Counting Points on Igusa Varieties of Hodge Type

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Abstract

Igusa varieties are algebraic varieties that arise in the study of special fibers of Shimura varieties, and have proved useful in the Langlands program via a Langlands–Kottwitz style point-counting formula in the case of PEL type. In this paper we formulate and prove an analogue of the Langlands–Rapoport conjecture for Igusa varieties of Hodge type, building off the work of Kisin and Kisin–Shin–Zhu on the Langlands–Rapoport conjecture for Shimura varieties of abelian type. We then use this description of the points to derive a point-counting formula for Igusa varieties of Hodge type, generalizing the formula in PEL type of Shin.

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1 Introduction

1.1 Context

In this paper we investigate the representations appearing in the cohomology of Igusa varieties, with a view towards applications in the cohomology of Shimura varieties and Langlands program.

A great deal of inspiration comes from the Langlands–Kottwitz method, pioneered by Langlands [Lan73; Lan76; Lan77; Lan79a; Lan79b] and developed further by Kottwitz in [Kot90; Kot92], which uses geometric and group-theoretic techniques to obtain a trace formula for the cohomology of Shimura varieties that can be compared to the automorphic trace formula, which comparison eventually allows us to relate Galois and automorphic representations. The case treated in [Kot90; Kot92] is that of PEL type and hyperspecial level at p. This case is favorable since Shimura varieties have good reduction and represent a moduli problem in terms of abelian varieties with extra structure.

To see ramified representations we must go beyond hyperspecial level at p, and the resulting Shimura varieties have bad reduction. An approach in the case of modular curves (i.e., GL_2) was described in Deligne's letter to Piatetski-Shapiro. This approach was extended to some simple Shimura varieties by Harris–Taylor [HT01], where the role of Igusa varieties became clear; and further developed by Mantovan [Man04; Man05] and Shin [Shi09; Shi10; Shi11; Shi12]. In short, Mantovan's formula [Man05, Thm. 22] allows us to express the cohomology of Shimura varieties in terms of that of Igusa varieties and Rapoport–Zink spaces, with the bad reduction going to the Rapoport–Zink space and the remaining global information going to the Igusa variety. Then a Langlands–Kottwitz style analysis of Igusa varieties [Shi09; Shi10] allows us to draw conclusions about Shimura varieties [Shi11] and Rapoport–Zink spaces [Shi12].

Beyond PEL type, the construction of integral models of Shimura varieties no longer represents moduli problems in a similar fashion, so more work is needed to describe points in the special fiber. In particular, the Langlands–Rapoport conjecture describes the points on the special fiber of more general Shimura varieties in a way that is suitable for counting points. This conjecture has been proven by Kisin [Kis17] for Shimura varieties of abelian type, up to a possible twist in a group action; subsequent work of Kisin–Shin–Zhu [KSZ21] has shown that the twist can be controlled enough to derive the correct point-counting formula.

Igusa varieties and Mantovan's formula have been generalized to Hodge type by Hamacher and Hamacher–Kim [Ham19; HK19]. The goal of the present work is to derive a trace formula for the cohomology of Igusa varieties of Hodge type, analogous to those for Shimura varieties given in [Kot90; Kot92; KSZ21] and generalizing the formula for Igusa varieties of PEL type [Shi09]. This provides an important missing tool for developing our understanding of Shimura varieties of Hodge type, and fits in well with many other recent works in the same thread (cf. §1.3).

1.2 Main results and methods

Our methods draw heavily from [Shi09; Kis17; KSZ21] mentioned above. For readers who are not intimate with these sources, some details are presented at greater length in [MC21].

As in the case of Shimura varieties, Igusa varieties of Hodge type do not admit representable moduli problems for counting points. Our first main theorem, the subject of §3, addresses this problem by establishing an analogue of the Langlands–Rapoport (LR) conjecture for Igusa varieties of Hodge type.

To explain the theorem and put it in the context, we begin by recalling (e.g., [Kis17, Conj. 3.3.7]) that the LR conjecture for Shimura varieties predicts a Frobenius-Hecke equivariant bijection

$$\mathscr{S}_{K_p}(G,X)(\overline{\mathbb{F}}_p) \xrightarrow{\sim} \coprod_{[\phi]:\text{adm.}} I_\phi(\mathbb{Q})\backslash\big(X^p(\phi)\times X_p(\phi)\big), \tag{1.2.1}$$

where $\mathscr{S}_{K_p} = \mathscr{S}_{K_p}(G,X)$ is the special fiber of the Shimura variety associated to a datum (G,X) with hyperspecial level K_p at p and infinite level away from p. On the right hand side, the objects are defined in terms of Galois gerbs; the disjoint union is over conjugacy classes $[\phi]$ of admissible morphisms of certain Galois gerbs. Intuitively each conjugacy class represents an isogeny class on the Shimura variety. The sets $X^p(\phi)$ and $X_p(\phi)$ represent away-from-p isogenies and p-power isogenies respectively preserving the "G-structures"; $X^p(\phi)$ is a right torsor under the finite adelic group $G(\mathbb{A}_f^p)$, and $X_p(\phi)$ is a set with a Frobenius action as recalled below. The group $I_\phi(\mathbb{Q})$ represents self-isogenies, acting on the sets $X^p(\phi)$ and $X_p(\phi)$ on the left.

The main difference between Shimura varieties and Igusa varieties is the structure at p. Igusa varieties Ig_Σ lie over $\mathscr{S}_{K_p}$, and augment the moduli description by restricting to a fixed an isomorphism class Σ of p-divisible groups (with extra structures) and adding the data of a trivialization of the p-divisible group associated to the abelian variety at each point. To formulate an analogue of the LR conjecture for Igusa varieties, we expect this difference to reflect in the set $X_p(\phi)$.

In the case of Shimura varieties we have the affine Deligne–Lusztig set

$$X_p(\phi) \cong X_v(b) = \{g \in G(L)/G(\mathcal{O}_L) : gb\sigma(g)^{-1} \in G(\mathcal{O}_L)v(p)G(\mathcal{O}_L)\}$$

(cf. 3.1.1), where v is a cocharacter of G arising from the Shimura datum, and $b \in G(L)$ is an element essentially recording the Frobenius on the isocrystal associated to a chosen point on the Shimura variety (here $L = \breve{\mathbb{Q}}_p$ is the completion of the maximal unramified extension of $\mathbb{Q}_p$).

For Igusa varieties we replace this $X_v(b)$ by a set $X_p^{\mathrm{Ig}}(\phi)$ (denoted $X_p^{\mathbf{b}}(\phi)$ in the main text), which is a right torsor under the group

$$J_b(\mathbb{Q}_p) = \{g \in G(L) : gb\sigma(g)^{-1} = b\}$$

(cf. 2.2.1). There is a natural "'forgetting trivialization" map $X_p^{\mathrm{Ig}}(\phi) \to X_p(\phi)$. Intuitively, replacing the condition $gb\sigma(g)^{-1} \in G(\mathcal{O}_L)v(p)G(\mathcal{O}_L)$ by the condition $gb\sigma(g)^{-1} = b$ corresponds to fixing an isomorphism class of p-divisible group or Dieudonné module (rather than fixing an isogeny class or an isocrystal); and replacing $G(L)/G(\mathcal{O}_L)$ by $G(L)$ corresponds to adding a trivialization of the p-divisible group (i.e., choosing a basis rather than simply a lattice). We have no need to modify the term $X^p(\phi) \cong G(\mathbb{A}_f^p)$ away from p.

