\omega = v \circ \Upsilon$ and $\omega' = v \circ \Upsilon'$. By Lemma 4.1.8, there is a unique isomorphism $\eta : \omega \xrightarrow{\sim} \omega'$ of fiber functors over $\mathcal{O}_{K_0}$ taking $\iota$ to $\iota'$. In particular the isomorphism $\eta(L) : \omega(L) \xrightarrow{\sim} \omega'(L)$ is compatible with the Frobenii on $\omega(L)[1/p]$ and on $\omega'(L)[1/p]$. Since for any $Q$ in $\mathrm{Rep}_{\mathbb{Z}_p} G$, the $\mathbb{Q}_p$-representation $Q \otimes \mathbb{Q}_p$ of $G$ is a subquotient of $(L \otimes \mathbb{Q}_p)^{\otimes}$, this implies that the isomorphism $\eta(Q) : \omega(Q) \xrightarrow{\sim} \omega'(Q)$ is compatible with the Frobenii on $\omega(Q)[1/p]$ and on $\omega'(Q)[1/p]$. Hence the isomorphism $\eta : \omega \xrightarrow{\sim} \omega'$ comes from a (necessarily unique) isomorphism $\Upsilon \xrightarrow{\sim} \Upsilon'$ in $G\text{-}\mathrm{Isoc}^\circ_{K_0}$. $\square$

**Lemma 4.2.4.** *Keep the setting of §4.2.2, and let $(\Upsilon, \iota)$ be a pair as in Lemma 4.2.3. Let $\Upsilon'$ be an $F$-isocrystal with $G$-structure, and let $\iota'$ be an isomorphism $\Upsilon'(L[1/p]) \xrightarrow{\sim} (D[1/p], \varphi_D)$ in $\mathrm{Isoc}_{K_0}$ such that $\iota'$ maps $\Upsilon'(s_\alpha) \in \Upsilon'(L[1/p])^{\otimes}$ to $s_{\alpha,0} \in D[1/p]^{\otimes}$ for each $\alpha \in \boldsymbol{\alpha}$. Then there is a unique isomorphism $\Upsilon[1/p] \xrightarrow{\sim} \Upsilon'$ in the category $G\text{-}\mathrm{Isoc}_{K_0}$ which takes $\iota$ to $\iota'$.*

*Proof.* Let $\omega = v \circ \Upsilon[1/p]$ and $\omega' = v \circ \Upsilon'[1/p]$. These are fiber functors $\mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Mod}^{\mathrm{fp}}_{K_0}$. By Lemma 4.1.10, there is a unique $\otimes$-isomorphism $\omega \xrightarrow{\sim} \omega'$ which takes $\iota$ to $\iota'$. By exactly the same argument as in the proof of Lemma 4.2.3, this isomorphism comes from a unique isomorphism $\Upsilon[1/p] \xrightarrow{\sim} \Upsilon'$ in $G\text{-}\mathrm{Isoc}_{K_0}$. $\square$

**4.2.5.** Let $G$ be a smooth affine group scheme over $\mathbb{Z}_p$ with connected fibers. Let $\Upsilon$ be an integral $F$-isocrystal with $G$-structure over $K_0$, and write $\omega$ for $v \circ \Upsilon$. By Lemma 4.1.11, $P_\omega(\mathcal{O}_{K_0})$ is a non-empty $G(\mathcal{O}_{K_0})$-torsor. Then we have the $G(K_0)$-torsor $Y(\omega)$, and a canonical $G(\mathcal{O}_{K_0})$-torsor $Y(\omega)^\circ \subset Y(\omega)$, as in §4.1.12. In the sequel, we shall write $Y(\Upsilon)$ and $Y(\Upsilon)^\circ$ for $Y(\omega)$ and $Y(\omega)^\circ$ respectively.

Now consider an element $y \in Y(\Upsilon)$. For each $L$ in $\mathrm{Rep}_{\mathbb{Z}_p} G$, the $K_0$-linear isomorphism $L \otimes_{\mathbb{Z}_p} K_0 \xrightarrow{\sim} \Upsilon(L)[1/p]$ induced by $y$ allows us to view the Frobenius on $\Upsilon(L)[1/p]$ as being given by $\delta_{y,L} \cdot \sigma$ for some $\delta_{y,L} \in \mathrm{GL}(L \otimes K_0)$. Since every representation in $\mathrm{Rep}_{\mathbb{Q}_p} G$ contains a $G$-stable $\mathbb{Z}_p$-lattice (see §4.1.1), the reconstruction theorem over a field implies that the elements $\delta_{y,L}$ for all $L$ come from a unique element $\delta_y \in G(K_0)$. Thus we have obtained a map

$$Y(\Upsilon) \longrightarrow G(K_0), \quad y \longmapsto \delta_y.$$

It is clear that the set $\{\delta_y \mid y \in Y(\Upsilon)\}$ is a $G(K_0)$-orbit under $\sigma$-conjugation, and the set $\{\delta_y \mid y \in Y(\Upsilon)^\circ\}$ is a $G(\mathcal{O}_{K_0})$-orbit under $\sigma$-conjugation. We call the last $G(\mathcal{O}_{K_0})$-orbit the *invariant* of $\Upsilon$, and denote it by $\mathrm{inv}(\Upsilon)$.

**Lemma 4.2.6.** *Keep the assumptions in §4.2.5. The construction $\Upsilon \mapsto \mathrm{inv}(\Upsilon)$ induces a bijection from the set of isomorphism classes in the category $G\text{-}\mathrm{Isoc}^\circ_{K_0}$, to the set of $G(\mathcal{O}_{K_0})$-$\sigma$-conjugacy classes in $G(K_0)$.*

*Proof.* It is easy to check that the map described in the lemma is well defined. The inverse of the map is induced by the following construction: Given $\delta \in G(K_0)$, we define a functor $\Upsilon_\delta : \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Isoc}^\circ_{K_0}$ by sending $L$ to $L \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_0}$ and equipping $L \otimes_{\mathbb{Z}_p} K_0$ with $\delta\sigma$ as the Frobenius. □

**Definition 4.2.7.** Let $G$ be a reductive group scheme over $\mathbb{Z}_p$, and let $\upsilon : \mathfrak{G}_m \to G_{\mathcal{O}_{K_0}}$ be a homomorphism of $\mathcal{O}_{K_0}$-group schemes. Let $\Upsilon$ be an integral $F$-isocrystal with $G$-structure over $K_0$. With the notation as in §4.2.5, we define

$$Y_\upsilon(\Upsilon) := \{y \in Y(\Upsilon) \mid \delta_y \in G(\mathcal{O}_{K_0})\upsilon(p)G(\mathcal{O}_{K_0})\}.$$

This is a subset of $Y(\Upsilon)$ stable under the $G(\mathcal{O}_{K_0})$-action. We define

$$X_\upsilon(\Upsilon) := Y_\upsilon(\Upsilon)/G(\mathcal{O}_{K_0}).$$

*Remark* 4.2.8. In the setting of Definition 4.2.7, if we fix an element $y_0 \in Y(\Upsilon)$ and use it to identify the $G(K_0)$-torsor $Y(\Upsilon)$ with $G(K_0)$, then $X_\upsilon(\Upsilon)$ is identified with

$$\left\{g \in G(K_0)/G(\mathcal{O}_{K_0}) \mid g^{-1}\delta_{y_0}\sigma(g) \in G(\mathcal{O}_{K_0})\upsilon(p)G(\mathcal{O}_{K_0})\right\}.$$

When $K_0 = \mathbb{Q}_p^{\mathrm{ur}}$, the above is the affine Deligne–Lusztig set $X_\upsilon(\delta_{y_0})$ introduced in §2.2.7.

## 4.3. Crystalline representations with $G$-structure.

**4.3.1.** Let $G$ be a connected reductive group over $\mathbb{Q}_p$. Let $K/\mathbb{Q}_p$ be a finite extension inside $\overline{\mathbb{Q}}_p$, and let $K_0$ be the maximal unramified extension of $\mathbb{Q}_p$ inside $K$.[22] We denote by $\mathrm{RZ}_G(K)$ the set of pairs $(\mu, \delta)$, where $\mu$ is a $K$-homomorphism $\mathbb{G}_{m,K} \to G_K$, and $\delta \in G(K_0)$. These pairs are considered by Rapoport–Zink in [RZ96, §1].

Let $\mathrm{MF}^\varphi_K$ be the category of filtered $\varphi$-modules over $K$. This is a Tannakian category, equipped with the fiber functor $v : \mathrm{MF}^\varphi_K \to \mathrm{Mod}^{\mathrm{fp}}_{K_0}$ taking each filtered $\varphi$-module to its underlying $K_0$-vector space. There is a bijection

$$(\mu, \delta) \longmapsto \mathcal{I}_{\mu,\delta}$$

from the set $\mathrm{RZ}_G(K)$ to the set of faithful exact $\otimes$-functors $\mathcal{I} : \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{MF}^\varphi_K$ such that $v \circ \mathcal{I}$ is equal to the standard fiber functor

$$\mathbb{1}_{K_0} : \mathrm{Rep}_{\mathbb{Q}_p} G \longrightarrow \mathrm{Mod}^{\mathrm{fp}}_{K_0}, \quad V \longmapsto V \otimes_{\mathbb{Q}_p} K_0.$$

We refer the reader to [RZ96, §1] for details.

We say that two elements $(\mu_1, \delta_1)$ and $(\mu_2, \delta_2)$ of $\mathrm{RZ}_G(K)$ are *equivalent*, if $\mathcal{I}_{\mu_1,\delta_1}$ and $\mathcal{I}_{\mu_2,\delta_2}$ are $\otimes$-isomorphic. We denote this equivalence relation on $\mathrm{RZ}_G(K)$ by $\sim$. We call an element $(\mu, \delta)$ of $\mathrm{RZ}_G(K)$ *admissible*, if $\mathcal{I}_{\mu,\delta}$ lands in the subcategory $\mathrm{MF}^{\varphi,\mathrm{a}}_K$ of admissible filtered $\varphi$-modules. We call an element $(\mu, \delta)$ of $\mathrm{RZ}_G(K)$ *neutral*, if $\kappa_G(\delta) = [\mu] \in \pi_1(G)_{\Gamma_p}$. Here $\kappa_G$ is the Kottwitz map $\mathrm{B}(G) \to \pi_1(G)_{\Gamma_p}$ as in §1.4.2. We denote by $\mathrm{RZ}^{\mathrm{a}}_G(K)$ (resp. $\mathrm{RZ}^{\mathrm{n}}_G(K)$) the subset of admissible (resp. neutral) elements of $\mathrm{RZ}_G(K)$. We also write $\mathrm{RZ}^{\mathrm{a,n}}_G(K)$ for $\mathrm{RZ}^{\mathrm{a}}_G(K) \cap \mathrm{RZ}^{\mathrm{n}}_G(K)$.

It is easy to see (cf. [RZ96, Def. 1.23]) that two elements $(\mu_1, \delta_1), (\mu_2, \delta_2) \in \mathrm{RZ}_G(K)$ are equivalent if and only if there exists $g \in G(K_0)$ such that $g\delta_1\sigma(g)^{-1} = \delta_2$ and such that $\mathrm{Int}(g) \circ \mu_1$ and $\mu_2$ define the same filtration (in the sense of [SR72, IV, §2]) on the fiber functor $\mathbb{1}_K : \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Mod}^{\mathrm{fp}}_K$. By [SR72, IV, 2.2.5 (2)], the last condition on $\mu_1$ and $\mu_2$ implies that $\mu_1$ and $\mu_2$ are conjugate by $G(K)$. From

[22] The notations $K$ and $K_0$ are standard in $p$-adic Hodge theory. In the context of Shimura varieties, we often use $K$ to denote the level. In such a case we will use notations such as $F$ and $F_0$ to denote $p$-adic fields.

this we see that the subset $\mathrm{RZ}_G^{\mathrm{a}} \subset \mathrm{RZ}_G$ is invariant under the equivalence relation $\sim$. It is also clear that the subset $\mathrm{RZ}_G^{\mathrm{n}} \subset \mathrm{RZ}_G$ is invariant under $\sim$.

When $G$ is a torus, the equivalence relation and the admissibility condition on $\mathrm{RZ}_G(K)$ can be made more explicit as follows.

**Proposition 4.3.2.** *Let $T$ be a torus over $\mathbb{Q}_p$. The following statements hold.*

- *(i) Two elements $(\mu_1, \delta_1), (\mu_2, \delta_2) \in \mathrm{RZ}_T(K)$ are equivalent if and only if $\mu_1 = \mu_2$ and $\delta_1$ is $\sigma$-conjugate to $\delta_2$ in $T(K_0)$.*
- *(ii) We have $\mathrm{RZ}_T^{\mathrm{a}}(K) = \mathrm{RZ}_T^{\mathrm{n}}(K)$.*

*Proof.* Part (i) follows easily from the discussion in §4.3.1. For part (ii), we fix an embedding $\overline{\mathbb{Q}}_p \to \overline{\breve{\mathbb{Q}}}_p$ as usual, and write $\breve{K}$ for the compositum of $K$ and $\breve{\mathbb{Q}}_p$ inside $\overline{\breve{\mathbb{Q}}}_p$. The condition of being neutral is equivalent to the second condition in [RZ96, Prop. 1.21]. Thus by that proposition, an element $(\mu, \delta) \in \mathrm{RZ}_T(K)$ is neutral if and only if the $\otimes$-functor $\mathrm{Rep}_{\mathbb{Q}_p} T \to \mathrm{MF}_{\breve{K}}^{\varphi}$ obtained by base changing $\mathcal{I}_{\mu,\delta}$ lands in $\mathrm{MF}_{\breve{K}}^{\varphi,\mathrm{a}}$. Since the admissibility of a filtered $\varphi$-module over $K$ is equivalent to the admissibility of its base change to $\breve{K}$, part (ii) follows. $\square$

**4.3.3.** Keep the setting of §4.3.1. If $K'/K$ is a finite extension inside $\overline{\mathbb{Q}}_p$, then there is a natural map $\mathrm{RZ}_G(K) \to \mathrm{RZ}_G(K')$, which sends equivalent elements to equivalent elements, and preserves neutrality and admissibility. We define

$$\mathrm{RZ}_G := \varinjlim_{K} \mathrm{RZ}_G(K),$$

where $K$ runs through finite extensions of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$, and the transition maps are the ones mentioned above. Similarly, we define the three subsets $\mathrm{RZ}_G^{\mathrm{a}}$, $\mathrm{RZ}_G^{\mathrm{n}}$, and $\mathrm{RZ}_G^{\mathrm{a,n}}$ of $\mathrm{RZ}_G$ by taking direct limits. We write $\sim$ for the inherited equivalence relation on $\mathrm{RZ}_G$. The three subsets of $\mathrm{RZ}_G$ introduced above are all stable under $\sim$.

**Corollary 4.3.4.** *Let $T$ be a torus over $\mathbb{Q}_p$. We have $\mathrm{RZ}_T^{\mathrm{a}} = \mathrm{RZ}_T^{\mathrm{n}}$. There is a natural bijection between $\mathrm{RZ}_T^{\mathrm{a}}/\sim$ and the set*

$$\left\{(\mu, [\delta]) \mid \mu \in X_*(T), [\delta] \in T(\mathbb{Q}_p^{\mathrm{ur}})/(1-\sigma), \kappa_T([\delta]) = [\mu]\right\}.$$

*Proof.* This follows from Proposition 4.3.2. $\square$

**4.3.5.** Keep the setting of §4.3.1. We say that a homomorphism

$$\rho : \Gamma_K \longrightarrow G(\mathbb{Q}_p)$$

is a *$G(\mathbb{Q}_p)$-valued crystalline representation of* $\Gamma_K$, if for some faithful representation $V$ in $\mathrm{Rep}_{\mathbb{Q}_p} G$, the homomorphism $\Gamma_K \to \mathrm{GL}(V)(\mathbb{Q}_p)$ arising from $\rho$ is a crystalline representation. This condition is in fact independent of the choice of $V$. We denote by $\mathrm{Crys}_G(K)$ the set of all such $\rho$.

We define

$$\mathrm{Crys}_G := \varinjlim_{K} \mathrm{Crys}_G(K),$$

where $K$ runs through finite extensions of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$, and the transition maps are given by restriction. We have a natural $G(\mathbb{Q}_p)$-action on $\mathrm{Crys}_G$ by conjugation. We denote the resulting equivalence relation on $\mathrm{Crys}_G$ by $\sim$.

For $\rho \in \mathrm{Crys}_G(K)$, we denote the image of $\rho$ in $\mathrm{Crys}_G$ by $[\rho]$. From $\rho$, we obtain a faithful exact $\otimes$-functor $\mathcal{I}_\rho : \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{MF}_K^{\varphi}$ taking each $V$ to

$$D_{\mathrm{cris}}(V) = (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{\Gamma_K},$$

where $V$ is viewed as a crystalline representation of $\Gamma_K$ via $\rho$. (The fact that $\mathcal{I}_\rho$ is a faithful exact $\otimes$-functor follows from the fundamental properties of $D_{\mathrm{cris}}$ [Fon79, §3.4].) We thus obtain a fiber functor $v \circ \mathcal{I}_\rho : \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Mod}_{K_0}^{\mathrm{fp}}$. By Steinberg's theorem, the $G_{K_0}$-torsor $\underline{\mathrm{Isom}}^{\otimes}(\mathbb{1}_{K_0}, v \circ \mathcal{I}_\rho)$ becomes trivial after a finite unramified extension of $K_0$. Hence if we replace $K$ by a suitable finite extension, then we may assume that this torsor is trivial. In this case, $\mathcal{I}_\rho$ is $\otimes$-isomorphic to $\mathcal{I}_{\mu,\delta}$ for some $(\mu, \delta) \in \mathrm{RZ}_G(K)$ which is unique up to equivalence; see §4.3.1. The image of $(\mu, \delta)$ in $\mathrm{RZ}_G/\sim$ is independent of all choices, and it depends on $\rho$ only via its image in $\mathrm{Crys}_G/\sim$. Mapping $\rho$ to $(\mu, \delta)$, we have obtained a well-defined map

$$D^G_{\mathrm{cris}} : \mathrm{Crys}_G/\sim \longrightarrow \mathrm{RZ}_G/\sim.$$

**Proposition 4.3.6.** *The map $D^G_{\mathrm{cris}}$ is injective with image $\mathrm{RZ}_G^{\mathrm{a,n}}/\sim$.*

*Proof.* Firstly, it is clear from the definitions that $\mathrm{im}(D^G_{\mathrm{cris}}) \subset \mathrm{RZ}^{\mathrm{a}}_G/\sim$.

For each finite extension $K/\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$, we let $\mathrm{Rep}^{\mathrm{cris}}_{\Gamma_K}$ be the category of crystalline representations of $\Gamma_K$ over $\mathbb{Q}_p$. Then $D_{\mathrm{cris}} : \mathrm{Rep}^{\mathrm{cris}}_{\Gamma_K} \to \mathrm{MF}^{\varphi,\mathrm{a}}_K$ is a $\otimes$-equivalence of $\otimes$-categories, with a quasi-inverse given by the functor

$$V_{\mathrm{cris}} : D \longmapsto \mathrm{Fil}^0(B_{\mathrm{cris}} \otimes_{K_0} D)^{\varphi=1}$$

(which is also a $\otimes$-functor). Let $u : \mathrm{Rep}^{\mathrm{cris}}_{\Gamma_K} \to \mathrm{Mod}^{\mathrm{fp}}_{\mathbb{Q}_p}$ be the functor sending a crystalline representation to its underlying $\mathbb{Q}_p$-vector space.

By a result of Wintenberger [Win97], we know that an element $(\mu, \delta) \in \mathrm{RZ}^{\mathrm{a}}_G(K)$ is neutral if and only if the composite $\otimes$-functor

$$u \circ V_{\mathrm{cris}} \circ \mathcal{I}_{\mu,\delta} : \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Mod}^{\mathrm{fp}}_{\mathbb{Q}_p}$$

is $\otimes$-isomorphic to the standard fiber functor $\mathbb{1}_{\mathbb{Q}_p}$.[23] From this result, it is clear that $\mathrm{im}(D^G_{\mathrm{cris}}) \subset \mathrm{RZ}^{\mathrm{a,n}}_G/\sim$.

Now we construct a map $\mathrm{RZ}^{\mathrm{a,n}}_G/\sim \to \mathrm{Crys}_G/\sim$ inverse to $D^G_{\mathrm{cris}}$. Let $(\mu, \delta) \in \mathrm{RZ}^{\mathrm{a,n}}_G(K)$. By the result of Wintenberger mentioned above, the $\otimes$-functor $\mathcal{F} := u \circ V_{\mathrm{cris}} \circ \mathcal{I}_{\mu,\delta}$ is $\otimes$-isomorphic to $\mathbb{1}_{\mathbb{Q}_p}$. By composing the tautological homomorphism $\Gamma_K \to \underline{\mathrm{Aut}}^{\otimes}(u)(\mathbb{Q}_p)$ with the natural homomorphism $\underline{\mathrm{Aut}}^{\otimes}(u)(\mathbb{Q}_p) \to \underline{\mathrm{Aut}}^{\otimes}(\mathcal{F})(\mathbb{Q}_p)$, we obtain a homomorphism $\rho' : \Gamma_K \to \underline{\mathrm{Aut}}^{\otimes}(\mathcal{F})(\mathbb{Q}_p)$. By choosing a $\otimes$-isomorphism between $\mathbb{1}_{\mathbb{Q}_p}$ and $\mathcal{F}$, we identify the $\mathbb{Q}_p$-group $\underline{\mathrm{Aut}}^{\otimes}(\mathcal{F})$ with $G$, and identify $\rho'$ with a homomorphism $\rho : \Gamma_K \to G(\mathbb{Q}_p)$. Clearly the $G(\mathbb{Q}_p)$-conjugacy class of $\rho$ is independent of choices. It is straightforward to check that the construction $(\mu, \delta) \mapsto \rho$ gives rise to the desired inverse map of $D^G_{\mathrm{cris}}$. $\square$

**Corollary 4.3.7.** *Let $T$ be a torus over $\mathbb{Q}_p$. The map $D^T_{\mathrm{cris}}$ induces a bijection*

$$\mathrm{Crys}_T \xrightarrow{\sim} \left\{ (\mu, [\delta]) \mid \mu \in X_*(T), [\delta] \in T(\mathbb{Q}_p^{\mathrm{ur}})/(1-\sigma), \kappa_T([\delta]) = [\mu] \right\}.$$

*Proof.* This follows from Corollary 4.3.4 and Proposition 4.3.6. (Note that $\mathrm{Crys}_T = \mathrm{Crys}_T/\sim$.) $\square$

[23] This was originally a conjecture of Rapoport–Zink; see the paragraph below [RZ96, Prop. 1.20]. Wintenberger showed in [Win97] that it is a consequence of the Colmez–Fontaine Theorem [CF00].

**4.3.8.** Let $T$ be a cuspidal torus over $\mathbb{Q}$ (see Definition 1.5.4). We call an element $\delta \in T(\mathbb{Q}_p^{\mathrm{ur}})$ *motivic*, if for some $n \geq 1$, the element $\gamma = \delta\sigma(\delta)\dots\sigma^{n-1}(\delta)$ is in $T(\mathbb{Q})$, and is a $p$-unit (i.e., $\gamma$ lies in a compact open subgroup of $T(\mathbb{A}_f^p)$). We denote by $T(\mathbb{Q}_p^{\mathrm{ur}})^{\mathrm{mot}} \subset T(\mathbb{Q}_p^{\mathrm{ur}})$ the subset of motivic elements. Note that $T(\mathbb{Q}_p^{\mathrm{ur}})^{\mathrm{mot}}$ is stable under $\sigma$-conjugation by $T(\mathbb{Q}_p^{\mathrm{ur}})$. We denote by $\mathcal{T}^\circ$ the connected Néron model of $T_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$. Let $\overset{\circ}{\sim}$ be the equivalence relation on $T(\mathbb{Q}_p^{\mathrm{ur}})^{\mathrm{mot}}$ defined by $\sigma$-conjugation by $\mathcal{T}^\circ(\mathbb{Z}_p^{\mathrm{ur}})$. As in §1.4.2, we denote by $w_T : T(\breve{\mathbb{Q}}_p) \to X_*(T)_{\Gamma_{p,0}}$ the Kottwitz homomorphism, which is surjective. By [Rap05, Rmk. 2.2 (iii)], we have $\ker(w_T) = \mathcal{T}^\circ(\breve{\mathbb{Z}}_p)$. Hence the restriction of $w_T$ to $T(\mathbb{Q}_p^{\mathrm{ur}})^{\mathrm{mot}}$ factors through $T(\mathbb{Q}_p^{\mathrm{ur}})^{\mathrm{mot}}/\overset{\circ}{\sim}$.

**Lemma 4.3.9.** *In the setting of §4.3.8, the map $w_T$ induces a bijection*

$$T(\mathbb{Q}_p^{\mathrm{ur}})^{\mathrm{mot}}/\overset{\circ}{\sim} \xrightarrow{\sim} X_*(T)_{\Gamma_{p,0}}. \tag{4.3.9.1}$$

*Proof.* The surjectivity of (4.3.9.1) follows from the second construction in [Kis17, §4.3.9]. We explain this in more detail. Let $\mu \in X_*(T)$, and let $L/\mathbb{Q}$ be a finite Galois extension inside $\overline{\mathbb{Q}}$ splitting $T$. Let $L_p$ be the completion of $L$ at the place above $p$ determined by the fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and let $\pi$ be a uniformizer of $L_p$. Then the same argument as in *loc. cit.* (with $\mu_{h_T}$ replaced by $\mu$) shows that there exist $s \in \mathbb{Z}_{\geq 1}$ and a $p$-unit $\gamma \in T(\mathbb{Q})$ such that $\mathrm{N}_{L_p/\mathbb{Q}_p}(\mu(\pi))^s\gamma^{-1} \in \mathcal{T}^\circ(\mathbb{Z}_p)$. By Greenberg's theorem [Gre63, Prop. 3], the map $\mathcal{T}^\circ(\breve{\mathbb{Z}}_p) \to \mathcal{T}^\circ(\breve{\mathbb{Z}}_p), c \mapsto c\sigma(c)^{-1}$ is surjective. Hence we can find $c \in \mathcal{T}^\circ(\breve{\mathbb{Z}}_p)$ such that[24]

$$c\,\mathrm{N}_{L_p/\mathbb{Q}_p}(\mu(\pi))^s\sigma(c)^{-1} = \gamma.$$

Let

$$\delta := c\,\mathrm{N}_{L_p/L_{p,0}}(\mu(\pi))\sigma(c)^{-1} \in T(\breve{\mathbb{Q}}_p),$$

where $L_{p,0}$ is the maximal unramified extension of $\mathbb{Q}_p$ inside $L_p$. As in *loc. cit.,* we have $\delta\sigma(\delta)\cdots\sigma^{n-1}(\delta) = \gamma$, and we have $\delta \in T(\mathbb{Q}_{p^n})$, where $n = s[L_{p,0} : \mathbb{Q}_p]$. Hence $\delta \in T(\mathbb{Q}_p^{\mathrm{ur}})^{\mathrm{mot}}$. We now check that $w_T(\delta)$ equals the image of $\mu$ in $X_*(T)_{\Gamma_{p,0}}$, which will prove the surjectivity of (4.3.9.1). For this, it suffices to show that $w_T(\mathrm{N}_{L_p/L_{p,0}}(\mu(\pi)))$ equals the image of $\mu$, since $w_T(c)$ is trivial. Writing $F$ for $L_{p,0}$, we have $X_*(T)_{\Gamma_F} = X_*(T)_{\Gamma_{p,0}}$. Let $\sigma_F = \sigma^{[F:\mathbb{Q}_p]}$. By [Kot85, §2.5], $w_T$ induces a bijection

$$\mathrm{B}_F(T) := \Big\{\sigma_F\text{-conjugacy classes in } T(\breve{\mathbb{Q}}_p)\Big\} \xrightarrow{\sim} X_*(T)_{\Gamma_F},$$

whose inverse is induced by $\mu \mapsto \mathrm{N}_{L_p/F}(\mu(\pi))$. (Here we use that $T$ splits over $L_p$ and that $L_p/F$ is totally ramified.) This gives what we want.

For the injectivity of (4.3.9.1), let $\delta_1, \delta_2 \in T(\mathbb{Q}_p^{\mathrm{ur}})^{\mathrm{mot}}$ be such that $w_T(\delta_1) = w_T(\delta_2)$. Let $\delta = \delta_1\delta_2^{-1}$. Then $\delta$ is also motivic, so we can choose $n$ such that $\gamma := \delta\sigma(\delta)\cdots\sigma^{n-1}(\delta)$ is a $p$-unit in $T(\mathbb{Q})$. Since $w_T(\delta) = 0$, we have $\delta \in \mathcal{T}^\circ(\breve{\mathbb{Z}}_p)$, and in particular $\gamma \in \mathcal{T}^\circ(\mathbb{Z}_p)$. Therefore $\gamma$ lies in a congruence subgroup of $T(\mathbb{Q})$, and has finite order by Lemma 1.5.5. Up to enlarging $n$, we may assume that $\gamma = 1$. Again by Greenberg's theorem, we can write $\delta \in \mathcal{T}^\circ(\breve{\mathbb{Z}}_p)$ as $c\sigma(c)^{-1}$, for some $c \in \mathcal{T}^\circ(\breve{\mathbb{Z}}_p)$. Since $\gamma = 1$, we have $c\sigma^n(c)^{-1} = 1$, i.e., $c \in \mathcal{T}^\circ(\mathbb{Z}_{p^n})$. Hence $\delta_1 = \delta\delta_2 = c\delta_2\sigma(c)^{-1}$ with $c \in \mathcal{T}^\circ(\mathbb{Z}_p^{\mathrm{ur}})$, which means $\delta_1 \overset{\circ}{\sim} \delta_2$. ∎

[24] In the last paragraph of [Kis17, §4.3.9], it is used that such $c$ can be found in $T(\breve{\mathbb{Q}}_p)$.

**Lemma 4.3.10.** *In the setting of §4.3.8, the map $w_T$ induces a bijection from the set of $T(\mathbb{Q}_p^{\mathrm{ur}})$-$\sigma$-conjugacy classes in $T(\mathbb{Q}_p^{\mathrm{ur}})^{\mathrm{mot}}$ to $X_*(T)_{\Gamma_p}$.*

*Proof.* The map $w_T$ induces a group isomorphism $T(\breve{\mathbb{Q}}_p)/\mathcal{T}^\circ(\breve{\mathbb{Z}}_p) \xrightarrow{\sim} X_*(T)_{\Gamma_{p,0}}$. In view of Lemma 1.6.8, this induces a group isomorphism $T(\mathbb{Q}_p^{\mathrm{ur}})/\mathcal{T}^\circ(\mathbb{Z}_p^{\mathrm{ur}}) \xrightarrow{\sim} X_*(T)_{\Gamma_{p,0}}$, which is equivariant for the natural actions of $\sigma$ on the two sides. The lemma follows from this fact and Lemma 4.3.9. □

**Definition 4.3.11.** Let $T$ be a cuspidal torus over $\mathbb{Q}$. Let $\mathrm{Mot}_T$ be the subset of $\mathrm{Crys}_{T_{\mathbb{Q}_p}}$ consisting of those $[\rho]$ whose image under the bijection in Corollary 4.3.7 is of the form $(\mu, [\delta])$ where $[\delta] \in T(\mathbb{Q}_p^{\mathrm{ur}})/(1-\sigma)$ is in the image of $T(\mathbb{Q}_p^{\mathrm{ur}})^{\mathrm{mot}}$. Note that the definition of $\mathrm{Mot}_T$ depends on $T$ over $\mathbb{Q}$, not just $T_{\mathbb{Q}_p}$.

**Proposition 4.3.12.** *In the setting of Definition 4.3.11, the $\mu$-component of the map $D_{\mathrm{cris}}^{T_{\mathbb{Q}_p}}$ induces a bijection*

$$\mathscr{M}_T : \mathrm{Mot}_T \xrightarrow{\sim} X_*(T).$$

*Proof.* This follows from Corollary 4.3.7 and Lemma 4.3.10. □

**4.3.13.** Let $T$ be a cuspidal torus over $\mathbb{Q}$. We now use class field theory to construct certain $T(\mathbb{Q}_p)$-valued global Galois representations, and show that their localizations at places above $p$ give rise to elements of $\mathrm{Mot}_T$.

Let $\mu \in X_*(T)$, and let $E_\mu \subset \overline{\mathbb{Q}}$ be the field of definition of $\mu$. Similar to §1.5.3, we consider the composite homomorphism of $\mathbb{Q}$-algebraic groups

$$r(\mu)^{\mathrm{alg}} : \mathrm{Res}_{E_\mu/\mathbb{Q}}\,\mathbb{G}_m \xrightarrow{\mathrm{Res}_{E_\mu/\mathbb{Q}}\,\mu} \mathrm{Res}_{E_\mu/\mathbb{Q}}\,T \xrightarrow{\mathrm{N}_{E_\mu/\mathbb{Q}}} T.$$

We have an induced homomorphism between topological groups

$$E_\mu^\times \backslash \mathbb{A}_{E_\mu}^\times \longrightarrow T(\mathbb{Q})\backslash T(\mathbb{A}). \tag{4.3.13.1}$$

By Lemma 1.5.5, $T(\mathbb{Q})$ is discrete in $T(\mathbb{A}_f)$, and so we have

$$T(\mathbb{Q})\backslash T(\mathbb{A}_f) = \varprojlim_U T(\mathbb{Q})\backslash T(\mathbb{A}_f)/U,$$

where $U$ runs through compact open subgroups of $T(\mathbb{A}_f)$. For each such $U$, we have a natural map $\pi_0(T(\mathbb{Q})\backslash T(\mathbb{A})) \to T(\mathbb{Q})\backslash T(\mathbb{A}_f)/U$, cf. §1.5.3. In the limit we obtain a map

$$\pi_0(T(\mathbb{Q})\backslash T(\mathbb{A})) \longrightarrow T(\mathbb{Q})\backslash T(\mathbb{A}_f). \tag{4.3.13.2}$$

The composition

(4.3.13.3)

$$E_\mu^\times \backslash \mathbb{A}_{E_\mu}^\times \xrightarrow{(4.3.13.1)} T(\mathbb{Q})\backslash T(\mathbb{A}) \longrightarrow \pi_0(T(\mathbb{Q})\backslash T(\mathbb{A})) \xrightarrow{(4.3.13.2)} T(\mathbb{Q})\backslash T(\mathbb{A}_f)$$

factors through the global Artin map $E_\mu^\times \backslash \mathbb{A}_{E_\mu}^\times \to \pi_0(E_\mu^\times \backslash \mathbb{A}_{E_\mu}^\times) \cong \mathrm{Gal}(E_\mu^{\mathrm{ab}}/E_\mu)$. (Recall from §1.5.3 that we take the geometric normalization of the global Artin map.) We thus obtain a map

$$r(\mu) : \mathrm{Gal}(E_\mu^{\mathrm{ab}}/E_\mu) \longrightarrow T(\mathbb{Q})\backslash T(\mathbb{A}_f).$$

Let $U \subset T(\mathbb{A}_f)$ be a neat compact open subgroup. Since $T(\mathbb{Q})$ is discrete in $T(\mathbb{A}_f)$, we have $T(\mathbb{Q}) \cap U = \{1\}$ (cf. the proof of Lemma 1.5.7). The kernel of the projection

$$\pi(U) : T(\mathbb{Q})\backslash T(\mathbb{A}_f) \to T(\mathbb{Q})\backslash T(\mathbb{A}_f)/U$$

is $T(\mathbb{Q})\backslash T(\mathbb{Q})U$, which we identify with $U$, using that $T(\mathbb{Q})\cap U = \{1\}$. Let $E_{\mu,U}/E_\mu$ be the finite extension inside $E_\mu^{\mathrm{ab}}/E_\mu$ such that $\mathrm{Gal}(E_\mu^{\mathrm{ab}}/E_{\mu,U})$ is the kernel of $\pi_U \circ r(\mu)$. Then $r(\mu)$ induces a homomorphism

$$r(\mu)_U : \mathrm{Gal}(E_\mu^{\mathrm{ab}}/E_{\mu,U}) \longrightarrow \ker \pi_U \cong U.$$

We denote by $r(\mu)_{U,p}$ the composite homomorphism

$$\mathrm{Gal}(E_\mu^{\mathrm{ab}}/E_{\mu,U}) \xrightarrow{r(\mu)_U} U \hookrightarrow T(\mathbb{A}_f) \xrightarrow{\text{projection}} T(\mathbb{Q}_p).$$

The fixed embeddings $E_{\mu,U} \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ give rise to a place $v$ of $E_{\mu,U}$ above $p$. Let $K = E_{\mu,U,v} \subset \overline{\mathbb{Q}}_p$. We denote by $r(\mu)_{U,p,\mathrm{loc}}$ the composite map

$$\Gamma_K = \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \longrightarrow \mathrm{Gal}(E_\mu^{\mathrm{ab}}/E_{\mu,U}) \xrightarrow{r(\mu)_{U,p}} T(\mathbb{Q}_p).$$

**Proposition 4.3.14.** *In the setting of §4.3.13, assume in addition that $U$ is of the form $U^pU_p$, where $U^p$ is a neat compact open subgroup of $T(\mathbb{A}_f^p)$ and $U_p$ is a compact open subgroup of $T(\mathbb{Q}_p)$. Then $r(\mu)_{U,p,\mathrm{loc}} : \Gamma_K \to T(\mathbb{Q}_p)$ is a $T(\mathbb{Q}_p)$-valued crystalline representation. The element $[r(\mu)_{U,p,\mathrm{loc}}] \in \mathrm{Crys}_{T_{\mathbb{Q}_p}}$ lies in $\mathrm{Mot}_T$. Moreover, the image of $[r(\mu)_{U,p,\mathrm{loc}}]$ under the bijection $\mathscr{M}_T$ in Proposition 4.3.12 is $-\mu$.*

*Proof.* Let $f$ be the composite map of topological groups

$$K^\times \xrightarrow{\mathrm{Art}_K} \Gamma_K^{\mathrm{ab}} \longrightarrow \mathrm{Gal}(E_\mu^{\mathrm{ab}}/E_{\mu,U}) \xrightarrow{r(\mu)_U} U,$$

where $\mathrm{Art}_K$ is the local Artin map (normalized geometrically). Let $F$ be the completion of $E_\mu$ inside $K = E_{\mu,U,v}$. Let $f_1$ be the composite map of $\mathbb{Q}_p$-algebraic groups

$$\mathrm{Res}_{K/\mathbb{Q}_p}\mathbb{G}_m \xrightarrow{\mathrm{N}_{K/F}} \mathrm{Res}_{F/\mathbb{Q}_p}\mathbb{G}_m \hookrightarrow (\mathrm{Res}_{E_\mu/\mathbb{Q}}\mathbb{G}_m)\otimes_{\mathbb{Q}}\mathbb{Q}_p \xrightarrow{r(\mu)^{\mathrm{alg}}\otimes_{\mathbb{Q}}\mathbb{Q}_p} T_{\mathbb{Q}_p}.$$

Then $f_1$ induces a map

$$K^\times = (\mathrm{Res}_{K/\mathbb{Q}_p}\mathbb{G}_m)(\mathbb{Q}_p) \xrightarrow{f_1} T(\mathbb{Q}_p) \hookrightarrow T(\mathbb{A}_f),$$

which we again denote by $f_1$.

We claim that $f$ and $f_1$ induce the same map $\mathcal{O}_K^\times \to T(\mathbb{A}_f)$. In fact, by the definition of $r(\mu)_U$ and the compatibility of the local and global Artin maps, we have

$$f_1(x) \in f(x)T(\mathbb{Q}) \subset T(\mathbb{A}_f), \quad \forall x \in K^\times. \tag{4.3.14.1}$$

Take $x \in \mathcal{O}_K^\times$, and let $\gamma \in T(\mathbb{Q})$ be such that $f_1(x) = f(x)\gamma$. Note that $f_1(x)$ lies inside the maximal compact subgroup $U_{p,\max}$ of $T(\mathbb{Q}_p)$, by the compactness of $\mathcal{O}_K^\times$. Hence $\gamma$ lies in $U^pU_{p,\max}$, which is a neat compact open subgroup of $T(\mathbb{A}_f)$ by the neatness of $U^p$. Since $T(\mathbb{Q})$ is discrete in $T(\mathbb{A}_f)$, we have $\gamma = 1$. The claim is proved.

Now to check that $r(\mu)_{U,p,\mathrm{loc}}$ is crystalline, we take an arbitrary representation $V$ of $T_{\mathbb{Q}_p}$ and check that $\Gamma_K \xrightarrow{r(\mu)_{U,p,\mathrm{loc}}} T(\mathbb{Q}_p) \to \mathrm{GL}(V)(\mathbb{Q}_p)$ is crystalline. For this, it suffices to check that the composition $K^\times \xrightarrow{f} U \to T(\mathbb{Q}_p) \to \mathrm{GL}(V)(\mathbb{Q}_p)$ agrees with a $\mathbb{Q}_p$-algebraic group homomorphism $\mathrm{Res}_{K/\mathbb{Q}_p}\mathbb{G}_m \to \mathrm{GL}(V)$ on $\mathcal{O}_K^\times$, by a well-known criterion in $p$-adic Hodge theory (see for instance [Con11, Prop. B.4 (i)] and the remark following it). But this follows immediately from our claim proved above.

We now check that $[r(\mu)_{U,p,\mathrm{loc}}] \in \mathrm{Crys}_{T_{\mathbb{Q}_p}}$ lies in $\mathrm{Mot}_T$. Let $[\delta] \in T(\mathbb{Q}_p^{\mathrm{ur}})/(1-\sigma)$ be the element attached to $[r(\mu)_{U,p,\mathrm{loc}}]$ as in Corollary 4.3.7, and let $\delta \in T(\mathbb{Q}_{p^n})$ be a representative of $[\delta]$. By taking a faithful representation of $T_{\mathbb{Q}_p}$ and applying [Con11, Prop. B.4 (ii)], we know that up to enlarging $n$ the element

$$\gamma_n := \delta\sigma(\delta)\cdots\sigma^{n-1}(\delta) \in T(\mathbb{Q}_p)$$

is equal to[25]

$$[f(\pi)_p^{-1} f_1(\pi)]^{-n/n_K} \in T(\mathbb{Q}_p).$$

Here $\pi$ is a uniformizer of $K$, $n_K$ is the residue degree of $K$, and $f(\pi)_p$ denotes the component at $p$ of $f(\pi) \in U$. It remains to show that $f(\pi)_p f_1(\pi)^{-1}$ is a $p$-unit in $T(\mathbb{Q})$. By (4.3.14.1) there exists $\gamma \in T(\mathbb{Q})$ such that $f_1(\pi) = f(\pi)\gamma \in T(\mathbb{A}_f)$. Then $f(\pi)_p^{-1} f_1(\pi)$ equals the image of $\gamma$ in $T(\mathbb{Q}_p)$. In addition, $\gamma$ and $f(\pi)^{-1}$ have the same image in $T(\mathbb{A}_f^p)$, which shows that $\gamma \in U^p$. Hence $\gamma$ is a $p$-unit.

Finally, we check that $\mathscr{M}_T([r(\mu)_{U,p,\mathrm{loc}}]) = -\mu$. Let $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}}_p$. For each faithful representation $V$ of $T_{\mathbb{Q}_p}$, the $\Gamma_K$-representation on $V$ induced by $r(\mu)_{U,p,\mathrm{loc}}$ is Hodge–Tate, and we have the cocharacter $h_V : \mathbb{G}_{m,\mathbb{C}_p} \to \mathrm{GL}(V)_{\mathbb{C}_p}$ as in [Ser79, §1.4]. We know that $h_V$ factors through $T_{\mathbb{C}_p}$ (see *loc. cit.*), and the resulting *Hodge–Tate cocharacter* $\mu_{\mathrm{HT}} \in X_*(T)$ is independent of the choice of $V$ by the functoriality of the construction. Since the filtration on $D_{\mathrm{dR}}(\mathbb{Q}_p(1))$ jumps at $-1$, it is easy to see that $\mathscr{M}_T([r(\mu)_{U,p,\mathrm{loc}}]) = -\mu_{\mathrm{HT}}$. We are left to check that $\mu_{\mathrm{HT}} = \mu$.

Let $T' = \mathrm{Res}_{K/\mathbb{Q}_p} \mathbb{G}_m$. We identify $T'_{\overline{\mathbb{Q}}_p}$ with $\prod_{\tau \in \mathrm{Hom}_{\mathbb{Q}_p}(K,\overline{\mathbb{Q}}_p)} \mathbb{G}_{m,\overline{\mathbb{Q}}_p}$, and define $\mu' \in X_*(T')$ by $\mu'(z) = (z, 1, \cdots, 1)$, where the first spot corresponds to the canonical embedding $K \hookrightarrow \overline{\mathbb{Q}}_p$. Let $r' : \Gamma_{K,0} \to \mathcal{O}_K^\times$ be the Lubin–Tate character (cf. [Ser79, §2.1]). Then $r'$ is a $T'(\mathbb{Q}_p)$-valued crystalline representation. Since $\mathrm{Art}_K \circ r'$ is the inclusion $\Gamma_{K,0} \hookrightarrow \Gamma_K$ (thanks to the geometric normalization of $\mathrm{Art}_K$), we know that the restriction of $r(\mu)_{U,p}$ to $\Gamma_{K,0}$ equals the composition

$$\Gamma_{K,0} \xrightarrow{r'} \mathcal{O}_K^\times \xrightarrow{f} U \to T(\mathbb{Q}_p).$$

Moreover, by our previous claim that $f$ and $f_1$ induce the same map $\mathcal{O}_K^\times \to T(\mathbb{A}_f)$, we know that the above composition is equal to the composition of $r'$ with $f_1 : K^\times \to T(\mathbb{Q}_p)$. Therefore if we let $\mu'_{\mathrm{HT}} \in X_*(T')$ be the Hodge–Tate cocharacter of $r'$, then $\mu_{\mathrm{HT}}$ equals $f_1 \circ \mu'_{\mathrm{HT}}$. (Recall that $f_1$ is an algebraic homomorphism $T' \to T_{\mathbb{Q}_p}$.) By the last paragraph of [Ser79, §2.1], we have $\mu'_{\mathrm{HT}} = \mu'$. Therefore $\mu_{\mathrm{HT}} = f_1 \circ \mu'$, which is easily seen to be equal to $\mu$. □

### 4.4. **Crystalline lattices with $G$-structure.**

**4.4.1.** Let $K$ be a finite extension of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$, with residue field $k$. Let $K_0$ be the maximal unramified extension of $\mathbb{Q}_p$ inside $K_0$, and let $\sigma \in \mathrm{Aut}(K_0)$ be the arithmetic $p$-Frobenius.

We write $W$ for $W(k) = \mathcal{O}_{K_0}$. Fix a uniformizer $\pi$ of $K$, and let $E = E(u)$ be its Eisenstein polynomial over $K_0$. We set $\mathfrak{S} = W[\![u]\!]$, and let $\varphi$ be the endomorphism

[25]Note that in [Con11] the arithmetic normalization of the local Artin map is used, which is opposite to our normalization. This results in the sign difference in the exponent in the expression below.

of $\mathfrak{S}$ that restricts to $\sigma$ on $W$ and sends $u$ to $u^p$. We have a $W$-algebra isomorphism $\mathfrak{S}/(E) \xrightarrow{\sim} \mathcal{O}_K$, which sends $u \mod (E)$ to $\pi$. Thus we have specialization maps $\mathfrak{S} \to \mathcal{O}_K$ and $\mathfrak{S} \to W$, sending $u$ to $\pi$ and 0 respectively. Using these two maps we view $K$ and $K_0$ as $\mathfrak{S}$-algebras respectively. (These $\mathfrak{S}$-algebra structures are not compatible with the inclusion $K_0 \hookrightarrow K$.)

For any height 1 prime ideal $\mathfrak{p} \subset \mathfrak{S}$, the localization $\mathfrak{S}_{\mathfrak{p}}$ is a DVR, and we write $\widehat{\mathfrak{S}}_{\mathfrak{p}}$ for its completion.

We denote by $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ the category of pairs $(\mathfrak{M}, \varphi_{\mathfrak{M}})$, where $\mathfrak{M}$ is a finite free $\mathfrak{S}$-module, and $\varphi_{\mathfrak{M}}$ is a $\mathfrak{S}$-module isomorphism $\varphi^*\mathfrak{M}[1/E] \xrightarrow{\sim} \mathfrak{M}[1/E]$. For such a pair $(\mathfrak{M}, \varphi_{\mathfrak{M}})$, the $\mathfrak{S}$-module $\varphi^*\mathfrak{M}$ carries a filtration, given by $\mathrm{Fil}^i(\varphi^*\mathfrak{M}) = \varphi_{\mathfrak{M}}^{-1}(E^i\mathfrak{M}) \cap \varphi^*\mathfrak{M} \subset \varphi^*\mathfrak{M}[1/E]$ for $i \in \mathbb{Z}$.

Let $\mathrm{Rep}^{\mathrm{cris}\,\circ}_{\Gamma_K}$ be the category of $\Gamma_K$-stable $\mathbb{Z}_p$-lattices in crystalline representations of $\Gamma_K$ over $\mathbb{Q}_p$. Recall from [Kis10, §1.2] that there is a faithful $\otimes$-functor

$$\mathfrak{M} : \mathrm{Rep}^{\mathrm{cris}\,\circ}_{\Gamma_K} \longrightarrow \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}.$$

For each $L$ in $\mathrm{Rep}^{\mathrm{cris}\,\circ}_{\Gamma_K}$, the following statements hold (see *loc. cit.*).

(i) There is a canonical isomorphism

$$\mathfrak{M}(L) \otimes_{\mathfrak{S}} K_0 \cong D_{\mathrm{cris}}(L \otimes \mathbb{Q}_p)$$

of isocrystals over $K_0$. The Frobenius on the left is induced by $\varphi_{\mathfrak{M}(L)}$. This isomorphism is functorial in $L$ and compatible with tensor products.

(ii) There is a canonical isomorphism

$$\varphi^*\mathfrak{M}(L) \otimes_{\mathfrak{S}} K \cong D_{\mathrm{dR}}(L \otimes \mathbb{Q}_p) \tag{4.4.1.1}$$

of filtered $K$-vector spaces. The filtration on the left is induced by the filtration on $\varphi^*\mathfrak{M}(L)$. This isomorphism is functorial in $L$ and compatible with tensor products.

(iii) There is a faithfully flat and formally étale $\widehat{\mathfrak{S}}_{(p)}$-algebra $\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{ur}}}$, and a canonical $\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{ur}}}$-linear isomorphism

$$\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathbb{Z}_p} L \cong \widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathfrak{S}} \mathfrak{M}(L).$$

This isomorphism is functorial in $L$ and compatible with tensor products.

We set $M_{\mathrm{cris}}(L) := \mathfrak{M}(L) \otimes_{\mathfrak{S}} W$. Thus $M_{\mathrm{cris}}$ is a $\otimes$-functor

$$\mathrm{Rep}^{\mathrm{cris}\,\circ}_{\Gamma_K} \longrightarrow \mathrm{Isoc}^{\circ}_{K_0}.$$

By (i) above, $M_{\mathrm{cris}}(L)$ is a $W$-lattice in the $K_0$-vector space $D_{\mathrm{cris}}(L \otimes \mathbb{Q}_p)$. The following property is proved by Tong Liu in [Liu18, §4].

(iv) Inside $D_{\mathrm{cris}}(L \otimes \mathbb{Q}_p)$, the $W$-lattice $M_{\mathrm{cris}}(L)$ is independent of the choice of a uniformizer in $F$ (which is needed to define the functor $\mathfrak{M}$). Moreover, if $K'/K$ is a finite extension in $\overline{\mathbb{Q}}_p$ and $L' \in \mathrm{Rep}^{\mathrm{cris}\,\circ}_{\Gamma_{K'}}$ denotes $L$ equipped with the inherited $\Gamma_{K'}$-action, then we have a canonical identification

$$M_{\mathrm{cris}}(L') \cong M_{\mathrm{cris}}(L) \otimes_{W(k)} W(k').$$

This is compatible with the usual identification

$$D_{\mathrm{cris}}(L') \cong D_{\mathrm{cris}}(L) \otimes_{K_0} K_0'.$$

Here $k'$ denotes the residue field of $K'$, and $K_0'$ denotes $W(k')[1/p]$.

The filtered isomorphism (4.4.1.1) is induced by another canonical filtered isomorphism, which we now recall. Note that $\widehat{\mathfrak{S}}_{(E)}$ is a complete DVR with residue field $K$, which has characteristic zero. Hence $\widehat{\mathfrak{S}}_{(E)}$ is canonically a $K$-algebra, and the $K$-algebra structure is compatible with the natural $W$-algebra structure. The following statements follow from the proof of [Kis10, Thm. 1.2.1], and [Kis06, Lem. 1.2.12 (4)]. For each $L$ in $\mathrm{Rep}^{\mathrm{cris}\,\circ}_{\Gamma_K}$, there is a canonical filtered isomorphism

$$\varphi^*\mathfrak{M}(L)\otimes_{\mathfrak{S}}\widehat{\mathfrak{S}}_{(E)}\xrightarrow{\sim} D_{\mathrm{dR}}(L\otimes\mathbb{Q}_p)\otimes_K\widehat{\mathfrak{S}}_{(E)}. \tag{4.4.1.2}$$

Here the filtration on the right is the tensor product filtration, coming from the filtration on $D_{\mathrm{dR}}(L\otimes\mathbb{Q}_p)$ and the $E$-adic filtration on $\widehat{\mathfrak{S}}_{(E)}$. The filtration on the left is the one induced by the filtration on $\varphi^*\mathfrak{M}(L)$ (which is also the same as the tensor product filtration coming from the filtration on $\varphi^*\mathfrak{M}(L)$ and the $E$-adic filtration on $\widehat{\mathfrak{S}}_{(E)}$). Now (4.4.1.1) is induced by (4.4.1.2) by passing to the residue field $K$ of $\widehat{\mathfrak{S}}_{(E)}$.

**4.4.2.** Let $G$ be a flat, finite-type, affine group scheme over $\mathbb{Z}_p$. We say that a homomorphism

$$\rho:\Gamma_K\longrightarrow G(\mathbb{Z}_p)$$

is a *$G(\mathbb{Z}_p)$-valued crystalline representation*, if the composition of $\rho$ with the inclusion $G(\mathbb{Z}_p)\subset G(\mathbb{Q}_p)$ is a $G(\mathbb{Q}_p)$-valued crystalline representation as in §4.3.5.

Given a $G(\mathbb{Z}_p)$-valued crystalline representation $\rho$, we obtain a tautological functor

$${}_\rho\mathbb{1}:\mathrm{Rep}_{\mathbb{Z}_p}G\longrightarrow\mathrm{Rep}^{\mathrm{cris}\,\circ}_{\Gamma_K},$$

sending each $L$ to the $\Gamma_K$-stable lattice $L$ in the crystalline representation $L\otimes\mathbb{Q}_p$.

We shall need a generalization of ${}_\rho\mathbb{1}$. Let $\Gamma_K$ act on the left on $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ by

$$\gamma(gG(\mathbb{Z}_p)):=\rho(\gamma)gG(\mathbb{Z}_p),\quad\forall\gamma\in\Gamma_K,g\in G(\mathbb{Q}_p).$$

Since $\rho(\Gamma_K)\subset G(\mathbb{Z}_p)$, the coset of 1 in $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ is fixed by $\Gamma_p$. Now let $\lambda\in G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ be a point fixed by $\Gamma_K$. Let

$${}_\rho\mathbb{1}^\lambda:\mathrm{Rep}_{\mathbb{Z}_p}G\longrightarrow\mathrm{Rep}^{\mathrm{cris}\,\circ}_{\Gamma_K}$$

be the functor sending each $L$ to the $\Gamma_K$-stable lattice $\lambda\cdot L$ inside $L\otimes\mathbb{Q}_p$.

We define $\otimes$-functors

$$\omega_\rho^\lambda:\mathrm{Rep}_{\mathbb{Z}_p}G\xrightarrow{\mathfrak{M}\circ{}_\rho\mathbb{1}^\lambda}\mathrm{Mod}^\varphi_{/\mathfrak{S}}\to\mathrm{Mod}^{\mathrm{fp}}_{\mathfrak{S}},\quad L\mapsto\mathfrak{M}(\lambda\cdot L)$$

and

$$\omega_{\rho,0}^\lambda:\mathrm{Rep}_{\mathbb{Z}_p}G\xrightarrow{M_{\mathrm{cris}}\circ{}_\rho\mathbb{1}^\lambda}\mathrm{Isoc}^\circ_{K_0}\to\mathrm{Mod}^{\mathrm{fp}}_W,\quad L\mapsto M_{\mathrm{cris}}(\lambda\cdot L).$$

Here the last arrows in both cases are the natural forgetful functors. Clearly

$$\omega_{\rho,0}^\lambda=(\omega_\rho^\lambda)_W.$$

We also write $\Upsilon_\rho^\lambda$ for the $\otimes$-functor

$$M_{\mathrm{cris}}\circ{}_\rho\mathbb{1}^\lambda:\mathrm{Rep}_{\mathbb{Z}_p}G\longrightarrow\mathrm{Isoc}^\circ_{K_0}.$$

When $\lambda$ is the coset of 1, we omit it from the superscripts. Note that the definition of $\omega_\rho^\lambda$ depends on the choice of a uniformizer in $F$, but the definitions of $\Upsilon_\rho^\lambda$ and $\omega_{\rho,0}^\lambda$ are independent of such a choice (up to canonical $\otimes$-isomorphism), by property (iv) in §4.4.1, As in §4.1.1, we denote by $P_{\omega_\rho^\lambda}$ the $\mathfrak{S}$-scheme $\underline{\mathrm{Isom}}^{\otimes}(\mathbb{1}_{\mathfrak{S}}, \omega_\rho^\lambda)$. Since $\underline{\mathrm{Aut}}^{\otimes}(\mathbb{1}_{\mathfrak{S}}) \cong G_{\mathfrak{S}}$ by the reconstruction theorem, we know that $P_{\omega_\rho^\lambda}$ is a pseudo-torsor under $G_{\mathfrak{S}}$ (i.e., for each $\mathfrak{S}$-scheme $S$, the set $P_{\omega_\rho^\lambda}(S)$ is either empty or a principal homogeneous space under $G(S)$). However, in general $\omega_\rho^\lambda$ may not be a fiber functor (as it may not be exact), and $P_{\omega_\rho^\lambda}$ may not be a $G_{\mathfrak{S}}$-torsor.

**Lemma 4.4.3.** *Let $G, \rho, \lambda$ be as in §4.4.2. Let $U$ be the complement of the closed point in* $\operatorname{Spec}\mathfrak{S}$*. Then $P_{\omega_\rho^\lambda}|_U$ is a $G_U$-torsor over $U$.*

*Proof.* We write $\omega$ for $\omega_\rho^\lambda$. By [KP18, Thm. 3.3.2], the functor $\omega_U : L \mapsto \omega(L)|_U$ is an exact faithful $\otimes$-functor from $\mathrm{Rep}_{\mathbb{Z}_p} G$ to the category of vector bundles on $U$. As in §4.1.1, we may also regard $\omega_U$ as a fiber-wise faithful exact functor between the fibered categories $\mathbf{Rep}\, G$ and $\mathbf{Bun}_U$, where the fibers of $\mathbf{Bun}_U$ over $\operatorname{Spec}\mathbb{Z}_p$ and $\operatorname{Spec}\mathbb{Q}_p$ are respectively the categories of vector bundles on $U$ and on $U \times_{\operatorname{Spec}\mathbb{Z}_p} \operatorname{Spec}\mathbb{Q}_p$. Since $P_\omega|_U$ is identified with $P_{\omega_U}$, we know that it is a $G_U$-torsor by [Bro13, Thm. 4.8]. □

**4.4.4.** Let $G$ be a smooth affine group scheme over $\mathbb{Z}_p$ with connected fibers. Let $U$ be the complement of the closed point in $\operatorname{Spec}\mathfrak{S}$. We say that $G$ *satisfies property KL*, if every $G$-torsor over $U$ extends to $\operatorname{Spec}\mathfrak{S}$.[26] (Here "KL" stands for "Key Lemma".) Since $G$ is smooth with connected fibers and since the residue field of the closed point in $\operatorname{Spec}\mathfrak{S}$ is the finite field $k$, by Lang's theorem we know that all $G$-torsors on $\operatorname{Spec}\mathfrak{S}$ are trivial. Thus property KL is equivalent to the property that all $G$-torsors on $U$ are trivial.

It has been proved by Anschütz [Ans18, Cor. 1.2] that all parahoric group schemes $G$ over $\mathbb{Z}_p$ satisfy property KL, generalizing earlier results in [CTS79] and [KP18]. We will make use of this result mainly when $G$ is either a reductive group scheme over $\mathbb{Z}_p$, or the connected Néron model of a torus. (In the former case this result already follows from [CTS79], as explained in Step 5 in the proof of [Kis10, Prop. 1.3.4].) In Corollary 4.4.16 below we will also apply the result of Anschütz to some other parahoric group schemes.

**Lemma 4.4.5.** *Let $G$ be a smooth affine group scheme over $\mathbb{Z}_p$ with connected fibers, satisfying property KL. Let $\rho : \Gamma_K \to G(\mathbb{Z}_p)$ be a $G(\mathbb{Z}_p)$-valued crystalline representation, and let $\lambda \in G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ be a point fixed by $\Gamma_K$. Then $\omega_\rho^\lambda$ : $\mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}_{\mathfrak{S}}^{\mathrm{fp}}$ defined in §4.4.2 is $\otimes$-isomorphic to $\mathbb{1}_{\mathfrak{S}}$ (non-canonically). In particular, $\omega_\rho^\lambda$ and $\omega_{\rho,0}^\lambda$ are fiber functors, and $\Upsilon_\rho^\lambda$ is an object in $G$-$\mathrm{Isoc}_{K_0}^{\circ}$.*

*Proof.* We write $\omega$ for $\omega_\rho^\lambda$. By Lemma 4.4.3 and by the discussion in §4.4.4, we know that $P_\omega|_U$ is a trivial $G_U$-torsor. Fix a section of it over $U$. Then for each $L$ in $\mathrm{Rep}_{\mathbb{Z}_p} G$, we obtain an isomorphism $\iota_L : \mathbb{1}_{\mathfrak{S}}(L)|_U \xrightarrow{\sim} \omega(L)|_U$ between vector bundles on $U$, which is functorial in $L$ and compatible with tensor products. Since $\mathfrak{S}$ is a noetherian normal domain and since the closed point in it has codimension 2, the isomorphism $\iota_L$ extends uniquely to an isomorphism $\widetilde{\iota}_L : \mathbb{1}_{\mathfrak{S}}(L) \xrightarrow{\sim} \omega(L)$ between finite projective $\mathfrak{S}$-modules. By the uniqueness, we know that $\widetilde{\iota}_L$ is functorial in $L$

[26] By descent, $G$ satisfies property KL if and only if all $G$-torsors on the complement of the closed point of $\operatorname{Spec} W(\overline{\mathbb{F}}_p)[\![u]\!]$ extend to $\operatorname{Spec} W(\overline{\mathbb{F}}_p)[\![u]\!]$, cf. Step 4 in the proof of [KP18, Prop. 1.4.3]. Thus property KL is intrinsic to the group $G$, and is independent of the finite extension $k/\mathbb{F}_p$ appearing in the definition $\mathfrak{S} = W(k)[\![u]\!]$.

and compatible with tensor products. Thus we have constructed a $\otimes$-isomorphism $\mathbb{1}_{\mathfrak{S}} \xrightarrow{\sim} \omega$ between $\otimes$-functors.

Since $\omega_\rho^\lambda$ is $\otimes$-isomorphic to $\mathbb{1}_{\mathfrak{S}}$, it is a fiber functor. Since $\omega_{\rho,0}^\lambda \cong (\omega_\rho^\lambda)_W$, it is also a fiber functor. It follows that $\Upsilon_\rho^\lambda$ is in $G\text{-Isoc}^\circ_{K_0}$. ☐

**4.4.6.** Keep the setting of Lemma 4.4.5, and take $\lambda$ to be trivial. Fix a $\otimes$-isomorphism $\eta : \mathbb{1}_{\mathfrak{S}} \xrightarrow{\sim} \omega_\rho$ as in Lemma 4.4.5. Then for $L$ in $\mathrm{Rep}_{\mathbb{Z}_p} G$, the isomorphism $L \otimes_{\mathbb{Z}_p} \mathfrak{S} \xrightarrow{\sim} \omega_\rho(L)$ induced by $\eta$ carries the Frobenius on $\omega_\rho(L)[1/E]$ to a $\varphi$-semi-linear endomorphism on $L \otimes_{\mathbb{Z}_p} \mathfrak{S}[1/E]$, which is of the form $\delta_{\mathfrak{S},L} \otimes \varphi$ for some $\delta_{\mathfrak{S},L} \in \mathrm{GL}(L)(\mathfrak{S}[1/E])$. By the reconstruction theorem, the elements $\delta_{\mathfrak{S},L}$ for all $L$ come from a common, unique element $\delta_{\mathfrak{S}} \in G(\mathfrak{S}[1/E])$ (cf. the similar argument in §4.2.5). If we change the choice of $\eta$, then $\delta_{\mathfrak{S}}$ gets $\varphi$-conjugated by an element of $G(\mathfrak{S})$.

Let $\delta \in G(K_0)$ be the image of $\delta_{\mathfrak{S}}$ under the specialization $u \mapsto 0$. Then the $G(\mathcal{O}_{K_0})$-$\sigma$-conjugacy class of $\delta$ is independent of the choice of $\eta$, and it coincides with $\mathrm{inv}(\Upsilon_{\rho,0})$ defined in §4.2.5. More precisely, $\eta$ naturally induces a point $y \in Y(\Upsilon_{\rho,0})^\circ$, and we have $\delta = \delta_y$, where $\delta_y$ is defined in §4.2.5.

Let $[\rho] \in \mathrm{Crys}_{G_{\mathbb{Q}_p}}$ be the element represented by $\rho$. We can apply the map $D^G_{\mathrm{cris}}$ in §4.3.5 to $[\rho]$ and obtain an element of $\mathrm{RZ}_G/\sim$. In particular, we obtain a cocharacter $\mu$ of $G_{\overline{\mathbb{Q}}_p}$, well defined up to $G(\overline{\mathbb{Q}}_p)$-conjugacy (cf. the discussion on the equivalence relation $\sim$ in §4.3.1).

For brevity, we write $\pi_1(G)$ for $\pi_1(G_{\mathbb{Q}_p})$. As in Definition 1.3.8, we have the Kottwitz homomorphism

$$\kappa^{v_p}_{G_{K_0}} : G(K_0) \longrightarrow \pi_1(G)_{\Gamma_{K_0}}$$

associated with the $p$-adic valuation on $K_0$. We write $[\mu]$ for the image of $\mu$ in $\pi_1(G)_{\Gamma_{K_0}}$, which depends only on $[\rho]$.

**Proposition 4.4.7.** *With the notation in §4.4.6, we have* $\kappa^{v_p}_{G_{K_0}}(\delta) = [\mu]$.

For the proof of the proposition we need the following result which will be used again later.

**Lemma 4.4.8.** *Let $f$ be a non-zero irreducible element of $\mathfrak{S}$ with $f(0) \in W$ having $p$-adic valuation 1. Let $v_f$ be the $f$-adic valuation on* $\mathrm{Frac}\,\mathfrak{S}$*, and let*

$$\kappa^{v_f}_{G_{K_0}} : G(\mathrm{Frac}\,\mathfrak{S}) \longrightarrow \pi_1(G)_{\Gamma_{K_0}}$$

*be the associated Kottwitz homomorphism as in Definition 1.3.8. Let $g \in G(\mathfrak{S}[1/f])$ and let $g_0$ be the image of $g$ in $G(K_0)$ via the specialization $u \mapsto 0$. Then we have $\kappa^{v_f}_{G_{K_0}}(g) = \kappa^{v_p}_{G_{K_0}}(g_0)$ in $\pi_1(G)_{\Gamma_{K_0}}$.*

*Proof.* Let $v_f, v_p$ be the discrete valuations on $\mathfrak{S}[1/fp]$ attached to the primes $(f), (p)$. Let $v_0$ be the discrete valuation on $\mathfrak{S}[1/fp]$ given by

$$\mathfrak{S}[1/fp] \xrightarrow{u \mapsto 0} K_0 \xrightarrow{v_p} \mathbb{Z} \cup \{\infty\}.$$

Since $f, p \in \mathfrak{S}$ are prime elements, any unit $w \in \mathfrak{S}[1/fp]^\times$ has the form $w = f^i p^j y$ with $i, j \in \mathbb{Z}$, and $y \in \mathfrak{S}^\times$. Hence we have $v_f(w) + v_p(w) = v_0(w) = i + j$. By Proposition 1.3.10, for any $g \in G(\mathfrak{S}[1/fp])$ we have

$$\kappa^{v_f}_{G_{K_0}}(g) + \kappa^{v_p}_{G_{K_0}}(g) = \kappa^{v_0}_{G_{K_0}}(g) = \kappa^{v_p}_{G_{K_0}}(g_0).$$

Now if $g \in G(\mathfrak{S}[1/f])$, then $g \in G(\mathfrak{S}_{(p)})$, and we have

$$\kappa^{v_p}_{G_{K_0}}(g) = 0,$$

by Corollary 1.3.12. The lemma follows. □

*Proof of Proposition 4.4.7.* We write $\omega$ for $\omega_\rho$. For each $L \in \mathrm{Rep}_{\mathbb{Z}_p} G$ and $V \in \mathrm{Rep}_{\mathbb{Q}_p} G$, we understand that $\Gamma_K$ acts on $L$ and $V$ via $\rho$.

Let $\omega' : \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}^{\mathrm{fp}}_{\mathfrak{S}}$ be the base change of $\omega$ along $\varphi : \mathfrak{S} \to \mathfrak{S}$, that is, $\omega'(L) := \varphi^*\mathfrak{M}(L)$. As in §4.1.2, we have a functor

$$\omega'_K : \mathrm{Rep}_{\mathbb{Q}_p} G \longrightarrow \mathrm{Mod}^{\mathrm{fp}}_K$$

induced by $\omega'$. By (ii) in §4.4.1, the functor $\omega'_K$ is canonically identified with the functor $V \mapsto D_{\mathrm{dR}}(V)$. Note that $D_{\mathrm{dR}}(V)$ is an admissible filtered $\varphi$-module for each $V \in \mathrm{Rep}_{\mathbb{Q}_p} G$. Now the filtrations on $D_{\mathrm{dR}}(V)$ for all $V$ give rise to a $\otimes$-filtration on $\omega'_K$. Since $\omega'_K$ is exact, and since exact sequences of admissible filtered $\varphi$-modules are automatically strict with respect to the filtrations, the $\otimes$-filtration on $\omega'_K$ is exact in the sense of [SR72, IV, §2.1] (cf. [Zie15, §4.2]). Now since $\mathbb{Q}_p$ and $K$ are fields of characteristic zero and since $G_{\mathbb{Q}_p}$ is of finite type, a theorem of Deligne (see [SR72, IV, §2.4]) implies that the filtration on $\omega'_K$ is induced by a cocharacter $\mu_{\mathrm{dR}} : \mathbb{G}_{m,K} \to \underline{\mathrm{Aut}}^{\otimes}(\omega'_K)$. The $\otimes$-isomorphism $\eta : \mathbb{1}_{\mathfrak{S}} \xrightarrow{\sim} \omega$ fixed in §4.4.6 induces a $\otimes$-isomorphism $\eta' : \mathbb{1}_{\mathfrak{S}} \xrightarrow{\sim} \omega'$ via pull-back along $\varphi$. (Note that we have a canonical identification $\varphi^*(\mathbb{1}_{\mathfrak{S}}) \cong \mathbb{1}_{\mathfrak{S}}$.) We use $\eta'$ to identify $\underline{\mathrm{Aut}}^{\otimes}(\omega'_K)$ with $G_K$, and thereby identify $\mu_{\mathrm{dR}}$ with a cocharacter $\mu'$ of $G_K$.

We claim that $\mu'$ lies in the $G(\overline{\mathbb{Q}}_p)$-conjugacy class of $\mu$, and in articular $[\mu'] = [\mu] \in \pi_1(G)_{\Gamma_{K_0}}$. In fact, by the definition of $\mu$, there exists a finite extension $F/K$ and an element of $\mathrm{RZ}_G(F)$ of the form $(\mu, \gamma)$ such that the $\otimes$-functor

$$\mathrm{Rep}_{\mathbb{Q}_p} G \longrightarrow \{\text{finite-dimensional filtered } F\text{-vector spaces}\}, \quad V \longmapsto D_{\mathrm{dR}}(V)$$

is $\otimes$-isomorphic with

(4.4.8.1)

$$\mathrm{Rep}_{\mathbb{Q}_p} G \xrightarrow{\mathcal{I}_{\mu,\gamma}} \mathrm{MF}^{\varphi}_K \xrightarrow{(\cdot)\otimes_{F_0} F} \{\text{finite-dimensional filtered } F\text{-vector spaces}\}.$$

Now (4.4.8.1) lifts $\mathbb{1}_F$, and gives an (exact) $\otimes$-filtration on $\mathbb{1}_F$. This $\otimes$-filtration is (tautologically) induced by the cocharacter $\mu$ of $\underline{\mathrm{Aut}}^{\otimes}(\mathbb{1}_F) = G_F$. The claim immediately follows.

Now consider an object $L$ in $\mathrm{Rep}_{\mathbb{Z}_p} G$. We write $\mathfrak{M}$ for $\omega(L) = \mathfrak{M}(L)$, and write $L_R$ for $L \otimes_{\mathbb{Z}_p} R$, for any $\mathbb{Z}_p$-algebra $R$. We write $B^+$ for $\widehat{\mathfrak{S}}_{(E)}$, which is a $\mathfrak{S}[1/p]$-algebra, and write $B$ for $B^+[1/E] = \mathrm{Frac}\, B^+$. Then $\eta'$ induces a $B$-linear isomorphism

$$\mathscr{F} : L_B \xrightarrow{\sim} \varphi^*\mathfrak{M} \otimes_{\mathfrak{S}} B.$$

We equip $L_B = L_K \otimes_K B$ with the tensor product filtration of the filtration on $L_K$ defined by $\mu'$ and the $E$-adic filtration on $B$. We equip $\varphi^*\mathfrak{M} \otimes_{\mathfrak{S}} B$ with the tensor product filtration of the filtration on $\varphi^*\mathfrak{M}$ and the $E$-adic filtration on $B$. Since the isomorphism (4.4.1.1) is induced by the filtered isomorphism (4.4.1.2), we know that $\mathscr{F}$ is a filtered isomorphism. In particular, we have

$$\mathscr{F}(\mu'(E)^{-1} \cdot L_{B^+}) = \mathscr{F}(\mathrm{Fil}^0 L_B) = \mathrm{Fil}^0(\varphi^*\mathfrak{M} \otimes_{\mathfrak{S}} B). \tag{4.4.8.2}$$

Here $\mu'(E) \in G(B)$ acts on $L_B$.

By the definition of $\delta_{\mathfrak{S}}$, we have a commutative diagram

$$\begin{tikzcd} L_{\mathfrak{S}[1/E]} \arrow[r, "\eta'", "\cong"'] \arrow[d, "\delta_{\mathfrak{S}}"] & \varphi^*\mathfrak{M}[1/E] \arrow[d, "\varphi_{\mathfrak{M}}"] \\ L_{\mathfrak{S}[1/E]} \arrow[r, "\eta", "\cong"'] & \mathfrak{M}[1/E] \end{tikzcd}$$

Base changing from $\mathfrak{S}[1/E]$ to $B$, we obtain the commutative diagram

$$\begin{tikzcd} L_B \arrow[r, "\mathscr{F}", "\cong"'] \arrow[d, "\delta_{\mathfrak{S}}"] & \varphi^*\mathfrak{M} \otimes_{\mathfrak{S}} B \arrow[d, "\varphi_{\mathfrak{M}}"] \\ L_B \arrow[r, "\eta", "\cong"'] & \mathfrak{M} \otimes_{\mathfrak{S}} B \end{tikzcd}$$

It is easy to see that in the above diagram $\varphi_{\mathfrak{M}}$ maps $\mathrm{Fil}^0(\varphi^*\mathfrak{M} \otimes_{\mathfrak{S}} B)$ into $\mathfrak{M} \otimes_{\mathfrak{S}} B^+ \subset \mathfrak{M} \otimes_{\mathfrak{S}} B$. Hence by (4.4.8.2) we have

$$\delta_{\mathfrak{S}} \cdot \mu'(E)^{-1} \cdot L_{B^+} \subset L_{B^+}.$$

Since $L$ is arbitrary, by the reconstruction theorem we know that the element $\delta_{\mathfrak{S}} \cdot \mu'(E)^{-1} \in G(B)$ lies in $G(B^+)$. Applying Corollary 1.3.15 to $F = B$ and $\mathcal{O}_F = B^+$, we obtain that $\kappa^{v_E}_{G_{K_0}}(\delta_{\mathfrak{S}}) = [\mu']$ in $\pi_1(G)_{\Gamma_{K_0}}$, where $\kappa^{v_E}_{G_{K_0}}$ is as in Lemma 4.4.8. By Lemma 4.4.8, we have $\kappa^{v_p}_{G_{K_0}}(\delta) = [\mu']$. But we have seen that $[\mu'] = [\mu]$. This finishes the proof. □

**4.4.9.** Let $K/\mathbb{Q}_p$ be a finite extension (inside $\overline{\mathbb{Q}}_p$). Consider $G$, $\rho$, and $\lambda$ as in Lemma 4.4.5. There is a natural *base change* functor $\mathrm{Isoc}^\circ_{K_0} \to \mathrm{Isoc}^\circ_{\mathbb{Q}_p^{\mathrm{ur}}}, D \mapsto D \otimes_{\mathcal{O}_{K_0}} \mathbb{Z}_p^{\mathrm{ur}}$. This induces a base change functor $G\text{-}\mathrm{Isoc}^\circ_{K_0} \to G\text{-}\mathrm{Isoc}^\circ_{\mathbb{Q}_p^{\mathrm{ur}}}$. We set

$$\Upsilon^\lambda_{\rho,\mathrm{ur}} \in G\text{-}\mathrm{Isoc}^\circ_{\mathbb{Q}_p^{\mathrm{ur}}}$$

to be the base change of $\Upsilon^\lambda_\rho \in G\text{-}\mathrm{Isoc}^\circ_{K_0}$, namely, $\Upsilon^\lambda_{\rho,\mathrm{ur}}$ is the composite $\otimes$-functor

$$\mathrm{Rep}_{\mathbb{Z}_p} G \xrightarrow{\Upsilon^\lambda_\rho} \mathrm{Isoc}^\circ_{K_0} \xrightarrow{(\cdot)\otimes_{\mathcal{O}_{K_0}} \mathbb{Z}_p^{\mathrm{ur}}} \mathrm{Isoc}^\circ_{\mathbb{Q}_p^{\mathrm{ur}}}.$$

If $K'/K$ is a finite extension inside $\overline{\mathbb{Q}}_p$ and if $\rho'$ is the restriction of $\rho$ to $\Gamma_{K'}$, then the same construction gives rise to

$$\Upsilon^\lambda_{\rho',\mathrm{ur}} \in G\text{-}\mathrm{Isoc}^\circ_{\mathbb{Q}_p^{\mathrm{ur}}}.$$

By property (iv) in §4.4.1, we know that $\Upsilon^\lambda_{\rho',\mathrm{ur}}$ is canonically $\otimes$-isomorphic with $\Upsilon^\lambda_{\rho,\mathrm{ur}}$. Therefore up to canonical isomorphism in $G\text{-}\mathrm{Isoc}^\circ_{\mathbb{Q}_p^{\mathrm{ur}}}$, the definition of $\Upsilon^\lambda_{\rho,\mathrm{ur}} \in G\text{-}\mathrm{Isoc}^\circ_{\mathbb{Q}_p^{\mathrm{ur}}}$ depends on $\rho$ only via its germ at $1 \in \Gamma_p$,

Now suppose we are given a general element $[\rho] \in \mathrm{Crys}_{G_{\mathbb{Q}_p}}$ (see §4.3.5). Let $\rho \in \mathrm{Crys}_{G_{\mathbb{Q}_p}}(K)$ be a representative of $[\rho]$, where $K/\mathbb{Q}_p$ is a finite extension inside $\overline{\mathbb{Q}}_p$. Since the homomorphism $\rho : \Gamma_K \to G(\mathbb{Q}_p)$ is continuous, there is a finite extension $K'/K$ in $\overline{\mathbb{Q}}_p$ for which $\rho(\Gamma_{K'}) \subset G(\mathbb{Z}_p)$. Moreover, if $\lambda \in G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ is an arbitrarily given element, we can further enlarge $K'$ if necessary to arrange that $\lambda$ is fixed by $\rho(\Gamma_{K'}) \subset G(\mathbb{Z}_p)$. (This can be achieved because the stabilizer of $\lambda$ in $G(\mathbb{Z}_p)$ is the open subgroup $G(\mathbb{Z}_p) \cap \lambda G(\mathbb{Z}_p)\lambda^{-1}$.) We then define

$$\Upsilon^{\lambda}_{[\rho]} := \Upsilon^{\lambda}_{\rho|_{\Gamma_{K'}},\mathrm{ur}} \in G\text{-}\mathrm{Isoc}^{\circ}_{\mathbb{Q}_p^{\mathrm{ur}}}.$$

This definition depends only on $[\rho] \in \mathrm{Crys}_{G_{\mathbb{Q}_p}}$ and $\lambda \in G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$, up to canonical isomorphism.

As usual, if $\lambda$ is trivial, we write $\Upsilon_{[\rho]}$ for $\Upsilon^{\lambda}_{[\rho]}$.

**4.4.10.** Let $G$ be a smooth affine group scheme over $\mathbb{Z}_p$ with connected fibers, satisfying property KL. We fix $[\rho] \in \mathrm{Crys}_{G_{\mathbb{Q}_p}}$, and obtain $\Upsilon_{[\rho]} \in G\text{-}\mathrm{Isoc}^{\circ}_{\mathbb{Q}_p^{\mathrm{ur}}}$ as in §4.4.9. In §4.2.5, we defined the $G(\mathbb{Z}_p^{\mathrm{ur}})$-$\sigma$-conjugacy class $\mathrm{inv}(\Upsilon_{[\rho]})$ attached to $\Upsilon_{[\rho]}$. Let $\delta_{\mathrm{ur}} \in G(\mathbb{Q}_p^{\mathrm{ur}})$ be a representative of $\mathrm{inv}(\Upsilon_{[\rho]})$. Let $\mu$ be as in §4.4.6 (which depends only on $[\rho] \in \mathrm{Crys}_{G_{\mathbb{Q}_p}}$). As usual, write $w_G : G(\breve{\mathbb{Q}}_p) \to \pi_1(G)_{\Gamma_{p,0}}$ for the Kottwitz homomorphism associated with the $p$-adic valuation on $\breve{\mathbb{Q}}_p$. Then $w_G$ is trivial on $G(\mathbb{Z}_p^{\mathrm{ur}})$ (by Corollary 1.3.12), and so $w_G(\delta_{\mathrm{ur}})$ depends only on $\Upsilon_{[\rho]}$ and not on the choice of $\delta_{\mathrm{ur}}$.

**Corollary 4.4.11.** *In the setting of §4.4.10, the element $w_G(\delta_{\mathrm{ur}}) \in \pi_1(G)_{\Gamma_{p,0}}$ is equal to the image of $\mu$.*

*Proof.* As explained in §4.4.9, we can pick a $G(\mathbb{Z}_p)$-valued crystalline representation $\rho : \Gamma_K \to G(\mathbb{Z}_p)$ that represents $[\rho]$. Up to replacing $K$ by a finite (unramified) extension, we may assume that $\pi_1(G)_{\Gamma_{p,0}} = \pi_1(G)_{\Gamma_{K_0}}$. We may also assume that $\delta_{\mathrm{ur}} = \delta$, where $\delta$ is as in §4.4.6 (defined with respect to $\rho$). The corollary then follows from Proposition 4.4.7. □

**Corollary 4.4.12.** *Let $T$ be a cuspidal torus over $\mathbb{Q}$, and let $[\rho_T]$ be an element of* $\mathrm{Mot}_T \subset \mathrm{Crys}_{T_{\mathbb{Q}_p}}$ *(see Definition 4.3.11). Let $\mu = \mathscr{M}_T([\rho_T]) \in X_*(T)$, where $\mathscr{M}_T$ is the bijection in Proposition 4.3.12. Let $\mathcal{T}^{\circ}$ be the connected Néron model of $T_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$, and let $\Upsilon_{[\rho_T]}$ be the object in $\mathcal{T}^{\circ}$-$\mathrm{Isoc}^{\circ}_{\mathbb{Q}_p^{\mathrm{ur}}}$ associated with $[\rho_T]$ as in §4.4.9. Let $\delta_T \in T(\mathbb{Q}_p^{\mathrm{ur}})$ be a representative of* $\mathrm{inv}(\Upsilon_{[\rho_T]})$ *(see §4.2.5). Then $\delta_T$ lies in $T(\mathbb{Q}_p^{\mathrm{ur}})^{\mathrm{mot}}$, and $w_{T_{\mathbb{Q}_p}}(\delta_T) \in X_*(T)_{\Gamma_{p,0}}$ is equal to the image of $\mu$.*

*Proof.* Since $[\rho_T]$ lies in $\mathrm{Mot}(T)$, we have $\delta_T \in T(\mathbb{Q}_p^{\mathrm{ur}})^{\mathrm{mot}}$. The claim about $w_{T_{\mathbb{Q}_p}}(\delta_T)$ follows from Corollary 4.4.11 applied to $G = \mathcal{T}^{\circ}$. Here we have used that $\mathcal{T}^{\circ}$ satisfies KL; see §4.4.4. □

*Remark* 4.4.13. By Lemma 4.3.9, the two properties satisfied by $\delta_T$ claimed in Corollary 4.4.12 uniquely characterize the $\mathcal{T}^{\circ}(\mathbb{Z}_p^{\mathrm{ur}})$-orbit of $\delta$ under $\sigma$-conjugation.