The other change in our LR conjecture for Igusa varieties is to define a notion of **b**-*admissible morphism* (Definition 3.3.1) to replace the admissible morphisms appearing in the LR conjecture. Since Ig_Σ fixes an isomorphism class of p-divisible groups, in particular it fixes an isogeny class, and therefore it lies over a single Newton stratum of $\mathscr{S}_{K_p}$ labeled by a Kottwitz isocrystal, which we denote by $\mathbf{b}$. Restricting to $\mathbf{b}$-admissible morphisms corresponds to restricting to isogeny classes in the $\mathbf{b}$-stratum. Indeed, one of the main ideas of the proof (undertaken in §3.2) is to relate isogeny classes on Ig_Σ and $\mathscr{S}_{K_p}$. Namely, we show that taking the preimage of an isogeny class on $\mathscr{S}_{K_p}$ along the natural map $\mathrm{Ig}_\Sigma \to \mathscr{S}_{K_p}$ gives a bijection between the set of isogeny classes on Ig_Σ and the set of isogeny classes contained in the $\mathbf{b}$-stratum of $\mathscr{S}_{K_p}$ (Corollary 3.2.5). This relation allows us to use the methods of [Kis17] to establish a bijection between the set of isogeny classes on Ig_Σ and the set of conjugacy classes of $\mathbf{b}$-admissible morphisms of Galois gerbs, as well as bijections between each individual isogeny class and its parametrizing set. Thus it is reasonable to formulate (see 3.6.2 below for a comment on relaxing the assumptions in the conjecture):

Conjecture A (LR conjecture for Igusa varieties)**.** *Let (G, X) be a Shimura datum of Hodge type with G unramified at p, and Ig_Σ an associated Igusa variety. Then there exists a $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$-equivariant bijection*

$$\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p) \xrightarrow{\sim} \coprod_{[\phi]:\mathbf{b}\text{-}adm.} I_\phi(\mathbb{Q})\backslash\big(X^p(\phi) \times X_p^{\mathrm{Ig}}(\phi)\big),$$

where the disjoint union ranges over conjugacy classes of **b**-*admissible morphisms.*

Our first goal is to prove the conjecture up to an ambiguity that is harmless for the computation of cohomology. In [Kis17] the LR conjecture for Shimura varieties (1.2.1) is proven only up to possibly twisting the action of $I_\phi(\mathbb{Q})$ on $X^p(\phi) \times X_p(\phi)$, which factors through an action of $I_\phi(\mathbb{A}_f)$, by an inner automorphism $\tau(\phi) \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$ for each ϕ. That is, [Kis17] proves the conjecture with the quotient by the twisted action

$$I_\phi(\mathbb{Q})_{\tau(\phi)}\backslash\big(X^p(\phi) \times X_p(\phi)\big)$$

on the right hand side of (1.2.1). The subscript $\tau(\phi)$ indicates the twist, which may well interfere with point-counting. The problem will go away if the LR conjecture is proved in full strength, namely if $\tau(\phi)$ is shown to be trivial for every ϕ; however this seems infeasible at this time even for Siegel modular varieties.

Here [KSZ21] comes in crucially to show that the family of twisting data $\phi \mapsto \tau(\phi)$ can be taken to satisfy tori-rationality and a certain compatibility for the family to lie in $\Gamma(\mathcal{H})_0$ (see 3.5). This turns out to be enough constraints on $\tau(\phi)$ for them to deduce the expected point-counting formula for Shimura varieties.

We adapt the methods of [KSZ21] to Igusa varieties of Hodge type to prove our first main theorem towards Conjecture A, where the twist τ is an exact analogue of the twist for Shimura varieties; as such, τ will be harmless for point-counting.

Theorem B (cf. Theorem 3.6.1)**.** *In the setting of Conjecture A, there exists a family of tori-rational elements $\tau(\phi)$ lying in $\Gamma(\mathcal{H})_0$ such that there is a $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$-equivariant bijection*

$$\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p) \xrightarrow{\sim} \coprod_{[\phi]:\mathbf{b}\text{-}adm.} I_\phi(\mathbb{Q})_{\tau(\phi)} \backslash \left(X^p(\phi) \times X_p^{\mathrm{Ig}}(\phi) \right).$$

Here is the idea of proof. As already mentioned above, there is a natural bijection between isogeny classes of Ig_Σ and those of $\mathscr{S}_{K_p}$ contained in the $\mathbf{b}$-stratum. Combined with [Kis17] or [KSZ21], this leads to a (non-canonical) bijection between isogeny classes of Ig_Σ and conjugacy classes of $\mathbf{b}$-admissible morphisms. Thus the core problem is to identify each isogeny class of Ig_Σ with the quotient as in the theorem for the corresponding $[\phi]$. The desired bijection for Igusa varieties is not deduced from the main results of [KSZ21] but obtained by delicately reassembling ingredients from *loc. cit.*

From Theorem B, we derive the point-counting trace formula for Igusa varieties of Hodge type in §4. Here ξ is a finite-dimensional representation of G, $\mathscr{L}_\xi$ the associated ℓ-adic local system on Ig_Σ, and $H_c(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)$ the compactly supported ℓ-adic cohomology of Ig_Σ viewed in a suitable Grothendieck group of $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$-modules. Refer to §4 for the notion and notation in the theorem that are not yet defined.

Theorem C (cf. Theorem 4.5.11)**.** *For every ϱ-acceptable function $f \in C_c^\infty(G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p))$, we have*

$$\mathrm{tr}(f \mid H_c(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)) = \sum_{\gamma_0} \sum_{(a,[b_0])} \frac{|\text{Ш}_G(\mathbb{Q}, G_{\gamma_0}^\circ)|}{|(G_{\gamma_0}/G_{\gamma_0}^\circ)(\mathbb{Q})|} \mathrm{vol}\left(I_{\mathfrak{c}}^\circ(\mathbb{Q}) \backslash I_{\mathfrak{c}}^\circ(\mathbb{A}_f) \right) O_{\gamma \times \delta}^{G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)}(f)\, \mathrm{tr}\, \xi(\gamma_0),$$

where γ_0 runs over the set of stable conjugacy classes in $G(\mathbb{Q})$ that are $\mathbb{R}$-elliptic, and $(a, [b_0])$ runs over pairs such that $(\gamma_0, a, [b_0])$ is a Kottwitz parameter. The group $I_{\mathfrak{c}}$ is the inner form of $G_{\gamma_0}^\circ$ associated to the Kottwitz parameter $\mathfrak{c} = (\gamma_0, a, [b_0])$ as in 4.5.4, and $\gamma \times \delta \in G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ are the elements belonging to the classical Kottwitz parameter $(\gamma_0, \gamma, \delta)$ assigned to $\mathfrak{c}$ as in 4.2.18.

Conceptually the argument consists of two main steps. The first step proves a preliminary counting formula as a sum over the so-called Langlands–Rapoport (LR) pairs. To get started, we interpret a sufficiently large class of test functions as correspondences on our Igusa varieties, and use Fujiwara–Varshavsky's trace formula to convert the problem of computing traces of the action on cohomology to the problem of computing fixed points of these correspondences. We can apply our first main theorem above to describe the fixed points of these correspondences, resulting in a preliminary form of our point-counting formula as a linear combination of orbital integrals over the conjugacy classes of LR pairs (ϕ, ε) consisting of $\mathbf{b}$-admissible ϕ and a conjugacy class in $I_\phi(\mathbb{Q})$, cf. Definition 4.2.1. This is undertaken in §4.1 and the early part of §4.5.

The second step is to re-parametrize LR pairs in the more group-theoretic terms of Kottwitz parameters (cf. 4.2.8). For this we adapt the techniques of [KSZ21, §3]. The theory required is quite analogous, but the relevant class of LR pairs is different; instead of their p^n-admissible pairs, we define notions of $\mathbf{b}$-*admissible* and *acceptable* pairs (Definitions 4.2.1 and 4.2.5). Then we need to re-work a substantial part of the theory under these new hypotheses. Fortunately we manage to prove essentially the same results, though the arguments are often quite different. This is undertaken in §§4.2–4.4 and the rest of §4.5.

1.3 Applications

For applications, it is necessary to stabilize our formula. This is work by Bertoloni Meli and Shin [BMS], generalizing [Shi10].

As described in §1.1, we expect our formula to be useful in combination with Mantovan's formula (due to Hamacher–Kim [HK19] in Hodge type) to investigate the cohomology of Shimura varieties and Rapoport–Zink spaces, as has been done to great effect in [HT01; Shi11; Shi12] for the case of PEL type. A particular example is the recent work of Bertoloni Meli and Nguyen [BMN21] who prove the Kottwitz conjecture for a certain class of unitary similitude groups; their method uses the point counting formula for Igusa varieties of PEL type. Therefore our formula in Hodge type is expectedly an important ingredient in extending their results.

Another promising application is to generalize the results of Caraiani–Scholze on torsion cohomology of Shimura varieties of PEL type [CS17; CS19]. A crucial part of their approach transfers problems from a Shimura variety to an associated flag variety via the Hodge–Tate period map. The fibers of this map are essentially Igusa varieties, whose cohomology can be understood thanks to the point-counting formula in the PEL case [Shi09; Shi10]. Thus our second main theorem above is needed to generalize their arguments to Hodge type.

More recently, Kret and Shin [KS23] combined our point-counting formula with automorphic trace formula techniques to give a description of the H^0 cohomology of Igusa varieties in terms of automorphic representations, with an application to the discrete part of the Chai–Oort Hecke orbit conjecture. With the work of d'Addezio and van Hoften [DvH], this settled the conjecture for Hodge-type Shimura varieties under a mild hypothesis.

1.4 Notation

When k is a field, $\bar{k}$ denotes an algebraic closure, and write Gal_k for the full Galois group over k. For each place v of $\mathbb{Q}$, we fix an embedding $i_v : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_v$. Throughout we fix distinct primes p and ℓ as well as algebraic closures $\overline{\mathbb{Q}}_p$, $\overline{\mathbb{F}}_p$, and $\overline{\mathbb{Q}}_\ell$. For $r \in \mathbb{Z}_{\geq 1}$, write $\mathbb{Q}_{p^r}$ for the finite unramified extension of $\mathbb{Q}_p$ of degree r (in $\overline{\mathbb{Q}}_p$). When T is a torus or pro-torus over k, write $X_*(T)$ (resp. $X^*(T)$) for the group of cocharacters (resp. characters) defined over $\bar{k}$. When H is an algebraic group over k, let H° denote the connected component of the identity. For a locally profinite group H, write $C_c^\infty(H)$ for the space of locally constant compactly supported functions on H; such functions will have values in $\overline{\mathbb{Q}}_\ell$ in this paper. Given a finite set S, write $|S|$ or $\#S$ for its cardinality. When X is an object, e.g., a module or an algebraic group, over a base ring R (determined in the context), write $X_{R'}$ for the base change of X from R to R'. Finally we often confuse the set of equivalence or isomorphism classes with a set of representatives in favor of simpler language.

2 Background

2.1 Isocrystals and p-divisible groups

2.1.1 Let $L = \breve{\mathbb{Q}}_p$ the completion of the maximal unramified extension of $\mathbb{Q}_p$ and σ the lift of Frobenius on L (coming from $\breve{\mathbb{Z}}_p = W(\overline{\mathbb{F}}_p)$). An *isocrystal* over $\overline{\mathbb{F}}_p$ is a finite-dimensional vector space V over L equipped with a σ-semilinear bijection $F : V \to V$, which we call its Frobenius map. A morphism of isocrystals is an L-linear map intertwining their Frobenius maps. For G a reductive group over $\mathbb{Q}_p$, an *isocrystal with G-structure* is an exact faithful tensor functor

$$\mathrm{Rep}_{\mathbb{Q}_p}(G) \to \mathrm{Isoc}$$

from the category of finite-dimensional representations of G over $\mathbb{Q}_p$ to the category of isocrystals [RR96, Def. 3.3].

2.1.2 An element $b \in G(L)$ gives rise to an isocrystal with G-structure

$$\begin{aligned} \mathcal{E}_b : \mathrm{Rep}_{\mathbb{Q}_p}(G) &\to \mathrm{Isoc} \\ (V, \varrho) &\mapsto (V_L, \varrho(b)(\mathrm{id}_V \otimes \sigma)), \end{aligned}$$

where $V_L := V \otimes_{\mathbb{Q}_p} L$. The association $b \mapsto \mathcal{E}_b$ identifies the set of isomorphism classes of isocrystals with G-structure with the set of σ-conjugacy classes in $G(L)$, where $b_0, b_1 \in G(L)$ are said to be σ-conjugate if $b_1 = g b_0 \sigma(g)^{-1}$ for some $g \in G(L)$. We denote this common set by $B(G)$, and write $[b] \in B(G)$ for the σ-conjugacy class of $b \in G(L)$. If G is connected, then every element of $B(G)$ has a representative in $G(\mathbb{Q}_{p^r})$ for some finite unramified extension $\mathbb{Q}_{p^r}$ of $\mathbb{Q}_p$ [Kot85, p. 4.3]. Given a cocharacter μ of G over $\overline{\mathbb{Q}}_p$, there is a distinguished finite subset $B(G, \mu)$ of *μ-admissible* classes, defined in [Kot97, §6].