**4.4.14.** Keep the setting of §4.4.10. Let $\lambda \in G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ be an arbitrary element. We then obtain $\Upsilon_{[\rho]} = \Upsilon^{1}_{[\rho]}$ and $\Upsilon^{\lambda}_{[\rho]}$. As in §4.2.5, associated with $\Upsilon^{\lambda}_{[\rho]} \in G\text{-}\mathrm{Isoc}^{\circ}_{\mathbb{Q}_p^{\mathrm{ur}}}$ we have the $G(\mathbb{Q}_p^{\mathrm{ur}})$-torsor $Y(\Upsilon^{\lambda}_{[\rho]})$, together with a $G(\mathbb{Z}_p^{\mathrm{ur}})$-torsor

$$Y(\Upsilon^{\lambda}_{[\rho]})^{\circ} \subset Y(\Upsilon^{\lambda}_{[\rho]}).$$

Thus we can canonically identify $Y(\Upsilon^{\lambda}_{[\rho]})/G(\mathbb{Z}_p^{\mathrm{ur}})$ with $G(\mathbb{Q}_p^{\mathrm{ur}})/G(\mathbb{Z}_p^{\mathrm{ur}})$. Similarly, we identify $Y(\Upsilon_{[\rho]})/G(\mathbb{Z}_p^{\mathrm{ur}})$ with $G(\mathbb{Q}_p^{\mathrm{ur}})/G(\mathbb{Z}_p^{\mathrm{ur}})$.

There is a tautological $\otimes$-isomorphism $\Upsilon^{\lambda}_{[\rho]}[1/p] \xrightarrow{\sim} \Upsilon_{[\rho]}[1/p]$, induced by the tautological isomorphism $L \otimes \mathbb{Q}_p \xrightarrow{\sim} (\lambda \cdot L) \otimes \mathbb{Q}_p$ in $\mathrm{Rep}_{\mathbb{Q}_p} G$ for each $L$ in $\mathrm{Rep}_{\mathbb{Z}_p} G$.

This induces a tautological isomorphism $Y(\Upsilon^{\lambda}_{[\rho]}) \xrightarrow{\sim} Y(\Upsilon_{[\rho]})$. Thus we have an induced map

$$Y(\Upsilon^{\lambda}_{[\rho]})/G(\mathbb{Z}_p^{\mathrm{ur}}) \longrightarrow Y(\Upsilon_{[\rho]})/G(\mathbb{Z}_p^{\mathrm{ur}}),$$

which we identify as a map

$$G(\mathbb{Q}_p^{\mathrm{ur}})/G(\mathbb{Z}_p^{\mathrm{ur}}) \to G(\mathbb{Q}_p^{\mathrm{ur}})/G(\mathbb{Z}_p^{\mathrm{ur}}).$$

Let $\lambda_0 \in G(\mathbb{Q}_p^{\mathrm{ur}})/G(\mathbb{Z}_p^{\mathrm{ur}})$ be the image of 1 under the last map. Since the Kottwitz homomorphism $w_G : G(\breve{\mathbb{Q}}_p) \to \pi_1(G)_{\Gamma_{p,0}}$ is trivial on $G(\mathbb{Z}_p^{\mathrm{ur}})$ (by Corollary 1.3.12), we obtain well-defined elements $w_G(\lambda), w_G(\lambda_0) \in \pi_1(G)_{\Gamma_{p,0}}$.

**Proposition 4.4.15.** *In the setting of §4.4.14, we have $w_G(\lambda) = w_G(\lambda_0)$.*

*Proof.* As explained in §4.4.9, we may pick a $G(\mathbb{Z}_p)$-valued crystalline representation $\rho : \Gamma_K \to G(\mathbb{Z}_p)$ representing $[\rho]$, where $K/\mathbb{Q}_p$ is a finite extension inside $\overline{\mathbb{Q}}_p$, such that the $\Gamma_K$-action on $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ induced by $\rho$ fixes $\lambda$. For later purposes we shall also suitably enlarge $K$ to assume that $\pi_1(G)_{\Gamma_{K_0}} = \pi_1(G)_{\Gamma_{p,0}}$. (As always, $K_0$ denotes the maximal unramified extension of $\mathbb{Q}_p$ inside $K$.)

As in §4.4.2, we have $\otimes$-functors $\omega_\rho$ and $\omega_\rho^\lambda : \mathrm{Rep}_{\mathbb{Z}_p} G \to \mathrm{Mod}^{\mathrm{fp}}_{\mathfrak{S}}$ (defined with respect to $K$ and a chosen uniformizer.) For brevity we denote them by $\omega$ and $\omega^\lambda$ respectively. By Lemma 4.4.5, $P_\omega(\mathfrak{S})$ and $P_{\omega^\lambda}(\mathfrak{S})$ are non-empty. In particular, $Y(\omega)/G(\mathfrak{S})$ and $Y(\omega^\lambda)/G(\mathfrak{S})$ admit canonical base points, and can both be identified with $G(\mathfrak{S}[1/p])/G(\mathfrak{S})$ canonically (see §4.1.12). The tautological isomorphism $\omega^\lambda[1/p] \xrightarrow{\sim} \omega[1/p]$ induces a map $Y(\omega^\lambda)/G(\mathfrak{S}) \to Y(\omega)/G(\mathfrak{S})$, which we identify as a map $G(\mathfrak{S}[1/p])/G(\mathfrak{S}) \to G(\mathfrak{S}[1/p])/G(\mathfrak{S})$. Denote the image of 1 under the last map by $\lambda_{\mathfrak{S}}$. It is clear that $\lambda_0$ is equal to the image of $\lambda_{\mathfrak{S}}$ under $G(\mathfrak{S}[1/p])/G(\mathfrak{S}) \to G(K_0)/G(\mathcal{O}_{K_0}) \to G(\mathbb{Q}_p^{\mathrm{ur}})/G(\mathbb{Z}_p^{\mathrm{ur}})$, where the first map is induced by the specialization $u \mapsto 0$.

Now write $C$ for $\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{ur}}}$. By (iii) in §4.4.1, the base change $\omega_C^\lambda$ of $\omega^\lambda$ to $C$ is canonically $\otimes$-isomorphic to the functor

$$\mathbb{1}_C^\lambda : \mathrm{Rep}_{\mathbb{Z}_p} G \longrightarrow \mathrm{Mod}_C^{\mathrm{fp}}, \quad L \mapsto (\lambda \cdot L) \otimes_{\mathbb{Z}_p} C.$$

Similarly, $\omega_C$ is canonically $\otimes$-isomorphic to $\mathbb{1}_C$. Moreover, the canonical $\otimes$-isomorphisms $\mathbb{1}_C^\lambda \cong \omega_C^\lambda$ and $\mathbb{1}_C \cong \omega_C$ are compatible with the tautological isomorphisms $\mathbb{1}_C^\lambda[1/p] \cong \mathbb{1}_C[1/p]$ and $\omega_C^\lambda[1/p] \cong \omega_C[1/p]$. It follows that the image of $\lambda_{\mathfrak{S}}$ under $G(\mathfrak{S}[1/p])/G(\mathfrak{S}) \to G(C[1/p])/G(C)$ is equal to the image of $\lambda$ under $G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \to G(C[1/p])/G(C)$. Since the map $\mathrm{Gr} \to G(C[1/p])/G(C)$ is injective, we conclude that the image of $\lambda_{\mathfrak{S}}$ in $G(\widehat{\mathfrak{S}}_{(p)}[1/p])/G(\widehat{\mathfrak{S}}_{(p)})$ is equal to the image of $\lambda$.

Consider the Kottwitz homomorphism

$$\kappa^{v_p}_{G_{K_0}} : G(\mathrm{Frac}\,\mathfrak{S}) \longrightarrow \pi_1(G)_{\Gamma_{K_0}} = \pi_1(G)_{\Gamma_{p,0}}$$

associated with the $p$-adic valuation $v_p$ on $\mathrm{Frac}\,\mathfrak{S}$. By Lemma 1.3.12 and by the functoriality of the Kottwitz homomorphism, $\kappa^{v_p}_{G_{K_0}}$ factors through a map $G(\widehat{\mathfrak{S}}_{(p)}[1/p])/G(\widehat{\mathfrak{S}}_{(p)}) \to \pi_1(G)_{\Gamma_{p,0}}$, whose restriction to $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ is equal to $w_G$. Since $\lambda_{\mathfrak{S}}$ and $\lambda$ have the same image in

$$G(\widehat{\mathfrak{S}}_{(p)}[1/p])/G(\widehat{\mathfrak{S}}_{(p)}),$$

we conclude that $\kappa^{v_p}_{G_{K_0}}(\lambda_{\mathfrak{S}}) = w_G(\lambda)$. Now by Lemma 4.4.8, we have $\kappa^{v_p}_{G_{K_0}}(\lambda_{\mathfrak{S}}) = w_G(\lambda_0)$ since $\lambda_0$ is the image of $\lambda_{\mathfrak{S}}$ under the specialization $u \mapsto 0$. Therefore $w_G(\lambda) = w_G(\lambda_0)$ as desired. □

**Corollary 4.4.16.** *Let $F/\mathbb{Q}_p$ be a finite extension. Let $G_0$ be a parahoric group scheme over $\mathcal{O}_F$, and let $G = \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}_p} G_0$. Then $G$ satisfies KL, and in particular the conclusion of Proposition 4.4.15 holds for $G$.*

*Proof.* It suffices to show that $G$ is parahoric and apply the result of Anschütz [Ans18] (recalled in §4.4.4). The fact that $G$ is parahoric is well known; see for instance [HR20, Prop. 4.7]. □

## 4.5. **Crystalline representations factoring through a maximal torus.**

**4.5.1.** Let $\mathcal{G}$ be a parahoric group scheme over $\mathbb{Z}_p$, and write $G$ for $\mathcal{G}_{\mathbb{Q}_p}$. (Note the change of notations from §4.4.) We fix $[\rho] \in \mathrm{Crys}_G$. Let $T \subset G$ be a maximal torus. We assume that $[\rho]$ is equal to the image of an element $[\rho_T] \in \mathrm{Crys}_T$ under the natural injection $\mathrm{Crys}_T \hookrightarrow \mathrm{Crys}_G$. Let $[\rho^{\mathrm{ab}}]$ be the image of $[\rho_T]$ under $\mathrm{Crys}_G \to \mathrm{Crys}_{G^{\mathrm{ab}}}$.

Let $\mathcal{T}^\circ$ (resp. $\mathcal{G}^{\mathrm{ab}}$) be the connected Néron model of $T$ (resp. $G^{\mathrm{ab}}$) over $\mathbb{Z}_p$. As recalled in §4.4.4, the $\mathbb{Z}_p$-group schemes $\mathcal{G}, \mathcal{T}^\circ, \mathcal{G}^{\mathrm{ab}}$ satisfy KL. As in §4.4.9 we obtain

$$\begin{aligned} \Upsilon_{[\rho]} &\in \mathcal{G}\text{-}\mathrm{Isoc}^\circ_{\mathbb{Q}_p^{\mathrm{ur}}}, \\ \Upsilon_{[\rho_T]} &\in \mathcal{T}^\circ\text{-}\mathrm{Isoc}^\circ_{\mathbb{Q}_p^{\mathrm{ur}}}, \\ \Upsilon_{[\rho^{\mathrm{ab}}]} &\in \mathcal{G}^{\mathrm{ab}}\text{-}\mathrm{Isoc}^\circ_{\mathbb{Q}_p^{\mathrm{ur}}}. \end{aligned}$$

As in §4.2.5 we obtain the $G(\mathbb{Q}_p^{\mathrm{ur}})$-torsor $Y(\Upsilon_{[\rho]})$, the $T(\mathbb{Q}_p^{\mathrm{ur}})$-torsor $Y(\Upsilon_{[\rho_T]})$, and the $G^{\mathrm{ab}}(\mathbb{Q}_p^{\mathrm{ur}})$-torsor $Y(\Upsilon_{[\rho^{\mathrm{ab}}]})$. For $* \in \left\{\Upsilon_{[\rho]}, \Upsilon_{[\rho_T]}, \Upsilon_{[\rho^{\mathrm{ab}}]}\right\}$, we have the set of integral points $Y(*)^\circ \subset Y(*)$.

By definition, $Y(\Upsilon_{[\rho]})$ depends only on the fiber functor

$$(v \circ \Upsilon_{[\rho]})[1/p] : \mathrm{Rep}_{\mathbb{Q}_p} G \longrightarrow \mathrm{Mod}^{\mathrm{fp}}_{\mathbb{Q}_p^{\mathrm{ur}}}.$$

Similarly, $Y(\Upsilon_{[\rho_T]})$ depends only on the fiber functor

$$(v \circ \Upsilon_{[\rho_T]})[1/p] : \mathrm{Rep}_{\mathbb{Q}_p} T \longrightarrow \mathrm{Mod}^{\mathrm{fp}}_{\mathbb{Q}_p^{\mathrm{ur}}}.$$

Observe that $(v \circ \Upsilon_{[\rho]})[1/p]$ is canonically isomorphic to the composition

$$\mathrm{Rep}_{\mathbb{Q}_p} G \xrightarrow{\mathrm{Res}} \mathrm{Rep}_{\mathbb{Q}_p} T \xrightarrow{(v \circ \Upsilon_{[\rho_T]})[1/p]} \mathrm{Mod}^{\mathrm{fp}}_{\mathbb{Q}_p^{\mathrm{ur}}},$$

as they are both identified with the functor $V \mapsto D_{\mathrm{cris}}(V) \otimes_{K_0} \mathbb{Q}_p^{\mathrm{ur}}$, where we view $V$ as a $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$-representation via $[\rho]$ (for a sufficiently large finite extension $K/\mathbb{Q}_p$). Hence we obtain a canonical map $Y(\Upsilon_{[\rho_T]}) \to Y(\Upsilon_{[\rho]})$, which is equivariant for the $T(\mathbb{Q}_p^{\mathrm{ur}})$-action on the two sides. In particular this map is injective.

Similarly, we obtain a natural map $Y(\Upsilon_{[\rho_T]}) \to Y(\Upsilon_{[\rho^{\mathrm{ab}}]})$ which is equivariant with respect to $T(\mathbb{Q}_p^{\mathrm{ur}}) \to G^{\mathrm{ab}}(\mathbb{Q}_p^{\mathrm{ur}})$, and a natural map $Y(\Upsilon_{[\rho]}) \to Y(\Upsilon_{[\rho^{\mathrm{ab}}]})$ which is equivariant with respect to $G(\mathbb{Q}_p^{\mathrm{ur}}) \to G^{\mathrm{ab}}(\mathbb{Q}_p^{\mathrm{ur}})$.

Since the inclusion $T \hookrightarrow G$ does not necessarily extend to a map $\mathcal{T}^\circ \to \mathcal{G}$ over $\mathbb{Z}_p$, one cannot expect that the map $Y(\Upsilon_{[\rho_T]}) \to Y(\Upsilon_{[\rho]})$ sends integral points to integral points. Nevertheless, we have the following compatibility result.

**Proposition 4.5.2.** *The natural maps*

$$Y(\Upsilon_{[\rho_T]}) \longrightarrow Y(\Upsilon_{[\rho^{\mathrm{ab}}]})$$

*and*

$$Y(\Upsilon_{[\rho]}) \longrightarrow Y(\Upsilon_{[\rho^{\mathrm{ab}}]})$$

*send integral points to integral points. If we assume that $\mathcal{G}$ is a reductive group scheme, then the image of $Y(\Upsilon_{[\rho_T]})^\circ$ in $Y(\Upsilon_{[\rho]})$ is contained in $Y(\Upsilon_{[\rho]})^\circ \cdot G_{\mathrm{der}}(\mathbb{Q}_p^{\mathrm{ur}})$.*

*Proof.* The first statement follows immediately from the functoriality of the constructions, and the fact that the $\mathbb{Q}_p$-homomorphisms $T \to G^{\mathrm{ab}}$ and $G \to G^{\mathrm{ab}}$ extend to $\mathbb{Z}_p$-homomorphisms $\mathcal{T}^\circ \to \mathcal{G}^{\mathrm{ab}}$ and $\mathcal{G} \to \mathcal{G}^{\mathrm{ab}}$ respectively. The second statement follows from the first statement once we know that the map $Y(\Upsilon_{[\rho]})^\circ \to Y(\Upsilon_{[\rho^{\mathrm{ab}}]})^\circ$ (provided by the first statement) is surjective. For the last fact, it suffices to observe that the map $\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}) \to \mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p^{\mathrm{ur}})$ is surjective when $\mathcal{G}$ is a reductive group scheme. In fact, in this case we even know that $\mathcal{G}(\mathbb{Z}_{p^n}) \to \mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_{p^n})$ is surjective for all $n \in \mathbb{Z}_{\geq 1}$, by Lang's theorem applied to $\mathcal{G}_{\mathrm{der},\mathbb{Z}_{p^n}}$ which is smooth over $\mathbb{Z}_{p^n}$ and has connected fibers. □

*Remark* 4.5.3. Let $K/\mathbb{Q}_p$ be a large enough finite extension such that $[\rho_T]$ is induced by $\mathcal{T}^\circ(\mathbb{Z}_p)$-valued crystalline representation $\rho_T : \mathrm{Gal}(\overline{K}/K) \to \mathcal{T}^\circ(\mathbb{Z}_p)$ which factors through $\mathcal{T}^\circ(\mathbb{Z}_p) \cap \mathcal{G}(\mathbb{Z}_p)$. Let $\rho$ (resp. $\rho^{\mathrm{ab}}$) be the induced $\mathcal{G}(\mathbb{Z}_p)$-valued (resp. $\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p)$-valued) crystalline representation. Let $K_0$ be the maximal unramified extension of $\mathbb{Q}_p$ inside $K$. Then we have the variant of Proposition 4.5.2, where $Y(\Upsilon_{[\rho_T]})$, $Y(\Upsilon_{[\rho]})$, $Y(\Upsilon_{[\rho^{\mathrm{ab}}]})$, and $G_{\mathrm{der}}(\mathbb{Q}_p^{\mathrm{ur}})$ are replaced by the $T(K_0)$-torsor $Y(\Upsilon_{\rho_T})$, the $G(K_0)$-torsor $Y(\Upsilon_\rho)$, the $G^{\mathrm{ab}}(K_0)$-torsor $Y(\Upsilon_{\rho^{\mathrm{ab}}})$, and $G_{\mathrm{der}}(K_0)$ respectively.

## 5. Shimura varieties of Hodge type

### 5.1. **Abelian schemes and related structures on the Shimura variety.**

**5.1.1.** Throughout this section we keep the following setting. Let $(G, X, p, \mathcal{G})$ be an unramified Shimura datum as in §2.4.1, and assume that $(G, X)$ is of Hodge type. Let $E \subset \mathbb{C}$ be the reflex field of $(G, X)$, viewed as a subfield of $\overline{\mathbb{Q}}$ via our fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let $\mathfrak{p}$ be the prime of $E$ determined by the embedding $E \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. We write $K_p$ for $\mathcal{G}(\mathbb{Z}_p)$.

Since $\mathcal{G}$ is smooth over $\mathbb{Z}_p$, its Hopf algebra $\mathcal{O}_{\mathcal{G}}(\mathcal{G})$ injects into $\mathcal{O}_{G_{\mathbb{Q}_p}}(G_{\mathbb{Q}_p})$. The intersection $\mathcal{O}_{\mathcal{G}}(\mathcal{G}) \cap \mathcal{O}_G(G)$ inside $\mathcal{O}_{G_{\mathbb{Q}_p}}(G_{\mathbb{Q}_p})$ has the natural structure of a Hopf algebra over $\mathbb{Z}_{(p)}$, and defines a $\mathbb{Z}_{(p)}$-group scheme $G_{(p)}$. Thus $G_{(p)}$ is the unique (up to unique isomorphism) reductive group scheme over $\mathbb{Z}_{(p)}$ whose generic fiber is identified with $G$ and whose base change to $\mathbb{Z}_p$ is identified with $\mathcal{G}$.

We fix an embedding of Shimura data $(G, X) \hookrightarrow (\mathrm{GSp}(V_{\mathbb{Q}}), S^{\pm})$, where $V_{\mathbb{Q}}$ is a symplectic space over $\mathbb{Q}$, and $(\mathrm{GSp}(V_{\mathbb{Q}}), S^{\pm})$ is the corresponding Siegel Shimura datum. As in [Kis17, §1.3.3], we may choose the symplectic space $V_{\mathbb{Q}}$ and the embedding $G \hookrightarrow \mathrm{GSp}(V_{\mathbb{Q}})$ suitably such that the latter is induced by a closed embedding of $\mathbb{Z}_{(p)}$-group schemes $G_{(p)} \hookrightarrow \mathrm{GL}(V_{\mathbb{Z}_{(p)}})$, for some self-dual $\mathbb{Z}_{(p)}$-lattice

$V_{\mathbb{Z}_{(p)}}$ in $V_{\mathbb{Q}}$.[27] We fix such choices of $V_{\mathbb{Q}}$, $G \hookrightarrow \mathrm{GSp}(V_{\mathbb{Q}})$, and $V_{\mathbb{Z}_{(p)}}$. For any $\mathbb{Z}_{(p)}$-algebra $R$ we write $V_R$ for $V_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} R$. We write GSp for the $\mathbb{Q}$-algebraic group $\mathrm{GSp}(V_{\mathbb{Q}})$, and write $\mathbb{K}_p$ for the compact open subgroup $\mathrm{GSp}(V_{\mathbb{Z}_{(p)}})(\mathbb{Z}_p)$ of $\mathrm{GSp}(\mathbb{Q}_p)$. Thus the embedding $G(\mathbb{Q}_p) \hookrightarrow \mathrm{GSp}(\mathbb{Q}_p)$ maps $K_p$ into $\mathbb{K}_p$.

As in [Kis17, §1.3.6], we fix once and for all a collection of tensors $(s_\alpha)_{\alpha\in\boldsymbol{\alpha}} \subset V_{\mathbb{Z}_{(p)}}^{\otimes}$ such that the image of $G_{(p)} \hookrightarrow \mathrm{GL}(V_{\mathbb{Z}_{(p)}})$ is the scheme-theoretic stabilizer of these tensors. In the sequel, we shall assume that the collection $(s_\alpha)_{\alpha\in\boldsymbol{\alpha}}$ is maximal, i.e., it consists of all elements of $V_{\mathbb{Z}_{(p)}}^{\otimes}$ that are stabilized by $G_{(p)}$.

Let $V_{\mathbb{Z}_p}^*$ be the $\mathbb{Z}_p$-linear dual of $V_{\mathbb{Z}_p}$. We shall view it as a representation of $\mathcal{G}$ over $\mathbb{Z}_p$, i.e., the contragredient of $V_{\mathbb{Z}_p}$. Similarly, we view the $\mathbb{Q}$-linear dual $V_{\mathbb{Q}}^*$ of $V_{\mathbb{Q}}$ as a representation of $G$ over $\mathbb{Q}$. We view $(s_\alpha)_{\alpha\in\boldsymbol{\alpha}}$ also as tensors over $V_{\mathbb{Z}_p}^*$ or $V_{\mathbb{Q}}^*$.

**Lemma 5.1.2.** *For any Shimura datum $(G, X)$ of Hodge type, the following statements hold.*

*(i) The $\mathbb{Q}$-tori $Z_G^0$ and $G^{\mathrm{ab}}$ are cuspidal (see Definition 1.5.4).*
*(ii) The anti-cuspidal part $(Z_G^0)_{ac}$ of $Z_G^0$ (see Definition 1.5.4) is trivial.*
*(iii) Let $(T, i, h)$ be a special point datum for $(G, X)$ (see Definition 3.3.1). Then $T$ is cuspidal, and $i(T_{\mathbb{R}})$ is an elliptic maximal torus in $G_{\mathbb{R}}$.*

*Proof.* Let $w$ be the weight cocharacter of $(G, X)$. Since $(G, X)$ is of Hodge type, we know that $w$ is defined over $\mathbb{Q}$, and that $\mathrm{Int}(h(\sqrt{-1}))$ induces a Cartan involution on $G_{\mathbb{R}}/w(\mathbb{G}_{m,\mathbb{R}})$ for each $h \in X$ (by directly checking the similar properties for the Siegel Shimura datum). Recall from [Del79, §2.1.1] that $w$ is central. Thus $(Z_G^0)_{\mathbb{R}}/w(\mathbb{G}_{m,\mathbb{R}})$ is anisotropic. Since $w(\mathbb{G}_{m,\mathbb{R}})$ is defined and split over $\mathbb{Q}$, we see that $Z_G^0$ is cuspidal. Since $G^{\mathrm{ab}}$ is isogenous to $Z_G^0$ over $\mathbb{Q}$, it is also cuspidal. This proves (i). Statement (ii) follows from (i) in view of Lemma 1.5.5. For (iii), we have $i \circ h \in X$, and the Cartan involution $\mathrm{Int}((i \circ h)(\sqrt{-1}))$ on $G_{\mathbb{R}}/w(\mathbb{G}_{m,\mathbb{R}})$ restricts to the identity on $i(T_{\mathbb{R}})/w(\mathbb{G}_{m,\mathbb{R}})$. Hence $i(T_{\mathbb{R}})/w(\mathbb{G}_{m,\mathbb{R}})$ is anisotropic, and the desired statements follow. $\square$

**5.1.3.** For each compact open subgroup $K \subset G(\mathbb{A}_f)$ (resp. $\mathbb{K} \subset \mathrm{GSp}(\mathbb{A}_f)$), we write $\mathrm{Sh}_K$ (resp. $\mathrm{Sh}_{\mathbb{K}}$) for the Shimura variety $\mathrm{Sh}_K(G, X)$ (resp. $\mathrm{Sh}_{\mathbb{K}}(\mathrm{GSp}, S^{\pm})$). Below we recall the construction of the canonical smooth integral model $\mathscr{S}_{K_p}$ of $\mathrm{Sh}_{K_p} = \varprojlim_{K^p} \mathrm{Sh}_{K_pK^p}$ in Theorem 2.5.3.

Fix once and for all a neat compact open subgroup $K_1^p \subset G(\mathbb{A}_f^p)$ whose image in $\mathrm{GSp}(\mathbb{A}_f^p)$ is a neat compact open subgroup $\mathbb{K}_1^p$. We write $\mathscr{K}^p$ for the set of open subgroups of $K_1^p$. Clearly all members of $\mathscr{K}^p$ are neat. Let $K^p \in \mathscr{K}^p$. By [Kis10, Lem. 2.1.2], there exists an open subgroup $\mathbb{K}^p \subset \mathbb{K}_1^p$ such that the image of $K^p$ in $\mathrm{GSp}(\mathbb{A}_f^p)$ is contained in $\mathbb{K}^p$ and such that the natural map

$$\mathrm{Sh}_{K_pK^p} \longrightarrow \mathrm{Sh}_{\mathbb{K}_p\mathbb{K}^p} \times_{\mathbb{Q}} E$$

is a closed embedding of $E$-schemes.

Now $\mathrm{Sh}_{\mathbb{K}_p\mathbb{K}^p}$ has a canonical model $\mathscr{S}_{\mathbb{K}_p\mathbb{K}^p}$ over $\mathbb{Z}_{(p)}$, which represents the usual Siegel moduli problem. We define $\mathscr{S}_{K_pK^p}$ to be the normalization of the closure

[27] This uses [Kis10, Lem. 2.3.1] and Zarhin's trick. In the former result, there is an extra assumption on $G$ when $p = 2$. However this extra assumption can be removed, as explained in the proof of [KMP16, Lem. 4.7].

of $\mathrm{Sh}_{K_pK^p}$ inside $\mathscr{S}_{\mathbb{K}_p\mathbb{K}^p} \times_{\mathbb{Z}_{(p)}} \mathcal{O}_{E,(\mathfrak{p})}$. (It is shown in [Xu20] that taking the normalization is redundant.) By the main results of [Kis10] and [KMP16], $\mathscr{S}_{K_pK^p}$ is smooth over $\mathcal{O}_{E,(\mathfrak{p})}$, and the inverse limit $\mathscr{S}_{K_p} = \varprojlim_{K^p \in \mathscr{K}^p} \mathscr{S}_{K_pK^p}$ is the canonical smooth integral model of $\mathrm{Sh}_{K_p}$. For each $K^p \in \mathscr{K}^p$, we have $\mathscr{S}_{K_pK^p} = \mathscr{S}_{K_p}/K^p$, so the notation here is consistent with that in Definition 2.5.2.

In the sequel, we write

$$K_1 := K_pK_1^p.$$

Over $\mathscr{S}_{K_1}$ we have an abelian scheme up to prime-to-$p$ isogeny $\mathcal{A}$ that is the pull-back of the universal abelian scheme up to prime-to-$p$ isogeny over $\mathscr{S}_{\mathbb{K}_p\mathbb{K}_1^p}$. For any $\mathcal{O}_{E,(\mathfrak{p})}$-scheme $Y$ and any $\mathcal{O}_{E,(\mathfrak{p})}$-morphism $x : Y \to \mathscr{S}_{K_1}$, we denote by $\mathcal{A}_x$ the pull-back of $\mathcal{A}$ along $x$. For any $Y$ as above and any $\mathcal{O}_{E,(\mathfrak{p})}$-morphism $x : Y \to \mathscr{S}_{K_p}$, we again write $\mathcal{A}_x$ for the pull-back of $\mathcal{A}$ along the composite map $Y \xrightarrow{x} \mathscr{S}_{K_p} \to \mathscr{S}_{K_1}$.

**5.1.4.** For a scheme $Y$ and a prime number $l$, we write $\mathrm{Lisse}_{\mathbb{Z}_l}(Y)$ (resp. $\mathrm{Lisse}_{\mathbb{Q}_l}(Y)$) for the $\otimes$-category of lisse $\mathbb{Z}_l$-sheaves (resp. lisse $\mathbb{Q}_l$-sheaves) on $Y$. We define the category of lisse $\mathbb{A}_f^p$-sheaves $\mathrm{Lisse}_{\mathbb{A}_f^p}(Y)$ to be the $\mathbb{Q}$-isogeny category associated with the product category of $\mathrm{Lisse}_{\mathbb{Z}_l}(Y)$ for all primes $l \neq p$. Thus $\mathrm{Lisse}_{\mathbb{A}_f^p}(Y)$ is an $\mathbb{A}_f^p$-linear $\otimes$-category, with unit object given by the product of the unit objects in $\mathrm{Lisse}_{\mathbb{Z}_l}(Y)$. For $R \in \{\mathbb{Z}_l, \mathbb{Q}_l, \mathbb{A}_f^p\}$, we denote the unit objects in $\mathrm{Lisse}_R(Y)$ by $R$. More generally, given any finite free $R$-module $W$ we still write $W$ for the "constant sheaf" in $\mathrm{Lisse}_R(Y)$ represented by $W$.

By Lemma 1.5.7, (1.5.8.2), and Lemma 5.1.2 (ii), we have

$$\mathrm{Gal}(\mathrm{Sh} / \mathrm{Sh}_{K_1}) \cong K_1.$$

For each $W \in \mathrm{Rep}_{\mathbb{Z}_p}\mathcal{G}$, we can view $W$ as a continuous $\mathbb{Z}_p$-representation of $\mathrm{Gal}(\mathrm{Sh} / \mathrm{Sh}_{K_1})$ via $\mathrm{Gal}(\mathrm{Sh} / \mathrm{Sh}_{K_1}) = K_1 \xrightarrow{\mathrm{proj}} K_p = \mathcal{G}(\mathbb{Z}_p)$. By the $\mathbb{Z}_p$-linear variant of the construction in [HT01, §III.3] (the $\overline{\mathbb{Q}}_p$-linear version was used in §1.5.8), we can attach to the $\mathrm{Gal}(\mathrm{Sh} / \mathrm{Sh}_{K_1})$-representation $W$ a lisse $\mathbb{Z}_p$-sheaf $\mathbb{L}_p(W)$ on $\mathrm{Sh}_{K_1}$. This construction defines a faithful exact $\otimes$-functor

$$\mathbb{L}_p : \mathrm{Rep}_{\mathbb{Z}_p}\mathcal{G} \longrightarrow \mathrm{Lisse}_{\mathbb{Z}_p}(\mathrm{Sh}_{K_1}). \tag{5.1.4.1}$$

Similarly, by Lemma 1.5.7, §2.5.5, and Lemma 5.1.2 (ii), we have

$$\mathrm{Gal}(\mathscr{S}_{K_p}/\mathscr{S}_{K_1}) \cong K_1^p,$$

and we obtain a faithful exact $\otimes$-functor

$$\mathbb{L}^p : \mathrm{Rep}_{\mathbb{Q}}G \longrightarrow \mathrm{Lisse}_{\mathbb{A}_f^p}(\mathscr{S}_{K_1}) \tag{5.1.4.2}$$

by viewing $W \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ as a continuous $\mathbb{A}_f^p$-representation of $\mathrm{Gal}(\mathscr{S}_{K_p}/\mathscr{S}_{K_1})$ for each $W \in \mathrm{Rep}_{\mathbb{Q}}G$.

The functors $\mathbb{L}_p$ and $\mathbb{L}^p$ have complex analytic analogues defined using the complex uniformization. For each $W \in \mathrm{Rep}_{\mathbb{Z}_p}\mathcal{G}$, define a subspace $\mathcal{E}(W) \subset W_{\mathbb{Q}_p} \times X \times G(\mathbb{A}_f)$ by

$$\mathcal{E}(W) = \left\{ (w, h, (g_v)_v) \mid w \in g_pW \subset W_{\mathbb{Q}_p} \right\}.$$

We let $G(\mathbb{Q})$ act on $\mathcal{E}(W)$ on the left by

$$g \cdot (w, h, (g_v)_v) = (gw, gh, (gg_v)_v),$$

and let $K$ act on $\mathcal{E}(W)$ on the right by

$$(w, h, (g_v)_v) \cdot k = (w, h, (g_v)_v \cdot k).$$

Let $\mathcal{L}_p(W)$ be the sheaf on the complex manifold $\mathrm{Sh}_{K_1}(\mathbb{C})$ consisting of local sections of

$$G(\mathbb{Q})\backslash\mathcal{E}(W)/K_1 \to \mathrm{Sh}_{K_1}(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_1, \quad [(w,h,(g_v)_v)] \mapsto [h,(g_v)_v].$$

Similarly, for each $W \in \mathrm{Rep}_{\mathbb{Q}} G$, let $\mathcal{L}_{\mathbb{Q}}(W)$ be the sheaf on $\mathrm{Sh}_{K_1}(\mathbb{C})$ consisting of local sections of

$$G(\mathbb{Q})\backslash W \times X \times G(\mathbb{A}_f)/K_1 \to \mathrm{Sh}_{K_1}(\mathbb{C}), \quad [(w,h,(g_v)_v)] \mapsto [h,(g_v)_v],$$

where on the left hand side $G(\mathbb{Q})$ acts diagonally on the three factors and $K_1$ acts by right multiplication on $G(\mathbb{A}_f)$. We obtain faithful exact $\otimes$-functors

$$\mathcal{L}_p : \mathrm{Rep}_{\mathbb{Z}_p}\mathcal{G} \longrightarrow \{\mathbb{Z}_p\text{-local systems on } \mathrm{Sh}_{K_1}(\mathbb{C})\},$$
$$\mathcal{L}_{\mathbb{Q}} : \mathrm{Rep}_{\mathbb{Q}} G \longrightarrow \{\mathbb{Q}\text{-local systems on } \mathrm{Sh}_{K_1}(\mathbb{C})\}.$$

For each $W \in \mathrm{Rep}_{\mathbb{Z}_p}\mathcal{G}$, there is a natural isomorphism between $\mathcal{L}_p(W)$ and the analytification $\mathbb{L}_p(W)^{\mathrm{an}}$ of $\mathbb{L}_p(W)$ which is compatible with tensor products. Similarly, for each $W \in \mathrm{Rep}_{\mathbb{Q}} G$, there is a natural isomorphism between $\mathcal{L}_{\mathbb{Q}}(W) \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ and the analytification $\mathbb{L}^p(W)^{\mathrm{an}}$ of $\mathbb{L}^p(W)$ which is compatible with tensor products. These observations go back to Langlands [Lan73, §3], cf. for instance [Mor05, §2.1.4].

**5.1.5.** Denote the structure morphism $\mathcal{A} \to \mathscr{S}_{K_1}$ by $h$, and denote the structure morphism $\mathcal{A}|_{\mathrm{Sh}_{K_1}} \to \mathrm{Sh}_{K_1}$ by $h_\eta$.

Over the complex manifold $\mathrm{Sh}_{K_1}(\mathbb{C})$, we have a $\mathbb{Z}_p$-local system $\mathcal{V}_{B,p}$ (resp. a $\mathbb{Q}$-local system $\mathcal{V}_{B,\mathbb{Q}}$) given by the first relative Betti cohomology of the analytification of $h_\eta$ with coefficients in $\mathbb{Z}_p$ (resp. $\mathbb{Q}$). By the moduli interpretation of the complex uniformization of the Siegel Shimura variety, we have canonical identifications

$$\mathcal{V}_{B,p} \cong \mathcal{L}_p(V_{\mathbb{Z}_p}^*), \quad \mathcal{V}_{B,\mathbb{Q}} \cong \mathcal{L}_{\mathbb{Q}}(V_{\mathbb{Q}}^*), \tag{5.1.5.1}$$

cf. [Kis17, §1.4.11].[28] For each $\alpha \in \boldsymbol{\alpha}$, we can view $s_\alpha$ as a morphism $\mathbb{Z}_p \to (V_{\mathbb{Z}_p}^*)^{\otimes}$ between $\mathcal{G}$-representations. By the identifications in (5.1.5.1) we obtain a tensor $s_{\alpha,B,p} := \mathcal{L}_p(s_\alpha)$ over $\mathcal{V}_{B,p}$ and a tensor $s_{\alpha,B,\mathbb{Q}} := \mathcal{L}_{\mathbb{Q}}(s_\alpha)$ over $\mathcal{V}_{B,\mathbb{Q}}$.

Let

$$\mathcal{V}_p := R^1 h_{\eta,\text{ét},*}\mathbb{Z}_p \in \mathrm{Lisse}_{\mathbb{Z}_p}(\mathrm{Sh}_{K_1}),$$

and

$$\mathcal{V}^p := R^1 h_{\text{ét},*}\mathbb{A}_f^p \in \mathrm{Lisse}_{\mathbb{A}_f^p}(\mathscr{S}_{K_1}).$$

Analogous to (5.1.5.1), we have canonical identifications

$$\mathcal{V}_p \cong \mathbb{L}_p(V_{\mathbb{Z}_p}^*), \quad \mathcal{V}^p \cong \mathbb{L}^p(V_{\mathbb{Q}}^*), \tag{5.1.5.2}$$

arising from the fact that the tower of Siegel Shimura varieties relatively represents the moduli of level structures.

By (5.1.5.2), for each $\alpha \in \boldsymbol{\alpha}$, we obtain a tensor $s_{\alpha,p} := \mathbb{L}_p(s_\alpha)$ over $\mathcal{V}_p$, and a tensor $s_{\alpha,\mathbb{A}_f^p} := \mathbb{L}^p(s_\alpha)$ over $\mathcal{V}^p$.

[28] In [Kis17, §1.4.11], the line "$H_1(\mathcal{A}(\mathbb{C}),\mathbb{Z}_{(p)}) \xrightarrow{\sim} g_p \cdot V_{\mathbb{Z}_{(p)}} \subset V_{\mathbb{Q}}$" should be corrected to "$H_1(\mathcal{A}(\mathbb{C}),\mathbb{Z}_p) \xrightarrow{\sim} g_p \cdot V_{\mathbb{Z}_p} \subset V_{\mathbb{Q}_p}$".

*Remark* 5.1.6. The natural isomorphism $\mathbb{L}_p(V^*_{\mathbb{Z}_p})^{\mathrm{an}} \cong \mathcal{L}_p(V^*_{\mathbb{Z}_p})$ at the end of §5.1.4 coincides with the comparison isomorphism $\mathcal{V}_p^{\mathrm{an}} \cong \mathcal{V}_{B,p}$, if we identify the two sides with $\mathcal{V}_p^{\mathrm{an}}$ and $\mathcal{V}_{B,p}$ respectively. Since the analytification functor is faithful, we see that $s_{\alpha,p}$ is uniquely characterized by the fact that under the comparison isomorphism $\mathcal{V}_p^{\mathrm{an}} \cong \mathcal{V}_{B,p}$ it corresponds to $s_{\alpha,B,p}$. In a similar way, $s_{\alpha,\mathbb{A}_f^p}$ is characterized by $s_{\alpha,B,\mathbb{Q}}$. This shows that the current definitions of $s_{\alpha,p}$ and $s_{\alpha,\mathbb{A}_f^p}$ agree with those in [Kis17, §1.3.6]. [29]

**5.1.7.** Let $\kappa$ be a subfield of $\mathbb{C}$ containing $E$, and let $\overline{\kappa}$ be the algebraic closure of $\kappa$ in $\mathbb{C}$. For $z \in \mathrm{Sh}_{K_1}(\kappa)$, we write $\mathcal{V}_p(z)$ for the stalk of $\mathcal{V}_p$ at $z$ viewed as a $\bar{\kappa}$-point. Thus $\mathcal{V}_p(z)$ is a finite free $\mathbb{Z}_p$-module equipped with a continuous $\mathrm{Gal}(\bar{\kappa}/\kappa)$-action, and it is identified with $\mathbf{H}^1_{\text{ét}}(\mathcal{A}_{z,\bar{\kappa}}, \mathbb{Z}_p)$. For each $\alpha \in \boldsymbol{\alpha}$, write $s_{\alpha,p,z}$ for the tensor over $\mathcal{V}_p(z)$ induced by $s_{\alpha,p}$. Then $s_{\alpha,p,z}$ is invariant under $\mathrm{Gal}(\bar{\kappa}/\kappa)$. We write $\mathcal{G}_z$ for the closed subgroup scheme of the $\mathbb{Z}_p$-group scheme $\mathrm{GL}(\mathcal{V}_p(z))$ fixing $s_{\alpha,p,z}$ for all $\alpha \in \boldsymbol{\alpha}$. Thus we have a natural continuous homomorphism $\rho(z) : \mathrm{Gal}(\bar{\kappa}/\kappa) \to \mathcal{G}_z(\mathbb{Z}_p)$.

**Lemma 5.1.8.** *In the setting of §5.1.7, there exists a (non-canonical) $\mathbb{Z}_p$-module isomorphism $V^*_{\mathbb{Z}_p} \xrightarrow{\sim} \mathcal{V}_p(z)$ taking each $s_\alpha$ to $s_{\alpha,p,z}$. In particular, there is an isomorphism of $\mathbb{Z}_p$-group schemes $\mathcal{G} \xrightarrow{\sim} \mathcal{G}_z$, canonical up to conjugation by $\mathcal{G}(\mathbb{Z}_p)$.*

*Proof.* Let $z_{\mathbb{C}} \in \mathrm{Sh}_{K_1}(\mathbb{C})$ be the point induced by $z$. By Remark 5.1.6, it suffices to show the existence of a $\mathbb{Z}_p$-module isomorphism $f$ from $V^*_{\mathbb{Z}_p}$ to the stalk $\mathcal{V}_{B,p}(z_{\mathbb{C}})$ of $\mathcal{V}_{B,p}$ at $z_{\mathbb{C}}$ such that $f$ takes each $s_\alpha$ to the tensor on $\mathcal{V}_{B,p}(z_{\mathbb{C}})$ induced by $s_{\alpha,B,p}$. This follows from [Kis17, §1.4.11]. ☐

**Lemma 5.1.9.** *On $\mathscr{S}_{K_p}$, there is a canonical isomorphism between the pull-back of $\mathcal{V}^p$ and the constant lisse $\mathbb{A}_f^p$-sheaf $V^*_{\mathbb{A}_f^p}$. This isomorphism takes $s_{\alpha,\mathbb{A}_f^p}$ to $s_\alpha$ (viewed as a tensor on the constant sheaf $V^*_{\mathbb{A}_f^p}$) for each $\alpha \in \boldsymbol{\alpha}$.*

*Proof.* The composition of $\mathbb{L}^p : \mathrm{Rep}_{\mathbb{Q}} G \to \mathrm{Lisse}_{\mathbb{A}_f^p}(\mathscr{S}_{K_1})$ with the pull-back functor $\mathrm{Lisse}_{\mathbb{A}_f^p}(\mathscr{S}_{K_1}) \to \mathrm{Lisse}_{\mathbb{A}_f^p}(\mathscr{S}_{K_p})$ is canonically identified with the functor sending each $W \in \mathrm{Rep}_{\mathbb{Q}} G$ to the constant sheaf $W \otimes_{\mathbb{Q}} \mathbb{A}_f^p$. The lemma follows from this fact and the second identification in (5.1.5.2). ☐

## 5.2. **Crystalline tensors.**

**5.2.1.** Let $A$ be an abelian variety up to prime-to-$p$ isogeny over $\overline{\mathbb{F}}_p$. We define

$$\mathcal{V}_0(A) := \mathbf{H}^1_{\mathrm{cris}}(A_0/W(k)) \otimes_{W(k)} \mathbb{Z}_p^{\mathrm{ur}},$$

where $A_0$ is a model of $A$ over some finite field $k \subset \overline{\mathbb{F}}_p$. Then $\mathcal{V}_0(A)$ has the natural structure of an integral $F$-isocrystal over $\mathbb{Q}_p^{\mathrm{ur}}$ (see §4.2.1), and it is independent of the choices of $k$ and $A_0$ up to canonical isomorphism. We denote the Frobenius on $\mathcal{V}_0(A)[1/p]$ simply by $\varphi$.

For $x \in \mathscr{S}_{K_1}(\overline{\mathbb{F}}_p)$, we write $\mathcal{V}_0(x)$ for $\mathcal{V}_0(\mathcal{A}_x)$. For $x \in \mathscr{S}_{K_pK^p}(\overline{\mathbb{F}}_p)$ with $K^p \in \mathscr{K}^p$ or $x \in \mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$, we define $\mathcal{V}_0(x)$ to be $\mathcal{V}_0(y)$, where $y \in \mathscr{S}_{K_1}(\overline{\mathbb{F}}_p)$ is the image of $x$.

[29] In [Kis17, §1.3.6], only the $l$-adic components of $\mathcal{V}^p$ and $s_{\alpha,\mathbb{A}_f^p}$ are considered.

**5.2.2.** Let $x$ be a closed point of the special fiber of $\mathscr{S}_{K_1}$, with residue field $k(x)$. We then obtain the abelian variety $\mathcal{A}_x$ up to prime-to-$p$ isogeny over $k(x)$. As a key construction in [Kis10] and [Kis17], for each $\alpha \in \boldsymbol{\alpha}$ we have a tensor $s_{\alpha,0,x}$ over the $W(k(x))$-module $\mathbf{H}^1_{\mathrm{cris}}(\mathcal{A}_x/W(k(x)))$, which is furthermore $\varphi$-invariant (after inverting $p$). Below we recall the construction of $s_{\alpha,0,x}$ given in [Kis17, §1.3], and explain why the assumption $p > 2$ in *loc. cit.* can be removed.

Let $F$ be a finite extension of $E_{\mathfrak{p}}$ in $\overline{\mathbb{Q}}_p$ whose residue field $k$ contains $k(x)$. Let $\widetilde{x} \in \mathrm{Sh}_{K_1}(F)$ be a point that extends (necessarily uniquely) to a point in $\mathscr{S}_{K_1}(\mathcal{O}_F)$ which specializes to $x$. (Clearly such a pair $(F, \widetilde{x})$ exists for any prescribed $x$ and finite extension $k/k(x)$; one can moreover take $F$ to be $W(k(x))[1/p]$.) Since $\mathcal{A}_{x,k} := \mathcal{A}_x \times_{k(x)} k$ and $\mathcal{A}_{\widetilde{x}}$ are the special and generic fibers of an abelian scheme over $\mathcal{O}_F$, we know that the $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$-representation $\mathcal{V}_p(\widetilde{x})[1/p] \cong \mathbf{H}^1_{\text{ét}}(\mathcal{A}_{\widetilde{x},\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ is crystalline. Moreover, by the *integral comparison isomorphism* we have a canonical isomorphism of integral $F$-isocrystals

$$M_{\mathrm{cris}}(\mathcal{V}_p(\widetilde{x})) = M_{\mathrm{cris}}(\mathbf{H}^1_{\text{ét}}(\mathcal{A}_{\widetilde{x},\overline{\mathbb{Q}}_p}, \mathbb{Z}_p)) \xrightarrow{\sim} \mathbf{H}^1_{\mathrm{cris}}(\mathcal{A}_{x,k}/W(k)), \tag{5.2.2.1}$$

which refines the crystalline comparison isomorphism

$$D_{\mathrm{cris}}\big(\mathbf{H}^1_{\text{ét}}(\mathcal{A}_{\widetilde{x},\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)\big) \xrightarrow{\sim} \mathbf{H}^1_{\mathrm{cris}}(\mathcal{A}_{x,k}/W(k))[1/p].$$

Here the functor $M_{\mathrm{cris}}$ is defined in §4.4.1, and recall that $M_{\mathrm{cris}}(L)$ is a $W(k)$-lattice in the $W(k)[1/p]$-vector space $D_{\mathrm{cris}}(L \otimes \mathbb{Q}_p)$, for any $L \in \mathrm{Rep}^{\mathrm{cris},\circ}_{\Gamma_K}$. This integral comparison isomorphism is proved in [Kis10, Thm. 1.4.2] (cf. [Kis17, Thm. 1.1.6] for a correction in the normalization) for $p > 2$, and proved in [Kim12, Prop. 4.2] for $p = 2$. An independent proof valid for all $p$ is given by Lau [Lau14, Lau19].[30]

Now since $s_{\alpha,p,\widetilde{x}}$ is $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$-invariant, we have the $\varphi_{M_{\mathrm{cris}}(\mathcal{V}_p(\widetilde{x}))}$-invariant tensor $M_{\mathrm{cris}}(s_{\alpha,p,\widetilde{x}})$ over $M_{\mathrm{cris}}(\mathcal{V}_p(\widetilde{x}))$ by the functoriality of $M_{\mathrm{cris}}$. Under (5.2.2.1), $M_{\mathrm{cris}}(s_{\alpha,p,\widetilde{x}})$ corresponds to a $\varphi$-invariant tensor $s_{\alpha,0,\widetilde{x}}$ over $\mathbf{H}^1_{\mathrm{cris}}(\mathcal{A}_{x,k}/W(k))$. Note that $\mathbf{H}^1_{\mathrm{cris}}(\mathcal{A}_{x,k}/W(k))$ is canonically identified with

$$\mathbf{H}^1_{\mathrm{cris}}(\mathcal{A}_x/W(k(x))) \otimes_{W(k(x))} W(k).$$

It is shown in the proof of [Kis10, Prop. 2.3.5] (cf. [Kis17, Prop. 1.3.9]) that $s_{\alpha,0,\widetilde{x}}$ in fact comes from a tensor $s_{\alpha,0,x}$ on $\mathbf{H}^1_{\mathrm{cris}}(\mathcal{A}_x/W(k(x)))$ that depends only on $x$ and not on the choices of $F$ and $\widetilde{x}$.

For any $x_1 \in \mathscr{S}_{K_1}(\overline{\mathbb{F}}_p)$ we have a canonical identification (see §5.2.1)

$$\mathcal{V}_0(x_1) \cong \mathbf{H}^1_{\mathrm{cris}}(\mathcal{A}_x/W(k(x))) \otimes_{W(k(x))} \otimes \mathbb{Z}_p^{\mathrm{ur}},$$

where $x$ is the closed point of $\mathscr{S}_{K_1}$ given by the image of $x_1$. The tensor $s_{\alpha,0,x}$ on $\mathbf{H}^1_{\mathrm{cris}}(\mathcal{A}_x/W(k(x)))$ thus induces a tensor $s_{\alpha,0,x_1}$ on $\mathcal{V}_0(x_1)$, for each $\alpha \in \boldsymbol{\alpha}$. For any $y \in \mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$ mapping to $x_1 \in \mathscr{S}_{K_1}(\overline{\mathbb{F}}_p)$, we have $\mathcal{V}_0(y) = \mathcal{V}_0(x_1)$ by definition. In this case we also write $s_{\alpha,0,y}$ for the tensor $s_{\alpha,0,x_1}$ on $\mathcal{V}_0(y)$.

**Lemma 5.2.3.** *For each $x_1 \in \mathscr{S}_{K_1}(\overline{\mathbb{F}}_p)$, there exists a $\mathbb{Z}_p^{\mathrm{ur}}$-module isomorphism $V^*_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{ur}} \xrightarrow{\sim} \mathcal{V}_0(x_1)$ which takes $s_\alpha$ to $s_{\alpha,0,x_1}$ for each $\alpha \in \boldsymbol{\alpha}$.*

[30] In all these references, the integral comparison isomorphism is proved more generally for $p$-divisible groups over $\mathcal{O}_F$. See the proof of [KMP16, Thm. 2.12] for a historical account of the different proofs. The integral comparison is now known for arbitrary proper smooth formal schemes (under a certain torsion-free assumption) by the work of Bhatt–Morrow–Scholze [BMS18, Thm. 14.6.3 (iii)].

*Proof.* This is proved in [Kis10, Cor. 1.3.4]. Below we recast the proof using the formalism in §4.4. Let $x$ be the closed point of $\mathscr{S}_{K_1}$ induced by $x_1$, and let $\widetilde{x} \in \mathrm{Sh}_{K_1}(F)$ be a lift of $x$ to some finite extension $F/\mathbb{Q}_p$ as in §5.2.2. Fix an isomorphism $V^*_{\mathbb{Z}_p} \xrightarrow{\sim} \mathcal{V}_p(\widetilde{x})$ as in Lemma 5.1.8, and use this isomorphism to view $V^*_{\mathbb{Q}_p}$ as a $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$-representation. This Galois representation is crystalline, and $V^*_{\mathbb{Z}_p}$ is a Galois-stable lattice. Moreover, each tensor $s_\alpha$ on $V^*_{\mathbb{Z}_p}$ is $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$-invariant, since each $s_{\alpha,p,\widetilde{x}}$ is invariant under $\rho(\widetilde{x})$ (see §5.1.7). The action map $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{GL}(V^*_{\mathbb{Z}_p})(\mathbb{Z}_p)$ thus factors through a $\mathcal{G}(\mathbb{Z}_p)$-valued crystalline representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathcal{G}(\mathbb{Z}_p)$. Let $k$ be the residue field of $F$. By the construction of $s_{\alpha,0,x}$ recalled in §5.2.2 and by Lemma 5.1.8, we only need to find a $\mathbb{Z}_p^{\mathrm{ur}}$-module isomorphism

$$M_{\mathrm{cris}}(V^*_{\mathbb{Z}_p}) \otimes_{W(k)} \mathbb{Z}_p^{\mathrm{ur}} \xrightarrow{\sim} V^*_{\mathbb{Z}_p^{\mathrm{ur}}}$$

which takes each $M_{\mathrm{cris}}(s_\alpha)$ to $s_\alpha$. For this, it suffices to find a $W(k)$-module isomorphism

$$\omega_{\rho,0}(V^*_{\mathbb{Z}_p}) \xrightarrow{\sim} V^*_{W(k)}$$

which takes each $\omega_{\rho,0}(s_\alpha)$ to $s_\alpha$. (See §4.4.2 for $\omega_{\rho,0}$.) For this it suffices to know that $\omega_{\rho,0}$ is $\otimes$-isomorphic with $\mathbb{1}_{W(k)}$. This is indeed the case by the fact that $\mathcal{G}$ satisfies KL (§4.4.4) and by Lemma 4.4.5. □

**Lemma 5.2.4.** *For each $x \in \mathscr{S}_{K_1}(\overline{\mathbb{F}}_p)$, there is an integral $F$-isocrystal with $\mathcal{G}$-structure*

$$\Upsilon_x : \mathrm{Rep}_{\mathbb{Z}_p}\mathcal{G} \longrightarrow \mathrm{Isoc}^{\circ}_{\mathbb{Q}_p^{\mathrm{ur}}},$$

*together with an isomorphism $\iota_x : \Upsilon_x(V^*_{\mathbb{Z}_p}) \xrightarrow{\sim} \mathcal{V}_0(x)$ in $\mathrm{Isoc}^{\circ}_{\mathbb{Q}_p^{\mathrm{ur}}}$ which takes $\Upsilon_x(s_\alpha)$ to $s_{\alpha,0,x}$ for each $\alpha \in \boldsymbol{\alpha}$. Moreover, the pair $(\Upsilon_x, \iota_x)$ is unique up to unique isomorphism, in the same sense as in Lemma 4.2.3.*

*Proof.* By Lemma 5.2.3, the object $(D, \varphi_D) := \mathcal{V}_0(x)$ in $\mathrm{Isoc}^{\circ}_{\mathbb{Q}_p^{\mathrm{ur}}}$, together with the tensors $s_{\alpha,0,x}$ on it, satisfies the hypotheses in §4.2.2 with respect to the defining datum $(V^*_{\mathbb{Z}_p}, (s_\alpha)_{\alpha\in\boldsymbol{\alpha}})$ for $\mathcal{G}$. The lemma then follows from Lemma 4.2.3. □

**Definition 5.2.5.** For each $x \in \mathscr{S}_{K_1}(\overline{\mathbb{F}}_p)$, we fix the choice of a pair $(\Upsilon_x, \iota_x)$ as in Lemma 5.2.4 once and for all. If $y \in \mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$ maps to $x$, we also write $(\Upsilon_y, \iota_y)$ for $(\Upsilon_x, \iota_x)$.

5.3. **Kottwitz triples.**

**Definition 5.3.1.** Let $n \in \mathbb{Z}_{\geq 1}$. Define $\mathfrak{T}_n^{\mathrm{str}}$ to be the set of triples $(\gamma_0, \gamma, \delta)$, where $\gamma_0 \in G(\mathbb{Q})$, $\gamma = (\gamma_l)_{l\neq p} \in G(\mathbb{A}_f^p)$, and $\delta \in G(\mathbb{Q}_{p^n})$, satisfying the following conditions:

(i) $\gamma_0$ is conjugate to $\gamma$ in $G(\bar{\mathbb{A}}_f^p)$.
(ii) $\gamma_0$ is conjugate to $\delta\sigma(\delta)\cdots\sigma^{n-1}(\delta)$ in $G(\overline{\mathbb{Q}}_p)$.
(iii) the image of $\gamma_0$ in $G(\mathbb{R})$ is elliptic over $\mathbb{R}$.

**5.3.2.** Note that for $n, t \in \mathbb{Z}_{\geq 1}$, there is a natural map

$$\mathfrak{T}_n^{\mathrm{str}} \longrightarrow \mathfrak{T}_{nt}^{\mathrm{str}}, \quad (\gamma_0, \gamma, \delta) \mapsto (\gamma_0^t, \gamma^t, \delta).$$

We set

$$\mathfrak{T}^{\mathrm{str}} := \varinjlim_{n\in\mathbb{Z}_{\geq 1}} \mathfrak{T}^{\mathrm{str}}_n,$$

where $\mathbb{Z}_{\geq 1}$ is a directed set under divisibility.

Let $\mathfrak{k} = (\gamma_0, \gamma, \delta) \in \mathfrak{T}^{\mathrm{str}}_n$ for some $n \in \mathbb{Z}_{\geq 1}$. Recall from [Kis17, §4.3] that $\mathfrak{k}$ gives rise to the following objects:

- a $\mathbb{Q}$-subgroup $I_0$ of $G$, defined to be the centralizer in $G$ of a sufficiently divisible power of $\gamma_0$.
- a $\mathbb{Q}_l$-subgroup $I_l$ of $G$ for each finite places $l \neq p$, defined to be the centralizer in $G$ of a sufficiently divisible power of $\gamma_l$.
- a $\mathbb{Q}_p$-algebraic group $I_p$ defined by

  $$I_p(R) = \left\{ g \in G(\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} R) \mid g^{-1}\delta\sigma(g) = \delta \right\}$$

  for any $\mathbb{Q}_p$-algebra $R$. We shall view $I_p$ as a subfunctor of the functor $\mathrm{Res}_{\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p}(G)$ which sends every $\mathbb{Q}_p$-algebra $R$ to the group $G(\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} R)$. When $t \in \mathbb{Z}_{\geq 1}$ is sufficiently divisible, we have

  $$I_p(R) = \left\{ g \in G(\mathbb{Q}_{p^{nt}} \otimes_{\mathbb{Q}_p} R) \mid g^{-1}\delta\sigma(g) = \delta \right\} \tag{5.3.2.1}$$

  for any $\mathbb{Q}_p$-algebra.

For each finite place $v$, there is a natural equivalence class (see Definition 1.2.1) of inner twistings $\eta_v : I_{0,\mathbb{Q}_v} \simeq I_v$; see [Kis17, §4.3.1]. The datum $(I_0, (I_v)_v, (\eta_v)_v)$ depends on $\mathfrak{k}$ only through the image of $\mathfrak{k}$ in $\mathfrak{T}^{\mathrm{str}}$. In other words, we can attach $(I_0, (I_v)_v, (\eta_v)_v)$ to any element of $\mathfrak{T}^{\mathrm{str}}$.

**Definition 5.3.3.** Let $\mathfrak{k} \in \mathfrak{T}^{\mathrm{str}}$, with associated datum $(I_0, (I_v)_v, (\eta_v)_v)$ as in §5.3.2. A *refinement* of $\mathfrak{k}$ is a tuple $(I, \iota_0, \iota = (\iota_v)_v)$, where

- $I$ is a $\mathbb{Q}$-group and $\iota_0$ is an inner twisting $I_{0,\overline{\mathbb{Q}}} \to I_{\overline{\mathbb{Q}}}$.
- For each finite place $v$, $\iota_v$ is a $\mathbb{Q}_v$-isomorphism $I_{\mathbb{Q}_v} \to I_v$ such that $\iota_v \circ \iota_0$ as an inner twisting between $\mathbb{Q}_v$-groups lies in the equivalence class $\eta_v$.
- $(I/\iota_0(Z_G))(\mathbb{R})$ is compact.

We denote by $\mathfrak{K}\mathfrak{T}^{\mathrm{str}}$ the subset of $\mathfrak{T}^{\mathrm{str}}$ consisting of elements which admit refinements. Elements of $\mathfrak{K}\mathfrak{T}^{\mathrm{str}}$ are called *strict Kottwitz triples.*

**Definition 5.3.4.** Two strict Kottwitz triples $\mathfrak{k}, \mathfrak{k}' \in \mathfrak{K}\mathfrak{T}^{\mathrm{str}}$ are called *equivalent* (resp. *congruent*), written as $\mathfrak{k} \sim \mathfrak{k}'$ (resp. $\mathfrak{k} \equiv \mathfrak{k}'$), if there exist $n \in \mathbb{Z}_{\geq 1}$ and respective representatives $(\gamma_0, \gamma, \delta), (\gamma_0', \gamma', \delta') \in \mathfrak{T}^{\mathrm{str}}_n$ of $\mathfrak{k}, \mathfrak{k}'$, satisfying the following conditions.

- $\gamma_0$ and $\gamma_0'$ are conjugate in $G(\overline{\mathbb{Q}})$.
- $\gamma$ and $\gamma'$ are conjugate in $G(\mathbb{A}_f^p)$ (resp. $\gamma = \gamma'$).
- $\delta$ and $\delta'$ are $\sigma$-conjugate in $G(\mathbb{Q}_{p^n})$ (resp. $\delta = \delta'$).

**5.3.5.** Recall that $\mathbb{A}_f^* := \mathbb{A}_f^p \times \mathbb{Q}_p^{\mathrm{ur}}$. Let $g = (g^p, g_p) \in G(\mathbb{A}_f^*) = G(\mathbb{A}_f^p) \times G(\mathbb{Q}_p^{\mathrm{ur}})$. For each $n \in \mathbb{Z}_{\geq 1}$, we have a bijection

$$\mathfrak{T}^{\mathrm{str}}_n \longrightarrow \mathfrak{T}^{\mathrm{str}}_n, \quad (\gamma_0, \gamma, \delta) \mapsto (\gamma_0, (g^p)^{-1}\gamma g^p, g_p^{-1}\delta\sigma(g_p)).$$

These maps for all $n$ induce a bijection $\mathfrak{K}\mathfrak{T}^{\mathrm{str}} \xrightarrow{\sim} \mathfrak{K}\mathfrak{T}^{\mathrm{str}}$. In this way we obtain a right action of $G(\mathbb{A}_f^*)$ on $\mathfrak{K}\mathfrak{T}^{\mathrm{str}}$, which descends to an action on $\mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\equiv$.

**Definition 5.3.6.** We denote the orbit space $(\mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\equiv)/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$ by $\mathfrak{K}\mathfrak{T}$. Elements of $\mathfrak{K}\mathfrak{T}$ are called *Kottwitz triples*. (This terminology agrees with [Kis17].) The equivalence relation $\sim$ on $\mathfrak{K}\mathfrak{T}^{\mathrm{str}}$ descends to an equivalence relation on $\mathfrak{K}\mathfrak{T}$, still denoted by $\sim$.

*Remark* 5.3.7. The natural map $\mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\sim\ \to \mathfrak{K}\mathfrak{T}/\sim$ is a bijection.

**5.3.8.** We summarize the various definitions in the following diagram:

$$\begin{tikzcd}
\mathfrak{T}^{\mathrm{str}} = \varinjlim_n \mathfrak{T}_n^{\mathrm{str}} & & \mathfrak{K}\mathfrak{T}^{\mathrm{str}} \arrow[ll, hook', "\text{subset of refinable elts}"'] \arrow[d, two heads] \arrow[loop right, "G(\mathbb{A}_f^*)"] \\
\mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\sim \arrow[d, "1:1"] & & \mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\equiv \arrow[ll, two heads] \arrow[d, two heads] \arrow[loop right, "G(\mathbb{A}_f^*)"] \\
\mathfrak{K}\mathfrak{T}/\sim & & \mathfrak{K}\mathfrak{T} = (\mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\equiv)/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}) \arrow[ll, two heads]
\end{tikzcd}$$

**5.3.9.** As in Definition 3.3.1, we denote by $\mathcal{SPD}(G, X)$ the set of special point data for $(G, X)$. Let $\mathfrak{s} = (T, i, h)$ be an element of $\mathcal{SPD}(G, X)$. By Lemma 5.1.2 (iii), $T$ is a cuspidal torus.

Let $\mu_h \in X_*(T)$ be the Hodge cocharacter of $h$. By Lemma 4.3.9, we obtain from the image of $-\mu_h$ in $X_*(T)_{\Gamma_{p,0}}$ a canonical $\overset{\circ}{\sim}$-equivalence class in $T(\mathbb{Q}_p^{\mathrm{ur}})^{\mathrm{mot}}$. Let $\delta_T$ be an element of this equivalence class. Then for $n$ sufficiently divisible, the element

$$\gamma_{0,T,n} := \delta_T \sigma(\delta_T) \cdots \sigma^{n-1}(\delta_T)$$

lies in $T(\mathbb{Q})$. Note that the triple

$$(i(\gamma_{0,T,n}), (i(\gamma_{0,T,n}))_{l \neq p}, i(\delta_T)) \tag{5.3.9.1}$$

is an element of $\mathfrak{T}_n^{\mathrm{str}}$. In fact, condition (iii) in Definition 5.3.1 is satisfied since $T_{\mathbb{R}}$ is elliptic in $G_{\mathbb{R}}$, and the other two conditions are trivial. For $n$ sufficiently divisible, the image of (5.3.9.1) under $\mathfrak{T}_n^{\mathrm{str}} \to \mathfrak{T}^{\mathrm{str}}$ is an element

$$\mathfrak{k}(\mathfrak{s}, \delta_T) = \mathfrak{k}(T, i, h, \delta_T) \in \mathfrak{T}^{\mathrm{str}}$$

which depends only on $\mathfrak{s}$ and $\delta_T$, not on $n$. By [Kis17, Lem. 4.3.11], $\mathfrak{k}(\mathfrak{s}, \delta_T)$ lies in $\mathfrak{K}\mathfrak{T}^{\mathrm{str}} \subset \mathfrak{T}^{\mathrm{str}}$.

Note that the $\overset{\circ}{\sim}$-equivalence class of $\delta_T$ is determined by $\mathfrak{s}$. If we $\sigma$-conjugate $\delta_T$ by an element of $\mathcal{T}^{\circ}(\mathbb{Z}_p^{\mathrm{ur}})$ (or even $T(\mathbb{Q}_p^{\mathrm{ur}})$), the element $\gamma_{0,T,n}$ remains unchanged as long as $n$ is sufficiently divisible. It follows that the image of $\mathfrak{k}(\mathfrak{s}, \delta_T)$ in $\mathfrak{K}\mathfrak{T}/\sim$ is a well-defined invariant of $\mathfrak{s} \in \mathcal{SPD}(G, X)$.[31] We denote this element of $\mathfrak{K}\mathfrak{T}/\sim$ by $\mathfrak{k}(\mathfrak{s})$.

**Definition 5.3.10.** An equivalence class of Kottwitz triples in $\mathfrak{K}\mathfrak{T}/\sim$ is called *special*, if it is of the form $\mathfrak{k}(\mathfrak{s})$ for some $\mathfrak{s} \in \mathcal{SPD}$.

[31] Note that the same cannot be said for the image of $\mathfrak{k}(\mathfrak{s}, \delta_T)$ in $\mathfrak{K}\mathfrak{T} = (\mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\equiv)/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$. This is because two elements of $G(\mathbb{Q}_p^{\mathrm{ur}})$ that are $\sigma$-conjugate by an element of $\mathcal{T}^{\circ}(\mathbb{Z}_p^{\mathrm{ur}})$ need not be $\sigma$-conjugate by an element of $\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$.

## 5.4. **Isogeny classes.**

**5.4.1.** For each $x \in \mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$, we write $\mathcal{V}^p(x)$ for the stalk at $x$ of (the pull-back to $\mathscr{S}_{K_p}$ of) $\mathcal{V}^p$. This is a finite free $\mathbb{A}_f^p$-module. For each $\alpha \in \boldsymbol{\alpha}$, we write $s_{\alpha,\mathbb{A}_f^p,x}$ for the tensor on $\mathcal{V}^p(x)$ induced by the tensor $s_{\alpha,\mathbb{A}_f^p}$ on $\mathcal{V}^p$.

Let $x, x' \in \mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$. Let $R$ be a $\mathbb{Q}$-algebra, and let

$$f \in \mathrm{Hom}(\mathcal{A}_x, \mathcal{A}_{x'}) \otimes_{\mathbb{Z}_{(p)}} R$$

be an $R$-isogeny. Then $f$ induces an $\mathbb{A}_f^p \otimes_{\mathbb{Q}} R$-linear isomorphism

$$f_{\mathcal{V}^p} : \mathcal{V}^p(x') \otimes_{\mathbb{Q}} R \xrightarrow{\sim} \mathcal{V}^p(x) \otimes_{\mathbb{Q}} R,$$

since the two sides are identified with $\mathbf{H}^1_{\text{ét}}(\mathcal{A}_{x'}, \mathbb{A}_f^p) \otimes_{\mathbb{Q}} R$ and $\mathbf{H}^1_{\text{ét}}(\mathcal{A}_x, \mathbb{A}_f^p) \otimes_{\mathbb{Q}} R$ respectively. Similarly, $f$ induces a $\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}} R$-linear isomorphism

$$f_{\mathcal{V}_0} : \mathcal{V}_0(x') \otimes_{\mathbb{Z}_p^{\mathrm{ur}}} \mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}} R \xrightarrow{\sim} \mathcal{V}_0(x) \otimes_{\mathbb{Z}_p^{\mathrm{ur}}} \mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}} R.$$

We say that $f$ *preserves $G$-structures*, if $f_{\mathcal{V}^p}$ takes $s_{\alpha,\mathbb{A}_f^p,x'}$ to $s_{\alpha,\mathbb{A}_f^p,x}$ and $f_{\mathcal{V}_0}$ takes $s_{\alpha,0,x'}$ to $s_{\alpha,0,x}$ for each $\alpha \in \boldsymbol{\alpha}$. We denote by $I_{x,x'}(R)$ the set of all such $f$ preserving $G$-structures. The functor $R \mapsto I_{x,x'}(R)$ is represented by a $\mathbb{Q}$-scheme $I_{x,x'}$. If $x = x'$, then we write $I_x$ for $I_{x,x'}$, which is a $\mathbb{Q}$-algebraic group. Two points $x, x' \in \mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$ are said to be *isogenous*, if $I_{x,x'}(\mathbb{Q}) \neq \emptyset$. This defines an equivalence relation on $\mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$, and the equivalence classes are called *isogeny classes*.

*Remark* 5.4.2. We explain that the above definition of isogeny classes is equivalent to the definition in [Kis17]. The perfect symplectic form on $V_{\mathbb{Z}_{(p)}}$ induces an isomorphism $\iota : V_{\mathbb{Z}_{(p)}} \xrightarrow{\sim} V^*_{\mathbb{Z}_{(p)}}$, from which we obtain an element

$$s_{\mathrm{pol}} := \iota \otimes \iota^{-1} \in \mathrm{Hom}(V_{\mathbb{Z}_{(p)}}, V^*_{\mathbb{Z}_{(p)}}) \otimes \mathrm{Hom}(V^*_{\mathbb{Z}_{(p)}}, V_{\mathbb{Z}_{(p)}}) \subset V^{\otimes}_{\mathbb{Z}_{(p)}}.$$

Note that the scheme-theoretic stabilizer of $s_{\mathrm{pol}}$ in the $\mathbb{Z}_p$-group scheme $\mathrm{GL}(V_{\mathbb{Z}_{(p)}})$ is precisely $\mathrm{GSp}(V_{\mathbb{Z}_{(p)}})$. By our maximality assumption on $(s_\alpha)_{\alpha\in\boldsymbol{\alpha}}$ in §5.1.1, $s_{\mathrm{pol}}$ is one of the $s_\alpha$. On $\mathcal{A}$ we have a canonical weak polarization (i.e., a $\mathbb{Z}_{(p)}^\times$-orbit of $\mathbb{Z}_{(p)}$-isogenies $\mathcal{A} \to \mathcal{A}^\vee$ which can be represented by a polarization) arising from the moduli interpretation of $\mathscr{S}_{\mathbb{K}_p\mathbb{K}_1^p}$. See [Kis17, §1.3.4] for details. This weak polarization induces an isomorphism $\jmath : \mathcal{V}^p \xrightarrow{\sim} (\mathcal{V}^p)^*$, which is well defined up to $(\mathbb{A}_f^p)^\times$. We can then form the tensor $\jmath \otimes \jmath^{-1}$ on $\mathcal{V}^p$, which is well defined on the nose. Recall that $s_{\mathrm{pol},\mathbb{A}_f^p}$ is defined to be $\mathbb{L}^p(s_{\mathrm{pol}})$ via the identification (5.1.5.2). Since the level structure on the tower of Siegel Shimura varieties $\varprojlim_{\mathbb{K}^p} \mathscr{S}_{\mathbb{K}_p\mathbb{K}^p}$ respects weak polarizations, we have $s_{\mathrm{pol},\mathbb{A}_f^p} = \jmath \otimes \jmath^{-1}$. It follows that each $f \in I_{x,x'}(\mathbb{Q})$ necessarily respects the canonical weak polarizations on $\mathcal{A}_x$ and $\mathcal{A}_{x'}$. Thus $x$ and $x'$ are isogenous in our sense if and only if they satisfy the conditions in [Kis17, Prop. 1.4.15]. By that proposition, our definition of isogeny classes is equivalent to the definition in [Kis17, §1.4.14]. In particular, each isogeny class is stable under the $G(\mathbb{A}_f^p)$-action on $\mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$.

## 5.5. **Connected components.**

**5.5.1.** Recall that $G(\mathbb{Q})_+$ denotes $G(\mathbb{Q}) \cap G(\mathbb{R})_+$, where $G(\mathbb{R})_+$ is the preimage of $G^{\mathrm{ad}}(\mathbb{R})^+$ in $G(\mathbb{R})$. As in [Kis17, Lem. 3.6.2], we set

$$\pi(G) := G(\mathbb{Q})_+^- \backslash G(\mathbb{A}_f)/K_p,$$

where $G(\mathbb{Q})_+^-$ denotes the closure of $G(\mathbb{Q})_+$ in $G(\mathbb{A}_f)$. Since $K_p$ is compact, the projection $G(\mathbb{A}_f) \to G(\mathbb{A}_f)/K_p$ is a closed map. Hence the image of $G(\mathbb{Q})_+^-$ in $G(\mathbb{A}_f)/K_p$ is closed. It easily follows that we have canonical isomorphisms

$$\pi(G) \cong \varprojlim_{K^p} G(\mathbb{Q})_+^- \backslash G(\mathbb{A}_f)/K_pK^p \cong \varprojlim_{K^p} G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K_pK^p, \tag{5.5.1.1}$$

where $K^p$ runs through compact open subgroups of $G(\mathbb{A}_f^p)$.

**Lemma 5.5.2.** *The set $\pi(G)$ has the natural structure of an abelian group. The natural map $G(\mathbb{A}_f^p) \to \pi(G)$ is a surjective group homomorphism.*

*Proof.* By strong approximation (see [Del79, §2.5.1]), $G(\mathbb{Q})_+^-$ contains the image of $G_{\mathrm{sc}}(\mathbb{A}_f) \to G(\mathbb{A}_f)$. Since this image is a normal subgroup of $G(\mathbb{A}_f)$ with abelian quotient (see [Del79, §2.0.2]), we see that $\pi(G)$ is naturally an abelian group. The second statement follows from [Kis10, Lem. 2.2.6] (which uses that $K_p$ is hyperspecial). $\square$

**Lemma 5.5.3.** *The subgroup $G(\mathbb{Q})_+^-$ of $G(\mathbb{A}_f)$ is generated by $G(\mathbb{Q})_+$ and the image of $G_{\mathrm{sc}}(\mathbb{A}_f) \to G(\mathbb{A}_f)$.*

*Proof.* Write $Q$ for the image of $G_{\mathrm{sc}}(\mathbb{A}_f) \to G(\mathbb{A}_f)$. We first show that $G(\mathbb{Q})Q$ is closed in $G(\mathbb{A}_f)$. The proof is similar to the argument in [Del79, §2.0.15]. By Lemma 5.1.2, $G^{\mathrm{ab}}$ is a cuspidal torus. By Lemma 1.5.5, $G^{\mathrm{ab}}(\mathbb{Q})$ is discrete in $G^{\mathrm{ab}}(\mathbb{A}_f)$. It follows that $G_{\mathrm{der}}(\mathbb{Q})Q$ is an open subgroup of $G(\mathbb{Q})Q$. On the other hand, by [Del79, Cor. 2.0.8], $G_{\mathrm{der}}(\mathbb{Q})Q$ is closed in $G_{\mathrm{der}}(\mathbb{A}_f)$. Hence $G(\mathbb{Q})Q$ is indeed closed in $G(\mathbb{A}_f)$ (as it is locally closed).

Consequently, $G(\mathbb{Q})_+^- \subset G(\mathbb{Q})Q$. Now by strong approximation applied to $G_{\mathrm{sc}}$ and by the connectedness of $G_{\mathrm{sc}}(\mathbb{R})$ (Cartan's theorem), we know that $Q \subset G(\mathbb{Q})_+^-$. Thus we have reduced the proof of the lemma to showing that

$$G(\mathbb{Q}) \cap G(\mathbb{Q})_+^- \subset G(\mathbb{Q})_+.$$

Let $g \in G(\mathbb{Q}) \cap G(\mathbb{Q})_+^-$. By [Del79, Cor. 2.0.7], there exists an open subgroup $U$ of $G(\mathbb{A}_f)$ such that $U \cap G(\mathbb{Q}) \subset G(\mathbb{R})^+$. Since $g \in G(\mathbb{Q})_+^-$, there exists $g' \in gU \cap G(\mathbb{Q})_+$. Then $g^{-1}g' \in U \cap G(\mathbb{Q}) \subset G(\mathbb{R})^+$, and so $g \in g'G(\mathbb{R})^+$. Since $g \in G(\mathbb{Q})$ and $g' \in G(\mathbb{Q})_+$, we conclude that $g \in G(\mathbb{Q})_+$ as desired. $\square$

**5.5.4.** Now consider the set (cf. [Del79, §2.1.3])

$$\pi(G,X) := \varprojlim_{K^p} \pi_0(\mathrm{Sh}_{K_pK^p}(\mathbb{C})) = \varprojlim_{K^p} G(\mathbb{Q})\backslash \left(\pi_0(X) \times G(\mathbb{A}_f)/K^pK_p\right),$$

where $K^p$ runs through compact open subgroups of $G(\mathbb{A}_f^p)$. There is a natural map $\pi_0(X) \to \pi(G,X)$, induced by

$$\pi_0(X) \longrightarrow \pi_0(X) \times G(\mathbb{A}_f), \quad C \longmapsto (C,1).$$

This allows us to speak of the image of an element of $X$ or $\pi_0(X)$ inside $\pi(G,X)$.

By (5.5.1.1) we see that $\pi(G,X)$ is a $\pi(G)$-torsor. We have a natural $G(\mathbb{A}_f^p)$-equivariant map

$$\mathscr{S}_{K_p}(\overline{\mathbb{F}}_p) \longrightarrow \pi(G,X),$$

defined as the inverse limit of the natural maps

$$\mathscr{S}_{K_pK^p}(\overline{\mathbb{F}}_p) \longrightarrow \pi_0(\mathscr{S}_{K_pK^p,\overline{\mathbb{F}}_p}) \cong \pi_0(\mathrm{Sh}_{K_pK^p}(\mathbb{C})).$$

In particular, for each isogeny class $\mathscr{I}$ in $\mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$ (see §5.4), we have a natural $G(\mathbb{A}_f^p)$-equivariant map

$$c_{\mathscr{I}} : \mathscr{I} \longrightarrow \pi(G, X).$$

By the $G(\mathbb{A}_f^p)$-equivariance and Lemma 5.5.2, the above map is surjective.

In the following definition, recall that $\mathbb{A}_f^* := \mathbb{A}_f^p \times \mathbb{Q}_p^{\mathrm{ur}}$.

**Definition 5.5.5.** We set $\pi^*(G) := G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f^*)/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})G_{\mathrm{der}}(\mathbb{A}_f^*)$. This is naturally an abelian group, and is a quotient group of the subgroup $G(\mathbb{A}_f^*)/G_{\mathrm{der}}(\mathbb{A}_f^*)$ of $G^{\mathrm{ab}}(\mathbb{A}_f^*)$.

**5.5.6.** By Lemma 5.5.3, the natural inclusion map $G(\mathbb{A}_f) \hookrightarrow G(\mathbb{A}_f^*)$ induces a group homomorphism $\pi(G) \to \pi^*(G)$. We define the $\pi^*(G)$-torsor $\pi^*(G, X)$ to be the push-out of the $\pi(G)$-torsor $\pi(G, X)$ along $\pi(G) \to \pi^*(G)$. Thus we have a canonical map $\pi(G, X) \to \pi^*(G, X)$, and we shall use this map to speak of the image of an element of $X$ or $\pi_0(X)$ inside $\pi^*(G, X)$.

## 5.6. **Uniformization on the geometric side.**

**Definition 5.6.1.** Let $\mathscr{I}$ be a small connected groupoid category, i.e., a small category where all morphisms are isomorphisms and all objects are isomorphic. Let $H$ be a group. By a *right $H$-torsor over $\mathscr{I}$*, we mean a functor from $\mathscr{I}$ to the category of right $H$-torsors. Let $Y$ be such a functor, and let $x$ be an object in $\mathscr{I}$. Then $\mathrm{Aut}(x)$ naturally acts (on the left) on $Y(x)$ via $H$-equivariant automorphisms, and the right $H$-set $\mathrm{Aut}(x)\backslash Y(x)$ is independent of $x$ up to canonical $H$-isomorphism. We denote this right $H$-set by $\bar{Y}(\mathscr{I})$.

**5.6.2.** For $x \in \mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$, let

$$Y(x) := Y_p(x) \times Y^p(x),$$

where $Y_p(x)$ is the right $G(\mathbb{Q}_p^{\mathrm{ur}})$-torsor $Y(\Upsilon_x)$, and $Y^p(x)$ is the right $G(\mathbb{A}_f^p)$-torsor consisting of $\mathbb{A}_f^p$-module isomorphisms $V^*_{\mathbb{A}_f^p} \xrightarrow{\sim} \mathcal{V}^p(x)$ taking $s_\alpha$ to $s_{\alpha,\mathbb{A}_f^p,x}$ for each $\alpha \in \boldsymbol{\alpha}$. Thus $Y(x)$ is a right $G(\mathbb{A}_f^*)$-torsor. In fact, by Lemma 5.1.9, $Y^p(x)$ has a canonical trivialization.