2.1.3 Given $b \in G(L)$, each representation $(V, \varrho) \in \mathrm{Rep}_{\mathbb{Q}_p}(G)$ produces an isocrystal on $V_L = V \otimes_{\mathbb{Q}_p} L$ via $\mathcal{E}_b$. The slope decomposition $V_L = \oplus_{\lambda \in \mathbb{Q}} V_\lambda$ by the Dieudonné-Manin classification determines a fractional cocharacter $\nu_\varrho : \mathbb{D} \to \mathrm{GL}(V_L)$ such that $\mathbb{D}$ acts on V_λ by the character $\lambda \in \mathbb{Q} = X^*(\mathbb{D})$. The *slope homomorphism* (a.k.a. Newton cocharacter) of b is the unique fractional cocharacter $\nu_b : \mathbb{D} \to G$ over L satisfying $\nu_\varrho = \varrho \circ \nu_b$ for all p-adic representations ϱ of G. Tautologically $\nu_\varrho = \nu_{\varrho(b)}$, by considering $\varrho(b) \in \mathrm{GL}(V_L)$.

Alternatively, ν_b can be defined (cf. [Kot85, p. 4.3]) as the unique element of $\mathrm{Hom}_L(\mathbb{D}, G)$ for which there exist $n > 0$ and $c \in G(L)$ such that

- $n\nu_b \in \mathrm{Hom}_L(\mathbb{G}_m, G)$,
- $\mathrm{Int}(c) \circ n\nu_b$ is defined over a finite unramified extension $\mathbb{Q}_{p^n}$ of $\mathbb{Q}_p$, and
- $c(b\sigma)^n c^{-1} = c \cdot n\nu_b(p) \cdot c^{-1} \cdot \sigma^n$ (considered in $G(L) \rtimes \langle \sigma \rangle$).

From this we see how ν_b changes under σ and conjugation:

$$\nu_{\sigma(b)} = \sigma(\nu_b), \qquad \nu_{gb\sigma(g)^{-1}} = \mathrm{Int}(g) \circ \nu_b, \quad g \in G(L).$$

The following lemma states that, to check if an element $g \in G(L)$ commutes with ν_b, it suffices to check on a single faithful representation.

Lemma 2.1.4. *Let $b \in G(L)$, defining an isocrystal with G-structure. Let $\varrho : G \to \mathrm{GL}(V)$ be a faithful p-adic representation, and $(V_L, \varrho(b)\sigma)$ the associated isocrystal. If $g \in G(L)$ (acting via $\varrho(g)$) preserves the slope decomposition of this isocrystal, then g commutes with the slope homomorphism ν_b.*

Proof. We have

$$\varrho \circ \nu_b = \nu_\varrho = \mathrm{Int}(\varrho(g)) \circ \nu_\varrho = \varrho \circ \mathrm{Int}(g) \circ \nu_b,$$

where the first and third equalities are by definition of ν_b, and the second follows from the assumption on g. Since ϱ is a monomorphism, this shows $\nu_b = \mathrm{Int}(\varrho(g))\nu_b$. □

2.1.5 Following [RZ96] (and references therein), we will freely use the notion of p-divisible groups over a general base scheme S equipped with isogenies and quasi-isogenies between them. In particular we consider the isogeny category of p-divisible groups over S in which quasi-isogenies are isomorphisms.

We write $\mathscr{G} \mapsto \mathbb{D}(\mathscr{G})$ for the contravariant Dieudonné module functor, which gives a contravariant equivalence between the category of p-divisible groups over $\overline{\mathbb{F}}_p$ and the category of Dieudonné modules (e.g., [Dem72]). By composing with the functor from Dieudonné modules to isocrystals, we get a contravariant functor $\mathscr{G} \to \mathbb{V}(\mathscr{G})$, which is an equivalence of categories between the isogeny category of p-divisible groups over $\overline{\mathbb{F}}_p$ and the category of isocrystals.

2.1.6 A p-divisible group is *isoclinic* if it has only a single slope (possibly with multiplicity). A *slope filtration* for a p-divisible group $\mathscr{G}$ is a filtration $0 = \mathscr{G}_0 \subset \cdots \subset \mathscr{G}_r = \mathscr{G}$ such that each successive quotient $\mathscr{G}_i / \mathscr{G}_{i-1}$ is isoclinic of slope λ_i with $\lambda_1 > \cdots > \lambda_r$. If it exists, it is unique. A slope filtration always exists for a p-divisible group over a field of positive characteristic, and the filtration splits canonically if the field is perfect [Gro74].

We say $\mathscr{G}$ is *completely slope divisible* if it has a slope filtration such that for each successive quotient X of slope $\lambda = \frac{a}{b}$, the quasi-isogeny $p^{-a}\,\mathrm{Frob}^b : X \to X^{(p^b)}$ is an isogeny. Over $\overline{\mathbb{F}}_p$, this is equivalent to being a direct sum of isoclinic p-divisible groups defined over finite fields [OZ02].

2.2 Acceptable Elements of $J_b(\mathbb{Q}_p)$

2.2.1 For $b \in G(L)$, define an algebraic group J_b (or J_b^G when it is helpful to remember G) over $\mathbb{Q}_p$ by defining its points for a $\mathbb{Q}_p$-algebra R by

$$J_b(R) = \{g \in G(R \otimes_{\mathbb{Q}_p} L) : gb\sigma(g)^{-1} = b\},$$

and define M_b to be the centralizer in G of ν_b. By definition $J_b(\mathbb{Q}_p) \subset M_b(L)$. Define P_b to be the unique parabolic subgroup of G over L such that every (nonzero) root α of A_{M_b} in Lie P_b satisfies $\langle \alpha, \nu \rangle > 0$. The opposite parabolic is denoted by P_b^{op}. Write N_b and N_b^{op} for the unipotent radicals of P_b and P_b^{op}.

Changing b by σ-conjugation in $G(L)$ does not essentially change the situation: if $b_0 = gb_1\sigma(g)^{-1}$, then $M_{b_0} = \mathrm{Int}(g)M_{b_1}$ and we have a canonical isomorphism

$$\begin{aligned} J_{b_1} &\xrightarrow{\sim} J_{b_0} \\ x &\longmapsto gxg^{-1}. \end{aligned} \tag{2.2.2}$$

Since G is quasi-split, we may and will change b inside its σ-conjugacy class to ensure that M_b is defined over $\mathbb{Q}_p$ and that b is decent. (By the proof of [Kot85, Prop. 6.2] b can be σ-conjugated in G such that ν_b is defined over $\mathbb{Q}_p$ and b is basic in $M_b(L)$. A further σ-conjugation in $M_b(L)$ ensures that b is decent without changing ν_b.) Then J_b is the automorphism group of the isocrystal with G-structure defined by b (indeed, the condition $gb\sigma(g)^{-1} = b$ precisely means that g commutes with $b\sigma$), and furthermore J_b is an inner form of M_b. Since $\nu_b \in X_*(M_b)_{\mathbb{Q}}$ is a central fractional cocharacter, ν_b can also be viewed as a central fractional cocharacter of J_b.

2.2.3 Choose a faithful representation $\varrho : G \hookrightarrow \mathrm{GL}(V)$ over $\mathbb{Q}_p$. Then our isocrystal with G-structure associated to b produces an isocrystal $(V_L, \varrho(b)\sigma)$ The group $J_b(\mathbb{Q}_p)$ acts on this isocrystal by linear automorphisms via its natural inclusion in $G(L)$. Write $V_L = \bigoplus_{i=1}^r V_{\lambda_i}$ for the slope decomposition of our isocrystal, with slopes in decreasing order $\lambda_1 > \lambda_2 > \cdots > \lambda_r$.

Definition 2.2.4. Define an element $\delta \in J_b(\mathbb{Q}_p)$ to be *ϱ-acceptable* (or say δ is *ϱ-acceptable with respect to b*) if, regarding $\varrho(\delta) = (\delta_i) \in \prod_i \mathrm{GL}(V_{\lambda_i})$, any eigenvalues e_i of δ_i and e_j of δ_j with $i < j$ (i.e., $\lambda_i > \lambda_j$) satisfy $v_p(e_i) < v_p(e_j)$.

A more general definition can be given for $\delta \in M_b(L)$ only in terms of ν_b (without requiring ν_b be defined over $\mathbb{Q}_p$); in that case $\varrho(\delta) \in \prod_i \mathrm{GL}(V_{\lambda_i})$ since $\varrho(\delta)$ commutes with $\varrho \circ \nu_b$, so the definition can be given by the same condition as above. In particular, if $\nu_b = \nu_{b'}$ for $b, b' \in G(L)$ then δ is ϱ-acceptable for b if and only if it is for b'. (The converse is Lemma 2.2.11 below.) Next we verify that ϱ-acceptability is invariant under σ-conjugacy.

Lemma 2.2.5. *Suppose that $x_0 \in J_{b_1}(\mathbb{Q}_p)$ and $x_1 \in J_{b_0}(\mathbb{Q}_p)$ correspond via (2.2.2). Then x_1 is ϱ-acceptable in $J_{b_1}(\mathbb{Q}_p)$ if and only if x_0 is ϱ-acceptable in $J_{b_0}(\mathbb{Q}_p)$.*

Proof. The isocrystals $(V_L, \varrho(b_1)\sigma)$ and $(V_L, \varrho(b_0)\sigma)$ are isomorphic by $\varrho(g)$; in particular the isomorphism respects the slope decomposition. The isomorphism is equivariant for the $J_{b_1}(\mathbb{Q}_p)$-action on the former isocrystal and the $J_{b_0}(\mathbb{Q}_p)$-action on the latter via (2.2.2). Now the lemma follows from inspecting Definition 2.2.4. □

Definition 2.2.6. Define an element $\delta \in J_b(\mathbb{Q}_p)$ to be *acceptable* if the adjoint action of δ (as an element of $M_b(L)$) on $\mathrm{Lie}\, N_b(L)$ is dilating, i.e., has eigenvalues λ with $|\lambda| > 1$.

If $\delta, \delta' \in J_b(\mathbb{Q}_p)$ are conjugate in $J_b(\overline{\mathbb{Q}}_p)$ then clearly δ is acceptable if and only if δ' is. It is also obvious that the isomorphism (2.2.2) maps acceptable elements in one group to those in the other.

Lemma 2.2.7. *Let b, ϱ be as above. If $\delta \in J_b(\mathbb{Q}_p)$ is ϱ-acceptable then it is acceptable.*

Proof. Since $\varrho(\delta) \in \prod_i \mathrm{GL}(V_{\lambda_i})$ acts on $\mathrm{Lie}\, N_{\varrho(b)}$ by eigenvalues $e_i e_j^{-1}$ with $\lambda_i < \lambda_j$, if $\delta \in J_b(\mathbb{Q}_p)$ is ϱ-acceptable then $\varrho(\delta)$ is acceptable as an element of $J_{\varrho(b)}(\mathbb{Q}_p)$ by definition. On the other hand, $\nu_{\varrho(b)} = \varrho \circ \nu_b$, so ϱ maps P_b into $P_{\varrho(b)}$, inducing an injection $\mathrm{Lie}\, N_b(L) \hookrightarrow \mathrm{Lie}\, N_{\varrho(b)}(L)$. The latter is compatible with the adjoint actions of $J_b(\mathbb{Q}_p)$ and $J_{\varrho(b)}(\mathbb{Q}_p)$ via $\varrho : J_b(\mathbb{Q}_p) \hookrightarrow J_{\varrho(b)}(\mathbb{Q}_p)$. Since $\varrho(\delta)$ is dilating on $\mathrm{Lie}\, N_{\varrho(b)}(L)$, it follows that δ is dilating on $\mathrm{Lie}\, N_b(L)$. □

2.2.8 An important example of an acceptable element is defined as follows. Choose an integer $s \in \mathbb{Z}$ such that $s\nu_b$ is a cocharacter of G_L, i.e., $s\nu_b : \mathbb{D} \to G_L$ factors through $\mathbb{G}_m \to G_L$. In view of our observation on ν_b in 2.2.1, $s\nu_b$ is a central cocharacter of M_b over $\mathbb{Q}_p$, also viewed as a central cocharacter of J_b over $\mathbb{Q}_p$. For each faithful representation $\varrho : G \to \mathrm{GL}(V)$, the slopes λ_i of $\mathcal{E}_b(V, \varrho)$ satisfy $s\lambda_i \in \mathbb{Z}$ (as they correspond to the weight decomposition of V_L by $s\nu_b$). Define a central element

$$fr^s := s\nu_b(p) \in J_b(\mathbb{Q}_p).$$

Lemma 2.2.9. *For s as above, suppose $s > 0$. The element fr^s is ϱ-acceptable for every $\varrho : G \hookrightarrow \mathrm{GL}(V)$. It is also acceptable. Given ϱ and $\delta \in J_b(\mathbb{Q}_p)$, there exists $s_0 \in \mathbb{Z}$ such that $fr^{-s}\delta$ is ϱ-acceptable for every $s \geq s_0$.*

Proof. Write $V_L = \oplus_i V_{\lambda_i}$ as before. Then $\varrho(fr^s)$ acts on V_{λ_i} by $p^{s\lambda_i}$, so fr^s is ϱ-acceptable by definition; the existence of s_0 as in the lemma is also clear from this. The acceptability of fr^s follows from Lemma 2.2.7 (or can be verified directly from the definition). □

Lemma 2.2.10. *Let $\varepsilon \in G(L)$ be a semi-simple element contained in the subset $J_b(\mathbb{Q}_p)$. If ε is ϱ-acceptable as an element of $J_b(\mathbb{Q}_p)$ then $G_\varepsilon \subset M_b$.*