The set $Y_p(x)$ can be interpreted without reference to $\Upsilon_x$ as follows. Recall from Definition 5.2.5 that we have a canonical isomorphism $\iota_x : \Upsilon_x(V^*_{\mathbb{Z}_p}) \xrightarrow{\sim} \mathcal{V}_0(x)$ taking $\Upsilon_x(s_\alpha)$ to $s_{\alpha,0,x}$. Now each element of $Y(\Upsilon_x)$ gives rise to an isomorphism $V^*_{\mathbb{Q}_p^{\mathrm{ur}}} \xrightarrow{\sim} \Upsilon_x[1/p](V^*_{\mathbb{Z}_p})$ taking $s_\alpha$ to $\Upsilon_x(s_\alpha)$. Composing this with $\iota_x[1/p]$, we obtain an isomorphism $V^*_{\mathbb{Q}_p^{\mathrm{ur}}} \xrightarrow{\sim} \mathcal{V}_0(x)[1/p]$ taking $s_\alpha$ to $s_{\alpha,0,x}$. In this way, $Y(\Upsilon_x)$ is in canonical bijection with the set of $\mathbb{Q}_p^{\mathrm{ur}}$-linear isomorphisms $V^*_{\mathbb{Q}_p^{\mathrm{ur}}} \xrightarrow{\sim} \mathcal{V}_0(x)[1/p]$ taking $s_\alpha$ to $s_{\alpha,0,x}$. The $G(\mathbb{Q}_p^{\mathrm{ur}})$-action on the latter set is given by the $G(\mathbb{Q}_p^{\mathrm{ur}})$-action on $V^*_{\mathbb{Q}_p^{\mathrm{ur}}}$. Note that under this bijection, the subset $Y(\Upsilon_x)^\circ \subset Y(\Upsilon_x)$ corresponds to those isomorphisms $V^*_{\mathbb{Q}_p^{\mathrm{ur}}} \xrightarrow{\sim} \mathcal{V}_0(x)[1/p]$ that take $s_\alpha$ to $s_{\alpha,0,x}$ and map $V^*_{\mathbb{Z}_p^{\mathrm{ur}}}$ to $\mathcal{V}_0(x)$.

Now let $y = (y_p, y^p) \in Y(x)$. Let $n \in \mathbb{Z}_{\geq 1}$ be sufficiently divisible such that the image of $x$ in $\mathscr{S}_{K_1}(\overline{\mathbb{F}}_p)$ comes from a $\mathbb{F}_{p^n}$-rational point $x_n$ (and $\mathbb{F}_{p^n} \supset \mathcal{O}_{E,\mathfrak{p}}/\mathfrak{p}$). We then have the $p^n$-Frobenius acting on $\mathcal{V}^p(x) \cong \mathbf{H}^1_{\text{ét}}(\mathcal{A}_{x_n,\overline{\mathbb{F}}_p}, \mathbb{A}_f^p)$, which fixes $s_{\alpha,\mathbb{A}_f^p,x}$ for all $\alpha \in \boldsymbol{\alpha}$. Via $y^p : V^*_{\mathbb{A}_f^p} \xrightarrow{\sim} \mathcal{V}^p(x)$, this automorphism of $\mathcal{V}^p(x)$ corresponds to an automorphism of $V^*_{\mathbb{A}_f^p}$ fixing all $s_\alpha$, namely an element $\gamma_n \in G(\mathbb{A}_f^p)$. On the other hand, attached to the element $y_p \in Y_p(x) = Y(\Upsilon_x)$ we have the element $\delta_{y_p} \in G(\mathbb{Q}_p^{\mathrm{ur}})$ as in §4.2.5. More concretely, $y_p$ gives rise to an isomorphism $V^*_{\mathbb{Q}_p^{\mathrm{ur}}} \xrightarrow{\sim} \mathcal{V}_0(x)[1/p]$ as in the above paragraph, and the Frobenius acting on the right hand side corresponds to $\delta_{y_p}\sigma$ acting on the left hand side.

It is shown in [Kis17, §2.3] that up to replacing $n$ by a multiple, the pair $(\gamma_n, \delta)$ extends to an element $(\gamma_{0,n}, \gamma_n, \delta) \in \mathfrak{T}_n^{\mathrm{str}}$ whose image in $\mathfrak{T}^{\mathrm{str}}$ lies in $\mathfrak{K}\mathfrak{T}^{\mathrm{str}}$. The image of $(\gamma_{0,n}, \gamma_n, \delta)$ in $\mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\equiv$ depends only on $y$ and not on the choices of $n$ and $\gamma_{0,n}$. Thus we have obtained a map

$$Y(x) \longrightarrow \mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\equiv, \quad y \mapsto \mathfrak{k}(y). \tag{5.6.2.1}$$

This map is easily seen to be $G(\mathbb{A}_f^*)$-equivariant. (See §5.3.5 for the right $G(\mathbb{A}_f^*)$-action on $\mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\equiv$.)

**5.6.3.** Now let $\mathscr{I} \subset \mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$ be an isogeny class. We view $\mathscr{I}$ as a small groupoid, where the set of morphisms between $x$ and $x' \in \mathscr{I}$ is given by $I_{x,x'}(\mathbb{Q})$ defined in §5.4. Then $\mathscr{I}$ is a connected groupoid category. For each $f \in I_{x,x'}(\mathbb{Q})$, we have isomorphisms $f_{\mathcal{V}^p} : \mathcal{V}^p(x') \xrightarrow{\sim} \mathcal{V}^p(x)$ and $f_{\mathcal{V}_0} : \mathcal{V}_0(x')[1/p] \xrightarrow{\sim} \mathcal{V}_0(x)[1/p]$ as in §5.4. By definition, $f_{\mathcal{V}^p}$ takes $s_{\alpha,\mathbb{A}_f^p,x'}$ to $s_{\alpha,\mathbb{A}_f^p,x}$, and so it induces a $G(\mathbb{A}_f^p)$-equivariant bijection

$$Y^p(f) : Y^p(x) \xrightarrow{\sim} Y^p(x').$$

Similarly, $f_{\mathcal{V}_0}$ takes $s_{\alpha,0,x'}$ to $s_{\alpha,0,x}$, and so it induces an isomorphism $\Upsilon_x[1/p] \xrightarrow{\sim} \Upsilon_{x'}[1/p]$ by Lemma 4.2.4. By functoriality, this then induces a $G(\mathbb{Q}_p^{\mathrm{ur}})$-equivariant bijection

$$Y_p(f) : Y_p(x) \xrightarrow{\sim} Y_p(x').$$

We define $Y(f)$ to be the bijection

$$(Y_p(f), Y^p(f)) : Y(x) \xrightarrow{\sim} Y(x').$$

The associations $\mathscr{I} \ni x \mapsto Y(x)$ and $I_{x,x'}(\mathbb{Q}) \ni f \mapsto Y(f)$ define a functor from $\mathscr{I}$ to the category of right $G(\mathbb{A}_f^*)$-torsors. In other words, we have obtained a right $G(\mathbb{A}_f^*)$-torsor $Y$ over $\mathscr{I}$ in the sense of Definition 5.6.1. As in that definition, we obtain a right $G(\mathbb{A}_f^*)$-set $\bar{Y}(\mathscr{I})$, together with canonical isomorphisms

$$I_x(\mathbb{Q})\backslash Y(x) \xrightarrow{\sim} \bar{Y}(\mathscr{I}) \tag{5.6.3.1}$$

for all $x \in \mathscr{I}$.

For $x, x' \in \mathscr{I}$ and $f \in I_{x,x'}(\mathbb{Q})$, we claim that the bijection $Y(f) : Y(x) \xrightarrow{\sim} Y(x')$ commutes with the maps $Y(x) \to \mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\equiv$ and $Y(x') \to \mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\equiv$ as in (5.6.2.1). In fact, since $Y_p(f) : Y(\Upsilon_x) \xrightarrow{\sim} Y(\Upsilon_{x'})$ is induced by an isomorphism $\Upsilon_x[1/p] \xrightarrow{\sim} \Upsilon_{x'}[1/p]$, it commutes with the maps $Y_p(x) \to G(\mathbb{Q}_p^{\mathrm{ur}}), y \mapsto \delta_y$ and $Y_p(x') \to G(\mathbb{Q}_p^{\mathrm{ur}}), y \mapsto \delta_y$ in §4.2.5. For sufficiently divisible $n$, the isogeny $f$ is defined over $\mathbb{F}_{p^n}$, and so the element $\gamma_n \in G(\mathbb{A}_f^p)$ attached to any $y \in Y^p(x)$ is equal to its counterpart attached to $Y^p(f)(y) \in Y^p(x')$. Our claim follows.

It follows that for each $x \in \mathscr{I}$, the map $Y(x) \to \mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\equiv$ descends to a map

$$I_x(\mathbb{Q})\backslash Y(x) \longrightarrow \mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\equiv,$$

which is independent of $x$ if we identify the left hand side with $\bar{Y}(\mathscr{I})$ as in (5.6.3.1). Hence we have obtained a canonical map

$$\bar{Y}(\mathscr{I}) \longrightarrow \mathfrak{KT}^{\mathrm{str}}/\equiv, \quad \bar{y} \longmapsto \mathfrak{k}(\bar{y}). \tag{5.6.3.2}$$

For $x \in \mathscr{I}$ and $y \in Y(x)$, the left $I_x(\mathbb{Q})$-action on the right $G(\mathbb{A}_f^*)$-torsor $Y(x)$ gives rise to a homomorphism

$$\iota_y : I_x(\mathbb{Q}) \longrightarrow G(\mathbb{A}_f^*) \tag{5.6.3.3}$$

defined by

$$j \cdot y = y \cdot \iota_y(j), \quad \forall j \in I_x(\mathbb{Q}).$$

Thus we have a map $\iota_{y,v} : I_x(\mathbb{Q}) \to G(\mathbb{Q}_v)$ for each prime $v \neq p$, and a map $\iota_{y,p} : I_x(\mathbb{Q}) \to G(\mathbb{Q}_p^{\mathrm{ur}})$. These maps have the following extra structures, by the results in [Kis17, §2.3]. Let $\dot{\mathfrak{k}}(y) \in \mathfrak{KT}^{\mathrm{str}}$ be a representative of the image of $\mathfrak{k}(y) \in \mathfrak{KT}^{\mathrm{str}}/\equiv$. Let $(I_0, (I_v)_v, (\eta_v)_v)$ be the datum attached to $\mathfrak{k}$ as in §5.3.2. For each prime $v \neq p$, the map $\iota_{y,v}$ comes from an isomorphism of $\mathbb{Q}_v$-groups $I_{x,\mathbb{Q}_v} \xrightarrow{\sim} I_v$, which we still denote by $\iota_{y,v}$. Also, the map $\iota_{y,p}$ comes from an isomorphism of $\mathbb{Q}_p$-groups $I_{x,\mathbb{Q}_p} \xrightarrow{\sim} I_p$, which we still denote by $\iota_{y,p}$. (Here recall that $I_v(\mathbb{Q}_v) \subset G(\mathbb{Q}_v)$ for $v \neq p$ and $I_p(\mathbb{Q}_p) \subset G(\mathbb{Q}_p^{\mathrm{ur}})$.) In particular, the map (5.6.3.3) is injective. Moreover, the isomorphisms $\iota_{y,v}$ for all primes $v$ can be extended to a refinement of $\dot{\mathfrak{k}}(y)$ of the form $(I_x, \iota_0, (\iota_{y,v})_v)$. (See Definition 5.3.3 for the notion of a refinement.) This in particular implies that $I_x$ is a reductive group over $\mathbb{Q}$ such that $I_{x,\mathbb{R}}$ is anisotropic mod center.

**5.6.4.** Let $\mathscr{I} \subset \mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$ be an isogeny class. Set

$$\mathscr{I}^* = \bar{Y}(\mathscr{I})/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}).$$

Then we have a natural right $G(\mathbb{A}_f^p)$-action on $\mathscr{I}^*$. It is immediate that the map (5.6.3.2) induces a map

$$\mathscr{I}^* \longrightarrow \mathfrak{KT} = (\mathfrak{KT}^{\mathrm{str}}/\equiv)/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}). \tag{5.6.4.1}$$

For each $x \in \mathscr{I}$, inside

$$Y(x)/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}) \cong (Y(\Upsilon_x)/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})) \times G(\mathbb{A}_f^p)$$

we have a canonical base point, whose first coordinate is given by the image of the $\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$-torsor $Y(\Upsilon_x)^\circ \subset Y(\Upsilon_x)$, and whose second coordinate is $1 \in G(\mathbb{A}_f^p)$. This base point determines an element $x^*$ of $\mathscr{I}^* \cong I_x(\mathbb{Q})\backslash Y(x)/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$ (where the canonical isomorphism is induced by (5.6.3.1)). Sending $x$ to $x^*$, we have obtained a map

$$\mathscr{I} \longrightarrow \mathscr{I}^*. \tag{5.6.4.2}$$

By [Kis17, Prop. 2.1.3], the map (5.6.4.2) is injective and $G(\mathbb{A}_f^p)$-equivariant. The image of (5.6.4.2) is described as follows. As in §2.4.1, we choose $\mu_X \in \mathfrak{p}_X(E_\mathfrak{p})$ such that $\mu_X$ extends to a cocharacter of $\mathcal{G}_{\mathcal{O}_{E,\mathfrak{p}}}$ over $\mathcal{O}_{E,\mathfrak{p}}$. Let

$$\upsilon = \sigma(-\mu_X)$$

Let $x \in \mathscr{I}$. Recall from Definition 4.2.7 that inside $Y(\Upsilon_x)$ we have the $\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$-stable subset $Y_\upsilon(\Upsilon_x)$, and that we denote the quotient $Y_\upsilon(\Upsilon_x)/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$ by $X_\upsilon(\Upsilon_x)$. The subset $\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})p^\upsilon\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}) \subset G(\mathbb{Q}_p^{\mathrm{ur}})$ depends on $\upsilon$ only via its $\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$-conjugacy class, and the latter is independent of the choice of $\mu_X$. Hence $Y_\upsilon(\Upsilon_x)$ and $X_\upsilon(\Upsilon_x)$ are independent of the choice of $\mu_X$ (cf. the independence of $\mu_X$ in §2.4.1). Moreover, if $x'$ is another element of $\mathscr{I}$ and if $f \in I_{x,x'}(\mathbb{Q})$, we have seen in §5.6.3 that the isomorphism $Y_p(f) : Y(\Upsilon_x) \xrightarrow{\sim} Y(\Upsilon_{x'})$ commutes with the maps $Y(\Upsilon_x) \to G(\mathbb{Q}_p^{\mathrm{ur}}), y \mapsto \delta_y$ and $Y(\Upsilon_{x'}) \to G(\mathbb{Q}_p^{\mathrm{ur}}), y \mapsto \delta_y$. It follows that $Y_p(f)$ induces a bijection $Y_\upsilon(\Upsilon_x) \xrightarrow{\sim} Y_\upsilon(\Upsilon_{x'})$. Therefore inside $\bar{Y}(\mathscr{I})$ we have a canonical subset of the form

$$\bar{Y}(\mathscr{I})^\natural \cong I_x(\mathbb{Q})\backslash Y_\upsilon(\Upsilon_x) \times Y^p(x),$$

which is independent of the choice of $x$.

**Proposition 5.6.5.** *The image of (5.6.4.2) is equal to the image of $\bar{Y}(\mathscr{I})^\natural$ under the projection $\bar{Y}(\mathscr{I}) \to \mathscr{I}^*$.*

*Proof.* By [Kis17, §1.4.1], we have

$$Y(\Upsilon_x)^\circ \subset Y_\upsilon(\Upsilon_x), \quad \forall x \in \mathscr{I}. \tag{5.6.5.1}$$

It follows that the image of (5.6.4.2) is contained in $\bar{Y}(\mathscr{I})^\natural$. The reverse containment follows from the definition of isogeny classes in [Kis17, §1.4.14], which we have seen is equivalent to our definition (see §5.4). ☐

*Remark* 5.6.6. Keep the setting of §5.6.4. From Proposition 5.6.5, we see that the choice of an element $x \in \mathscr{I}$ gives rise to a bijection

$$I_x(\mathbb{Q})\backslash X_\upsilon(\Upsilon_x) \times Y^p(x) \xrightarrow{\sim} \mathscr{I}.$$

As explained in Remark 4.2.8, if we choose $y \in Y_\upsilon(\Upsilon_x)$, then $X_\upsilon(\Upsilon_x)$ is identified with the affine Deligne–Lusztig set $X_\upsilon(\delta_y)$. The natural action of $I_x(\mathbb{Q})$ on $X_\upsilon(\Upsilon_x)$ corresponds to the natural action of $\iota_{y,p}(I_x(\mathbb{Q}))$ on $X_\upsilon(\delta_y)$. (Recall from §5.6.3 that $\iota_{y,p}(I_x(\mathbb{Q}_p))$ is the $\sigma$-centralizer of $\delta_y$ in $G(\mathbb{Q}_p^{\mathrm{ur}})$. This group acts on $X_\upsilon(\delta_y)$ by left multiplication). Similarly, under the canonical identification $Y^p(x) \cong G(\mathbb{A}_f^p)$, the natural action of $I_x(\mathbb{Q})$ on $X_\upsilon(\Upsilon_x) \times Y^p(x)$ corresponds to the left-multiplication action of $\iota_y(I_x(\mathbb{Q}))$ on $X_\upsilon(\delta_y) \times G(\mathbb{A}_f^p)$. Thus we obtain a bijection

$$\iota_y(I_x(\mathbb{Q}))\backslash X_\upsilon(\delta_y) \times G(\mathbb{A}_f^p) \xrightarrow{\sim} \mathscr{I} \tag{5.6.6.1}$$

associated with the choices of $x$ and $y$. This bijection is the same as the map [Kis17, (2.1.4)].

**5.6.7.** We summarize the various constructions in the following commutative diagram.

$$\begin{tikzcd}
I_x(\mathbb{Q})\backslash Y_\upsilon(x) \times Y^p(x) \arrow[r, hook] \arrow[d, "\forall x\in\mathscr{I}"', "\cong"] & I_x(\mathbb{Q})\backslash Y(x) \arrow[d, "\forall x\in\mathscr{I}"', "{\cong,(5.6.3.1)}"] & \\
\bar{Y}(\mathscr{I})^\natural \arrow[r, hook] \arrow[d, two heads, "\text{quot. by } \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})"'] & \bar{Y}(\mathscr{I}) \arrow[r, "(5.6.3.2)"] \arrow[d, two heads, "\text{quot. by } \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})"'] & \mathfrak{KT}^{\mathrm{str}}/\equiv \arrow[d, two heads, "\text{quot. by } \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})"] \\
\mathscr{I} \arrow[r, hook, "(5.6.4.2)"] & \mathscr{I}^* \arrow[r, "(5.6.4.1)"] & \mathfrak{KT}
\end{tikzcd}$$

**5.6.8.** It follows from the Cartan decomposition that the Kottwitz homomorphism $w_G : G(\breve{\mathbb{Q}}_p) \to \pi_1(G)$ induces a bijection

$$G(\mathbb{Q}_p^{\mathrm{ur}})/G_{\mathrm{sc}}(\mathbb{Q}_p^{\mathrm{ur}})\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}) \xrightarrow{\sim} \pi_1(G)$$

and an injection

$$G(\mathbb{Q}_p)/G_{\mathrm{sc}}(\mathbb{Q}_p)\mathcal{G}(\mathbb{Z}_p) \hookrightarrow \pi_1(G)^{\Gamma_p} = \pi_1(G)^{\sigma}.$$

By [Kot97, §7.7] (cf. [Kis17, Lem. 1.2.3]), the above injection is also a bijection. It follows that the natural map

$$G(\mathbb{Q}_p)/G_{\mathrm{sc}}(\mathbb{Q}_p)\mathcal{G}(\mathbb{Z}_p) \longrightarrow G(\mathbb{Q}_p^{\mathrm{ur}})/G_{\mathrm{sc}}(\mathbb{Q}_p^{\mathrm{ur}})\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$$

is injective, and that its image is precisely the preimage of $\pi_1(G)^{\sigma}$ in

$$G(\mathbb{Q}_p^{\mathrm{ur}})/G_{\mathrm{sc}}(\mathbb{Q}_p^{\mathrm{ur}})\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}).$$

From this observation, we see that each element $r_p \in G(\mathbb{Q}_p^{\mathrm{ur}})$ satisfying $w_G(r_p) \in \pi_1(G)^{\sigma}$ uniquely determines an element $r'_p \in G(\mathbb{Q}_p)/G_{\mathrm{sc}}(\mathbb{Q}_p)\mathcal{G}(\mathbb{Z}_p)$.

**Lemma 5.6.9.** *Let $\mathscr{I} \subset \mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$ be an isogeny class. Let $y \in \bar{Y}(\mathscr{I})$. Let $r = (r^p, r_p) \in G(\mathbb{A}_f^*) = G(\mathbb{A}_f^p) \times G(\mathbb{Q}_p^{\mathrm{ur}})$, and let $y' = yr \in \bar{Y}(\mathscr{I})$. Assume that both $y$ and $y'$ lie in $\bar{Y}(\mathscr{I})^{\natural}$. Then the following statements hold.*

*(i) The element $r_p \in G(\mathbb{Q}_p^{\mathrm{ur}})$ satisfies $w_G(r_p) \in \pi_1(G)^{\sigma}$. In particular, $r_p$ determines an element $r'_p \in G(\mathbb{Q}_p)/G_{\mathrm{sc}}(\mathbb{Q}_p)\mathcal{G}(\mathbb{Z}_p)$ as in §5.6.8. By Lemma 5.5.3, the image of $(r^p, r'_p)$ in $\pi(G)$ is well defined. We denote this image by $r_{\pi(G)}$.*

*(ii) The images of $y$ and $y'$ under the composite map*

$$\bar{Y}(\mathscr{I})^{\natural} \longrightarrow \mathscr{I} \xrightarrow{c_{\mathscr{I}}} \pi(G, X)$$

*differ by the element $r_{\pi(G)} \in \pi(G)$ in part (i). More precisely, $c_{\mathscr{I}}(y') = c_{\mathscr{I}}(y) \cdot r_{\pi(G)}$.*

*Proof.* Since (5.6.4.2) is $G(\mathbb{A}_f^p)$-equivariant, so is the map $\bar{Y}(\mathscr{I})^{\natural} \to \mathscr{I}$. The map $c_{\mathscr{I}}$ is also $G(\mathbb{A}_f^p)$-equivariant. The proof is thus reduced to the case $r^p = 1$. In the following we assume $r^p = 1$, and write $r$ for $r_p$.

Let $x \in \mathscr{I}$ be the image of $y$. Then under the identification

$$\bar{Y}(\mathscr{I}) \cong I_x(\mathbb{Q})\backslash Y_p(x) \times Y^p(x) = I_x(\mathbb{Q})\backslash Y(\Upsilon_x) \times G(\mathbb{A}_f^p),$$

the element $y$ is represented by $(y_p, 1) \in Y(\Upsilon_x) \times G(\mathbb{A}_f^p)$ with $y_p \in Y(\Upsilon_x)^{\circ}$. The element $y'$ is represented by $(y_p r, 1)$.

Let $\delta = \delta_{y_p} \in G(\mathbb{Q}_p^{\mathrm{ur}})$. Let $\upsilon$ be as in §5.6.4. By (5.6.5.1), we have $y_p \in Y_{\upsilon}(\Upsilon_x)$. Since $y' \in \bar{Y}(\mathscr{I})^{\natural}$, we also have $y_p r \in Y_{\upsilon}(\Upsilon_x)$ by the discussion in §5.6.4. As in Remark 4.2.8, we have an identification $X_{\upsilon}(\Upsilon_x) = Y_{\upsilon}(\Upsilon_x)/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}) \xrightarrow{\sim} X_{\upsilon}(\delta)$, under which the image of $y_p$ (resp. $y_p r$) in $X_{\upsilon}(\Upsilon_x)$ corresponds to the element $\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$ (resp. $r\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$) of $G(\mathbb{Q}_p^{\mathrm{ur}})/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$. Clearly all elements of $X_{\upsilon}(\delta)$ have the same image under the composite map

$$G(\mathbb{Q}_p^{\mathrm{ur}})/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}) \xrightarrow{w_G} \pi_1(G) \xrightarrow{1-\sigma} \pi_1(G).$$

Since $\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$ and $r\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$ both lie in $X_{\upsilon}(\delta)$, statement (i) follows.

Let $C$ denote the composite map

$$X_{\upsilon}(\delta) \cong X_{\upsilon}(\Upsilon_x) \xrightarrow{z \mapsto (z,1)} I_x(\mathbb{Q})\backslash X_{\upsilon}(\Upsilon_x) \times G(\mathbb{A}_f^p) \cong \mathscr{I} \xrightarrow{c_{\mathscr{I}}} \pi(G, X).$$

We are left to show that $C(r)$ and $C(1)$ differ by $r_{\pi(G)}$.

We follow [Kis17, §§1.2–1.4] closely. Let $\mathscr{G}$ be the $p$-divisible group $\mathcal{A}_x[p^\infty]$ over $\overline{\mathbb{F}}_p$. By [Kis17, §1.2.16, Lem. 1.2.18], for any finite extension $K/\breve{\mathbb{Q}}_p$ and any $\mathcal{G}_{\breve{\mathbb{Z}}_p}$-adapted lifting $\widetilde{\mathscr{G}}$ of $\mathscr{G}$ to $\mathcal{O}_K$, there is an associated map (which is denoted by $g \mapsto g_0$ in *loc. cit.*)

$$f_{\widetilde{\mathscr{G}}} : G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p) \longrightarrow X_v(\delta).$$

In fact, in the language of §4.4, the choice of $\widetilde{\mathscr{G}}$ gives rise to an element $[\rho_{\widetilde{\mathscr{G}}}] \in \mathrm{Crys}_G$, and the map $f_{\widetilde{\mathscr{G}}}$ by definition sends each $\lambda \in G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p)$ to the element $\lambda_0 \in G(\mathbb{Q}_p^{\mathrm{ur}})/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$ associated with $[\rho_{\widetilde{\mathscr{G}}}]$ and $\lambda$ as in §4.4.14.

By [Kis17, Prop. 1.2.23], we can choose $\widetilde{\mathscr{G}}$ such that the composite map

$$G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p) \xrightarrow{f_{\widetilde{\mathscr{G}}}} X_v(\delta) \longrightarrow \pi_0(X_v(\delta))$$

is surjective. Here $\pi_0(X_v(\delta))$ is the set of *connected components* of the affine Deligne–Lusztig set $X_v(\delta)$, defined in [CKV15] (cf. [Kis17, §1.2]). We fix such a choice of $\widetilde{\mathscr{G}}$, and choose $g \in G(\mathbb{Q}_p)$ such that $f_{\widetilde{\mathscr{G}}}(g)$ lies in the same connected component of $X_v(\delta)$ as $r$. In what follows we write $g_0$ for $f_{\widetilde{\mathscr{G}}}(g) \in X_v(\delta)$.

By [Kis17, Cor. 1.4.12], we have

$$C(g_0) = C(1) \cdot g.$$

Here on the right hand side we again write $g$ for the natural image of $g \in G(\mathbb{Q}_p)$ in $\pi^*(G)$. Thus we are left to show that

$$C(r) = C(g_0) \cdot g^{-1} r_{\pi(G)}.$$

Since $r$ and $g_0$ lie in the same connected component of $X_v(\delta)$, we have $C(r) = C(g_0)$.[32] Hence it suffices to show that the image of $g$ in $\pi(G)$ is equal to $r_{\pi(G)}$. For this, it suffices to show that $w_G(g) = w_G(r)$.

The fact that $r$ and $g_0$ lie in the same connected component of $X_v(\delta)$ implies that $w_G(r) = w_G(g_0)$, in view of [CKV15, Lem. 2.3.6]. By [Kis17, Lem. 1.2.18] or Proposition 4.4.15, we have $w_G(g_0) = w_G(g)$. Hence $w_G(g) = w_G(r)$ as desired. $\square$

**Proposition 5.6.10.** *Let $\mathscr{I} \subset \mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$ be an isogeny class. There is a unique map*

$$c^*_{\mathscr{I}} : \mathscr{I}^* \longrightarrow \pi^*(G, X)$$

*such that the diagram*

$$\begin{tikzcd} \mathscr{I} \arrow[r, hook, "(5.6.4.2)"] \arrow[d, "c_{\mathscr{I}}"] & \mathscr{I}^* \arrow[d, "c^*_{\mathscr{I}}"] \\ \pi(G,X) \arrow[r] & \pi^*(G,X) \end{tikzcd}$$

*commutes, and such that the composite*

$$\bar{Y}(\mathscr{I}) \to \mathscr{I}^* \xrightarrow{c^*_{\mathscr{I}}} \pi^*(G,X)$$

[32] In fact, if one modifies the setting of [Kis17, §1.4.10] by replacing the integral model of the Shimura variety with a connected component of the geometric special fiber, then exactly the same argument there shows that the map $X_v(\delta) \cong X_v(\Upsilon_x) \to I_x(\mathbb{Q})\backslash X_v(\Upsilon_x) \times G(\mathbb{A}_f^p) \cong \mathscr{I}$ sends each connected component of $X_v(\delta)$ into one fiber of $c_{\mathscr{I}}$.

*is $G(\mathbb{A}_f^*)$-equivariant. Here $G(\mathbb{A}_f^*)$ acts on $\pi^*(G,X)$ via the natural homomorphism $G(\mathbb{A}_f^*) \to \pi^*(G)$.*

*Proof.* We have $\bar{Y}(\mathscr{I})^\natural \cdot G(\mathbb{Q}_p^{\mathrm{ur}}) = \bar{Y}(\mathscr{I})$, from which the uniqueness of $c^*_{\mathscr{I}}$ follows. To show the existence, we fix $x \in \mathscr{I}$. The choice of $x$ gives an identification $I_x(\mathbb{Q})\backslash Y(x) \cong \bar{Y}(\mathscr{I})$. As discussed in §5.6.3, the left action of $I_x(\mathbb{Q})$ on $Y(x)$ can be reconstructed as the composition of the inversion map $I_x(\mathbb{Q}) \to I_x(\mathbb{Q})$, the embedding $\iota_y : I_x(\mathbb{Q}) \to G(\mathbb{A}_f^*)$ (see (5.6.3.3)), and the right multiplication of $G(\mathbb{A}_f^*)$ on itself. By Lemma 5.6.9, the composed homomorphism $I_x(\mathbb{Q}) \xrightarrow{\iota_y} G(\mathbb{A}_f^*) \to \pi^*(G)$ is trivial. By this fact and by Lemma 5.6.9, we know that the map $\bar{Y}(\mathscr{I})^\natural \to \pi^*(G,X)$, obtained as the composition of the map $\bar{Y}(\mathscr{I})^\natural \to \pi(G,X)$ considered in Lemma 5.6.9 and the natural map $\pi(G,X) \to \pi^*(G,X)$, extends to a $G(\mathbb{A}_f^*)$-equivariant map $c : \bar{Y}(\mathscr{I}) \to \pi^*(G,X)$. Now $c$ necessarily factors through the projection $\bar{Y}(\mathscr{I}) \to \mathscr{I}^* = \bar{Y}(\mathscr{I})/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$, because $\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$ acts trivially on $\pi^*(G,X)$. This finishes the proof. □

## 5.7. **Special points on the geometric side.**

**5.7.1.** Let $\mathfrak{s} = (T,i,h) \in \mathcal{SPD}(G,X)$ be a special point datum. From $\mathfrak{s}$ we can produce a canonical element
$$x_{\mathfrak{s}} \in \mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$$
as follows. For each neat compact open subgroup $K \subset G(\mathbb{A}_f)$, the subgroup $i^{-1}(K) \subset T(\mathbb{A}_f)$ is neat compact open. We have the Shimura variety $\mathrm{Sh}_{i^{-1}(K)}(T,h)$, which is a zero-dimensional $E(T,h)$-scheme. We write $\mathrm{Sh}_K(T,h)$ for $\mathrm{Sh}_{i^{-1}(K)}(T,h)$, while we still write $\mathrm{Sh}_K$ for $\mathrm{Sh}_K(G,X)$. We write $\mu$ for the Hodge cocharacter $\mu_h \in X_*(T)$ associated with $h$. The reflex field of the Shimura datum $(T,h)$ is by definition the field of definition of $\mu$, and we shall denote it by $E_\mu \subset \mathbb{C}$ in accordance with the notation in §4.3.13.

We have $E \subset E_\mu$. Let $K^p \in \mathscr{K}^p$. The morphism $i : (T,h) \to (G,X)$ between Shimura data induces an $E_\mu$-scheme morphism

$$\mathrm{Sh}_{K_pK^p}(T,h) \longrightarrow \mathrm{Sh}_{K_pK^p} \times_{\mathrm{Spec}\, E} \mathrm{Spec}\, E_\mu. \tag{5.7.1.1}$$

For each neat compact open subgroup $U \subset T(\mathbb{A}_f)$ we have the finite abelian extension $E_{\mu,U}/E_\mu$ as defined in §4.3.13. By the explicit description in §1.5.3 of the Shimura varieties associated with $(T,h)$, we know that all geometric connected components of the $E_\mu$-scheme $\mathrm{Sh}_{K_pK^p}(T,h)$ have the same field of definition $E_{\mu,i^{-1}(K_pK^p)}$. To simplify notation, we write $E_{\mathfrak{s},K^p}$ for $E_{\mu,i^{-1}(K_pK^p)}$. The restriction of (5.7.1.1) to the neutral geometric connected component, namely the one corresponding to the neutral $\mathbb{C}$-point
$$1 \in \mathrm{Sh}_{K_pK^p}(T,h)(\mathbb{C}) = T(\mathbb{Q})\backslash T(\mathbb{A}_f)/i^{-1}(K_pK^p),$$
gives rise to an $E$-scheme morphism
$$\widetilde{x}_{\mathfrak{s},K^p} : \mathrm{Spec}\, E_{\mathfrak{s},K^p} \longrightarrow \mathrm{Sh}_{K_pK^p}\,.$$

Let $F_{\mathfrak{s},K^p}$ be the topological closure of $E_{\mathfrak{s},K^p}$ inside $\overline{\mathbb{Q}}_p$ (with respect to the fixed embedding $E_{\mathfrak{s},K^p} \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$). We thus have a tower of field extensions $(F_{\mathfrak{s},K^p})_{K^p\in\mathscr{K}^p}$, and we let $F_{\mathfrak{s}}$ be the union of these fields. Note that for every place $w$ of $E_\mu$ above $p$, the kernel of the map
$$\mathcal{O}_{E_\mu,w}^\times \to E_\mu^\times\backslash\mathbb{A}_{E_\mu}^\times \xrightarrow{(4.3.13.3)} T(\mathbb{Q})\backslash T(\mathbb{A}_f) \to T(\mathbb{Q})\backslash T(\mathbb{A}_f)/i^{-1}(K_pK^p)$$

is independent of $K^p$, which follows easily from the discreteness of $T(\mathbb{Q})$ in $T(\mathbb{A}_f)$ (by Lemma 1.5.5 and Lemma 5.1.2 (iii)) and the neatness of $i^{-1}(K^p)$. This implies that the transition maps in the tower $(F_{\mathfrak{s},K^p})_{K^p\in\mathscr{K}^p}$ are unramified field extensions. In particular, $\mathcal{O}_{F_\mathfrak{s}}$ is a regular local ring.

The morphisms $\widetilde{x}_{\mathfrak{s},K^p}$ are compatible when $K^p$ varies in $\mathscr{K}^p$, so they give rise to a morphism of $E_\mathfrak{p}$-schemes

$$\operatorname{Spec} F_\mathfrak{s} = \varprojlim_{K^p\in\mathscr{K}^p} \operatorname{Spec} F_{\mathfrak{s},K^p} \longrightarrow \mathscr{S}_{K_p,E_\mathfrak{p}}. \tag{5.7.1.2}$$

Since $\mathcal{O}_{F_\mathfrak{s}}$ is a regular local ring, the extension property of $\mathscr{S}_{K_p}$ implies that (5.7.1.2) extends to a unique $\mathcal{O}_{E,\mathfrak{p}}$-morphism $\operatorname{Spec}\mathcal{O}_{F_\mathfrak{s}} \to \mathscr{S}_{K_p}$. Passing to the special fiber we obtain a point

$$x_\mathfrak{s} \in \mathscr{S}_{K_p}(\overline{\mathbb{F}}_p).$$

**5.7.2.** Keep the setting and notation of §5.7.1. We write $K_1$ for $K_pK_1^p$ as in §5.1.3. Fix $\mathfrak{s}\in\mathcal{SPD}(G,X)$. To simplify notation, we write $F$ for $F_{\mathfrak{s},K_1^p}$, and write $\widetilde{x}\in\operatorname{Sh}_{K_1}(F)$ for the point induced by $\widetilde{x}_{\mathfrak{s},K_1^p}$. We also simply write $x$ for $x_\mathfrak{s}$. By construction, the image of $x$ in $\mathscr{S}_{K_1}$ is the specialization of $\widetilde{x}$.

We write $\mathcal{V}_{B,\mathbb{Q}}(\widetilde{x})$ for the stalk of $\mathcal{V}_{B,\mathbb{Q}}$ at $\widetilde{x}$, viewed as a point in $\operatorname{Sh}_{K_1}(\mathbb{C})$. Thus $\mathcal{V}_{B,\mathbb{Q}}(\widetilde{x}) \cong \mathbf{H}^1_B(\mathcal{A}_{\widetilde{x}}(\mathbb{C}),\mathbb{Q})$. For each $\alpha\in\boldsymbol{\alpha}$, we write $s_{\alpha,B,\mathbb{Q},\widetilde{x}}$ for the tensor over $\mathcal{V}_{B,\mathbb{Q}}(\widetilde{x})$ induced by the tensor $s_{\alpha,B,\mathbb{Q}}$ over $\mathcal{V}_{B,\mathbb{Q}}$ (see §5.1.5).

Since $\widetilde{x}$ comes from the neutral point $1\in\operatorname{Sh}_{K_1}(T,h)(\mathbb{C})$, there is a canonical $\mathbb{Q}$-linear isomorphism

$$\operatorname{triv}_\mathfrak{s} : V^*_\mathbb{Q} \xrightarrow{\sim} \mathcal{V}_{B,\mathbb{Q}}(\widetilde{x}).$$

This satisfies the following properties:

(i) $\operatorname{triv}_\mathfrak{s}$ takes $s_\alpha$ to $s_{\alpha,B,\mathbb{Q},\widetilde{x}}$ for each $\alpha\in\boldsymbol{\alpha}$.
(ii) $\operatorname{triv}_\mathfrak{s}$ restricts to a $\mathbb{Z}_{(p)}$-module isomorphism $V^*_{\mathbb{Z}_{(p)}} \xrightarrow{\sim} \mathbf{H}^1_B(\mathcal{A}_{\widetilde{x}}(\mathbb{C}),\mathbb{Z}_{(p)})$.
(iii) We view $\mathcal{V}_{B,\mathbb{Q}}(\widetilde{x})$ as a faithful representation of $T$ via $\operatorname{triv}_\mathfrak{s}$ and the representation $T\xrightarrow{i} G\to\operatorname{GL}(V^*_\mathbb{Q})$. Then the action of the Mumford–Tate group of $\mathcal{A}_{\widetilde{x}}$ on $\mathcal{V}_{B,\mathbb{Q}}(\widetilde{x})$ is via an embedding into $T$. The Hodge structure on $\mathcal{V}_{B,\mathbb{Q}}(\widetilde{x})$ is given by $h:\mathbb{S}\to T_\mathbb{R}$.

Using the comparison isomorphisms, we obtain from $\operatorname{triv}_\mathfrak{s}$ canonical isomorphisms

$$\operatorname{triv}_{\mathfrak{s},p} : V^*_{\mathbb{Z}_p} \xrightarrow{\sim} \mathcal{V}_p(\widetilde{x}),$$
$$\operatorname{triv}_{\mathfrak{s},\mathbb{A}_f^p} : V^*_{\mathbb{A}_f^p} \xrightarrow{\sim} \mathcal{V}^p(x_\mathfrak{s}).$$

We use these isomorphisms to define a $T_{\mathbb{Q}_p}$-representation on $\mathcal{V}_p(\widetilde{x})[1/p]$ and a $T_{\mathbb{A}_f^p}$-representation on $\mathcal{V}^p(x)$. The isomorphism $\operatorname{triv}_{\mathfrak{s},p}$ induces an isomorphism

$$\mathcal{G} \xrightarrow{\sim} \mathcal{G}_{\widetilde{x}} \tag{5.7.2.1}$$

which lies in the canonical $\mathcal{G}(\mathbb{Z}_p)$-conjugacy class of such isomorphisms as in Lemma 5.1.8. (Here $\mathcal{G}_{\widetilde{x}}$ is defined as in §5.1.7.) The isomorphism $\operatorname{triv}_{\mathfrak{s},\mathbb{A}_f^p}$ coincides with the stalk at $x$ of the canonical isomorphism in Lemma 5.1.9.

Since the Mumford–Tate group of $\mathcal{A}_{\widetilde{x}}$ is contained in $T$, we know that $\mathcal{A}_{\widetilde{x}}$ has complex multiplication by some CM field $H$, and that the action of $H^\times$ on $\mathcal{V}_{B,\mathbb{Q}}(\widetilde{x})$ induces embeddings of $\mathbb{Q}$-algebraic groups

$$T \hookrightarrow \operatorname{Res}_{H/\mathbb{Q}}\mathbb{G}_m \hookrightarrow \operatorname{GL}(\mathcal{V}_{B,\mathbb{Q}}(\widetilde{x})).$$

In particular, we have a canonical embedding of $T$ into the $\mathbb{Q}$-algebraic group of self-isogenies of $\mathcal{A}_x$. This embedding factors through $I_x$, since the action of $T$ on $\mathcal{V}_{B,\mathbb{Q}}(\widetilde{x})$ fixes $s_{\alpha,B,\mathbb{Q},\widetilde{x}}$ for all $\alpha \in \boldsymbol{\alpha}$. Thus we have a canonical embedding

$$T \hookrightarrow I_x. \tag{5.7.2.2}$$

**5.7.3.** Keep the setting and notation of §5.7.1 and §5.7.2.

Analogous to the functor (5.1.4.1), we have a faithful exact $\otimes$-functor

$$\mathbb{L}' : \mathrm{Rep}_{\mathbb{Q}_p} T \longrightarrow \mathrm{Lisse}_{\mathbb{Q}_p}(\mathrm{Sh}_{K_1}(T,h)).$$

Analogous to (5.1.5.2), we have a canonical isomorphism between the pull-back to $\mathrm{Sh}_{K_1}(T,h)_E$ of $\mathcal{V}_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and (the pull-back of) $\mathbb{L}'(V^*_{\mathbb{Q}_p})$. In particular, we have a canonical $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$-equivariant $\mathbb{Q}_p$-linear isomorphism

$$\mathcal{V}_p(\widetilde{x})[1/p] \cong \mathbb{L}'(V^*_{\mathbb{Q}_p})(\widetilde{x}), \tag{5.7.3.1}$$

where the right hand side denotes the stalk of $\mathbb{L}'(V^*_{\mathbb{Q}_p})$ at $\widetilde{x}$ viewed as a $\overline{\mathbb{Q}}_p$-point of $\mathrm{Sh}_{K_1}(T,h)$. Using (5.7.3.1), for each $T_{\mathbb{Q}_p}$-invariant tensor $r_\beta \in (V^*_{\mathbb{Q}_p})^{\otimes}$, we obtain a tensor $r_{\beta,p,\widetilde{x}}$ over $\mathcal{V}_p(\widetilde{x})[1/p]$ induced by the tensor $\mathbb{L}'(r_\beta)$ over $\mathbb{L}'(V^*_{\mathbb{Q}_p})$. It is not hard to see that the isomorphism

$$\mathrm{triv}_{\mathfrak{s},p}[1/p] : V^*_{\mathbb{Q}_p} \xrightarrow{\sim} \mathcal{V}_p(\widetilde{x})[1/p]$$

takes each $r_\beta$ to $r_{\beta,p,\widetilde{x}}$. It follows that the image of the embedding

$$T_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}(\mathcal{V}_p(\widetilde{x})[1/p])$$

coincides with the stabilizer of all $r_{\beta,p,\widetilde{x}}$. In particular, the $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$-action on $\mathcal{V}_p(\widetilde{x})[1/p]$ is via a homomorphism

$$\rho_{(T,h)} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \longrightarrow T(\mathbb{Q}_p).$$

We now give an explicit description of $\rho_{(T,h)}$. Let $\mu = \mu_h \in X_*(T)$, and let $U = i^{-1}(K_1) \subset T(\mathbb{A}_f^p)$. As in §4.3.13, we have the field $E_{\mu,U,v}$, and the homomorphism

$$r(\mu)_{U,p,\mathrm{loc}} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/E_{\mu,U,v}) \longrightarrow T(\mathbb{Q}_p).$$

Note that $E_{\mu,U,v} \subset F$. The homomorphism $\rho_{(T,h)}$ is equal to the restriction of $r(\mu)_{U,p,\mathrm{loc}}$ to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$. This fact is just another way to look at the Shimura–Taniyama reciprocity law, and it easily follows from the explicit description of the tower of Shimura varieties attached to $(T,h)$ as in §1.5.3. Also cf. [Pin92a, §(5.5)].

Note that the hypothesis on $U$ in Proposition 4.3.14 is satisfied in the current situation. By that proposition and by the above discussion, we know that $\rho_{(T,h)}$ is crystalline, and that the element $[\rho_{(T,h)}] \in \mathrm{Crys}_{T_{\mathbb{Q}_p}}$ is equal to $\mathscr{M}_T^{-1}(-\mu_h) \in \mathrm{Mot}_T$.

**5.7.4.** Keep the setting and notation of §§5.7.1–5.7.3. From $[\rho_{(T,h)}] \in \mathrm{Crys}_{T_{\mathbb{Q}_p}}$, we obtain an element $[i \circ \rho_{(T,h)}] \in \mathrm{Crys}_{G_{\mathbb{Q}_p}}$. Denote by $\mathcal{T}^\circ$ the connected Néron model of $T_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$. Applying §4.5.1 to $[\rho] = [i \circ \rho_{(T,h)}]$ and $[\rho_T] = [\rho_{(T,h)}]$, we obtain $\Upsilon_{[\rho(T,h)]} \in \mathcal{T}^\circ\text{-}\mathrm{Isoc}^\circ_{\mathbb{Q}_p^{\mathrm{ur}}}$ and $\Upsilon_{[i\circ\rho_{(T,h)}]} \in \mathcal{G}\text{-}\mathrm{Isoc}^\circ_{\mathbb{Q}_p^{\mathrm{ur}}}$. To simplify notation we denote them by $\Upsilon_{(T,h)}$ and $\Upsilon_{\mathfrak{s}}$ respectively. As in §4.5.1, we have a natural injection from the $T(\mathbb{Q}_p^{\mathrm{ur}})$-torsor $Y(\Upsilon_{(T,h)})$ to the $G(\mathbb{Q}_p^{\mathrm{ur}})$-torsor $Y(\Upsilon_{\mathfrak{s}})$.

As before we write $x$ for the point $x_{\mathfrak{s}} \in \mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$. We have a canonical isomorphism $\Upsilon_{\mathfrak{s}} \cong \Upsilon_x$, which we now explain.

Letting $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ act on $V^*_{\mathbb{Q}_p}$ via

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \xrightarrow{\rho_{(T,h)}} T(\mathbb{Q}_p) \xrightarrow{i} G(\mathbb{Q}_p) \to \mathrm{GL}(V^*_{\mathbb{Q}_p}),$$

we have a canonical $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$-equivariant isomorphism $V^*_{\mathbb{Q}_p} \cong \mathcal{V}_p(\widetilde{x})[1/p]$ induced by $\mathrm{triv}_{\mathfrak{s},p}$. Since this isomorphism takes $V^*_{\mathbb{Z}_p}$ to $\mathcal{V}_p(\widetilde{x})$, it induces an isomorphism of integral $F$-isocrystals

$$\Upsilon_{\mathfrak{s}}(V^*_{\mathbb{Z}_p}) \xrightarrow{\sim} M_{\mathrm{cris}}(\mathcal{V}_p(\widetilde{x})) \otimes_{\mathcal{O}_{F_0}} \mathbb{Z}_p^{\mathrm{ur}}, \tag{5.7.4.1}$$

where $F_0$ denotes the maximal unramified extension of $\mathbb{Q}_p$ in $F$ as usual. The isomorphism (5.7.4.1) takes the tensor $\Upsilon_{\mathfrak{s}}(s_\alpha)$ over the left hand side to the tensor $M_{\mathrm{cris}}(s_{\alpha,p,\widetilde{x}})$ over the right hand side, since $\mathrm{triv}_{\mathfrak{s},p}$ takes $s_\alpha$ to $s_{\alpha,p,\widetilde{x}}$. Now the right hand side of (5.7.4.1) is identified with $\mathcal{V}_0(x)$ via the integral comparison isomorphism (5.2.2.1), and under this identification $s_{\alpha,p,\widetilde{x}}$ is identified with $s_{\alpha,0,x}$ by the discussion in §5.2.2. Hence we obtain a canonical isomorphism of integral $F$-isocrystals

$$\Upsilon_{\mathfrak{s}}(V^*_{\mathbb{Z}_p}) \xrightarrow{\sim} \mathcal{V}_0(x), \tag{5.7.4.2}$$

which takes $\Upsilon_{\mathfrak{s}}(s_\alpha)$ to $s_{\alpha,0,x}$ for all $\alpha \in \boldsymbol{\alpha}$. It then follows from Lemma 5.2.4 that there is a unique isomorphism

$$\Upsilon_{\mathfrak{s}} \xrightarrow{\sim} \Upsilon_x \tag{5.7.4.3}$$

in $\mathcal{G}\text{-}\mathrm{Isoc}^\circ_{\mathbb{Q}_p^{\mathrm{ur}}}$ taking the isomorphism (5.7.4.2) to the isomorphism $\iota_x : \Upsilon_x(V^*_{\mathbb{Z}_p}) \xrightarrow{\sim} \mathcal{V}_0(x)$.

Via (5.7.4.3), we have a canonical isomorphism $Y(\Upsilon_{\mathfrak{s}}) \cong Y(\Upsilon_x)$. Composing this with the canonical injection $Y(\Upsilon_{(T,h)}) \hookrightarrow Y(\Upsilon_{\mathfrak{s}})$, we obtain an injection

$$Y(\Upsilon_{(T,h)}) \hookrightarrow Y(\Upsilon_x). \tag{5.7.4.4}$$

Recall from §5.6.2 that $Y(x) = Y_p(x) \times Y^p(x)$, where $Y_p(x) = Y(\Upsilon_x)$, and $Y^p(x)$ is canonically identified with $G(\mathbb{A}_f^p)$. We write 1 for the canonical base point of $Y^p(x)$.

Recall from §4.2.5 that inside the $T(\mathbb{Q}_p^{\mathrm{ur}})$-torsor $Y(\Upsilon_{(T,h)})$ we have the subset of integral points $Y(\Upsilon_{(T,h)})^\circ$, which is a $\mathcal{T}^\circ(\mathbb{Z}_p^{\mathrm{ur}})$-torsor.

**Definition 5.7.5.** Let $\mathfrak{s} \in \mathcal{SPD}(G,X)$, and let $x = x_{\mathfrak{s}} \in \mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$. Let $y_p \in Y_p(x)$ be an element of the image of $Y(\Upsilon_{(T,h)})^\circ$ under (5.7.4.4), and let $y = (y_p, 1) \in Y(x)$. We call $y$ an *integral special point*[33] associated with $\mathfrak{s}$.

In the next proposition we prove fundamental properties of integral special points.

**Proposition 5.7.6.** *Let $x = x_{\mathfrak{s}}$ and let $y = (y_p, 1) \in Y(x)$ be an integral special point as in Definition 5.7.5. The following statements hold.*

*(i) Let $\delta_{y_p}$ be the image of $y_p$ under the map $Y(\Upsilon_{(T,h)}) \to T(\mathbb{Q}_p^{\mathrm{ur}}), z \mapsto \delta_z$ as in §4.2.5. Then $\delta_{y_p}$ lies in $T(\mathbb{Q}_p^{\mathrm{ur}})^{\mathrm{mot}}$, and the $\overset{\circ}{\sim}$-equivalence class of $\delta_{y_p}$ corresponds to the image of $-\mu_h$ in $X_*(T)_{\Gamma_{p,0}}$ under the bijection (4.3.9.1).*

[33] The images of the integral special points under $Y(x) \to \mathscr{I}^*$, where $\mathscr{I}$ is the isogeny class of $x$, are called "integral special points" in the Introduction.

(ii) *By (i), the element $\delta_{y_p}$ satisfies the assumptions on $\delta_T$ in §5.3.9. By the construction in §5.3.9, we obtain an element $\mathfrak{k}(\mathfrak{s}, \delta_{y_p}) \in \mathfrak{KT}^{\mathrm{str}}$. The image $\mathfrak{k}(y)$ of $y$ under the map $Y(x) \to \mathfrak{KT}^{\mathrm{str}}/\equiv$ as in (5.6.2.1) is equal to the image of $\mathfrak{k}(\mathfrak{s}, \delta_{y_p})$.*

(iii) *Let $\mathscr{I}$ be the isogeny class of $x$. The image of $y$ under the composite map*

$$Y(x) \to I_x(\mathbb{Q})\backslash Y(x)/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}) \cong \mathscr{I}^* \xrightarrow{c^*_{\mathscr{I}}} \pi^*(G, X)$$

*is equal to the image of $i \circ h \in X$ in $\pi^*(G, X)$. (See Proposition 5.6.10 for $c^*_{\mathscr{I}}$.)*

*Proof.* We have seen in §5.7.3 that $[\rho_{(T,h)}] = \mathscr{M}_T^{-1}(-\mu_h) \in \mathrm{Mot}_T$. Statement (i) follows from this fact and Corollary 4.4.12.

For (ii), tracing the definitions we see that the $\delta$-component of $\mathfrak{k}(y)$ is $i(\delta_{y_p})$. By the Shimura–Taniyama reciprocity law, for sufficiently divisible $n$ the geometric $p^n$-Frobenius in $I_x(\mathbb{Q})$ is given by an element of $T(\mathbb{Q})$, if we view $T$ as a $\mathbb{Q}$-subgroup of $I_x$ as in (5.7.2.2). This element has to be $\gamma_{0,T,n}$ introduced in §5.3.9. (Recall that $\gamma_{0,T,n}$ depends only on $(T, h)$ and $n$.) Now the composition

$$T_{\mathbb{A}_f^p} \xrightarrow{(5.7.2.2)} I_{x,\mathbb{A}_f^p} \to \mathrm{GL}(\mathcal{V}^p(x)) \xrightarrow{\sim} \mathrm{GL}(V^*_{\mathbb{A}_f^p}),$$

where the last isomorphism is induced by $\mathrm{triv}_{\mathfrak{s},\mathbb{A}_f^p} : V^*_{\mathbb{A}_f^p} \xrightarrow{\sim} \mathcal{V}^p(x)$, is equal to the base change to $\mathbb{A}_f^p$ of $i : T \to G$. Hence the $\gamma$-component of $\mathfrak{k}(y)$ is represented by $i(\gamma_{0,T,n})$ at level $n$. Statement (ii) follows.

For (iii), let $y'_p$ be an element of $Y(\Upsilon_x)^\circ \subset Y(\Upsilon_x) = Y_p(x)$, and set $y' := (y'_p, 1) \in Y_p(x) \times Y^p(x) = Y(x)$. Then the image of $y'$ in $\mathscr{I}^*$ is equal to the image of $x$ under the canonical injection $\mathscr{I} \hookrightarrow \mathscr{I}^*$. See §5.6.4 for details. By the construction of $x = x_{\mathfrak{s}}$ in §5.7.1, for each $K^p \in \mathscr{K}^p$ the image of $x$ in $\mathscr{S}_{K_pK^p}(\overline{\mathbb{F}}_p)$ is the reduction of a point of $\mathrm{Sh}_{K_pK^p}$ whose induced $\mathbb{C}$-point is the image of $(i \circ h, 1) \in X \times G(\mathbb{A}_f)$. Hence $x$ and $i \circ h \in X$ have the same image in $\pi(G, X)$ (cf. §5.5.4), and *a fortiori* they have the same image in $\pi^*(G, X)$. Therefore we only need find $y'$ as above such that $y$ and $y'$ have the same image in $\pi^*(G, X)$. By the second statement in Proposition 4.5.2, we can find $y'_p$ such that it lies in the $G_{\mathrm{der}}(\mathbb{Q}_p^{\mathrm{ur}})$-orbit of $y_p$. But then $y$ and $y'$ have the same image in $\pi^*(G, X)$, since the map $Y(x) \to \pi^*(G, X)$ in question is $G(\mathbb{A}_f^*)$-equivariant, and since the $G(\mathbb{A}_f^*)$-action on $\pi^*(G, X)$ restricts to the trivial action of $G_{\mathrm{der}}(\mathbb{A}_f^*)$. □

**5.7.7.** Let $x \in \mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$, and let $T$ be a maximal torus in the $\mathbb{Q}$-reductive group $I_x$. For each $\mu \in X_*(T)$, we define $\bar{\mu}^{T_{\mathbb{Q}_p}} \in X_*(T)$ as in §3.3.5 (with respect to the $\mathbb{Q}_p$-torus $T_{\mathbb{Q}_p}$).

Let $y \in Y(x)$. Then we have a $\mathbb{Q}_p$-isomorphism $\iota_{y,p} : I_{x,\mathbb{Q}_p} \xrightarrow{\sim} I_p$, where $I_p \subset \mathrm{Res}_{\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p} G$ is the reductive group over $\mathbb{Q}_p$ associated with $\mathfrak{k}(y) \in \mathfrak{KT}^{\mathrm{str}}$. Let $\delta_y$ be the $\delta$-component of $\mathfrak{k}(y)$, and let $\nu_{\delta_y}$ be the Newton cocharacter of $\delta_y \in G(\mathbb{Q}_p^{\mathrm{ur}})$ Then $\nu_{\delta_y}$ can be viewed as a central fractional cocharacter of $I_{x,\mathbb{Q}_p}$ via $\iota_{y,p}$. In particular we can view $\nu_{\delta_y}$ as an element of $X_*(T) \otimes \mathbb{Q}$. We say that a cocharacter $\mu \in X_*(T)$ is *$x$-admissible*, if the composition

$$\mathbb{G}_{m,\overline{\mathbb{Q}}_p} \xrightarrow{\mu} T_{\overline{\mathbb{Q}}_p} \hookrightarrow I_{\phi,\overline{\mathbb{Q}}_p} \xrightarrow{\iota_{y,p}} I_{p,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p} \tag{5.7.7.1}$$

lies in $\mathfrak{u}_X(\overline{\mathbb{Q}}_p)$, and if $\bar{\mu}^{T_{\mathbb{Q}_p}} = \nu_{\delta_y}$ as elements of $X_*(T) \otimes \mathbb{Q}$. Here the map $I_{p,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ is induced by the map $(\mathrm{Res}_{\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p} G)_{\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ induced by the inclusion $\mathbb{Q}_p^{\mathrm{ur}} \hookrightarrow \overline{\mathbb{Q}}_p$. It is straightforward to check that the definition of $x$-admissible cocharacters is independent of the choice of $y$. (Note the analogy between this definition and the definition in §3.3.8.) The following theorem is the geometric analogue of Theorem 3.3.9.

**Theorem 5.7.8.** *Let $x \in \mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$, and let $T$ be a maximal torus in $I_x$ defined over $\mathbb{Q}$. The following statements hold.*

*(i) There exists $\mu \in X_*(T)$ that is $x$-admissible.*

*(ii) Let $\mu \in X_*(T)$ be an $x$-admissible cocharacter. Then there exists a special point datum of the form $\mathfrak{s} = (T, i, h)$ satisfying the following conditions:*

*(a) $\mu = \mu_h$.*

*(b) The points $x$ and $x_{\mathfrak{s}}$ lie in the same isogeny class. Moreover, there exists $g \in I_{x,x_{\mathfrak{s}}}(\mathbb{Q})$ such that the isomorphism $g_* : I_x \xrightarrow{\sim} I_{x_{\mathfrak{s}}}$ induced by $g$ has the property that the composition*

$$T \hookrightarrow I_x \xrightarrow{g_*} I_{x_{\mathfrak{s}}}$$

*is equal to the canonical embedding $T \hookrightarrow I_{x_{\mathfrak{s}}}$ as in (5.7.2.2).*

*Proof.* Part (i) is proved in [Kis17, Lem. 2.2.2], and part (ii) is proved in [Kis17, Cor. 2.2.5]. □

## 5.8. Uniformization on the gerb side.

**5.8.1.** Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be an admissible morphism (see Definition 2.4.2). Set

$$\begin{aligned} Y_p(\phi) &:= \mathcal{UR}(\phi(p) \circ \zeta_p), \\ Y^p(\phi) &:= X^p(\phi), \\ Y(\phi) &:= Y_p(\phi) \times Y^p(\phi). \end{aligned}$$

See Definition 2.2.3 and §2.4.7 for the notations. Thus $Y_p(\phi)$ is a right $G(\mathbb{Q}_p^{\mathrm{ur}})$-torsor, $Y^p(\phi)$ is a right $G(\mathbb{A}_f^p)$-torsor, and $Y(\phi)$ is a right $G(\mathbb{A}_f^*)$-torsor.

The construction in [Kis17, §4.5.1] gives rise to a map

$$Y(\phi) \longrightarrow \mathfrak{KT}. \tag{5.8.1.1}$$

In fact, the construction is more precise, as we now explain. Let $y = (y_p, y^p) \in Y(\phi)$. Write $\phi(p)_y$ for the morphism $\mathrm{Int}(y_p)^{-1} \circ \phi(p) : \mathfrak{Q}(p) \to \mathfrak{G}_G(p)$. The $\overline{\mathbb{Q}}_p$-subgroup $\mathrm{im}(\phi(p)_y^{\Delta}) \subset G_{\overline{\mathbb{Q}}_p}$ is defined over $\mathbb{Q}_p^{\mathrm{ur}}$. Let

$$\mathscr{U}_{\phi,y} = \mathscr{U}_y := \mathrm{im}(\phi(p)_y^{\Delta})(\mathbb{Q}_p^{\mathrm{ur}}) \cap \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}),$$

where the intersection is inside $G(\mathbb{Q}_p^{\mathrm{ur}})$. The construction in *loc. cit.* attaches to $y$ a $\mathscr{U}_y$-orbit in $\mathfrak{KT}^{\mathrm{str}}/\equiv$, which we denote by

$$[\mathfrak{k}(y)] \subset \mathfrak{KT}^{\mathrm{str}}/\equiv.$$

(Here $\mathscr{U}_y$ acts on $\mathfrak{KT}^{\mathrm{str}}/\equiv$ by the embedding $\mathscr{U}_y \hookrightarrow \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$ and the $\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$-action in Definition 5.3.6.) If we just remember the $\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$-orbit in $\mathfrak{KT}^{\mathrm{str}}/\equiv$ induced by $[\mathfrak{k}(y)]$, then we obtain the map (5.8.1.1). (However, we caution the reader that there is no well-defined map $Y(\phi) \to \mathfrak{KT}^{\mathrm{str}}/\equiv$, which is unlike the situation in §5.6.2.)

Let $\mathfrak{k} \in [\mathfrak{k}(y)]$ and let $\delta_y$ be the $\delta$-component of $\mathfrak{k}$. Thus $\delta_y$ is canonical up to $\sigma$-conjugation by $\mathscr{U}_y$. It follows easily from the construction in [Kis17, §4.5.1] that $\delta_y$ satisfies the following conditions. (Note that both the conditions are invariant when we $\sigma$-conjugate $\delta_y$ by $\mathscr{U}_y$.)

(i) Define $b_y \in G(\mathbb{Q}_p^{\mathrm{ur}})$ such that the morphism $\mathfrak{D} \to \mathfrak{G}_G^{\mathrm{ur}}$ underlying $\phi(p)_y \circ \zeta_p : \mathfrak{G}_p \to \mathfrak{G}_G(p)$ sends $d_\sigma$ to $b_y \rtimes \sigma$, cf. Definition 2.2.5. Given any neighborhood $\mathscr{O}$ of 1 in $\mathrm{im}(\phi(p)_y^{\Delta})(\breve{\mathbb{Q}}_p)$ (for the $p$-adic topology), there exists $u \in \mathscr{U}_y$ such that
$$u\delta_y\sigma(u)^{-1} \in \mathscr{O} \cdot_\sigma b_y.$$
Here the right hand side denotes
$$\left\{ab_y\sigma(a)^{-1} \in G(\breve{\mathbb{Q}}_p) \mid a \in \mathscr{O}\right\},$$
where each element of $\mathscr{O}$ is viewed as an element of $G(\breve{\mathbb{Q}}_p)$. In particular, $b_y$ and $\delta_y$ are $\sigma$-conjugate by an element of $\mathcal{G}(\breve{\mathbb{Z}}_p) \cap \mathrm{im}(\phi(p)_y^{\Delta})(\breve{\mathbb{Q}}_p)$.

(ii) The canonical homomorphism $I_{\phi(p)_y, \overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ is defined over $\mathbb{Q}_p^{\mathrm{ur}}$, and induces an isomorphism
$$I_{\phi(p)_y}(R) = \left\{g \in G(\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} R) \mid g\delta_y\sigma(g)^{-1} = \delta_y\right\}$$
for each $\mathbb{Q}_p$-algebra $R$. (Here we view the left hand side as a subgroup of $I_{\phi(p)_y}(\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} R)$, which maps to $G(\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} R)$ via the $\mathbb{Q}_p^{\mathrm{ur}}$-homomorphism $I_{\phi(p)_y, \mathbb{Q}_p^{\mathrm{ur}}} \to G_{\mathbb{Q}_p^{\mathrm{ur}}}$.)

By property (ii) above, we know that although $\delta_y$ is only well defined up to $\sigma$-conjugation by $\mathscr{U}_y$, the $\sigma$-centralizer of $\delta_y$ in $G$ is unambiguous as a subfunctor of $\mathrm{Res}_{\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p} G$.

Let $y$ be as above. Let $g \in G(\mathbb{A}_f^*)$. Then we have $y \cdot g \in Y(\phi)$. Recall from §5.3.5 that $G(\mathbb{A}_f^*)$ acts on $\mathfrak{KT}^{\mathrm{str}}/{\equiv}$ on the right. By inspecting the construction in [Kis17, §4.5.1], we see that:

(iii) There exists $\mathfrak{k} \in [\mathfrak{k}(y)]$ such that $\mathfrak{k} \cdot g \in [\mathfrak{k}(y \cdot g)]$.

**5.8.2.** Let $\mathscr{J}$ be a conjugacy class of admissible morphisms $\mathfrak{Q} \to \mathfrak{G}_G$. We make $\mathscr{J}$ into a small category, where morphisms $\phi \to \phi'$ are given by elements $g \in G(\overline{\mathbb{Q}})$ such that $\phi' = \mathrm{Int}(g) \circ \phi$. The composition of morphisms $\phi \xrightarrow{g} \phi' \xrightarrow{h} \phi''$ is given by $\phi \xrightarrow{hg} \phi''$. Then $\mathscr{J}$ is a connected groupoid category. Each morphism $g : \phi \to \phi'$ in $\mathscr{J}$ induces a $G(\mathbb{A}_f^*)$-map $Y(g) : Y(\phi) \to Y(\phi')$ given by the left multiplication by $g$. This makes $Y$ a right $G(\mathbb{A}_f^*)$-torsor over $\mathscr{J}$ in the sense of Definition 5.6.1. As in that definition, we obtain a right $G(\mathbb{A}_f^*)$-set $\bar{Y}(\mathscr{J})$, together with canonical isomorphisms
$$I_\phi(\mathbb{Q})\backslash Y(\phi) \xrightarrow{\sim} \bar{Y}(\mathscr{J})$$
for all $\phi \in \mathscr{J}$.

Let $\phi \in \mathscr{J}$ and $y \in Y(\phi)$. It is easy to see that the subgroup $\mathscr{U}_{\phi,y} \subset \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$ depends only on the image $\bar{y}$ of $y$ in $\bar{Y}(\mathscr{J})$, not on $\phi$ and $y$. We therefore denote it also by $\mathscr{U}_{\bar{y}}$. Moreover, the $\mathscr{U}_{\phi,y}$-orbit $[\mathfrak{k}(y)]$ in $\mathfrak{KT}^{\mathrm{str}}/{\equiv}$ attached to $y$ as in §5.8.1 depends only on $\bar{y}$. We thus have a canonical $\mathscr{U}_{\bar{y}}$-orbit in $\mathfrak{KT}^{\mathrm{str}}/{\equiv}$ attached to each $\bar{y} \in \bar{Y}(\mathscr{J})$, which we denote by $[\mathfrak{k}(\bar{y})]$.

The following discussion is completely analogous to §5.6.3. For $\phi \in \mathscr{J}$ and $y = (y_p, y^p) \in Y(\phi)$, the left $I_\phi(\mathbb{Q})$-action on the right $G(\mathbb{A}_f^*)$-torsor $Y(\phi)$ gives rise to a homomorphism

$$\iota_y : I_\phi(\mathbb{Q}) \longrightarrow G(\mathbb{A}_f^*) \tag{5.8.2.1}$$

defined by

$$j \cdot y = y \cdot \iota_y(j), \quad \forall j \in I_\phi(\mathbb{Q}).$$

Thus we have a map $\iota_{y,v} : I_\phi(\mathbb{Q}) \to G(\mathbb{Q}_v)$ for each prime $v \neq p$, and a map $\iota_{y,p} : I_\phi(\mathbb{Q}) \to G(\mathbb{Q}_p^{\mathrm{ur}})$. For $v \neq p$, clearly $\iota_{y,v}$ is induced by $\mathrm{Int}(y_v^{-1})$, where $y_v \in G(\overline{\mathbb{Q}}_v)$ is the image of $y^p$ under $G(\bar{\mathbb{A}}_f^p) \to G(\overline{\mathbb{Q}}_v)$. Similarly $\iota_{y,p}$ is induced by $\mathrm{Int}(y_p^{-1})$. Let $\dot{\mathfrak{k}}(y) \in \mathfrak{KT}^{\mathrm{str}}$ be an arbitrary element whose image in $\mathfrak{KT}^{\mathrm{str}}/\equiv$ belongs to $[\mathfrak{k}(y)]$. Let $(I_0, (I_v)_v, (\eta_v)_v)$ be the datum attached to $\dot{\mathfrak{k}}(y)$ as in §5.3.2. For each prime $v \neq p$, the map $\iota_{y,v}$ comes from an isomorphism of $\mathbb{Q}_v$-groups $\iota_{y,v} : I_{\phi,\mathbb{Q}_v} \xrightarrow{\sim} I_v$, which is still induced by $\mathrm{Int}(y_v^{-1})$. Also, the map $\iota_{y,p}$ comes from an isomorphism of $\mathbb{Q}_p$-groups $\iota_{y,p} : I_{\phi,\mathbb{Q}_p} \xrightarrow{\sim} I_p$. The isomorphism $\iota_{y,p}$ is induced by $\mathrm{Int}(y_p^{-1})$, in the sense that the following diagram commutes

$$\begin{tikzcd} I_{\phi,\overline{\mathbb{Q}}_p} \arrow[r, hook] \arrow[d, "\iota_{y,p}"] & G_{\overline{\mathbb{Q}}_p} \arrow[r, "\mathrm{Int}(y_p^{-1})"] & G_{\overline{\mathbb{Q}}_p} \\ I_{p,\overline{\mathbb{Q}}_p} \arrow[rr, hook] & & (\mathrm{Res}_{\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p} G)_{\overline{\mathbb{Q}}_p} \arrow[u] \end{tikzcd} \tag{5.8.2.2}$$

Here the bottom arrow is the base change to $\overline{\mathbb{Q}}_p$ of the $\mathbb{Q}_p$-embedding $I_p \hookrightarrow \mathrm{Res}_{\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p} G$, and the vertical arrow on the right is given by the map $G(\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} R) \to G(R)$ induced by $\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} R \to R, a \otimes a' \mapsto aa'$ for all $\overline{\mathbb{Q}}_p$-algebras $R$, . (This description of $\iota_{y,p}$ is just a reformulation of property (ii) in §5.8.1.) Moreover, the isomorphisms $\iota_{y,v}$ for all primes $v$ can be extended to a refinement of $\dot{\mathfrak{k}}(y)$ of the form $(I_\phi, \iota_0, (\iota_{y,v})_v)$.

With the above notation, note that $\mathscr{U}_{\phi,y}$ is canonically identified with a subgroup of $Z_{I_p}(\mathbb{Q}_p^{\mathrm{ur}})$. If we change the choice of $\dot{\mathfrak{k}}(y)$, then both $I_p$ (as a subfunctor of $\mathrm{Res}_{\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p} G$) and the map $\iota_{y,p} : I_{\phi,\mathbb{Q}_p} \xrightarrow{\sim} I_p$ do not change. Thus for every prime $v$, the reductive group $I_v$ and the map $\iota_{y,v} : I_{\phi,\mathbb{Q}_v} \xrightarrow{\sim} I_v$ depend only on $y$, not on the choice of $\dot{\mathfrak{k}}(y)$.

The following lemma holds in our current setting of Hodge type.

**Lemma 5.8.3.** *Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be an admissible morphism, and let $\tau \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$. Let $K^p \subset G(\mathbb{A}_f^p)$ be a neat compact open subgroup. Then the action of $I_\phi(\mathbb{Q})_\tau := \mathrm{Int}(\tau)(I_\phi(\mathbb{Q})) \subset I_\phi(\mathbb{A}_f)$ on $X(\phi)/K^p$ is free. Moreover, the natural map*

$$I_\phi(\mathbb{Q})_\tau \backslash X(\phi) \longrightarrow S_\tau(\phi)$$

*is a bijection.*

*Proof.* Suppose $\gamma \in I_\phi(\mathbb{Q})_\tau$ has a fixed point in $X(\phi)/K^p$. By Lemma 3.7.2 (i), we have $\gamma \in Z_G(\mathbb{Q}) \cap K_p K^p$. By Lemma 1.5.7 and Lemma 5.1.2 (ii), we have $Z_G(\mathbb{Q}) \cap K_p K^p = \{1\}$. Hence $\gamma = 1$. This proves the first part.

By the definition of $S_\tau(\phi)$ (see §2.4.7), we have a natural surjection

$$I_\phi(\mathbb{Q})_\tau \backslash \left( \varprojlim_{K^p} X(\phi)/K^p \right) \longrightarrow S_\tau(\phi).$$

By the previous part this surjection is a bijection. Since $X^p(\phi)$ is a torsor under the locally profinite group $G(\mathbb{A}_f^p)$, the natural map $X(\phi) \to \varprojlim_{K^p} X(\phi)/K^p$ is a bijection. This proves the second part. $\square$

**5.8.4.** Let $\mathscr{J}$ be a conjugacy class of admissible morphisms $\mathfrak{Q} \to \mathfrak{G}_G$. Set

$$S^*(\mathscr{J}) = \bar{Y}(\mathscr{J})/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}).$$

For each $\phi \in \mathscr{J}$, by Lemma 5.8.3 we have

$$S(\phi) \cong I_\phi(\mathbb{Q})\backslash X(\phi) = I_\phi(\mathbb{Q})\backslash X_p(\phi) \times X^p(\phi).$$

In the future we shall view this as an equality. Recall from §2.2.7 and §2.4.1 that

$$X_p(\phi) = X_{-\mu_X}(\phi(p) \circ \zeta_p) = Y_{-\mu_X}(\phi(p) \circ \zeta_p)/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}),$$

where $Y_{-\mu_X}(\phi(p)\circ\zeta_p)$ is a subset of $\mathcal{UR}(\phi(p)\circ\zeta_p) = Y_p(\phi)$. Here $\mu_X$ is as in §2.4.1, and the subset $Y_{-\mu_X}(\phi(p) \circ \zeta_p) \subset Y_p(\phi)$ is independent of the choice of $\mu_X$. We have a natural injection

$$S(\phi) \hookrightarrow S^*(\mathscr{J}). \tag{5.8.4.1}$$

If $g : \phi \to \phi'$ is a morphism in $\mathscr{J}$, then the bijection $Y_p(g) : Y_p(\phi) \to Y_p(\phi')$ restricts to a bijection $Y_{-\mu_X}(\phi(p) \circ \zeta_p) \to Y_{-\mu_X}(\phi'(p) \circ \zeta_p)$. It follows that inside $\bar{Y}(\mathscr{J})$ we have a canonical subset of the form

$$\bar{Y}(\mathscr{J})^\natural \cong I_\phi(\mathbb{Q})\backslash Y_{-\mu_X}(\phi(p) \circ \zeta_p) \times Y^p(\phi),$$

which is independent of the choice of $\phi \in \mathscr{J}$. The image of the injection (5.8.4.1) is equal to the image of $\bar{Y}(\mathscr{J})^\natural$ under the projection $\bar{Y}(\mathscr{J}) \to S^*(\mathscr{J})$, namely $\bar{Y}(\mathscr{J})^\natural/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$. In particular, we have a canonical bijection

$$S(\phi) \cong \bar{Y}(\mathscr{J})^\natural/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}).$$

**5.8.5.** Recall the following constructions in [Kis17, §§3.6–3.7]. Associated with each admissible morphism $\phi : \mathfrak{Q} \longrightarrow \mathfrak{G}_G$, we have a $\pi(G)$-torsor $\pi(G,\phi)$, together with a $G(\mathbb{A}_f^p)$-equivariant map

$$\widetilde{c}_\phi : X(\phi) \longrightarrow \pi(G,\phi).$$

(Here $G(\mathbb{A}_f^p)$ acts on $\pi(G,\phi)$ via the natural surjection $G(\mathbb{A}_f^p) \to \pi(G)$ as in Lemma 5.5.2.) For each $\tau \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$, the map $c_\phi$ descends to a $G(\mathbb{A}_f^p)$-equivariant map

$$c_{\phi,\tau} : S_\tau(\phi) \longrightarrow \pi(G,\phi).$$

See [Kis17, Cor. 3.6.4, Lem. 3.7.4] for more details. It follows from the $G(\mathbb{A}_f^p)$-equivariance that the maps $\widetilde{c}_\phi$ and $c_{\phi,\tau}$ are surjective.

By [Kis17, Prop. 3.6.10], for each admissible morphism $\phi$ there is a canonical isomorphism of $\pi(G)$-torsors

$$\vartheta_\phi : \pi(G,\phi) \cong \pi(G,X). \tag{5.8.5.1}$$

In the following we shall use the above identification freely, sometimes omitting it from the notation.

**Lemma 5.8.6.** *Let $\mathscr{J}$ be a conjugacy class of admissible morphism $\mathfrak{Q} \to \mathfrak{G}_G$. Let $\phi \in \mathscr{J}$. The composite map*

$$\bar{Y}(\mathscr{J})^\natural \to \bar{Y}(\mathscr{J})^\natural/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}) \cong S(\phi) \xrightarrow{c_{\phi,1}} \pi(G,\phi) \xrightarrow{\vartheta_\phi} \pi(G,X) \tag{5.8.6.1}$$

*depends only on $\mathscr{J}$ and not on $\phi$.*

*Proof.* The proof is just by collecting various facts from [Kis17, §3.6, §3.7]. By [Kis17, Cor. 3.6.4], the $\pi(G)$-torsor $\pi(G,\phi)$ depends on $\phi$ only via the conjugacy class of $\phi$ (in fact, only via the conjugacy class of the composite morphism $\mathfrak{Q} \xrightarrow{\phi} \mathfrak{G}_G \to \mathfrak{G}_{G^{\mathrm{ad}}}$), up to canonical isomorphism. Also, by the characterization of $\vartheta_\phi$ in [Kis17, Prop. 3.6.10], $\vartheta_\phi$ depends on $\phi$ only via its conjugacy class. By the definition of $\widetilde{c}_\phi$ (see [Kis17, Lem. 3.7.4]), it is functorial in $\phi \in \mathscr{J}$. Namely, if $g : \phi \to \phi'$ is a morphism in $\mathscr{J}$, then we have a commutative diagram

$$\begin{tikzcd} X(\phi) \arrow[r, "X(g)"] \arrow[d, "\widetilde{c}_\phi"] & X(\phi') \arrow[d, "\widetilde{c}_{\phi'}"] \\ \pi(G,\phi) \arrow[r, "\cong"] & \pi(G,\phi') \end{tikzcd}$$

where the top map is the functorial map induced by $g$ and the bottom map is the canonical isomorphism mentioned above. The lemma follows from these facts. $\square$

**Lemma 5.8.7.** *Let $\mathscr{J}$ be a conjugacy class of admissible morphisms $\mathfrak{Q} \to \mathfrak{G}_G$. Let $y \in \bar{Y}(\mathscr{J})$. Let $r \in G(\mathbb{A}_f^*)$, and let $y' = yr \in \bar{Y}(\mathscr{J})$, Assume that both $y$ and $y'$ lie in $\bar{Y}(\mathscr{J})^\natural$. Then the images of $y$ and $y'$ under the composite map*

$$\bar{Y}(\mathscr{J})^\natural \xrightarrow{(5.8.6.1)} \pi(G,X) \to \pi^*(G,X)$$

*differ by the image of $r$ in $\pi^*(G)$ under the natural map $G(\mathbb{A}_f^*) \to \pi^*(G)$. Here the sign is similar to the one in Lemma 5.6.9 (ii).*

*Proof.* This follows from the proofs of [Kis17, Lem. 3.6.2, Cor. 3.6.4]. $\square$

**Proposition 5.8.8.** *Let $\mathscr{J}$ be a conjugacy class of admissible morphisms $\mathfrak{Q} \to \mathfrak{G}_G$. There is a unique map*

$$c^*_{\mathscr{J}} : S^*(\mathscr{J}) \longrightarrow \pi^*(G,X)$$

*such that for each $\phi \in \mathscr{J}$ the diagram*

$$\begin{tikzcd} S(\phi) \arrow[r, hook, "(5.8.4.1)"] \arrow[d, "c_{\phi,1}"] & S^*(\mathscr{J}) \arrow[d, "c^*_{\mathscr{J}}"] \\ \pi(G,X) \arrow[r] & \pi^*(G,X) \end{tikzcd}$$

*commutes, and such that the composite*

$$\bar{Y}(\mathscr{J}) \to S^*(\mathscr{J}) \xrightarrow{c^*_{\mathscr{J}}} \pi^*(G,X)$$

*is $G(\mathbb{A}_f^*)$-equivariant. Here $G(\mathbb{A}_f^*)$ acts on $\pi^*(G,X)$ via the natural homomorphism $G(\mathbb{A}_f^*) \to \pi^*(G)$.*

*Proof.* The proof is similar to Proposition 5.6.10. One applies Lemma 5.8.7 instead of Lemma 5.6.9. $\square$

5.9. **Special points on the gerb side.**

**5.9.1.** Let $\mathfrak{s} = (T, i, h) \in \mathcal{SPD}(G, X)$ be a special point datum. From $\mathfrak{s}$ we obtain a morphism $\Psi_{T,\mu_h} : \mathfrak{Q} \to \mathfrak{G}_T$ as in §2.2.9. To simplify notation we write $\Psi_{T,h}$ for $\Psi_{T,\mu_h}$. As in Definition 3.3.2, we write $\phi(\mathfrak{s}) = \phi(T, i, h)$ for the morphism $i \circ \Psi_{T,h} : \mathfrak{Q} \to \mathfrak{G}_G$. This morphism is admissible, as recalled in Theorem 3.3.3.

We make the following definitions which are analogous to §5.8.1.

$$\begin{aligned} Y_p(\Psi_{T,h}) &:= \mathcal{UR}(\Psi_{T,h}(p) \circ \zeta_p), \\ Y^p(\Psi_{T,h}) &:= X^p(\Psi_{T,h}), \\ Y(\Psi_{T,h}) &:= Y_p(\Psi_{T,h}) \times Y^p(\Psi_{T,h}). \end{aligned}$$

By Lemma 2.2.4, $Y_p(\Psi_{T,h})$ is a $T(\mathbb{Q}_p^{\mathrm{ur}})$-torsor. The definition of $X^p(\Psi_{T,h})$ is as in §2.4.7, but with $G$ replaced by $T$. *A priori* $X^p(\Psi_{T,h})$ is either empty or a $T(\mathbb{A}_f^p)$-torsor. By [Kis17, Prop. 3.6.7], it is a $T(\mathbb{A}_f^p)$-torsor.

There is a canonical injection

$$Y(\Psi_{T,h}) \hookrightarrow Y(\phi(\mathfrak{s})) \tag{5.9.1.1}$$

induced by $i$.

Each $t \in Y_p(\Psi_{T,h})$ determines an element $b_t^T \in T(\mathbb{Q}_p^{\mathrm{ur}})$ such that the morphism $\mathfrak{D} \to \mathfrak{G}_T^{\mathrm{ur}}$ underlying the unramified morphism $\mathrm{Int}(t^{-1}) \circ \Psi_{T,h}(p) \circ \zeta_p$ maps $d_\sigma$ : $\mathfrak{G}_p \to \mathfrak{G}_T(p)$ to $b_t^T \rtimes \sigma$, cf. Definition 2.2.5. Define

$$X_p(\Psi_{T,h}) := \left\{ t \in Y_p(\Psi_{T,h}) \mid w_{T_{\mathbb{Q}_p}}(b_t^T) = [-\mu_h] \in X_*(T)_{\Gamma_{p,0}} \right\}.$$

Here $w_{T_{\mathbb{Q}_p}} : T(\breve{\mathbb{Q}}_p) \to X_*(T)_{\Gamma_{p,0}}$ is the Kottwitz map. By [Kis17, Prop. 3.6.7], the set $X_p(\Psi_{T,h})$ is a $T(\mathbb{Q}_p)\mathcal{T}^\circ(\mathbb{Z}_p^{\mathrm{ur}})$-torsor, where $\mathcal{T}^\circ$ is the connected Néron model of $T_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$.[34] We set

$$X(\Psi_{T,h}) := X_p(\Psi_{T,h}) \times X^p(\Psi_{T,h}).$$

Thus $X(\Psi_{T,h})$ is a $T(\mathbb{A}_f)\mathcal{T}^\circ(\mathbb{Z}_p^{\mathrm{ur}})$-torsor.

By Lemma 1.5.5 and Lemma 5.1.2 (iii), $T(\mathbb{Q})$ is discrete in $T(\mathbb{A}_f)$, and hence closed in $T(\mathbb{A}_f)$. By this fact and by [Kis17, Prop. 3.6.7], there is a canonical $T(\mathbb{A}_f)$-equivariant bijection

$$T(\mathbb{Q})\backslash X(\Psi_{T,h})/\mathcal{T}^\circ(\mathbb{Z}_p^{\mathrm{ur}}) \xrightarrow{\sim} T(\mathbb{Q})\backslash T(\mathbb{A}_f)/\mathcal{T}^\circ(\mathbb{Z}_p).$$

We denote by $X(\Psi_{T,h})_{\mathrm{neu}}$ the $T(\mathbb{Q})\mathcal{T}^\circ(\mathbb{Z}_p^{\mathrm{ur}})$-orbit in $X(\Psi_{T,h})$ corresponding to the double coset of $1 \in T(\mathbb{A}_f)$ in the right hand side. (The subscript stands for "neutral".)