Proof. As in 2.2.3, the isocrystal $(V_L, \varrho(b)\sigma)$ admits a slope decomposition

$$V_L = \bigoplus_i V_{\lambda_i}.$$

Since ε is semi-simple, its action on V_L is diagonalizable. Since $\varepsilon \in J_b(\mathbb{Q}_p)$, it preserves the slope components V_{λ_i}. Thus each V_{λ_i} has a basis of eigenvectors for the action of ε. The ϱ-acceptable condition implies that ε has different eigenvalues on different slope components, so each slope component is a direct sum of full eigenspaces of ε. Now, each $x \in G_\varepsilon$ must preserve the eigenspaces of ε. Since the slope components are direct sums of eigenspaces of ε, we see that x preserves the slope decomposition. By Lemma 2.1.4 this implies $x \in M_b$. □

Lemma 2.2.11. *Let b_0, b_1 be σ-conjugate elements of $G(L)$. Suppose that there is a semi-simple element $\varepsilon \in G(L)$ such that ε lies in both $J_{b_0}(\mathbb{Q}_p)$ and $J_{b_1}(\mathbb{Q}_p)$ and furthermore is ϱ-acceptable with respect to both b_0 and b_1. Then $\nu_{b_0} = \nu_{b_1}$.*

Proof. The isocrystal structures $(V_L, \varrho(b_0)\sigma)$ and $(V_L, \varrho(b_1)\sigma)$ induce two slope decompositions

$$V_L = \bigoplus_i V_{\lambda_i,0} \qquad \text{and} \qquad V_L = \bigoplus_i V_{\lambda_i,1}.$$

Since b_0, b_1 are σ-conjugate, these two isocrystals are isomorphic. In particular we can use the same index set for i, and $\dim V_{\lambda_i,0} = \dim V_{\lambda_i,1}$ for each i.

Consider ε as in the lemma. As in the proof of Lemma 2.2.10, each slope component $V_{\lambda_i,j}$ is a direct sum of full eigenspaces of ε for $j \in \{0,1\}$. Since $\dim V_{\lambda_i,0} = \dim V_{\lambda_i,1}$ and since (by the ϱ-acceptable condition) the eigenspaces appearing in the various slope components must be ordered by the p-adic valuation of the corresponding eigenvalue, this shows that $V_{\lambda_i,0}$ and $V_{\lambda_i,1}$ must consist of the same eigenspaces. That is, $V_{\lambda_i,0} = V_{\lambda_i,1}$ for all slopes λ_i. Thus $\nu_{\varrho(b_0)} = \nu_{\varrho(b_1)}$. Since ϱ is a monomorphism, we conclude that $\nu_{b_0} = \nu_{b_1}$. □

2.3 Igusa Varieties of Siegel Type

We review the case of Siegel type, as it is required for understanding Hodge type. Henceforth we fix a field isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$. Many objects will be decorated with a tick $\bullet'$ to distinguish them from the analogous objects of Hodge type introduced in §2.4.

2.3.1 Let V be a free $\mathbb{Z}$-module of finite rank $2r$ and ψ a perfect symplectic pairing on V. Let $\mathrm{GSp} = \mathrm{GSp}(V)$ denote the corresponding symplectic similitude group over $\mathbb{Z}$. For any ring R, we write $V_R = V \otimes_{\mathbb{Z}} R$. Write $S^\pm$ for the Siegel double space as in [Kis10, (2.1.5)]. This data gives rise to a Siegel Shimura datum $(\mathrm{GSp}, S^\pm)$. Set $K'_p = \mathrm{GSp}(\mathbb{Z}_p) \subset \mathrm{GSp}(\mathbb{Q}_p)$, which is a hyperspecial subgroup. We write $K' = K'_p K'^p$, where $K'^p \subset \mathrm{GSp}(\mathbb{A}_f^p)$ is a sufficiently small compact open subgroup. The corresponding Shimura variety $\mathrm{Sh}_{K'}(\mathrm{GSp}, S^\pm)$, whose canonical model is defined over $\mathbb{Q}$, has a canonical integral model $\mathscr{S}_{K'}(\mathrm{GSp}, S^\pm)$ over $\mathbb{Z}_{(p)}$. The scheme $\mathscr{S}_{K'}(\mathrm{GSp}, S^\pm)$ is a smooth $\mathbb{Z}_{(p)}$-scheme representing the following moduli functor:

$$\begin{aligned} \mathbf{Schemes}_{/\mathbb{Z}_{(p)}} &\longrightarrow \mathbf{Sets} \\ X &\longmapsto \{(A, \lambda, \eta^p_{K'})\}/\sim \end{aligned}$$

where

- A is an abelian scheme over X up to prime-to-p isogeny;
- λ is a weak polarization of A, i.e., a prime-to-p quasi-isogeny $\lambda : A \to A^\vee$ modulo scaling by $\mathbb{Z}_{(p)}^\times$, some multiple of which is a polarization;

- $\eta^p_{K'} \in \Gamma(X, \underline{\mathrm{Isom}}(V_{\mathbb{A}^p_f}, \hat{V}^p(A))/K'^p)$ is a K'^p-level structure, where we regard $\hat{T}^p(A) = \varprojlim_{p\nmid n} A[n]$ and $\hat{V}^p(A) = \hat{T}^p(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ as étale sheaves on X, and define $\underline{\mathrm{Isom}}(V_{\mathbb{A}^p_f}, \hat{V}^p(A))$ to be the étale sheaf of isomorphisms compatible with the pairings induced by ψ and λ up to $\mathbb{A}^{p\times}_f$-scalar; and
- two triples are equivalent $(A_1, \lambda_1, \eta^p_{K',1}) \sim (A_2, \lambda_2, \eta^p_{K',2})$ if there is a prime-to-p quasi-isogeny $A_1 \to A_2$ sending λ_1 to λ_2 and $\eta^p_{K',1}$ to $\eta^p_{K',2}$.

We often abbreviate $\mathrm{Sh}_{K'}(\mathrm{GSp}, S^\pm)$ and $\mathscr{S}_{K'}(\mathrm{GSp}, S^\pm)$ as $\mathrm{Sh}_{K'}$ and $\mathscr{S}_{K'}$. By virtue of the moduli structure, it carries a universal polarized abelian scheme $\mathcal{A}' \to \mathscr{S}_{K'}$. Denote the special fiber by

$$\overline{\mathscr{S}}_{K'} = \overline{\mathscr{S}}_{K'}(\mathrm{GSp}, S^\pm) := \mathscr{S}_{K'} \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p.$$

2.3.2 The universal polarized abelian scheme gives rise to a universal polarized p-divisible group $(\mathcal{A}'[p^\infty], \lambda)$ and hence an isocrystal with GSp-structure over $\overline{\mathscr{S}}_{K'}$.

Fixing a class $\mathbf{b} \in B(\mathrm{GSp})$ whose Newton stratum is non-empty, let (Σ, λ_Σ) be a p-divisible group with GSp-structure of type $\mathbf{b}$, i.e.,

- Σ is a p-divisible group over $\overline{\mathbb{F}}_p$ and
- λ_Σ is a polarization of Σ, such that
- there is an isomorphism $\mathbb{D}(\Sigma) \xrightarrow{\sim} V_{\mathcal{O}_L}$ preserving the pairings induced by λ_Σ and ψ up to scaling by $\mathcal{O}_L^\times$, and taking the Frobenius on $\mathbb{D}(\Sigma)$ to an endomorphism $b\sigma$ on $V_{\mathcal{O}_L}$ with b a representative of the class $\mathbf{b} \in B(\mathrm{GSp})$.

The last point is independent of the choice of the isomorphism. Indeed, a different choice changes b by σ-conjugation by $\mathrm{GSp}(\mathcal{O}_L)$. We define the *Newton stratum*

$$\overline{\mathscr{S}}^{(\mathbf{b})}_{K'} := \{x \in \overline{\mathscr{S}}_{K'} : (\mathcal{A}'_x[p^\infty], \lambda_x) \times_{k(x)} \overline{k(x)} \text{ is isogenous to } (\Sigma, \lambda_\Sigma) \times_{\overline{\mathbb{F}}_p} \overline{k(x)}\},$$

where $k(x)$ is the residue field of x, and $\overline{k(x)}$ denotes its algebraic closure. Then $\overline{\mathscr{S}}^{(\mathbf{b})}_{K'}$ is a locally closed subset of $\overline{\mathscr{S}}_{K'}$, which we promote to a subscheme by taking the reduced subscheme structure. This definition is equivalent to the definition in terms of isocrystals as in 2.4.3.

The *central leaf* $C'_{\Sigma,K'}$ corresponding to (Σ, λ_Σ) is defined by changing "isogenous" to "isomorphic" in the definition of $\overline{\mathscr{S}}^{(\mathbf{b})}_{K'}$ above. Then $C'_{\Sigma,K'}$ is a closed subset of $\overline{\mathscr{S}}^{(\mathbf{b})}_{K'}$, which is smooth when equipped with the reduced subscheme structure, cf. [Man05, Prop. 1].

2.3.3 Now we assume that Σ is completely slope divisible. Such a p-divisible group is guaranteed to exist when $\mathbf{b}$ has a representative over $\mathbb{Q}_{p^r}$ for some r, which is always the case when our group is connected, as GSp is. Then the universal p-divisible group $\mathcal{A}'[p^\infty]$ over $C'_{\Sigma,K'}$, being isomorphic to Σ (over each geometric generic point of $C'_{\Sigma,K'}$), is also completely slope divisible. Let $\mathcal{A}'[p^\infty]^{(i)}$ be the successive quotients of the slope filtration, and define $\mathcal{A}'[p^\infty]^{\mathrm{sp}} = \bigoplus_i \mathcal{A}'[p^\infty]^{(i)}$ to be the associated split p-divisible group, which inherits a polarization from $\mathcal{A}'[p^\infty]$. Denote its p^m-torsion by $\mathcal{A}'[p^m]^{\mathrm{sp}}$.

The *level-m Igusa variety* of Siegel type $\mathrm{Ig}'_{\Sigma,K',m}$ is a smooth $\overline{\mathbb{F}}_p$-scheme, finite étale and Galois over $C'_{\Sigma,K'}$, defined by the moduli problem

$$\mathrm{Ig}'_{\Sigma,K',m}(X) = \Big\{(A, \lambda, \eta^p_{K'}, j_m) : (A, \lambda, \eta^p_{K'}) \in C'_{\Sigma,K'}(X),$$
$$j_m : \Sigma[p^m] \times_{\overline{\mathbb{F}}_p} X \xrightarrow{\sim} \mathcal{A}'[p^m]^{\mathrm{sp}} \times_{C'_{\Sigma,K'}} X\Big\},$$

where j_m is an isomorphism preserving polarizations up to $(\mathbb{Z}/p^m\mathbb{Z})^\times$-scalar, and extending étale locally to any higher level $m' \geq m$.

Define the Igusa variety at infinite level $\mathrm{Ig}'_{\Sigma,K'} := \varprojlim_m \mathrm{Ig}'_{\Sigma,K',m}$, and $\mathscr{J}'_{\Sigma,K'} := \mathrm{Ig}'^{(p^{-\infty})}_{\Sigma,K'}$ its perfection. Then $\mathscr{J}'_{\Sigma,K'}$ is the moduli space over $C'_{\Sigma,K'}$ parametrizing trivializations of the universal p-divisible group; that is, for an $\overline{\mathbb{F}}_p$-algebra R,

$$\mathscr{J}'_{\Sigma,K'}(R) = \left\{(x,j) \,:\, x \in C'_{\Sigma,K'}(R),\ j : \Sigma \times_{\overline{\mathbb{F}}_p} R \xrightarrow{\sim} \mathcal{A}'_x[p^\infty]\right\}$$

where j is an isomorphism preserving polarizations up to scaling [Ham19, Lem. 3.5], [CS17, Def. 4.3.1, Prop. 4.3.8]. The slope filtration splits canonically over a perfect base [Man20, p. 4.1], so we no longer need to impose the splitting on $\mathcal{A}'[p^\infty]$. The group $\mathrm{GSp}(\mathbb{A}_f^p)$ acts on the system of $\mathrm{Ig}'_{\Sigma,K'}$ over varying K', thus also on the system of $\mathscr{J}'_{\Sigma,K'}$, by acting on the level structure $\eta^p_{K'}$. This action is inherited from the Siegel modular variety, since it happens away from p (i.e., it does not interact with the Igusa level structure).

2.3.4 The system $\mathrm{Ig}'_{\Sigma,K'}$ also receives an action of a submonoid $S_b^{\mathrm{GSp}} \subset J_b^{\mathrm{GSp}}(\mathbb{Q}_p)$ which we now define. Recall from 2.3.2 that we have chosen Σ a p-divisible group of type $\mathbf{b}$ admitting an isomorphism $\mathbb{V}(\Sigma) \xrightarrow{\sim} V_L$ which identifies the Frobenius on $\mathbb{V}(\Sigma)$ with $b\sigma$ for a representative b of $\mathbf{b}$. In this setup we get an action of the group $J_b^{\mathrm{GSp}}(\mathbb{Q}_p)$ on Σ by quasi-isogenies. Let $\delta \in J_b^{\mathrm{GSp}}(\mathbb{Q}_p)$, and suppose that δ^{-1} is an isogeny. Regard $\delta = (\delta_i) \in \prod \mathrm{GL}(V_{\lambda_i})$ as in 2.2.3–2.2.4. For each i write $e_i(\delta)$ and $f_i(\delta)$ for the minimal and maximal integers such that

$$\ker p^{f_i(\delta)} \subset \ker \delta_i^{-1} \subset \ker p^{e_i(\delta)}.$$

Then S_b^{GSp} is the submonoid of $J_b^{\mathrm{GSp}}(\mathbb{Q}_p)$ defined by

$$S_b^{\mathrm{GSp}} := \{\delta \in J_b^{\mathrm{GSp}}(\mathbb{Q}_p) : \delta^{-1} \text{ an isogeny, and } f_{i-1}(\delta) \geq e_i(\delta) \text{ for all } i\}.$$

Note that $p^{-1}, fr^{-s} \in S_b^{\mathrm{GSp}}$, for s as in 2.2.8. Furthermore $J_b(\mathbb{Q}_p)$ is generated as a monoid by S_b together with p and fr^s. In other words, every element of $J_b(\mathbb{Q}_p)$ can be translated into S_b^{GSp} by multiplying by high enough powers of p^{-1} and fr^{-s}.