Elements of the image of $X(\Psi_{T,h})_{\mathrm{neu}}$ under (5.9.1.1) play a parallel role as the integral special points in Definition 5.7.5. In order to avoid complicated terminology we do not give these elements a name parallel to "integral special points". The fundamental properties of these elements are proved in the following proposition.

**Proposition 5.9.2.** *Keep the setting of §5.9.1. Let $y \in X(\Psi_{T,h})_{\mathrm{neu}}$. We denote the image of $y$ in $Y(\phi(\mathfrak{s}))$ under (5.9.1.1) still by $y$. The following statements hold.*

[34] In [Kis17, §3.6.6, Prop. 3.6.7], what is denoted by $X_p(\psi_{\mu_T})$ is $X_p(\Psi_{T,h})/\mathcal{T}^\circ(\mathbb{Z}_p)$ in our notation.

(i) *Write $\mathscr{J}$ for the conjugacy class of $\phi(\mathfrak{s})$. The image of $y$ under the composite map*

$$Y(\phi(\mathfrak{s})) \to \bar{Y}(\mathscr{J}) \to S^*(\mathscr{J}) \xrightarrow{c^*_{\mathscr{J}}} \pi^*(G,X) \tag{5.9.2.1}$$

*is equal to the image of $i \circ h \in X$ in $\pi^*(G,X)$. (See Proposition 5.8.8 for $c^*_{\mathscr{J}}$.)*

(ii) *The $\mathscr{U}_y$-orbit $[\mathfrak{k}(y)] \subset \mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\equiv$ (see §5.8.1) has a representative in $\mathfrak{K}\mathfrak{T}^{\mathrm{str}}$ of the form $\mathfrak{k}(\mathfrak{s}, \delta_T)$, for some $\delta_T$ lying in the $\overset{\circ}{\sim}$-equivalence class in $T(\mathbb{Q}_p^{\mathrm{ur}})^{\mathrm{mot}}$ determined by $-\mu_h$. (See §5.3.9 for $\mathfrak{k}(\mathfrak{s}, \delta_T) \in \mathfrak{K}\mathfrak{T}^{\mathrm{str}}$.)*

*Proof.* As is explained in [Kis17, §3.6.8], there is a natural map $f : X(\Psi_{T,h}) \to \pi(G,X)$. (More precisely, the target is $\pi(G,\phi(\mathfrak{s}))$, but we identify it with $\pi(G,X)$ via (5.8.5.1).) The composition of $f$ with the natural map $\pi(G,X) \to \pi^*(G,X)$ is equal to the composite map

$$X(\Psi_{T,h}) \subset Y(\Psi_{T,h}) \xrightarrow{(5.9.1.1)} Y(\phi(\mathfrak{s})) \xrightarrow{(5.9.2.1)} \pi^*(G,X).$$

By [Kis17, Prop. 3.6.10 (2)], $f$ sends $y$ to the image of $i \circ h$ in $\pi(G,X)$. Statement (i) follows.

We now prove (ii). We view $i : T \hookrightarrow G$ as the inclusion and omit it from the notation. Write $\phi$ for $\phi(\mathfrak{s})$. Define $b_y \in G(\mathbb{Q}_p^{\mathrm{ur}})$ as in property (i) in §5.8.1 (with respect to $\phi$). Since $y$ comes from $X_p(\Psi_{T,h})$, we have $b_y = b_y^T \in T(\mathbb{Q}_p^{\mathrm{ur}})$, where $b_y^T$ is determined by $y \in X_p(\Psi_{T,h})$ as in §5.9.1. By the definition of $X_p(\Psi_{T,h})$, we have $w_{T_{\mathbb{Q}_p}}(b_y) = [-\mu_h] \in X_*(T)_{\Gamma_{p,0}}$. Keep the notation $\phi(p)_y$ as in §5.8.1. As a subgroup of $G_{\overline{\mathbb{Q}}_p}$, $\mathrm{im}(\phi(p)_y^{\Delta})$ is contained in $T_{\overline{\mathbb{Q}}_p}$.

Let $\mathfrak{k}$ be an arbitrary element of $[\mathfrak{k}(y)] \subset \mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\equiv$. By the construction in [Kis17, §4.5.1] (also cf. [Kis17, §4.3.9]), $\mathfrak{k}$ has a representative in $\mathfrak{T}_n^{\mathrm{str}}$ (for suitable $n$) of the form $(\gamma_0, (\gamma_0)_{l \neq p}, \delta_{\mathfrak{k}})$, where $\delta_{\mathfrak{k}} \in T(\mathbb{Q}_{p^n})$ and $\gamma_0 = \delta_{\mathfrak{k}}\sigma(\delta_{\mathfrak{k}}) \cdots \sigma^{n-1}(\delta_{\mathfrak{k}}) \in T(\mathbb{Q}) \subset T(\mathbb{Q}_p)$. Moreover, $\gamma_0$ is a $p$-unit in $T(\mathbb{Q})$, so in particular $\delta_{\mathfrak{k}} \in T(\mathbb{Q}_p^{\mathrm{ur}})^{\mathrm{mot}}$. Note that $\delta_{\mathfrak{k}}$ is uniquely determined by $\mathfrak{k}$, which justifies our notation.

Now 1 has an open neighborhood

$$\mathscr{O} := \mathrm{im}(\phi(p)_y^{\Delta})(\breve{\mathbb{Q}}_p) \cap \mathcal{T}^{\circ}(\breve{\mathbb{Z}}_p)$$

in $\mathrm{im}(\phi(p)_y^{\Delta})(\breve{\mathbb{Q}}_p)$. By property (i) in §5.8.1, there exists $\mathfrak{k} \in [\mathfrak{k}(y)]$ such that $\delta_{\mathfrak{k}} \in \mathscr{O} \cdot_{\sigma} b_y$. Thus $\delta_{\mathfrak{k}}$ is $\sigma$-conjugate to $b_y$ by an element of $\mathcal{T}^{\circ}(\breve{\mathbb{Z}}_p)$, and in particular, $w_{T_{\mathbb{Q}_p}}(\delta_{\mathfrak{k}}) = w_{T_{\mathbb{Q}_p}}(b_y) = [-\mu_h]$. Therefore $\delta_{\mathfrak{k}}$ lies in the $\overset{\circ}{\sim}$-equivalence class in $T(\mathbb{Q}_p^{\mathrm{ur}})^{\mathrm{mot}}$ determined by $-\mu_h$. Letting $\delta_T = \delta_{\mathfrak{k}}$, we know from the previous paragraph that the current $\mathfrak{k} \in \mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\equiv$ is the image of $\mathfrak{k}(\mathfrak{s}, \delta_T) \in \mathfrak{K}\mathfrak{T}^{\mathrm{str}}$. □

**Lemma 5.9.3.** *Let $\mathfrak{s}, \mathfrak{s}_1$ be two special point data of the form $\mathfrak{s} = (T, i, h)$ and $\mathfrak{s}_1 = (T, i_1, h)$. Let $y \in X(\Psi_{T,h})_{\mathrm{neu}}$. We still write $y$ for the image of $y$ in $Y(\phi(\mathfrak{s}))$ under (5.9.1.1), and we write $y_1$ for the image of $y$ in $Y(\phi(\mathfrak{s}_1))$ under the obvious analogue of (5.9.1.1). Then there exists $\delta_T$ lying in the $\overset{\circ}{\sim}$-equivalence class in $T(\mathbb{Q}_p^{\mathrm{ur}})^{\mathrm{mot}}$ determined by $-\mu_h$ such that $\mathfrak{k}(\mathfrak{s}, \delta_T) \in \mathfrak{K}\mathfrak{T}^{\mathrm{str}}$ is a representative of (an element of) $[\mathfrak{k}(y)] \subset \mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\equiv$ and $\mathfrak{k}(\mathfrak{s}_1, \delta_T) \in \mathfrak{K}\mathfrak{T}^{\mathrm{str}}$ is a representative of (an element of) $[\mathfrak{k}(y_1)] \subset \mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\equiv$.*

*Proof.* Let $b_y^T$ be the element of $T(\mathbb{Q}_p^{\mathrm{ur}})$ determined by $y \in X_p(\Psi_{T,h})$ as in §5.9.1. By the construction in [Kis17, §4.5.1] (also cf. [Kis17, §4.3.9]), for every neighborhood $\mathscr{O}_T$ of 1 in $T(\breve{\mathbb{Q}}_p)$ contained in $i^{-1}(\mathcal{G}(\breve{\mathbb{Z}}_p)) \cap i_1^{-1}(\mathcal{G}(\breve{\mathbb{Z}}_p))$, there exists $\delta_T$ lying in the intersection of $T(\mathbb{Q}_p^{\mathrm{ur}})$ and

$$\mathscr{O}_T \cdot_\sigma b_y^T = \left\{ a b_y^T \sigma(a)^{-1} \mid a \in \mathscr{O}_T \right\}$$

satisfying the following conditions.

- For sufficiently divisible $n$, $[\mathfrak{k}(y)]$ has a representative in $\mathfrak{T}_n^{\mathrm{str}}$ of the form $(i(\gamma_0), (i(\gamma_0))_{l \neq p}, i(\delta_T))$, where
$$\gamma_0 = \delta_T \sigma(\delta_T) \cdots \sigma^{n-1}(\delta_T) \in T(\mathbb{Q}) \subset T(\mathbb{Q}_p).$$
Moreover, $\gamma_0$ is a $p$-unit.
- For sufficiently divisible $n$, $[\mathfrak{k}(y_1)]$ has a representative in $\mathfrak{T}_n^{\mathrm{str}}$ of the form $(i_1(\gamma_0), (i_1(\gamma_0))_{l \neq p}, i_1(\delta_T))$, where $\gamma_0$ is as above.

In fact, write $\theta_y^{T,\mathrm{ur}}$ for the morphism $\mathfrak{D} \to \mathfrak{G}_T^{\mathrm{ur}}$ underlying the unramified morphism $\mathrm{Int}(y^{-1}) \circ \Psi_{T,h}(p) \circ \zeta_p : \mathfrak{G}_p \to \mathfrak{G}_T(p)$. In [Kis17, §4.5.1], choose the element $c'$ sufficiently close to $c$ such that $\theta_y^{T,\mathrm{ur}}(c'c^{-1}) \in \mathscr{O}_T$. Write $a$ for $\theta_y^{T,\mathrm{ur}}(c'c^{-1})$. We can then take $\delta_T$ to be $ab_y^T\sigma(a)^{-1}$. Here the key point is that $c'$ is sufficiently close to $c$ with respect to both $\phi(\mathfrak{s})$ and $\phi(\mathfrak{s}_1)$ in the sense of *loc. cit.*, since $\mathscr{O}_T$ is contained in $i^{-1}(\mathcal{G}(\breve{\mathbb{Z}}_p)) \cap i_1^{-1}(\mathcal{G}(\breve{\mathbb{Z}}_p))$.

We now take $\mathscr{O}_T$ to be sufficiently small such that it is also contained in $\mathcal{T}^\circ(\breve{\mathbb{Z}}_p)$, and choose $\delta_T$ with respect to $\mathscr{O}_T$ as above. As in the proof of Proposition 5.7.6 (ii), this $\delta_T$ necessarily lies in the $\overset{\circ}{\sim}$-equivalence class in $T(\mathbb{Q}_p^{\mathrm{ur}})^{\mathrm{mot}}$ determined by $-\mu_h$. The above two conditions imply that $\mathfrak{k}(\mathfrak{s}, \delta_T)$ represents $[\mathfrak{k}(y)]$ and that $\mathfrak{k}(\mathfrak{s}_1, \delta_T)$ represents $[\mathfrak{k}(y_1)]$. $\square$

*Remark* 5.9.4. The analogue of Lemma 5.9.3 for finitely many special point data of the form $(T, i, h), (T, i_1, h), \cdots, (T, i_k, h)$ is also true. The choice of $\delta_T$ is not intrinsic to the Shimura datum $(T, h)$ and the point $y \in X(\Psi_{T,h})_{\mathrm{neu}}$, but depends on the given finite list of embeddings $i, i_1, \cdots, i_k$ of $T$ into $G$.

## 5.10. **Markings and amicable pairs.**

**Definition 5.10.1.** Let $\mathscr{I}$ be an isogeny class in $\mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$, and let $\mathscr{J}$ be a conjugacy class of admissible morphisms $\mathfrak{Q} \to \mathfrak{G}_G$. By a *marking* of $(\mathscr{I}, \mathscr{J})$, we mean a pair

$$(\bar{y}, \bar{y}') \in \bar{Y}(\mathscr{I}) \times \bar{Y}(\mathscr{J})$$

such that

$$\mathfrak{k}(\bar{y}) \in [\mathfrak{k}(\bar{y}')].$$

Here $\mathfrak{k}(\bar{y}) \in \mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\equiv$ is the image of $\bar{y}$ under (5.6.3.2), and $[\mathfrak{k}(\bar{y}')]$ is the $\mathscr{U}_{\bar{y}'}$-orbit in $\mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\equiv$ attached to $\bar{y}'$ as in §5.8.2. We say that the marking $(\bar{y}, \bar{y}')$ is $\pi^*$-*compatible*, if the image of $\bar{y}$ under

$$\bar{Y}(\mathscr{I}) \longrightarrow \mathscr{I}^* \xrightarrow{c^*_{\mathscr{I}}} \pi^*(G, X) \tag{5.10.1.1}$$

equals the image of $\bar{y}'$ under

$$\bar{Y}(\mathscr{J}) \longrightarrow S^*(\mathscr{J}) \xrightarrow{c^*_{\mathscr{J}}} \pi^*(G, X). \tag{5.10.1.2}$$

See Proposition 5.6.10 and Proposition 5.8.8 for $c^*_{\mathscr{I}}$ and $c^*_{\mathscr{J}}$ respectively.

We call the pair $(\mathscr{I}, \mathscr{J})$ *weakly amicable* (resp. *amicable*) if a it admits a marking (resp. a $\pi^*$-compatible marking).

**Lemma 5.10.2.** *Let $(\mathscr{I}, \mathscr{J})$ be a weakly amicable pair. For every $\bar{y}' \in \bar{Y}(\mathscr{J})$, there exists $\bar{y} \in \bar{Y}(\mathscr{I})$ such that $(\bar{y}, \bar{y}')$ is a marking of $(\mathscr{I}, \mathscr{J})$. Moreover, if $(\mathscr{I}, \mathscr{J})$ is amicable, then we can choose $\bar{y}$ such that $(\bar{y}, \bar{y}')$ is a $\pi^*$-compatible marking.*

*Proof.* Let $(\bar{z}, \bar{z}')$ be a marking of $(\mathscr{I}, \mathscr{J})$. Let $u \in G(\mathbb{A}_f^*)$ be such that $\bar{z}' \cdot u = \bar{y}'$. By assumption, $\mathfrak{k}(\bar{z}) \in [\mathfrak{k}(\bar{z}')]$. By property (iii) in §5.8.1, there exists $\mathfrak{k}_0 \in [\mathfrak{k}(\bar{z}')]$ such that $\mathfrak{k}_0 \cdot u \in [\mathfrak{k}(\bar{y}')]$. Since $\mathfrak{k}_0$ and $\mathfrak{k}(\bar{z})$ lie in the same $\mathscr{U}_{\bar{z}'}$-orbit, and since $\mathscr{U}_{\bar{z}'} \subset \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$, there exists $u_0 \in \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$ such that $\mathfrak{k}_0 = \mathfrak{k}(\bar{z}) \cdot u_0$. Then we have

$$\mathfrak{k}(\bar{z}) \cdot u_0 u \in [\mathfrak{k}(\bar{y}')].$$

Let $\bar{y} = \bar{z} \cdot u_0 u \in \bar{Y}(\mathscr{I})$. By the $G(\mathbb{A}_f^*)$-equivariance of the map (5.6.2.1), we have $\mathfrak{k}(\bar{y}) = \mathfrak{k}(\bar{z}) \cdot u_0 u$. Thus we have

$$\mathfrak{k}(\bar{y}) \in [\mathfrak{k}(\bar{y}')],$$

which means that $(\bar{y}, \bar{y}')$ is a marking of $(\mathscr{I}, \mathscr{J})$.

If we assume that $(\mathscr{I}, \mathscr{J})$ is amicable, then we can choose $(\bar{z}, \bar{z}')$ as above to be $\pi^*$-compatible. It remains to show that the marking $(\bar{y}, \bar{y}')$ produced above is $\pi^*$-compatible. But this follows from the $G(\mathbb{A}_f^*)$-equivariance of the maps (5.10.1.1) and (5.10.1.2), and the fact that $u_0$ has trivial image in $\pi^*(G)$ (since $u_0 \in \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$). □

**5.10.3.** Let $(\mathscr{I}, \mathscr{J})$ be a weakly amicable pair, and let $(\bar{y}, \bar{y}')$ be a marking of it. Fix $\dot{\mathfrak{k}}(\bar{y}) \in \mathfrak{K}\mathfrak{T}^{\mathrm{str}}$ representing $\mathfrak{k}(\bar{y}) \in \mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\equiv$, and let $(I_0, (I_v)_v, (\eta_v)_v)$ be the datum attached to $\dot{\mathfrak{k}}(\bar{y})$ as in §5.3.2. Let $x \in \mathscr{I}$ and $\phi \in \mathscr{J}$. We choose $y \in Y(x)$ lifting $\bar{y}$, and choose $y' \in Y(\phi)$ lifting $\bar{y}'$. Recall from §5.6.3 and §5.8.2 that there exist inner twistings $\iota_0 : I_{0,\overline{\mathbb{Q}}} \to I_{x,\overline{\mathbb{Q}}}$ and $\iota_0' : I_{0,\overline{\mathbb{Q}}} \to I_{\phi,\overline{\mathbb{Q}}}$ of $\mathbb{Q}$-groups such that the tuples $(I_x, \iota_0, (\iota_{y,v})_v)$ and $(I_\phi, \iota_0', (\iota_{y',v})_v)$ are both refinements of $\dot{\mathfrak{k}}(\bar{y})$. By the Hasse principle for adjoint groups and by the fact that $I_x$ and $I_\phi$ are both compact mod center at the place $\infty$, there is an isomorphism of $\mathbb{Q}$-groups

$$f : I_x \xrightarrow{\sim} I_\phi$$

such that for each finite place $v$ the two $\mathbb{Q}_v$-maps $\iota_{y,v} \circ f^{-1} : I_{\phi,\mathbb{Q}_v} \to I_v$ and $\iota_{y',v} : I_{\phi,\mathbb{Q}_v} \to I_v$ differ by an inner automorphism of $I_{\phi,\mathbb{Q}_v}$. The isomorphism $f$ is uniquely determined by $(x, \phi, y, y')$, up to composing with inner automorphisms defined over $\mathbb{Q}$. (Note that $I_v$ and the maps $\iota_{y,v}, \iota_{y',v}$ depend only on $(x, \phi, y, y')$, not on the choice of the lifting $\dot{\mathfrak{k}}(\bar{y})$ of $\mathfrak{k}(\bar{y})$; see the last paragraph of §5.8.2.) In Remark 5.10.10 below, we will see that $f$ is in fact uniquely determined by $(x, \phi)$ up to composing with inner automorphisms defined over $\mathbb{Q}$.

Now there is an element

$$\tau = (\tau_v)_v \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$$

such that for each finite place $v$ we have

$$\iota_{y,v} \circ f^{-1} = \iota_{y',v} \circ \mathrm{Int}(\tau_v) : I_{\phi,\mathbb{Q}_v} \longrightarrow I_v.$$

Clearly $\tau$ is uniquely determined by $(x, \phi, y, y')$ and $f$. Moreover, the image of $\tau$ in $I_\phi^{\mathrm{ad}}(\mathbb{A}_f)/I_\phi^{\mathrm{ad}}(\mathbb{Q})$ is determined by $(x, \phi, y, y')$ and independent of the choice of $f$.

We denote this element by

$$\tau_{x,\phi,y,y'} \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f)/I_\phi^{\mathrm{ad}}(\mathbb{Q}).$$

The image of $\tau_{x,\phi,y,y'}$ in $I_\phi(\mathbb{Q})\backslash I_\phi^{\mathrm{ad}}(\mathbb{A}_f)/I_\phi^{\mathrm{ad}}(\mathbb{Q})$ is determined by $(\phi,\bar{y},\bar{y}')$ and independent of the choices of $x,y,y'$. We denote this element by

$$\tau_{\bar{y},\bar{y}'} \in I_\phi(\mathbb{Q})\backslash I_\phi^{\mathrm{ad}}(\mathbb{A}_f)/I_\phi^{\mathrm{ad}}(\mathbb{Q}).$$

If $\phi'$ is another element of $\mathscr{J}$, then we have a canonical identification

$$I_\phi(\mathbb{Q})\backslash I_\phi^{\mathrm{ad}}(\mathbb{A}_f)/I_\phi^{\mathrm{ad}}(\mathbb{Q}) \cong I_{\phi'}(\mathbb{Q})\backslash I_{\phi'}^{\mathrm{ad}}(\mathbb{A}_f)/I_{\phi'}^{\mathrm{ad}}(\mathbb{Q})$$

induced by $\operatorname{Int} g$ for any $g\in G(\overline{\mathbb{Q}})$ conjugating $\phi$ to $\phi'$. (This identification is indeed independent of $g$, since $g$ is unique up to right multiplication by $I_\phi(\mathbb{Q})$.) If we identify $I_\phi(\mathbb{Q})\backslash I_\phi^{\mathrm{ad}}(\mathbb{A}_f)/I_\phi^{\mathrm{ad}}(\mathbb{Q})$ for all $\phi\in\mathscr{J}$ in this way, then $\tau_{\bar{y},\bar{y}'}$ is independent of $\phi$. This justifies our notation.

Let $\xi_y$ be the unique $G(\mathbb{A}_f^*)$-equivariant bijection $Y(x)\xrightarrow{\sim} G(\mathbb{A}_f^*)$ taking $y$ to 1, and let $\xi_{y'}$ be the unique $G(\mathbb{A}_f^*)$-equivariant bijection $Y(\phi)\xrightarrow{\sim} G(\mathbb{A}_f^*)$ taking $y'$ to 1. Let $\delta_y\in G(\mathbb{Q}_p^{\mathrm{ur}})$ be the $\delta$-component of $\mathfrak{k}(y)\in\mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\equiv$, and let $b_{y'}\in G(\mathbb{Q}_p^{\mathrm{ur}})$ be the element attached to $y'$ as in property (i) in §5.8.1. By the defining property of a marking and by property (i) in §5.8.1, there exists

$$e\in\mathcal{G}(\breve{\mathbb{Z}}_p)\cap\operatorname{im}(\phi(p)_y^{\Delta})(\breve{\mathbb{Q}}_p)\subset G(\breve{\mathbb{Q}}_p)$$

such that $e\delta_y\sigma(e)^{-1}=b_{y'}$. Fix such $e$. Define

$$\mathbf{f}_1: G(\mathbb{Q}_p^{\mathrm{ur}})/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})\longrightarrow G(\mathbb{Q}_p^{\mathrm{ur}})/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}),\quad g\longmapsto\sigma^{-1}(\delta_y^{-1}g).$$

This is a well-defined bijection because $\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$ is $\sigma$-stable. Recall from Lemma 1.6.8 that $G(\mathbb{Q}_p^{\mathrm{ur}})/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})\cong G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p)$. Using this, we define

$$\mathbf{f}_2: G(\mathbb{Q}_p^{\mathrm{ur}})/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})\cong G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p)\xrightarrow{g\mapsto ege^{-1}} G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p)\cong G(\mathbb{Q}_p^{\mathrm{ur}})/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}).$$

Here the middle map is a well-defined bijection because $e\in\mathcal{G}(\breve{\mathbb{Z}}_p)$. Let $\xi_{y,y',e}$ be the composite bijection

$$Y(x)/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})\xrightarrow{\xi_y} G(\mathbb{A}_f^*)/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})\xrightarrow{(\mathrm{id}_{G(\mathbb{A}_f^p)},\ \mathbf{f}_2\circ\mathbf{f}_1)} G(\mathbb{A}_f^*)/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})\xrightarrow{\xi_{y'}^{-1}} Y(\phi)/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}).$$

**Proposition 5.10.4.** *Keep the setting and notation of §5.10.3. The map $\xi_{y,y',e}$ descends to a bijection*

$$\mathscr{I}^*\xrightarrow{\sim} I_\phi(\mathbb{Q})_\tau\backslash Y(\phi)/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}). \tag{5.10.4.1}$$

*Here $I_\phi(\mathbb{Q})_\tau$ is the image of $I_\phi(\mathbb{Q})\hookrightarrow I_\phi(\mathbb{A}_f)\xrightarrow{\operatorname{Int}\tau} I_\phi(\mathbb{A}_f)$. Moreover, (5.10.4.1) restricts to a bijection*

$$\mathscr{I}\xrightarrow{\sim} S_\tau(\phi), \tag{5.10.4.2}$$

*which is compatible with the actions of $G(\mathbb{A}_f^p)$ and the $q$-Frobenius $\Phi$ on the two sides.*

*Proof.* The map $\xi_y$ induces a bijection

$$\bar{\xi}_y: I_x(\mathbb{Q})\backslash Y(x)/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})\xrightarrow{\sim}\iota_y(I_x(\mathbb{Q}))\backslash G(\mathbb{A}_f^*)/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}).$$

Similarly, $\xi_{y'}$ induces a bijection

$$\bar{\xi}_{y'}: I_\phi(\mathbb{Q})_\tau\backslash Y(\phi)/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})\xrightarrow{\sim}\iota_y(I_x(\mathbb{Q}))\backslash G(\mathbb{A}_f^*)/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}).$$

Here we have used that the image of $I_\phi(\mathbb{Q})_\tau \hookrightarrow I_\phi(\mathbb{A}_f) \xrightarrow{\iota_{y'}} G(\mathbb{A}_f^*)$ is equal to $\iota_y(I_x(\mathbb{Q})) \subset G(\mathbb{A}_f^*)$. To show that $\xi_{y,y'}$ descends to (5.10.4.1) it remains to show that for $i = 1, 2$ we have

$$\mathbf{f}_i(hg) = h\mathbf{f}_i(g)$$

for all $g \in G(\mathbb{Q}_p^{\mathrm{ur}})/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$ and all $h \in I_p(\mathbb{Q}_p)$. Now $\mathbf{f}_1$ satisfies this because $I_p(\mathbb{Q}_p)$ is the $\sigma$-centralizer of $\delta_y$ in $G(\mathbb{Q}_p^{\mathrm{ur}})$. Meanwhile $\mathbf{f}_2$ satisfies this because in $G(\breve{\mathbb{Q}}_p)$, $e$ commutes with every element in the image of $\iota_{y',p} : I_\phi(\mathbb{Q}_p) \to G(\mathbb{Q}_p^{\mathrm{ur}})$, and this image is equal to $I_p(\mathbb{Q}_p)$.

We now show that (5.10.4.1) induces $\mathscr{I} \xrightarrow{\sim} S_\tau(\phi)$. The image of $\mathscr{I} \hookrightarrow \mathscr{I}^*$ is described in Proposition 5.6.5 and Remark 5.6.6. By that description we know that the image of $\mathscr{I}$ under $\bar{\xi}_y$ is

$$\iota_y(I_x(\mathbb{Q}))\backslash X_\upsilon(\delta_y) \times G(\mathbb{A}_f^p). \tag{5.10.4.3}$$

In fact, the bijection from $\mathscr{I}$ onto the above set induced by $\bar{\xi}_y$ is just the inverse of (5.6.6.1). By the definition of $S_\tau(\phi)$ and by the discussion in §2.2.7, the image of $S_\tau(\phi)$ under $\bar{\xi}_{y'}$ is

$$\iota_y(I_x(\mathbb{Q}))\backslash X_{-\mu}(b_{y'}) \times G(\mathbb{A}_f^p), \tag{5.10.4.4}$$

with $\mu \in \mathfrak{p}_X$. It remains to show that the bijection $\mathbf{f}_2 \circ \mathbf{f}_1 : G(\mathbb{Q}_p^{\mathrm{ur}})/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}) \to G(\mathbb{Q}_p^{\mathrm{ur}})/\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$ restricts to a bijection

$$X_\upsilon(\delta_y) \xrightarrow{\sim} X_{-\mu}(b_{y'}). \tag{5.10.4.5}$$

Since $\upsilon = \sigma(-\mu)$ (see §5.6.4), it is immediate that $\mathbf{f}_1$ induces $X_\upsilon(\delta_y) \xrightarrow{\sim} X_{-\mu}(\delta_y)$. Using the presentation of affine Deligne–Lusztig sets as in (2.2.7.1), we see that $\mathbf{f}_2$ induces $X_{-\mu}(\delta_y) \xrightarrow{\sim} X_{-\mu}(b_{y'})$.

Finally, we need to show that (5.10.4.2) is compatible with the actions of $G(\mathbb{A}_f^p)$ and $\Phi$. The compatibility with $G(\mathbb{A}_f^p)$ is clear. The compatibility with $\Phi$ boils down to the following three compatibilities. Firstly, the bijection from $\mathscr{I}$ to (5.10.4.3) induced by $\bar{\xi}_y$ is compatible with $\Phi$ on $\mathscr{I}$ and the operator $(\delta_y \rtimes \sigma)^r$ on $X_\upsilon(\delta_y)$ (with $r = [\mathbb{F}_q : \mathbb{F}_p]$). As we have remarked above, this bijection is just the inverse of (5.6.6.1), which is the map [Kis17, (2.1.4)]. The compatibility follows from [Kis17, Prop. 1.4.4], cf. [Kis17, Cor. 1.4.13, Prop. 2.1.3, Prop. 4.4.14]. Secondly, the bijection (5.10.4.5) induced by $\mathbf{f}_2 \circ \mathbf{f}_1$ is compatible with $(\delta_y \rtimes \sigma)^r$ on the left hand side and $(b_{y'} \rtimes \sigma)^r$ on the right hand side. This is immediate from the definitions. Thirdly, the bijection from $S_\tau(\phi)$ to (5.10.4.4) induced by $\bar{\xi}_{y'}$ is compatible with $\Phi$ on $S_\tau(\phi)$ and $(b_{y'} \rtimes \sigma)^r$ on $X_{-\mu_X}(b_{y'})$. This follows from the discussion in §2.2.7. $\square$

**5.10.5.** Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be an admissible morphism. Since $\mathfrak{Q}$ satisfies the assumption on $\mathfrak{H}$ in the last paragraph of §2.1.14, we have reductive $\mathbb{Q}$-groups $\widetilde{I}_\phi$ and $I_\phi^\dagger$ associated with $\phi$. Recall that $I_\phi^\dagger$ is identified with the natural $\mathbb{Q}$-homomorphism $I_\phi \to G^{\mathrm{ab}}$. Note that $I_\phi \to G^{\mathrm{ab}}$ is surjective, because $I_{\phi,\overline{\mathbb{Q}}}$ contains a maximal torus in $G_{\overline{\mathbb{Q}}}$. We write $Z_\phi^\dagger$ for the center of $I_\phi^\dagger$. By Lemma 1.2.10 (i) applied to the map $I_\phi \to G^{\mathrm{ab}}$, we have

$$Z_\phi^\dagger = Z_{I_\phi} \cap I_\phi^\dagger, \tag{5.10.5.1}$$

and the embedding $I_\phi^\dagger \hookrightarrow I_\phi$ induces an isomorphism between the adjoint groups.

Recall from §2.6.11 that the canonical $\overline{\mathbb{Q}}$-embedding $I_{\phi,\overline{\mathbb{Q}}} \hookrightarrow G_{\overline{\mathbb{Q}}}$ and the set $\mathcal{W} = \{g \in G(\overline{\mathbb{Q}}) \mid \operatorname{Int} g \circ \phi \text{ is gg}\}$ form an inner transfer datum from $I_\phi$ to $G$. It follows that the canonical $\overline{\mathbb{Q}}$-embedding $I^\dagger_{\phi,\overline{\mathbb{Q}}} \hookrightarrow G_{\mathrm{der},\overline{\mathbb{Q}}}$ and the set $\mathcal{W} \cap G_{\mathrm{der}}(\overline{\mathbb{Q}})$ (which is clearly non-empty, given the non-emptiness of $\mathcal{W}$) form an inner transfer datum from $I^\dagger_\phi$ to $G_{\mathrm{der}}$. We use this inner transfer datum to define the map

$$\text{Ш}^\infty(\mathbb{Q}, I^\dagger_\phi) \longrightarrow \text{Ш}^\infty(\mathbb{Q}, G_{\mathrm{der}}), \tag{5.10.5.2}$$

as well as to define $\text{Ш}^\infty_{G_{\mathrm{der}}}(\mathbb{Q}, H)$ for any $\mathbb{Q}$-subgroup $H \subset I^\dagger_\phi$, as in §1.2.5.

Consider the boundary map

$$I^{\mathrm{ad}}_\phi(\mathbb{A}_f) \longrightarrow \mathbf{H}^1(\mathbb{A}_f, Z^\dagger_\phi) \tag{5.10.5.3}$$

arising from the short exact sequence $1 \to Z^\dagger_\phi \to I^\dagger_\phi \to I^{\mathrm{ad}}_\phi = (I^\dagger_\phi)^{\mathrm{ad}} \to 1$. As we explained in §2.6.18, $I^{\mathrm{ad}}_\phi(\mathbb{R})$ is connected, and hence $I^\dagger_\phi(\mathbb{R}) \to I^{\mathrm{ad}}_\phi(\mathbb{R})$ is surjective. Therefore the boundary map $I^{\mathrm{ad}}_\phi(\mathbb{R}) \to \mathbf{H}^1(\mathbb{R}, Z^\dagger_\phi)$ arising from the same short exact sequence is zero. It follows that (5.10.5.3) descends to a map

$$I^{\mathrm{ad}}_\phi(\mathbb{A}_f)/I^{\mathrm{ad}}_\phi(\mathbb{Q}) \longrightarrow \mathbf{H}^1(\mathbb{A}_f, Z^\dagger_\phi)/\text{Ш}^\infty_{I^\dagger_\phi}(\mathbb{Q}, Z^\dagger_\phi). \tag{5.10.5.4}$$

Consider the boundary map $\partial : G^{\mathrm{ab}}(\mathbb{Q}_p) \to \mathbf{H}^1(\mathbb{Q}_p, Z^\dagger_\phi)$ arising from the short exact sequence $1 \to Z^\dagger_\phi \to Z_{I_\phi} \to G^{\mathrm{ab}} \to 1$. We define the abelian group

$$\mathfrak{H}(\phi) := \operatorname{coker}(\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p) \xrightarrow{\partial} \mathbf{H}^1(\mathbb{A}_f, Z^\dagger_\phi)/\text{Ш}^\infty_{G_{\mathrm{der}}}(\mathbb{Q}, Z^\dagger_\phi)). \tag{5.10.5.5}$$

The map (5.10.5.4) induces a map

$$I^{\mathrm{ad}}_\phi(\mathbb{A}_f)/I^{\mathrm{ad}}_\phi(\mathbb{Q}) \longrightarrow \mathfrak{H}(\phi). \tag{5.10.5.6}$$

Suppose that $\phi_1 : \mathfrak{Q} \to \mathfrak{G}_G$ is another admissible morphism satisfying $\phi \approx \phi_1$ as in §2.6.16. Then the abelian groups $\mathfrak{H}(\phi)$ and $\mathfrak{H}(\phi_1)$ are canonically isomorphic. Indeed, $\mathfrak{H}(\phi)$ clearly depends only on the two-term complex $Z_{I_\phi} \to G^{\mathrm{ab}}$, and similarly for $\mathfrak{H}(\phi_1)$. Since $\phi \approx \phi_1$, there is a canonical equivalence class of inner twistings between $I_\phi$ and $I_{\phi_1}$, and they all induce the same isomorphism from the complex $Z_{I_\phi} \to G^{\mathrm{ab}}$ to the complex $Z_{I_{\phi_1}} \to G^{\mathrm{ab}}$. Thus we have a canonical isomorphism $\mathfrak{H}(\phi) \xrightarrow{\sim} \mathfrak{H}(\phi_1)$.

**5.10.6.** Now let $(\mathscr{I}, \mathscr{J})$ be a weakly amicable pair, and let $\phi \in \mathscr{J}$. For any marking $(\bar{y}, \bar{y}')$ of $(\mathscr{I}, \mathscr{J})$, recall that $\tau_{\bar{y},\bar{y}'}$ is an element of $I_\phi(\mathbb{Q})\backslash I^{\mathrm{ad}}_\phi(\mathbb{A}_f)/I^{\mathrm{ad}}_\phi(\mathbb{Q})$. Note that the map (5.10.5.6) factors through $I^{\mathrm{ad}}_\phi(\mathbb{Q})\backslash I^{\mathrm{ad}}_\phi(\mathbb{A}_f)/I^{\mathrm{ad}}_\phi(\mathbb{Q})$. (In fact, suppose that $g_1 \in I^\dagger_\phi(\overline{\mathbb{Q}})$ lifts an element of $I^{\mathrm{ad}}_\phi(\mathbb{Q})$ and $g_2 \in I^\dagger_\phi(\bar{\mathbb{A}}_f)$ lifts an element of $I^{\mathrm{ad}}_\phi(\mathbb{A}_f)$. Then for $\rho \in \Gamma$ we have $(g_1g_2)^{-1\rho}(g_1g_2) = (g_2g_1)^{-1\rho}(g_2g_1)$ since $g_1^{-1\rho}g_1$ and $g_2^{-1\rho}g_2$ are central in $I^\dagger_\phi$. This shows that $g_1g_2$ and $g_2g_1$ have the same image under (5.10.5.6).) We denote by

$$\tau^{\mathfrak{H}}_{\bar{y},\bar{y}'} \in \mathfrak{H}(\phi)$$

the image of $\tau_{\bar{y},\bar{y}'}$ in $\mathfrak{H}(\phi)$ under the map induced by (5.10.5.6). Recall from §5.10.3 that the set $I_\phi(\mathbb{Q})\backslash I^{\mathrm{ad}}_\phi(\mathbb{A}_f)/I^{\mathrm{ad}}_\phi(\mathbb{Q})$ is independent of $\phi$ up to canonical bijection, and that the element $\tau_{\bar{y},\bar{y}'}$ of this set is independent of $\phi$. In a similar sense, $\tau^{\mathfrak{H}}_{\bar{y},\bar{y}'}$ depends only on $(\bar{y}, \bar{y}')$ and is independent of $\phi$. More precisely, if $\phi'$ is another element of $\mathscr{J}$, then we have a canonical isomorphism of abelian groups $\mathfrak{H}(\phi) \cong$

$\mathfrak{H}(\phi')$ which is induced by $\mathrm{Int}\, g$ for any $g \in G(\overline{\mathbb{Q}})$ conjugating $\phi$ to $\phi'$. (This is a special case of the canonical isomorphism discussed in §5.10.5 as $\phi \approx \phi'$.) If we identify $\mathfrak{H}(\phi)$ for all $\phi \in \mathscr{J}$ in this way, then the element $\tau^{\mathfrak{H}}_{\bar{y},\bar{y}'}$ depends only on $(\bar{y}, \bar{y}')$ and not on $\phi$.

Recall from §2.6.13 that we defined

$$\mathcal{H}(\phi) := I_\phi(\mathbb{A}_f)\backslash I^{\mathrm{ad}}_\phi(\mathbb{A}_f)/I^{\mathrm{ad}}_\phi(\mathbb{Q}).$$

The element $\tau_{\bar{y},\bar{y}'}$ has a natural image in $\mathcal{H}(\phi)$, which we denote by

$$\tau^{\mathcal{H}}_{\bar{y},\bar{y}'} \in \mathcal{H}(\phi).$$

Again, for different $\phi \in \mathscr{J}$, the abelian groups $\mathcal{H}(\phi)$ are canonically identified (cf. §2.6.16). Under such identifications the element $\tau^{\mathcal{H}}_{\bar{y},\bar{y}'}$ is independent of $\phi$.

**Lemma 5.10.7.** *Let $(\mathscr{I}, \mathscr{J})$ be an amicable pair. Let $(\bar{y}_i, \bar{y}'_i), i = 1, 2,$ be two $\pi^*$-compatible markings of $(\mathscr{I}, \mathscr{J})$. Then*

$$\tau^{\mathfrak{H}}_{\bar{y}_1,\bar{y}'_1} = \tau^{\mathfrak{H}}_{\bar{y}_2,\bar{y}'_2},$$

*where the two sides are defined in §5.10.6.*

*Proof.* Fix $\phi \in \mathscr{J}$. We view $\tau^{\mathfrak{H}}_{\bar{y}_1,\bar{y}'_1}$ and $\tau^{\mathfrak{H}}_{\bar{y}_2,\bar{y}'_2}$ as elements of $\mathfrak{H}(\phi)$. Pick $x \in \mathscr{I}$, and pick $y_i \in Y(x)$ lifting $\bar{y}_i$ for $i = 1, 2$. Also pick $y'_i \in Y(\phi)$ lifting $\bar{y}'_i$ for $i = 1, 2$.

Let $u \in G(\mathbb{A}^*_f)$ be such that $y'_2 = y'_1 \cdot u$. As in the proof of Lemma 5.10.2, there exists $u_0 \in \mathscr{U}_{\phi,y'_1}$ such that $(\bar{y}_1 \cdot u_0 u, \bar{y}'_2)$ is a marking of $(\mathscr{I}, \mathscr{J})$. Using that $u_0$ has trivial image in $\pi^*(G)$, it is easy to see that $(\bar{y}_1 \cdot u_0 u, \bar{y}'_2)$ is $\pi^*$-compatible. For every finite place $v$, let $I_v$ denote the reductive group over $\mathbb{Q}_v$ associated with $\mathfrak{k}(\bar{y}_1) \in \mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\equiv$. Since $(\bar{y}_1, \bar{y}'_1)$ is a marking, we know that $I_v$ is also the reductive group over $\mathbb{Q}_v$ associated with any element of $[\mathfrak{k}(\bar{y}'_1)]$. Recall that $\mathscr{U}_{\phi,y'_1}$ is canonically embedded into the $\mathbb{Q}^{\mathrm{ur}}_p$-points of the center of $I_p$ (cf. the last paragraph of §5.8.2). It follows that $\iota_{y_1\cdot u_0 u,v} = \iota_{y_1\cdot u,v}$ as maps $I_{x,\mathbb{Q}_v} \to I_v$ for every finite place $v$. From this, it is easy to see that $\tau_{\bar{y}_1,\bar{y}'_1} = \tau_{\bar{y}_1\cdot u_0 u,\bar{y}'_2}$. We have thus reduced the proof of the lemma to the case where $\bar{y}'_1 = \bar{y}'_2$, since we can replace $(\bar{y}_1, \bar{y}'_1)$ by $(\bar{y}_1 \cdot u_0 u, \bar{y}'_2)$ .

We now assume that $\bar{y}'_1 = \bar{y}'_2$, and write $\bar{y}'$ for this element. Obviously we can arrange that $y'_1 = y'_2$. For each finite place $v$, the reductive groups over $\mathbb{Q}_v$ associated with $\mathfrak{k}(\bar{y}_1), \mathfrak{k}(\bar{y}_2)$ and any element of $[\mathfrak{k}(\bar{y}')]$ are all the same, and we denote it by $I_v$.

Write $y_1 = y_2 h$ for $h = (h_v)_v \in G(\mathbb{A}^*_f)$. Then $h_v$ lies in $I_v(\mathbb{Q}_v)$ for $v \neq p$, and $h_p$ lies in

$$\mathscr{U}_{\phi,y'} \cdot I_p(\mathbb{Q}_p) \subset \big(Z_{I_p}(\mathbb{Q}^{\mathrm{ur}}_p) \cap \mathcal{G}(\mathbb{Z}^{\mathrm{ur}}_p)\big) I_p(\mathbb{Q}_p).$$

Thus we can write $h = \mathrm{Int}(y')^{-1}(s) \cdot t$, with $s \in I_\phi(\mathbb{A}_f)$ and $t \in Z_{I_p}(\mathbb{Q}^{\mathrm{ur}}_p) \cap \mathcal{G}(\mathbb{Z}^{\mathrm{ur}}_p)$. (Here we view $y'$ and $s$ both as elements of $G(\mathbb{A}^*_f \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$ in writing $\mathrm{Int}(y')^{-1}(s)$. The element $\mathrm{Int}(y')^{-1}(s) \in G(\mathbb{A}^*_f \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$ in fact lies in $G(\mathbb{A}^*_f)$.) Then the elements $\tau_{x,\phi,y_1,y'}$ and $\tau_{x,\phi,y_2,y'}$ of $I^{\mathrm{ad}}_\phi(\mathbb{A}_f)/I^{\mathrm{ad}}_\phi(\mathbb{Q})$ differ by left multiplication by $s$. More precisely, if $f_1 : I_x \xrightarrow{\sim} I_\phi$ is a $\mathbb{Q}$-isomorphism and $\tau_1 = (\tau_{1,v})_v$ is an element of $I^{\mathrm{ad}}_\phi(\mathbb{A}_f)$ satisfying that

$$\iota_{y_1,v} \circ f_1^{-1} = \iota_{y',v} \circ \mathrm{Int}(\tau_{1,v}) : I_{\phi,\mathbb{Q}_v} \longrightarrow I_v,$$

then we have

$$\iota_{y_2,v} \circ f_1^{-1} = \iota_{y',v} \circ \mathrm{Int}(s\tau_{1,v}) : I_{\phi,\mathbb{Q}_v} \longrightarrow I_v. \tag{5.10.7.1}$$

It remains to show that the image of $s$ under (5.10.5.6) is zero. Since $y_1, y_2, y'$ all have the same image in $\pi^*(G, X)$, the image of $h \in G(\mathbb{A}_f^*)$ in $\pi^*(G)$ must be trivial. Note that $t$ has trivial image in $\pi^*(G)$ since $t \in \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$. Hence $\mathrm{Int}(y')^{-1}(s)$ has trivial image in $\pi^*(G)$. The natural map

$$G(\mathbb{A}_f^*) \longrightarrow G(\mathbb{Q})_+ \backslash G^{\mathrm{ab}}(\mathbb{A}_f^*)/\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p^{\mathrm{ur}})$$

factors through $\pi^*(G)$. Therefore the image $s^{\mathrm{ab}} \in G^{\mathrm{ab}}(\mathbb{A}_f^*)$ of $s$ under $I_\phi \to G^{\mathrm{ab}}$ lies in $[G(\mathbb{Q})_+]^{\mathrm{ab}} \mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p^{\mathrm{ur}})$, where $[G(\mathbb{Q})_+]^{\mathrm{ab}}$ denotes the image of $G(\mathbb{Q})_+ \to G^{\mathrm{ab}}(\mathbb{A}_f)$. Since $s^{\mathrm{ab}}$ in fact lies in $G^{\mathrm{ab}}(\mathbb{A}_f)$, we have $s_{\mathrm{ab}} \in [G(\mathbb{Q})_+]^{\mathrm{ab}} \mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p)$. By Lemma 1.2.10 applied to $I_\phi \to G^{\mathrm{ab}}$, the composite maps

$$I_\phi(\mathbb{A}_f) \to I_\phi^{\mathrm{ad}}(\mathbb{A}_f) \xrightarrow{\delta^1} \mathbf{H}^1(\mathbb{A}_f, Z_\phi^\dagger)$$

and

$$I_\phi(\mathbb{A}_f) \to G^{\mathrm{ab}}(\mathbb{A}_f) \xrightarrow{\delta^2} \mathbf{H}^1(\mathbb{A}_f, Z_\phi^\dagger)$$

differ by a sign. Here $\delta^1$ is associated with the short exact sequence $1 \to Z_\phi^\dagger \to I_\phi^\dagger \to I_\phi^{\mathrm{ad}} \to 1$, and $\delta^2$ is associated with $1 \to Z_\phi^\dagger \to Z_{I_\phi} \to G^{\mathrm{ab}} \to 1$. To prove our desired statement that the image of $s$ under (5.10.5.6) is zero, it suffices to prove that the image of $[G(\mathbb{Q})_+]^{\mathrm{ab}}$ under the boundary map $G^{\mathrm{ab}}(\mathbb{Q}) \to \mathbf{H}^1(\mathbb{Q}, Z_\phi^\dagger)$ analogous to $\delta^2$ is contained in $\text{Ш}^\infty_{G_{\mathrm{der}}}(\mathbb{Q}, Z_\phi^\dagger)$.

In fact, a stronger statement is true, namely that the image of $[G(\mathbb{Q})_+]^{\mathrm{ab}}$ under the boundary map $G^{\mathrm{ab}}(\mathbb{Q}) \to \mathbf{H}^1(\mathbb{Q}, Z_{G_{\mathrm{der}}})$ associated with $1 \to Z_{G_{\mathrm{der}}} \to Z_G \to G^{\mathrm{ab}} \to 1$ is contained in $\text{Ш}^\infty_{G_{\mathrm{der}}}(\mathbb{Q}, Z_{G_{\mathrm{der}}})$. (This is indeed stronger, as $Z_{G_{\mathrm{der}}} \subset Z_\phi^\dagger$.) This statement follows from Corollary 1.2.11 applied to $I = G$. □

**Lemma 5.10.8.** *Let $(\mathscr{I}, \mathscr{J})$ be a weakly amicable pair. Let $(\bar{y}_i, \bar{y}'_i), i = 1, 2$ be two markings of $(\mathscr{I}, \mathscr{J})$. Then*

$$\tau^{\mathcal{H}}_{\bar{y}_1, \bar{y}'_1} = \tau^{\mathcal{H}}_{\bar{y}_2, \bar{y}'_2},$$

*where the two sides are defined in §5.10.6.*

*Proof.* Fix $\phi \in \mathscr{J}$. We view $\tau^{\mathcal{H}}_{\bar{y}_1, \bar{y}'_1}$ and $\tau^{\mathcal{H}}_{\bar{y}_2, \bar{y}'_2}$ as elements of $\mathcal{H}(\phi)$. Pick $x \in \mathscr{I}$. By the same argument as in the proof of Lemma 5.10.7, we reduce to the case where $\bar{y}'_1 = \bar{y}'_2 = \bar{y}'$. In this case, pick $y_i \in Y(x)$ lifting $\bar{y}_i$ for $i = 1, 2$, and pick $y' \in Y(\phi)$ lifting $\bar{y}'$. Then by the same argument we know that the elements $\tau_{x,\phi,y_1,y'}$ and $\tau_{x,\phi,y_2,y'}$ of $I_\phi^{\mathrm{ad}}(\mathbb{A}_f)/I_\phi^{\mathrm{ad}}(\mathbb{Q})$ differ by left multiplication by an element of $I_\phi(\mathbb{A}_f)$. But this immediately implies what we want. □

**Definition 5.10.9.** Let $(\mathscr{I}, \mathscr{J})$ be an amicable pair. Let $\phi \in \mathscr{J}$. We define

$$\tau^{\mathfrak{H}}(\mathscr{I}, \mathscr{J}) \in \mathfrak{H}(\phi)$$

to be $\tau^{\mathfrak{H}}_{\bar{y}, \bar{y}'}$ (see §5.10.6) for any $\pi^*$-compatible marking $(\bar{y}, \bar{y}')$ of $(\mathscr{I}, \mathscr{J})$. By Lemma 5.10.7 this is well defined. Similarly, we define

$$\tau^{\mathcal{H}}(\mathscr{I}, \mathscr{J}) \in \mathcal{H}(\phi)$$

to be $\tau^{\mathcal{H}}_{\bar{y}, \bar{y}'}$ (see §5.10.6) for any marking $(\bar{y}, \bar{y}')$. This is well defined by Lemma 5.10.8. If we identify $\mathfrak{H}(\phi)$ (resp. $\mathcal{H}(\phi)$) for all $\phi \in \mathscr{J}$ as discussed in §5.10.6, then $\tau^{\mathfrak{H}}(\mathscr{I}, \mathscr{J})$ (resp. $\tau^{\mathcal{H}}(\mathscr{I}, \mathscr{J})$) is independent of $\phi$.

*Remark* 5.10.10. The proofs of Lemma 5.10.7 and Lemma 5.10.8 also show that the isomorphism $f : I_x \xrightarrow{\sim} I_\phi$ in §5.10.3 indeed depends only on $(x, \phi)$, up to composing with inner automorphisms defined over $\mathbb{Q}$. In fact, suppose we have two pairs $(y_1, y_1')$ and $(y_2, y_2')$ in $Y(x) \times Y(\phi)$, whose images $(\bar{y}_1, \bar{y}_1')$ and $(\bar{y}_2, \bar{y}_2')$ in $\bar{Y}(\mathscr{I}) \times \bar{Y}(\mathscr{J})$ are markings of $(\mathscr{I}, \mathscr{J})$. For $i = 1, 2$, the tuple $(x, \phi, y_i, y_i')$ gives rise to an isomorphism $f_i : I_x \xrightarrow{\sim} I_\phi$, which is well defined up to composing with inner automorphisms defined over $\mathbb{Q}$. In order to show that $f_1$ and $f_2$ differ only by an inner automorphism, we argue in the same way as in the proof of Lemma 5.10.7 and Lemma 5.10.8 to reduce to the case where $\bar{y}_1' = \bar{y}_2'$. In this case, clearly replacing $(y_1, y_1')$ by $(y_1, y_2')$ does not change $f_1$, so we further reduce to the case where $y_1' = y_2'$. Then as we showed in the proof of Lemma 5.10.7 (see especially (5.10.7.1)), we can choose $f_2$ to be equal to $f_1$.

5.11. **Gauges.**

**Definition 5.11.1.** By a *special fork*, we mean an ordered pair $(\mathfrak{s}, \mathfrak{s}')$ consisting of two special point data $\mathfrak{s}, \mathfrak{s}' \in \mathcal{SPD}(G, X)$ of the form $\mathfrak{s} = (T, i, h)$ and $\mathfrak{s}' = (T, i', h)$, satisfying the following conditions:

(i) The points $i \circ h$ and $i' \circ h$ lie in the same connected component of $X$.
(ii) The maps $i : T \to G$ and $i' : T \to G$ are conjugate by $G^{\mathrm{ad}}(\overline{\mathbb{Q}})$.

When we want to make explicit the ingredients, we also write $(T, h, i, i')$ for a special fork.

**5.11.2.** Given a special fork $(T, h, i, i')$, the two composite maps $T \xrightarrow{i} G \to G^{\mathrm{ab}}$ and $T \xrightarrow{i'} G \to G^{\mathrm{ab}}$ are equal. We denote the kernel by $T^\dagger$. The two maps $Ш^\infty(\mathbb{Q}, T^\dagger) \to Ш^\infty(\mathbb{Q}, G_{\mathrm{der}})$ induced by $i$ and $i'$ are equal (since $Ш^\infty(\mathbb{Q}, G_{\mathrm{der}}) \cong Ш^\infty_{\mathrm{ab}}(\mathbb{Q}, G_{\mathrm{der}})$ and since $i, i'$ are conjugate by $G^{\mathrm{ad}}(\overline{\mathbb{Q}})$), and we denote the kernel by $Ш^\infty_{G_{\mathrm{der}}}(\mathbb{Q}, T^\dagger)$. Similarly, we define $Ш^\infty_G(\mathbb{Q}, T)$ to be the kernel of $Ш^\infty(\mathbb{Q}, T) \to Ш^\infty(\mathbb{Q}, G)$ induced by either $i$ or $i'$.

Clearly there exists $g \in G_{\mathrm{der}}(\overline{\mathbb{Q}})$ such that $\mathrm{Int}(g) \circ i = i'$. Write $T'$ for $i'(T) \subset G$ write $T'^\dagger$ for $T' \cap G_{\mathrm{der}} = i'(T^\dagger)$. Since $T'$ is self-centralizing in $G$, the cocycle $(g^\rho g^{-1})_{\rho \in \Gamma}$ defines an element $\alpha_{i,i'} \in \mathbf{H}^1(\mathbb{Q}, T'^\dagger)$ which is independent of the choice of $g$.

**Lemma 5.11.3.** *The element $\alpha_{i,i'}$ lies in $Ш^\infty_{G_{\mathrm{der}}}(\mathbb{Q}, T'^\dagger)$.*

*Proof.* The only non-trivial condition to check is that $\alpha_{i,i'}$ has trivial image in $\mathbf{H}^1(\mathbb{R}, T'^\dagger)$. The argument is similar to the proof of [Kis17, Prop. 4.4.13]. Let $K'_\infty$ be the $\mathbb{R}$-algebraic group that is the stabilizer of $i' \circ h$ in $G_{\mathrm{der},\mathbb{R}}$. Since $i \circ h$ and $i' \circ h$ lie in the same connected component of $X$, they are conjugate by an element of $G_{\mathrm{der}}(\mathbb{R})$ (or even $G_{\mathrm{sc}}(\mathbb{R})$). It follows that $\alpha_{i,i'}$ has trivial image in $\mathbf{H}^1(\mathbb{R}, K'_\infty)$. By [Kis17, Lem. 4.4.5] applied to $H' = T'^\dagger$ and $H = K'_\infty$, the only element of $\mathbf{H}^1(\mathbb{R}, T'^\dagger)$ having trivial image in $\mathbf{H}^1(\mathbb{R}, K'_\infty)$ is the trivial element. Hence $\alpha_{i,i'}$ must have trivial image in $\mathbf{H}^1(\mathbb{R}, T'^\dagger)$, as desired. □

In the sequel, given any torus $T$ over $\mathbb{Q}$, we write $\mathcal{T}^\circ$ for the connected Néron model of $T_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$.

**Definition 5.11.4.** Let $\mathfrak{g}$ be a tuple $(T, h, i, i', y, y')$, where

- $(T, h, i, i')$ is a special fork.
- $y$ is an element of the $\mathcal{T}^\circ(\mathbb{Z}_p^{\mathrm{ur}})$-torsor $Y(\Upsilon_{(T,h)})^\circ$ (see §5.7.4).

- $y'$ is an element of $X(\Psi_{T,h})_{\mathrm{neu}}$ (see §5.9.1).

We define the following objects associated with $\mathfrak{g}$.

- Let $x_{\mathfrak{g}}$ be the point $x_{(T,i,h)} \in \mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$ (see §5.7.1), and let $\mathscr{I}_{\mathfrak{g}}$ be the isogeny class of $x_{\mathfrak{g}}$.
- Let $\phi_{\mathfrak{g}}$ be the admissible morphism $\phi(T, i', h) : \mathfrak{Q} \to \mathfrak{G}_G$ (see Definition 3.3.2 and Theorem 3.3.3), and let $\mathscr{J}_{\mathfrak{g}}$ be the conjugacy class of $\phi_{\mathfrak{g}}$.
- Let $\bar{y}_{\mathfrak{g}}$ be the image of $y$ under $Y(\Upsilon_{(T,h)})^{\circ} \to Y_p(x_{\mathfrak{g}}) \hookrightarrow Y(x_{\mathfrak{g}}) \to \bar{Y}(\mathscr{I}_{\mathfrak{g}})$, where the first map is as in (5.7.4.4), and the second map sends $y_p$ to $(y_p, 1)$ where 1 is the canonical base point of $Y^p(x_{\mathfrak{g}}) \cong G(\mathbb{A}_f^p)$.
- Let $\bar{y}'_{\mathfrak{g}}$ be the image of $y'$ under $X(\Psi_{T,h})_{\mathrm{neu}} \to Y(\phi_{\mathfrak{g}}) \to \bar{Y}(\mathscr{J}_{\mathfrak{g}})$, where the first map is as in (5.9.1.1).
- Let $\delta_{\mathfrak{g}}$ be the element of $T(\mathbb{Q}_p^{\mathrm{ur}})^{\mathrm{mot}}$ attached to $y$ as in Proposition 5.7.6 (i).

We say that $\mathfrak{g}$ is a *quasi-gauge*, if the following condition is satisfied:

- The element $\mathfrak{k}(T, i', h, \delta_{\mathfrak{g}}) \in \mathfrak{K}\mathfrak{T}^{\mathrm{str}}$ defined as in §5.3.9 represents an element of $[\mathfrak{k}(\bar{y}'_{\mathfrak{g}})] \subset \mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\equiv$, where $[\mathfrak{k}(\bar{y}'_{\mathfrak{g}})]$ is the $\mathscr{U}_{\bar{y}'_{\mathfrak{g}}}$-orbit in $\mathfrak{K}\mathfrak{T}^{\mathrm{str}}/\equiv$ associated with $\bar{y}'_{\mathfrak{g}}$ as in §5.8.2. (Note that $\mathfrak{k}(T, i', h, \delta_{\mathfrak{g}})$ is indeed defined, since $\delta_{\mathfrak{g}}$ lies in the $\overset{\circ}{\sim}$-equivalence class in $T(\mathbb{Q}_p^{\mathrm{ur}})^{\mathrm{mot}}$ corresponding to $-\mu_h$ by Proposition 5.7.6(i).)

**Lemma 5.11.5.** *Let $(T, h, i')$ and $(T, h, i'_1)$ be two elements of $\mathcal{SPD}(G, X)$. Let $y' \in X(\Psi_{T,h})_{\mathrm{neu}}$. Then there exists $y \in Y(\Upsilon_{(T,h)})^{\circ}$ satisfying the following conditions.*

*(i) For every special fork of the form $(T, h, i, i')$, the tuple $(T, h, i, i', y, y')$ is a quasi-gauge.*
*(ii) For every special fork of the form $(T, h, i_1, i'_1)$, the tuple $(T, h, i_1, i'_1, y, y')$ is a quasi-gauge.*

*Proof.* We still write $y'$ for the image of $y'$ under the map

$$X(\Psi_{T,h})_{\mathrm{neu}} \longrightarrow Y(\phi(T, i', h))$$

as in (5.9.1.1). We write $y'_1$ for the image of $y'$ under the analogous map

$$X(\Psi_{T,h})_{\mathrm{neu}} \to Y(\phi(T, i'_1, h)).$$

By Lemma 5.9.3 there exists $\delta_T$ in the $\overset{\circ}{\sim}$-equivalence class in $T(\mathbb{Q}_p^{\mathrm{ur}})^{\mathrm{mot}}$ determined by $-\mu_h$ such that $\mathfrak{k}(T, i', h, \delta_T) \in \mathfrak{K}\mathfrak{T}^{\mathrm{str}}$ represents $[\mathfrak{k}(y')]$ and $\mathfrak{k}(T, i'_1, h, \delta_T) \in \mathfrak{K}\mathfrak{T}^{\mathrm{str}}$ represents $[\mathfrak{k}(y'_1)]$. By Proposition 5.7.6 (i), the image of $Y(\Upsilon_{(T,h)})^{\circ}$ under the map $Y(\Upsilon_{(T,h)}) \to T(\mathbb{Q}_p^{\mathrm{ur}}), z \mapsto \delta_z$ is precisely the $\overset{\circ}{\sim}$-equivalence of $\delta_T$ in $T(\mathbb{Q}_p^{\mathrm{ur}})^{\mathrm{mot}}$. Thus we can find $y \in Y(\Upsilon_{(T,h)})^{\circ}$ such that $\delta_y = \delta_T$. This $y$ is our desired element. $\square$

**Definition 5.11.6.** Let $\mathfrak{g}$ be a quasi-gauge. By a *rectification of* $\mathfrak{g}$, we mean an element $u \in \ker(G(\mathbb{A}_f^*) \to \pi^*(G))$ satisfying the following conditions:

(i) The pair $(\bar{y}_{\mathfrak{g}} \cdot u, \bar{y}'_{\mathfrak{g}})$ is a marking of $(\mathscr{I}_{\mathfrak{g}}, \mathscr{J}_{\mathfrak{g}})$.
(ii) Let $u_p$ be the component of $u$ in $G(\mathbb{Q}_p^{\mathrm{ur}})$. We have

$$i'(\delta_{\mathfrak{g}}) = u_p^{-1} i(\delta_{\mathfrak{g}}) \sigma(u_p).$$

We call $\mathfrak{g}$ a *gauge*, if it admits a rectification.

**Lemma 5.11.7.** *A quasi-gauge $\mathfrak{g}$ is a gauge if and only if $(\mathscr{I}_{\mathfrak{g}}, \mathscr{J}_{\mathfrak{g}})$ is an amicable pair.*

*Proof.* Suppose $\mathfrak{g}$ has a rectification $u$. Write $\mathfrak{g} = (T, h, i, i', y, y')$. To show that $(\mathscr{I}_{\mathfrak{g}}, \mathscr{J}_{\mathfrak{g}})$ is amicable, we only need to check that the marking $(\bar{y}_{\mathfrak{g}} \cdot u, \bar{y}'_{\mathfrak{g}})$ is $\pi^*$-compatible. By Proposition 5.7.6 (iii) and by the fact that $u \in \ker(G(\mathbb{A}_f^*) \to \pi^*(G))$, the image of $\bar{y}_{\mathfrak{g}} \cdot u$ in $\pi^*(G, X)$ equals that of $i \circ h$. By Proposition 5.9.2 (i), the image of $\bar{y}'_{\mathfrak{g}}$ in $\pi^*(G, X)$ equals that of $i' \circ h$. But $i \circ h$ and $i' \circ h$ lie in the same connected component of $X$ by assumption, so they have the same image in $\pi^*(G, X)$. Thus $(\bar{y}_{\mathfrak{g}} \cdot u, \bar{y}'_{\mathfrak{g}})$ is indeed $\pi^*$-compatible.

Conversely, suppose that $(\mathscr{I}_{\mathfrak{g}}, \mathscr{J}_{\mathfrak{g}})$ is an amicable pair. By Lemma 5.10.2, there is a $\pi^*$-compatible marking of $(\mathscr{I}_{\mathfrak{g}}, \mathscr{J}_{\mathfrak{g}})$ of the form $(\bar{z}, \bar{y}_{\mathfrak{g}'})$ for some $\bar{z} \in \bar{Y}(\mathscr{I}_{\mathfrak{g}})$. Find $w \in G(\mathbb{A}_f^*)$ such that $\bar{z} = \bar{y}_{\mathfrak{g}} \cdot w$. Again by Proposition 5.7.6 (iii), Proposition 5.9.2 (i), and the assumption that $i \circ h$ and $i' \circ h$ lie in the same connected component of $X$, we know that $\bar{y}_{\mathfrak{g}}$ and $\bar{y}'_{\mathfrak{g}}$ have the same image in $\pi^*(G, X)$. Since $(\bar{y}_{\mathfrak{g}} \cdot w, \bar{y}'_{\mathfrak{g}})$ is $\pi^*$-compatible, we must have $w \in \ker(G(\mathbb{A}_f^*) \to \pi^*(G))$.

By the defining property of a quasi-gauge, $[\mathfrak{k}(\bar{y}'_{\mathfrak{g}})] \subset \mathfrak{KT}^{\mathrm{str}}/\equiv$ contains the image of $\mathfrak{k}(T, i', h, \delta_{\mathfrak{g}}) \in \mathfrak{KT}^{\mathrm{str}}$. Since $(\bar{z}, \bar{y}'_{\mathfrak{g}})$ is a marking of $(\mathscr{I}_{\mathfrak{g}}, \mathscr{J}_{\mathfrak{g}})$, we know that $[\mathfrak{k}(\bar{y}'_{\mathfrak{g}})]$ also contains $\mathfrak{k}(\bar{z})$, which is equal to $\mathfrak{k}(\bar{y}_{\mathfrak{g}}) \cdot w$. By Proposition 5.7.6 (ii), the $\delta$-component of $\mathfrak{k}(\bar{y}_{\mathfrak{g}})$ is $i(\delta_{\mathfrak{g}})$. By definition, the $\delta$-component of $\mathfrak{k}(T, i', h, \delta_{\mathfrak{g}})$ is $i'(\delta_{\mathfrak{g}})$. Thus there exists $w_0 \in \mathscr{U}_{\bar{y}'_{\mathfrak{g}}}$ such that

$$(w_p w_0)^{-1} i(\delta_{\mathfrak{g}}) \sigma(w_p w_0) = i'(\delta_{\mathfrak{g}}),$$

where $w_p$ denotes the component of $w$ in $G(\mathbb{Q}_p^{\mathrm{ur}})$.

Let $u = w w_0 \in G(\mathbb{A}_f^*)$. Since $w \in \ker(G(\mathbb{A}_f^*) \to \pi^*(G))$ and $w_0 \in \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$, we have $u \in \ker(G(\mathbb{A}_f^*) \to \pi^*(G))$. It is straightforward to check that $u$ is a rectification of $\mathfrak{g}$. □

**Definition 5.11.8.** Let $\mathfrak{s} = (T, i, h) \in \mathcal{SPD}(G, X)$. We write $\mathscr{I}_{\mathfrak{s}}$ for the isogeny class of $x_{\mathfrak{s}} \in \mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$, and write $\mathscr{J}_{\mathfrak{s}}$ for the conjugacy class of the admissible morphism $\phi(\mathfrak{s}) : \mathfrak{Q} \to \mathfrak{G}_G$.

**Corollary 5.11.9.** *Let $\mathfrak{s} \in \mathcal{SPD}(G, X)$. Then $(\mathscr{I}_{\mathfrak{s}}, \mathscr{J}_{\mathfrak{s}})$ is an amicable pair.*

*Proof.* By Lemma 5.11.5, we can extend the special fork $(\mathfrak{s}, \mathfrak{s})$ to a quasi-gauge of the form $\mathfrak{g} = (T, h, i, i, y, y')$. By the defining property of a quasi-gauge, the fact that the two embeddings in $\mathfrak{g}$ are both $i$, and Proposition 5.7.6 (ii), we know that $(\bar{y}_{\mathfrak{g}}, \bar{y}'_{\mathfrak{g}})$ is a marking of $(\mathscr{I}_{\mathfrak{g}}, \mathscr{J}_{\mathfrak{g}})$. It follows that $u = 1$ is a rectification of $\mathfrak{g}$, and therefore $(\mathscr{I}_{\mathfrak{s}}, \mathscr{J}_{\mathfrak{s}}) = (\mathscr{I}_{\mathfrak{g}}, \mathscr{J}_{\mathfrak{g}})$ is amicable by Lemma 5.11.7. □

**Definition 5.11.10.** Let $\mathfrak{g} = (T, h, i, i', y, y')$ be a gauge. We have a natural $\mathbb{Q}$-embedding $T \hookrightarrow I_{\phi_{\mathfrak{g}}}$ whose composition with $I_{\phi_{\mathfrak{g}}, \overline{\mathbb{Q}}} \hookrightarrow G_{\overline{\mathbb{Q}}}$ is $i'$, and a natural $\mathbb{Q}$-embedding $T \hookrightarrow I_{x_{\mathfrak{g}}}$ as in (5.7.2.2). We say that a $\mathbb{Q}$-isomorphism $f : I_{x_{\mathfrak{g}}} \xrightarrow{\sim} I_{\phi_{\mathfrak{g}}}$ is $\mathfrak{g}$-*adapted*, if it satisfies the following conditions.

(i) By Lemma 5.11.7, $(\mathscr{I}_{\mathfrak{g}}, \mathscr{J}_{\mathfrak{g}})$ is (weakly) amicable, so there is canonical $I^{\mathrm{ad}}_{\phi_{\mathfrak{g}}}(\mathbb{Q})$-conjugacy class of isomorphisms $I_{x_{\mathfrak{g}}} \xrightarrow{\sim} I_{\phi_{\mathfrak{g}}}$ as in §5.10.3 (cf. Remark 5.10.10). We require that $f$ lies in this conjugacy class.

(ii) $f$ commutes with the natural $\mathbb{Q}$-embeddings $T \hookrightarrow I_{\phi_{\mathfrak{g}}}$ and $T \hookrightarrow I_{x_{\mathfrak{g}}}$.

**5.11.11.** Let $\mathfrak{g} = (T, h, i, i', y, y')$ be a gauge. By Lemma 5.11.7, the pair $(\mathscr{I}_{\mathfrak{g}}, \mathscr{J}_{\mathfrak{g}})$ is amicable. Therefore by Definition 5.10.9 we have a well-defined element

$$\tau^{\mathfrak{H}}(\mathscr{I}_{\mathfrak{g}}, \mathscr{J}_{\mathfrak{g}}) \in \mathfrak{H}(\phi) = \operatorname{coker}(\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p) \to \mathbf{H}^1(\mathbb{A}_f, Z_\phi^\dagger)/\text{Ш}^\infty_{G_{\mathrm{der}}}(\mathbb{Q}, Z_\phi^\dagger)).$$

Define $T^\dagger$ and $\text{Ш}^\infty_{G_{\mathrm{der}}}(\mathbb{Q}, T^\dagger)$ as in §5.11.2. Then we have $Z_\phi^\dagger \subset T^\dagger \subset I_\phi^\dagger$ as subgroups of $I_\phi$. Thus we have a natural map from $\mathfrak{H}(\phi)$ to

$$\operatorname{coker}\left(\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p) \to \mathbf{H}^1(\mathbb{A}_f, T^\dagger)/\text{Ш}^\infty_{G_{\mathrm{der}}}(\mathbb{Q}, T^\dagger)\right). \tag{5.11.11.1}$$

Here the map in the definition of the cokernel is the restriction of the boundary map $G^{\mathrm{ab}}(\mathbb{Q}_p) \to \mathbf{H}^1(\mathbb{Q}_p, T^\dagger)$ arising from the short exact sequence $1 \to T^\dagger \to T \to G^{\mathrm{ab}} \to 1$.

**Proposition 5.11.12.** *Let $\mathfrak{g} = (T, h, i, i', y, y')$ be a gauge, and assume that there exists a $\mathfrak{g}$-adapted isomorphism $f : I_{x_{\mathfrak{g}}} \xrightarrow{\sim} I_{\phi_{\mathfrak{g}}}$. Then the image of $\tau^{\mathfrak{H}}(\mathscr{I}_{\mathfrak{g}}, \mathscr{J}_{\mathfrak{g}})$ in (5.11.11.1), as explained in §5.11.11, is trivial.*

*Proof.* **(I) Finding a representative of $\tau^{\mathfrak{H}}(\mathscr{I}_g, \mathscr{J}_g)$.**

In this part, we find an element $\tau \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$ representing $\tau^{\mathfrak{H}}(\mathscr{I}_{\mathfrak{g}}, \mathscr{J}_{\mathfrak{g}})$. Write $\phi$ and $x$ for $\phi_{\mathfrak{g}}$ and $x_{\mathfrak{g}}$. We denote the image of $y$ in $Y(x)$ still by $y$, and denote the image of $y'$ in $Y(\phi)$ still by $y'$. Fix a rectification $u$ of $\mathfrak{g}$. Let $z := y \cdot u \in Y(x)$, and let $\bar{z}$ be the image of $z$ in $\bar{Y}(\mathscr{I}_{\mathfrak{g}})$. For each finite place $v$, let $I_v$ be the $\mathbb{Q}_v$-group associated with $\mathfrak{k}(y)$, and let $I'_v$ be the $\mathbb{Q}_v$-group associated with (any element of) $[\mathfrak{k}(y')]$. Then $I'_v$ is also the $\mathbb{Q}_v$-group associated with $\mathfrak{k}(z)$ since $(\bar{z}, \bar{y}'_{\mathfrak{g}})$ is a marking. Fix a $\mathfrak{g}$-adapted isomorphism $f : I_x \xrightarrow{\sim} I_\phi$, which exists by our assumption. Then there exists $\tau = (\tau_v)_v \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$ such that

$$\iota_{z,v} \circ f^{-1} = \iota_{y',v} \circ \operatorname{Int}(\tau_v) : I_{\phi,\mathbb{Q}_v} \to I'_v. \tag{5.11.12.1}$$

As we showed in the proof of the “only if” direction in Lemma 5.11.7, the marking $(\bar{z}, \bar{y}'_{\mathfrak{g}})$ is $\pi^*$-compatible. Hence $\tau^{\mathfrak{H}}(\mathscr{I}_{\mathfrak{g}}, \mathscr{J}_{\mathfrak{g}})$ is the image of $\tau$. In the rest of the proof we show that the image of $\tau$ in (5.11.11.1) is trivial.

**(II) Constructing an element $t_v \in T(\overline{\mathbb{Q}}_v)$.**

Choose $g \in G_{\mathrm{der}}(\overline{\mathbb{Q}})$ such that

$$\operatorname{Int}(g) \circ i = i'. \tag{5.11.12.2}$$

Denote the natural embeddings $T \hookrightarrow I_\phi$ and $T \hookrightarrow I_x$ (cf. Definition 5.11.10) by $j'$ and $j$ respectively. Then $\iota_{y,v} \circ j = i$ and $\iota_{y',v} \circ j' = i'$. Let $v$ be a finite place, and let $t \in T(\overline{\mathbb{Q}}_v)$ be a test element. Write $u_v$ for the component of $u$ in $G(\mathbb{Q}_v)$ (resp. $G(\mathbb{Q}_p^{\mathrm{ur}})$) at $v \neq p$ (resp. $v = p$). When $v \neq p$, we have the following equalities between elements of $G(\overline{\mathbb{Q}}_v)$:

$$\begin{aligned} \iota_{y',v}\big(\tau_v j'(t) \tau_v^{-1}\big) &= \iota_{z,v} \circ f^{-1}(j'(t)) && \text{by (5.11.12.1)} \\ &= \iota_{z,v}(j(t)) && \text{since } f \text{ is } \mathfrak{g}\text{-adapted} \\ &= \operatorname{Int}(u_v)^{-1} \circ \iota_{y,v}(j(t)) && \text{since } z = y \cdot u \\ &= \operatorname{Int}(u_v)^{-1} \circ i(t). \end{aligned} \tag{5.11.12.3}$$

When $v = p$, the above computation is still valid if we interpret the equalities as between elements of $G(\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p)$, and view $u_p$ as an element thereof via the map $G(\mathbb{Q}_p^{\mathrm{ur}}) \to G(\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p)$ induced by $\mathbb{Q}_p^{\mathrm{ur}} \to \mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p, a \mapsto a \otimes 1$.

Choose $\widetilde{\tau}_v \in I_\phi^\dagger(\overline{\mathbb{Q}}_v)$ lifting $\tau_v \in I_\phi^{\mathrm{ad}}(\mathbb{Q}_v)$, and write $\widehat{\tau}_v$ for $\iota_{y',v}(\widetilde{\tau}_v) \in I'_v(\overline{\mathbb{Q}}_v)$. Then we have

(5.11.12.4)
$$\iota_{y',v}\big(\tau_v j'(t)\tau_v^{-1}\big) = \mathrm{Int}\,\big(\widehat{\tau}_v\big)\big(\iota_{y',v}(j'(t))\big) = \mathrm{Int}\,\big(\widehat{\tau}_v\big)(i'(t)) = \mathrm{Int}\,\big(\widehat{\tau}_v g\big)(i(t)).$$

For every finite place $v$, set
$$s_v = u_v\widehat{\tau}_v g.$$
Comparing the computations (5.11.12.3) and (5.11.12.4), we have
$$i(t) = \mathrm{Int}(s_v)(i(t)), \quad \forall t \in T(\overline{\mathbb{Q}}_v). \tag{5.11.12.5}$$
When $v \neq p$, $s_v$ is an element of $G(\overline{\mathbb{Q}}_v)$. It follows from (5.11.12.5) that $s_v \in i(T(\overline{\mathbb{Q}}_v))$. When $v = p$, $s_p$ is *a priori* an element of $G(\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p)$. Now $\widehat{\tau}_p$ lies in $I'_p(\overline{\mathbb{Q}}_p)$, and $I'_p$ is the $\sigma$-centralizer of $i'(\delta_{\mathfrak{g}})$. In the following computation, we let $\sigma$ act on $G(\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p)$ only via the first factor $\mathbb{Q}_p^{\mathrm{ur}}$. We have
$$\begin{aligned} s_p i(\delta_{\mathfrak{g}})\sigma(s_p)^{-1} &= u_p\widehat{\tau}_p g i(\delta_{\mathfrak{g}})g^{-1}\sigma(\tau_p)^{-1}\sigma(u_p)^{-1} && \text{because } \sigma(g) = g\\ &= u_p\widehat{\tau}_p i'(\delta_{\mathfrak{g}})\sigma(\widehat{\tau}_p)^{-1}\sigma(u_p)^{-1} && \text{by (5.11.12.2)}\\ &= u_p i'(\delta_{\mathfrak{g}})\sigma(u_p)^{-1} && \text{because } \widehat{\tau}_p \in I'_p(\overline{\mathbb{Q}}_p)\\ &= i(\delta_{\mathfrak{g}}) && \text{by (ii) in Definition 5.11.6.} \end{aligned}$$
Comparing this with (5.11.12.5), we see that $s_p$ is in fact $\sigma$-invariant, i.e., it lies in $G(\overline{\mathbb{Q}}_p)$ (which is embedded into $G(\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p)$ via $\overline{\mathbb{Q}}_p \to \mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p, a \mapsto 1 \otimes a$). It then follows from (5.11.12.5) that $s_p \in i(T(\overline{\mathbb{Q}}_p))$.

We have seen that $s_v \in i(T(\overline{\mathbb{Q}}_v))$ for every finite place $v$. Write
$$s_v = i(t_v), \quad t_v \in T(\overline{\mathbb{Q}}_v).$$

**(III) Relationship between $t_v$ and $\tau$.**

In this part, we show that the image of $\tau$ in (5.11.11.1) is represented by $(t_v)_v$ in a suitable sense.

In the sequel, for every $\mathbb{Q}$-algebra $R$ and every $r \in G(R)$, we write $r^{\mathrm{ab}}$ for the image of $r$ in $G^{\mathrm{ab}}(R)$. Note that $s_v^{\mathrm{ab}} = u_v^{\mathrm{ab}}$. Since $u_p \in G(\mathbb{Q}_p^{\mathrm{ur}})$ and since $s_p$ is $\sigma$-invariant, we have $u_p^{\mathrm{ab}} \in G^{\mathrm{ab}}(\mathbb{Q}_p)$.[35] Thus $u^{\mathrm{ab}} \in G^{\mathrm{ab}}(\mathbb{A}_f)$. Since $u \in \ker(G(\mathbb{A}_f^*) \to \pi^*(G))$, we can write $u = u^{(0)}u^{(1)}$, with $u^{(0)} \in G(\mathbb{Q})_+$ and $u^{(1)} \in G_{\mathrm{der}}(\mathbb{A}_f^*)\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$. Since $u^{\mathrm{ab}} \in G^{\mathrm{ab}}(\mathbb{A}_f)$, we have $u^{(1),\mathrm{ab}} \in \mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p)$.

We view the embeddings $j' : T \hookrightarrow I_\phi$ and $i' : T \hookrightarrow G$ both as inclusions and omit them from the notations. For each finite place $v$, the cocycle $(\widehat{\tau}_v^{-1}{}^\rho\widehat{\tau}_v)_{v\in\Gamma_v}$ is valued in $T^\dagger$, and the collection of these cocycles for all $v$ represents the image of $\tau$ in (5.11.11.1). Let $v$ be a finite place and let $\rho \in \Gamma_v$. In the following computation, if $v = p$ we let $\rho$ act on $G(\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p)$ only via the second factor $\overline{\mathbb{Q}}_p$. In particular $u_v$ is $\rho$-invariant even for $v = p$. We compute
$$\begin{aligned} \widehat{\tau}_v^{-1}{}^\rho\widehat{\tau}_v &= g s_v^{-1} u_v {}^\rho u_v^{-1} {}^\rho s_v {}^\rho g^{-1} = g s_v^{-1} {}^\rho s_v {}^\rho g^{-1}\\ &= g i(t_v^{-1} {}^\rho t_v) g^{-1} g {}^\rho g^{-1} = t_v^{-1} {}^\rho t_v g {}^\rho g^{-1}. \end{aligned} \tag{5.11.12.6}$$
Here for the last equality we used (5.11.12.2), and in the last term we wrote $t_v {}^\rho t_v^{-1}$ for $i'(t_v {}^\rho t_v^{-1})$ as we explained above. By Lemma 5.11.3, the cocycle $(g^\rho g^{-1})_{\rho\in\Gamma}$

[35] This fact also follows directly from condition (ii) in Definition 5.11.6.

represents an element of $Ш^\infty_{G_{\mathrm{der}}}(\mathbb{Q}, T^\dagger)$. Therefore the cocycle $(t_v^{-1\rho} t_v)_{\rho\in\Gamma_v}$ represents an element $\beta_v \in \mathbf{H}^1(\mathbb{Q}_v, T^\dagger)$, and the collection $(\beta_v)_v$ represents the image of $\tau$ in (5.11.11.1).

**(IV) Finishing the proof.**

To finish the proof it suffices to show that $(\beta_v)_v$ has trivial image in (5.11.11.1). Now $\beta_v$ is equal to the image of $t_v^{\mathrm{ab}} = s_v^{\mathrm{ab}} = u_v^{\mathrm{ab}} \in G^{\mathrm{ab}}(\mathbb{Q}_v)$ under the boundary map $G^{\mathrm{ab}}(\mathbb{Q}_v) \to \mathbf{H}^1(\mathbb{Q}_v, T^\dagger)$ associated with the short exact sequence $1 \to T^\dagger \to T \to G^{\mathrm{ab}} \to 1$. We have $u^{\mathrm{ab}} = u^{(1),\mathrm{ab}} u^{(0),\mathrm{ab}}$. By Corollary 1.2.11, the image of $u^{(0),\mathrm{ab}}$ in $\mathbf{H}^1(\mathbb{A}_f, T^\dagger)$ comes from $Ш^\infty_{G_{\mathrm{der}}}(\mathbb{Q}, T^\dagger)$. We have also seen that $u^{(1),\mathrm{ab}} \in \mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p)$ (with trivial components away from $p$). Hence the image of $(\beta_v)_v$ in (5.11.11.1) is indeed zero. $\square$

**Proposition 5.11.13.** *Assume that we have two quasi-gauges of the form*

$$\mathfrak{g} = (T, h, i, i', y, y'), \quad \mathfrak{g}_1 = (T, h, i_1, i_1', y, y').$$

*Let $\lambda, \lambda' \in G_{\mathrm{der}}(\overline{\mathbb{Q}})$. Assume the following conditions.*

- *(i) We have $i_1 = \mathrm{Int}(\lambda) \circ i$ and $i_1' = \mathrm{Int}(\lambda') \circ i'$.*
- *(ii) We have $\lambda \in G_{\mathrm{der}}(\mathbb{R})^+ i(T^\dagger(\mathbb{C}))$, and $\lambda' \in G_{\mathrm{der}}(\mathbb{R})^+ i'(T^\dagger(\mathbb{C}))$.*
- *(iii) Define $T^\dagger$ to be the kernel of the common map $T \to G^{\mathrm{ab}}$ induced by $i, i', i_1, i_1'$. There exists a cocycle $\beta_0(\cdot) \in Z^1(\mathbb{Q}, T^\dagger)$ such that*

$$i(\beta_0(\rho)) = \lambda^{-1\rho}\lambda, \quad i'(\beta_0(\rho)) = \lambda'^{-1\rho}\lambda', \quad \forall \rho \in \Gamma. \tag{5.11.13.1}$$

- *(iv) There exists a rectification $u$ of $\mathfrak{g}$ such that $u \in G_{\mathrm{der}}(\mathbb{A}_f^*)\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$. In particular, $\mathfrak{g}$ is a gauge.*
- *(v) There exists a $\mathfrak{g}$-adapted isomorphism $f : I_{x_\mathfrak{g}} \xrightarrow{\sim} I_{\phi_\mathfrak{g}}$. (This notion makes sense since $\mathfrak{g}$ is a gauge.)*

*Then $\mathfrak{g}_1$ admits a rectification $u_1 \in G_{\mathrm{der}}(\mathbb{A}_f^*)\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$, and there exists a $\mathfrak{g}_1$-adapted isomorphism $f_1 : I_{x_{\mathfrak{g}_1}} \xrightarrow{\sim} I_{\phi_{\mathfrak{g}_1}}$.*

*Remark* 5.11.14. By condition (ii), for each $\rho \in \Gamma$ we have $\lambda^{-1\rho}\lambda \in i(T^\dagger)(\overline{\mathbb{Q}})$ and $\lambda'^{-1\rho}\lambda' \in i'(T^\dagger)(\overline{\mathbb{Q}})$. In view of this, condition (iii) is equivalent to the requirement that $i^{-1}(\lambda^{-1\rho}\lambda) = i'^{-1}(\lambda'^{-1\rho}\lambda')$ for all $\rho \in \Gamma$.

*Proof.* **(I) Some notations.**

We write $x$ and $x_1$ for the points $x_\mathfrak{g} = x_{(T,i,h)}$ and $x_{\mathfrak{g}_1} = x_{(T,i_1,h)}$ in $\mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$ respectively, and write $\phi$ and $\phi_1$ for the admissible morphisms $\phi_\mathfrak{g} = \phi(T, i', h)$ and $\phi_{\mathfrak{g}_1} = \phi(T, i_1', h)$ respectively. Let $j : T \hookrightarrow I_x$, $j' : T \hookrightarrow I_\phi$, $j_1 : T \hookrightarrow I_{x_1}$, and $j_1' : T \hookrightarrow I_{\phi_1}$ be the canonical $\mathbb{Q}$-embeddings. We denote the image of $y$ in $Y(x)$ still by $y$, and denote the image of $y$ in $Y(x_1)$ by $y_1$. Similarly, we denote the image of $y'$ in $Y(\phi)$ still by $y'$, and denote the image of $y'$ in $Y(\phi_1)$ by $y_1'$.

By construction $\delta_{\mathfrak{g}_1} = \delta_\mathfrak{g}$. We write $\delta_T$ for this element. To simplify notation, for $\varpi \in \{i, i', i_1, i_1'\}$, we write $\mathfrak{k}(\varpi)$ for $\mathfrak{k}(T, \varpi, h, \delta_T) \in \mathfrak{K}\mathfrak{T}^{\mathrm{str}}$ (see §5.3.9). Since $\mathfrak{g}$ and $\mathfrak{g}_1$ are quasi-gauges, we know that $\mathfrak{k}(i')$ (resp. $\mathfrak{k}(i_1')$) represents $[\mathfrak{k}(\bar{y}'_\mathfrak{g})]$ (resp. $[\mathfrak{k}(\bar{y}'_{\mathfrak{g}_1})]$). By Proposition 5.7.6 (ii), we know that $\mathfrak{k}(i)$ (resp. $\mathfrak{k}(i_1)$) represents $\mathfrak{k}(\bar{y}_\mathfrak{g})$ (resp. $\mathfrak{k}(\bar{y}_{\mathfrak{g}_1})$).

Let $v$ be a finite place. Let $I_v, I_v', I_{v,1}, I_{v,1}'$ be the reductive groups over $\mathbb{Q}_v$ associated with $\mathfrak{k}(i), \mathfrak{k}(i'), \mathfrak{k}(i_1), \mathfrak{k}(i_1')$ respectively. Write $u_v$ for the component of $u$ in $G(\mathbb{Q}_v)$ (resp. $G(\mathbb{Q}_p^{\mathrm{ur}})$) for $v \neq p$ (resp. $v = p$). Then we have a $\mathbb{Q}_v$-isomorphism

$\mathrm{Int}(u_v)^{-1} : I_v \xrightarrow{\sim} I'_v$, and we have

$$\iota_{y\cdot u,v} = \mathrm{Int}(u_v)^{-1} \circ \iota_{y,v} : I_{x,\mathbb{Q}_v} \xrightarrow{\sim} I'_v, \tag{5.11.14.1}$$

cf. the third equality in (5.11.12.3).[36] If $I$ is one of the four groups $I_v, I'_v, I_{v,1}, I'_{v,1}$, there is a canonical $\mathbb{Q}_v$-map $I \to G^{\mathrm{ab}}_{\mathbb{Q}_v}$, and we denote the kernel by $I^\dagger$.

**(II) Commutative diagrams involving $u$.**

As in §5.10.3, let $\tau = (\tau_v)_v \in I^{\mathrm{ad}}_\phi(\mathbb{A}_f)$ be the element associated with $(x, \phi, y \cdot u, y')$ and $f$. Namely, for each finite place $v$ we have

$$\iota_{y\cdot u,v} \circ f^{-1} = \iota_{y',v} \circ \mathrm{Int}(\tau_v) : I_{\phi,\mathbb{Q}_v} \longrightarrow I'_v. \tag{5.11.14.2}$$

The diagram

$$\begin{tikzcd} I_v^\dagger \arrow[r, "\mathrm{Int}(u_v)^{-1}"] & I_v'^{,\dagger} \\ I_x^\dagger \arrow[u, "\iota_{y,v}"] \arrow[r, "\mathrm{Int}(\tau_v)\circ f"] & I_\phi^\dagger \arrow[u, "\iota_{y',v}"'] \end{tikzcd} \tag{5.11.14.3}$$

consists of $\mathbb{Q}_v$-isomorphisms, and it commutes by (5.11.14.1) and (5.11.14.2). Since $\mathrm{Int}(\tau_v)$ induces the identity on $\mathbf{H}^1(\mathbb{Q}_v, I^\dagger_\phi) \cong \mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}_v, I^\dagger_\phi)$, and since $f \circ j = j'$ (as $f$ is $\mathfrak{g}$-adapted), we obtain from (5.11.14.3) a commutative diagram

$$\begin{tikzcd} \mathbf{H}^1(\mathbb{Q}_v, I_v^\dagger) \arrow[r, "\mathrm{Int}(u_v)^{-1}"] & \mathbf{H}^1(\mathbb{Q}_v, I_v'^{,\dagger}) \\ \mathbf{H}^1(\mathbb{Q}_v, T^\dagger) \arrow[u, "\iota_{y,v}\circ j"] \arrow[r, equal] & \mathbf{H}^1(\mathbb{Q}_v, T^\dagger) \arrow[u, "\iota_{y',v}\circ j'"'] \end{tikzcd}$$

Since $\iota_{y,v} \circ j = i$ and $\iota_{y',v} \circ j' = i'$, the above commutative diagram implies that there exists $\Delta_v \in I_v'^{,\dagger}(\overline{\mathbb{Q}}_v)$ such that

$$u_v^{-1} i(\beta_0(\rho)) u_v = \Delta_v \cdot i'(\beta_0(\rho)) \cdot {}^\rho\Delta_v^{-1}, \quad \forall \rho \in \Gamma_v. \tag{5.11.14.4}$$

In view of (5.11.13.1), we can rewrite (5.11.14.4) as

$$u_v^{-1} \lambda^{-1\rho}\lambda u_v = \Delta_v \lambda'^{-1\rho}\lambda'^\rho \Delta_v^{-1}, \quad \forall \rho \in \Gamma_v. \tag{5.11.14.5}$$

**(III) Constructing $u_1$.**

We shall construct a rectification $u_1$ of $\mathfrak{g}_1$. Let $u_{1,v} := \lambda u_v \Delta_v \lambda'^{-1}$. Then $u_{1,v} \in G(\overline{\mathbb{Q}}_v)$ for $v \neq p$ and $u_{1,p} \in G(\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p)$. Here when $v = p$ we view $\lambda, \lambda' \in G(\overline{\mathbb{Q}}_p)$ as inside $G(\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p)$ via $\overline{\mathbb{Q}}_p \to \mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p, a \mapsto 1 \otimes a$, and view $u_p \in G(\mathbb{Q}_p^{\mathrm{ur}})$ as inside $G(\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p)$ via $\mathbb{Q}_p^{\mathrm{ur}} \to \mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p, a \mapsto a \otimes 1$. When $v = p$, we let $\Gamma_p$ act on $G(\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p)$ only via the second factor $\overline{\mathbb{Q}}_p$. Thus $u_v$ is $\Gamma_v$-invariant for every finite place $v$. It then immediately follows from (5.11.14.5) that $u_{1,v}$ is $\Gamma_v$-invariant. Thus we have $u_{1,v} \in G(\mathbb{Q}_v)$ for $v \neq p$ and $u_{1,p} \in G(\mathbb{Q}_p^{\mathrm{ur}})$. We may and shall also assume that the elements $\Delta_v$ have been chosen such that the element

[36] As in the proof of Proposition 5.11.12, when $v = p$ we view $u_p$ as an element of $(\mathrm{Res}_{\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p} G)(\mathbb{Q}_p) = G(\mathbb{Q}_p^{\mathrm{ur}})$. The isomorphism $\mathrm{Int}(u_p) : I_p \xrightarrow{\sim} I'_p$ is understood as the isomorphism between two subfunctors of $\mathrm{Res}_{\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p} G$ induced by the inner automorphism $\mathrm{Int}(u_p)$ of $\mathrm{Res}_{\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p} G$.

$(\Omega_v)_{v\neq p} := (\iota_{y',v}^{-1}(\Delta_v))_{v\neq p} \in \prod_{v\neq p} I_\phi^\dagger(\overline{\mathbb{Q}}_v)$ comes from $I_\phi^\dagger(\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{A}_f^p)$. We then see that the element $u_1 := (u_{1,v})_v \in \prod_{v\neq p} G(\mathbb{Q}_v) \times G(\mathbb{Q}_p^{\mathrm{ur}})$ lies in $G(\mathbb{A}_f^*)$.

For $v \neq p$, we have $u_v \in G_{\mathrm{der}}(\mathbb{Q}_v)$ by our assumption (i), and so $u_{1,v} \in G_{\mathrm{der}}(\mathbb{Q}_v)$. Again by the assumption (i), the image of $u_p$ in $G^{\mathrm{ab}}(\mathbb{Q}_p^{\mathrm{ur}})$ lies in $\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p^{\mathrm{ur}})$. Since $u_p$ and $u_{1,p}$ have the same image in $G^{\mathrm{ab}}(\mathbb{Q}_p^{\mathrm{ur}})$, we have $u_{1,p} \in G_{\mathrm{der}}(\mathbb{Q}_p^{\mathrm{ur}})\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$ by the surjectivity of $\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}) \to \mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p^{\mathrm{ur}})$ (which follows from Lang's theorem applied to $\mathcal{G}_{\mathrm{der}}$; see the proof of Proposition 4.5.2). We have thus constructed an element $u_1 \in G_{\mathrm{der}}(\mathbb{A}_f)\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$.

**(IV) Proof that $u_1$ is a rectification of $\mathfrak{g}_1$.**

Clearly $u_1 \in \ker(G(\mathbb{A}_f^*) \to \pi^*(G))$. For a sufficiently divisible $n \in \mathbb{Z}_{\geq 1}$, let $\gamma_{0,T,n} = \delta_T\sigma(\delta_T)\cdots\sigma^{n-1}(\delta_T) \in T(\mathbb{Q})$ (see §5.3.9). Then $\mathfrak{k}(\varpi) \in \mathfrak{K}\mathfrak{T}^{\mathrm{str}}$ is represented by

$$(\varpi(\gamma_{0,T,n}), (\varpi(\gamma_{0,T,n}))_{v\neq p}, \varpi(\delta_T)) \in \mathfrak{T}_n^{\mathrm{str}},$$

for $\varpi \in \{i, i', i_1, i_1'\}$. Clearly $\varpi(\gamma_{0,T,n})$ for all four choices of $\varpi$ are conjugate in $G(\overline{\mathbb{Q}})$. For a finite place $v \neq p$, we have

$$\begin{aligned} \mathrm{Int}(u_{1,v})^{-1}(i_1(\gamma_{0,T,n})) &= \mathrm{Int}\left(\lambda'\Delta_v^{-1}u_v^{-1}\lambda^{-1}\right)(i_1(\gamma_{0,T,n})) \\ &= \mathrm{Int}\left(\lambda'\Delta_v^{-1}u_v^{-1}\right)(i(\gamma_{0,T,n})) \\ &= \mathrm{Int}\left(\lambda'\Delta_v^{-1}\right)(i'(\gamma_{0,T,n})) \\ &= \mathrm{Int}\left(\lambda'\right)(i'(\gamma_{0,T,n})) \\ &= i_1'(\gamma_{0,T,n}). \end{aligned} \tag{5.11.14.6}$$

Here the second equality is because $i_1 = \mathrm{Int}(\lambda) \circ i$, the third equality is because $\mathfrak{k}(i) \cdot u \in \mathfrak{k}(i') \cdot \mathscr{U}_{\tilde{y}'_{\mathfrak{g}}}$, the fourth equality is because $\Delta_v \in I_v'$ centralizes $i'(\gamma_{0,T,n})$, and the fifth equality is because $i_1' = \mathrm{Int}(\lambda') \circ i'$. For $a, b \in G(\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p)$, we write $\mathrm{Int}_\sigma(a)(b)$ for $ab\sigma(a)^{-1}$, where $\sigma$ acts on $G(\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p)$ only via the first factor $\mathbb{Q}_p^{\mathrm{ur}}$. Analogously as in the above computation, we have

$$\begin{aligned} \mathrm{Int}_\sigma(u_{1,p})^{-1}(i_1(\delta_T)) &= \mathrm{Int}_\sigma\left(\lambda'\Delta_p^{-1}u_p^{-1}\lambda^{-1}\right)(i_1(\delta_T)) \\ &= \mathrm{Int}_\sigma\left(\lambda'\Delta_p^{-1}u_p^{-1}\right)(i(\delta_T)) \\ &= \mathrm{Int}_\sigma\left(\lambda'\Delta_p^{-1}\right)(i'(\delta_T)) \\ &= \mathrm{Int}_\sigma\left(\lambda'\right)(i'(\delta_T)) \\ &= i_1'(\delta_T). \end{aligned} \tag{5.11.14.7}$$

Here the second equality is because $i_1 = \mathrm{Int}(\lambda) \circ i = \mathrm{Int}_\sigma(\lambda) \circ i$, the third equality is by condition (ii) in Definition 5.11.6, the fourth equality is because $i'(\delta_T)$ is $\sigma$-centralized by $\Delta_p \in I_p'$, and the fifth equality is because $i_1' = \mathrm{Int}(\lambda') \circ i' = \mathrm{Int}_\sigma(\lambda') \circ i'$. The fact that $i_1(\gamma_{0,T,n})$ and $i_1'(\gamma_{0,T,n})$ are conjugate in $G(\overline{\mathbb{Q}})$, together with (5.11.14.6) and (5.11.14.7), implies that $u_1$ is a rectification of $\mathfrak{g}_1$.

**(V) Constructing $f_1$.**

We have a $\overline{\mathbb{Q}}$-isomorphism $\psi : I_{\phi,\overline{\mathbb{Q}}} \xrightarrow{\sim} I_{\phi_1,\overline{\mathbb{Q}}}$ induced by $\mathrm{Int}(\lambda')$. Write $\phi(q_\rho) = g_\rho \rtimes \rho$ and $\phi_1(q_\rho) = g_{1,\rho} \rtimes \rho$. Then $g_\rho \in i'(T)(\overline{\mathbb{Q}})$ and $g_{1,\rho} = \mathrm{Int}(\lambda')(g_\rho)$. We compute

$$\begin{aligned}{}^{\rho}\psi(\cdot) &= g_{1,\rho}\rho[\mathrm{Int}(\lambda')(g_{\rho^{-1}}\rho^{-1}(\cdot)g_{\rho^{-1}}^{-1})]g_{1,\rho}^{-1}\\ &= \mathrm{Int}(\lambda' g_\rho \lambda'^{-1\rho}\lambda'^{\rho} g_{\rho^{-1}})(\cdot)\\ &= \mathrm{Int}(\lambda' g_\rho \lambda'^{-1\rho}\lambda' g_\rho^{-1})(\cdot),\end{aligned}$$

where in the last step we use that $g_\rho{}^\rho g_{\rho^{-1}}$ is central in $I_\phi(\overline{\mathbb{Q}})$. By (5.11.13.1) and the fact that $g_\rho \in i'(T)$, we have ${}^\rho\psi(\cdot) = \mathrm{Int}({}^\rho\lambda')$. Thus $\psi$ is an inner twisting satisfying

$$\psi^{-1}\circ {}^\rho\psi = \mathrm{Int}(j'(\beta_0(\rho))) \in \mathrm{Aut}(I_{\phi,\overline{\mathbb{Q}}}), \quad \forall \rho\in\Gamma. \tag{5.11.14.8}$$

We now construct an inner twisting $\chi : I_{x,\overline{\mathbb{Q}}} \xrightarrow{\sim} I_{x_1,\overline{\mathbb{Q}}}$ analogous to $\psi$. As in §5.7.2, we let $\widetilde{x} = \widetilde{x}_{(T,i,h),K_1^p}$ and $\widetilde{x}_1 = \widetilde{x}_{(T,i_1,h),K_1^p}$. These are points of $\mathrm{Sh}_{K_1}(F)$ for some finite extension $F/E_{\mathfrak{p}}$. Under the canonical isomorphisms $\mathrm{triv}_{(T,i,h)} : V_{\mathbb{Q}}^* \xrightarrow{\sim} \mathcal{V}_{B,\mathbb{Q}}(\widetilde{x})$ and $\mathrm{triv}_{(T,i_1,h)} : V_{\mathbb{Q}}^* \xrightarrow{\sim} \mathcal{V}_{B,\mathbb{Q}}(\widetilde{x})$ as in §5.7.2, the element $\lambda\in G(\overline{\mathbb{Q}})$ induces an isomorphism of $\overline{\mathbb{Q}}$-Hodge structures $\mathcal{V}_{B,\mathbb{Q}}(\widetilde{x})\otimes_{\mathbb{Q}}\overline{\mathbb{Q}} \xrightarrow{\sim} \mathcal{V}_{B,\mathbb{Q}}(\widetilde{x}_1)\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}$ which sends the tensors $s_{\alpha,B,\mathbb{Q},\widetilde{x}}$ to $s_{\alpha,B,\mathbb{Q},\widetilde{x}_1}$. Thus $\lambda$ induces an element $\theta\in I_{x_1,x}(\overline{\mathbb{Q}})$, satisfying

$$\theta\circ\rho(\theta^{-1}) = j(\beta_0(\rho)) \in I_x(\overline{\mathbb{Q}}), \quad \forall\rho\in\Gamma. \tag{5.11.14.9}$$

Define $\chi : I_{x,\overline{\mathbb{Q}}} \xrightarrow{\sim} I_{x_1,\overline{\mathbb{Q}}}$ to be the isomorphism induced by $\theta$. From (5.11.14.9) it immediately follows that $\chi$ is an inner twisting satisfying

$$\chi^{-1}\circ{}^\rho\chi = \mathrm{Int}(j(\beta_0(\rho))) \in \mathrm{Aut}(I_{x,\overline{\mathbb{Q}}}), \quad \forall\rho\in\Gamma. \tag{5.11.14.10}$$

We define $f_1$ by the commutative diagram

$$\begin{tikzcd} I_{x,\overline{\mathbb{Q}}} \arrow[r, "f"] \arrow[d, "\chi"] & I_{\phi,\overline{\mathbb{Q}}} \arrow[d, "\psi"] \\ I_{x_1,\overline{\mathbb{Q}}} \arrow[r, "f_1"] & I_{\phi_1,\overline{\mathbb{Q}}} \end{tikzcd}$$

By (5.11.14.8), (5.11.14.10), and the fact that $f\circ j = j'$ (as $f$ is $\mathfrak{g}$-adapted), we know that $f_1$ is a $\mathbb{Q}$-isomorphism. Moreover, $\chi$ and $\psi$ commute with the canonical embeddings of $T$. Hence $f_1\circ j_1 = j_1'$.

**(VI) Constructing $\tau_1$.**

Let $v$ be a finite place. From the constructions of $\psi$ and $\chi$, we have the commutative diagrams

$$\begin{tikzcd} I_{\phi,\overline{\mathbb{Q}}_v} \arrow[r, "\iota_{y',v}"] \arrow[d, "\psi"] & (I_v')_{\overline{\mathbb{Q}}_v} \arrow[d, "\mathrm{Int}(\lambda')"] \\ I_{\phi_1,\overline{\mathbb{Q}}_v} \arrow[r, "\iota_{y_1',v}"] & (I_{v,1}')_{\overline{\mathbb{Q}}_v} \end{tikzcd} \tag{5.11.14.11}$$

and

$$\begin{tikzcd} I_{x,\overline{\mathbb{Q}}_v} \arrow[r, "\iota_{y,v}"] \arrow[d, "\chi"] & (I_v)_{\overline{\mathbb{Q}}_v} \arrow[d, "\mathrm{Int}(\lambda)"] \\ I_{x_1,\overline{\mathbb{Q}}_v} \arrow[r, "\iota_{y_1,v}"] & (I_{v,1})_{\overline{\mathbb{Q}}_v} \end{tikzcd} \tag{5.11.14.12}$$

As in part (III), let $\Omega_v := \iota_{y',v}^{-1}(\Delta_v) \in I_\phi^\dagger(\overline{\mathbb{Q}}_v)$. Let $\widetilde{\tau}_v \in I_\phi^\dagger(\overline{\mathbb{Q}}_v)$ be a lift of $\tau_v$. Let

$$\widetilde{\tau}_{1,v} := \psi(\Omega_v^{-1}\widetilde{\tau}_v) \in I_{\phi_1}^\dagger(\overline{\mathbb{Q}}_v).$$

Using $\iota_{y,v} \circ j = i$ and $\iota_{y',v} \circ j' = i'$, we can rewrite (5.11.14.4) as

$$\mathrm{Int}(u_v)^{-1} \circ \iota_{y,v}(j(\beta_0(\rho))) = \Delta_v \cdot \iota_{y',v}(j'(\beta_0(\rho))) \cdot {}^\rho\Delta_v^{-1}, \quad \forall \rho \in \Gamma_v.$$

By (5.11.14.1) and (5.11.14.2), we deduce from the above equality that

$$\iota_{y',v} \circ \mathrm{Int}(\tau_v) \circ f(j(\beta_0(\rho))) = \Delta_v \cdot \iota_{y',v}(j'(\beta_0(\rho))) \cdot {}^\rho\Delta_v^{-1}, \quad \forall \rho \in \Gamma_v.$$

Using $f \circ j = j'$ and the definition of $\Omega_v$, we get

$$\mathrm{Int}(\tau_v)(j'(\beta_0(\rho))) = \Omega_v(j'(\beta_0(\rho)))^\rho\Omega_v^{-1}, \quad \forall \rho \in \Gamma_v. \tag{5.11.14.13}$$

Write $\beta_\rho$ for $j'(\beta_0(\rho))$. We compute

$$\begin{aligned} \widetilde{\tau}_{1,v}^{-1}\,{}^\rho\widetilde{\tau}_{1,v} &= \psi(\widetilde{\tau}_v^{-1}\Omega_v) \cdot ({}^\rho\psi)({}^\rho\Omega_v^{-1}\,{}^\rho\widetilde{\tau}_v) \\ &= \psi(\widetilde{\tau}_v^{-1}\Omega_v) \cdot \psi(\beta_\rho\,{}^\rho\Omega_v^{-1}\,{}^\rho\widetilde{\tau}_v\beta_\rho^{-1}) && \text{by (5.11.14.8)} \\ &= \psi(\widetilde{\tau}_v^{-1}\Omega_v\beta_\rho\,{}^\rho\Omega_v^{-1}\,{}^\rho\widetilde{\tau}_v\beta_\rho^{-1}) \\ &= \psi(\beta_\rho\widetilde{\tau}_v^{-1}\,{}^\rho\widetilde{\tau}_v\beta_\rho^{-1}) && \text{by (5.11.14.13)} \\ &= \psi(\widetilde{\tau}_v^{-1}\,{}^\rho\widetilde{\tau}_v) && \text{because } \widetilde{\tau}_v^{-1}\,{}^\rho\widetilde{\tau}_v \in Z_\phi^\dagger. \end{aligned} \tag{5.11.14.14}$$

Since $\psi$ is an inner twisting, it follows from (5.11.14.14) that the image of $\widetilde{\tau}_{1,v}$ in $I_{\phi_1}^{\mathrm{ad}}(\overline{\mathbb{Q}}_v)$ lies in $I_{\phi_1}^{\mathrm{ad}}(\mathbb{Q}_v)$. We denote this element by $\tau_{1,v}$. Recall from part (III) that $(\Omega_v)_{v\neq p}$ comes from an element of $I_\phi^\dagger(\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{A}_f^p)$. It easily follows that $\tau_1 := (\tau_{1,v})_v \in \prod_v I_{\phi_1}^{\mathrm{ad}}(\mathbb{Q}_v)$ is in fact an element of $I_{\phi_1}^{\mathrm{ad}}(\mathbb{A}_f)$.

**(VII) Proof that $\tau_1$ is associated with $(x_1, \phi_1, y_1 \cdot u_1, y_1')$ and $f_1$.**

We compute

$$\begin{aligned} \iota_{y_1',v} &= \mathrm{Int}(\lambda') \circ \iota_{y',v} \circ \psi^{-1} \\ &= \mathrm{Int}(\lambda'\Delta_v^{-1}) \circ \iota_{y',v} \circ \mathrm{Int}(\Omega_v) \circ \psi^{-1} \\ &= \mathrm{Int}(\lambda'\Delta_v^{-1}) \circ \iota_{y\cdot u,v} \circ f^{-1} \circ \mathrm{Int}(\widetilde{\tau}_v^{-1}\Omega_v) \circ \psi^{-1} \\ &= \mathrm{Int}(\lambda'\Delta_v^{-1}u_v^{-1}) \circ \iota_{y,v} \circ f^{-1} \circ \mathrm{Int}(\widetilde{\tau}_v^{-1}\Omega_v) \circ \psi^{-1} \\ &= \mathrm{Int}(u_{1,v}^{-1}\lambda) \circ \iota_{y,v} \circ f^{-1}\,\mathrm{Int}(\widetilde{\tau}_v^{-1}\Omega_v) \circ \psi^{-1} \\ &= \mathrm{Int}(u_{1,v}^{-1}) \circ \iota_{y_1,v} \circ \chi \circ f^{-1} \circ \mathrm{Int}(\widetilde{\tau}_v^{-1}\Omega_v) \circ \psi^{-1} \\ &= \mathrm{Int}(u_{1,v}^{-1}) \circ \iota_{y_1,v} \circ f_1^{-1} \circ \psi \circ \mathrm{Int}(\widetilde{\tau}_v^{-1}\Omega_v) \circ \psi^{-1} \\ &= \mathrm{Int}(u_{1,v}^{-1}) \circ \iota_{y_1,v} \circ f_1^{-1} \circ \mathrm{Int}(\tau_{1,v}) \\ &= \iota_{y_1\cdot u_1,v} \circ f_1^{-1} \circ \mathrm{Int}(\tau_{1,v}). \end{aligned}$$

Here the first equality is by (5.11.14.11), the second by the definition of $\Omega_v$, the third by (5.11.14.2), the fourth by (5.11.14.1), the fifth by the definition of $u_{1,v}$, the sixth by (5.11.14.12), the seventh by the definition of $f_1$, and the ninth by the analogue of (5.11.14.1). This shows that $f_1 : I_{x_1} \xrightarrow{\sim} I_{\phi_1}$ is the canonical isomorphism (up to $I^{\mathrm{ad}}_{\phi_1}(\mathbb{Q})$-conjugation) as in §5.10.3, and that $\tau_1$ is the element of $I^{\mathrm{ad}}_{\phi_1}(\mathbb{A}_f)$ associated with $(x_1, \phi_1, y_1 \cdot u_1, y'_1)$ and $f_1$. We have already seen in part (V) that $f_1 \circ j_1 = j'_1$. We conclude that $f_1$ is $\mathfrak{g}_1$-adapted. □

**5.11.15.** Keep the assumptions of Proposition 5.11.13. Then $\mathfrak{g}$ and $\mathfrak{g}'$ both admit rectifications, and so the pairs $(\mathscr{I}_{\mathfrak{g}}, \mathscr{J}_{\mathfrak{g}})$ and $(\mathscr{I}_{\mathfrak{g}_1}, \mathscr{J}_{\mathfrak{g}_1})$ are amicable by Lemma 5.11.7. Thus we have the elements

$$(\tau^{\mathfrak{H}}(\mathscr{I}_{\mathfrak{g}}, \mathscr{J}_{\mathfrak{g}}), \tau^{\mathcal{H}}(\mathscr{I}_{\mathfrak{g}}, \mathscr{J}_{\mathfrak{g}})) \in \mathfrak{H}(\phi) \oplus \mathcal{H}(\phi) \tag{5.11.15.1}$$

and

$$(\tau^{\mathfrak{H}}(\mathscr{I}_{\mathfrak{g}_1}, \mathscr{J}_{\mathfrak{g}_1}), \tau^{\mathcal{H}}(\mathscr{I}_{\mathfrak{g}_1}, \mathscr{J}_{\mathfrak{g}_1})) \in \mathfrak{H}(\phi_1) \oplus \mathcal{H}(\phi_1) \tag{5.11.15.2}$$

as in Definition 5.10.9. Note that we have $\phi \approx \phi_1$, where $\approx$ is defined in §2.6.16. Thus the abelian groups $\mathcal{H}(\phi)$ and $\mathcal{H}(\phi_1)$ are canonically identified by §2.6.16, and similarly the abelian groups $\mathfrak{H}(\phi)$ and $\mathfrak{H}(\phi_1)$ are canonically identified by §5.10.5.

**Proposition 5.11.16.** *The canonical identification $\mathfrak{H}(\phi) \oplus \mathcal{H}(\phi) \cong \mathfrak{H}(\phi_1) \oplus \mathcal{H}(\phi_1)$ sends (5.11.15.1) to (5.11.15.2).*

*Proof.* Let $\tau$ and $\tau_1$ be as in the proof of Proposition 5.11.13. As we showed in the proof of Lemma 5.11.7, the marking $(\bar{y}_{\mathfrak{g}} \cdot u, \bar{y}'_{\mathfrak{g}})$ of $(\mathscr{I}_{\mathfrak{g}}, \mathscr{J}_{\mathfrak{g}})$ and the marking $(\bar{y}_{\mathfrak{g}_1} \cdot u_1, \bar{y}'_{\mathfrak{g}_1})$ of $(\mathscr{I}_{\mathfrak{g}_1}, \mathscr{J}_{\mathfrak{g}_1})$ are in fact $\pi^*$-compatible. Hence $\tau^{\mathfrak{H}}(\mathscr{I}_{\mathfrak{g}}, \mathscr{J}_{\mathfrak{g}})$ and $\tau^{\mathcal{H}}(\mathscr{I}_{\mathfrak{g}}, \mathscr{J}_{\mathfrak{g}})$ are the images of $\tau$ in $\mathfrak{H}(\phi)$ and in $\mathcal{H}(\phi)$ respectively, while $\tau^{\mathfrak{H}}(\mathscr{I}_{\mathfrak{g}_1}, \mathscr{J}_{\mathfrak{g}_1})$ and $\tau^{\mathcal{H}}(\mathscr{I}_{\mathfrak{g}_1}, \mathscr{J}_{\mathfrak{g}_1})$ are the images of $\tau_1$ in $\mathfrak{H}(\phi_1)$ and in $\mathcal{H}(\phi_1)$ respectively. It remains to show that the natural image of $\tau$ in $\mathfrak{H}(\phi) \oplus \mathcal{H}(\phi)$ corresponds to the natural image of $\tau_1$ in $\mathfrak{H}(\phi_1) \oplus \mathcal{H}(\phi_1)$. This follows from (5.11.14.14). □

## 5.12. **Galois cohomological properties of amicable pairs.**

**5.12.1.** Let $(\mathscr{I}, \mathscr{J})$ be an amicable pair. Let $\phi \in \mathscr{J}$. For any maximal torus $T \subset I_\phi$, we write $T^\dagger$ for $T \cap I^\dagger_\phi = \ker(T \to G^{\mathrm{ab}})$, which is a subtorus of $T$ defined over $\mathbb{Q}$. As in §5.10.5, we define $Ш^\infty_{G_{\mathrm{der}}}(\mathbb{Q}, H)$ for any $\mathbb{Q}$-subgroup $H \subset I^\dagger_\phi$, in particular for $H = T^\dagger$.

We have a boundary map $\partial : G^{\mathrm{ab}}(\mathbb{Q}_p) \to \mathbf{H}^1(\mathbb{Q}_p, T^\dagger)$ arising from the short exact sequence $1 \to T^\dagger \to T \to G^{\mathrm{ab}} \to 1$. Recall from Definition 5.10.9 that there is a canonical element $\tau^{\mathfrak{H}}(\mathscr{I}, \mathscr{J}) \in \mathfrak{H}(\phi)$. There is a natural homomorphism from $\mathfrak{H}(\phi)$ to the group

$$\operatorname{coker}\left(\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p) \xrightarrow{\partial} \mathbf{H}^1(\mathbb{A}_f, T^\dagger)/Ш^\infty_{G_{\mathrm{der}}}(\mathbb{Q}, T^\dagger)\right) \tag{5.12.1.1}$$

**Theorem 5.12.2.** *In the setting of §5.12.1, the image of $\tau^{\mathfrak{H}}(\mathscr{I}, \mathscr{J})$ in (5.12.1.1) is trivial.*

*Proof.* Let $j' : T \hookrightarrow I_\phi$ be the inclusion map. Pick a $\pi^*$-compatible marking $(\bar{y}, \bar{y}')$ of $(\mathscr{I}, \mathscr{J})$. Pick $x \in \mathscr{I}$. Pick $y \in Y(x)$ lifting $\bar{y}$, and pick $y' \in Y(\phi)$ lifting $\bar{y}'$. Choose an isomorphism $f : I_x \xrightarrow{\sim} I_\phi$ as in §5.10.3 and Remark 5.10.10. Let $j$ be the composition $T \xrightarrow{j'} I_\phi \xrightarrow{f^{-1}} I_x$. Via $j$ we view $T$ also as a maximal torus in $I_x$.

We claim that a cocharacter $\mu \in X_*(T)$ is $\phi$-admissible in the sense of §3.3.8 if and only if it is $x$-admissible in the sense of §5.7.7. In fact, using Theorem

3.3.9, Proposition 5.9.2, Theorem 5.7.8, and Proposition 5.7.6, one can show that the elements $(\phi(p) \circ \zeta_p)^{\Delta}$ and $\nu_{\delta_y}$ of $X_*(T) \otimes \mathbb{Q}$ (introduced in §3.3.8 and §5.7.7 respectively) are both equal to the Newton cocharacter of $[\delta_T] \in \mathrm{B}(T_{\mathbb{Q}_p})$, where $\delta_T$ is any element of the $\overset{\circ}{\sim}$-equivalence class in $T(\mathbb{Q}_p^{\mathrm{ur}})^{\mathrm{mot}}$ corresponding to $-\mu_h$. It is also clear that for $\mu \in X_*(T)$ the composition (5.7.7.1) lies in $\mathfrak{y}_X(\overline{\mathbb{Q}}_p)$ if and only if $j' \circ \mu$ lies in $\mathfrak{y}_X(\overline{\mathbb{Q}})$. The claim follows.

By the above claim, Theorem 3.3.9, and Theorem 5.7.8, we find the following objects:

- Two special point data of the form $\mathfrak{s} = (T, i, h), \mathfrak{s} = (T, i', h) \in \mathcal{SPD}(G, X)$ such that $\mathscr{I} = \mathscr{I}_{\mathfrak{s}}$ and $\mathscr{J} = \mathscr{J}_{\mathfrak{s}'}$.
- An isomorphism $w : I_x \xrightarrow{\sim} I_{x_{\mathfrak{s}}}$ which is induced by some element of $I_{x,x_{\mathfrak{s}}}(\mathbb{Q})$, satisfying that the composition $w \circ j : T \to I_{x_{\mathfrak{s}}}$ is the canonical embedding (i.e., the embedding (5.7.2.2)).
- An isomorphism $w' : I_\phi \xrightarrow{\sim} I_{\phi(\mathfrak{s}')}$ which is induced by some element of $G(\overline{\mathbb{Q}})$ conjugating $\phi$ to $\phi(\mathfrak{s})$, satisfying that the composition $w' \circ j' : T \to I_{\phi(\mathfrak{s})}$ is the canonical embedding (i.e., the one whose composition with $I_{\phi(\mathfrak{s}),\overline{\mathbb{Q}}} \hookrightarrow G_{\overline{\mathbb{Q}}}$ is $i'_{\overline{\mathbb{Q}}}$).

We claim that we can choose the above objects such that $(T, h, i, i')$ is a special fork (see Definition 5.11.1). In fact, from the fact that $(\mathscr{I}, \mathscr{J})$ is weakly amicable it already follows that $i$ and $i'$ are conjugate by $G^{\mathrm{ad}}(\overline{\mathbb{Q}}_v)$ for each finite place $v$. Then $i$ and $i'$ are conjugate by $G^{\mathrm{ad}}(\overline{\mathbb{Q}})$, since $i(T)$ and $i'(T)$ are $\mathbb{Q}$-maximal tori in $G$ and since the absolute Weyl group of $i(T)$ in $G$ is the same when considered over $\overline{\mathbb{Q}}_v$ and when considered over $\overline{\mathbb{Q}}$. Also note that we can freely replace $i'$ by $\mathrm{Int}(g) \circ i'$ for $g \in G(\mathbb{Q})$. By the real approximation theorem, we can choose $g$ such that $\mathrm{Int}(g) \circ i' \circ h$ lies in the same connected component of $X$ as $i \circ h$. The claim is proved.

The canonical isomorphism $\mathfrak{H}(\phi) \xrightarrow{\sim} \mathfrak{H}(\phi_{\mathfrak{s}'})$ commutes with the natural map from $\mathfrak{H}(\phi)$ to (5.12.1.1) induced by $j' : T \to I_\phi$ and the natural map from $\mathfrak{H}(\phi_{\mathfrak{s}'})$ to (5.12.1.1) induced by the canonical embedding $T \to I_{\phi(\mathfrak{s}')}$. Thus we can reduce the theorem to the following situation:

- We have a special fork $(\mathfrak{s}, \mathfrak{s}') = (T, h, i, i')$ such that $\mathscr{I} = \mathscr{I}_{\mathfrak{s}}$ and $\mathscr{J} = \mathscr{J}_{\mathfrak{s}'}$. Moreover, $\phi = \phi(\mathfrak{s}')$, and the inclusion $T \hookrightarrow I_\phi$ is the canonical embedding, namely the one whose composition with $I_{\phi,\overline{\mathbb{Q}}} \hookrightarrow G_{\overline{\mathbb{Q}}}$ is $i'$.

By Lemma 5.11.5 and Lemma 5.11.7, we can extend the special fork $(\mathfrak{s}, \mathfrak{s}')$ to a gauge $\mathfrak{g}$. Moreover, tracing the above reduction steps we see that there exists a $\mathfrak{g}$-adapted isomorphism $f : I_{x_{\mathfrak{g}}} \xrightarrow{\sim} I_{\phi_{\mathfrak{g}}}$. (This comes from the initial definition of $j$ in the first paragraph of the proof.) The theorem then follows from Proposition 5.11.12. □

**5.12.3.** Let $(\mathscr{I}, \mathscr{J})$ be a weakly amicable pair. Fix $x \in \mathscr{I}$ and $\phi \in \mathscr{J}$. Recall from §5.10.3 and Remark 5.10.10 that we have an isomorphism $f : I_x \xrightarrow{\sim} I_\phi$ which is canonical up to $I_\phi^{\mathrm{ad}}(\mathbb{Q})$-conjugation. In particular, we have a canonical isomorphism between the abelian groups $Ш_G^\infty(\mathbb{Q}, I_\phi)$ and $Ш_G^\infty(\mathbb{Q}, I_x)$ that is independent of all choices. Now fix an element $\beta \in Ш_G^\infty(\mathbb{Q}, I_\phi)$, also viewed as an element of $Ш_G^\infty(\mathbb{Q}, I_x)$. As in Definition 2.1.17 and Proposition 2.6.12, we obtain the twisted admissible morphism $\phi^\beta$, which is well defined up to conjugacy. We denote the conjugacy class of $\phi^\beta$ by $\mathscr{J}^\beta$. Note that for different choices of $\phi \in \mathscr{J}$, the abelian groups $Ш^{\infty}_G(\mathbb{Q}, I_\phi)$ are canonically identified. Define

$$Ш^{\infty}_G(\mathscr{J}) := \varprojlim_{\phi\in\mathscr{J}} Ш^{\infty}_G(\mathbb{Q}, I_\phi).$$

If we view $\beta$ as an element of $Ш^{\infty}_G(\mathscr{J})$, then $\mathscr{J}^\beta$ depends only on $\mathscr{J}$ and $\beta$, not on $\phi \in \mathscr{J}$.

Similarly, for different choices of $x \in \mathscr{I}$, the groups $Ш^{\infty}_G(\mathbb{Q}, I_x)$ are canonically identified. Similarly as above we define

$$Ш^{\infty}_G(\mathscr{I}) := \varprojlim_{x\in\mathscr{I}} Ш^{\infty}_G(\mathbb{Q}, I_x).$$

The canonical isomorphisms $Ш^{\infty}_G(\mathbb{Q}, I_\phi) \cong Ш^{\infty}_G(\mathbb{Q}, I_x)$ for all $\phi \in \mathscr{J}$ and $x \in \mathscr{I}$ induce a canonical isomorphism

$$Ш^{\infty}_G(\mathscr{I}) \cong Ш^{\infty}_G(\mathscr{J}).$$

We have the twisted isogeny class $\mathscr{I}^\beta$ constructed in [Kis17, §4.4.7, Prop. 4.4.8]. If we view $\beta$ as an element of $Ш^{\infty}_G(\mathscr{I})$, then $\mathscr{I}^\beta$ depends only on $\mathscr{I}$ and $\beta$.

By §2.6.16, the abelian groups $\mathcal{H}(\phi)$ and $\mathcal{H}(\phi^\beta)$ are canonically identified, since $\phi \approx \phi^\beta$. Similarly, by §5.10.5, the abelian groups $\mathfrak{H}(\phi)$ and $\mathfrak{H}(\phi^\beta)$ are canonically identified.

**5.12.4.** We take this opportunity to correct a mistake in [Kis10, §3.1], and we freely use the notation introduced there. Above we used a twisting construction defined in [Kis10, §3.1]. The statement of Lemma 3.1.5 of *loc. cit.* should include the condition that the $Z$-torsor $\mathcal{P}$, is trivial over $\mathbb{R}$, i.e., that $\mathcal{P}_\mathbb{R}$ is a trivial $Z_\mathbb{R}$-torsor. The result is not true without this condition, as the $\mathbb{Q}$-isogeny $\lambda^\mathcal{P}$ need not be a weak polarization.

Unfortunately, this error was imported into [Kis17, §4.1.6] and [KP18, Lemma 4.4.8] via citation. Fortunately, in all instances where this construction is applied to Shimura varieties in these papers, the condition of triviality at $\infty$ holds. Moreover, the result is always applied to give a moduli theoretic description of a construction defined via complex uniformization. Thus, logically, the fact that $\lambda^\mathcal{P}$ is a weak polarization is never used, but rather follows *a posteriori* in all these cases.

Nevertheless, let us explain why $\lambda^\mathcal{P}$ is a weak polarization if $\mathcal{P}_\mathbb{R}$ is trivial. Recall that for any $\mathbb{Q}$-algebra $R$, an $R$-isogeny $\phi : \mathcal{A} \to \mathcal{A}^*$ is an element of $\mathrm{Hom}(\mathcal{A}, \mathcal{A}^*)\otimes R$ which has an inverse in $\mathrm{Hom}(\mathcal{A}^*, \mathcal{A}) \otimes R$, cf. [Kot92b, §9]. If $R$ is a subring of $\mathbb{R}$, we say that $\phi$ is an $R$-polarization if it is an $R$-linear combination of polarizations, with positive coefficients. We say $\phi$ is a weak $R$-polarization if $\phi$ or $-\phi$ is a polarization. Thus a weak $\mathbb{Q}$-polarization is the same thing as a weak polarization. Now if $\phi_0 : \mathcal{A} \to \mathcal{A}^*$ is a $\mathbb{Q}$-isogeny, then $\phi_0$ is a weak polarization if and only if it is a weak $\mathbb{R}$-polarization when viewed as an $\mathbb{R}$-isogeny. This follows from that fact that the set of $\mathbb{Q}$-polarizations is a convex cone in $\mathrm{Hom}(\mathcal{A}, \mathcal{A}^*) \otimes \mathbb{Q}$. Similarly $\phi_0$ is a weak polarization if and only if $\phi_0$ is a weak $\mathbb{R}$-polarization. Moreover, the set of weak $\mathbb{R}$-polarizations is stable under multiplication by $\mathbb{R}^\times$.

We now return to the explanation that $\lambda^\mathcal{P}$ is a weak polarization if $\mathcal{P}_\mathbb{R}$ is trivial. Thus, suppose $\mathcal{P}_\mathbb{R}$ is trivial, and let $x \in \mathcal{P}(\mathbb{R})$. Specializing the commutative diagram [Kis10, (3.1.6)] by $x$, we see that $f_c(x)^{-1}\lambda^\mathcal{P}$ is a weak $\mathbb{R}$-polarization. Thus, by what we just saw, $\lambda^\mathcal{P}$ is a weak $\mathbb{R}$-polarization and hence a weak polarization.

**Theorem 5.12.5.** *Keep the setting of §5.12.3. Assume that $(\mathscr{I}, \mathscr{J})$ is amicable. Then $(\mathscr{I}^\beta, \mathscr{J}^\beta)$ is again amicable. Moreover, the elements*

$$(\tau^{\mathfrak{H}}(\mathscr{I}, \mathscr{J}), \tau^{\mathcal{H}}(\mathscr{I}, \mathscr{J})) \in \mathfrak{H}(\phi) \oplus \mathcal{H}(\phi)$$

*and*

$$(\tau^{\mathfrak{H}}(\mathscr{I}^\beta, \mathscr{J}^\beta), \tau^{\mathcal{H}}(\mathscr{I}^\beta, \mathscr{J}^\beta)) \in \mathfrak{H}(\phi^\beta) \oplus \mathcal{H}(\phi^\beta)$$

*(see Definition 5.10.9) correspond to each other under the canonical identification $\mathfrak{H}(\phi) \oplus \mathcal{H}(\phi) \cong \mathfrak{H}(\phi^\beta) \oplus \mathcal{H}(\phi^\beta)$.*

*Proof.* Let $T$ be a maximal torus in $I_\phi$ such that $\beta$ comes from a class $\beta_T$ in $\mathbf{H}^1(\mathbb{Q}, T)$. Such a maximal torus always exists by [Bor98, Thm. 5.10]. Let $j' : T \hookrightarrow I_\phi$ be the inclusion map. As in the proof of Theorem 5.12.2, we reduce the theorem to the following situation.

- There is a special fork $(\mathfrak{s}, \mathfrak{s}') = (T, h, i, i')$. We have $\phi = \phi(\mathfrak{s}')$, and $j'$ is the canonical embedding, namely the one whose composition with $I_{\phi,\overline{\mathbb{Q}}} \hookrightarrow G_{\overline{\mathbb{Q}}}$ is $i'$. We have $x_{\mathfrak{s}} \in \mathscr{I}$. Moreover, in the canonical $I_\phi^{\mathrm{ad}}(\mathbb{Q})$-orbit of isomorphisms $I_{x_{\mathfrak{s}}} \xrightarrow{\sim} I_\phi$, we can find an isomorphism $f$ such that $f^{-1} \circ j' : T \to I_{x_{\mathfrak{s}}}$ is the canonical embedding.

Define $Ш^\infty_G(\mathbb{Q}, T)$ and $Ш^\infty_{G_{\mathrm{der}}}(\mathbb{Q}, T^\dagger)$ as in §5.11.2, with respect to the special fork $(T, h, i, i')$. We will still need to modify our choice of $i$, but note that $Ш^\infty_G(\mathbb{Q}, T)$ and $Ш^\infty_{G_{\mathrm{der}}}(\mathbb{Q}, T^\dagger)$ are already determined by $(T, i')$. Also note that $Ш^\infty_G(\mathbb{Q}, T)$ is equal to the preimage of $Ш^\infty_G(\mathbb{Q}, I_\phi)$ under the map $Ш^\infty(\mathbb{Q}, T) \to Ш^\infty(\mathbb{Q}, I_\phi)$ induced by $j'$. By [Kis17, Lem. 4.4.5], the map of pointed sets $\mathbf{H}^1(\mathbb{R}, T) \to \mathbf{H}^1(\mathbb{R}, I_\phi)$ induced by $j'$ has trivial kernel. Hence $Ш^\infty_G(\mathbb{Q}, T)$ is equal to the preimage of $Ш^\infty_G(\mathbb{Q}, I_\phi)$ under the map $\mathbf{H}^1(\mathbb{Q}, T) \to \mathbf{H}^1(\mathbb{Q}, I_\phi)$ induced by $j'$. We conclude that $\beta_T \in Ш^\infty_G(\mathbb{Q}, T)$. Now by Lemma 1.2.6, we can find a class $\beta_0 \in Ш^\infty_{G_{\mathrm{der}}}(\mathbb{Q}, T^\dagger)$ lifting $\beta_T$. Fix a cocycle $\beta_0(\cdot)$ representing $\beta_0$.

Choose $\lambda' \in G_{\mathrm{der}}(\overline{\mathbb{Q}})$ such that

$$i'(\beta_0(\rho)) = \lambda'^{-1}\,{}^{\rho}\lambda', \quad \forall \rho \in \Gamma. \tag{5.12.5.1}$$

Since $\beta_0$ is trivial at infinity, we have $\lambda' \in G_{\mathrm{der}}(\mathbb{R}) i'(T^\dagger(\mathbb{C}))$. Since $G_{\mathrm{der}}(\mathbb{Q}) G_{\mathrm{der}}(\mathbb{R})^+$ is equal to $G_{\mathrm{der}}(\mathbb{R})$ (by real approximation), and since $\lambda'$ is determined by (5.12.5.1) up to left multiplication by $G_{\mathrm{der}}(\mathbb{Q})$, we may and shall assume that

$$\lambda' \in G_{\mathrm{der}}(\mathbb{R})^+ i'(T^\dagger(\mathbb{C})). \tag{5.12.5.2}$$

We define

$$i'_1 := \mathrm{Int}(\lambda') \circ i' : T \longrightarrow G.$$

By (5.12.5.1), $i'_1$ is defined over $\mathbb{Q}$. Choose an arbitrary $y' \in X(\Psi_{T,h})_{\mathrm{neu}}$, and choose $y \in Y(\Upsilon_{(T,h)})^\circ$ as in Lemma 5.11.5 with respect to $(T, h, i', i'_1, y')$. Then $\mathfrak{g} = (T, h, i, i', y, y')$ is a quasi-gauge, and $(\mathscr{I}_{\mathfrak{g}}, \mathscr{J}_{\mathfrak{g}}) = (\mathscr{I}, \mathscr{J})$. Since $(\mathscr{I}, \mathscr{J})$ is amicable, $\mathfrak{g}$ has a rectification $u$ by Lemma 5.11.7. Since $u \in \ker(G(\mathbb{A}_f^*) \to \pi^*(G))$, we can write $u = u^{(0)} u^{(1)}$, with $u^{(0)} \in G(\mathbb{Q})_+$ and $u^{(1)} \in G_{\mathrm{der}}(\mathbb{A}_f^*) \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$. Note that $(T, h, \mathrm{Int}(u^{(0)})^{-1} \circ i, i', y, y')$ is still a gauge, and that $u^{(1)}$ is a rectification of it. Moreover, writing $\mathfrak{s}_0$ for the special point datum $(T, \mathrm{Int}(u^{(0)})^{-1} \circ i, h)$, we have $x_{\mathfrak{s}_0} \in \mathscr{I}$, and in the canonical $I_\phi^{\mathrm{ad}}(\mathbb{Q})$-orbit of isomorphisms $I_{x_{\mathfrak{s}_0}} \xrightarrow{\sim} I_\phi$ we can find an isomorphism $f_0$ such that $f_0^{-1} \circ j' : T \to I_{x_{\mathfrak{s}_0}}$ is the canonical embedding. Therefore after replacing $i$ by $\mathrm{Int}(u^{(0)})^{-1} \circ i$ we can arrange that

$\mathfrak{g}$ admits a rectification $u \in G_{\mathrm{der}}(\mathbb{A}_f^*)\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$, and that there exists a $\mathfrak{g}$-adapted isomorphism $I_{x_{\mathfrak{g}}} \xrightarrow{\sim} I_{\phi_{\mathfrak{g}}}$.

By the same argument as before, there exists $\lambda \in G_{\mathrm{der}}(\overline{\mathbb{Q}})$ such that

$$i(\beta_0(\rho)) = \lambda^{-1\rho}\lambda, \quad \forall \rho \in \Gamma, \tag{5.12.5.3}$$

and

$$\lambda \in G_{\mathrm{der}}(\mathbb{R})^+ i(T^\dagger(\mathbb{C})). \tag{5.12.5.4}$$

We define

$$i_1 := \mathrm{Int}(\lambda) \circ i : T \longrightarrow G,$$

which is defined over $\mathbb{Q}$ by (5.12.5.3). By (5.12.5.2), (5.12.5.4), and the fact that $i \circ h$ and $i' \circ h$ lie in the same connected component of $X$, we know that all four points

$$i_1 \circ h,\ i_1' \circ h,\ i \circ h,\ i' \circ h$$

lie in the same connected component of $X$. It is also clear that $i_1$ and $i_1'$ are conjugate by $G^{\mathrm{ad}}(\overline{\mathbb{Q}})$. Hence $(T, h, i_1, i_1')$ is a special fork. It follows that the tuple $\mathfrak{g}_1 = (T, h, i_1, i_1', y, y')$ is a quasi-gauge, by our choice of $y$ (see Lemma 5.11.5). The same argument as in the proof of [Kis17, Cor. 4.6.5] shows that $(\mathscr{I}^\beta, \mathscr{J}^\beta) = (\mathscr{I}_{\mathfrak{g}_1}, \mathscr{J}_{\mathfrak{g}_1})$. Now $\mathfrak{g}, \mathfrak{g}_1, \lambda, \lambda'$ satisfy all the assumptions in Proposition 5.11.13, so the current theorem follows from Lemma 5.11.7, Proposition 5.11.13, and Proposition 5.11.16. □

## 5.13. **Construction and properties of a bijection.**

**5.13.1.** In the current setting of Hodge type, it is expected that there should be a canonical bijection between the set of conjugacy classes of admissible morphisms $\mathfrak{Q} \to \mathfrak{G}_G$ and the set of isogeny classes in $\mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$. One candidate for such a bijection is constructed in [Kis17]. However, even giving a general characterization of what "canonical" should mean for such a bijection seems to be out of current reach. In the following, we construct such a bijection in a way that is different from [Kis17] (cf. Remark 5.13.8 below). Throughout this subsection we keep the setting of §5.1.

We write $\mathbb{I}$ for the set of isogeny classes in $\mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$, and write $\mathbb{J}$ for the set of conjugacy classes of admissible morphisms $\mathfrak{Q} \to \mathfrak{G}_G$.[37]

In §2.6.16, we defined an equivalence relation $\approx$ on the set of admissible morphisms $\mathfrak{Q} \to \mathfrak{G}_G$. This descends to an equivalence relation on $\mathbb{J}$, which we still denote by $\approx$. By Proposition 2.6.12, for $\mathscr{J}_1, \mathscr{J}_2 \in \mathbb{J}$ we have $\mathscr{J}_1 \approx \mathscr{J}_2$ if and only if $\mathscr{J}_1 = \mathscr{J}_2^\beta$ for some (unique) $\beta \in Ш_G^\infty(\mathscr{J}_1)$. (See §5.12.3 for the notations $Ш_G^\infty(\mathscr{J}_1)$ and $\mathscr{J}_1^\beta$.) When this is the case, for any $\phi_1 \in \mathscr{J}_1$ and $\phi_2 \in \mathscr{J}_2$ we have a canonical equivalence class of inner twistings between $I_{\phi_1}$ and $I_{\phi_2}$. The induced equivalence class of inner twistings between $I_{\phi_1,\mathbb{R}}$ and $I_{\phi_2,\mathbb{R}}$ is trivial, i.e., it contains an $\mathbb{R}$-isomorphism. Thus we have an induced canonical isomorphism $\mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}, I_{\phi_1}) \cong \mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}, I_{\phi_2})$, which restricts to an isomorphism $Ш_G^\infty(\mathbb{Q}, I_{\phi_1}) \cong Ш_G^\infty(\mathbb{Q}, I_{\phi_2})$. If we view the last isomorphism as an isomorphism $Ш_G^\infty(\mathscr{J}_1) \cong Ш_G^\infty(\mathscr{J}_2)$, then it depends only on $\mathscr{J}_1$ and $\mathscr{J}_2$, not on any other choices.

[37] In §2.6.16, the set $\mathbb{J}$ was also denoted by $\mathcal{AM}/\mathrm{conj}$. In the current setting of Hodge type we choose the notation $\mathbb{J}$ to reflect the symmetry with the set $\mathbb{I}$ of isogeny classes.

Similarly, we define a binary relation $\approx$ on $\mathbb{I}$ by declaring $\mathscr{I}_1 \approx \mathscr{I}_2$ when there exists $\beta \in Ш_G^\infty(\mathscr{I}_1)$ such that $\mathscr{I}_2 = \mathscr{I}_1^\beta$. (In Corollary 5.13.4 below we will see that $\approx$ is an equivalence relation on $\mathbb{I}$.) Similarly as before, if $\mathscr{I}_1 \approx \mathscr{I}_2$ then for any $x_1 \in \mathscr{I}_1$ and $x_2 \in \mathscr{I}_2$ there is a canonical equivalence class of inner twistings between $I_{x_1}$ and $I_{x_2}$, and moreover the induced equivalence class of inner twistings between $I_{x_1,\mathbb{R}}$ and $I_{x_2,\mathbb{R}}$ is trivial. In fact, in this case the $\mathbb{Q}$-scheme $I_{x_1,x_2}$ considered in §5.4 is an $I_{x_1}$-torsor, and its class in $\mathbf{H}^1(\mathbb{Q}, I_{x_1})$ is the element $\beta \in Ш_G^\infty(\mathscr{I}_1)$ with $\mathscr{I}_2 = \mathscr{I}_1^\beta$; see the proof of [Kis17, Prop. 4.4.8]. The above-mentioned equivalence class of inner twistings between $I_{x_1}$ and $I_{x_2}$ is induced by elements of $I_{x_1,x_2}(\overline{\mathbb{Q}})$. In particular, the element of $\mathbf{H}^1(\mathbb{Q}, I_{x_1}^{\mathrm{ad}})$ corresponding to this equivalence class of inner twistings (see Remark 1.2.3) is the image of $\beta$ under the natural map $\mathbf{H}^1(\mathbb{Q}, I_{x_1}) \to \mathbf{H}^1(\mathbb{Q}, I_{x_1}^\beta)$. Since $\beta$ has trivial image in $\mathbf{H}^1(\mathbb{R}, I_{x_1})$, our assertion that the induced equivalence class of inner twistings between $I_{x_1,\mathbb{R}}$ and $I_{x_2,\mathbb{R}}$ is trivial follows.

From the above discussion, we have a canonical isomorphism $Ш_G^\infty(\mathbb{Q}, I_{x_1}) \cong Ш_G^\infty(\mathbb{Q}, I_{x_2})$. Again, if we view this isomorphism as an isomorphism $Ш_G^\infty(\mathbb{Q}, \mathscr{I}_1) \cong Ш_G^\infty(\mathbb{Q}, \mathscr{I}_2)$, then it depends only on $\mathscr{I}_1$ and $\mathscr{I}_2$.

**Lemma 5.13.2.** *Let $\mathscr{J}_1, \mathscr{J}_2 \in \mathbb{J}$. Assume that $\mathscr{J}_2 = \mathscr{J}_1^\beta$ for some $\beta \in Ш_G^\infty(\mathscr{J}_1)$. Let $\beta' \in Ш_G^\infty(\mathscr{J}_1)$, also viewed as an element of $Ш_G^\infty(\mathscr{J}_2)$ via the canonical isomorphism $Ш_G^\infty(\mathscr{J}_1) \cong Ш_G^\infty(\mathscr{J}_2)$. Then $\mathscr{J}_1^{\beta+\beta'} = \mathscr{J}_2^{\beta'}$.*

*Proof.* We first make a reduction step that is very similar to the proof of Theorem 5.12.5. Let $\phi \in \mathscr{J}_1$. By [Bor98, Thm. 5.10], there exists a maximal torus $T \subset I_\phi$ such that both $\beta$ and $\beta'$ come from elements $\beta_T$ and $\beta'_T$ of $\mathbf{H}^1(\mathbb{Q}, T)$. By Theorem 3.3.9, we reduce to the case where $\phi = \phi(T, i, h)$ for some $(T, i, h) \in \mathcal{SPD}(G, X)$, and where the inclusion $T \hookrightarrow I_\phi$ is the canonical inclusion (namely the one whose composition with $I_{\phi,\overline{\mathbb{Q}}} \hookrightarrow G_{\overline{\mathbb{Q}}}$ is $i$). Define $Ш_G^\infty(\mathbb{Q}, T)$ to be the kernel of $Ш^\infty(\mathbb{Q}, T) \to Ш^\infty(\mathbb{Q}, G)$ induced by $i$. By the same argument as in the proof of Theorem 5.12.5, we have $\beta_T, \beta'_T \in Ш_G^\infty(\mathbb{Q}, T)$. Now fix cocycles $\beta_T(\cdot)$ and $\beta'_T(\cdot)$ representing $\beta_T$ and $\beta'_T$, and find $\lambda_1, \lambda_2 \in G(\overline{\mathbb{Q}})$ such that

$$\lambda_1^{-1\rho}\lambda_1 = i(\beta_T(\rho)), \quad \lambda_2^{-1\rho}\lambda_2 = i(\beta_T(\rho)\beta'_T(\rho)), \quad \forall \rho \in \Gamma.$$

Define $i_1 = \mathrm{Int}(\lambda_1) \circ i$ and $i_2 = \mathrm{Int}(\lambda_2) \circ i$. Then $(T, i_1, h)$ and $(T, i_2, h)$ are special point data in $\mathcal{SPD}(G, X)$. Moreover, it is easy to check that $\mathscr{J}_2 = \mathscr{J}_{(T,i_1,h)}$ and $\mathscr{J}_1^{\beta+\beta'} = \mathscr{J}_{(T,i_2,h)}$.

To finish the proof, we need to show that $\mathscr{J}_{(T,i_2,h)}$ is equal to $\mathscr{J}_{(T,i_1,h)}^{\beta'}$. By the same argument as before, we need only show that there exists $\lambda \in G(\overline{\mathbb{Q}})$ satisfying the following conditions:

(i) We have $\lambda^{-1\rho}\lambda = i_1(\beta'_T(\rho)), \ \forall \rho \in \Gamma$.
(ii) We have $i_2 = \mathrm{Int}(\lambda) \circ i_1$.

Clearly $\lambda = \lambda_2\lambda_1^{-1}$ satisfies the second condition. To check the first condition, we compute

$$\begin{aligned}\lambda^{-1\rho}\lambda &= \lambda_1\lambda_2^{-1\rho}\lambda_2{}^\rho\lambda_1^{-1} = \lambda_1 i\big(\beta_T(\rho)\beta'_T(\rho)\big){}^\rho\lambda_1^{-1} \\ &= \lambda_1\lambda_1^{-1\rho}\lambda_1 i(\beta'_T(\rho)){}^\rho\lambda_1^{-1} = {}^\rho\lambda_1 i(\beta'_T(\rho)){}^\rho\lambda_1^{-1}.\end{aligned}$$

Since $i$ is defined over $\mathbb{Q}$, the above is equal to

$$^{\rho}\Big(\lambda_1 i\big(\rho^{-1}(\beta'_T(\rho))\big)\lambda_1^{-1}\Big) = {}^{\rho}\Big(i_1\big(\rho^{-1}(\beta'_T(\rho))\big)\Big).$$

But $i_1$ is also defined over $\mathbb{Q}$, so the above is equal to $i_1(\beta'_T(\rho))$, as desired. $\square$

**Lemma 5.13.3.** *Let $\mathscr{I}_1, \mathscr{I}_2 \in \mathbb{I}$. Assume that $\mathscr{I}_2 = \mathscr{I}_1^{\beta}$ for some $\beta \in \text{Ш}_G^{\infty}(\mathscr{I}_1)$. Let $\beta' \in \text{Ш}_G^{\infty}(\mathscr{I}_1)$, also viewed as an element of $\text{Ш}_G^{\infty}(\mathscr{I}_2)$ via the canonical isomorphism $\text{Ш}_G^{\infty}(\mathscr{I}_1) \cong \text{Ш}_G^{\infty}(\mathscr{I}_2)$. Then $\mathscr{I}_1^{\beta+\beta'} = \mathscr{I}_2^{\beta'}$.*

*Proof.* The proof is completely analogous to that of Lemma 5.13.2. Let $x \in \mathscr{I}_1$. By [Bor98, Thm. 5.10], there exists a maximal torus $T \subset I_x$ such that both $\beta$ and $\beta'$ come from elements $\beta_T$ and $\beta'_T$ of $\mathbf{H}^1(\mathbb{Q}, T)$. By Theorem 5.7.8, we reduce to the case where $x = x_{(T,i,h)}$ for some $(T, i, h) \in \mathcal{SPD}(G, X)$, and where the inclusion $T \hookrightarrow I_x$ is the canonical inclusion (as in (5.7.2.2)). Define $\text{Ш}_G^{\infty}(\mathbb{Q}, T)$ to be the kernel of the map $\text{Ш}^{\infty}(\mathbb{Q}, T) \to \text{Ш}^{\infty}(\mathbb{Q}, G)$ induced by $i$. As in the proof of Lemma 5.13.2, we have $\beta_T, \beta'_T \in \text{Ш}_G^{\infty}(\mathbb{Q}, T)$. Now fix cocycles $\beta_T(\cdot)$ and $\beta'_T(\cdot)$ representing $\beta_T$ and $\beta'_T$, and find $\lambda_1, \lambda_2 \in G(\overline{\mathbb{Q}})$ such that

$$\lambda_1^{-1\,\rho}\lambda_1 = i(\beta_T(\rho)), \quad \lambda_2^{-1\,\rho}\lambda_2 = i(\beta_T(\rho)\beta'_T(\rho)), \quad \forall \rho \in \Gamma.$$

Define $i_1 = \mathrm{Int}(\lambda_1) \circ i$ and $i_2 = \mathrm{Int}(\lambda_2) \circ i$. Then $(T, i_1, h)$ and $(T, i_2, h)$ are special point data in $\mathcal{SPD}(G, X)$. Moreover, the same argument as in the proof of [Kis17, Cor. 4.6.5] shows that $\mathscr{I}_2 = \mathscr{I}_{(T,i_1,h)}$ and $\mathscr{I}_1^{\beta+\beta'} = \mathscr{I}_{(T,i_2,h)}$.

To finish the proof, we need to show that $\mathscr{I}_{(T,i_2,h)}$ is equal to $\mathscr{I}_{(T,i_1,h)}^{\beta'}$. By the same argument as before, we need only show that there exists $\lambda \in G(\overline{\mathbb{Q}})$ satisfying the following conditions:

(i) We have $\lambda^{-1\,\rho}\lambda = i_1(\beta'_T(\rho))$, $\forall \rho \in \Gamma$.
(ii) We have $i_2 = \mathrm{Int}(\lambda) \circ i_1$.

As in the proof of Lemma 5.13.2, the element $\lambda = \lambda_2\lambda_1^{-1}$ satisfies the above conditions. $\square$

**Corollary 5.13.4.** *The relation $\approx$ on $\mathbb{I}$ is an equivalence relation.*

*Proof.* To see reflexivity, for each $\mathscr{I} \in \mathbb{I}$ we have by definition $\mathscr{I} = \mathscr{I}^{\beta}$ with $\beta = 0 \in \text{Ш}_G^{\infty}(\mathscr{I})$. The transitivity and symmetry follow directly from Lemma 5.13.3. $\square$

**5.13.5.** For $\mathscr{I} \in \mathbb{I}$ and $y \in \bar{Y}(\mathscr{I})$, the image of $\mathfrak{k}(y) \in \mathfrak{KT}^{\mathrm{str}}/\equiv$ under $\mathfrak{KT}^{\mathrm{str}}/\equiv \to \mathfrak{KT}/\sim$ depends only on $\mathscr{I}$. We denote this element by $\mathfrak{k}(\mathscr{I})$. Thus we have a map $\mathbb{I} \to \mathfrak{KT}/\sim$. Similarly, for $\mathscr{J} \in \mathbb{J}$ and $y \in \bar{Y}(\mathscr{J})$, the image of $[\mathfrak{k}(y)] \subset \mathfrak{KT}^{\mathrm{str}}/\equiv$ under $\mathfrak{KT}^{\mathrm{str}}/\equiv \to \mathfrak{KT}/\sim$ consists of a unique element, and this element depends only on $\mathscr{J}$. We denote this element by $\mathfrak{k}(\mathscr{J})$. Thus we have a map $\mathbb{J} \to \mathfrak{KT}/\sim$.

It follows from Theorem 3.3.9, Theorem 5.7.8, Proposition 5.7.6, and Proposition 5.9.2, that the images of $\mathbb{I} \to \mathfrak{KT}/\sim$ and $\mathbb{J} \to \mathfrak{KT}/\sim$ are both equal to $\{\mathfrak{k}(\mathfrak{s}) \mid \mathfrak{s} \in \mathcal{SPD}(G, X)\}$. (See §5.3.9 for $\mathfrak{k}(\mathfrak{s})$.) We denote this set by $(\mathfrak{KT}/\sim)^{\mathrm{sp}}$.

**5.13.6.** Let $(\mathscr{I}, \mathscr{J})$ be a weakly amicable pair. For each $x \in \mathscr{I}$ and $\phi \in \mathscr{J}$, we have an isomorphism $I_x \xrightarrow{\sim} I_{\phi}$ that is canonical up to composing with inner automorphisms defined over $\mathbb{Q}$; see §5.10.3 and Remark 5.10.10. We thus have a canonical isomorphism of abelian groups

$$\text{(5.13.6.1)} \qquad Ш^{\infty}_G(\mathbb{Q}, \mathscr{I}) \cong Ш^{\infty}_G(\mathbb{Q}, \mathscr{J})$$

that is independent of all choices (also cf. §5.12.3).

As a special case, let $\mathfrak{s} \in \mathcal{SPD}(G, X)$ be a special point datum and consider the pair $(\mathscr{I}_\mathfrak{s}, \mathscr{J}_\mathfrak{s})$. By Corollary 5.11.9, $(\mathscr{I}_\mathfrak{s}, \mathscr{J}_\mathfrak{s})$ is amicable. Hence we get a canonical identification

$$Ш^{\infty}_G(\mathscr{I}_\mathfrak{s}) \cong Ш^{\infty}_G(\mathscr{J}_\mathfrak{s}).$$

We denote the identified abelian group by $Ш^{\infty}_G(\mathfrak{s})$.

More generally, we call a pair $(\mathscr{I}', \mathscr{J}')$ consisting of $\mathscr{I}' \in \mathbb{I}$ and $\mathscr{J}' \in \mathbb{J}$ an *acquainted pair*, if there exists a weakly amicable pair $(\mathscr{I}, \mathscr{J})$ such that $\mathscr{I}' \approx \mathscr{I}$ and $\mathscr{J}' \approx \mathscr{J}$. Recall that in this case we have canonical isomorphisms $Ш^{\infty}_G(\mathscr{I}) \cong Ш^{\infty}_G(\mathscr{I}')$ and $Ш^{\infty}_G(\mathscr{J}) \cong Ш^{\infty}_G(\mathscr{J}')$. Composing these two with the isomorphisms (5.13.6.1) with respect to the weakly amicable pair $(\mathscr{I}, \mathscr{J})$, we obtain a canonical isomorphism

$$Ш^{\infty}_G(\mathscr{I}') \cong Ш^{\infty}_G(\mathscr{J}')$$

which depends only on the acquainted pair $(\mathscr{I}', \mathscr{J}')$.

**Lemma 5.13.7.** *There exists a (non-canonical) bijection $\mathscr{B} : \mathbb{J} \xrightarrow{\sim} \mathbb{I}$ satisfying the following conditions.*

- *(i) For each $\mathscr{J} \in \mathbb{J}$, there exists a special point datum $\mathfrak{s} \in \mathcal{SPD}(G, X)$ and an element $\beta \in Ш^{\infty}_G(\mathfrak{s})$ such that $\mathscr{J} = \mathscr{J}_\mathfrak{s}^\beta$ and $\mathscr{B}(\mathscr{J}) = \mathscr{I}_\mathfrak{s}^\beta$.*
- *(ii) Whenever $\mathscr{I} = \mathscr{B}(\mathscr{J})$, the pair $(\mathscr{I}, \mathscr{J})$ is an acquainted pair. In particular we have a canonical isomorphism $Ш^{\infty}_G(\mathscr{I}) \cong Ш^{\infty}_G(\mathscr{J})$.*
- *(iii) Suppose we have $\mathscr{I} = \mathscr{B}(\mathscr{J})$. For any $\beta \in Ш^{\infty}_G(\mathscr{I}) \cong Ш^{\infty}_G(\mathscr{J})$, we have $\mathscr{B}(\mathscr{J}^\beta) = \mathscr{I}^\beta$.*

*Remark* 5.13.8. A similar bijection $\mathbb{J} \xrightarrow{\sim} \mathbb{I}$ is implicitly used in [Kis17]; see the proof of [Kis17, Cor. 4.6.5]. However, the bijection in [Kis17] satisfies a different set of conditions than those in Lemma 5.13.7. More specifically, the groups $Ш^{\infty}_G(\mathfrak{s})$ and $Ш^{\infty}_G(\mathscr{J})$ in conditions (i) and (iii) in Lemma 5.13.7 are replaced by the subgroups consisting of elements that come from the centers of $I_{\phi(\mathfrak{s})}$ and $I_\phi$ (for $\phi \in \mathscr{J}$).

*Proof of Lema 5.13.7.* As we explained in §5.13.5, the maps $\mathbb{J} \to \mathfrak{KT}/\sim$ and $\mathbb{I} \to \mathfrak{KT}/\sim$ have the same image $(\mathfrak{KT}/\sim)^{\mathrm{sp}} \subset \mathfrak{KT}/\sim$. By [Kis17, Prop. 4.5.7], each fiber of the map $\mathbb{J} \to (\mathfrak{KT}/\sim)^{\mathrm{sp}}$ is contained in one equivalence class of $\approx$. By [Kis17, Prop. 4.4.13], each fiber of the map $\mathbb{I} \to (\mathfrak{KT}/\sim)^{\mathrm{sp}}$ is contained in one equivalence class of $\approx$.

By [Kis17, Lem. 4.4.11, Prop. 4.4.13, Lem. 4.5.6, Prop. 4.5.7], we know that there is an equivalence relation on $(\mathfrak{KT}/\sim)^{\mathrm{sp}}$ whose pull-back to $\mathbb{J}$ is $\approx$ and whose pull-back to $\mathbb{I}$ is $\approx$. We denote this equivalence relation on $(\mathfrak{KT}/\sim)^{\mathrm{sp}}$ also by $\approx$. In the current proof we do not need an explicit description of this equivalence relation. [38]

[38] In the notation of [Kis17, §4], we have $\mathfrak{k}_1 \approx \mathfrak{k}_2$ in $(\mathfrak{KT}/\sim)^{\mathrm{sp}}$ if and only if there exists $\beta \in Ш^{\infty}_G(\mathbb{Q}, I)$ such that $\mathfrak{k}_2 = \mathfrak{k}_1^\beta$. Here $I$ is the reductive group over $\mathbb{Q}$ associated with $\mathfrak{k}_1$, which is unique up to an isomorphism that is canonical up to composing with inner automorphisms defined over $\mathbb{Q}$.

From the above discussion, we have natural bijections $\mathbb{J}/\approx \xrightarrow{\sim} (\mathfrak{K}\mathfrak{T}/\sim)^{\mathrm{sp}}/\approx$ and $\mathbb{I}/\approx \xrightarrow{\sim} (\mathfrak{K}\mathfrak{T}/\sim)^{\mathrm{sp}}/\approx$. We now fix a set of representatives $\{\mathfrak{k}_j \mid j \in J\} \subset (\mathfrak{K}\mathfrak{T}/\sim)^{\mathrm{sp}}$ for the equivalence relation $\approx$ on $(\mathfrak{K}\mathfrak{T}/\sim)^{\mathrm{sp}}$. For each $j \in J$, we choose a special point datum $\mathfrak{s}_j \in \mathcal{SPD}(G, X)$ such that $\mathfrak{k}_j = \mathfrak{k}(\mathfrak{s}_j)$. We then let $\mathscr{J}_j := \mathscr{J}_{\mathfrak{s}_j}$ and $\mathscr{I}_j := \mathscr{I}_{\mathfrak{s}_j}$. Consider the subset $B := \{\mathscr{J}_j \mid j \in J\}$ of $\mathbb{J}$, and the subset $A := \{\mathscr{I}_j \mid j \in J\}$ of $\mathbb{I}$. For each $j \in J$, we have a canonical identification $\text{Ш}^{\infty}_G(\mathscr{I}_j) \cong \text{Ш}^{\infty}_G(\mathscr{J}_j)$. As in §5.13.6, we denote the identified group by $\text{Ш}^{\infty}_G(\mathfrak{s}_j)$.

Define the set

$$D := \{(j, \beta) \mid j \in J, \beta \in \text{Ш}^{\infty}_G(\mathfrak{s}_j)\}.$$

Then we have a map $\mathscr{G} : D \to \mathbb{J}$ sending $(j, \beta)$ to $\mathscr{J}_j^{\beta}$, and a map $\mathscr{F} : D \to \mathbb{I}$ sending $(j, \beta)$ to $\mathscr{I}_j^{\beta}$. Clearly $B$ (resp. $A$) is a set of representatives for the equivalence relation $\approx$ on $\mathbb{J}$ (resp. on $\mathbb{I}$). It follows that both $\mathscr{G}$ and $\mathscr{F}$ are surjective. Moreover, by [Kis17, Prop. 4.4.8, Lem. 4.5.6], both $\mathscr{G}$ and $\mathscr{F}$ are injective, and so they are bijections.

We now define the desired bijection $\mathscr{B}$ to be $\mathscr{F} \circ \mathscr{G}^{-1}$. Then condition (i) follows from the construction. Condition (ii) follows from condition (i). We are left to check condition (iii). Suppose we have $\mathscr{B}(\mathscr{J}) = \mathscr{I}$. Let $(j, \beta') = \mathscr{G}^{-1}(\mathscr{J})$. Then $\mathscr{J} = \mathscr{J}_j^{\beta'}$ and $\mathscr{I} = \mathscr{I}_j^{\beta'}$. Let $\beta$ be an arbitrary element of $\text{Ш}^{\infty}_G(\mathscr{I}) \cong \text{Ш}^{\infty}_G(\mathscr{J})$. By Lemma 5.13.2, we have $\mathscr{J}^{\beta} = \mathscr{J}_j^{\beta+\beta'}$. Hence $\mathscr{B}(\mathscr{J}^{\beta}) = \mathscr{I}_j^{\beta+\beta'}$ by the definition of $\mathscr{B}$. But by Lemma 5.13.3, we have $\mathscr{I}_j^{\beta+\beta'} = (\mathscr{I}_j^{\beta'})^{\beta}$, and this is equal to $\mathscr{I}^{\beta}$. Therefore $\mathscr{B}(\mathscr{J}^{\beta}) = \mathscr{I}^{\beta}$, as desired. □

**Theorem 5.13.9.** *There exists a (non-canonical) bijection $\mathscr{B} : \mathbb{J} \xrightarrow{\sim} \mathbb{I}$ satisfying the following conditions.*

*(i) Whenever $\mathscr{I} = \mathscr{B}(\mathscr{J})$, the pair $(\mathscr{I}, \mathscr{J})$ is an amicable pair. In particular we have a canonical isomorphism $\text{Ш}^{\infty}_G(\mathscr{I}) \cong \text{Ш}^{\infty}_G(\mathscr{J})$.*

*(ii) Suppose we have $\mathscr{I} = \mathscr{B}(\mathscr{J})$. For any $\beta \in \text{Ш}^{\infty}_G(\mathscr{I}) \cong \text{Ш}^{\infty}_G(\mathscr{J})$, we have $\mathscr{B}(\mathscr{J}^{\beta}) = \mathscr{I}^{\beta}$.*

*Proof.* It suffices to show that the bijection $\mathscr{B}$ as in Lemma 5.13.7 satisfies the conditions. Condition (i) in the theorem follows from condition (i) in Lemma 5.13.7, Corollary 5.11.9, and Theorem 5.12.5. Condition (ii) in the theorem is just condition (iii) in Lemma 5.13.7. □

**5.13.10.** Fix a bijection $\mathscr{B}$ as in Theorem 5.13.9. Let $\mathscr{J} \in \mathbb{J}$ and let $\phi \in \mathscr{J}$. Since $(\mathscr{B}(\mathscr{J}), \mathscr{J})$ is an amicable pair, we have the elements $\tau^{\mathfrak{H}}(\mathscr{B}(\mathscr{J}), \mathscr{J}) \in \mathfrak{H}(\phi)$ and $\tau^{\mathcal{H}}(\mathscr{B}(\mathscr{J}), \mathscr{J}) \in \mathcal{H}(\phi)$ as in Definition 5.10.9. We also write $\underline{\tau}_{\mathscr{B}}(\phi)$ for $\tau^{\mathcal{H}}(\mathscr{B}(\mathscr{J}), \mathscr{J})$. The assignment $\phi \mapsto \underline{\tau}_{\mathscr{B}}(\phi)$ is thus an element of $\Gamma(\mathcal{H})$, in the notation of Definition 2.6.17. We denote this element by $\underline{\tau}_{\mathscr{B}}$. Since $\underline{\tau}_{\mathscr{B}}(\phi)$ depends on $\phi$ only via its conjugacy class (that is, after we canonically identify all $\mathcal{H}(\phi')$ for $\phi'$ in the conjugacy class of $\phi$), we know that $\underline{\tau}_{\mathscr{B}} \in \Gamma(\mathcal{H})_1$.

**Corollary 5.13.11.** *We have $\underline{\tau}_{\mathscr{B}} \in \Gamma(\mathcal{H})_0$.*

*Proof.* This follows from the two conditions satisfied by $\mathscr{B}$ in Theorem 5.13.9 and Theorem 5.12.5. □

## 6. The Langlands–Rapoport–$\tau$ Conjecture in case of abelian type

### 6.1. **More sheaves on the set of admissible morphisms.**

**6.1.1.** Let $(G, X, p, \mathcal{G})$ be an unramified Shimura datum. We let $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ be the adjoint Shimura datum. Let $\mathcal{G}^{\mathrm{ad}}$ be the adjoint group of $\mathcal{G}$, which is a reductive model of $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$.

As in §2.6.16, we write $\mathcal{AM}(G, X, p, \mathcal{G})$ and $\mathcal{AM}(G^{\mathrm{ad}}, X^{\mathrm{ad}}, p, \mathcal{G}^{\mathrm{ad}})$ for the sets of admissible morphisms with respect to $(G, X, p, \mathcal{G})$ and $(G^{\mathrm{ad}}, X^{\mathrm{ad}}, p, \mathcal{G}^{\mathrm{ad}})$ respectively. For simplicity, in the sequel we denote these two sets by $\mathcal{AM}(G)$ and $\mathcal{AM}(G^{\mathrm{ad}})$ respectively. Since $(G, X)$ is arbitrary, our discussion below regarding $\mathcal{AM}(G)$ will also be valid for $\mathcal{AM}(G^{\mathrm{ad}})$.

**Definition 6.1.2.** For each $\phi \in \mathcal{AM}(G)$, we define the $\mathbb{Q}$-reductive group $I_\phi^\dagger$, the abelian group $\mathfrak{H}(\phi)$, and the map $I_\phi^{\mathrm{ad}}(\mathbb{A}_f)/I_\phi^{\mathrm{ad}}(\mathbb{Q}) \to \mathfrak{H}(\phi)$ in exactly the same way as in §5.10.5. Taking the direct sum of the map $I_\phi^{\mathrm{ad}}(\mathbb{A}_f)/I_\phi^{\mathrm{ad}}(\mathbb{Q}) \to \mathfrak{H}(\phi)$ and the quotient map $I_\phi^{\mathrm{ad}}(\mathbb{A}_f)/I_\phi^{\mathrm{ad}}(\mathbb{Q}) \to \mathcal{H}(\phi) = I_\phi(\mathbb{A}_f)\backslash I_\phi^{\mathrm{ad}}(\mathbb{A}_f)/I_\phi^{\mathrm{ad}}(\mathbb{Q})$, we obtain a natural map $I_\phi^{\mathrm{ad}}(\mathbb{A}_f)/I_\phi^{\mathrm{ad}}(\mathbb{Q}) \longrightarrow \mathfrak{H}(\phi) \oplus \mathcal{H}(\phi)$. We denote its image by $\mathcal{H}^+(\phi)$. We denote by $\mathcal{H}^\dagger(\phi)$ the abelian group $\mathfrak{E}(Z_\phi^\dagger, I_\phi^\dagger; \mathbb{A}_f)$. Here $Z_\phi^\dagger$ denotes the center of $I_\phi^\dagger$ as usual.

**6.1.3.** Let $\phi \in \mathcal{AM}(G)$. We have a natural map $\mathcal{H}^\dagger(\phi) \to \mathfrak{H}(\phi)$ induced by the identity map on $\mathbf{H}^1(\mathbb{A}_f, Z_\phi^\dagger)$. We also have a natural map $\mathcal{H}^\dagger(\phi) \to \mathcal{H}(\phi)$ induced by the inclusions $Z_\phi^\dagger \hookrightarrow Z_{I_\phi}$ and $I_\phi^\dagger \hookrightarrow I_\phi$, in view of the presentation of $\mathcal{H}(\phi)$ as a quotient of $\mathfrak{E}(Z_{I_\phi}, I_\phi; \mathbb{A}_f)$ as in Lemma 2.6.14. We thus have a natural map

$$\mathcal{H}^\dagger(\phi) \longrightarrow \mathfrak{H}(\phi) \oplus \mathcal{H}(\phi). \tag{6.1.3.1}$$

As in (5.10.5.3), we have the boundary map $I_\phi^{\mathrm{ad}}(\mathbb{A}_f) \to \mathbf{H}^1(\mathbb{A}_f, Z_\phi^\dagger)$ arising from the short exact sequence $1 \to Z_\phi^\dagger \to I_\phi^\dagger \to I_\phi^{\mathrm{ad}} \to 1$. The image of the map is $\mathfrak{D}(Z_\phi^\dagger, I_\phi^\dagger; \mathbb{A}_f) \cong \mathfrak{E}(Z_\phi^\dagger, I_\phi^\dagger; \mathbb{A}_f) = \mathcal{H}^\dagger(\phi)$. It follows that the image of (6.1.3.1) is precisely $\mathcal{H}^+(\phi)$. In particular, $\mathcal{H}^+(\phi)$ is a subgroup of the abelian group $\mathfrak{H}(\phi) \oplus \mathcal{H}(\phi)$ since (6.1.3.1) is a group homomorphism.

As in §2.6.16, we view $\mathcal{AM}(G)$ as a discrete topological space. We have already defined the sheaf of abelian groups $\mathcal{H}$ on $\mathcal{AM}(G)$, whose stalk at each $\phi \in \mathcal{AM}(G)$ is $\mathcal{H}(\phi)$. We now define similarly sheaves of abelian groups $\mathfrak{H}, \mathcal{H}^+, \mathcal{H}^\dagger$ on $\mathcal{AM}(G)$. In §2.6.16 we saw that $\mathcal{H}$ descends canonically to a sheaf on $\mathcal{AM}(G)/\approx$, since we have a canonical isomorphism $\mathcal{H}(\phi_1) \cong \mathcal{H}(\phi_2)$ whenever $\phi_1 \approx \phi_2$, and such canonical isomorphisms satisfy the cocycle relation. Similarly, the sheaves $\mathfrak{H}, \mathcal{H}^+, \mathcal{H}^\dagger$ on $\mathcal{AM}(G)$ all descend canonically to sheaves on $\mathcal{AM}(G)/\approx$.

In the next definition we introduce notations analogous to those in Definition 2.6.17.

**Definition 6.1.4.** For $\mathcal{F} \in \{\mathfrak{H}, \mathcal{H}^+, \mathcal{H}^\dagger\}$, let $\Gamma(\mathcal{F})$ denote the set of global sections of the sheaf $\mathcal{F}$ on $\mathcal{AM}(G)$. Let $\mathcal{F}_\approx$ denote the canonical descent of $\mathcal{F}$ over $\mathcal{AM}(G)/\approx$. Let $\Gamma(\mathcal{F})_0$ denote the subset of $\Gamma(\mathcal{F})$ consisting of those global sections that descend to global sections of $\mathcal{F}_\approx$.

**6.1.5.** For each $\phi \in \mathcal{AM}(G)$, we have natural maps $\mathcal{H}^\dagger(\phi) \to \mathcal{H}^+(\phi) \to \mathfrak{H}(\phi) \oplus \mathcal{H}(\phi)$, and the map $\mathcal{H}^\dagger(\phi) \to \mathcal{H}^+(\phi)$ is surjective. It follows that we have natural maps $\Gamma(\mathcal{H}^\dagger) \to \Gamma(\mathcal{H}^+) \to \Gamma(\mathfrak{H}) \oplus \Gamma(\mathcal{H})$, which restrict to natural maps $\Gamma(\mathcal{H}^\dagger)_0 \to \Gamma(\mathcal{H}^+)_0 \to \Gamma(\mathfrak{H})_0 \oplus \Gamma(\mathcal{H})_0$. Clearly the maps $\Gamma(\mathcal{H}^\dagger) \to \Gamma(\mathcal{H}^+)$ and $\Gamma(\mathcal{H}^\dagger)_0 \to \Gamma(\mathcal{H}^+)_0$ are surjective.

**Definition 6.1.6.** We say that an element $\underline{\tau}^{\mathfrak{H}} \in \Gamma(\mathfrak{H})$ is *tori-rational*, if for each $\phi \in \mathcal{AM}(G)$ and each maximal torus $T$ in $I_\phi$, the element $\underline{\tau}^{\mathfrak{H}}(\phi) \in \mathfrak{H}(\phi)$ has trivial image in

$$\operatorname{coker}(\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p) \xrightarrow{\partial} \mathbf{H}^1(\mathbb{A}_f, T^\dagger)/\text{Ш}^\infty_{G_{\mathrm{der}}}(\mathbb{Q}, T^\dagger)).$$

Here the definition of the above group, as well as the definition of the natural map from $\mathfrak{H}(\phi)$ to the above group, are the same as in §5.12.1. We say that an element $\underline{\tau}^+ \in \Gamma(\mathcal{H}^+)$ is *tori-rational*, if its image in $\Gamma(\mathfrak{H})$ is tori-rational.

**Lemma 6.1.7.** *Let $\underline{\tau}^+ \in \Gamma(\mathcal{H}^+)$ be a tori-rational element. Then its image in $\mathcal{H}(\phi)$ is tori-rational (see Definition 2.6.19).*

*Proof.* It suffices to note that for each $\phi \in \mathcal{AM}(G)$ and each maximal torus $T$ in $I_\phi$, the composition $\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p) \xrightarrow{\partial} \mathbf{H}^1(\mathbb{A}_f, T^\dagger) \to \mathbf{H}^1(\mathbb{A}_f, T)$ is the zero map. This is indeed the case, since $\partial$ is induced by the boundary map $G^{\mathrm{ab}}(\mathbb{Q}_p) \to \mathbf{H}^1(\mathbb{Q}_p, T^\dagger)$ associated with the short exact sequence $1 \to T^\dagger \to T \to G^{\mathrm{ab}} \to 1$. □

**6.1.8.** We let $\mathcal{H}^{\dagger,\mathrm{ad}}, \mathcal{H}^{+,\mathrm{ad}}, \mathfrak{H}^{\mathrm{ad}}, \mathcal{H}^{\mathrm{ad}}$ be the sheaves on $\mathcal{AM}(G^{\mathrm{ad}})$ defined in the same way as $\mathcal{H}^\dagger, \mathcal{H}^+, \mathfrak{H}, \mathcal{H}$, with $(G, X, p, \mathcal{G})$ replaced by $(G^{\mathrm{ad}}, X^{\mathrm{ad}}, p, \mathcal{G}^{\mathrm{ad}})$. We shall apply Definition 6.1.4 and Definition 6.1.6 to these sheaves.

For each $\phi \in \mathcal{AM}(G)$, it is easy to see that the composition of $\phi$ with the natural morphism $\mathfrak{G}_G \to \mathfrak{G}_{G^{\mathrm{ad}}}$ is an admissible morphism $\phi^{\mathrm{ad}} : \mathfrak{Q} \to \mathfrak{G}_{G^{\mathrm{ad}}}$ in $\mathcal{AM}(G^{\mathrm{ad}})$. Thus we have a natural map $\mathcal{AM}(G) \to \mathcal{AM}(G^{\mathrm{ad}}), \phi \mapsto \phi^{\mathrm{ad}}$. This further induces a map $\mathcal{AM}(G)/\approx \to \mathcal{AM}(G^{\mathrm{ad}})/\approx$.[39]

**Lemma 6.1.9.** *Let $\phi_0 \in \mathcal{AM}(G^{\mathrm{ad}})$. Let $\mathcal{U}(\phi_0)$ be the set of conjugacy classes of $\phi \in \mathcal{AM}(G)$ such that $\phi^{\mathrm{ad}}$ is conjugate to $\phi_0$. Then $\mathcal{U}(\phi_0)$ is non-empty and is acted on transitively by the abelian group $\text{Ш}^\infty_G(\mathbb{Q}, Z_G)$, where the action is given by the usual twisting construction (see Proposition 2.6.12).*

*Proof.* We claim that

$$\text{Ш}^\infty_G(\mathbb{Q}, Z_G) = \text{Ш}^\infty(\mathbb{Q}, Z_G) \cap \operatorname{im}(\mathbf{H}^1(\mathbb{Q}, Z_{G_{\mathrm{sc}}}) \to \mathbf{H}^1(\mathbb{Q}, Z_G)).$$

In fact, as in the proof of [Kis17, Lem. 3.4.8], the group on the right hand side is the same as

$$\ker(\mathbf{H}^1(\mathbb{Q}, Z_G) \to \mathbf{H}^1(\mathbb{R}, Z_G) \oplus \mathbf{H}^1_{\mathrm{ab}}(\mathbb{Q}, G)),$$

which is equal to

$$\ker(\text{Ш}^\infty(\mathbb{Q}, Z_G) \to \text{Ш}^\infty_{\mathrm{ab}}(\mathbb{Q}, G)),$$

and equal to $\text{Ш}^\infty_G(\mathbb{Q}, Z_G)$. The claim is proved. The lemma now follows from the claim and [Kis17, Lem. 3.4.8, Prop. 3.4.11]. □

**Lemma 6.1.10.** *Let $\phi \in \mathcal{AM}(G)$. The natural map $I_\phi \to I_{\phi^{\mathrm{ad}}}$ is surjective with kernel $Z_G$ (which is canonically a $\mathbb{Q}$-subgroup of $I_\phi$). In particular, we have a natural identification $I_\phi^{\mathrm{ad}} \cong I_{\phi^{\mathrm{ad}}}^{\mathrm{ad}}$, and a natural surjective map*

$$\mathcal{H}(\phi) = I_\phi(\mathbb{A}_f)\backslash I_\phi^{\mathrm{ad}}(\mathbb{A}_f)/I_\phi^{\mathrm{ad}}(\mathbb{Q}) \longrightarrow \mathcal{H}(\phi^{\mathrm{ad}}) = I_{\phi^{\mathrm{ad}}}(\mathbb{A}_f)\backslash I_{\phi^{\mathrm{ad}}}^{\mathrm{ad}}(\mathbb{A}_f)/I_{\phi^{\mathrm{ad}}}^{\mathrm{ad}}(\mathbb{Q}).$$

[39] Here we have used the same symbol $\approx$ for equivalence relations on $\mathcal{AM}(G)$ and $\mathcal{AM}(G^{\mathrm{ad}})$. They are defined separately, with respect to the two unramified Shimura data $(G, X, p, \mathcal{G})$ and $(G^{\mathrm{ad}}, X^{\mathrm{ad}}, p, \mathcal{G}^{\mathrm{ad}})$.

*Proof.* As a subgroup of $G_{\overline{\mathbb{Q}}}$, $I_{\phi,\overline{\mathbb{Q}}}$ is the centralizer of $\mathrm{im}(\phi^{\Delta})$. Note that $\mathrm{im}(\phi^{\Delta})$ is a subtorus of $G_{\overline{\mathbb{Q}}}$, since $\mathfrak{Q} = (\mathfrak{Q}^L)_L$ and each $\mathfrak{Q}^{L,\Delta} = Q^L_{\overline{\mathbb{Q}}}$ is a torus (see §2.2.8). Similarly, $I_{\phi^{\mathrm{ad}},\overline{\mathbb{Q}}}$ is the centralizer of the subtorus $\mathrm{im}(\phi^{\mathrm{ad},\Delta})$ of $G^{\mathrm{ad}}_{\overline{\mathbb{Q}}}$.

Clearly $\mathrm{im}(\phi^{\mathrm{ad},\Delta})$ is the image of $\mathrm{im}(\phi^{\Delta})$ under the natural map $G \to G^{\mathrm{ad}}$. It is a standard result (see for instance [Bor91, p. 153, Cor. 2]) that if $S$ is a subtorus of $G_{\overline{\mathbb{Q}}}$ with image $S'$ in $G^{\mathrm{ad}}_{\overline{\mathbb{Q}}}$, then the natural map $Z_{G_{\overline{\mathbb{Q}}}}(S) \to Z_{G^{\mathrm{ad}}_{\overline{\mathbb{Q}}}}(S')$ is surjective with kernel $(Z_G)_{\overline{\mathbb{Q}}}$. Since we already know that the natural map $I_\phi \to I_{\phi^{\mathrm{ad}}}$ is defined over $\mathbb{Q}$ and that $Z_G$ is contained in the kernel, this finishes the proof. □

**Lemma 6.1.11.** *The natural map $\mathcal{AM}(G)/\approx \to \mathcal{AM}(G^{\mathrm{ad}})/\approx$ is a bijection.*

*Proof.* The surjectivity follows from Lemma 6.1.9. We show injectivity. Let $\phi, \phi' \in \mathcal{AM}(G)$ be such that $\phi^{\mathrm{ad}} \approx \phi'^{,\mathrm{ad}}$. We need to show that $\phi \approx \phi'$.

By assumption, there exists $\beta_0 \in Ш^{\infty}_{G^{\mathrm{ad}}}(\mathbb{Q}, I_{\phi^{\mathrm{ad}}})$ such that $\phi'^{,\mathrm{ad}}$ is conjugate to $(\phi^{\mathrm{ad}})^{\beta_0}$. By Lemma 6.1.10, we have $I_{\phi^{\mathrm{ad}}} \cong I_\phi/Z_G$. By this fact and by Corollary 1.2.9 (applied to $I = I_\phi$ and $Z = Z_G$, where $I$ indeed has the same absolute rank as $G$), the natural map $Ш^{\infty}_{G}(\mathbb{Q}, I_\phi) \to Ш^{\infty}_{G^{\mathrm{ad}}}(\mathbb{Q}, I_{\phi^{\mathrm{ad}}})$ is surjective. Thus we can find $\beta \in Ш^{\infty}_{G}(\mathbb{Q}, I_\phi)$ lifting $\beta_0$. Then $(\phi^\beta)^{\mathrm{ad}}$ is conjugate to $\phi'^{,\mathrm{ad}}$. (Here $\phi^\beta$ is only well defined up to conjugacy, but the ambiguity does not affect the conjugacy class of $(\phi^\beta)^{\mathrm{ad}}$.) By the transitivity in Lemma 6.1.11, we have $\phi^\beta \approx \phi'$. But $\phi \approx \phi^\beta$, so $\phi \approx \phi'$. □

**Proposition 6.1.12.** *We have a natural surjection $\Gamma(\mathcal{H})_0 \to \Gamma(\mathcal{H}^{\mathrm{ad}})_0$.*

*Proof.* We can identify $\Gamma(\mathcal{H})_0$ with the group of global sections of the sheaf $\mathcal{H}/\approx$ on $\mathcal{AM}(G)/\approx$, and identify $\Gamma(\mathcal{H}^{\mathrm{ad}})_0$ with the group of global sections of the sheaf $\mathcal{H}^{\mathrm{ad}}/\approx$ on $\mathcal{AM}(G^{\mathrm{ad}})/\approx$. By Lemma 6.1.11, we identify the spaces $\mathcal{AM}(G)/\approx$ and $\mathcal{AM}(G^{\mathrm{ad}})/\approx$. It follows from Lemma 6.1.10 that we have a natural surjection $\mathcal{H}/\approx \to \mathcal{H}^{\mathrm{ad}}/\approx$ between sheaves on the identified space. The proposition follows. □

**6.1.13.** Consider two unramified Shimura data $(G, X, p, \mathcal{G})$ and $(G_2, X_2, p, \mathcal{G}_2)$ together with an isomorphism

$$\iota : (G^{\mathrm{ad}}, X^{\mathrm{ad}}) \xrightarrow{\sim} (G_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$$

between the adjoint Shimura data. Assume that $\iota : G^{\mathrm{ad}} \xrightarrow{\sim} G_2^{\mathrm{ad}}$ lifts to a (unique) central isogeny $\widetilde{\iota} : G_{\mathrm{der}} \to G_{2,\mathrm{der}}$, and that $\iota_{\mathbb{Q}_p} : G^{\mathrm{ad}}_{\mathbb{Q}_p} \xrightarrow{\sim} G^{\mathrm{ad}}_{2,\mathbb{Q}_p}$ extends to an isomorphism $\mathcal{G}^{\mathrm{ad}} \xrightarrow{\sim} \mathcal{G}_2^{\mathrm{ad}}$. We still use the symbols $\mathcal{H}^\dagger, \mathcal{H}^+, \mathfrak{H}, \mathcal{H}$ to denote the sheaves on $\mathcal{AM}(G) = \mathcal{AM}(G, X, p, \mathcal{G})$ as in §6.1.3. Their counterparts on $\mathcal{AM}(G_2) = \mathcal{AM}(G_2, X_2, p, \mathcal{G}_2)$ will be denoted with a subscript 2. On $\mathcal{AM}(G^{\mathrm{ad}}) = \mathcal{AM}(G^{\mathrm{ad}}, X^{\mathrm{ad}}, p, \mathcal{G}^{\mathrm{ad}})$ we define the sheaves $\mathcal{H}^{\dagger,\mathrm{ad}}$, $\mathcal{H}^{+,\mathrm{ad}}$, $\mathfrak{H}^{\mathrm{ad}}$, $\mathcal{H}^{\mathrm{ad}}$ as in §6.1.8, and on $\mathcal{AM}(G_2^{\mathrm{ad}}) = \mathcal{AM}(G_2^{\mathrm{ad}}, X_2^{\mathrm{ad}}, p, \mathcal{G}_2^{\mathrm{ad}})$ we have the counterparts denoted with a subscript 2. Then $\iota$ induces an isomorphism

$$\mathcal{AM}(\iota) : \mathcal{AM}(G^{\mathrm{ad}}) \xrightarrow{\sim} \mathcal{AM}(G_2^{\mathrm{ad}})$$

under which the four sheaves on one space are identified with the four on the other respectively.

Let $\mu^{\mathrm{ab}} \in X_*(G^{\mathrm{ab}})$ denote the composite cocharacter

$$\mathbb{G}_m \xrightarrow{\mu} G_{\overline{\mathbb{Q}}} \to G^{\mathrm{ab}}_{\overline{\mathbb{Q}}},$$

where $\mu$ is any element of $\mathfrak{p}_X(\overline{\mathbb{Q}})$. Clearly $\mu^{\mathrm{ab}}$ is independent of the choice of $\mu$.

Let $\underline{\tau}^+ \in \Gamma(\mathcal{H}^+)_0$, and let $\underline{\tau} \in \Gamma(\mathcal{H})_0$ be the image of $\underline{\tau}^+$. By Proposition 6.1.12, we have surjections $\Gamma(\mathcal{H})_0 \to \Gamma(\mathcal{H}^{\mathrm{ad}})_0$ and $\Gamma(\mathcal{H}_2)_0 \to \Gamma(\mathcal{H}^{\mathrm{ad}})_0$. Let $\underline{\tau}_0$ be the image of $\underline{\tau}$ in $\Gamma(\mathcal{H}^{\mathrm{ad}})_0$.

**Proposition 6.1.14.** *Keep the setting of §6.1.13. Assume that $\underline{\tau}^+$ is tori-rational, and assume that $X_*(G^{\mathrm{ab}})$ is generated by $\mu^{\mathrm{ab}}$ as a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module. Then there exists a tori-rational element $\underline{\tau}_2 \in \Gamma(\mathcal{H}_2)_0$ mapping to $\underline{\tau}_0$ under the composite map*

$$\Gamma(\mathcal{H}_2)_0 \to \Gamma(\mathcal{H}_2^{\mathrm{ad}})_0 \xrightarrow{\sim} \Gamma(\mathcal{H}^{\mathrm{ad}})_0, \tag{6.1.14.1}$$

*where the last isomorphism is induced by $\mathcal{AM}(\iota)^{-1}$.*

*Proof.* Consider $\phi \in \mathcal{AM}(G)$ and $\phi_2 \in \mathcal{AM}(G_2)$ such that $\mathcal{AM}(\iota)$ sends $\phi^{\mathrm{ad}} \in \mathcal{AM}(G^{\mathrm{ad}})$ to $\phi_2^{\mathrm{ad}} \in \mathcal{AM}(G_2^{\mathrm{ad}})$. By Lemma 6.1.10, $\iota$ induces a $\mathbb{Q}$-isomorphism $I_\phi^{\mathrm{ad}} \xrightarrow{\sim} I_{\phi_2}^{\mathrm{ad}}$. Clearly this isomorphism lifts to a unique central isogeny $I_\phi^\dagger \to I_{\phi_2}^\dagger$, and the latter is induced by the central isogeny $\widetilde{\iota} : G_{\mathrm{der}} \to G_{2,\mathrm{der}}$. In particular, we have a natural map $\mathcal{H}^\dagger(\phi) \to \mathcal{H}_2^\dagger(\phi_2)$ induced by $\iota$. By this observation, the same argument as the proof of Proposition 6.1.12 shows that there is a natural map $\Gamma(\mathcal{H}^\dagger)_0 \to \Gamma(\mathcal{H}_2^\dagger)_0$ induced by $\iota$.

Recall that the natural map $\Gamma(\mathcal{H}^\dagger)_0 \to \Gamma(\mathcal{H}^+)_0$ is surjective. We fix an element $\underline{\tau}^\dagger \in \Gamma(\mathcal{H}^\dagger)_0$ lifting $\underline{\tau}^+ \in \Gamma(\mathcal{H}^+)_0$. Let $\underline{\tau}_2^\dagger$ be the image of $\underline{\tau}^\dagger$ under the natural map $\Gamma(\mathcal{H}^\dagger)_0 \to \Gamma(\mathcal{H}_2^\dagger)_0$ in the above paragraph. By construction, $\underline{\tau}_2$ is sent to $\underline{\tau}_0$ under the map in the proposition. We are left to check that $\underline{\tau}_2$ is tori-rational.

Let $\underline{\tau}_2^+$ and $\underline{\tau}_2$ be the images of $\underline{\tau}_2^\dagger$ in $\Gamma(\mathcal{H}_2^+)_0$ and $\Gamma(\mathcal{H}_2)_0$ respectively. Let $\phi_2 \in \mathcal{AM}(G_2)$ and let $T_2$ be a maximal torus in $I_{\phi_2}$. We need to show that the image of $\underline{\tau}_2(\phi_2)$ in $\mathbf{H}^1(\mathbb{A}_f, T_2)/\text{Ш}^\infty_{G_2}(\mathbb{Q}, T_2)$ is trivial.

By Lemma 6.1.11 we find $\phi \in \mathcal{AM}(G)$ such that $\mathcal{AM}(\iota)(\phi^{\mathrm{ad}}) = \phi_2^{\mathrm{ad}}$. To simplify notation in the rest of the proof we treat $\iota$ as the identity and omit it from the notation. Write $\phi_0$ for $\phi^{\mathrm{ad}}$, which we identify with $\phi_2^{\mathrm{ad}}$. Let $T_0$ be the image of $T_2$ under $I_{\phi_2} \to I_{\phi_0}$, and let $T$ be the preimage of $T_0$ under $I_\phi \to I_{\phi_0}$. It follows from Lemma 6.1.10 that $T_0$ (resp. $T$) is a maximal torus in $I_{\phi_0}$ (resp. $I_\phi$). As usual, set

$$T^\dagger := \ker(T \to G^{\mathrm{ab}}) = T \cap I_\phi^\dagger$$
$$T_2^\dagger := \ker(T_2 \to G_2^{\mathrm{ab}}) = T_2 \cap I_{\phi_2}^\dagger.$$

Then there is a natural map $u : T^\dagger \to T_2^\dagger$ induced by the central isogeny $I_\phi^\dagger \to I_{\phi_2}^\dagger$ discussed at the beginning of the proof.

Define the torus $U := (T \times_{T_0} T_2)^0$. Denote the natural maps $U \to T$ and $U \to T_2$ by $p_1$ and $p_2$ respectively. The inclusion map $T^\dagger \hookrightarrow T$ and the composite map $T^\dagger \xrightarrow{u} T_2^\dagger \hookrightarrow T_2$ together define a map $T^\dagger \to U$, which we denote by $\Delta$. Let $V$ be the quotient torus $U/\Delta(T^\dagger)$. Then we have a commutative diagram with exact rows in the category of tori over $\mathbb{Q}$:

(6.1.14.2)
$$\begin{tikzcd}
1 \arrow[r] & T^{\dagger} \arrow[r] & T \arrow[r] & G^{\mathrm{ab}} \arrow[r] & 1 \\
1 \arrow[r] & T^{\dagger} \arrow[r, "\Delta"] \arrow[u, equal] \arrow[d, "u"] & U \arrow[r] \arrow[u, "p_1"] \arrow[d, "p_2"] & V \arrow[r] \arrow[u, "\bar{p}_1"] \arrow[d, "\bar{p}_2"] & 1 \\
1 \arrow[r] & T_2^{\dagger} \arrow[r] & T_2 \arrow[r] & G_2^{\mathrm{ab}} \arrow[r] & 1
\end{tikzcd}$$

Here $\bar{p}_1$ and $\bar{p}_2$ are induced by $p_1$ and $p_2$ respectively.

We claim that the image of the map $X_*(V) \to X_*(G^{\mathrm{ab}})$ induced by $\bar{p}_1$ contains $\mu^{\mathrm{ab}}$. To show the claim, let $\mu_T$ be an arbitrary element of $\mathfrak{p}_X(\overline{\mathbb{Q}})$ that factors through $T_{\overline{\mathbb{Q}}}$ (which is embedded into $G_{\overline{\mathbb{Q}}}$ via $I_{\phi,\overline{\mathbb{Q}}} \hookrightarrow G_{\overline{\mathbb{Q}}}$). Similarly, pick $\mu_{T_2} \in \mathfrak{p}_{X_2}(\overline{\mathbb{Q}})$ that factors through $T_{2,\overline{\mathbb{Q}}}$. The two cocharacters of $T_{0,\overline{\mathbb{Q}}}$ induced by $\mu_T$ and $\mu_{T_2}$ respectively are conjugate by $G^{\mathrm{ad}}(\overline{\mathbb{Q}})$, and are hence in the same orbit under the Weyl group of $(G^{\mathrm{ad}}_{\overline{\mathbb{Q}}}, T_{0,\overline{\mathbb{Q}}})$ (cf. [Kot84a, Lem. (1.1.3)]). Since the Weyl group of $(G^{\mathrm{ad}}_{\overline{\mathbb{Q}}}, T_{0,\overline{\mathbb{Q}}})$ and that of $(G_{2,\overline{\mathbb{Q}}}, T_{2,\overline{\mathbb{Q}}})$ are canonically isomorphic, we can replace $\mu_{T_2}$ by a Weyl-conjugate and assume that $\mu_T$ and $\mu_{T_2}$ induce the same cocharacter of $T_{0,\overline{\mathbb{Q}}}$. We then obtain from $\mu_T$ and $\mu_{T_2}$ an element of $X_*(U)$. The image of this element under the composite map $X_*(U) \to X_*(V) \to X_*(G^{\mathrm{ab}})$ is $\mu_T$, by the upper right commutative square in (6.1.14.2). The claim is proved.

By the claim and by our assumption on $X_*(G^{\mathrm{ab}})$, we know that $X_*(V) \to X_*(G^{\mathrm{ab}})$ is surjective. It follows that the kernel of $\bar{p}_1 : V \to G^{\mathrm{ab}}$ is a torus, which we denote by $V^{\dagger}$. Now it is easy to see that the map $(\bar{p}_1, \bar{p}_2) : V \to G^{\mathrm{ab}} \times G_2^{\mathrm{ab}}$ is an isogeny between tori over $\mathbb{Q}$. Since $G^{\mathrm{ab}}$ and $G_2^{\mathrm{ab}}$ are both unramified over $\mathbb{Q}_p$, we deduce that $V$ and $V^{\dagger}$ are both unramified tori over $\mathbb{Q}_p$. Let $\mathcal{V}$ denote the $\mathbb{Z}_p$-torus extending $V_{\mathbb{Q}_p}$. The kernel of $\mathcal{V} \to \mathcal{G}^{\mathrm{ab}}$ is a torus over $\mathbb{Z}_p$, namely the one extending the unramified $\mathbb{Q}_p$-torus $V^{\dagger}_{\mathbb{Q}_p}$. By Lang's theorem applied to that kernel (which is smooth over $\mathbb{Z}_p$ and has connected fibers), we know that the map $\mathcal{V}(\mathbb{Z}_p) \to \mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p)$ is surjective.

Using this surjectivity result and the commutative diagram (6.1.14.2), we see that the natural map

$$\mathbf{H}^1(\mathbb{A}_f, T^{\dagger})/\text{Ш}^{\infty}_{G_{\mathrm{der}}}(\mathbb{Q}, T^{\dagger}) \longrightarrow \mathbf{H}^1(\mathbb{A}_f, T_2^{\dagger})/\text{Ш}^{\infty}_{G_{2,\mathrm{der}}}(\mathbb{Q}, T_2^{\dagger})$$

induced by $u : T^{\dagger} \to T_2^{\dagger}$ descends to a map

$$\operatorname{coker}\left(\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p) \to \mathbf{H}^1(\mathbb{A}_f, T^{\dagger})/\text{Ш}^{\infty}_{G_{\mathrm{der}}}(\mathbb{Q}, T^{\dagger})\right) \longrightarrow$$
$$\operatorname{coker}\left(\mathcal{G}_2^{\mathrm{ab}}(\mathbb{Z}_p) \to \mathbf{H}^1(\mathbb{A}_f, T_2^{\dagger})/\text{Ш}^{\infty}_{G_{2,\mathrm{der}}}(\mathbb{Q}, T_2^{\dagger})\right).$$

(The point is that the first cokernel does not change if $\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p)$ is replaced by $\mathcal{V}(\mathbb{Z}_p)$.) From this, we see that tori-rationality of $\underline{\tau}^+$ (which is our assumption) implies tori-rationality of $\underline{\tau}_2^+$. Finally, we apply Lemma 6.1.7 to deduce tori-rationality of $\underline{\tau}_2$ from tori-rationality of $\underline{\tau}_2^+$. $\square$

*Remark* 6.1.15. The significance of Proposition 6.1.14 is that it allows the propagation of the tori-rational condition between the unramified Shimura data $(G, X, p, \mathcal{G})$ and $(G_2, X_2, p, \mathcal{G}_2)$, at least when $X_*(G^{\mathrm{ab}})$ satisfies the technical condition. In §6.3 below we shall apply the lemma to an arbitrary $(G_2, X_2, p, \mathcal{G}_2)$ of abelian type and an auxiliary $(G, X, p, \mathcal{G})$ of Hodge type.

6.2. **Reformulation of results from** [Kis17]**.** Throughout this subsection, we fix a prime $p$, and fix an unramified Shimura datum $(G_2, X_2, p, \mathcal{G}_2)$ where $(G_2, X_2)$ is of abelian type. For every Shimura datum $(G, X)$, we have an embedding of fields $E(G, X) \hookrightarrow \overline{\mathbb{Q}}_p$ as in §2.4.1. We denote the completion of $E(G, X)$ with respect to this embedding by $E(G, X)_p$. (In §2.4.1 this was denoted by $E(G, X)_{\mathfrak{p}}$.)

**Definition 6.2.1.** By a *nice lifting* of $(G_2, X_2, p, \mathcal{G}_2)$, we mean an unramified Shimura datum of the form $(G, X, p, \mathcal{G})$ together with an isomorphism of Shimura data $\iota : (G^{\mathrm{ad}}, X^{\mathrm{ad}}) \xrightarrow{\sim} (G_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$ satisfying the following conditions:

(i) $(G, X)$ is of Hodge type.
(ii) We have $E(G, X)_p = E(G^{\mathrm{ad}}, X^{\mathrm{ad}})_p$.
(iii) $\iota : G^{\mathrm{ad}} \xrightarrow{\sim} G_2^{\mathrm{ad}}$ lifts to a (unique) central isogeny $\widetilde{\iota} : G_{\mathrm{der}} \to G_{2,\mathrm{der}}$.
(iv) $\iota_{\mathbb{Q}_p} : G^{\mathrm{ad}}_{\mathbb{Q}_p} \xrightarrow{\sim} G^{\mathrm{ad}}_{2,\mathbb{Q}_p}$ extends to an isomorphism $\mathcal{G}^{\mathrm{ad}} \xrightarrow{\sim} \mathcal{G}_2^{\mathrm{ad}}$

**Lemma 6.2.2.** *A nice lifting exists.*

*Proof.* By [Kis17, Lem. 4.6.6][40], there exist a Shimura datum $(G, X)$ and an isomorphism $\iota : (G^{\mathrm{ad}}, X^{\mathrm{ad}}) \xrightarrow{\sim} (G_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$ satisfying conditions (i), (ii), (iii) in Definition 6.2.1. The fact that we can extend $(G, X)$ to an unramified Shimura datum $(G, X, p, \mathcal{G})$ satisfying condition (iv) is shown in the proof of [Kis10, Cor. 3.4.14]. (The assumption that $p > 2$ in *loc. cit.* is not used for the construction of $\mathcal{G}$.) □

**6.2.3.** Fix a nice lifting $(G, X, p, \mathcal{G}, \iota)$ of $(G_2, X_2, p, \mathcal{G}_2)$. We apply the notation in §6.1.13 to our current $(G, X, p, \mathcal{G})$ and $(G_2, X_2, p, \mathcal{G}_2)$. Fix a bijection $\mathscr{B}$ as in Theorem 5.13.9, and define the corresponding element $\underline{\tau}_{\mathscr{B}} \in \Gamma(\mathcal{H})_0$ as in Definition 5.13.10 and Corollary 5.13.11. Let $\underline{\tau}_2 \in \Gamma(\mathcal{H}_2)_0$ be an arbitrary element whose image in $\Gamma(\mathcal{H}^{\mathrm{ad}})_0$ under (6.1.14.1) is equal to that of $\underline{\tau}_{\mathscr{B}}$.

**Theorem 6.2.4.** *In the setting of §6.2.3, the statement* $\mathsf{LR}(G_2, X_2, p, \mathcal{G}_2, \underline{\tau}_2)$ *(see §2.7.1) holds.*

*Proof.* The existence of a canonical smooth integral model having well-behaved $\mathbf{H}_c^*$ follows from Theorem 2.5.3 and Theorem 2.5.7. Hence the question is only about the bijection in the statement $\mathsf{LR}(G_2, X_2, p, \mathcal{G}_2, \underline{\tau}_2)$. We first explain why this bijection is essentially proved in [Kis17, Prop. 4.6.2, Cor. 4.6.5, Thm. 4.6.7], at least when $Z_G$ is a torus and $p > 2$. We then explain how to remove the last two assumptions.[41]

For the first part, there are two points that deserve clarification. The first point is that by Proposition 5.10.4, the elements $\underline{\tau}_{\mathscr{B}}(\phi_0) \in \mathcal{H}^{\mathrm{ad}}(\phi_0)$, for $\phi_0 \in \mathcal{AM}(G^{\mathrm{ad}})$, are

[40] In [Kis17, Lem. 4.6.6] it is used that every totally real field $F$ admits a totally imaginary quadratic extension $K/F$ such that every prime of $F$ over $p$ splits in $K$. This fact is an immediate consequence of Thm. 5 or Thm. 6 in [AT09, §X.2], regardless of the parity of $p$.

[41] For the purpose of [Kis17], the assumption that $Z_G$ is a torus is harmless. This is because by [Kis17, Lem. 4.6.6], for the fixed $(G_2, X_2, p, \mathcal{G}_2)$ one can always choose a nice lifting $(G, X, p, \mathcal{G}, \iota)$ such that $Z_G$ is a torus. However, in the current paper we will need to consider choices of $(G, X, p, \mathcal{G})$ which do not necessarily satisfy this condition. See Remark 6.3.4 below.

indeed the same as the elements denoted by $\tau$ in [Kis17, Prop. 4.6.2] (except that in *loc. cit.* $\tau$ is viewed as in $I_{\phi_0}^{\mathrm{ad}}(\mathbb{A}_f)$, instead of $\mathcal{H}^{\mathrm{ad}}(\phi_0) = I_{\phi_0}(\mathbb{A}_f)\backslash I_{\phi_0}^{\mathrm{ad}}(\mathbb{A}_f)/I_{\phi_0}^{\mathrm{ad}}(\mathbb{Q})$). The second point is that in [Kis17, §4.6] the following property of the bijection $\mathscr{B}$ is assumed (where $\mathbb{J}$ is defined as in §5.13.1, with respect to $(G, X, p, \mathcal{G})$):

(*) For every $\mathscr{J} \in \mathbb{J}$ , there exists $\mathfrak{s} \in \mathcal{SPD}(G, X)$ such that $\mathscr{J} = \mathscr{J}_{\mathfrak{s}}$ and $\mathscr{B}(\mathscr{J}) = \mathscr{I}_{\mathfrak{s}}$.

This condition is indeed satisfied by the bijection $\mathscr{B}' : \mathbb{J} \xrightarrow{\sim} \mathbb{I}$ that is implicitly used in [Kis17][42], but it may not be satisfied by $\mathscr{B}$ in our current discussion. Nevertheless, it is clear from the proof of [Kis17, Prop. 4.6.2] that the hypothesis in that proposition can be weakened as follows: Instead of requiring $\mathscr{J}$ and $\mathscr{I}$ to be associated with one same $\mathfrak{s} \in \mathcal{SPD}(G, X)$, we only require that $(\mathscr{I}, \mathscr{J})$ is amicable. From this variant of [Kis17, Prop. 4.6.2], the conclusion of [Kis17, Cor. 4.6.5] easily follows. More specifically, in the proof of [Kis17, Cor. 4.6.5], instead of applying [Kis17, Prop. 4.6.2] to all pairs $(\mathscr{J}, \mathscr{I})$ with $\mathscr{J} = \mathscr{J}_{T,i^\beta,h}, \mathscr{I} = \mathscr{I}_{T,i^\beta,h}$, we apply the above-mentioned variant of [Kis17, Prop. 4.6.2] to all pairs $(\mathscr{J}, \mathscr{I})$ with arbitrary $\mathscr{J} \in \mathbb{J}$ and $\mathscr{I} = \mathscr{B}(\mathscr{J})$. This is valid because $(\mathscr{J}, \mathscr{B}(\mathscr{J}))$ is indeed amicable by Theorem 5.13.9.

We now explain how to remove the assumption that $Z_G$ is a torus, which is made in [Kis17, §4.6]. This assumption comes from [Kis17, Prop. 4.4.17]. As is explained in the proof of that proposition (see especially footnote 24), this assumption can be removed once we know that [Kis17, Lem. 1.2.18] can be generalized to $\mathbb{Z}_p$-group schemes of the form $\mathcal{G}' = \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}_p} \mathcal{G}$, where $F/\mathbb{Q}_p$ is an arbitrary finite extension and $\mathcal{G}$ is a reductive group scheme over $\mathcal{O}_F$. (The result [Kis17, Lem. 1.2.18] is only proved for reductive group schemes over $\mathbb{Z}_p$, but $\mathcal{G}'$ is not reductive unless $F/\mathbb{Q}_p$ is unramified.) This desired generalization is provided by Corollary 4.4.16.

Finally, we explain why the assumption $p > 2$ in [Kis17] is no longer needed. In fact, there are two reasons why this assumption is made in [Kis10] and [Kis17]. Firstly, this assumption is made in [Kis10, Lem. 2.3.1]. That this is unnecessary is explained in the proof of [KMP16, Lem. 4.7]. (We already mentioned this in §5.1.1.) Secondly, the assumption $p > 2$ is needed for the integral comparison isomorphism (5.2.2.1), which is key to both the papers [Kis10] and [Kis17]. We have already explained in §5.2.2 why $p > 2$ is no longer needed for the integral comparison isomorphism. □

## 6.3. **Proof of the Langlands–Rapoport–$\tau$ Conjecture.**

**6.3.1.** In the following, by a *CM field* we mean a totally imaginary quadratic extension of a totally real field contained in our fixed $\overline{\mathbb{Q}}$. We denote by $\iota$ the complex conjugation in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, defined with respect to our fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. For any $\mathbb{Q}$-torus $T$ and $\mu \in X_*(T)$, we write $E_\mu$ for the field of definition of $\mu$ inside $\overline{\mathbb{Q}}$.

Following [Del82, §A, (a)], we consider the category $I$ of pairs $(T, \mu)$, where $T$ is a $\mathbb{Q}$-torus and $\mu \in X_*(T)$, satisfying that $w := \mu + \iota(\mu)$ is defined over $\mathbb{Q}$ and that $(T/w(\mathbb{G}_m))_{\mathbb{R}}$ is anisotropic. By definition, a morphism $(T, \mu) \to (T', \mu')$ in $I$ is a $\mathbb{Q}$-homomorphism $T \to T'$ taking $\mu$ to $\mu'$. For each $(T, \mu)$ in $I$, we know that $T$ is a cuspidal torus since it satisfies condition (ii) in Lemma 1.5.5. By condition (vii) in that lemma, $T$ splits over a CM or totally real field, so $E_\mu$ is either CM or

[42] In [Kis17] this bijection is not explicitly defined, but it is any bijection as in Remark 5.13.8. That such a bijection satisfies condition (*) is shown in the proof of [Kis17, Cor. 4.6.5].

totally real. In the latter case we have $E_\mu = \mathbb{Q}$, since $\mu + \iota(\mu)$ is defined over $\mathbb{Q}$ and is equal to $2\mu$.

As in [Del82, §A, (a)], for every $(T, \mu)$ in $I$, there exists an object in $I$ of the form $(S^L, \mu^L)$ which maps to $(T, \mu)$. Here $L$ is a CM field with maximal totally real subfield $L_0$, and $S^L$ is the $\mathbb{Q}$-torus[43]

$$\mathrm{Res}_{L/\mathbb{Q}}\, \mathbb{G}_m / \ker(\mathrm{N}_{L_0/\mathbb{Q}} : \mathrm{Res}_{L_0/\mathbb{Q}}\, \mathbb{G}_m \to \mathbb{G}_m).$$

The cocharacter $\mu^L \in X_*(S^L)$ is the one induced by the cocharacter of $\mathrm{Res}_{L/\mathbb{Q}}\, \mathbb{G}_m$ corresponding to the canonical embedding $L \hookrightarrow \overline{\mathbb{Q}}$. In fact, as discussed above, $E_\mu$ is either a CM field or $\mathbb{Q}$. We can take $L$ to any CM field containing $E_\mu$, and take the homomorphism $S^L \to T$ to be the one induced by the composite homomorphism

$$\mathrm{Res}_{L/\mathbb{Q}}\, \mathbb{G}_m \xrightarrow{\mathrm{Res}_{L/\mathbb{Q}}\, \mu} \mathrm{Res}_{L/\mathbb{Q}}\, T \xrightarrow{\mathrm{N}_{L/\mathbb{Q}}} T.$$

The fact that the above homomorphism factors through $S^L$ follows easily from the fact that $(T, \mu)$ satisfies the defining conditions for objects in $I$.

**Lemma 6.3.2.** *Keep the setting and notation of §6.2. There exists nice lifting $(G, X, p, \mathcal{G}, \iota)$ of $(G_2, X_2, p, \mathcal{G}_2)$ such that $X_*(G^{\mathrm{ab}})$ is generated by $\mu^{\mathrm{ab}}$ as a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module. Here $\mu^{\mathrm{ab}}$ is as in §6.1.13.*

*Remark* 6.3.3. The purpose of this lemma is to ensure that the technical assumption on $X_*(G^{\mathrm{ab}})$ in Proposition 6.1.14 can be met, so that we can apply that proposition to propagate the tori-rational condition.

*Proof.* By Lemma 6.2.2 we can find a nice lifting $(G_1, X_1, p, \mathcal{G}_1, \iota_1)$ of $(G_2, X_2, p, \mathcal{G}_2)$. Let $\mu_1^{\mathrm{ab}} \in X_*(G_1^{\mathrm{ab}})$ denote the composite cocharacter $\mathbb{G}_m \xrightarrow{\mu_1} G_{1,\overline{\mathbb{Q}}} \to G^{\mathrm{ab}}_{1,\overline{\mathbb{Q}}}$ where $\mu_1 \in \mathbb{\mu}_{X_1}(\overline{\mathbb{Q}})$. Since $(G_1, X_1)$ is of Hodge type, as in the proof of Lemma 5.1.2 we know that the weight cocharacter $w_1$ of $X_1$ is defined over $\mathbb{Q}$, and that $(Z^0_{G_1}/w_1(\mathbb{G}_m))_{\mathbb{R}}$ is anisotropic. Since $G_1^{\mathrm{ab}}$ is isogenous to $Z^0_{G_1}$ and since $\mu_1^{\mathrm{ab}} + \iota(\mu_1^{\mathrm{ab}}) \in X_*(G_1^{\mathrm{ab}})$ is induced by $w_1 \in X_*(Z^0_{G_1})$, we know that the pair $(G_1^{\mathrm{ab}}, \mu_1^{\mathrm{ab}})$ is in the category $I$ in §6.3.1. As discussed in §6.3.1, the field $E_{\mu_1^{\mathrm{ab}}}$ is either CM or $\mathbb{Q}$. In the former case, we let $L$ be $E_{\mu_1^{\mathrm{ab}}}$. In the latter case, we let $L$ be an arbitrary imaginary quadratic field in which $p$ splits. We construct the morphism $(S^L, \mu^L) \to (G_1^{\mathrm{ab}}, \mu_1^{\mathrm{ab}})$ in the category $I$ as in §6.3.1. Since we have an unramified Shimura datum $(G_1, X_1, p, \mathcal{G}_1)$, the torus $G_1^{\mathrm{ab}}$ is unramified over $\mathbb{Q}_p$. Hence every conjugate of the subgroup $\Gamma_{p,0} \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ fixes $\mu_1^{\mathrm{ab}}$. It follows that the Galois closure of $E_{\mu_1^{\mathrm{ab}}}/\mathbb{Q}$ is unramified over $p$, and so is the Galois closure of $L/\mathbb{Q}$. Hence the torus $S^L$ is unramified over $\mathbb{Q}_p$ since it splits over this Galois closure of $L/\mathbb{Q}$.

We now view $(S^L, \mu^L)$ as a Shimura datum $(S^L, \{h^L\})$. (Recall that to specify a Shimura datum for a given torus over $\mathbb{Q}$ is the same as to specify the Hodge cocharacter, which can be an arbitrary cocharacter over $\overline{\mathbb{Q}}$.) By the paragraph preceding [Del82, §A, (a)] and by [Del82, Lem. A.2], we know that $S^L$ admits a faithful representation over $\mathbb{Q}$ such that the Hodge structure on that representation determined by $h^L$ is of type $\{(-1,0),(0,-1)\}$. By [Del79, Prop. 2.3.2], the last fact implies that $(S^L, \{h^L\})$ is a Shimura datum of Hodge type.

Let $G = G_1 \times_{G_1^{\mathrm{ab}}} S^L$. Fix an element $h_1 \in X_1$. Note that $h_1$ and $h^L$ induce the same map $\mathbb{S} \to G^{\mathrm{ab}}_{1,\mathbb{R}}$. Hence we obtain a homomorphism $h = (h_1, {}^L h) : \mathbb{S} \to G_{\mathbb{R}}$.

[43] In [Del82, §A, (a)], the CM field is denoted by $K$ and our $S^L$ is denoted by ${}^K S$. We have avoided the usage of $K$, and avoided the notation ${}^L S$ as this conflicts with the L-group notation.

Let $X$ be the $G(\mathbb{R})$-conjugacy class of $h$. It is clear that $X$ is a Shimura datum for $G$.[44] Since $(G_1, X_1)$ and $(S^L, \{h^L\})$ are both of Hodge type, we know that $(G, X)$ is of Hodge type (by taking the direct sum of the faithful symplectic representations of the two factors). There is a canonical identification $\iota : G^{\mathrm{ad}} \cong G_1^{\mathrm{ad}}$ determined by $\iota_1$, under which $X^{\mathrm{ad}}$ is identified with $X_1^{\mathrm{ad}}$. Then $(G, X)$ and $\iota$ satisfy conditions (i) and (iii) in Definition 6.2.1. Note that $E_{\mu^L}$ is contained in $L$, and the completion of $L$ with respect to $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ is equal to the completion of $E_{\mu_1^{\mathrm{ab}}}$ with respect to $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, which is a subfield of $E(G_1, X_1)_p$. It follows that $E(G, X)_p = E(G_1, X_1)_p$. Since $(G_1, X_1)$ satisfies condition (ii) in Definition 6.2.1, so does $(G, X)$. Since $G_1$ and $S^L$ are both unramified over $\mathbb{Q}_p$, so is $G$. Thus as in the proof of [Kis10, Cor. 3.4.14] we can extend $(G, X)$ to an unramified Shimura datum $(G, X, p, \mathcal{G})$ satisfying condition (iv) in Definition 6.2.1.

We have thus produced a nice lifting $(G, X, p, \mathcal{G}, \iota)$ of $(G_2, X_2, p, \mathcal{G}_2)$. It remains to show that $X_*(G^{\mathrm{ab}})$ is generated by $\mu^{\mathrm{ab}}$ as a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module. Now $G^{\mathrm{ab}}$ is canonically identified with $S^L$, and $\mu^{\mathrm{ab}}$ is identified with $\mu^L$. It is clear from the definition that $X_*(S^L)$ is generated by $\mu^L$ as a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module. □

*Remark* 6.3.4. In the proof of Lemma 6.2.2, even if $Z_{G_1}$ is a torus, it can happen that $Z_G$ is not a torus. This is why in Theorem 6.2.4 we needed to remove the assumption that $Z_G$ is a torus which is made in [Kis17, §4.6] .

**Theorem 6.3.5.** *Let $(G_2, X_2, p, \mathcal{G}_2)$ be an unramified Shimura datum such that $(G_2, X_2)$ is of abelian type. Then Conjecture 2.7.3 holds for $(G_2, X_2, p, \mathcal{G}_2)$.*

*Proof.* We choose a nice lifting $(G, X, p, \mathcal{G}, \iota)$ of $(G_2, X_2, p, \mathcal{G}_2)$ as in Lemma 6.3.2. We apply the notation in §6.1.13 to our current $(G, X, p, \mathcal{G})$ and $(G_2, X_2, p, \mathcal{G}_2)$. Fix a bijection $\mathscr{B}$ as in Theorem 5.13.9 with respect to $(G, X, p, \mathcal{G})$, and define the corresponding element $\underline{\tau}_{\mathscr{B}} \in \Gamma(\mathcal{H})_0$ as in Definition 5.13.10 and Corollary 5.13.11. In view of Theorem 6.2.4, we only need to show that there exists a tori-rational element $\underline{\tau}_2 \in \Gamma(\mathcal{H}_2)_0$ whose image in $\Gamma(\mathcal{H}^{\mathrm{ad}})_0$ under (6.1.14.1) is equal to that of $\underline{\tau}_{\mathscr{B}}$.

It is clear that the assignment

$$\mathcal{AM}(G) \ni \phi \longmapsto \left(\tau^{\mathfrak{H}}(\mathscr{B}(\mathscr{J}), \mathscr{J}), \tau^{\mathcal{H}}(\mathscr{B}(\mathscr{J}), \mathscr{J})\right) \in \mathfrak{H}(\phi) \oplus \mathcal{H}(\phi)$$

as in §5.13.10, where $\mathscr{J}$ is the conjugacy class of $\phi$, defines an element $\underline{\tau}^+ \in \Gamma(\mathcal{H}^+)$. By the conditions satisfied by $\mathscr{B}$ in Theorem 5.13.9 and by Theorem 5.12.5, we know that $\underline{\tau}^+ \in \Gamma(\mathcal{H}^+)_0$. Also, by Theorem 5.12.2, we know that $\underline{\tau}^+$ is tori-rational.

It is clear that $\underline{\tau}^+$ maps to $\underline{\tau}_{\mathscr{B}}$ under the natural map $\Gamma(\mathcal{H}^+)_0 \to \Gamma(\mathcal{H})_0$. Since $\underline{\tau}^+$ is tori-rational and since $X_*(G^{\mathrm{ab}})$ is generated by $\mu^{\mathrm{ab}}$ as a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module, the existence of the desired tori-rational $\underline{\tau}_2 \in \Gamma(\mathcal{H}_2)_0$ follows from Proposition 6.1.14. □

**Theorem 6.3.6** (cf. Theorem 2 in the Introduction)**.** *Conjecture 1.8.8 holds for all Shimura varieties of abelian type.*

*Proof.* This follows from Theorems 2.7.4 and 6.3.5. □

[44] Here $X$ may depend on the choice of $h_1$, but it does not matter for our proof.

# Part 3. Stabilization

## 7. Preliminaries for stabilization

### 7.1. Central character data and the trace formula.

**7.1.1.** To stabilize the point counting formula for Shimura varieties (1.8.8.1) in general, it is necessary to work with fixed central characters. To this end, we are going to introduce the formalism of central character data following [Art13, Ch. 3.1]. The point counting formula can be understood through the lens of a particular central character datum, but it is useful to allow flexible central character data to accommodate $z$-extensions during the stabilization process.

The rest of §7.1 is devoted to discussing the invariant trace formula with fixed central character. Though this is not logically needed for the stabilization in §8, it puts central character data into context, motivates the definition of stable distributions with fixed central characters, and also has an application in §9 below.

Throughout §7.1, let $G$ be a connected reductive group over $\mathbb{Q}$ with center $Z$. Write $A_Z$ for the maximal $\mathbb{Q}$-split torus in $Z$ and set $A_{Z,\infty} := A_Z(\mathbb{R})^0$.

**Definition 7.1.2.** A *central character datum* for $G$ is a pair $(\mathfrak{X}, \chi)$, where $\mathfrak{X}$ is a closed subgroup of $Z(\mathbb{A})$ containing $A_{Z,\infty}$ such that $Z(\mathbb{Q})\mathfrak{X}$ is closed in $Z(\mathbb{A})$, with a choice of Haar measure on $\mathfrak{X}$ (often implicit), and $\chi : \mathfrak{X} \to \mathbb{C}^\times$ a continuous character which is trivial on $\mathfrak{X}_\mathbb{Q} := \mathfrak{X} \cap Z(\mathbb{Q})$.

*Remark* 7.1.3. The two extreme cases where $\mathfrak{X} = A_{Z,\infty}$ or $\mathfrak{X} = Z(\mathbb{A})$ are already interesting, but it is important to allow intermediate groups for our purpose.

**7.1.4.** Let $\Gamma_{\mathrm{ell}}(G)$ denote the set of elliptic conjugacy classes in $G(\mathbb{Q})$. Given a central character datum $(\mathfrak{X}, \chi)$. Denote by $\Gamma_{\mathfrak{X},\mathrm{ell}}(G)$ the set of $\mathfrak{X}_\mathbb{Q}$-orbits in $\Gamma_{\mathrm{ell}}(G)$ with respect to the multiplication action. Write $\mathrm{Stab}_\mathfrak{X}(\gamma)$ for the stabilizer subgroup of $\mathfrak{X}_\mathbb{Q}$ fixing $\gamma \in \Gamma_{\mathrm{ell}}(G)$. It is not hard to see that $\mathrm{Stab}_\mathfrak{X}(\gamma)$ is finite, for instance by reducing to the case of a product of general linear groups via a (possibly reducible) faithful representation of $G$.

Fix a maximal compact subgroup $K_\infty \subset G(\mathbb{R})$. Let $v$ be a place of $\mathbb{Q}$, and $\mathfrak{Z}$ a closed subgroup of $Z(\mathbb{Q}_v)$. For a continuous character $\omega : \mathfrak{Z} \to \mathbb{C}^\times$ define $\mathcal{H}(G(\mathbb{Q}_v), \omega^{-1})$ to be the Hecke algebra of smooth functions on $G(\mathbb{Q}_v)$ which transform under $\mathfrak{Z}$ by $\omega^{-1}$ and have compact support modulo $\mathfrak{Z}$; we also require $K_\infty$-finiteness if $v = \infty$. Let $\pi$ be an admissible representation of $G(\mathbb{Q}_v)$ which has central character on $\mathfrak{Z}$ equal to $\omega$. For $f \in \mathcal{H}(G(\mathbb{Q}_v), \omega^{-1})$ define

$$\pi(f)(u) \quad := \quad \int_{G(\mathbb{Q}_v)/\mathfrak{Z}} f(g)\pi(g)u \cdot dg, \quad u \in \pi.$$

The trace of the trace-class operator $\pi(f)$ is denoted by $\mathrm{tr}\,(f|\pi)$ or $\mathrm{tr}\,\pi(f)$. The orbital integrals for $f \in \mathcal{H}(G(\mathbb{Q}_v), \omega^{-1})$ are defined by the same formula as for $\mathcal{H}(G(\mathbb{Q}_v))$, cf. §1.8.2. These definitions obviously extend to the adelic setting.

**7.1.5.** We recall the invariant distributions

$$I_{\mathrm{geom},\chi_0},\ I_{\mathrm{spec},\chi_0},\ I_{\mathrm{ell},\chi_0},\ I_{\mathrm{disc},\chi_0},\ T_{\mathrm{ell},\chi_0},\ T_{\mathrm{disc},\chi_0}$$

in the classical setup where the central character datum consists of $\mathfrak{X} = A_{Z,\infty}$ and $\chi_0 : A_{Z,\infty} \to \mathbb{C}^\times$. First off, $I_{\mathrm{geom},\chi_0}$ and $I_{\mathrm{spec},\chi_0}$ are Arthur's invariant distributions given in sections 3 and 4 of [Art88], respectively. Define $I_{\mathrm{ell},\chi_0}$ to be the $M = G$

part of formula (3.3) and $I_{\mathrm{disc},\chi_0}$ to be formula (4.4), both referenced to *loc. cit.* All the four distributions are distributions on $\mathcal{H}(G(\mathbb{A}),\chi_0^{-1})$.[45]

For $\gamma\in\Gamma_{\mathrm{ell}}(G)$, write $I_\gamma$ for the connected centralizer of $\gamma$ in $G$. We put

$$\begin{aligned} T_{\mathrm{ell},\chi_0}(f) &:= \sum_{\gamma\in\Gamma_{\mathrm{ell}}(G)} \iota(\gamma)^{-1}\mathrm{vol}(I_\gamma(\mathbb{Q})\backslash I_\gamma(\mathbb{A})/A_{Z,\infty})O_\gamma(f),\\ T_{\mathrm{disc},\chi_0}(f) &:= \mathrm{tr}\,(f\,|\,L^2_{\mathrm{disc},\chi_0}(G(\mathbb{Q})\backslash G(\mathbb{A}))), \qquad f\in\mathcal{H}(G(\mathbb{A}),\chi_0^{-1}).\end{aligned}$$

In general $I_\star(f)$ is more complicated than $T_\star(f)$ for $\star\in\{\mathrm{ell},\mathrm{disc}\}$. Arthur's invariant trace formula is the equality

$$I_{\mathrm{geom},\chi_0}=I_{\mathrm{spec},\chi_0}. \tag{7.1.5.1}$$

When $G/Z$ is anisotropic over $\mathbb{Q}$,

$$T_{\mathrm{ell},\chi_0}=I_{\mathrm{ell},\chi_0}=I_{\mathrm{geom},\chi_0}=I_{\mathrm{spec},\chi_0}=I_{\mathrm{disc},\chi_0}=T_{\mathrm{disc},\chi_0}. \tag{7.1.5.2}$$

**7.1.6.** Next we introduce the trace formula with respect to a fixed character on a closed central subgroup. This must be well known to experts but for the lack of a convenient reference we state the formula here.[46]

Let $(\mathfrak{X},\chi)$ be a central character datum for $G$. Suppose that $\chi_0:A_{Z,\infty}\to\mathbb{C}^\times$ is the restriction of $\chi$. We will obtain the $\chi$-versions of the above invariant distributions by averaging.

**Lemma 7.1.7.** *Let $D$ be a multiplicative group over $\mathbb{Q}$, $A_D$ its maximal $\mathbb{Q}$-split subtorus, and $A_{D,\infty}:=A_D(\mathbb{R})^0$. Then $D(\mathbb{Q})\backslash D(\mathbb{A})/A_{D,\infty}$ is compact.*

*Proof.* Replacing $D$ by $D^0$, we may assume that $D$ is a torus. Via a closed embedding, we reduce to the case where $D$ is a finite product of tori of the form $\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$ for a finite extension $F$ over $\mathbb{Q}$. When $D=\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$, the lemma is clear since $F^\times\backslash\mathbb{A}_F^1$ is compact, where $\mathbb{A}_F^1$ denotes the group of ideles of norm 1. □

**Corollary 7.1.8.** *The quotient $\mathfrak{X}_\mathbb{Q}\backslash\mathfrak{X}/A_{Z,\infty}$ is compact.*

*Proof.* This is clear since the inclusion $\mathfrak{X}\subset Z(\mathbb{A})$ induces a closed embedding from $\mathfrak{X}_\mathbb{Q}\backslash\mathfrak{X}/A_{Z,\infty}$ into $Z(\mathbb{Q})\backslash Z(\mathbb{A})/A_{Z,\infty}$. (The image is closed since $Z(\mathbb{Q})\mathfrak{X}$ is closed in $Z(\mathbb{A})$.) □

**7.1.9.** There is a surjection $\mathcal{H}(G(\mathbb{A}),\chi_0^{-1})\to\mathcal{H}(G(\mathbb{A}),\chi^{-1})$ sending $f$ to the function

$$g\mapsto\overline{f}_\chi(g):=\int_{\mathfrak{X}/A_{Z,\infty}} f(gz)\chi(z)dz,$$

where the integral converges because $z\mapsto f(gz)$ has compact support in $\mathfrak{X}/A_{Z,\infty}$. Given a function $f$ on $G(\mathbb{A})$ and $z\in Z(\mathbb{R})$, write $f_z$ for the translated function $g\mapsto f(gz)$. For each $\star\in\{\mathrm{geom},\mathrm{spec},\mathrm{ell},\mathrm{disc}\}$ define

(7.1.9.1)

$$I_{\star,\chi}(f):=\frac{1}{\mathrm{vol}(\mathfrak{X}_\mathbb{Q}\backslash\mathfrak{X}/A_{Z,\infty})}\int_{\mathfrak{X}_\mathbb{Q}\backslash\mathfrak{X}/A_{Z,\infty}}\chi(z)I_{\star,\chi_0}(f_z)dz, \quad f\in\mathcal{H}(G(\mathbb{A}),\chi_0^{-1}),$$

[45] Arthur defines them as distributions on a certain space of functions on $G(\mathbb{A})^1$ named $\mathcal{H}(G(\mathbb{A})^1)$, but this space is isomorphic to $\mathcal{H}(G(\mathbb{A}),\chi_0^{-1})$ via the product decomposition $G(\mathbb{A})=G(\mathbb{A})^1\times A_{Z,\infty}$. We do not mention $\mathcal{H}(G(\mathbb{A})^1)$ again.

[46] Chapter 3.1 of [Art13] discusses such a variant in the discrete part of the trace formula. Sections 2 and 3 of [Art02] present both the spectral and geometric expansions of the trace formula with fixed central character on an induced torus. We treat a more general case than *loc. cit.* but proceed in a similar spirit.

as well as $T_{\mathrm{ell},\chi}$ and $T_{\mathrm{disc},\chi}$ in a similar manner. In the special case where $G/Z$ is anisotropic over $\mathbb{Q}$, it is clear from (7.1.5.2) that

$$T_{\mathrm{ell},\chi}(f) = I_{\mathrm{ell},\chi}(f) = I_{\mathrm{geom},\chi}(f) = I_{\mathrm{spec},\chi}(f) = I_{\mathrm{disc},\chi}(f) = T_{\mathrm{disc},\chi}(f).$$

Write $L^2_{\mathrm{disc},\chi}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ for the discrete spectrum in the $L^2$-space of complex-valued functions $\phi$ on $G(\mathbb{Q})\backslash G(\mathbb{A})$ such that

- $\phi(gz) = \chi(z)\phi(g)$ for every $g \in G(\mathbb{A})$ and every $z \in \mathfrak{X}$,
- $\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1/\mathfrak{X}\cap G(\mathbb{A})^1} |\phi(g)|^2 dg < \infty$ (for any Haar measure), where $G(\mathbb{A})^1$ denote the "norm one" subgroup of $G(\mathbb{A})$ as defined in [Art78, p. 917].

**Proposition 7.1.10.** *For $f \in \mathcal{H}(G(\mathbb{A}), \chi_0^{-1})$ the following equalities hold.*

$$\begin{aligned} T_{\mathrm{ell},\chi}(f) &= \sum_{\gamma\in\Gamma_{\mathrm{ell},\mathfrak{X}}(G)} |\mathrm{Stab}_{\mathfrak{X}}(\gamma)|^{-1}\iota(\gamma)^{-1}\mathrm{vol}(I_\gamma(\mathbb{Q})\backslash I_\gamma(\mathbb{A})/\mathfrak{X}) O_\gamma(\overline{f}_\chi), \\ T_{\mathrm{disc},\chi}(f) &= \mathrm{tr}\,(\overline{f}_\chi \,|\, L^2_{\mathrm{disc},\chi}(G(\mathbb{Q})\backslash G(\mathbb{A}))). \end{aligned}$$

*Proof.* We compute $T_{\mathrm{ell},\chi}(f)$ as follows:

$$\begin{aligned} &\int_{\mathfrak{X}_\mathbb{Q}\backslash\mathfrak{X}/A_{Z,\infty}} \sum_{\gamma\in\Gamma_{\mathrm{ell}}(G)} \chi(z)\iota(\gamma)^{-1}\frac{\mathrm{vol}(I_\gamma(\mathbb{Q})\backslash I_\gamma(\mathbb{A})/A_{Z,\infty})}{\mathrm{vol}} O_{\gamma z}(f)dz \\ = &\sum_{\gamma\in\Gamma_{\mathrm{ell}}(G)} \int_{\mathfrak{X}_\mathbb{Q}\backslash\mathfrak{X}/A_{Z,\infty}} \chi(z)\iota(\gamma)^{-1}\mathrm{vol}(I_\gamma(\mathbb{Q})\backslash I_\gamma(\mathbb{A})/\mathfrak{X}) O_{\gamma z}(f)dz \\ = &\sum_{\gamma\in\Gamma_{\mathrm{ell},\mathfrak{X}}(G)} \frac{\mathrm{vol}(I_\gamma(\mathbb{Q})\backslash I_\gamma(\mathbb{A})/\mathfrak{X})}{|\mathrm{Stab}_{\mathfrak{X}}(\gamma)|\iota(\gamma)} \int_{\mathfrak{X}/A_{Z,\infty}} \chi(z) O_{\gamma z}(f)dz \\ = &\sum_{\gamma\in\Gamma_{\mathrm{ell},\mathfrak{X}}(G)} \frac{\mathrm{vol}(I_\gamma(\mathbb{Q})\backslash I_\gamma(\mathbb{A})/\mathfrak{X})}{|\mathrm{Stab}_{\mathfrak{X}}(\gamma)|\iota(\gamma)} \int_{\mathfrak{X}/A_{Z,\infty}} O_\gamma(\overline{f}_\chi). \end{aligned}$$

The equality for $T_{\mathrm{disc},\chi}(f)$ follows from the following two observations. First, given an admissible representation $\pi$ of $G(\mathbb{A})$ with central character $\chi_\pi$ on $\mathfrak{X}$,

$$\int_{\mathfrak{X}_\mathbb{Q}\backslash\mathfrak{X}/A_{Z,\infty}} \chi(z)\pi(f_z)dz = \pi(f)\int_{\mathfrak{X}_\mathbb{Q}\backslash\mathfrak{X}/A_{Z,\infty}} \chi(z)\chi_\pi^{-1}(z)dz,$$

which equals 0 if $\chi \neq \chi_\pi$ and $\mathrm{vol}(\mathfrak{X}_\mathbb{Q}\backslash\mathfrak{X}/A_{Z,\infty})\pi(f)$ if $\chi = \chi_\pi$. Second, if $\chi = \chi_\pi$,

$$\begin{aligned} \pi(\overline{f}_\chi) &= \int_{G(\mathbb{A})/\mathfrak{X}} \pi(g)\int_{\mathfrak{X}/A_{Z,\infty}} f(gz)\chi(z)dzdg \\ &= \int_{G(\mathbb{A})/\mathfrak{X}} \int_{\mathfrak{X}/A_{Z,\infty}} \pi(gz)f(gz)dzdg = \pi(f). \end{aligned}$$

□

**7.1.11.** We record a simplification of the trace formula with fixed central character when the test function is a stable cuspidal function at the real place.

Given a central character datum $(\mathfrak{X}, \chi)$, suppose that $\mathfrak{X} = \mathfrak{X}^\infty \times \mathfrak{X}_\infty$ with $\mathfrak{X}^\infty \subset Z(\mathbb{A}_f)$ and $\mathfrak{X}_\infty \subset Z(\mathbb{R})$. (In particular $\mathfrak{X}_\infty$ contains $A_{Z,\infty}$.) Accordingly we decompose $\chi = \chi^\infty\chi_\infty$. We also assume that $G(\mathbb{R})$ contains a maximal torus which is compact modulo $\mathfrak{X}_\infty$.

Let $\xi$ be an irreducible algebraic representation of $G_\mathbb{C}$. The inverse of the central character of $\xi$ on $Z(\mathbb{R})$ is denoted by $\chi_\xi$. Let $\Pi_2(\xi)$ denote the set of isomorphism classes of (irreducible) discrete series representations of $G(\mathbb{R})$ whose central and infinitesimal characters are the same as those of the contragredient of $\xi$. Define $f_\xi \in \mathcal{H}(G(\mathbb{R}), \chi_\xi^{-1})$ to be the sum of pseudocoefficients of $\pi_\infty$ as $\pi_\infty$ runs over $\Pi_2(\xi)$, cf. [Art89, Lem. 3.1], [CD90].

A *stable cuspidal function* on $G(\mathbb{R})$ (relative to $\mathfrak{X}_\infty$) is defined to be $f_\infty \in C_c^\infty(G(\mathbb{R}), \chi_\infty^{-1})$ such that for every irreducible tempered representation $\pi_\infty$ of $G(\mathbb{R})$ whose central character restricts to $\chi_\infty$ on $\mathfrak{X}_\infty$, we have (i) $\mathrm{tr}\, \pi_\infty(f_\infty) = 0$ unless $\pi_\infty$ is a discrete series representation, and (ii) $\mathrm{tr}\, \pi_\infty(f_\infty)$ has a constant value as $\pi_\infty$ runs over each discrete series $L$-packet, cf. [Art89, §4, p. 266]. An example is $f_\xi$ in the last paragraph. In general, a stable cuspidal function is a finite linear combination of character twists of functions of the form $f_\xi$ (for different $\xi$'s), up to a function whose orbital integrals are identically zero.

**Proposition 7.1.12.** *If $f_\infty$ is a stable cuspidal function then*

$$I_{\mathrm{spec},\chi}(f) = I_{\mathrm{disc},\chi}(f) = T_{\mathrm{disc},\chi}(f).$$

*Proof.* This is proved in [Art89, §3] when $\mathfrak{X}_\infty = A_{Z,\infty}$. The same proof extends. □

*Remark* 7.1.13. When $f_\infty$ is stable cuspidal, a simple expansion for $I_{\mathrm{geom},\chi_0}(f)$ is obtained in [Art89, Thm. 6.1]. A similar expansion for $I_{\mathrm{geom},\chi}(f)$ is given in [Dal19, 6.4, 6.5] for more general central character data.

### 7.2. **Endoscopic data and $z$-extensions.**

**7.2.1.** From here throughout §7, let $G$ be a connected reductive group over a local or global field $F$ of characteristic zero.

Langlands–Shelstad [LS87, §1.2] and Kottwitz–Shelstad [KS99, §2.1] have defined endoscopic data and related notions in the untwisted and twisted settings. Here we recall the untwisted case as well as a specific kind of local twisted endoscopy (generalizing the unramified base change) as studied in [Mor10, App. A].

**Definition 7.2.2.** Let $F, G$ be as above. An *endoscopic datum* for $G$ is a quadruple $\mathfrak{e} = (H, \mathcal{H}, s, \eta)$, where

- $H$ is a quasi-split reductive group over $F$,
- $\mathcal{H}$ is a split extension of $W_F$ by $\widehat{H}$ such that the $L$-action of $W_F$ on $\widehat{H}$ determined by $\mathcal{H}$ coincides with the $L$-action of the $L$-group ${}^L H$,
- $s$ is a semi-simple element of $\widehat{G}$,
- $\eta : \mathcal{H} \to {}^L G$ is an $L$-morphism inducing an isomorphism $\widehat{H} \cong \mathrm{Cent}(s, \widehat{G})^0$ (via $\eta$ we view $s$ also as an element of $\widehat{H}$),

such that $\mathrm{Int}(s) \circ \eta = a \cdot \eta$ for a 1-cocycle $a : W_F \to Z(\widehat{G})$ which is trivial (resp. locally trivial) if $F$ is local (resp. global). In this case $H$ is said to be an *endoscopic group* for $G$.

The datum $\mathfrak{e}$ is said to be *elliptic* if $\eta(Z(\widehat{H})^\Gamma)^0 \subset Z(\widehat{G})$. When $F$ is non-archimedean, we say that $\mathfrak{e}$ is *unramified* if $H$ and $G$ are unramified groups over $F$ and if $\eta$ is inflated from an $L$-morphism with respect to the Weil group of $F^{\mathrm{ur}}$ over $F$. An *isomorphism* from $\mathfrak{e} = (H, \mathcal{H}, s, \eta)$ to $\mathfrak{e}' = (H', \mathcal{H}', s', \eta')$ is an element $g \in \widehat{G}$ such that $g\eta(\mathcal{H})g^{-1} = \eta'(\mathcal{H}')$ in ${}^L G$ and $gsg^{-1} = s'$ in $\widehat{G}/Z(\widehat{G})$.

**7.2.3.** Automorphisms of an endoscopic datum $\mathfrak{e}$ induce outer automorphisms of $H$ as in [KS99, (2.1.8)]. By $\mathrm{Out}_F(\mathfrak{e})$ we mean the image subgroup of the outer automorphism group $\mathrm{Out}_F(H) := \mathrm{Aut}_F(H)/H_{\mathrm{ad}}(F)$. Set

$$\lambda(\mathfrak{e}) := |\mathrm{Out}_F(\mathfrak{e})| \in \mathbb{Z}_{>0}.$$

The set of endoscopic data is denoted by $E(G)$. Write $E_{\mathrm{ell}}(G)$ for the subset of elliptic endoscopic data. Write $\mathcal{E}(G)$ and $\mathcal{E}_{\mathrm{ell}}(G)$ for the corresponding sets of isomorphism classes.

For $\mathfrak{e} = (H, \mathcal{H}, s, \eta) \in E(G)$, there exists a canonical injection over $F$

$$Z_G \hookrightarrow Z_H \tag{7.2.3.1}$$

as we now explain. Since an inner twisting induces a canonical isomorphism of centers, we may assume that $G$ is quasi-split over $F$. Choose a maximal torus $T_H$ of $H$ over $F$. Then there exist a maximal torus $T$ of $G$ and an isomorphism $T_H \cong T$, both defined over $F$ [LS87, p. 226] such that the composite embedding $T \cong T_H \subset H$ is canonical up to $H(\overline{F})$-conjugacy. Restricting from $T_H \cong T$, we get the desired map $Z_G \hookrightarrow Z_H$, which is independent of the choices involved.

**7.2.4.** If $G_{\mathrm{der}} = G_{\mathrm{sc}}$ then by [Lan79c, Prop. 1], every $\mathfrak{e} \in \mathcal{E}(G)$ is represented by $(H, {}^L H, s, \eta)$, that is, we can take $\mathcal{H} = {}^L H$. In general there is no guarantee that this is possible, so we use $z$-extensions to reduce to this case.

A *$z$-extension* of $G$ over $F$ is defined to be a connected reductive group $G_1$ equipped with a short exact sequence

$$1 \to Z_1 \to G_1 \to G \to 1 \tag{7.2.4.1}$$

such that $G_{1,\mathrm{der}} = G_{1,\mathrm{sc}}$, $Z_1 \subset Z_{G_1}$, and $Z_1 \cong \prod_{i \in I} \mathrm{Res}_{F_i/F}\, \mathbb{G}_m$ for finite extensions $F_i$ of $F$ over a finite index set $I$. We also call such a short exact sequence itself a $z$-extension of $G$.

**Lemma 7.2.5.** *If $F$ is non-archimedean and if $G$ is unramified, then there exists a $z$-extension $G_1$ of $G$ that is unramified. Similarly, if $F$ is a number field and if $G$ is unramified at a finite set of finite places $S$, then there exists a $z$-extension $G_1$ of $G$ that is unramified at $S$.*

*Proof.* The lemma follows from [MS82, Prop. 3.1], possibly except the point, pertaining to the global case, that there exists a maximal torus $T$ of $G$ such that $T$ splits over an extension of $F$ unramified at $S$. Let us verify it.

Since $G$ is unramified over $F_v$ for each $v \in S$, so is $G_{\mathrm{sc}}$. Thus there exists an unramified maximal torus $T_{\mathrm{sc},v}$ in $G_{\mathrm{sc},F_v}$. Write $T_{\mathrm{sc},v}(F_v)_{\mathrm{rs}} \subset T_{\mathrm{sc},v}(F_v)$ for the open subset consisting of regular semi-simple elements. Then the non-empty subset

$$Y_v := \bigcup_{g_v \in G_{\mathrm{sc}}(F_v)} g_v \cdot T_{\mathrm{sc},v}(F_v)_{\mathrm{rs}} \cdot g_v^{-1} \subset G_{\mathrm{sc}}(F_v)$$

is open in $G_{\mathrm{sc}}(F_v)$ by Harish-Chandra's submersion principle [HC80]. By weak approximation for $G_{\mathrm{sc}}$, there exists an element $\gamma_0 \in G_{\mathrm{sc}}(F) \cap (\prod_{v \in S} Y_v)$. Let $T'$ denote the connected centralizer in $G_{\mathrm{sc}}$ of $\gamma_0$. Then $T'$ is unramified at $S$ as it is conjugate to $T_{\mathrm{sc},v}$ at each $v \in S$. The obvious image of $T' \times Z_G^0$ in $G$ is then a maximal torus of $G$ which is unramified at $S$. $\square$

**Lemma 7.2.6.** *Fix a $z$-extension of $G$ as in* (7.2.4.1). *For each $\mathfrak{e} = (H, \mathcal{H}, s, \eta) \in E(G)$, there exists a central extension $1 \to Z_1 \to H_1 \to H \to 1$ over $F$, with $H_1$*

*connected reductive, such that the induced short exact sequence* $1 \to Z_1 \to Z_{H_1} \to Z_H \to 1$ *fits in the following row-exact commutative diagram:*

$$\begin{tikzcd} 1 \arrow[r] & Z_1 \arrow[r] \arrow[d, equal] & Z_{G_1} \arrow[r] \arrow[d] & Z_G \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & Z_1 \arrow[r] & Z_{H_1} \arrow[r] & Z_H \arrow[r] & 1. \end{tikzcd} \tag{7.2.3.1}$$

*Moreover, when $F$ is non-archimedean, we can choose $H_1$ to be unramified if $H$ and $G_1$ are unramified. When $F$ is a number field, $H_1$ can be chosen to be unramified at a finite set of places $S$ if $H$ and $G_1$ are unramified at $S$.*

*Remark* 7.2.7. We do not claim that $H_1$ is a $z$-extension of $H$, that is, the derived subgroup of $H_1$ need not be simply connected.

*Proof.* Choose maximal tori $T_H \subset H$ and $T \subset G$ over $F$ together with an isomorphism $T_H \cong T$ over $F$ as in §7.2.3. We pull back the resulting embedding $Z_H \subset T_H \cong T \subset G$ via the surjection $G_1 \to G$ to obtain the preimage $\widetilde{Z}_1 \subset G_1$ fitting in the exact sequence

$$1 \to Z_1 \to \widetilde{Z}_1 \to Z_H \to 1. \tag{7.2.7.1}$$

Recall from [Del79, §2.0.1] and the notation therein that $H = H_{\mathrm{sc}} *_{Z(H_{\mathrm{sc}})} Z_H$. Setting $H_1 := H_{\mathrm{sc}} *_{Z(H_{\mathrm{sc}})} \widetilde{Z}_1$, we have a surjection $H_1 \to H$ induced by $\widetilde{Z}_1 \to Z_H$, whose kernel is identified with $Z_1$. The construction yields $Z_{H_1} = \widetilde{Z}_1$ and the commutative diagram of the lemma. □

**7.2.8.** Let $F$ be a local or global field of characteristic 0. Fix a $z$-extension of $G$ as in (7.2.4.1). Let us explain how each $\mathfrak{e} = (H, \mathcal{H}, s, \eta) \in E(G)$ gives rise to an endoscopic datum for $G_1$.

Fix inner twistings to quasi-split inner forms $H_{\overline{F}} \cong H^*_{\overline{F}}$ and $G_{\overline{F}} \cong G^*_{\overline{F}}$ together with $F$-pinnings for $H^*$ and $G^*$. Along the central extension $H_1 \to H$ provided by the preceding lemma, we can lift inner twistings to obtain $H_{1,\overline{F}} \cong H^*_{1,\overline{F}}$ and $G_{1,\overline{F}} \cong G^*_{1,\overline{F}}$ as well as $F$-pinnings for $H^*_1$ and $G^*_1$. As explained in [Kot84b, 1.8], we obtain $\Gamma$-equivariant maps $\widehat{H} \to \widehat{H}_1$, $\widehat{G} \to \widehat{G}_1$, $\widehat{H}_1 \to \widehat{Z}_1$, and $\widehat{G}_1 \to \widehat{Z}_1$. We will consider the natural extension of the last three maps to $L$-morphisms $\zeta_{G_1} : {}^LG \to {}^LG_1$, ${}^LH_1 \to {}^LZ_1$, and ${}^LG_1 \to {}^LZ_1$, respectively. (As for $\widehat{H} \to \widehat{H}_1$, we have $\zeta_{H_1}$ in the lemma below.) The composition $\widehat{H} \to \widehat{G} \to \widehat{G}_1$ factors through the embedding $\widehat{H} \to \widehat{H}_1$ to yield the following commutative diagram.

$$\begin{tikzcd} \widehat{H} \arrow[r] \arrow[d] & \widehat{H}_1 \arrow[d] \\ \widehat{G} \arrow[r] & \widehat{G}_1 \end{tikzcd} \tag{7.2.8.1}$$

**Lemma 7.2.9.** *Maintain the notation of §7.2.8.*

(i) *The embedding $\widehat{H} \hookrightarrow \widehat{H}_1$ can be extended to an $L$-morphism $\zeta_{H_1} : \mathcal{H} \to {}^LH_1$ such that $\zeta_{H_1}$ induces a homeomorphism from $\mathcal{H}$ onto its image.*

*(ii) There exists an L-morphism $\eta_1 : {}^L H_1 \to {}^L G_1$ such that the following is a commutative diagram extending* (7.2.8.1).

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\zeta_{H_1}} & {}^L H_1 \\ \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta_1} \\ {}^L G & \xrightarrow{\zeta_{G_1}} & {}^L G_1 \end{array}$$

*(iii) The quadruple $\mathfrak{e}_1 := (H_1, {}^L H_1, s_1, \eta_1)$, with $s_1 := \zeta_{G_1}(s)$, is an endoscopic datum for $G_1$. The isomorphism class of $\mathfrak{e}_1$ is independent of the choices in (i) and (ii).*

*Proof.* To verify (i), consider the split extension of $\widehat{H}_1$ by $W_F$ given by $\mathcal{H}_1 := (Z(\widehat{H}_1) \rtimes \mathcal{H})/Z(\widehat{H})$, where $Z(\widehat{H})$ embeds in the semi-direct product diagonally. The assignment $h \mapsto (1 \rtimes h)$ induces an embedding $\mathcal{H} \hookrightarrow \mathcal{H}_1$ extending the map $\widehat{H} \hookrightarrow \widehat{H}_1$. As remarked in §7.2.4, $\mathcal{H}_1 \cong {}^L H_1$ since $G_{1,\mathrm{sc}} = G_{1,\mathrm{der}}$, so we obtain the desired map $\zeta_{H_1}$ by composition. Let us verify (ii). Since $G_{1,\mathrm{sc}} = G_{1,\mathrm{der}}$, we can extend $\widehat{H}_1 \to \widehat{G}_1$ to $\eta_0 : {}^L H_1 \to {}^L G_1$ by [Lan79c, Prop. 1]. The two $L$-morphisms $\zeta_{G_1}\eta$ and $\eta_0\zeta_{H_1}$ differ by a 1-cocycle $a : W_F \to C$, where $C := \mathrm{Cent}(\widehat{H}, \widehat{G}_1)$. Clearly $Z(\widehat{H}_1) \subset C$. We also note that $Z(\widehat{H}_1) \cap (C \cap \widehat{G}) = Z(\widehat{H}_1) \cap \widehat{G} = Z(\widehat{H})$. Thus

$$\widehat{Z}_1 = Z(\widehat{H}_1)/Z(\widehat{H}) \subset C/C \cap \widehat{G} \subset \widehat{G}_1/\widehat{G} = \widehat{Z}_1,$$

implying that $C = Z(\widehat{H}_1)$. As $a$ is valued in $Z(\widehat{H}_1)$, one can twist $\eta_0$ by $a$ to obtain $\eta_1$, which then makes the diagram commute. Lastly (iii) is a routine check. □

**7.2.10.** Given a central extension $H_1$ of $H$ as above, choose a splitting $W_F \to \mathcal{H}$ to consider the composition

$$W_F \to \mathcal{H} \xrightarrow{\zeta_{H_1}} {}^L H_1 \to {}^L Z_1.$$

Write $\lambda_1 : Z_1(F) \to \mathbb{C}^\times$ if $F$ is local, or $\lambda_1 : Z_1(F)\backslash Z_1(\mathbb{A}_F) \to \mathbb{C}^\times$ if $F$ is global, for the corresponding continuous character, which is independent of the choice of splitting. This character naturally shows up in endoscopic transfer.

We show that the assignment $\mathfrak{e} \to \mathfrak{e}_1$ admits an inverse map.

**Lemma 7.2.11.** *The map $\mathfrak{e} \mapsto \mathfrak{e}_1$ defined by Lemma 7.2.9 induces a bijection from $\mathcal{E}(G)$ onto $\mathcal{E}(G_1)$.*

*Proof.* The injectivity is easy to see since $\widehat{G}_1$ is generated by $\widehat{G}$ and $Z(\widehat{G}_1)$.

To prove the surjectivity, let $\mathfrak{e}_1 = (H_1, {}^L H_1, s_1, \eta_1) \in E(G_1)$. Replacing $\mathfrak{e}_1$ by an isomorphic datum, we may assume that $\eta_1(W_F)$ lies in the subgroup ${}^L G$ of ${}^L G_1$. Indeed, consider the exact sequence of continuous cohomology

$$\mathbf{H}^1(W_F, \widehat{G}) \to \mathbf{H}^1(W_F, \widehat{G}_1) \to \mathbf{H}^1(W_F, \widehat{Z}_1).$$

The image of $\eta_1$ under the second map lifts to a 1-cocycle $c$ valued in $Z(\widehat{G}_1)$, up to a 1-coboundary, via the map $\mathbf{H}^1(W_F, Z(\widehat{G}_1)) \to \mathbf{H}^1(W_F, \widehat{Z}_1)$, which is surjective by [Lan79c, Lem. 4]. Then $c \cdot \eta_1$ comes from a 1-cocycle valued in $\widehat{G}$ up to a 1-coboundary.

We define $H$ to be the cokernel of the composite map $Z_1 \hookrightarrow Z_{G_1} \hookrightarrow Z_{H_1} \hookrightarrow H_1$. We can write $s_1 = sz$ for some $s \in \widehat{G}$ and $z \in Z(\widehat{G}_1)$. By pulling back $\eta_1 : {}^L H_1 \to$

${}^LG_1$ via ${}^LG \hookrightarrow {}^LG_1$, we obtain an injection $\eta : \mathcal{H} \to {}^LG$ and see that $\mathcal{H}$ is a split extension of $W_F$ by $\widehat{H}$ (as $\mathcal{H}$ is generated by $\eta_1(W_F)$ and $\widehat{H}$).

It is enough to verify that $\mathfrak{e} := (H, \mathcal{H}, s, \eta)$ is an endoscopic datum for $G$, since it would then be obvious that $\mathfrak{e} \mapsto \mathfrak{e}_1$ by construction, and we will be done. The main point to show is that $\mathrm{Int}(s) \circ \eta = a\eta$ with trivial (resp. locally trivial) 1-cocycle $a : W_F \to Z(\widehat{G})$ if $F$ is local (resp. global). Let us check this when $F$ is global as the local case is only simpler. Since $\mathfrak{e}_1 \in \mathcal{E}(G_1)$ we know that $\mathrm{Int}(s) \circ \eta_1 = \mathrm{Int}(s_1) \circ \eta_1 = a_1\eta_1$ with a locally trivial 1-cocycle $a_1 : W_F \to Z(\widehat{G}_1)$. Since $\eta_1(W_F) \subset \widehat{G}$ we have that $a_1(W_F) \subset Z(\widehat{G}_1) \cap \widehat{G} = Z(\widehat{G})$ and that $\mathrm{Int}(s) \circ \eta = a_1\eta$. As $G_1$ is a $z$-extension of $G$, the map $\mathbf{H}^1(W_{F_v}, Z(\widehat{G})) \to \mathbf{H}^1(W_{F_v}, Z(\widehat{G}_1))$ is injective at each place $v$ (e.g., by [Kot84b, Cor. 2.3]). Hence $a_1$ is locally trivial as a cocycle valued in $Z(\widehat{G})$. $\square$

**7.2.12.** From here until the end of §7.2, $F$ is assumed to be non-archimedean. Write $F_m$ for the unramified extension of $F$ of degree $m \in \mathbb{Z}_{\geq 1}$ in a fixed algebraic closure $\overline{F}$. Denote by $\sigma \in \mathrm{Gal}(F_m/F)$ the arithmetic Frobenius generator. Fix a $z$-extension $1 \to Z_1 \to G_1 \to G \to 1$. Set $R := \mathrm{Res}_{F_m/F}\, G$ and $R_1 := \mathrm{Res}_{F_m/F}\, G_1$. Write $\theta$ (resp. $\theta_1$) for the automorphism of $R$ (resp. $R_1$) induced by $\sigma$. Identify

$$\widehat{R} = \widehat{G}^{\mathrm{Hom}(F_m, \overline{F})} = \prod_{j=0}^{m-1} \widehat{G}$$

such that the $j$-th component corresponds to the inclusion $F_m \subset \overline{F}$ precomposed by $\sigma^j$, and similarly for $\widehat{R}_1$. There are unique embeddings $i : {}^LG \hookrightarrow {}^LR$ and $i_1 : {}^LG_1 \hookrightarrow {}^LR_1$ such that the maps are diagonal embeddings on the dual groups and the identity map on the Weil groups.

The following is a variant of Lemma 7.2.6. In practice $G_1$ and $H_1$ over $F$ will come from central extensions over $\mathbb{Q}$ independently of $m$. By contrast, the extensions $G_1'$ and $H_1'$ below depend on $m$ and will be considered only in a local setting.

**Lemma 7.2.13.** *Suppose that $G$ and $\mathfrak{e} = (H, \mathcal{H}, s, \eta)$ are unramified. Consider $z$-extensions $1 \to Z_1 \to G_1 \to G \to 1$ and $1 \to Z_1 \to H_1 \to H \to 1$ as constructed in Lemma 7.2.6 (disregarding the last assertion) such that $G_1$ and $H_1$ are unramified. Let $F'/F$ be a finite unramified extension. Then there exist*

*(i) a $z$-extension $1 \to Z_1' \to G_1' \to G \to 1$ such that $Z_1' \cong \prod_{j \in J} \mathrm{Res}_{F_j'/F}\, \mathbb{G}_m$ with $J$ a finite index set and $F_j' \supset F'$, and*

*(ii) a central extension $1 \to Z_1' \to H_1' \to H \to 1$ arising from (i) as in Lemma 7.2.6,*

*such that $G_1'$ and $H_1'$ are unramified over $F$, and such that there is a commutative diagram with an injective middle vertical arrow*

$$\begin{tikzcd} 1 \arrow[r] & Z_1 \arrow[r] \arrow[d] & G_1 \arrow[r] \arrow[d, hook] & G \arrow[r] \arrow[d, equal] & 1, \\ 1 \arrow[r] & Z_1' \arrow[r] & G_1' \arrow[r] & G \arrow[r] & 1 \end{tikzcd} \tag{7.2.13.1}$$

*as well as the analogous diagram with $H, H_1, H_1'$ in place of $G, G_1, G_1'$.*

*Proof.* We have $Z_1 \cong \prod_{j \in J} \mathrm{Res}_{E_j/F}\, \mathbb{G}_m$ for finite unramified extensions $E_j/F$. Take $Z_1' := \prod_{j \in J} \mathrm{Res}_{E_jF'/F}\, \mathbb{G}_m$. Along the canonical inclusion $Z_1 \hookrightarrow Z_1'$, we make a pushout diagram from the top row of Lemma 7.2.6 as follows:

$$\begin{tikzcd} 1 \arrow[r] & Z_1 \arrow[r] \arrow[d] & Z_{G_1} \arrow[r] \arrow[d] & Z_G \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & Z_1' \arrow[r] & Z'_{G_1} \arrow[r] & Z_G \arrow[r] & 1 \end{tikzcd}$$

With the bottom row in place of (7.2.7.1) we construct a $z$-extension $1 \to Z_1' \to G_1' \to G \to 1$ such that $Z_{G_1'} = Z'_{G_1}$ as in the proof of Lemma 7.2.6. By construction, we have (i) and (7.2.13.1). Applying Lemma 7.2.6 to $G_1'$, we obtain (ii). Since $Z_{H_1}$ and $Z_{H_1'}$ are preimages of $Z_H \subset G$ under $G_1 \to G$ and $G_1' \to G$, we have $Z_{H_1} \subset Z_{H_1'}$. This in turn induces $H_1 = H_{\mathrm{sc}} *_{Z(H_{\mathrm{sc}})} Z_{H_1} \subset H_1' = H_{\mathrm{sc}} *_{Z(H_{\mathrm{sc}})} Z_{H_1'}$. With this inclusion as the middle vertical arrow, we see that there is a commutative diagram analogous to (7.2.13.1) for $H, H_1, H_1'$. □

**7.2.14.** Under a temporary assumption on $\mathfrak{e} = (H, \mathcal{H}, s, \eta) \in E(G)$ that

$$s \in Z(\widehat{H})^{\Gamma_F}$$

(in general $s \in Z(\widehat{H})^{\Gamma_F} Z(\widehat{G})$), we construct some twisted endoscopic data to be used in the stabilization (§8). Put $\widetilde{s} := (s, 1, ..., 1) \in \widehat{R}$, which lies in $\mathcal{Z} := \mathrm{Cent}(i\eta(\mathcal{H}), \widehat{R})$. Define the $L$-morphism $\widetilde{\eta} : \mathcal{H} \to {}^L R$ to be the twist of $i\eta$ by the unramified 1-cocycle $a : W_F \to \mathcal{Z}$ mapping $\sigma$ to $\widetilde{s}$. Then $\widetilde{\mathfrak{e}} = (H, \mathcal{H}, \widetilde{s}, \widetilde{\eta})$ is checked to be a twisted endoscopic datum for $(R, \theta)$, cf. [Mor10, A.1.3, A.2.6] or [Kot90, §7]. Replacing $\mathfrak{e}$ by $\mathfrak{e}_1$ (noting that the temporary assumption is still satisfied for $\mathfrak{e}_1$, i.e., $s_1 \in Z(\widehat{H}_1)^{\Gamma_F}$) we construct a twisted endoscopic datum $\widetilde{\mathfrak{e}}_1 = (H_1, {}^L H_1, \widetilde{s}_1, \widetilde{\eta}_1)$ for $(R_1, \theta_1)$. With $H_1'$ playing the role of $H_1$, we also construct $\mathfrak{e}_1'$ and $\widetilde{\mathfrak{e}}_1'$.

7.3. **Cohomological lemmas.**

**7.3.1.** Let $F$, $G$, and $\mathfrak{e}$ be as in §7.2. (The field $F$ is either local or global.) Take a $z$-extension $G_1$ as in Lemma 7.2.6. By Hilbert 90 the map $G_1(F) \to G(F)$ is onto. Let $\gamma \in G(F)_{\mathrm{ss}}$ and choose a lift $\gamma_1 \in G_1(F)$. We have a commutative diagram of reductive groups

$$\begin{tikzcd} 1 \arrow[r] & Z_1 \arrow[r] \arrow[d, equal] & I_{\gamma_1} \arrow[r] \arrow[d] & I_\gamma \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & Z_1 \arrow[r] & G_1 \arrow[r] & G \arrow[r] & 1, \end{tikzcd}$$

which gives rise to a $\Gamma$-equivariant commutative diagram by [Kot84b, 1.8]:

$$\begin{tikzcd} 1 \arrow[r] & Z(\widehat{I}_\gamma) \arrow[r] & Z(\widehat{I}_{\gamma_1}) \arrow[r] & \widehat{Z}_1 \arrow[r] & 1 \\ 1 \arrow[r] & Z(\widehat{G}) \arrow[r] \arrow[u] & Z(\widehat{G}_1) \arrow[r] \arrow[u] & \widehat{Z}_1 \arrow[r] \arrow[u, equal] & 1. \end{tikzcd} \tag{7.3.1.1}$$

In particular we get a $\Gamma$-equivariant isomorphism

$$Z(\widehat{I}_\gamma)/Z(\widehat{G}) \cong Z(\widehat{I}_{\gamma_1})/Z(\widehat{G}_1).$$

**Lemma 7.3.2.** *We have*

(i) *If $F$ is global or local, there is a canonical isomorphism*

$$\mathfrak{K}(I_\gamma/F) \cong \mathfrak{K}(I_{\gamma_1}/F).$$

*(ii) If $F$ is local non-archimedean, there is a canonical bijection*

$$\mathfrak{D}(I_{\gamma_1}, G_1; F) \cong \mathfrak{D}(I_\gamma, G; F).$$

*Proof.* Let us check (i) in the global case. We obtain the following commutative diagram from [Kot84b, Cor. 2.3] and diagram (7.3.1.1):

$$\begin{tikzcd}
\pi_0((Z(\widehat{I}_\gamma)/Z(\widehat{G}))^{\Gamma_F}) \arrow[r] \arrow[d, "\sim"] & \mathbf{H}^1(F, Z(\widehat{G})) \arrow[d] \\
\pi_0((Z(\widehat{I}_{\gamma_1})/Z(\widehat{G}_1))^{\Gamma_F}) \arrow[r] & \mathbf{H}^1(F, Z(\widehat{G}_1)).
\end{tikzcd}$$

The left vertical map is an isomorphism. The right vertical map is injective by [Kot84b, Cor. 2.3] since $\widehat{Z}_1^{\Gamma_F}$ is connected (a product of copies of $\mathbb{C}^\times$). Further observe that the injective right vertical map induces an isomorphism $\ker^1(F, Z(\widehat{G})) \cong \ker^1(F, Z(\widehat{G}_1))$. To see this, notice that the cokernel is mapped injectively into $\ker^1(F, \widehat{Z}_1)$, which is trivial since $Z_1$ is a product of induced tori. Since $\mathfrak{K}(I_\gamma/F)$ (resp. $\mathfrak{K}(I_{\gamma_1}/F)$) is the preimage of $\ker^1(F, Z(\widehat{G}))$ (resp. $\ker^1(F, Z(\widehat{G}_1))$) under the top (resp. bottom) horizontal arrow, the left vertical map induces the desired isomorphism of (i). In the case of local fields, the same argument works if we replace the $\ker^1$-groups with the trivial group.

For (ii) consider the following commutative diagram of pointed sets (or of abelian groups by identifying $\mathbf{H}^1$ with $\mathbf{H}^1_{\mathrm{ab}}$)

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] \\
1 \arrow[r] & \mathfrak{D}(I_{\gamma_1}, G_1; F) \arrow[r] \arrow[d] & \mathbf{H}^1(F, I_{\gamma_1}) \arrow[r] \arrow[d] & \mathbf{H}^1(F, G_1) \arrow[d] \\
1 \arrow[r] & \mathfrak{D}(I_\gamma, G; F) \arrow[r] & \mathbf{H}^1(F, I_\gamma) \arrow[r] \arrow[d] & \mathbf{H}^1(F, G) \arrow[d] \\
 & & \mathbf{H}^2(F, Z_1) \arrow[r, equal] & \mathbf{H}^2(F, Z_1),
\end{tikzcd}$$

where the middle and right columns come from the exact sequences preceding the lemma and the fact that $\mathbf{H}^1(F, \cdot) = \mathbf{H}^1_{\mathrm{ab}}(F, \cdot)$ when $F$ is non-archimedean (see 1.1.6). Assertion (ii) now follows from a diagram chase. $\square$

**7.3.3.** Let $(G, X)$ be a Shimura datum. We study Kottwitz parameters and their invariants with respect to a $z$-extension $1 \to Z_1 \to G_1 \to G \to 1$ over $\mathbb{Q}$. Let $T \subset G_\mathbb{R}$ be an elliptic maximal torus. There exists $h \in X$ such that $\mu_h : \mathbb{G}_m \to G_\mathbb{C}$ factors through $T_\mathbb{C}$. In the notation of §2.4.1, $\mu_h \in \mathbb{\mu}_X(\mathbb{C})$. Write $T_1 \subset G_{1,\mathbb{R}}$ for the preimage of $T$. Then $\mu_h : \mathbb{G}_m \to T_\mathbb{C}$ lifts to a cocharacter $\mu_1 : \mathbb{G}_m \to T_{1,\mathbb{C}}$, which we fix henceforth and view also as a cocharacter of $G_1$ over $\mathbb{C}$. As noted in [MS82, 3.4], the conjugacy class of $\mu_1$ comes from a Shimura datum $(G_1, X_1)$ for a suitable $X_1$, that is, $\mu_1 \in \mathbb{\mu}_{X_1}(\mathbb{C})$. In particular the discussion of cohomological invariants (§§1.7.5–1.7.7) applies to $(G_1, X_1)$ and $\mu_1$.

Let $\gamma_{0,1} \in G_1(\mathbb{A}_f^p)$ and $\gamma_0 \in G(\mathbb{A}_f^p)$ such that $\gamma_{0,1}$ maps to $\gamma_0$. Write $I_{0,1}$ and $I_0$ for the connected centralizers of $\gamma_{0,1}$ and $\gamma_0$ in $G_1$ and $G$ over $\mathbb{A}_f^p$, respectively.

**Lemma 7.3.4.** *Suppose that $\gamma_0 \in G(\mathbb{A}_f^p)$ is the image of an element $\gamma_{0,1} \in G_1(\mathbb{A}_f^p)_{\mathrm{ss}}$. Then the natural map $\mathfrak{D}(I_{0,1}, G_1; \mathbb{A}_f^p) \to \mathfrak{D}(I_0, G; \mathbb{A}_f^p)$ is a bijection.*

*Proof.* This follows from (1.1.6.1) and part (ii) of Lemma 7.3.2. □

**7.3.5.** Now consider a $z$-extension $1 \to Z_1 \to G_1 \to G \to 1$ over $\mathbb{Q}_p$ (which need not come from $\mathbb{Q}$-groups via base change). Let $\mu : \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$ be a cocharacter over $\overline{\mathbb{Q}}_p$. Let $\gamma_{0,1} \in G_1(\mathbb{Q}_p)_{\mathrm{ss}}$, and $\gamma_0 \in G(\mathbb{Q}_p)$ the image of $\gamma_{0,1}$. As usual, $I_0$ and $I_{0,1}$ are the connected centralizers of $\gamma_0$ and $\gamma_{0,1}$ in $G$ and $G_1$ over $\mathbb{Q}_p$, respectively. Fix a level $n \in \mathbb{Z}_{>0}$.

Denote by $\mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p)$ the set of all $[b] \in \mathrm{B}(I_0)$ satisfying condition **KP1** of Definition 1.6.5 with the given $\gamma_0$ and $n$. (Here we do not require **KP0**, which will come into play in Corollary 7.4.18 below.) This means that for some (thus every) representative $b \in I_0(\breve{\mathbb{Q}}_p)$ of $[b]$, there exists $c \in G(\breve{\mathbb{Q}}_p)$ such that

$$c^{-1}\gamma_0 c = c^{-1} b\sigma(b)\cdots\sigma^{n-1}(b)\sigma^n(c).$$

Given $[b] \in \mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p)$, we will often write $\delta_{[b]} \in G(\mathbb{Q}_{p^n})$ for the element arising from $[b]$ (well defined up to $\sigma$-conjugacy in $G(\mathbb{Q}_{p^n})$) as in Lemma 1.6.7. Then $\gamma_0$ is a degree $n$ norm of $\delta_{[b]}$.

Likewise we define $\mathfrak{D}_n(\gamma_{0,1}, G_1; \mathbb{Q}_p)$. The natural map $I_{0,1} \to I_0$ induces a map

$$\mathfrak{D}_n(\gamma_{0,1}, G_1; \mathbb{Q}_p) \longrightarrow \mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p). \tag{7.3.5.1}$$

**Lemma 7.3.6.** *Let $\gamma_0, \gamma_{0,1}$ be as above. Suppose that $Z_1 \cong \prod_{j \in J} \mathrm{Res}_{F_j/\mathbb{Q}_p} \mathbb{G}_m$ with all $F_j$ containing $\mathbb{Q}_{p^n}$ and $J$ a finite index set. Then the map* (7.3.5.1) *is a bijection.*

*Proof.* The map (7.3.5.1) fits in the following commutative diagram

$$\begin{tikzcd}
\mathfrak{D}_n(\gamma_{0,1}, G_1; \mathbb{Q}_p) \arrow[r, hook] \arrow[d, "(7.3.5.1)"'] & \mathrm{B}(I_{0,1}) \arrow[r] \arrow[d] & \mathrm{B}(R_1) \arrow[d] \\
\mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p) \arrow[r, hook] & \mathrm{B}(I_0) \arrow[r] & \mathrm{B}(R),
\end{tikzcd}$$

where the vertical maps are induced by the natural maps $I_{0,1} \to I_0$ and $R_1 \to R$.

We may assume that $\mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p)$ is non-empty since the lemma is vacuously true otherwise. We claim that $\mathfrak{D}_n(\gamma_{0,1}, G_1; \mathbb{Q}_p)$ is also non-empty. To see this, fix $[b] \in \mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p)$ and pick a lift $\delta_1' \in G_1(\mathbb{Q}_{p^n})$ of $\delta_{[b]}$. Let $\gamma_{0,1}' \in G_1(\mathbb{Q}_p)$ be a degree $n$ norm of $\delta_1'$. Then the norm of $z\delta_1'$ is $z^\sigma z \cdots {}^{\sigma^{n-1}} z \gamma_{0,1}'$ with $z \in Z_1(\mathbb{Q}_{p^n})$. The norm map $Z_1(\mathbb{Q}_{p^n}) \to Z_1(\mathbb{Q}_p)$ is onto by the hypothesis on $Z_1$, so we may choose $z$ such that $\gamma_{0,1}$ is a norm of $\delta_1 := z\delta_1'$. This implies that there is $c_1 \in G_1(\breve{\mathbb{Q}}_p)$ such that $c_1^{-1}\gamma_{0,1}c_1 = \delta_1{}^\sigma\delta_1 \cdots {}^{\sigma^{n-1}}\delta_1$. Setting $b_1 := c_1\delta_1{}^\sigma c_1^{-1}$, we see that $b_1 \in I_{0,1}(\breve{\mathbb{Q}}_p)$ (since $G_1$ has simply connected derived subgroup) and that $[b_1] \in \mathfrak{D}_n(\gamma_{0,1}, G_1; \mathbb{Q}_p)$, proving the claim.

We fix $[b]$, $\delta_1$, and $[b_1]$ as in the last paragraph. In particular $\delta_1$ maps to $\delta := \delta_{[b]}$, and $[b_1]$ to $[b]$. By Corollary 1.6.11, $[b_1] \in \mathrm{B}(I_{0,1})$ and $[b] \in \mathrm{B}(I_0)$ are basic. Define $\mathfrak{D}([b_1], R_1)$ to be the set of basic elements $[b_1'] \in \mathrm{B}(I_{0,1})$ such that $\kappa_{I_{0,1}}([b_1']) - \kappa_{I_{0,1}}([b_1])$ lies in $\ker(\pi_1(I_{0,1})_{\Gamma_p,\mathrm{tors}} \to \pi_1(R_1)_{\Gamma_p,\mathrm{tors}})$. Define $\mathfrak{D}([b], R)$ exactly in the same way with $I_0$ and $R$ in place of $I_{0,1}$ and $R_1$.

We claim that $\mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p)$ is a subset of $\mathfrak{D}([b], R)$. Indeed, consider an element $[b'] \in \mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p)$, which gives rise to $\delta' \in G(\mathbb{Q}_{p^n})$ as in Lemma 1.6.7. Corollary

1.6.11 implies that $\kappa_{I_{0,1}}([b_1']) - \kappa_{I_{0,1}}([b_1])$ is a torsion element in $\pi_1(I_{0,1})_{\Gamma_p}$ via the right half of the commutative diagram on [RR96, p. 162]. We know from Lemma 1.6.7 that $\delta$ and $\delta'$ have the same norm, implying that $\delta$ and $\delta'$ are stably $\sigma$-conjugate by [Kot82, Prop. 5.7]. Therefore $[\delta]$ and $[\delta']$ are equal in $\mathrm{B}(R)$. Since $b$ and $b'$ are $\sigma$-conjugate to $\delta$ and $\delta'$ in $G(\breve{\mathbb{Q}}_p)$, respectively, it follows that $[b]=[b']$ in $\mathrm{B}(R)$. Hence $\kappa_{I_0}([b'])$ and $\kappa_{I_0}([b])$ have the same image in $\pi_1(R)_{\Gamma_p}$, completing the proof of the claim.

What we have shown is summarized in the following commutative diagram.

$$\begin{tikzcd} \mathfrak{D}_n(\gamma_{0,1}, G_1; \mathbb{Q}_p) \arrow[r, hook] \arrow[d] & \mathfrak{D}([b_1], R_1) \arrow[d, "\text{bij.}"] \\ \mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p) \arrow[r, hook] & \mathfrak{D}([b], R) \end{tikzcd} \tag{7.3.6.1}$$

The right vertical map is a bijection by the proof of [Kot82, Lem. 5.6.(2)]. (This relies on the assumption of the lemma on $Z_1$.) Indeed, the proof there shows a canonical bijection from

$$\ker(\mathbf{H}^1(\mathbb{Q}_p, I_{\delta_1}) \to \mathbf{H}^1(\mathbb{Q}_p, R_1))$$

to

$$\ker(\mathbf{H}^1(\mathbb{Q}_p, I_{\delta}) \to \mathbf{H}^1(\mathbb{Q}_p, R)),$$

but this is exactly the right vertical map above via the functorial bijection between $\mathbf{H}^1(\mathbb{Q}_p, H)$ and $\pi_1(H)_{\Gamma_p,\mathrm{tors}}$ for an arbitrary connected reductive group $H$ over $\mathbb{Q}_p$, cf. [Lab99, Prop. 1.6.7] and [RR96, Thm. 1.15.(i)].

Now we verify that the top horizontal map in (7.3.6.1) is a bijection. As we have seen in the last paragraph, $\mathfrak{D}([b_1], R_1)$ is in a canonical bijection with $\ker(\mathbf{H}^1(\mathbb{Q}_p, I_{\delta_1}) \to \mathbf{H}^1(\mathbb{Q}_p, R_1))$, which in turn is canonically bijective onto the set of $\sigma$-conjugacy classes in the stable $\sigma$-conjugacy class of $\delta_1$ in $G_1(\mathbb{Q}_{p^n})$. Recall that $\gamma_{0,1}$ is a norm of $\delta_1$. Therefore each $\delta_1' \in G_1(\mathbb{Q}_{p^n})$ stably $\sigma$-conjugate to $\delta_1$ gives rise to an element of $\mathfrak{D}_n(\gamma_{0,1}, G_1; \mathbb{Q}_p)$ as described in the second paragraph of the proof of the current lemma (more detailed on p. 167 of [Kot90]). It is routine to check that this map gives the inverse of the top horizontal map in (7.3.6.1).

Going back to diagram (7.3.6.1), it is now clear that the left vertical map (as well as the lower horizontal map) is a bijection. □

**Corollary 7.3.7.** *Assume that $\mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p)$ is non-empty. Then the map $[b] \mapsto \delta_{[b]}$ gives a surjection from $\mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p)$ onto the set of $\sigma$-conjugacy classes in the stable $\sigma$-conjugacy class in $G(\mathbb{Q}_{p^n})$ whose norm is $\gamma_0$. If $G$ has simply connected derived subgroup then this map is a bijection.*

*Proof.* Thanks to Lemma 7.2.13, we can choose a $z$-extension $G_1$ such that the condition on $Z_1$ in Lemma 7.3.6 is satisfied. Then the corollary follows from the bijectivity of the lower horizontal map of (7.3.6.1), together with the interpretation of $\mathfrak{D}([b], R)$ in terms of $\sigma$-conjugacy classes below (7.3.6.1). □

**7.3.8.** Now we turn to the real place. Let $(G, X)$ and $(G_1, X_1)$ be as in §7.3.3. Following §1.7.5 and §1.7.6, for each elliptic element $\gamma_0 \in G(\mathbb{R})$, we have $\widetilde{\beta}_\infty(\gamma_0) \in \pi_1(I_0) = X^*(Z(\widehat{I}_0))$ mapping to $[\mu] \in \pi_1(G)$ for any $\mu \in \mathfrak{p}_X(\overline{\mathbb{Q}})$. The definition involves an extra choice, but the restriction of $\widetilde{\beta}_\infty(\gamma_0)$ to $Z(\widehat{I}_0)^{\Gamma_\infty} Z(\widehat{G})$ is independent of the choice. Analogously, for each elliptic element $\gamma_{0,1} \in G_1(\mathbb{R})$, we have $\widetilde{\beta}_\infty(\gamma_{0,1}) \in \pi_1(I_{0,1}) = X^*(Z(\widehat{I}_{0,1}))$ mapping to $[\mu_1] \in \pi_1(G_1)$ with $\mu_1 \in \mathbb{\mu}_{X_1}(\overline{\mathbb{Q}})$.

**Lemma 7.3.9.** *Assume that $\gamma_{0,1} \in G_1(\mathbb{R})$ and $\gamma_0 \in G(\mathbb{R})$ are elliptic elements such that $\gamma_{0,1}$ maps to $\gamma_0$. Then the image of $\widetilde{\beta}_\infty(\gamma_{0,1})$ under the natural map $\pi_1(I_{0,1}) \to \pi_1(I_0)$ coincides with $\widetilde{\beta}_\infty(\gamma_0)$ on $Z(\widehat{I}_0)^{\Gamma_\infty} Z(\widehat{G})$.*

*Proof.* In §1.7.5 and §1.7.6 (adapted to $(G_1, X_1)$), choose any $\mathbb{R}$-elliptic maximal torus $T_1 \subset G_1$ and $h_1 \in X_1$ factoring through $T_1$ to define $\widetilde{\beta}_\infty(\gamma_{0,1})$. Take $T \subset G$ and $h \in X$ to be the images of $T_1$ and $h_1$ in the definition of $\widetilde{\beta}_\infty(\gamma_0)$. Then the lemma is true on the nose. □

7.4. **Langlands–Shelstad–Kottwitz transfer.**

**7.4.1.** Let $F$, $G$, and $\mathfrak{e} = (H, \mathcal{H}, s, \eta)$ be as in Definition 7.2.2. Throughout §7.4, assume $F$ to be a local field (of characteristic 0). Let $\psi : G^*_{\overline{F}} \to G_{\overline{F}}$ be an inner twisting of $F$-groups with $G^*$ quasi-split over $F$. We will use the following notation for $G$ (and likewise for other reductive groups).

- $\Gamma(G) = \Gamma(G(F))$ is the set of semi-simple $G(F)$-conjugacy classes in $G(F)$,
- $\Sigma(G) = \Sigma(G(F))$ is the set of stable semi-simple conjugacy classes in $G(F)$,
- $\mathcal{H}(G) = \mathcal{H}(G(F))$ and $\mathcal{H}(G, \omega^{-1}) = \mathcal{H}(G(F), \omega^{-1})$ as in §7.1.4,
- When $F$ is non-archimedean and $G$ is unramified, we fix a hyperspecial subgroup $K \subset G(F)$ and define $\mathcal{H}^{\mathrm{ur}}(G) \subset \mathcal{H}(G)$ and $\mathcal{H}^{\mathrm{ur}}(G, \omega^{-1}) \subset \mathcal{H}(G, \omega^{-1})$ to be the subalgebra consisting of $K$-bi-invariant functions.

**7.4.2.** We explain the transfer of conjugacy classes in endoscopy following [Kot86, §3.1]. See also [LS87, §1.3].

Let $\gamma_H \in H(F)_{\mathrm{ss}}$. Choose a maximal torus $T_H$ of $H$ over $F$ containing $\gamma_H$. There exists a canonical $G^*(\overline{F})$-conjugacy class of embeddings $j : T_H \to G^*$ over $\overline{F}$. Fix a choice of $j$ and put $T^* := j(T_H)$. Denoting the set of absolute roots of $T_H$ in $H$ (resp. $T^*$ in $G^*$) by $R(T_H, H)$ (resp. $R(T^*, G^*)$), we have $R(T_H, H) \subset R(T^*, G^*)$. The element $\gamma_H$ is said to be $(G, H)$-*regular* if $\alpha(\gamma_H) \neq 1$ for all $\alpha \in R(T^*, G^*) \backslash R(T_H, H)$. The definition depends only on the $H(\overline{F})$-conjugacy class of $\gamma_H$ and not on the extra choices. The $(G, H)$-regular subset of $H(F)_{\mathrm{ss}}$ and $\Sigma(H)$ will be denoted by $H(F)_{(G,H)\text{-reg}}$ and $\Sigma(H)_{(G,H)\text{-reg}}$, respectively.

Let $\gamma_H \in H(F)_{(G,H)\text{-reg}}$. The $\overline{F}$-embedding $\psi \circ j : T_H \to G$ is canonical up to $G(\overline{F})$-conjugacy, so $\gamma_H$ determines a semi-simple $G(\overline{F})$-conjugacy class in $G(\overline{F})$ defined over $F$. If this conjugacy class contains an element $\gamma$ of $G(F)$, then we take the stable conjugacy class of $\gamma$ to be the image of $\gamma_H$. Otherwise the image of $\gamma_H$ is formally denoted by the empty set symbol $\emptyset$. (Such a $\gamma$ always exists by [Kot82, Thm. 4.4] if $G$ is quasi-split and has simply connected derived subgroup, but not in general.) To summarize, we obtain a map

$$\Sigma(H)_{(G,H)\text{-reg}} \longrightarrow \Sigma(G) \cup \{\emptyset\}. \tag{7.4.2.1}$$

We say that $\gamma_H$ and $\gamma \in G(F)_{\mathrm{ss}}$ have *matching conjugacy classes*, or simply that $\gamma$ is an *image* of $\gamma_H$. If the centralizer of $\gamma$ in $G$ is connected (e.g., if $G_{\mathrm{der}} = G_{\mathrm{sc}}$) then the centralizer of $\gamma_H$ in $H$ is also connected by [Kot86, Lem. 3.2].

If $G_1$ and $H_1$ are as in (7.2.4.1) and Lemma 7.2.6, then the above construction can be performed for $G_1$ and $H_1$ in place of $G$ and $H$. This is visibly compatible with the surjections $G_1 \to G$ and $H_1 \to H$, leading to the following commutative diagram (where the symbol $\emptyset$ maps to itself under the right vertical map).

$$\begin{tikzcd} \Sigma(H_1)_{(G_1,H_1)\text{-reg}} \arrow[r] \arrow[d] & \Sigma(G_1) \cup \{\emptyset\} \arrow[d] \\ \Sigma(H)_{(G,H)\text{-reg}} \arrow[r] & \Sigma(G) \cup \{\emptyset\} \end{tikzcd}$$

By slight abuse of language (as $H_1$ is not an endoscopic group of $G$), $\gamma \in G(F)_{\mathrm{ss}}$ is said to be an image of $\gamma_{H_1} \in H_1(F)_{(G_1,H_1)\text{-reg}}$ if the stable conjugacy class of $\gamma_{H_1}$ maps to that of $\gamma$ in the above diagram.

**7.4.3.** We introduce $\kappa$-orbital integrals, of which stable orbital integrals are the special case. Let us assume that $F$ is local for convenience. The main definitions here extend to the adelic setting in the obvious manner.

Let $\gamma \in G(F)_{\mathrm{ss}}$ and $x \in G(\overline{F})$. Suppose that $\gamma_x := x^{-1}\gamma x \in G(F)$ and $x\rho(x)^{-1} \in I_\gamma(\overline{F})$ for every $\rho \in \Gamma_F$. Then $x$ and the 1-cocycle $\rho \mapsto x\rho(x)^{-1}$ define an element of $\mathbf{H}^0(F, I_\gamma\backslash G)$, to be denoted by $\dot{x}$. The map $x \mapsto \gamma_x$ factors through $\mathbf{H}^0(F, I_\gamma\backslash G)$, namely there is an induced map

$$\mathbf{H}^0(F, I_\gamma\backslash G) \longrightarrow G(F), \quad \dot{x} \longmapsto \gamma_{\dot{x}}.$$

Recall that there is a short exact sequence

$$1 \to I_\gamma(F)\backslash G(F) \to \mathbf{H}^0(F, I_\gamma\backslash G) \to \mathfrak{D}(I_\gamma, G; F) \to 1,$$

coming from a long exact sequence. Given Haar measures on $I_\gamma(F)$ and $G(F)$ and the counting measure on $\mathfrak{D}(I_\gamma, G; F)$, there is a unique way to equip $\mathbf{H}^0(F, I_\gamma\backslash G)$ with a compatible measure. The map $\dot{x} \mapsto \gamma_{\dot{x}}$ induces a map

$$\mathfrak{D}(I_\gamma, G; F) \longrightarrow \Gamma(G), \qquad [x] \longmapsto \gamma_{[x]},$$

whose image consists of conjugacy classes in the stable conjugacy class of $\gamma$. If $G_\gamma = \mathrm{Cent}(\gamma, G)$ is connected (so that it equals $I_\gamma$) then $[x] \mapsto \gamma_{[x]}$ is a bijection onto the image. In general the fiber over the conjugacy class of $\gamma' \in G(F)$ (stably conjugate to $\gamma$) is in bijection with $\ker(\mathbf{H}^1(F, I_{\gamma'}) \to \mathbf{H}^1(F, G_{\gamma'}))$. Recall the map $\mathfrak{D}(I_\gamma, G; F) \to \mathfrak{E}(I_\gamma, G; F) = \mathfrak{K}(I_\gamma/F)$ from §1.1.7. Thus we have a pairing

$$\langle \cdot, \cdot \rangle : \ \mathfrak{D}(I_\gamma, G; F) \times \mathfrak{K}(I_\gamma/F) \longrightarrow \mathbb{C}^\times.$$

Given $\kappa \in \mathfrak{K}(I_\gamma/F)$ and $f \in \mathcal{H}(G)$ we define the $\kappa$-orbital integral of $\gamma$ by

$$\begin{aligned} O^{G(F),\kappa}_\gamma(f) \quad &:= \quad \int_{\mathbf{H}^0(F, I\backslash G)} e(I_{\gamma_{\dot{x}}})\langle \dot{x}, \kappa\rangle O^{G(F)}_{\gamma_{\dot{x}}}(f) d\dot{x} \\ &= \quad \sum_{[x] \in \mathfrak{D}(I_\gamma, G; F)} e(I_{\gamma_{[x]}})\langle [x], \kappa\rangle O^{G(F)}_{\gamma_{[x]}}(f). \end{aligned}$$

When $\kappa$ is trivial, one has the stable orbital integral

$$SO^{G(F)}_\gamma(f) := O^{G(F),1}_\gamma(f).$$

The superscript $G(F)$ will be omitted if there is no danger of confusion. The above definition of $\kappa$-orbital integrals works verbatim for $f \in \mathcal{H}(G(F), \omega^{-1})$.

**7.4.4.** We recall the Langlands–Shelstad transfer in the local untwisted case. See §7.4.12 below for the twisted case.

When $\mathcal{H} = {}^L H$ in the endoscopic datum (this assumption will be removed via $z$-extensions), Langlands–Shelstad [LS87, LS90] define the transfer factor

$$\Delta(\cdot,\cdot) : H(F)_{(G,H)\text{-reg}} \times G(F)_{\text{ss}} \longrightarrow \mathbb{C}$$

which vanishes on $(\gamma_H, \gamma)$ unless $\gamma$ is an image of $\gamma_H$.

When $G$ is quasi-split, the canonical transfer factor $\Delta_0$ was given by Langlands–Shelstad depending only on a choice of $F$-pinning for $G$. Another natural normalization in the quasi-split case is the Whittaker normalization in [KS99, §5.3]. While there is no direct analogue of either when $G$ is not quasi-split, the Whittaker normalization can be extended to $G$ given a suitable rigidification of an inner twisting of $G$ against its quasi-split inner form. See [Kal16, Kal18] for Kaletha's notion of rigid inner forms and a discussion of other rigidifications. In this paper, we do not attempt to choose a rigidification or a canonical normalization of transfer factor at every place. However when $G$ is defined over $\mathbb{Q}$, we may and will always choose a global normalization as in [LS87, §6.4] so that the product formula (Corollary 6.4.B therein) holds true.

In fact there are more than one sign conventions for (untwisted and twisted) transfer factors as explained in [KS12]. We work with the factor $\Delta'$ in *loc. cit.*, which coincides with the one in [Kot90] (see p. 178 therein) but differs from the definition of [LS87] by the map $(H, \mathcal{H}, s, \eta) \mapsto (H, \mathcal{H}, s^{-1}, \eta)$. The reason for our choice is that the former is better suited for extension to the twisted setting.

**7.4.5.** There exists a smooth character $\lambda_H : Z_G(F) \to \mathbb{C}^\times$ such that

$$\Delta(z\gamma_H, z\gamma) = \lambda_H(z)\Delta(\gamma_H, \gamma), \qquad z \in Z_G(F). \tag{7.4.5.1}$$

This is [LS90, Lem. 3.5.A]. We can describe $\lambda_H$ explicitly on $Z_G^0(F)$ as follows.

**Lemma 7.4.6.** *When $\mathcal{H} = {}^L H$, the restriction of $\lambda_H$ to $Z_G^0(F)$ corresponds to the composite Langlands parameter*

$$W_F \to {}^L H \stackrel{\eta}{\to} {}^L G \stackrel{\zeta}{\to} {}^L Z_G^0,$$

*where the first map is the distinguished splitting, and $\zeta$ is dual to the inclusion $Z_G^0 \hookrightarrow G$.*

*Proof.* Consider $\gamma_H \in H(F)$, $\gamma \in G(F)$, and maximal tori $T \subset G$ and $T_H \subset H$ as in [LS87, §3]. In particular we are given an isomorphism $i : T \cong T_H$, inducing an isomorphism ${}^L i : {}^L T_H \cong {}^L T$. They construct $L$-morphisms $\xi_{T_H} : {}^L T_H \to {}^L H$ and $\xi : {}^L T \to {}^L G$ (depending on some additional choices) as well as a 1-cocycle $\mathbf{a} : W_F \to \widehat{T}_H$ such that $\eta\xi_{T_H} = \mathbf{a} \cdot \xi_T {}^L i$ as $L$-morphisms from ${}^L T_H$ to ${}^L G$. Restricting the equality via the splitting $W_F \to {}^L T_H$, we obtain

$$\eta(\xi_{T_H}(w)) = \mathbf{a}(w)\xi_T(w) \in {}^L G, \qquad w \in W_F. \tag{7.4.6.1}$$

The first paragraph of [LS87, p. 253] tells us that $\lambda_H|_{Z_G^0(F)}$ corresponds to $\mathbf{a}$ composed with $\widehat{T}_H \cong \widehat{T} \to \widehat{Z}_G^0$, which is dual to $Z_G^0 \subset T \cong T_H$. To prove the lemma, it is thus enough to verify that the composite map in the lemma is $w \mapsto \zeta(\mathbf{a}(w)) \rtimes w$.

To this end, write $\xi_{T_H}(w) = b(w) \rtimes w$ and $\xi_T(w) = c(w) \rtimes w$. From the construction of $\xi_T$ in [LS87, §2.6], it follows that $\zeta(c(w)) = 1$. Indeed, the two main points are that the image of every morphism $\mathrm{SL}_2 \to \widehat{G}$ maps trivially in $\widehat{Z}_G^0$ (thus also $n(\omega_T(\sigma))$ therein) and that the coroots of $\widehat{G}$ map trivially in $\widehat{Z}^0_G$ (thus also $r_p(w)$ therein). Similarly, $b(w)$ maps to $1 \in \widehat{Z}^0_H$, so $\zeta(b(w)) = 1$. Now we apply $\zeta$ to (7.4.6.1) to see that $\zeta(\eta(w)) = \zeta(\mathbf{a}(w)) \rtimes w$, as desired. □

**7.4.7.** We introduce transfer factors in general by reducing to the case $G_{\mathrm{der}} = G_{\mathrm{sc}}$, in which case we can always assume that $\mathcal{H} = {}^L H$, cf. §7.2.4 and §7.4.4. Let $\mathfrak{e} = (H, \mathcal{H}, s, \eta) \in E(G)$. Take a $z$-extension $1 \to Z_1 \to G_1 \to G \to 1$ and define $\mathfrak{e}_1 = (H_1, {}^L H_1, s_1, \eta_1) \in E(G_1)$ as in §7.2.8. Then we have $\Delta(\gamma_{H_1}, \gamma_1) \in \mathbb{C}$ defined on $\gamma_{H_1} \in H_1(F)_{(G_1,H_1)\text{-reg}}$ and $\gamma_1 \in G_1(F)_{\mathrm{ss}}$. Langlands–Shelstad define

$$\Delta(\cdot,\cdot) : H_1(F)_{(G_1,H_1)\text{-reg}} \times G(F)_{\mathrm{ss}} \longrightarrow \mathbb{C}$$

as follows. Set $\Delta(\gamma_{H_1}, \gamma) = 0$ unless $\gamma$ is an image of $\gamma_{H_1}$ (§7.4.2), that is, unless $\gamma$ lifts to $\gamma_1 \in G_1(F)$ which is an image of $\gamma_{H_1}$, in which case $\Delta(\gamma_{H_1}, \gamma) := \Delta(\gamma_{H_1}, \gamma_1)$. By (7.4.5.1),

$$\Delta(z\gamma_{H_1}, \gamma) = \lambda_{H_1}(z)\Delta(\gamma_{H_1}, \gamma), \qquad z \in Z^0_{G_1}(F).$$

Notice that $Z_1 \subset Z^0_{G_1}$. Lemma 7.4.6 implies that $\lambda_{H_1}|_{Z_1(F)} = \lambda_1$, where $\lambda_1$ was given in §7.2.10. (This is also checked in [LS87, p. 254].)

**7.4.8.** Keep on assuming that $F$ is a local field. We state the Langlands–Shelstad transfer and the fundamental lemma for connected reductive groups $G_1$ with $G_{1,\mathrm{der}} = G_{1,\mathrm{sc}}$. Let $\mathfrak{e}_1 = (H_1, {}^L H_1, s_1, \eta_1) \in E(G_1)$. If $G_1$ and $\mathfrak{e}_1$ are unramified, then $\eta_1$ induces a $\mathbb{C}$-algebra map via the Satake transform:

$$\eta_1^* : \mathcal{H}^{\mathrm{ur}}(G_1) \longrightarrow \mathcal{H}^{\mathrm{ur}}(H_1). \tag{7.4.8.1}$$

**Proposition 7.4.9.** *For each $f_1 \in \mathcal{H}(G_1)$, there exists $f_1^{H_1} \in \mathcal{H}(H_1)$ enjoying the following property: If $\gamma_{H_1} \in H_1(F)_{(G_1,H_1)\text{-reg}}$ has no image in $G_1(F)_{\mathrm{ss}}$ then $SO_{\gamma_{H_1}}(f_1^{H_1}) = 0$. If $\gamma_{H_1}$ has an image $\gamma_1 \in G(F)_{\mathrm{ss}}$ then*

$$SO_{\gamma_{H_1}}(f_1^{H_1}) = \sum_{[x] \in \mathfrak{D}(I_{\gamma_1}, G_1; F)} e(I_{\gamma_{1,[x]}})\Delta(\gamma_{H_1}, \gamma_{1,[x]})O_{\gamma_{1,[x]}}(f_1). \tag{7.4.9.1}$$

*Moreover, the fundamental lemma (FL) holds true, i.e., if $G_1$ and $\mathfrak{e}_1$ are unramified and if $f_1 \in \mathcal{H}^{\mathrm{ur}}(G_1)$ then the above holds with $f_1^{H_1} = \eta_1^* f_1$.*

*Proof.* The second assertion (FL) follows from work of Ngô as well as Cluckers-Loeser, Hales, and Waldspurger [CL10, Hal95, Ngô10, Wal06]. The first assertion (the transfer conjecture) is implied by FL [Wal97]. □

**7.4.10.** Let us adapt Proposition 7.4.9 to the setting with fixed central characters. Let $\mathfrak{X}$ be a closed subgroup of $Z(F)$ and $\chi : \mathfrak{X} \to \mathbb{C}^\times$ a continuous character. We view $\mathfrak{X}$ also as a closed subgroup of $Z_H(F)$ via $Z \hookrightarrow Z_H$. Denote by $\mathfrak{X}_{H_1}$ and $\mathfrak{X}_1$ the preimages of $\mathfrak{X}$ under $H_1(F) \to H(F)$ and $G_1(F) \to G(F)$, respectively, so that $\mathfrak{X}_{H_1} \cong \mathfrak{X}_1$ canonically. Choose compatible Haar measures on $\mathfrak{X}_{H_1}$ and $\mathfrak{X}_1$. Let $\chi_1$ denote the character of $\mathfrak{X}_{H_1}$ or $\mathfrak{X}_1$ pulled back from $\chi : \mathfrak{X} \to \mathbb{C}^\times$. Restricting $\lambda_{H_1}$ as in §7.4.7, we obtain another character

$$\lambda_{H_1}|_{\mathfrak{X}_{H_1}} : \mathfrak{X}_{H_1} \longrightarrow \mathbb{C}^\times.$$

By slight abuse of notation, we will often denote the above still by $\lambda_{H_1}$ (as we will not consider $\lambda_{H_1}$ on a larger domain). In the special case when $\mathfrak{X} = \{1\}$, notice that $\mathfrak{X}_1 = \mathfrak{X}_{H_1} = Z_1(F)$, $\chi_1 = 1$, and $\lambda_{H_1} = \lambda_1$.

If $F$ is non-archimedean, $\chi : \mathfrak{X} \to \mathbb{C}^\times$ is said to be *unramified* if the character is trivial on the maximal compact subgroup of $\mathfrak{X}$. The same definition works with

$\mathfrak{X}_{H_1}$ in place of $\mathfrak{X}$. When $G$, $\mathfrak{e}$, $\chi$, and the $z$-extension $G_1$ are unramified, we have $G_1$ and $\mathfrak{e}_1$ also unramified. In this case, the map $\eta_1^* : \mathcal{H}^{\mathrm{ur}}(G_1) \to \mathcal{H}^{\mathrm{ur}}(H_1)$ from (7.4.8.1) is averaged to give a map

$$\mathcal{H}^{\mathrm{ur}}(G, \chi^{-1}) \longrightarrow \mathcal{H}^{\mathrm{ur}}(H_1, \chi_1^{-1}\lambda_{H_1}) \tag{7.4.10.1}$$

as follows. Let $f_1 \in \mathcal{H}^{\mathrm{ur}}(G_1)$ be any lift of $f$ along the natural surjective map, where the first arrow is averaging against $\chi_1$:

$$\mathcal{H}^{\mathrm{ur}}(G_1) \to \mathcal{H}^{\mathrm{ur}}(G_1, \chi_1^{-1}) \cong \mathcal{H}^{\mathrm{ur}}(G, \chi^{-1}).$$

For $z \in \mathfrak{X}_{H_1}$, write $f_{1,z}(h) := f_1(zh)$. Define a function $f^{H_1}$ on $H_1(F)$ by

$$f^{H_1}(h) := \int_{\mathfrak{X}_{H_1}} \chi_1(z)\eta_1^* f_{1,z}(h)dz = \int_{\mathfrak{X}_{H_1}} \chi_1(z)\lambda_{H_1}^{-1}(z)\eta_1^* f_1(zh)dz.$$

Then $f^{H_1}$ belongs to $\mathcal{H}^{\mathrm{ur}}(H_1, \chi_1^{-1}\lambda_{H_1})$ and is independent of the choice of $f_1$. The resulting map (7.4.10.1) is again denoted by $\eta_1^*$ as there is little danger of confusion.

**Proposition 7.4.11.** *For each $\chi : \mathfrak{X} \to \mathbb{C}^\times$ and $f \in \mathcal{H}(G, \chi^{-1})$, there exists*

$$f^{H_1} \in \mathcal{H}(H_1, \chi_1^{-1}\lambda_{H_1})$$

*such that for every $\gamma_{H_1} \in H_1(F)_{(G_1,H_1)\text{-reg}}$, if $\gamma \in G(F)_{\mathrm{ss}}$ is an image of $\gamma_{H_1}$ then*

$$SO^{H_1(F)}_{\gamma_{H_1}}(f^{H_1}) = \sum_{[x]\in\mathfrak{D}(I_\gamma,G;F)} e(I_{\gamma_{[x]}})\Delta(\gamma_{H_1}, \gamma_{[x]})O^{G(F)}_{\gamma_{[x]}}(f).$$

*If $\gamma_{H_1}$ admits no image in $G(F)$ then $SO^{H_1(F)}_{\gamma_{H_1}}(f^{H_1}) = 0$.*

*Moreover when $F$ is non-archimedean, if $G_1$, $\mathfrak{e}$, and $\chi$ are unramified, and if $f \in \mathcal{H}^{\mathrm{ur}}(G, \chi^{-1})$, then $\chi_1, \lambda_{H_1}|_{\mathfrak{X}_{H_1}}$ are unramified and the above holds true with $f^{H_1} = \eta_1^*(f)$.*

*Proof.* Given $f \in \mathcal{H}(G, \chi^{-1})$, choose a lifting $f_1 \in \mathcal{H}(G_1)$ under the surjective composite map $\mathcal{H}(G_1) \to \mathcal{H}(G_1, \chi_1^{-1}) = \mathcal{H}(G, \chi^{-1})$. Let $f_1^{H_1} \in \mathcal{H}(H_1)$ be a transfer of $f_1$ as in Proposition 7.4.9. Define $f^{H_1} \in \mathcal{H}(H_1, \chi_1^{-1}\lambda_{H_1})$ by

$$f^{H_1}(\gamma_{H_1}) := \frac{1}{\mathrm{vol}(\mathfrak{X}_{H_1})} \int_{\mathfrak{X}_{H_1}} f_1^{H_1}(z\gamma_{H_1})\chi_1(z)\lambda_{H_1}^{-1}(z)dz, \quad \gamma_{H_1} \in H_1(F),$$

so that for $\gamma_{H_1} \in H_1(F)_{\mathrm{ss}}$,

$$SO_{\gamma_{H_1}}(f^{H_1}) = \frac{1}{\mathrm{vol}(\mathfrak{X}_{H_1})} \int_{\mathfrak{X}_{H_1}} \chi_1(z)\lambda_{H_1}^{-1}(z)SO_{z_1\gamma_{H_1}}(f_1^{H_1})dz. \tag{7.4.11.1}$$

If $\gamma_{H_1}$ has no image in $G(F)$ then it has no image in $G_1(F)$ either, so $SO_{\gamma_{H_1}}(f^{H_1})$ vanishes. Otherwise, let $\gamma_1 \in G_1(F)$ be an image of $\gamma_{H_1}$. Then $z\gamma_1$ is an image of $z\gamma_{H_1}$ for each $z \in \mathfrak{X}_{H_1}$. Applying Proposition 7.4.9 to (7.4.11.1), we see that $SO_{\gamma_{H_1}}(f^{H_1})$ equals

$$\int_{\mathfrak{X}_{H_1}} \frac{\chi_1(z)}{\mathrm{vol}(\mathfrak{X}_{H_1})} \sum_{[x]\in\mathfrak{D}(I_{\gamma_1},G_1;F)} e(I_{\gamma_{1,[x]}})\frac{\Delta(z\gamma_{H_1}, z\gamma_{1,[x]})}{\lambda_{H_1}(z)}O_{z\gamma_{1,[x]}}(f_1)dz,$$

where Lemma 7.3.2 gives a bijection $\mathfrak{D}(I_{z\gamma_1}, G_1; F) = \mathfrak{D}(I_{\gamma_1}, G_1; F) \cong \mathfrak{D}(I_\gamma, G; F)$. (We view $[x]$ also as an element of $\mathfrak{D}(I_{z\gamma_1}, G_1; F)$ or $\mathfrak{D}(I_\gamma, G; F)$.) By definition and (7.4.5.1), we have

$$\Delta(z\gamma_{H_1}, z\gamma_{1,[x]}) = \Delta(z\gamma_{H_1}, \gamma_{[x]}) = \lambda_{H_1}(z)\Delta(\gamma_{H_1}, \gamma_{[x]}), \qquad z \in \mathfrak{X}_{H_1}.$$

By [Kot83, Cor. (2)] $e(I_{\gamma_{1,[x]}}) = e(I_{\gamma_{[x]}})$. All in all, as the sums run over $[x] \in \mathfrak{D}(I_\gamma, G; F)$ below,

$$\begin{aligned} SO_{\gamma_{H_1}}(f^{H_1}) &= \sum_{[x]} e(I_{\gamma_{[x]}})\Delta(\gamma_{H_1}, \gamma_{[x]}) \left( \frac{1}{\mathrm{vol}(\mathfrak{X}_1)} \int_{\mathfrak{X}_1} \chi_1(z) O_{z\gamma_{1,[x]}}(f_1)dz \right) \\ &= \sum_{[x]} e(I_{\gamma_{[x]}})\Delta(\gamma_{H_1}, \gamma_{[x]}) O_{\gamma_{[x]}}(f). \end{aligned}$$

It remains to prove the last assertion when $G$, $\mathfrak{e}_1$, and $\chi$ are unramified. Let $z$ be an element in the maximal compact subgroup of $\mathfrak{X}_{H_1}$. In the notation above, if $f_1$ is replaced with a translate $f_{1,z}$ then $f^{H_1}$ is multiplied by $\lambda_{H_1}(z)$ according to §7.4.7. On the other hand, $f_1$ is unchanged if translated by $z$ since $f_1$ is in the unramified Hecke algebra. Combining the two facts, we see that the stable orbital integrals of $f^{H_1}$ do not change values under multiplication by $\lambda_{H_1}(z)$. Therefore $\lambda_{H_1}$ is unramified. The fact that we can take $f^{H_1} = \eta_1^*(f)$ follows from the earlier part of the current proof, where we can pick $f_1 \in \mathcal{H}^{\mathrm{ur}}(G_1)$ and choose $f_1^{H_1}$ to be the image of $f_1$ under (7.4.8.1). □

**7.4.12.** Here we work out a small generalization of some results in local twisted endoscopy by Morel and Kottwitz [Mor10, §9, App. A] to the setting where $\mathcal{H}$ in the endoscopic datum cannot be taken to be an $L$-group. We put ourselves in the setting of §7.2.12 with $F = \mathbb{Q}_p$. Let $\mathfrak{e} = (H, \mathcal{H}, s, \eta) \in E(G)$, which gives rise to $\mathfrak{e}_1 = (H_1, {}^L H_1, s_1, \eta_1) \in E(G_1)$ as in Lemma 7.2.9. We make an additional hypothesis that

$$s \in Z(\widehat{H})^{\Gamma_p}.$$

The group $G$ is assumed quasi-split over $\mathbb{Q}_p$ so that [Mor10, App. A] applies. If $G$ and $H$ are unramified (which we are not assuming) then we may and will choose $G_1$ and $H_1$ to be also unramified.

In twisted endoscopy, the norm map is defined by Kottwitz and Shelstad [KS99] for strongly regular elements and by Labesse [Lab04] for elliptic elements (which may not be strongly regular). The norm map in untwisted endoscopy is simply the transfer of conjugacy classes as in §7.4.2. In the special case of base change, Kottwitz [Kot82] defines the norm map for general elements. (These norm maps coincide when there are more than one definitions available in a given setting.) For our purpose, we define the norm map from $R_1(\mathbb{Q}_p)$ to $H_1(\mathbb{Q}_p)$ to be the degree $n$ base change norm from $R_1(\mathbb{Q}_p)$ to $G_1(\mathbb{Q}_p)$ followed by the transfer of semi-simple conjugacy classes (§7.4.2) from $G_1(\mathbb{Q}_p)$ to $H_1(\mathbb{Q}_p)$. It is an exercise to check that this is consistent with the norm map by Kottwitz–Shelstad and Labesse.

Let $\delta_1 \in G_1(\mathbb{Q}_{p^n}) = R_1(\mathbb{Q}_p)$. It has a degree $n$ norm $\gamma_{0,1} \in G_1(\mathbb{Q}_p)$. We assume $\gamma_{0,1}$ to be semi-simple. Then $\delta_1$ corresponds to $[b_1] \in \mathfrak{D}_n(\gamma_{0,1}, G_1; \mathbb{Q}_p)$ according to Corollary 7.3.7. So we write $\delta_{[b_1]}$ for $\delta_1 = \delta_{[b_1]}$. Recall $\beta_p(\gamma_{0,1}, [b_1]) = \kappa_{I_{0,1}}([b_1])$. Suppose that $\gamma_{H_1} \in H_1(\mathbb{Q}_p)_{\mathrm{ss}}$ is a norm of $\delta_1$. Then $\gamma_{0,1}$ is an image of $\gamma_{H_1}$.

When $\gamma_{H_1}$ is strongly $G_1$-regular (equivalently when $\gamma_{0,1}$ is strongly regular), Kottwitz [Mor10, Cor. A.2.10] proved that

$$\Delta_0(\gamma_{H_1}, \delta_{[b_1]}) = \Delta_0(\gamma_{H_1}, \gamma_{0,1})\langle \beta_p(\gamma_{0,1}, [b_1]), s_1 \rangle, \tag{7.4.12.1}$$

where we have taken the sign correction of [KS12, §5.6] into account. To make sense of the pairing, we view $s_1 \in Z(\widehat{H}_1)^{\Gamma_p}$ as an element of $Z(\widehat{I}_{0,1})^{\Gamma_p}$ via $Z(\widehat{H}_1)^{\Gamma_p} \subset Z(\widehat{I}_{\gamma_{H_1}})^{\Gamma_p} = Z(\widehat{I}_{0,1})^{\Gamma_p}$, cf. [Mor10, A.3.11].

When $\gamma_{H_1}$ is $(G_1, H_1)$-regular but not strongly $G_1$-regular, we take (7.4.12.1) as the definition of $\Delta_0(\gamma_{H_1}, \delta_{[b_1]})$, cf. [Mor10, (A.3.11.1)].

Let $\gamma_0 \in G(\mathbb{Q}_p)$ and $[b] \in \mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p)$ be the images of $\gamma_{0,1}$ and $[b_1]$, respectively. Then $[b]$ gives rise to $\delta_{[b]} \in G(\mathbb{Q}_{p^n})$ via Corollary 7.3.7, and $\gamma_0$ is a degree $n$ norm of $\delta_{[b]}$. If the hypothesis on $Z_1$ in Lemma 7.3.6 is satisfied, and if $[b_1] \in \mathfrak{D}_n(\gamma_{0,1}, G_1; \mathbb{Q}_p)$ and $[b] \in \mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p)$ correspond under the bijection of that lemma, then the $\sigma$-conjugacy class of $\delta_{[b_1]}$ maps to that of $\delta_{[b]}$. We defined $\beta_p(\gamma_0, [b])$ to be $\kappa_{I_0}([b])$, which equals the image of $\beta_p(\gamma_{0,1}, [b_1])$. Thus we have

$$\langle \beta_p(\gamma_{0,1}, [b_1]), s_1 \rangle = \langle \beta_p(\gamma_0, [b]), s \rangle.$$

In case $G_1$ and $\mathfrak{e}$ are unramified, the twisted datum $\widetilde{\mathfrak{e}}_1 = (H_1, \mathcal{H}_1, \widetilde{s}_1, \widetilde{\eta}_1)$ of §7.2.14 is also unramified, and $\widetilde{\eta}_1$ induces $\mathbb{C}$-algebra morphisms

$$\mathcal{H}^{\mathrm{ur}}(R_1) \to \mathcal{H}^{\mathrm{ur}}(H_1) \quad \text{and} \quad \mathcal{H}^{\mathrm{ur}}(R) \to \mathcal{H}^{\mathrm{ur}}(H_1, \lambda_{H_1})$$

as in the untwisted case, cf. §7.4.8 and §7.4.10. By slight abuse of notation we call both maps $\widetilde{\eta}_1^*$. The following is a twisted analogue of Proposition 7.4.11.

**Proposition 7.4.13.** *Let $f \in \mathcal{H}(R(\mathbb{Q}_p))$. Then there exists $f^{H_1} \in \mathcal{H}(H_1, \lambda_{H_1})$ such that the following holds: Let $\gamma_{H_1} \in H_1(\mathbb{Q}_p)_{(G_1,H_1)\text{-reg}}$. We have $SO_{\gamma_{H_1}}(f^{H_1}) = 0$ if $\gamma_{H_1}$ is not a norm from $R_1(\mathbb{Q}_p)$. If $\gamma_{H_1}$ is a norm of $\delta_1 \in R_1(\mathbb{Q}_p) = G_1(\mathbb{Q}_{p^n})$ then whenever $\gamma_0 \in G(\mathbb{Q}_p)$ is an image of $\gamma_{H_1}$, we have*

(7.4.13.1)
$$SO_{\gamma_{H_1}}(f^{H_1}) = \Delta_0(\gamma_{H_1}, \gamma_0) \sum_{[b] \in \mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p)} e(I_{\delta_{[b]}}) \langle \beta_p(\gamma_0, [b]), s \rangle TO_{\delta_{[b]}}(f).$$

*Moreover if $G_1$ and $\mathfrak{e}$ are unramified, and if $f \in \mathcal{H}^{\mathrm{ur}}(R)$ then the above is true for $f^{H_1} = \widetilde{\eta}_1^*(f) \in \mathcal{H}^{\mathrm{ur}}(H_1, \lambda_{H_1})$.*

*Remark* 7.4.14. In the essential case when $G_1$ and $\mathfrak{e}$ are unramified with $f$ in the unramified Hecke algebra, if $\gamma_{H_1}$ is restricted to be a strongly $G_1$-regular element, the proposition is a special case of the twisted fundamental lemma (TFL) for the full unramified Hecke algebra. When the residue characteristic $p$ is large, TFL for the unit element is true thanks to Ngô, Waldspurger, Cluckers-Loeser, and others ([Ngô10, Wal08] with [Wal06] or [CL10]). For general elements of unramified Hecke algebras, including the case of small $p$, TFL was recently established by Lemaire, Moeglin, and Waldspurger [LMW18] and [LW17]. Note that TFL states an orbital integral identity only for strongly regular elements; in fact the twisted transfer factors are not defined for other elements in general. However we want an orbital integral identity for $(G_1, H_1)$-regular semi-simple elements which may not be strongly regular. The transfer factors for such elements are available in our setting, and Proposition 7.4.13 is proved for such elements in [Mor10, Ch. 9, App. A], under the hypotheses that $G_{\mathrm{der}} = G_{\mathrm{sc}}$ and that $\mathcal{H} = {}^L H$ in the endoscopic datum. So the point of our proof below is to remove the hypotheses of [Mor10]. The basic idea is to employ $z$-extensions, but there is a technical problem: no single $z$-extension $G_1$ satisfies the assumption of Lemma 7.3.6 for all (sufficiently large) $n$. We choose an auxiliary $z$-extension as in Lemma 7.2.13 to get around the issue.

*Proof of Proposition 7.4.13.* We present a proof when $G_1$ and $\mathfrak{e}$ are unramified assuming that $f \in \mathcal{H}^{\mathrm{ur}}(R)$. The general case will be taken care of in the last paragraph of this proof.

Put $\widetilde{Z}_1 := \mathrm{Res}_{\mathbb{Q}_{p^n}/\mathbb{Q}_p}(Z_1)_{\mathbb{Q}_{p^n}}$. Choose $f_1$ to be any preimage of $f$ under the averaging map $\mathcal{H}^{\mathrm{ur}}(R_1) \to \mathcal{H}^{\mathrm{ur}}(R)$ over $\ker(R_1(\mathbb{Q}_p) \to R(\mathbb{Q}_p)) = \widetilde{Z}_1(\mathbb{Q}_p)$. Suppose that $z\gamma_{H_1}$ is not a norm from $R_1(\mathbb{Q}_p)$ for any $z \in Z_1(\mathbb{Q}_p)$. Then [Mor10, Prop. 9.5.1, Prop. A.3.14] tells us that $SO_{z\gamma_{H_1}}(\widetilde{\eta}_1^*(f_1)) = 0$. Hence

$$SO_{\gamma_{H_1}}(\widetilde{\eta}_1^*(f)) = \int_{Z_1(\mathbb{Q}_p)} \lambda_{H_1}^{-1}(z) SO_{z\gamma_{H_1}}(\widetilde{\eta}_1^*(f_1))dz = 0.$$

Now assume the existence of $z \in Z_1(\mathbb{Q}_p)$ such that $\gamma'_{H_1} := z\gamma_{H_1}$ is a norm of some element $\delta'_1 \in R_1(\mathbb{Q}_p)$. Let $\gamma_{0,1} \in G_1(\mathbb{Q}_p)$ be an image of $\gamma'_{H_1}$. (Such a $\gamma_{0,1}$ exists since $G_1$ is quasi-split and has simply connected derived subgroup, cf. [Kot82, Thm. 4.4].) Write $\gamma_H \in H(\mathbb{Q}_p)$ and $\gamma_0 \in G(\mathbb{Q}_p)$ for the respective images of $\gamma_{H_1}$ and $\gamma_{0,1}$. Take central extensions $G'_1$ and $H'_1$ as in Lemma 7.2.13 and write $Z'_1$ for the common kernel of the surjections $G'_1 \to G$ and $H'_1 \to H$. Thus $Z'_1 \cong \prod_{i=1}^r \mathrm{Res}_{F_i/\mathbb{Q}_p} \mathbb{G}_m$ for $r \in \mathbb{Z}_{\geq 1}$ and $F_i \supset \mathbb{Q}_{p^n}$. Set

$$\widetilde{Z}'_1 := \mathrm{Res}_{\mathbb{Q}_{p^n}/\mathbb{Q}_p}(Z'_1)_{\mathbb{Q}_{p^n}} \cong \prod_{i=1}^{r} \prod_{\mathbb{Q}_{p^n} \hookrightarrow F_i} \mathrm{Res}_{F_i/\mathbb{Q}_p} \mathbb{G}_m,$$

where the second product runs over the set of $\mathbb{Q}_p$-embeddings. Fix an embedding $\mathbb{Q}_{p^n} \hookrightarrow F_i$ for each $i$ and define $Y_1$ to be the subtorus of $\widetilde{Z}'_1$ whose components outside the set of fixed embeddings are trivial. The Frobenius automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}_{p^n}/\mathbb{Q}_p)$ acts on $\widetilde{Z}'_1$ by permuting the $\mathbb{Q}_p$-embeddings.

The norm map $N : \widetilde{Z}'_1(\mathbb{Q}_p) \to Z'_1(\mathbb{Q}_p)$ is obviously onto and restricts to an isomorphism

$$N : Y_1(\mathbb{Q}_p) \xrightarrow{\sim} Z'_1(\mathbb{Q}_p).$$

From $\mathfrak{e}$, we have $\mathfrak{e}'_1 = (H'_1, {}^L H'_1, s'_1, \eta'_1) \in E(G_1)$ as in Lemma 7.2.9. Since $s \in Z(\widehat{H})^\Gamma$ we have that $s'_1 \in Z(\widehat{H}'_1)^\Gamma$. One builds a twisted endoscopic datum $\widetilde{\mathfrak{e}}'_1$ for $(R'_1, \theta'_1)$ as in §7.2.12 and §7.2.14.

Choose $f'_1 \in \mathcal{H}^{\mathrm{ur}}(R'_1)$ to be a preimage of $f$ under the averaging surjection $\mathcal{H}^{\mathrm{ur}}(R'_1) \to \mathcal{H}^{\mathrm{ur}}(R)$. We have

$$(\widetilde{\eta}'_1)^*(f'_1) \in \mathcal{H}^{\mathrm{ur}}(H'_1), \qquad f^{H'_1} := (\widetilde{\eta}'_1)^*(f) \in \mathcal{H}^{\mathrm{ur}}(H'_1, \lambda_{H'_1}).$$

The inclusion $H_1 \hookrightarrow H'_1$ induces an isomorphism $\mathcal{H}^{\mathrm{ur}}(H_1, \lambda_{H_1}) \cong \mathcal{H}^{\mathrm{ur}}(H'_1, \lambda_{H'_1})$ since the character $\lambda_{H'_1}$ restricts to $\lambda_{H_1}$. The functions $f^{H'_1}$ and $f^{H_1}$ correspond under the isomorphism. Clearly

$$SO_{\gamma'_{H_1}}(f^{H'_1}) = SO_{\gamma'_{H_1}}(f^{H_1}) = SO_{\gamma_{H_1}}(f^{H_1}).$$

By [Mor10, Prop. 9.5.1, Prop. A.3.14],

$$SO_{\gamma'_{H_1}}((\widetilde{\eta}'_1)^*(f'_1)) = \sum_{\delta'_1} e(I_{\delta'_1}) \Delta_0(\gamma'_{H_1}, \delta'_1) TO_{\delta'_1}(f'_1), \tag{7.4.14.1}$$

where the sum runs over a set of representatives for the $\sigma$-conjugacy classes in $G'_1(\mathbb{Q}_{p^n}) = R'_1(\mathbb{Q}_p)$ whose norm is $\gamma_{0,1}$. (Here we view $\gamma'_{H_1}$ and $\gamma_{0,1}$ as elements of $H'_1(\mathbb{Q}_p)$ and $G'_1(\mathbb{Q}_p)$ via $H_1 \subset H'_1$ and $G_1 \subset G'_1$.) Let $\delta \in R(\mathbb{Q}_p)$ denote the image of $\delta'_1$, and $y \in \widetilde{Z}'_1(\mathbb{Q}_p)$ an arbitrary element. We collect the following facts.

- the set of representatives $\delta'_1$ in the sum is in bijection with $\mathfrak{D}_n(\gamma_{0,1}, G'_1; \mathbb{Q}_p)$, and also with $\mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p)$ (Lemma 7.3.6 and Corollary 7.3.7),
- $e(I_{y\delta'_1}) = e(I_\delta)$ by [Kot83, Cor. (2)],

- $N(y)\gamma_{0,1}$ (resp. $N(y)\gamma'_{H_1}$) is a norm of $y\delta'_1$ in $G'_1$ (resp. $H'_1$),
- (7.4.14.1) holds with $N(y)\gamma_{0,1}$ and $N(y)\gamma'_{H_1}$ in place of $\gamma_{0,1}$ and $\gamma'_{H_1}$,
- $\Delta_0(N(y)\gamma'_{H_1}, y\delta'_1) = \lambda_{H'_1}(N(y))\Delta_0(\gamma'_{H_1}, \delta'_1)$ by [KS99, p. 53],[47]
- $\Delta_0(\gamma'_{H_1}, \gamma_{0,1})$ remains the same whether it is viewed with respect to the transfer between $H'_1$ and $G'_1$ or between $H_1$ and $G_1$.[48]

Putting all this together, we deduce that $SO_{\gamma'_{H_1}}(f^{H'_1})$ equals

$$\begin{aligned}
& \int_{Z'_1(\mathbb{Q}_p)} \lambda^{-1}_{H'_1}(z') SO_{z'\gamma_{H'_1}}((\widetilde{\eta}'_1)^*(f'_1))dz' \\
= \quad & \int_{Y_1(\mathbb{Q}_p)} \lambda^{-1}_{H'_1}(y) \sum_{\delta'_1} e(I_{y\delta'_1})\Delta_0(N(y)\gamma'_{H_1}, y\delta'_1) TO_{y\delta'_1}(f'_1)dy. \\
= \quad & \sum_{\delta'_1} e(I_{\delta'_1})\Delta_0(\gamma'_{H_1}, \delta'_1) \int_{Y_1(\mathbb{Q}_p)} TO_{y\delta'_1}(f'_1)dy. \\
= \quad & \sum_{[b'_1]\in \mathfrak{D}_n(\gamma_{0,1}, G'_1; \mathbb{Q}_p)} e(I_{\delta_{[b'_1]}})\Delta_0(\gamma'_{H_1}, \gamma_{0,1})\langle \beta_p(\gamma_{0,1}, [b'_1]), s'_1\rangle TO_{\delta_{[b]}}(f). \\
= \quad & \sum_{[b]\in \mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p)} e(I_{\delta_{[b]}})\Delta_0(\gamma_{H_1}, \gamma_0)\langle \beta_p(\gamma_0, [b]), s\rangle TO_{\delta_{[b]}}(f).
\end{aligned}$$

This finishes the proof in the unramified case.

The general case (when either $G_1$ or $\mathfrak{e}$ is ramified) works in the same way. If $z\gamma_{H_1}$ is not a norm from $R_1(\mathbb{Q}_p)$ for any $z \in Z_1(\mathbb{Q}_p)$ then $SO_{\gamma_{H_1}}(f^{H_1}) = 0$ as before. Otherwise we assume that $\gamma'_{H,1} := z\gamma_{H,1}$ is a norm for some $z$. Then we repeat the preceding argument, with the difference occurring in the choice of $f^{H'_1}$. Namely by [Mor10, Prop. A.3.14], there exists $f^{H'_1} \in \mathcal{H}(H'_1(\mathbb{Q}_p))$ such that (7.4.14.1) holds true with $f^{H'_1}$ in place of $(\widetilde{\eta}'_1)^*(f'_1)$, and such that $SO_{\gamma'_{H_1}}(f^{H'_1}) = 0$ if $\gamma$ is not a norm. Setting $f^{H'_1}(h') := \int_{Z_{H'_1}(\mathbb{Q}_p)} \lambda^{-1}_{H'_1}(z') f^{H'_1}(z'h')dz'$, we see from the above computation that Proposition 7.4.13 holds. The proof is complete.

□

**7.4.15.** The last part of Proposition 7.4.13 can be slightly generalized, following [Kot90, p. 181]. Assume that $H$ and $G$ are unramified, thus choose $H_1$ and $G_1$ to be unramified, but allow $\mathfrak{e}_1$ to be ramified. In that case, one can write $\eta_1 = c\cdot\eta_1^\circ$ with a continuous 1-cocycle $c : W_{\mathbb{Q}_p} \to Z(\widehat{H}_1)$ such that $(H_1, \mathcal{H}_1, s_1, \eta_1^\circ)$ is unramified. (To see this, apply [Lan79c, Prop. 1] for an unramified extension to find $\eta_1^\circ$, and observe that $\eta_1$ and $\eta_1^\circ$ must differ by a continuous 1-cocycle valued in $Z(\widehat{H}_1)$.) Via local class field theory, $c$ determines a smooth character $\chi_c : H_1(\mathbb{Q}_p) \to \mathbb{C}^\times$. Writing $f^{H_1,\circ} := \widetilde{\eta}_1^{\circ,*}(f)$, we see that $f^{H_1} := \chi_c \cdot f^{H_1,\circ}$ satisfies the desired condition of Proposition 7.4.13. This follows from the fact that $\Delta_0(\gamma_{H_1}, \gamma_{0,1})$ gets multiplied by

[47]Unlike the formula in *loc. cit.*, we do not put an inverse over $\lambda_{H'_1}$ since we are following the convention for $\Delta'$ in [KS12], which inverts $\Delta_{III}$ in [KS99].

[48]To see this, one reduces to the strongly regular case by [LS87, §4.3] and [LS90, §2.4]. The same $a$-data and $\chi$-data may be chosen in the two cases to compare transfer factors, as the centralizer of $\gamma_{0,1}$ determines the same root system in $G_1$ and $G'_1$ respectively. Then one sees that each of $\Delta_I$, $\Delta_{II}$, $\Delta_{III_1}$, $\Delta_{III_2}$, and $\Delta_{IV}$ is the same by inspecting the definition in [LS87, §3].

$\chi_c(\gamma_{H,1})$ when changing from $\eta_1^\circ$ to $\eta_1$. (This factor comes from $\Delta_2$ in [LS87, §3.5] as the 1-cocycle $a$ there is replaced with $ac$.)

**Proposition 7.4.16.** *In the setting of Proposition 7.4.13, assume that $G$ is unramified over $\mathbb{Q}_p$ and $f \in \mathcal{H}^{\mathrm{ur}}(R(\mathbb{Q}_p))$. If $H$ is ramified over $\mathbb{Q}_p$ then we can take $f^{H_1} = 0$ (i.e., the right hand side of* (7.4.13.1) *always vanishes).*

*Proof.* The untwisted analogue of this proposition is proven in [Kot86, Prop. 7.5]. We adapt this proof, referred to as "*loc. cit.*" below, to our twisted setting. (A proof in the twisted case is alluded to on p. 189 of [Kot90]. We are elaborating on the details.) We write $G'$ for the group $G_1$ in *loc. cit.*, as we reserve the symbol $G_1$ to stand for a $z$-extension. Via $z$-extensions we reduce to the case where $G_{\mathrm{der}} = G_{\mathrm{sc}}$ and $\mathcal{H} = {}^L H$. Recall we are also assuming $s \in Z(\widehat{H})^{\Gamma_p}$.

Fix $[b] \in \mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p)$ and a representative $b \in G(\breve{\mathbb{Q}}_p)$ of $[b]$. Fix $c \in G(\breve{\mathbb{Q}}_p)$ as in condition **KP1** of Definition 1.6.5 so that $c^{-1}\gamma_0 c = \delta\sigma(\delta)\cdots\sigma^{n-1}(\delta)$ with $\delta = c^{-1}b\sigma(c) \in G(\mathbb{Q}_{p^n})$ (which is also denoted $\delta_{[b]}$). As in the proof of Lemma 7.3.6, the following sets are in natural bijections with each other:

(i) the set of $\sigma$-conjugacy classes in the stable $\sigma$-conjugacy class of $\delta$,
(ii) $\ker(H^1(\mathbb{Q}_p, I_\delta) \to H^1(\mathbb{Q}_p, R))$, and
(iii) $\mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p)$.

To go from (i) to (ii), let $\delta' = x\delta\theta(x)^{-1}$ with $x \in R(\overline{\mathbb{Q}}_p)$, where $\theta$ denotes the automorphism of $R$ induced by $\sigma \in \mathrm{Gal}(\mathbb{Q}_{p^n}/\mathbb{Q}_p)$. Sending $\delta'$ to the cocycle $z_{\delta,\delta'} : \tau \mapsto x^{-1}{}^\tau x$ induces the bijection from (i) onto (ii). To go from (i) to (iii), take $c' \in G(\breve{\mathbb{Q}}_p)$ such that $(c')^{-1}\gamma_0 c' = \delta'\sigma(\delta')\cdots\sigma^{n-1}(\delta')$. Then we send $\delta'$ to $[c'\delta'\sigma(c')^{-1}] \in B(I_0)$, which lies in the set (iii). Moreover we have the following compatibility: it follows from the bottom commutative diagram in [Kot97, p. 273] (with $I_\delta$ and $I_0$ in place of $J^h$ and $H$ there, and the cocycle $h$ determined by $\sigma \mapsto c\delta\sigma(c)^{-1}$) that the image of $z_{\delta,\delta'}$ under the composite map

$$H^1(\mathbb{Q}_p, I_\delta) \cong \pi_1(I_\delta)_{\Gamma_p,\mathrm{tors}} \cong \pi_1(I_0)_{\Gamma_p,\mathrm{tors}} \hookrightarrow \pi_1(I_0)_{\Gamma_p}$$

coincides with the image of $[b'] - [b]$ under the composite map

$$\mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p) \hookrightarrow \mathrm{B}(I_0) \xrightarrow{\kappa_{I_0}} \pi_1(I_0)_{\Gamma_p}.$$

Write $\mathrm{inv}(\delta, \delta')$ for the image in $\pi_1(I_0)_{\Gamma_p}$. With the preparation so far, we follow *loc. cit.* to construct an exact sequence of unramified reductive groups over $\mathbb{Q}_p$

$$1 \to G \to G' \to C \to 1,$$

where $C$ is a non-trivial unramified torus. Define $R' := \mathrm{Res}_{\mathbb{Q}_{p^n}/\mathbb{Q}_p} G'$. Via the dual map $\widehat{G'} \to \widehat{G}$, we can pull-back $\eta : \widehat{H} \to \widehat{G}$ to define $\widehat{H}'$ equipped with an embedding $\widehat{H}' \hookrightarrow \widehat{G}'$. We equip $\widehat{H}'$ with a $\Gamma_p$-action as in *loc. cit.*

Write $I'_\delta$ for the connected $\sigma$-centralizer of $\delta$ in $R'$. As in *loc. cit.* we have an exact sequence $1 \to I_\delta \to I'_\delta \to C \to 1$, whose dual exact sequence fits in the following commutative diagram, where rows are $\Gamma_p$-equivariant and exact:

$$\begin{tikzcd}
1 \arrow[r] & \widehat{C} \arrow[r] \arrow[d, equal] & Z(\widehat{H}') \arrow[r] \arrow[d] & Z(\widehat{H}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \widehat{C} \arrow[r] & Z(\widehat{I}'_\delta) \arrow[r] & Z(\widehat{I}_\delta) \arrow[r] & 1
\end{tikzcd}$$

We have $s \in Z(\widehat{H})^{\Gamma_p} \subset Z(\widehat{I}_\delta)^{\Gamma_p}$. Write $\chi_s$ for the image of $s$ in $H^1(\mathbb{Q}_p, \widehat{C})$ under the connecting homomorphism arising from the first row. Then $\chi_s$ determines a smooth character $C(\mathbb{Q}_p) \to \mathbb{C}^\times$, still denoted by $\chi_s$. As in the proof of [Kot86, Prop. 7.3.5], the character $\chi_s$ is *ramified*, i.e., non-trivial on $C(\mathbb{Z}_p)$. (As $C$ is unramified, it extends uniquely to a $\mathbb{Z}_p$-torus.) This is crucial for our proof.

Now consider $g_1 \in G'(\mathbb{Q}_p)$ and put $\delta' := g_1\delta\theta(g_1)^{-1} = g_1\delta g_1^{-1} \in G(\mathbb{Q}_{p^n})$. We claim that $\delta'$ is stably $\sigma$-conjugate to $\delta$. Indeed, writing $c_1 \in C(\mathbb{Q}_p)$ for the image of $g_1$, take a lift $g_2 \in I'_\delta(\breve{\mathbb{Q}}_p)$ of $c_1^{-1}$ via the surjection $I'_\delta \to C$. Set $g := g_1 g_2 \in R(\breve{\mathbb{Q}}_p)$. Then we have $\delta' = g\delta\theta(g)^{-1}$, which proves the claim.

Let $[b']$ denote the image of $\delta'$ under the bijection from (i) to (iii) above. Then we assert that

$$\langle \beta_p(\gamma_0, [b']), s\rangle \langle \beta_p(\gamma_0, [b]), s\rangle^{-1} = \langle \mathrm{inv}(\delta, \delta'), s\rangle = \chi_s(c_1^{-1}).$$

The first equality follows from the aforementioned compatibility. The second equality comes from [Kot86, Lem. 1.6] (applied to $I = C$ and $G = I_\delta$), where the image of $c_1^{-1}$ in $H^1(\mathbb{Q}_p, I_\delta)$ is represented by the cocycle $z_{\delta,\delta'}$. Indeed, after applying the injection from $H^1(\mathbb{Q}_p, I_\delta)$ into $B(I_\delta) = H^1(W_{\mathbb{Q}_p}, I_\delta(\overline{\breve{\mathbb{Q}}_p}))$, they are represented by the same cocycle since $g_2^{-1\tau} g_2 = g^{-1\tau} g$ for $\tau \in W_{\mathbb{Q}_p}$.

For $f \in \mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))$, define $f_0 \in \mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))$ by $f(g_1 x g_1^{-1})$ with $g_1$ as above. (The analogue of $f_0$ is denoted by $f_1$ in *loc. cit.*) Write $f^H$ and $f_0^H$ for their twisted transfers to $H$. On the one hand, we have $f = f_0$ by the argument of *loc. cit.*, so we can take $f^H = f_0^H$. On the other hand, comparing the right hand sides of (7.4.13.1), we have

$$SO_{\gamma_H}(f_0^H) = \langle \beta_p(\gamma_0, [b']), s\rangle \langle \beta_p(\gamma_0, [b]), s\rangle^{-1} SO_{\gamma_H}(f^H) = \chi_s(c_1^{-1}) SO_{\gamma_H}(f^H)$$

if $\gamma_H$ is a norm of some $\delta$ as above, and $SO_{\gamma_H}(f^H) = 0$ if $\gamma_H$ is not a norm of any such $\delta$. In order to verify that the stable orbital integral of $f^H$ is identically zero, it is thus enough to exhibit a suitable $c_1$ such that $\chi_s(c_1^{-1}) \neq 1$.

To this end, let $T$ be a maximally split maximal $\mathbb{Q}_p$-torus in $G$, and take $T'$ to be the centralizer of $T$ in $G'$. The resulting exact sequence of unramified tori

$$1 \to T \to T' \to C \to 1$$

extends uniquely to an exact sequence of tori over $\mathbb{Z}_p$, with $1 \to T(\mathbb{Z}_p) \to T'(\mathbb{Z}_p) \to C(\mathbb{Z}_p) \to 1$ exact. Fix any $c_1 \in C(\mathbb{Z}_p)$ such that $\chi_s(c_1) \neq 1$ and choose $g_1 \in T'(\mathbb{Z}_p)$ to be a lift of $c_1$. Running through the above argument, we conclude that the stable orbital integral of $f^H$ vanishes everywhere.

□

**7.4.17.** Finally we drop the assumption that $s \in Z(\widehat{H})^{\Gamma_p}$ and consider the general case where $s \in Z(\widehat{H})^{\Gamma_p} Z(\widehat{G})$. Write $s = s's''$ with $s' \in Z(\widehat{H})^{\Gamma_p}$ and $s'' \in Z(\widehat{G})$. As in §7.2.14, $(H, \mathcal{H}, s', \eta) \in E(G)$ yields a twisted endoscopic datum $(H_1, {}^L H_1, \widetilde{s}'_1, \widetilde{\eta}'_1)$ for $(R, \theta)$, with $s'$ playing the role of $s$.

Consider the setting of §1.8.2, where $\phi_n \in \mathcal{H}^{\mathrm{ur}}(R(\mathbb{Q}_p))$ was introduced. If $\mathfrak{e}_1$ is unramified, then take $f_n^{H_1} := \mu^{-1}(s'')\widetilde{\eta}_1'^{,*}(\phi_n)$. If $H$ is unramified but $\mathfrak{e}_1$ is ramified, then we take $f_n^{H_1}$ as in §7.4.15. If $H$ (thus also $H_1$) is ramified over $\mathbb{Q}_p$ then take $f_n^{H_1} := 0$. We check that Proposition 7.4.13 extends to this case.

**Corollary 7.4.18.** *Let $\gamma_{H_1} \in H_1(\mathbb{Q}_p)_{(G_1,H_1)\text{-reg}}$. We have $SO_{\gamma_{H_1}}(f_n^{H_1}) = 0$ if $\gamma_{H_1}$ is not a norm from $R_1(\mathbb{Q}_p)$. If $\gamma_{H_1}$ is a norm of $\delta_1 \in R_1(\mathbb{Q}_p)$ then denoting by*

*$\gamma_0 \in G(\mathbb{Q}_p)$ an image of $\gamma_{H_1}$, we have*

$$SO_{\gamma_{H_1}}(f_n^{H_1}) = \sum_{[b]\in\mathfrak{D}_n(\gamma_0,G;\mathbb{Q}_p)} e(I_{\delta_{[b]}})\Delta_0(\gamma_{H_1},\gamma_0)\langle\widetilde{\beta}_p(\gamma_0,[b]),s\rangle TO_{\delta_{[b]}}(\phi_n),$$

*where the summand is understood to be zero if $TO_{\delta_{[b]}}(\phi_n) = 0$. If $TO_{\delta_{[b]}}(\phi_n) \neq 0$ then $[b]$ satisfies **KP0** and **KP1** in Definition 1.6.5 so $\widetilde{\beta}_p(\gamma_0,[b])$ is well defined as explained in §1.7.5.*

*Proof.* By definition $[b] \in \mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p)$ always satisfies **KP1**. Let $\mu \in \mathfrak{p}_X^{\mathcal{G}}$ as in §2.4.1. If $TO_{\delta_{[b]}}(\phi_n) \neq 0$ for some $[b] \in \mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p)$ then $X_{-\mu}(\delta_{[b]})$ is non-empty by definition (2.2.7.1), so $[\delta_{[b]}] \in \mathrm{B}(G, -\mu)$ by [RR96]. Since $[b] = [\delta_{[b]}]$ in $\mathrm{B}(G)$, we see that **KP0** holds true. Thus

$$\langle\widetilde{\beta}_p(\gamma_0,[b]),s\rangle = \mu^{-1}(s'')\langle\widetilde{\beta}_p(\gamma_0,[b]),s'\rangle = \mu^{-1}(s'')\langle\beta_p(\gamma_0,[b]),s'\rangle.$$

(The second equality holds because $s' \in Z(\widehat{H})^{\Gamma_p} \subset Z(\widehat{I}_0)^{\Gamma_p}$ and $\beta_p(\gamma_0,[b])$ is an element of $\pi_1(I_0)_{\Gamma_p}$.) Now the proof follows from Proposition 7.4.13 with $\phi_n$ and $s'$ in place of $f$ and $s$, respectively, as $\mu^{-1}(s'') \in \mathbb{C}^\times$ cancels out. □

## 8. Stabilization

We return to the point counting formula for Shimura varieties. Taking Conjecture 1.8.8 for granted, we carry out the stabilization of the formula (1.8.8.1), with a view towards a representation-theoretic description of the cohomology.

### 8.1. Initial steps.

**8.1.1.** We start by fixing a central character datum, rewrite the coefficients in (1.8.8.1), and apply a Fourier transform on the finite abelian group $\mathfrak{K}(I_0/\mathbb{Q})$.

We freely use the setting of §1.8. Throughout stabilization, we fix an unramified Shimura datum $(G, X, p, \mathcal{G})$, which determines $K_p = \mathcal{G}(\mathbb{Z}_p)$, and an open compact subgroup $K^p \subset G(\mathbb{A}_f^p)$ such that $K = K_pK^p$ is a neat subgroup.

**8.1.2.** Recall that $Z$ is the center of $G$. We endow $A_{Z,\infty} = A_{G,\infty}$, which is isomorphic to a finite product of copies of $\mathbb{R}_{>0}^\times$, with the standard multiplicative Haar measure. Fix Haar measures on $Z(\mathbb{A}_f)$ and $Z(\mathbb{R})$, thereby also on $\mathfrak{X} := (Z(\mathbb{A}_f)\cap K)\cdot Z(\mathbb{R})$ and $\mathfrak{X}_\mathbb{Q}\backslash\mathfrak{X}/A_{Z,\infty}$, relative to the counting measure on $\mathfrak{X}_\mathbb{Q}$ (which is discrete and compact in $\mathfrak{X}/A_{Z,\infty}$). With respect to the set of places of $\mathbb{Q}$, we have the decomposition $\mathfrak{X} = \mathfrak{X}^{p,\infty}\mathfrak{X}_p\mathfrak{X}_\infty$. We put a Haar measure on $Z(\mathbb{Q})\backslash Z(\mathbb{A})/\mathfrak{X}$ via the following exact sequence of topological groups

$$1 \to \mathfrak{X}_\mathbb{Q}\backslash\mathfrak{X}/A_{Z,\infty} \to Z(\mathbb{Q})\backslash Z(\mathbb{A})/A_{Z,\infty} \to Z(\mathbb{Q})\backslash Z(\mathbb{A})/\mathfrak{X} \to 1,$$

where the space in the middle is given the Tamagawa measure.

We promote $\mathfrak{X}$ to a central character datum $(\mathfrak{X},\chi)$ by defining the finite part $\chi^\infty$ to be trivial on $Z(\mathbb{A}_f)\cap K$ and the infinite part $\chi_\infty := \omega_\xi^{-1}$, where $\omega_\xi$ is the central character of $\xi$ on $Z(\mathbb{R})$.

For a connected reductive subgroup $G_0$ of $G$ over $\mathbb{Q}$ containing $Z$ such that $A_{G_0} = A_Z$, we use the quotient measure to define

$$\tau_\mathfrak{X}(G_0) := \mathrm{vol}(G_0(\mathbb{Q})\backslash G_0(\mathbb{A})/\mathfrak{X})$$

by viewing the double coset space as the quotient of $G_0(\mathbb{Q})\backslash G_0(\mathbb{A})/A_{G_0,\infty}$ by $\mathfrak{X}_\mathbb{Q}\backslash\mathfrak{X}/A_{Z,\infty}$, where the former is equipped with the Tamagawa measure and the latter with the measure explained above. For any inner form $G_0'$ of $G_0$ over $\mathbb{Q}$ we make sense of $\tau_{\mathfrak{X}}(G_0')$ by transporting the measure. Since the Tamagawa volumes are equal for $G_0$ and $G_0'$ by [Kot88], we have $\tau_{\mathfrak{X}}(G_0) = \tau_{\mathfrak{X}}(G_0')$.

Once and for all, fix a $z$-extension $G_1$ of $G$ over $\mathbb{Q}$ which is unramified over $\mathbb{Q}_p$. This is possible by Lemma 7.2.5 (recalling that $G$ is unramified over $\mathbb{Q}_p$). Write $\mathfrak{X}_1 \subset Z_{G_1}(\mathbb{A})$ for the preimage of $\mathfrak{X}$ under $G_1 \to G$. Write $\chi_1$ for the pull-back of $\chi$ to $\mathfrak{X}_1$ (thus the finite part of $\chi_1$ is trivial). Then $(\mathfrak{X}_1, \chi_1)$ is a central character datum for $G_1$, and $\mathfrak{X}_1 = \mathfrak{X}_1^{p,\infty}\mathfrak{X}_{1,p}\mathfrak{X}_{1,\infty}$ analogously as the decomposition of $\mathfrak{X}$. We put the unique Haar measure on $\mathfrak{X}_1$, and thus also on $\mathfrak{X}_{1,\mathbb{Q}}\backslash\mathfrak{X}_1/A_{G_1,\infty}$ similarly as above, such that

$$\mathrm{vol}(\mathfrak{X}_{1,\mathbb{Q}}\backslash\mathfrak{X}_1/A_{G_1,\infty})/\mathrm{vol}(\mathfrak{X}_{\mathbb{Q}}\backslash\mathfrak{X}/A_{G,\infty}) = \tau(G_1)/\tau(G), \tag{8.1.2.1}$$

so that

$$\tau_{\mathfrak{X}_1}(G_1) = \tau_{\mathfrak{X}}(G). \tag{8.1.2.2}$$

Let $\mathfrak{c} = (\gamma_0, a, [b]) \in \mathfrak{K}\mathfrak{P}_{\mathrm{a}}(p^n)$ for some $n \in \mathbb{Z}_{\geq 1}$. Recall from §1.7.11 that $\mathfrak{c}$ determines an inner form $I_v$ of $I_0$ over $\mathbb{Q}_v$. Note that $I_p$ depends only on $\gamma_0$ and $[b]$ (not on $a$). Since $I_\infty$ is always a compact-mod-center inner form of $I_{0,\mathbb{R}}$ we often write $I_0^{\mathrm{cpt}}$ for $I_\infty$. If the Kottwitz invariant $\alpha(\mathfrak{c})$ (defined in §1.7.5) vanishes, then we have an inner form $I$ of $I_0$ over $\mathbb{Q}$ which localizes to $I_v$ at each $v$ (Proposition 1.7.12). We defined the constants $c_1(\mathfrak{c}, K^p, di_p di^p)$ and $c_2(\gamma_0)$ in §1.8.6.

**Lemma 8.1.3.** *If $\alpha(\mathfrak{c})$ vanishes, then*

$$c_1(\mathfrak{c}, K^p, di_p di^p)c_2(\gamma_0) = \tau_{\mathfrak{X}}(G)\cdot|\mathfrak{K}(I_0/\mathbb{Q})|\cdot\mathrm{vol}(Z(\mathbb{R})\backslash I_\infty(\mathbb{R}))^{-1}.$$

*Proof.* With the choice of measures as above,

$$\begin{aligned} c_1(\mathfrak{c}, K^p, di_p di^p) &= \mathrm{vol}(I(\mathbb{Q})\backslash I(\mathbb{A})/Z_K I_\infty(\mathbb{R})) \\ &= \frac{\mathrm{vol}(I(\mathbb{Q})\backslash I(\mathbb{A})/Z_K Z(\mathbb{R}))}{\mathrm{vol}(Z(\mathbb{R})\backslash I_\infty(\mathbb{R}))} = \frac{\tau_{\mathfrak{X}}(I_0)}{\mathrm{vol}(Z(\mathbb{R})\backslash I_\infty(\mathbb{R}))}. \end{aligned}$$

On the other hand one deduces as in [Kot86, p. 395] that

$$c_2(\gamma_0) = \tau(I_0)^{-1}\tau(G)|\mathfrak{K}(I_0/\mathbb{Q})| = \tau_{\mathfrak{X}}(I_0)^{-1}\tau_{\mathfrak{X}}(G)|\mathfrak{K}(I_0/\mathbb{Q})|.$$

We conclude by taking product of the two equations. $\square$

**8.1.4.** Let $\mathfrak{c} = (\gamma_0, a, [b]) \in \mathfrak{K}\mathfrak{P}_{\mathrm{a}}(p^n)$. Define $e(\mathfrak{c}) := \prod_v e(I_v) \in \{\pm 1\}$, the product of Kottwitz signs over all places. If $\alpha(\mathfrak{c})$ vanishes, then $e(\mathfrak{c}) = 1$ since $I_v$'s come from a $\mathbb{Q}$-group $I$. Hence

$$\sum_{\kappa \in \mathfrak{K}(I_0/\mathbb{Q})} e(\mathfrak{c})\langle\alpha(\mathfrak{c}), \kappa\rangle = \begin{cases} |\mathfrak{K}(I_0/\mathbb{Q})|, & \text{if } \alpha(\mathfrak{c}) = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{8.1.4.1}$$

Applying Lemma 8.1.3 and (8.1.4.1) to (1.8.8.1), we have

$$T(\Phi_{\mathfrak{p}}^m, f^p dg^p) = \tau_{\mathfrak{X}}(G)\sum_{\gamma_0 \in \Sigma_{\mathfrak{X},\mathbb{R}\text{-ell}}(G)} \bar{\iota}_G(\gamma_0)^{-1}\sum_{\kappa \in \mathfrak{K}(I_0/\mathbb{Q})}\sum_{(a,[b])} N(\gamma_0, \kappa, a, [b]), \tag{8.1.4.2}$$

where the third sum runs over $\mathfrak{D}(I_0, G; \mathbb{A}_f^p) \times \mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p)$ and

$$N(\gamma_0, \kappa, a, [b]) = \langle\alpha(\gamma_0, a, [b]), \kappa\rangle e(\gamma_0, a, [b])\frac{O_{\gamma_a}(f^p)TO_{\delta_{[b]}}(\phi_n)\mathrm{tr}\,\xi(\gamma_0)}{\mathrm{vol}(Z(\mathbb{R})\backslash I_\infty(\mathbb{R}))}.$$

This is straightforward possibly except for the following point. In (1.8.8.1), every $\mathfrak{c} = (\gamma_0, a, [b])$ is a $p^n$-admissible Kottwitz parameter. In the formula above, since condition **KP0** is not imposed on $[b] \in \mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p)$, a priori $(\gamma_0, a, [b])$ may not be a Kottwitz parameter. The necessary observation is that $[b]$ satisfies **KP0** as soon as $\delta_{[b]}$ lies in the support of $\phi_n$.

## 8.2. **Local transfer of orbital integrals.**

**8.2.1.** Here we prove adelic orbital integral identities for each $\mathfrak{e} = (H, \mathcal{H}, s, \eta) \in E_{\mathrm{ell}}(G)$. The fixed $z$-extension $G_1$ (which is unramified over $\mathbb{Q}_p$) gives rise to a central extension $H_1$ of $H$ via Lemma 7.2.6. In particular, if $H$ is unramified over $\mathbb{Q}_p$ then so is $H_1$. As before, $\mathfrak{X}_{H_1}$ is the preimage of $\mathfrak{X} \subset Z_G(\mathbb{A}) \subset Z_H(\mathbb{A})$ under $Z_{H_1} \to Z_H$. We have a continuous character $Z^0_{G_1}(\mathbb{Q}) \backslash Z^0_{G_1}(\mathbb{A}) \to \mathbb{C}^\times$ corresponding to the global parameter $W_{\mathbb{Q}} \to {}^L H_1 \xrightarrow{\eta_1} {}^L G_1 \to {}^L Z^0_{G_1}$ as in Lemma 7.4.6. Restricting to $\mathfrak{X}_1$, we obtain a character

$$\lambda_{H_1} : \mathfrak{X}_{1,\mathbb{Q}} \backslash \mathfrak{X}_1 \longrightarrow \mathbb{C}^\times, \tag{8.2.1.1}$$

which can be viewed as a character of $\mathfrak{X}_{H_1}$ via $\mathfrak{X}_1 \cong \mathfrak{X}_{H_1}$. According as $\mathfrak{X}_1 = \mathfrak{X}_1^{p,\infty} \mathfrak{X}_{1,p} \mathfrak{X}_{1,\infty}$ we decompose $\lambda_{H_1} = \lambda^{p,\infty}_{H_1} \lambda_{H_1,p} \lambda_{H_1,\infty}$. With the Haar measure on $\mathfrak{X}_1$ transferred to $\mathfrak{X}_{H_1}$ via the isomorphism, the analogue of (8.1.2.2) holds.

**Lemma 8.2.2.** *We have* $\tau_{\mathfrak{X}_{H_1}}(H_1) = \tau_{\mathfrak{X}}(H)$.

*Proof.* Once we prove that $\tau(G_1)/\tau(G) = \tau(H_1)/\tau(H)$, the lemma follows in the same way as (8.1.2.2) is implied by (8.1.2.1). By [Kot84b, (5.2.3), §5.3] ($\tau_1(\cdot)$ therein is $\tau(\cdot)$ by [Kot88]), we have

$$\tau(G_1)/\tau(G) = \left| \mathrm{coker}(X_*(Z(\widehat{G}_1))^\Gamma \to X_*(\widehat{Z}_1)^\Gamma) \right|,$$
$$\tau(H_1)/\tau(H) = \left| \mathrm{coker}(X_*(Z(\widehat{H}_1))^\Gamma \to X_*(\widehat{Z}_1)^\Gamma) \right|.$$

So it is enough to show that the two cokernels are isomorphic. Consider the commutative diagram below, where the rows are coming from the exact sequence of [Kot84b, Cor. 2.3].

$$\begin{tikzcd}
1 \arrow[r] & X_*(Z(\widehat{G}))^\Gamma \arrow[r, "i_G"] \arrow[d, hook] & X_*(Z(\widehat{G}_1))^\Gamma \arrow[r] \arrow[d, hook] & X_*(\widehat{Z}_1)^\Gamma \arrow[d, equal] \\
1 \arrow[r] & X_*(Z(\widehat{H}))^\Gamma \arrow[r, "i_H"] \arrow[d, two heads] & X_*(Z(\widehat{H}_1))^\Gamma \arrow[r] \arrow[d, two heads] & X_*(\widehat{Z}_1)^\Gamma \\
& X_*(Z(\widehat{H})/Z(\widehat{G}))^\Gamma \arrow[r, equal] & X_*(Z(\widehat{H}_1)/Z(\widehat{G}_1))^\Gamma &
\end{tikzcd}$$

We obtain $\mathrm{coker}\, i_G \cong \mathrm{coker}\, i_H$ by diagram chase, and thereby the two cokernels above are isomorphic. $\square$

**8.2.3.** For the moment we assume that $s \in Z(\widehat{H})^{\Gamma_p}$, and we are going to drop this assumption in §8.2.6 below. By Proposition 7.4.11 there exists a transfer

$$f^{H_1,p,\infty} \in \mathcal{H}(H_1(\mathbb{A}_f^{p,\infty}), \chi_1^{-1} \lambda^{p,\infty}_{H_1})$$

of $f^{p,\infty} \in \mathcal{H}(G(\mathbb{A}_f^p)/\!\!/K^p)$ with the following property. For each

$$\gamma_{H_1} \in H_1(\mathbb{A}_f^p)_{(G_1,H_1)\text{-reg}},$$

if it has no image in $G_1(\mathbb{A}_f^p)_{\mathrm{ss}}$ then $SO_{\gamma_{H_1}}(f^{H_1,p,\infty}) = 0$. If there exists an image $\gamma_{0,1} \in G_1(\mathbb{A}_f^p)_{\mathrm{ss}}$ of $\gamma_{H_1}$, then writing $\gamma_0 \in G(\mathbb{A}_f^p)$ for the projection of $\gamma_{0,1}$, we have

$$\begin{aligned} SO_{\gamma_{H_1}}(f^{H_1,p,\infty}) &= \sum_{a_1} e(a_1)\Delta(\gamma_{H_1},\gamma_{0,1,a_1})O_{\gamma_{0,1,a_1}}(f^{p,\infty}) \\ &= \sum_{a_1} e(a_1)\Delta(\gamma_{H_1},\gamma_{0,1})\langle\beta^{p,\infty}(\gamma_{0,1},a_1),s_1\rangle O_{\gamma_{0,1,a_1}}(f^{p,\infty}) \\ &= \sum_{a}\left(\prod_{v\neq p,\infty} e(I_v)\right)\Delta(\gamma_{H_1},\gamma_{0,1})\langle\beta^{p,\infty}(\gamma_0,a),s\rangle O_{\gamma_{0,a}}(f^{p,\infty}), \end{aligned} \tag{8.2.3.1}$$

where the sums for $a_1$ and $a$ run over $\mathfrak{D}(I_{0,1},G_1;\mathbb{A}_f^p)$ and $\mathfrak{D}(I_0,G;\mathbb{A}_f^p)$ respectively. Recall that $\beta^{p,\infty}(\cdot,\cdot)$ was introduced in §1.7.6. The second equality above follows from a basic property of transfer factors regarding the change of $\gamma_{0,1}$ within its stable conjugacy class as stated in [Kot86, Conj. 5.5], which can be proved by arguing as in the proof of [LS87, Lem. 4.1.C ] in the $G_1$-regular case and extended to the $(G_1,H_1)$-regular case by [LS90].

**8.2.4.** From §7.4.17 and Corollary 7.4.18, we obtain $f_p^{H_1} \in \mathcal{H}(H_1(\mathbb{Q}_p),\lambda_{H_1,p})$ (renaming $f_n^{H_1}$) with the following property. Let $\gamma_{H_1} \in H_1(\mathbb{Q}_p)_{(G_1,H_1)\text{-reg}}$. If $\gamma_{H_1}$ is not a norm from an element of $R_1(\mathbb{Q}_p)$ then $SO_{\gamma_{H_1}}(f_p^{H_1}) = 0$. If it is a norm, there exists $\gamma_{0,1} \in G_1(\mathbb{Q}_p)$ whose conjugacy class matches $\gamma_{H_1}$. Writing $\gamma_0 \in G(\mathbb{Q}_p)$ for the projection of $\gamma_{0,1}$, we have

$$SO_{\gamma_{H_1}}(f_p^{H_1}) = \sum_{[b]\in\mathfrak{D}_n(\gamma_0,G;\mathbb{Q}_p)} e(I_{\delta_{[b]}})\Delta_0(\gamma_{H_1},\gamma_{0,1})\langle\widetilde{\beta}_p(\gamma_0,[b]),s\rangle TO_{\delta_{[b]}}(\phi_n). \tag{8.2.4.1}$$

By definition $I_{\delta_{[b]}} = I_p$ if $[b]$ comes from $\mathfrak{c} = (\gamma_0,a,[b]) \in \mathfrak{KP}_{\mathrm{a}}(p^n)$. Thanks to Proposition 7.4.13 we may and will take $f_p^{H_1} = 0$ if $H$ is ramified over $\mathbb{Q}_p$.

**8.2.5.** Our starting point for real orbital integrals is the argument of [Kot90, §7] based on Shelstad's real endoscopy [She82] and the pseudo-coefficients of Clozel–Delorme [CD90]. We incorporate central characters and $z$-extensions.

We have the characters $\chi_{1,\infty}$ and $\lambda_{H_1,\infty}$ on $\mathfrak{X}_{H_1,\infty}$. We will drop $\infty$ from the subscript and write $\lambda_{H_1}$ and $\chi_1$ when the context is clearly local archimedean. Let $\xi_1$ be the irreducible representation of $G_1$ obtained from $\xi$ via the surjection $G_1 \to G$. As in §7.3.8, fix a Shimura datum $(G_1,X_1)$ and $\mu_1 \in \mathbb{\mu}_{X_1}(\overline{\mathbb{Q}})$. For each elliptic $\gamma_{0,1} \in G_1(\mathbb{R})$ we have an element $\widetilde{\beta}_\infty(\gamma_{0,1}) \in \pi_1(I_{0,1})$ which maps to $[\mu_1] \in \pi_1(G_1)$, cf. §1.7.6.

We simply set $f_\infty^{H_1} := 0$ if elliptic maximal tori of $G_\mathbb{R}$ do not come from those of $H_\mathbb{R}$ via transfer (equivalently, if elliptic maximal tori of $G_{1,\mathbb{R}}$ do not come from those of $H_{1,\mathbb{R}}$). In particular $f_\infty^{H_1} = 0$ if $H_\mathbb{R}$ contains no elliptic maximal tori. From now on, we assume that $H_\mathbb{R}$ contains elliptic maximal tori and they transfer to elliptic maximal tori of $G_\mathbb{R}$. In particular $Z^0_{G_\mathbb{R}}$ and $Z^0_{H_\mathbb{R}}$ have the same $\mathbb{R}$-rank.

Following Kottwitz, we construct a smooth function $f_\infty^{H_1}$ on $H_1(\mathbb{R})$ that is compactly supported modulo center, by taking a suitable finite linear combination of pseudo-coefficients of discrete series representations (explicitly as on page 186 of

[Kot90] with our $H_1$ and $G_1$ playing the roles of his $H$ and $G$) such that the following hold: if $\gamma_{H_1} \in H_1(\mathbb{R})_{\mathrm{ss}}$ is elliptic and $(G_1, H_1)$-regular, which by the above assumption has an elliptic element $\gamma_{0,1} \in G_1(\mathbb{R})_{\mathrm{ss}}$ as image, then

$$SO_{\gamma_{H_1}}(f_\infty^{H_1}) = \frac{e(I_{0,1}^{\mathrm{cpt}})\langle \widetilde{\beta}_\infty(\gamma_{0,1}), s_1\rangle \Delta_\infty(\gamma_{H_1}, \gamma_{0,1}) \operatorname{tr} \xi_1(\gamma_{0,1})}{\operatorname{vol}(Z_{G_1}(\mathbb{R})\backslash I_{0,1}^{\mathrm{cpt}}(\mathbb{R}))}, \tag{8.2.5.1}$$

whereas if $\gamma_{H_1}$ is non-elliptic and $(G_1, H_1)$-regular then $SO_{\gamma_{H_1}}(f_\infty^{H_1}) = 0$. If $\gamma_{H_1} \in H_1(\mathbb{R})_{\mathrm{ss}}$ is not $(G_1, H_1)$-regular, then by [Mor10, Prop. 3.3.4, Rem 3.3.5] adapted to our setting we have $SO_{\gamma_{H_1}}(f_\infty^{H_1}) = 0$. The $\chi_1^{-1}\lambda_{H_1}$-equivariance of $f_\infty^{H_1}$ follows from (8.2.5.1) and the equivariance of transfer factors (§7.4.7), recalling that the central character of $\xi_1$ is $\chi_1^{-1}$.

We claim that if $\gamma_{H_1}$ is elliptic and $(G_1, H_1)$-regular then (8.2.5.1) implies that

$$SO_{\gamma_{H_1}}(f_\infty^{H_1}) = \frac{e(I_0^{\mathrm{cpt}})\langle \widetilde{\beta}_\infty(\gamma_0), s\rangle \Delta_\infty(\gamma_{H_1}, \gamma_0) \operatorname{tr} \xi(\gamma_0)}{\operatorname{vol}(Z_G(\mathbb{R})\backslash I_0^{\mathrm{cpt}}(\mathbb{R}))}, \tag{8.2.5.2}$$

where $\gamma_0$ is the image of $\gamma_{0,1}$ in $G(\mathbb{R})$. It is routine to check term-by-term equalities between the right hand sides of (8.2.5.1) and (8.2.5.2). We illustrate the idea by showing that $\langle \widetilde{\beta}_\infty(\gamma_{0,1}), s_1\rangle = \langle \widetilde{\beta}_\infty(\gamma_0), s\rangle$, leaving the rest to the reader. Let us write $s$ for $s_1$ since the latter is the image of $s$ under the inclusion $\widehat{G} \to \widehat{G}_1$. We have a decomposition $s = s's''$ with $s' \in Z(\widehat{H})^{\Gamma_\infty}$ and $s'' \in Z(\widehat{G})$. By definition ([Kot90, §2]) the character $\widetilde{\beta}_\infty(\gamma_0)$ (resp. $\widetilde{\beta}_\infty(\gamma_{0,1})$) restricts to a character on $Z(\widehat{I}_0)^{\Gamma_\infty}$ (resp. $Z(\widehat{I}_{0,1})^{\Gamma_\infty}$) and a character on $Z(\widehat{G})$ (resp. $Z(\widehat{G}_1)$), each of which is determined by $\mu$ (resp. $\mu_1$). They are related via the following commutative diagrams, from which it is obvious that $\langle \widetilde{\beta}_\infty(\gamma_{0,1}), s's''\rangle = \langle \widetilde{\beta}_\infty(\gamma_0), s's''\rangle$.

$$\begin{tikzcd}
s' \in Z(\widehat{H})^{\Gamma_\infty} \arrow[r] \arrow[d] & Z(\widehat{I}_0)^{\Gamma_\infty} \arrow[r, "\widetilde{\beta}_\infty(\gamma_0)"] \arrow[d] & \mathbb{C}^\times \\
Z(\widehat{H}_1)^{\Gamma_\infty} \arrow[r] & Z(\widehat{I}_{0,1})^{\Gamma_\infty} \arrow[ur, "\widetilde{\beta}_\infty(\gamma_{0,1})"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
s'' \in Z(\widehat{G}) \arrow[r, "\widetilde{\beta}_\infty(\gamma_0)"] \arrow[d] & \mathbb{C}^\times \\
Z(\widehat{G}_1) \arrow[ur, "\widetilde{\beta}_\infty(\gamma_{0,1})"'] &
\end{tikzcd}$$

**8.2.6.** So far we have constructed the function $f^{H_1} := f^{H_1,p,\infty} f_p^{H_1} f_\infty^{H_1}$ on $H_1(\mathbb{A})$. If $H$ is ramified over $\mathbb{Q}_p$ or if elliptic tori of $G_{\mathbb{R}}$ do not come from those of $H_{\mathbb{R}}$, then we have $f^{H_1} = 0$ since $f_p^{H_1} = 0$ or $f_\infty^{H_1} = 0$ in each case. When neither is the case, $f^{H_1}$ depends only on the image of $s$ modulo $Z(\widehat{G})$. To see this, suppose that $s$ is replaced with $sz$ for $z \in Z(\widehat{G})$. Then $f^{H_1,p,\infty}$ as well as the identity (8.2.3.1) remains unchanged as $\prod_{v\neq p,\infty} \widetilde{\beta}_v(\gamma_0, a)$ is trivial on $Z(\widehat{G})$. The function $f_p^{H_1}$ is multiplied by $\mu(z)^{-1}$ according to §7.4.17. Since $\langle \widetilde{\beta}_\infty(\gamma_0), s\rangle$ is the only term in (8.2.5.2) to change and it is multiplied by $\langle \widetilde{\beta}_\infty(\gamma_0), z\rangle = \mu(z)$, the function $f_\infty^{H_1}$ is multiplied by $\mu(z)$ to keep (8.2.5.2) valid. All in all, $f^{H_1}$ indeed remains invariant.

**8.2.7.** Let us summarize the above results in terms of adelic orbital integral identities. To this end, we slightly extend the definition of $N(\gamma_0, \kappa, a, [b])$ in §8.1.4 from rational to adelic elements. Let $\mathfrak{e} = (H, \mathcal{H}, s, \eta) \in \mathcal{E}_{\mathrm{ell}}(G)$. Let $\gamma_0 \in G(\mathbb{A})$ and suppose that $\gamma_0$ is an image of $\gamma_{H_1} \in H_1(\mathbb{A})_{(G_1,H_1)\text{-reg}}$ at every place $v$. Define a similar quantity $N'(\gamma_0, \gamma_{H_1}, s, a, [b])$ for $s \in Z(\widehat{H})$, $a \in \mathfrak{D}(I_0, G; \mathbb{A}_f^p)$, and

$[b] \in \mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p)$ by

$$N'(\gamma_0, \gamma_{H_1}, s, a, [b]) = \langle \widetilde{\beta}^{p,\infty}(\gamma_0, a), s\rangle \langle \widetilde{\beta}_p(\gamma_0, [b]), s\rangle \langle \widetilde{\beta}_\infty(\gamma_0), s\rangle \Delta_{\mathbb{A}}(\gamma_{H_1}, \gamma_0) \times O_{\gamma_a}(f^p) TO_{\delta_{[b]}}(\phi_n) \mathrm{tr}\, \xi(\gamma_0) \mathrm{vol}(Z(\mathbb{R}) \backslash I_\infty(\mathbb{R}))^{-1}.$$

To compare with (8.1.4), if $\gamma_0 \in G(\mathbb{Q})_{\mathbb{R}\text{-ell}}$ then

$$N'(\gamma_0, \gamma_{H_1}, s, a, [b]) = N(\gamma_0, s, a, [b]) \quad (8.2.7.1)$$

by definition of $\alpha(\gamma_0, a, [b])$ and the product formula that $\Delta_{\mathbb{A}}(\gamma_{H_1}, \gamma_0) = 1$.

**Lemma 8.2.8.** *Let $\gamma_{H_1} \in H_1(\mathbb{A})_{(G_1,H_1)\text{-reg}}$. If $\gamma_0 \in G(\mathbb{A})_{\mathrm{ss}}$ is $\mathbb{R}$-elliptic and an image of $\gamma_{H_1}$ at every place $v$ then*

$$SO_{\gamma_{H_1}}(f^{H_1}) = \sum_{\substack{a \in \mathfrak{D}(I_0, G; \mathbb{A}_f^p) \\ [b] \in \mathfrak{D}_n(\gamma_0, G; \mathbb{Q}_p)}} N'(\gamma_0, \gamma_{H_1}, s, a, [b]).$$

*If no $\mathbb{R}$-elliptic $\gamma_0 \in G(\mathbb{A})_{\mathrm{ss}}$ is an image of $\gamma_{H_1}$ then $SO_{\gamma_{H_1}}(f^{H_1}) = 0$. If $\gamma_{H_1} \in H_1(\mathbb{Q})_{\mathrm{ss}}$ is not $(G_1, H_1)$-regular then again $SO_{\gamma_{H_1}}(f^{H_1}) = 0$.*

*Proof.* If elliptic maximal tori of $G_{\mathbb{R}}$ do not come from those of $H_{\mathbb{R}}$ then no $\mathbb{R}$-elliptic element of $G(\mathbb{R})$ is an image of an element of $H(\mathbb{R})$. Then $f_\infty^{H_1} = 0$ by construction, so the lemma holds. If $\gamma_{H_1,\infty}$ is not $(G_1, H_1)$-regular then we saw $SO_{\gamma_{H_1,\infty}}(f_\infty^{H_1}) = 0$, so in particular $SO_{\gamma_{H_1}}(f^{H_1}) = 0$ for non-$(G_1, H_1)$-regular elements $\gamma_{H_1} \in H_1(\mathbb{Q})_{\mathrm{ss}}$.

From now on, let $\gamma_{H_1} \in H_1(\mathbb{A})_{(G_1,H_1)\text{-reg}}$, and assume that elliptic maximal tori of $G_{\mathbb{R}}$ come from those of $H_{\mathbb{R}}$. If $\gamma_{H_1}$ is not $\mathbb{R}$-elliptic (or equivalently if $\gamma_{H_1}$ has no image in $G(\mathbb{R})_{\mathrm{ell}}$) then $SO_{\gamma_{H_1,\infty}}(f_\infty^{H_1}) = 0$ by [Kot92a, Lem. 3.1] (asserting that the orbital integrals of pseudocoefficients of discrete series representations are supported on elliptic elements). Thus the lemma is verified is this case. If $\gamma_{H_1,v}$ does not have an image in $G_1(F_v)_{\mathrm{ss}}$ at some finite place $v$ then $SO_{\gamma_{H_1}}(f_v^{H_1}) = 0$ by §8.2.3 and §8.2.4 so the lemma is again true. In the remaining case, there exists $\gamma_0$ as in the lemma. Then the desired equality follows from (8.2.3.1), (8.2.4.1), and (8.2.5.2). ☐

## 8.3. **Final steps.**

**8.3.1.** Resuming from the formula (8.1.4.2), we apply the adelic transfer of orbital integrals to finish stabilization. To re-parametrize the sum over $(\gamma_0, \kappa)$ in (8.1.4.2), consider the set of equivalence classes

$$\Sigma\mathfrak{K}_{\mathrm{ell}}(G) := \{(\gamma_0, \kappa) \mid \gamma_0 \in G(\mathbb{Q})_{\mathrm{ell}},\ \kappa \in \mathfrak{K}(I_0/\mathbb{Q})\}/\sim,$$

where $(\gamma_0, \kappa) \sim (\gamma_0', \kappa')$ if there exists $g \in G(\overline{\mathbb{Q}})$ such that (i) $g\gamma_0 g^{-1} = \gamma_0'$, (ii) $g^{-1}{}^{\tau}g \in I_0(\overline{\mathbb{Q}})$ for every $\tau \in \Gamma$, and (iii) the inner twisting $I_{0,\overline{\mathbb{Q}}} \cong I_{\gamma_0',\overline{\mathbb{Q}}}$ induced by $\mathrm{Int}(g)$ carries $\kappa$ to $\kappa'$. Define another set of equivalence classes

$$\mathcal{E}\Sigma_{\mathrm{ell}}(G) := \{(\mathfrak{e}, \gamma_H) :\ \mathfrak{e} = (H, \mathcal{H}, s, \eta) \in E_{\mathrm{ell}}(G),\ \gamma_H \in \Sigma_{\mathrm{ell}}(H)_{(G,H)\text{-reg}}\}/\sim,$$

where $\Sigma_{\mathrm{ell}}(H)_{(G,H)\text{-reg}}$ is the set of stable conjugacy classes of $(G,H)$-regular semi-simple elements of $H(\mathbb{Q})$, and $(\mathfrak{e}, \gamma_H) \sim (\mathfrak{e}', \gamma_H')$ if there is an isomorphism between endoscopic data $\mathfrak{e} = (H, \mathcal{H}, s, \eta)$ and $\mathfrak{e}' = (H', \mathcal{H}', s', \eta')$ such that $\gamma_H$ is carried to

$\gamma'_H$. (In particular $(\mathfrak{e},\gamma_H)\sim(\mathfrak{e},\gamma'_H)$ if there is an outer automorphism of $\mathfrak{e}$ mapping $\gamma_H$ to $\gamma'_H$.) Define an analogous set with the obvious surjection

$$(8.3.1.1)\qquad \mathcal{E}\Sigma^{\sim}_{\mathrm{ell}}(G) := \left\{(\mathfrak{e},\gamma_H)\mid \begin{array}{c}\mathfrak{e}=(H,\mathcal{H},s,\eta)\in\mathcal{E}_{\mathrm{ell}}(G),\\ \gamma_H\in\Sigma_{\mathrm{ell}}(H)_{(G,H)\text{-reg}}\end{array}\right\}\to\mathcal{E}\Sigma_{\mathrm{ell}}(G).$$

The outer automorphism group $\mathrm{Out}_F(\mathfrak{e})$ acts transitively on each fiber of the map. Let us define a map

$$\widetilde{\mathfrak{E}}:\mathcal{E}\Sigma^{\sim}_{\mathrm{ell}}(G)\longrightarrow\Sigma\mathfrak{K}_{\mathrm{ell}}(G)\cup\{\emptyset\}.$$

We explained in §7.4.2 how $(\mathfrak{e},\gamma_H)$ determines either a stable conjugacy class of $\gamma_0\in G(\mathbb{Q})$ or $\emptyset$ (when there is no matching conjugacy class in $G(\mathbb{Q})$). In the latter case, set $\widetilde{\mathfrak{E}}:(\mathfrak{e},\gamma_H)\mapsto\emptyset$. In the former, we map $(\mathfrak{e},\gamma_H)$ to $(\gamma_0,\kappa)$, where $\kappa\in\mathfrak{K}(I_0/\mathbb{Q})$ is determined by the image of $s$ under the composition $Z(\widehat{H})\hookrightarrow Z(\widehat{I}_{\gamma_H})\cong Z(\widehat{I}_0)$. Here the canonical isomorphism comes from the fact that $I_0$ is an inner form of $I_{\gamma_H}$. The ellipticity of $\gamma_0$ follows from that of $\gamma_H$. Letting $Z(\mathbb{Q})$ act on each of $\Sigma_{\mathrm{ell}}(G)$ and $\Sigma_{\mathrm{ell}}(H)$ by multiplication, we see that $\widetilde{\mathfrak{E}}$ is $Z(\mathbb{Q})$-equivariant.

By [Kot86, Lem. 9.7], when $G_{\mathrm{der}}=G_{\mathrm{sc}}$, the map $\widetilde{\mathfrak{E}}$ factors through a unique map $\mathfrak{E}:\mathcal{E}\Sigma^{\sim}_{\mathrm{ell}}(G)\to\mathcal{E}\Sigma_{\mathrm{ell}}(G)$, and the image of $\widetilde{\mathfrak{E}}$ contains $\Sigma\mathfrak{K}_{\mathrm{ell}}(G)$. When $G_{\mathrm{der}}$ is not simply connected, write $\widetilde{\mathfrak{E}}_1$ and $\mathfrak{E}_1$ for the analogous maps for the $z$-extension $G_1$. We have a commutative diagram (*a priori* without $\mathfrak{E}$)

$$(8.3.1.2)\qquad \begin{tikzcd} \mathcal{E}\Sigma^{\sim}_{\mathrm{ell}}(G_1) \arrow[r, two heads] \arrow[d] & \mathcal{E}\Sigma_{\mathrm{ell}}(G_1) \arrow[r, "\mathfrak{E}_1"] \arrow[d] & \Sigma\mathfrak{K}_{\mathrm{ell}}(G_1)\cup\{\emptyset\} \arrow[d] \\ \mathcal{E}\Sigma^{\sim}_{\mathrm{ell}}(G) \arrow[r, two heads] \arrow[rr, bend right, "\widetilde{\mathfrak{E}}"'] & \mathcal{E}\Sigma_{\mathrm{ell}}(G) \arrow[r, dashed, "\exists!\,\mathfrak{E}"] & \Sigma\mathfrak{K}_{\mathrm{ell}}(G)\cup\{\emptyset\}, \end{tikzcd}$$

where the vertical maps are induced by $G_1\to G$, using $\mathcal{E}_{\mathrm{ell}}(G_1)\cong\mathcal{E}_{\mathrm{ell}}(G)$ (Lemma 7.2.11) and part (i) of Lemma 7.3.2. We add that the right vertical is required to send $\emptyset$ to itself. By the $Z_{G_1}(\mathbb{Q})$-equivariance of $\mathfrak{E}_1$, there exists a unique $Z_G(\mathbb{Q})$-equivariant map $\mathfrak{E}$ making the entire diagram commute. (The image of $\mathfrak{E}$ does not contain $\emptyset$ if $G$ is quasi-split over $\mathbb{Q}$, since every $\gamma_H$ then has an image in $G(\mathbb{Q})$.)

**Lemma 8.3.2.** *The map $\mathfrak{E}$ is $Z(\mathbb{Q})$-equivariant and contains $\Sigma\mathfrak{K}_{\mathrm{ell}}(G)$ in its image. Moreover for $(\gamma_0,\kappa)\in\Sigma\mathfrak{K}_{\mathrm{ell}}(G)$ and $\mathfrak{e}\in E_{\mathrm{ell}}(G)$,*

$$\bar\iota_G(\gamma_0)^{-1}=\lambda(\mathfrak{e})^{-1}\sum_{\substack{\gamma_H\in\Sigma(H)\ \text{s.t.}\\ \mathfrak{E}:(\mathfrak{e},\gamma_H)\mapsto(\gamma_0,\kappa)}}\bar\iota_H(\gamma_H)^{-1}.$$

*Remark* 8.3.3. We remark that, in the strongly regular case the lemma follows from [KS99, Lem. 7.2.A], which covers twisted endoscopy.

*Proof.* The $Z(\mathbb{Q})$-equivariance was observed above. Since the right vertical map is surjective in (8.3.1.2) by Lemma 7.3.2, the containment of $\Sigma\mathfrak{K}_{\mathrm{ell}}(G)$ in the image of $\mathfrak{E}$ reduces to the case for $G_1$, which is proved in [Kot86, Lem. 9.7] (since $G_{1,\mathrm{der}}=G_{1,\mathrm{sc}}$). Finally the equality asserted in the lemma follows from [Lab04, Cor. IV.3.6]. □

**8.3.4.** Recall that $K^p$ is a neat subgroup and that $\mathfrak{X}=(Z(\mathbb{A}_f)\cap K)\cdot Z(\mathbb{R})$. In particular

$$(8.3.4.1)\qquad \mathfrak{X}\cap G_{\mathrm{der}}(\overline{\mathbb{Q}})=\{1\}.$$

Let $\mathcal{E}\Sigma^{\sim}_{\mathrm{ell},\mathfrak{X}}(G)$ denote the quotient set of $\mathcal{E}\Sigma^{\sim}_{\mathrm{ell}}(G)$ by the obvious multiplication of $\mathfrak{X}_{\mathbb{Q}} = \mathfrak{X} \cap Z(\mathbb{Q})$. Likewise, define $\Sigma_{\mathfrak{X}}(H)$ and $\Sigma\mathfrak{K}_{\mathrm{ell},\mathfrak{X}}(G)$ from $\Sigma(H)$ and $\Sigma\mathfrak{K}_{\mathrm{ell}}(G)$. Let

$$\widetilde{\mathfrak{E}}_{\mathfrak{X}} : \mathcal{E}\Sigma^{\sim}_{\mathrm{ell},\mathfrak{X}}(G) \to \Sigma\mathfrak{K}_{\mathrm{ell},\mathfrak{X}}(G) \cup \{\emptyset\}$$

denote the map $\mathfrak{E}$ pre-composed with (8.3.1.1) modulo the $\mathfrak{X}_{\mathbb{Q}}$-action.

**Corollary 8.3.5.** *The image of $\widetilde{\mathfrak{E}}_{\mathfrak{X}}$ is equal to $\Sigma\mathfrak{K}_{\mathrm{ell},\mathfrak{X}}(G)$ if $G$ is quasi-split and contains $\Sigma\mathfrak{K}_{\mathrm{ell},\mathfrak{X}}(G)$ in general. Moreover for fixed $(\gamma_0, \kappa)$ and $\mathfrak{e}$,*

$$\bar{\iota}_G(\gamma_0)^{-1} = \lambda(\mathfrak{e})^{-1} \sum_{\substack{\gamma_H \in \Sigma_{\mathfrak{X}}(H) \text{ s.t.} \\ \mathfrak{E}_{\mathfrak{X}} : (\mathfrak{e}, \gamma_H) \mapsto (\gamma_0, \kappa)}} \bar{\iota}_H(\gamma_H)^{-1}.$$

*Proof.* The only non-trivial point is to deduce the equality from Lemma 8.3.2. Let $\mathfrak{e} = (H, \mathcal{H}, s, \eta) \in E_{\mathrm{ell}}(G)$, $\gamma_H, \gamma'_H \in H(\mathbb{Q})_{(G,H)\text{-reg}}$. Suppose that $\mathfrak{E}$ maps both $(\mathfrak{e}, \gamma_H)$ and $(\mathfrak{e}, \gamma'_H)$ to $(\gamma_0, \kappa)$. What we need to show is that, if $\gamma'_H$ is stably conjugate to $z\gamma_H$ for some $z \in \mathfrak{X}$, then $\gamma'_H$ is stably conjugate to $\gamma_H$.

By assumption, $\mathfrak{E}$ maps $(\mathfrak{e}, z\gamma_H)$ to $(z\gamma_0, \kappa)$. Hence $z\gamma_0$ is stably conjugate to $\gamma_0$. In particular $gz\gamma_0 g^{-1} = \gamma_0$ for some $g \in G(\overline{\mathbb{Q}})$. Thanks to (8.3.4.1) the element $z$ is trivial, so $\gamma'_H$ is stably conjugate to $\gamma_H$ as desired. □

**Lemma 8.3.6.** *Let $f^{H_1}$ be as in §8.3.4. Assume that $\widetilde{\mathfrak{E}}_{\mathfrak{X}}$ maps $(\mathfrak{e}, \gamma_H)$ to $\emptyset$. Then $SO_{\gamma_{H_1}}(f^{H_1}) = 0$ for every lift $\gamma_{H_1} \in H_1(\mathbb{Q})$ of $\gamma_H$.*

*Proof.* Suppose that $SO_{\gamma_{H_1}}(f^{H_1}) \neq 0$. By Lemma 8.2.8 there exists $\gamma_{0,1} \in G_1(\mathbb{A})$ such that $\gamma_{0,1,v}$ is an image of $\gamma_{H_1,v}$ at every place $v$. To show that $(\mathfrak{e}, \gamma_H)$ is not mapped to $\emptyset$ under $\mathfrak{E}$, it suffices to show the existence of $\gamma'_{0,1} \in G_1(\mathbb{Q})$ which is stably conjugate to $\gamma_{0,1}$ in $G_1(\mathbb{Q}_v)$ for every place $v$. Indeed, $(\mathfrak{e}_1, \gamma_{H_1})$ then does not map to $\emptyset$, thus $(\mathfrak{e}, \gamma_H)$ does not either, cf. (8.3.1.2). Thus the proof boils down to the case where $G = G_1$ and $H = H_1$ (with $G_{\mathrm{der}} = G_{\mathrm{sc}}$). Henceforth we will write $\gamma_0$ for $\gamma_{0,1}$, $\gamma_H$ for $\gamma_{H_1}$, and so on.

Let $G^*$ denote a quasi-split inner form of $G$. Write $\gamma_0^* \in G^*(\mathbb{Q})_{\mathrm{ss}}$ for the image of $\gamma_H$ under (7.4.2.1). Now recall that Labesse (see [Lab99, §2.6], with $L = G, H = G^*$) constructs a non-empty subset

$$\mathrm{obs}_{\gamma_0^*}(\gamma_0) \subset \mathfrak{E}(I_{\gamma_0^*}, G^*; \mathbb{A}/\mathbb{Q}) \xlongequal{\text{Cor. 1.7.4}} \mathfrak{K}(I_{\gamma_0^*}/\mathbb{Q})^D,$$

generalizing the construction of Kottwitz in [Kot86].

As in the second paragraph of [Kot90, p. 188], the Chebotarev density theorem implies that the natural map $\mathfrak{K}(I_{\gamma_0^*}/\mathbb{Q}_w)^D \to \mathfrak{K}(I_{\gamma_0^*}/\mathbb{Q})^D$ is surjective for some finite place $w$. In Labesse's construction, if we twist $\gamma_{0,w}$ within its stable conjugacy class by a class $c \in \mathfrak{K}(I_{\gamma_0^*}/\mathbb{Q}_w)^D$, then $\mathrm{obs}_{\gamma_0^*}(\gamma_0)$ gets shifted by the image of $c$ in the abelian group $\mathfrak{K}(I_{\gamma_0^*}/\mathbb{Q})^D$. Hence, if we replace $\gamma_{0,w}$ by a stably conjugate element and keep the components of $\gamma_0$ outside $w$ unchanged, then we may arrange that $0 \in \mathrm{obs}_{\gamma_0^*}(\gamma_0) \subset \mathfrak{K}(I_{\gamma_0^*}/\mathbb{Q})^D$. By [Lab99, Thm. 2.6.3], the $G(\mathbb{A})$-conjugacy class of $\gamma_0$ contains an element of $G(\mathbb{Q})$, which we can take to be $\gamma'_0$. □

**8.3.7.** For each $\mathfrak{e} = (H, \mathcal{H}, s, \eta) \in \mathcal{E}_{\mathrm{ell}}(G)$, set

$$\iota(G, H) := \tau(G)\tau(H)^{-1}\lambda(\mathfrak{e})^{-1} = \tau_{\mathfrak{X}}(G)\tau_{\mathfrak{X}}(H)^{-1}\lambda(\mathfrak{e})^{-1}. \tag{8.3.7.1}$$

Given $\gamma_H \in \Sigma_{\mathrm{ell},\mathfrak{X}}(H)$ we define $\mathrm{Stab}_{\mathfrak{X}}(\gamma_H)$ to be the group of $z \in \mathfrak{X}_{\mathbb{Q}}$ such that $z\gamma_H = \gamma_H$ in $\Sigma_{\mathrm{ell},\mathfrak{X}}(H)$. This group is finite by the same argument as in §7.1.4 showing the finiteness of $\mathrm{Stab}_{\mathfrak{X}}(\gamma_H)$.

Let us introduce the stable analogue of $T_{\mathrm{ell}}$ (see §7.1.6) for $H_1$ with respect to the central character datum $(\mathfrak{X}_{H_1}, \chi_{H_1})$, where $\chi_{H_1} := \chi_1 \lambda_{H_1}^{-1}$. To check that it is indeed a central character datum, note that both $\chi_1$ and $\lambda_{H_1}$ are trivial on $\mathfrak{X}_{H_1,\mathbb{Q}}$ by construction, cf. §8.1.2 and (8.2.1.1). For $h \in \mathcal{H}(H_1(\mathbb{A}), \chi_{H_1}^{-1})$, set

$$ST^{H_1}_{\mathrm{ell},\chi_{H_1}}(h) := \tau_{\mathfrak{X}_{H_1}}(H_1) \sum_{\gamma_{H_1} \in \Sigma_{\mathrm{ell},\mathfrak{X}_{H_1}}(H_1)} |\mathrm{Stab}_{\mathfrak{X}_{H_1}}(\gamma_{H_1})|^{-1} SO_{\gamma_{H_1}}(h). \tag{8.3.7.2}$$

There is no need for the factor $\bar{\iota}(\gamma_{H_1})^{-1}$, which is equal to 1 since $H_1$ has simply connected derived subgroup.

**Lemma 8.3.8.** *For $\gamma_{H_1} \in H_1(\mathbb{Q})_{(G_1,H_1)\text{-reg}}$,*

$$|\mathrm{Stab}_{\mathfrak{X}_{H_1}}(\gamma_{H_1})| = |\mathrm{Stab}_{\mathfrak{X}}(\gamma_H)| \bar{\iota}_H(\gamma_H).$$

*Proof.* It suffices to construct a short exact sequence of groups

$$1 \to (H_{\gamma_H}/H^0_{\gamma_H})(\mathbb{Q}) \to \mathrm{Stab}_{\mathfrak{X}_{H_1}}(\gamma_{H_1}) \to \mathrm{Stab}_{\mathfrak{X}}(\gamma_H) \to 1.$$

The third arrow from the left is the map induced by the projection $\mathfrak{X}_{H_1} \to \mathfrak{X}$ and is clearly surjective. To construct the second arrow, given $\bar{h} \in (H_{\gamma_H}/H^0_{\gamma_H})(\mathbb{Q})$, choose a lift $h \in H_{\gamma_H}(\overline{\mathbb{Q}})$ and a further lift $h_1 \in H_1(\overline{\mathbb{Q}})$. Then $x_1 := h_1\gamma_{H_1}h_1^{-1}\gamma_{H_1}$ belongs to $Z_1(\mathbb{Q})$, and moreover $x_1 \in \mathrm{Stab}_{\mathfrak{X}_{H_1}}(\gamma_{H_1})$ since $h_1\gamma_{H_1}h_1^{-1} = x_1\gamma_{H_1}$. The assignment $\bar{h} \mapsto x_1$ is a well-defined homomorphism. To check this map is injective, suppose $h_1\gamma_{H_1}h_1^{-1} = \gamma_{H_1}$. Then $h_1$ lies in the centralizer of $\gamma_{H_1}$ in $H_1$, which is connected. (To see this, choose a place $v$ such that $G_{1,\mathbb{Q}_v}$ is quasi-split, so that $\gamma_{H_1,v}$ has an image $\gamma_{1,v} \in G_{1,\mathbb{Q}_v}$. Since the centralizer of $\gamma_{1,v}$ is connected, and since $\gamma_{H_1}$ is $(G_1,H_1)$-regular, the same is true for $\gamma_{H_1,v}$ by [Kot86, Lem. 3.2].) Thus the image of $h_1$ in $H_{\gamma_H}$ lies in $H^0_{\gamma_H}$, implying that $\bar{h}$ is trivial.

The composition of the two maps above is clearly trivial. Finally suppose that $x_1 \in \mathrm{Stab}_{\mathfrak{X}_{H_1}}(\gamma_{H_1})$ maps trivially into $\mathrm{Stab}_{\mathfrak{X}}(\gamma_H)$. Then $x_1 \in Z_1(\mathbb{Q})$. To check that $x_1$ comes from $(H_{\gamma_H}/H^0_{\gamma_H})(\mathbb{Q})$, choose $h_1 \in H_1(\overline{\mathbb{Q}})$ such that $h_1\gamma_{H_1}h_1^{-1} = x_1\gamma_{H_1}$. Its image $h$ in $H(\overline{\mathbb{Q}})$ clearly centralizes $\gamma_H$. Writing $\bar{h}$ for the image of $h$ in $(H_{\gamma_H}/H^0_{\gamma_H})(\overline{\mathbb{Q}})$, we see that $\bar{h}$ is $\mathbb{Q}$-rational and maps to $x_1$ by construction. □

*Remark* 8.3.9. In our situation $|\mathrm{Stab}_{\mathfrak{X}}(\gamma_H)| = 1$. Indeed the stabilizer group is a finite subgroup of $K$ via $\mathfrak{X}_{\mathbb{Q}} \subset Z(\mathbb{Q})_K \subset K$, but $K$ has no non-trivial torsion elements as $K$ is neat.

**Theorem 8.3.10.** *Assume that Conjecture 1.8.8 is true (cf. Remark 1.8.10). With $f^{H_1}$ constructed as in §8.3.4 for each $\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(G)$, we have*

$$\sum_i (-1)^i \mathrm{tr}\left(\Phi^m_{\mathfrak{p}} \times (f^p dg^p) \mid \mathbf{H}^i_c(\mathrm{Sh}_{\overline{E}}, \xi)^{K_p}\right) = \sum_{\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(G)} \iota(G,H) ST^{H_1}_{\mathrm{ell},\chi_{H_1}}(f^{H_1})$$

*for all sufficiently large $m$.*

*Proof.* We compute the right hand side as follows.

$$
\begin{aligned}
& \sum_{\mathfrak{e}\in\mathcal{E}_{\mathrm{ell}}(G)} \iota(G,H) ST^{H_1}_{\mathrm{ell},\chi_{H_1}}(f^{H_1}) \\
= \ & \tau_{\mathfrak{X}_H}(G) \sum_{\mathfrak{e}\in\mathcal{E}_{\mathrm{ell}}(G)} \lambda(\mathfrak{e})^{-1} \sum_{\gamma_{H_1}\in\Sigma_{\mathrm{ell},\mathfrak{X}_{H_1}}(H_1)} |\mathrm{Stab}_{\mathfrak{X}_{H_1}}(\gamma_{H_1})|^{-1} SO_{\gamma_{H_1}}(f^{H_1}) \\
= \ & \tau_{\mathfrak{X}_H}(G) \sum_{\mathfrak{e}\in\mathcal{E}_{\mathrm{ell}}(G)} \lambda(\mathfrak{e})^{-1} \sum_{\gamma_H\in\Sigma_{\mathrm{ell},\mathfrak{X}}(H)} \bar{\iota}(\gamma_H)^{-1} SO_{\gamma_{H_1}}(f^{H_1}) \\
= \ & \tau_{\mathfrak{X}}(G) \sum_{(\mathfrak{e},\gamma_H)\in\mathcal{E}\Sigma^{\sim}_{\mathrm{ell},\mathfrak{X}}(G)} \lambda(\mathfrak{e})^{-1}\bar{\iota}(\gamma_H)^{-1} SO_{\gamma_{H_1}}(f^{H_1}) \\
= \ & \tau_{\mathfrak{X}}(G) \sum_{(\gamma_0,\kappa)\in\Sigma\mathfrak{K}_{\mathrm{ell},\mathfrak{X}}(G)} \sum_{\substack{(\mathfrak{e},\gamma_H)\in\mathcal{E}\Sigma^{\sim}_{\mathrm{ell},\mathfrak{X}}(G) \\ \mathfrak{E}_{\mathfrak{X}}:(\mathfrak{e},\gamma_H)\mapsto(\gamma_0,\kappa)}} \lambda(\mathfrak{e})^{-1}\bar{\iota}(\gamma_H)^{-1} SO_{\gamma_{H_1}}(f^{H_1}) \\
= \ & \tau_{\mathfrak{X}}(G) \sum_{\substack{(\gamma_0,\kappa)\in\Sigma\mathfrak{K}_{\mathrm{ell},\mathfrak{X}}(G) \\ \gamma_0:\,\mathbb{R}\text{-elliptic}}} \sum_{\substack{a\in\mathfrak{D}(I_0,G;\mathbb{A}^p_f) \\ [b]\in\mathfrak{D}_n(\gamma_0,G;\mathbb{Q}_p)}} \bar{\iota}(\gamma_0)^{-1} N(\gamma_0,\kappa,a,[b]).
\end{aligned}
$$

In the third, fourth, and fifth lines, $\gamma_{H_1} \in H_1(\mathbb{Q})_{\mathrm{ss}}$ is an arbitrary lift of $\gamma_H \in H(\mathbb{Q})_{\mathrm{ss}}$. Each summand is independent of the choice since $f_1$ transforms under the character $\chi_{H_1}^{-1}$, which is trivial on $Z_1(\mathbb{Q})$.

We justify these equalities. The first equality uses Lemma 8.2.2 and (8.3.7.1). The next one is based on Lemma 8.3.8 and the bijection $\Sigma_{\mathrm{ell},\mathfrak{X}_{H_1}}(H_1) \to \Sigma_{\mathrm{ell},\mathfrak{X}}(H)$ induced by the surjection $H_1(\mathbb{Q}) \to H(\mathbb{Q})$. We also used $|\mathrm{Stab}_{\mathfrak{X}}(\gamma_H)| = 1$, and the vanishing of the summand if $\gamma_{H_1}$ is not $(G_1, H_1)$-regular by Lemma 8.2.8. To continue, the third equality is justified by Lemma 8.2.8 telling us that only $(G, H)$-regular $\gamma_H$ contributes to the sum. The fourth equality follows from Lemma 8.3.6. The last equality is deduced from Lemma 8.2.8 Corollary 8.3.5, and (8.2.7.1), noting that $\gamma_{0,1}$ can be taken from $G_1(\mathbb{Q})$ whenever $SO_{\gamma_{H_1}}(f^{H_1}) \neq 0$ as shown in the proof of Lemma 8.3.6.

The proof is complete as the last expression in the displayed formula is exactly the left hand side of the theorem by §8.1.4. □

**Theorem 8.3.11** (cf. Theorem 1 in the Introduction)**.** *Assume that* $(G, X)$ *is of abelian type. With notation as in Theorem 8.3.10, we have*

$$
\sum_i (-1)^i \mathrm{tr}\left(\Phi^m_{\mathfrak{p}} \times (f^p dg^p) \mid \mathbf{H}^i_c(\mathrm{Sh}_{\overline{E}}, \xi)^{K_p}\right) = \sum_{\mathfrak{e}\in\mathcal{E}_{\mathrm{ell}}(G)} \iota(G,H) ST^{H_1}_{\mathrm{ell},\chi_{H_1}}(f^{H_1})
$$

*for all sufficiently large* $m$.

*Proof.* This follows from Theorems 6.3.6 and 8.3.10. □

## 9. Spectral interpretation

In order to read off spectral information from Theorem 8.3.10, we need to turn the geometric stable distribution into a spectral expansion. After discussing the stable trace formula in §9.1 when the test function is stable cuspidal at $\infty$, we will remark on the prospect for unconditional spectral interpretation in §9.2.

9.1. **The stable trace formula.**

**9.1.1.** Let $G$ be a quasi-split connected reductive group over $\mathbb{Q}$ with a fixed $z$-extension $G_1$. (We do not assume that $G$ is part of a Shimura datum in §9.1.) Let $(\mathfrak{X},\chi)$ be a central character datum such that $\mathfrak{X}\supset A_{Z,\infty}$. For each elliptic endoscopic datum $\mathfrak{e}=(H,\mathcal{H},s,\eta)\in\mathcal{E}_{\mathrm{ell}}(G)$, choose a central extension $H_1$ and define $\mathfrak{e}_1\in\mathcal{E}_{\mathrm{ell}}(G_1)$ as well as characters $\chi_1$ and $\lambda_{H_1}$ as at the start of §8.2. We have a central character datum $(\mathfrak{X}_{H_1},\chi_{H_1})$ with $\chi_{H_1}:=\chi_1\lambda_{H_1}^{-1}$ as in §8.3.7. Let us recall relationships between certain stable distributions on $G(\mathbb{A})$.

**9.1.2.** Set $\mathfrak{X}_0:=A_{Z,\infty}$. Define $\chi_0:A_{Z,\infty}\to\mathbb{C}^\times$ to be the restriction of $\mathfrak{X}$ to $\mathfrak{X}_0$. Then $(\mathfrak{X}_0,\chi_0)$ is a central character datum. Let $S_{\chi_0}=S^G_{\chi_0}$ denote Arthur's stable distribution on $\mathcal{H}(G(\mathbb{A}),\chi_0^{-1})$ inductively defined in [Art02, §9]. Write $S_{\mathrm{disc},\chi_0}$ for the discrete part of $S_{\chi_0}$ (see (7.11) in *loc. cit.*). Define $S_\chi$ and $S_{\mathrm{disc},\chi}$ in terms of $S_{\chi_0}$ and $S_{\mathrm{disc},\chi_0}$ exactly as in (7.1.9.1). The equality defining $S^G_{\chi_0}$ inductively leads to the analogous equality (compare with [Art13, (3.2.3)])

$$S^G_\chi(f)=I^G_\chi(f)-\sum_{\substack{\mathfrak{e}=(H,\mathcal{H},s,\eta)\in\mathcal{E}_{\mathrm{ell}}(G)\\ H\neq G}}\iota(G,H)S^{H_1}_{\chi_{H_1}}(f^{H_1}),\tag{9.1.2.1}$$

where $I^G_\chi$ means either $I^G_{\mathrm{spec},\chi}$ or $I^G_{\mathrm{geom},\chi}$, which are equal, and $f^{H_1}\in\mathcal{H}(H_1(\mathbb{A}),\chi_{H_1}^{-1})$ denotes a Langlands–Shelstad transfer of $f$ (Proposition 7.4.11). If $f_\infty$ is stable cuspidal then $f_\infty^{H_1}$ is also stable cuspidal (possibly trivial). Indeed, this is reduced via $z$-extensions to the case that $G_{\mathrm{der}}=G_{\mathrm{sc}}$, where this fact follows from work of Shelstad and Clozel–Delorme by the argument as in [Kot90, pp. 182–186]. (This argument is also at the basis of constructing $f_\infty^{H_1}$ in §8.2.5.)

Likewise the analogue of (9.1.2.1) holds true with $S_{\mathrm{disc}}$ and $I_{\mathrm{disc}}$ in place of $S$ and $I$. What follows is the stable version of Proposition 7.1.12.

**Lemma 9.1.3.** *Let $f=f^\infty f_\infty\in\mathcal{H}(G(\mathbb{A}),\chi^{-1})$ with $f_\infty$ stable cuspidal. Then*

$$S^G_\chi(f)=S^G_{\mathrm{disc},\chi}(f).$$

*Proof.* This follows from Proposition 7.1.12, which implies that

$$I_{\mathrm{spec},\chi}(f)=I_{\mathrm{disc},\chi}(f)$$

via the inductive definition above. □

**9.1.4.** We are going to state a stabilization of the geometric side. Let $(\mathfrak{X},\chi)$ be a central character datum for $G$. Write $A_{G_{\mathbb{R}}}$ for the maximal $\mathbb{R}$-split torus in $Z_{G_{\mathbb{R}}}$. (In general $A_{G_{\mathbb{R}}}\neq(A_G)_{\mathbb{R}}$.) Consider the following hypotheses:

(H1) $G_{\mathbb{R}}$ contains an elliptic maximal torus,

(H2) $\mathfrak{X}=\mathfrak{X}^\infty\times\mathfrak{X}_\infty$ with $\mathfrak{X}^\infty\subset Z(\mathbb{A}_f)$ and $A_{G_{\mathbb{R}},\infty}\subset\mathfrak{X}_\infty\subset Z(\mathbb{R})$.

The two conditions are satisfied by the groups contributing to the right hand side of Theorem 8.3.10, so (H1) and (H2) are harmless to assume for our purpose.

We adapt the definition of the stable distribution $ST^G_M$ in [Mor10, §5.4] to the case of fixed central character. Let $T_\infty$ be an elliptic maximal torus of $G_{\mathbb{R}}$ and write $T_{\mathrm{sc},\infty}\subset G_{\mathrm{sc},\mathbb{R}}$ for the preimage. Write $G^{\mathrm{cpt}}$ for an inner form of $G_{\mathbb{R}}$ which is anisotropic modulo $A_{G_{\mathbb{R}}}$; such a $G^{\mathrm{cpt}}$ exists by (H1). The Haar measure on $G^{\mathrm{cpt}}(\mathbb{R})$ is always chosen to be compatible with that of $G(\mathbb{R})$. Define

$$\begin{aligned}k(G_{\mathbb{R}})&:=|\mathrm{im}(\mathbf{H}^1(\mathbb{R},T_{\mathrm{sc},\infty})\to H^1(\mathbb{R},T_\infty))|,\\ \overline{v}(G_{\mathbb{R}})&:=e(G^{\mathrm{cpt}})\mathrm{vol}(G^{\mathrm{cpt}}(\mathbb{R})/A_{G_{\mathbb{R}}}(\mathbb{R})^0).\end{aligned}$$

The two numbers depend only on $G_{\mathbb{R}}$ and a Haar measure on $G(\mathbb{R})$.

Let $M_{\mathbb{R}} \subset G_{\mathbb{R}}$ be an $\mathbb{R}$-rational Levi subgroup containing an elliptic maximal torus. (So the torus is anisotropic modulo $A_{M_{\mathbb{R}}}$.) Let $\Pi$ be a discrete series $L$-packet of $G(\mathbb{R})$ with fixed central character $\chi_\infty$ on $\mathfrak{X}$, and write $\Theta_\Pi$ for the associated stable character (either as a function on regular elements or as a distribution on the space of test functions). Let $D^{G_{\mathbb{R}}}_{M_{\mathbb{R}}}$ denote the Weyl discriminant. Write $\Phi^{G_{\mathbb{R}}}_{M_{\mathbb{R}}}(\cdot, \Theta_\Pi)$ for the unique function on the set of elliptic elements in $M(\mathbb{R})$ which extends the function $\gamma \mapsto |D^{G_{\mathbb{R}}}_{M_{\mathbb{R}}}(\gamma)|^{1/2}\Theta_\Pi(\gamma)$ on $M(\mathbb{R}) \cap G(\mathbb{R})_{\mathrm{reg}}$; such an extension exists by [Art89, Lem. 4.2]. Let $f_\infty \in C_c^\infty(G(\mathbb{R}), \chi_\infty^{-1})$. For elliptic elements $\gamma$ in $M(\mathbb{R})$, define

$$S\Phi^{G_{\mathbb{R}}}_{M_{\mathbb{R}}}(\gamma, f_\infty) := (-1)^{\dim A_{M_{\mathbb{R}}}/A_{G_{\mathbb{R}}}} \cdot \overline{v}(M^0_{\mathbb{R},\gamma})^{-1} \frac{k(M_{\mathbb{R}})}{k(G_{\mathbb{R}})} \sum_{\Pi} \Phi^{G_{\mathbb{R}}}_{M_{\mathbb{R}}}(\gamma^{-1}, \Phi_\Pi)\Theta_\Pi(f_\infty),$$

where $M^0_{\mathbb{R},\gamma}$ is the connected centralizer of $\gamma$ in $M_{\mathbb{R}}$, and the sum runs over discrete series $L$-packets with central character $\chi_\infty$. Set $S\Phi^{G_{\mathbb{R}}}_{M_{\mathbb{R}}}(\gamma, f_\infty) = 0$ if $\gamma \in M(\mathbb{R})$ is not elliptic.

Turning back to the global setting, assuming (H1) and (H2) for $G$, let $M$ be a $\mathbb{Q}$-rational Levi subgroup of $G$, which is said to be *$G$-cuspidal* if $M_{\mathbb{R}}$ contains an elliptic maximal torus and if $\dim A_M/A_G = \dim A_{M_{\mathbb{R}}}/A_{G_{\mathbb{R}}}$. This relativizes the notion of cuspidal reductive groups over $\mathbb{Q}$. Note that $M = G$ is always $G$-cuspidal even if $G$ is not cuspidal over $\mathbb{Q}$. Let $f = f^\infty f_\infty$ with $f^\infty \in \mathcal{H}(G(\mathbb{A}_f), (\chi^\infty)^{-1})$ and $f_\infty \in \mathcal{H}(G(\mathbb{R}), \chi_\infty^{-1})$. Denote by $f^\infty_M \in \mathcal{H}(M(\mathbb{A}_f), (\chi^\infty)^{-1})$ the constant term of $f^\infty$ defined by [GKM97, (7.13.2)]. (The same definition works regardless of fixed central character. As explained therein, it is not $f^\infty_M$ itself but its orbital integral that has well-defined values.) If $M$ is $G$-cuspidal, put

$$ST^G_{M,\chi}(f) := \tau_{\mathfrak{X}}(M) \sum_{\gamma} \bar{\iota}_M(\gamma)^{-1} SO_\gamma(f^\infty_M) S\Phi^{G_{\mathbb{R}}}_{M_{\mathbb{R}}}(\gamma, f_\infty),$$

where the sum runs over the set of stable semi-simple conjugacy classes in $M(\mathbb{Q})$. (The summand is zero unless $\gamma$ is elliptic in $M(\mathbb{R})$.) Define $ST^G_{M,\chi}$ to be identically zero if $M$ is not $G$-cuspidal. Finally, define

$$ST^G_\chi(f) := \sum_M |(N_G(M)/M)(\mathbb{Q})|^{-1} ST^G_{M,\chi}(f),$$

where $M$ runs over the set of $G(\mathbb{Q})$-conjugacy classes of $\mathbb{Q}$-rational Levi subgroups.

**Lemma 9.1.5.** *Let $G$ be a quasi-split reductive group over $\mathbb{Q}$ with central character datum $(\mathfrak{X}, \chi)$. Assume (H1) and (H2) above, and let $f = f^\infty f_\infty$ as above with $f_\infty$ stable cuspidal. Then*

$$S^G_\chi(f) = ST^G_\chi(f).$$

*Proof.* As in (9.1.2.1), we have

$$I^G_\chi(f) = \sum_{\mathfrak{e}=(H,\mathcal{H},s,\eta)\in\mathcal{E}_{\mathrm{ell}}(G)} \iota(G,H) S^{H_1}_{\chi_{H_1}}(f^{H_1}).$$

The assertion is trivial if $G$ is a torus. We induct on the semi-simple rank. Then

$$S^{H_1}_{\chi_{H_1}}(f^{H_1}) = ST^{H_1}_{\chi_{H_1}}(f^{H_1})$$

for all $\mathfrak{e}$ such that $H \ncong G$. Indeed, if $H_{1,\mathbb{R}}$ contains no elliptic maximal torus or if $A_{G_{\mathbb{R}}} \subsetneq A_{H_{\mathbb{R}}}$, then the transfer $f^{H_1}_\infty$ vanishes so the equality holds trivially.

Otherwise $H_{1,\mathbb{R}}$ and $\mathfrak{X}_{H_1}$ satisfy the analogue of (H1) and (H2), so the above equality is true by the induction hypothesis.

To conclude, it is enough to show that

$$I^G_\chi(f) = \sum_{\mathfrak{e}=(H,\mathcal{H},s,\eta)\in\mathcal{E}_{\mathrm{ell}}(G)} \iota(G,H) ST^{H_1}_{\chi_{H_1}}(f^{H_1})$$

when $f_\infty$ is a stable cuspidal function. This was proven by Peng [Pen19, Thm. 9.2] without fixed central character. The desired equality follows from it by averaging with respect to $(\mathfrak{X},\chi)$. (When $G$ is cuspidal over $\mathbb{Q}$ with simply connected derived subgroup, Peng's result was obtained in an unpublished manuscript by Kottwitz [Kot, Thm. 5.1], cf. [Mor10, Thm. 5.4.1].) □

## 9.2. **Speculations.**

**9.2.1.** We return to the setting of Theorem 8.3.10 for the compactly supported cohomology of Shimura varieties, where (H1) and (H2) hold true for $G$ and $\mathfrak{X}$ as well as for $H_1$ and $\mathfrak{X}_{H_1}$ contributing non-trivially to the right hand side.

The analogue of Theorem 8.3.10 is expected to be true for the intersection cohomology of the Baily–Borel compactification. Writing $T^{\mathrm{IH}}(\Phi^m_{\mathfrak{p}}, f^p dg^p)$ for the intersection cohomology analogue of $T(\Phi^m_{\mathfrak{p}}, f^p dg^p)$, the conjectural stabilization should have the form (cf. [Kot90, (10.1)])

$$T^{\mathrm{IH}}(\Phi^m_{\mathfrak{p}}, f^p dg^p) = \sum_{\mathfrak{e}\in\mathcal{E}_{\mathrm{ell}}(G)} \iota(G,H) ST^{H_1}_{\chi_{H_1}}(f^{H_1}). \tag{9.2.1.1}$$

The point is that the non-elliptic terms in $ST^{H_1}_{\chi_{H_1}}$ (coming from proper Levi subgroups) should be accounted for exactly by the boundary strata of the Baily–Borel compactification. For non-proper Shimura varieties, (9.2.1.1) is known for certain special orthogonal group and unitary similitude groups in addition to general symplectic groups in [LR92, Mor08, Mor10, Mor11, Zhu18]. On the other hand, Lemmas 9.1.3 and 9.1.5 imply that

$$ST^{H_1}_{\chi_{H_1}}(f^{H_1}) = S^{H_1}_{\mathrm{disc},\chi_{H_1}}(f^{H_1}). \tag{9.2.1.2}$$

Combined with (9.2.1.1), this yields a trace formula for the intersection cohomology in terms of the stable distributions $S^{H_1}_{\mathrm{disc},\chi_{H_1}}$, which are of a spectral nature. Then one can follow Kottwitz [Kot90, §§9–10] to unravel $S^{H_1}_{\mathrm{disc},\chi_{H_1}}(f^{H_1})$ to obtain a conjectural description of the intersection cohomology (in each degree, by purity) as a $G(\mathbb{A}_f)\times \mathrm{Gal}(\overline{E}/E)$-module, in terms of automorphic representations of $G(\mathbb{A})$ and their endoscopic classification; see p. 201 therein.[49] The endoscopic classification for classical groups is worked out in [Art13, Mok15, KMSW14, Taï19], in the quasi-split case and some more. However, little is known for groups of higher rank beyond classical groups, except for partial results on general symplectic and orthogonal groups in [Xu18, Xu21].

[49] We do not reproduce Kottwitz's argument or his conjectural description here. We content ourselves with remarking that the destabilization process in [Kot90] may also be carried out by applying the conjectural stable multiplicity formula, cf. [Art13, Thm. 4.1.2, (4.8.5)]. Still the key computation at $p$ and $\infty$ of [Kot90, §9] is irreplaceable as it reflects the features of test functions at $p$ and $\infty$ specific to the context of Shimura varieties.

**9.2.2.** We return to compactly supported cohomology. In the special case that $G/Z$ is anisotropic over $\mathbb{Q}$, the Shimura variety $\mathrm{Sh}_K$ is proper over $E$ for each $K$. Thus the intersection cohomology coincides with the compactly supported cohomology. In particular $T^{\mathrm{IH}}(\Phi_{\mathfrak{p}}^m, f^p dg^p) = T(\Phi_{\mathfrak{p}}^m, f^p dg^p)$, and the above consideration suggests that

$$ST^{H_1}_{\chi_{H_1}}(f^{H_1}) \overset{?}{=} S^{H_1}_{\mathrm{ell},\chi_{H_1}}(f^{H_1}). \tag{9.2.2.1}$$

We stress that this equality is not intrinsic to $H_1$. Indeed, a quasi-split inner form $G^*$ of $G$ over $\mathbb{Q}$ shares the same elliptic endoscopic data as $G$. When $G^*$ can be promoted to a Shimura datum, (9.2.2.1) would be false for $f^{H_1}$ constructed in the context of the Shimura variety for $G^*$ (since the latter does have a non-empty boundary). Once (9.2.2.1) is verified, we obtain (9.2.1.1), and the preceding paragraph explains how to extract the spectral information for the compactly supported cohomology in this case.

If $G/Z$ is isotropic over $\mathbb{Q}$, then the description of the $G(\mathbb{A}_f) \times \mathrm{Gal}(\overline{E}/E)$-module structure on the compactly supported cohomology is expected to be very complicated. Indeed, this is confirmed by Franke's formula [Fra98] even if the Galois action is forgotten. See also [Lau97] for the case of $\mathrm{GSp}_4$. Moreover, there may be cancellations between different degrees since the compactly supported cohomology need not be pure. We think that it is better to study the intersection cohomology by proving (9.2.1.1) in this case.

**9.2.3.** We end by summarizing the prospect of unconditional results on the cohomology of Shimura varieties associated with $(G, X)$. In the case of abelian type, our main result is that the identity in Theorem 8.3.10 holds unconditionally whenever $K_p$ is a hyperspecial subgroup of $G(\mathbb{Q}_p)$ and $m$ is sufficiently large. When $G/Z$ is anisotropic over $\mathbb{Q}$, in order to make the conjectural description in the style of [Kot90, p. 201] unconditional, the two main missing ingredients are the endoscopic classification of automorphic representations (for $G$ and the groups $H_1$'s contributing to the stabilization) and the equality (9.2.2.1). When $G/Z$ is isotropic, instead of (9.2.2.1), one should attempt to prove (9.2.1.1) by extending the methods of [Mor10] and [Zhu18]. On top of that, the same endoscopic classification is needed to arrive at the final description of cohomology.

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