The details of the action of S_b^{GSp} on $\mathrm{Ig}'_{\Sigma,K'}$ are given in [Man05, Lem. 5]. (Our J_b^{GSp}, resp. $\mathrm{Ig}'_{\Sigma,K',m}$ are denoted T_b, resp. $J_{b,m}$ there.) The action of S_b^{GSp} on $\mathrm{Ig}_{\Sigma,K'}$ extends to an action of the full group $J_b^{\mathrm{GSp}}(\mathbb{Q}_p)$ on the perfections $\mathscr{J}'_{\Sigma,K'}$ and on étale cohomology, cf. [CS17, §4.3].

2.4 Igusa Varieties of Hodge Type

2.4.1 Let (G, X) be a Shimura datum of Hodge type: G is a connected reductive $\mathbb{Q}$-group, which we further assume to be unramified at p; X is a $G(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$; and there exists a closed embedding $G \hookrightarrow \mathrm{GSp}$ which sends X to $S^{\pm}$. Denote by E the reflex field of (G, X). We permanently fix a Hodge embedding $G \hookrightarrow \mathrm{GSp}$. By composing with the standard embedding $\mathrm{GSp} \subset \mathrm{GL}(V)$, we obtain the embedding $\varrho : G \hookrightarrow \mathrm{GL}(V)$.

Each $x \in X$ is associated with $h_x : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$ by definition. Each h_x gives rise to a cocharacter $\mu_x = \mu_{h_x} : \mathbb{G}_m \to G_{\mathbb{C}}$, cf. [KMPS22, p. 1.2.3]. Denote by $\{\mu_X\}$ the canonical $G(\mathbb{C})$-conjugacy class of such cocharacters, which is also viewed as a conjugacy class of cocharacters of $G_{\overline{\mathbb{Q}}_p}$ via the fixed isomorphism $\mathbb{C} \cong \mathbb{Q}_p$. Unless specified, write μ for a representative of $\{\mu_X\}$.

As $G_{\mathbb{Q}_p}$ is unramified, it has a reductive model $G_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$ and corresponding hyperspecial subgroup $K_p = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$. As in [Kis17, p. 1.3.3], there is a $\mathbb{Z}_{(p)}$-lattice $V_{\mathbb{Z}_{(p)}} \subset V_{\mathbb{Q}}$ such that the embedding $G \hookrightarrow \mathrm{GSp}$ is induced by an embedding $G_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{GL}(V_{\mathbb{Z}_{(p)}})$. Enlarging our symplectic space V if necessary, we may assume ψ induces a perfect pairing on $V_{\mathbb{Z}(p)}$, so we can define a hyperspecial subgroup $K'_p = \mathrm{GSp}(V_{\mathbb{Z}_{(p)}})(\mathbb{Z}_p) \subset \mathrm{GSp}(\mathbb{Q}_p)$ which is compatible in the sense that the embedding $G \hookrightarrow \mathrm{GSp}$ takes K_p into K'_p.

For each compact open $K^p \subset G(\mathbb{A}_f^p)$ write $\mathrm{Sh}_K(G, X)$ for the canonical model over E. Given K^p, there is a $K'^p \subset \mathrm{GSp}(\mathbb{A}_f^p)$ such that $K = K_p K^p \subset K' = K'_p K'^p$ and the natural map

$$\mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{K'} \times_{\mathbb{Q}} E$$

is a closed embedding over E. Our integral models are not defined by a moduli problem. Instead, consider the composition

$$\mathrm{Sh}_K(G,X) \to \mathrm{Sh}_{K'} \times_{\mathbb{Q}} E \to \mathscr{S}_{K'} \times_{\mathbb{Z}_{(p)}} \mathcal{O}_{E,(p)},$$

and let $\mathscr{S}_K(G,X)$ be the closure of $\mathrm{Sh}_K(G,X)$ in $\mathscr{S}_{K'} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{E,(p)}$, where $\mathcal{O}_{E,(p)} = \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Then $\mathscr{S}_K(G,X)$ is the canonical integral model of $\mathrm{Sh}_K(G,X)$. (By [Kis10], the normalization of $\mathscr{S}_K(G,X)$ is the canonical integral model, and recently it has been shown [Xu20] that the normalization is unnecessary).

Pulling back the universal abelian scheme $\mathscr{A}' \to \mathscr{S}_{K'}$ along the map $\mathscr{S}_K(G,X) \to \mathscr{S}_{K'}$ we obtain a universal abelian scheme $\mathcal{A} \to \mathscr{S}_K(G,X)$ and p-divisible group $\mathcal{A}[p^\infty]$. Henceforth we usually write Sh_K and $\mathscr{S}_K$ for $\mathrm{Sh}_K(G,X)$ and $\mathscr{S}_K(G,X)$.

2.4.2 For a vector space or module W, we write $W^\otimes$ be the direct sum of all finite combinations of tensor powers, duals, and symmetric and exterior powers of W. Since we include duals, we can identify $W^\otimes$ with $(W^\vee)^\otimes$.

As in [Kis10, p. 2.3.2], our reductive model $G_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$ can be defined as the subgroup of $\mathrm{GL}(V_{\mathbb{Z}_{(p)}})$ stabilizing a finite collection of tensors $\{s_\alpha\} \subset V_{\mathbb{Z}_{(p)}}^\otimes$. Fix such a collection of tensors $\{s_\alpha\}$. Our G-structures essentially amount to transporting the tensors $\{s_\alpha\}$ to all relevant spaces. Let S be an $\mathcal{O}_{E,(p)}$-scheme. Following [Kis17, pp. 1.3.6–10], to each point $x \in \mathscr{S}_K(S)$ we assign a finite set of tensors

$$\{s_{\alpha,\ell,x}\} \subset H^1_{\text{ét}}(\mathcal{A}_x, \mathbb{Q}_\ell)^\otimes \cong V_\ell(\mathcal{A}_x)^\otimes, \qquad \ell \neq p$$

and to each point $x \in \mathscr{S}_K(\overline{\mathbb{F}}_p)$ a finite set of tensors

$$\{s_{\alpha,0,x}\} \subset H^1_{\text{crys}}(\mathcal{A}_x/\mathcal{O}_L)^\otimes \cong \mathbb{D}(\mathcal{A}_x[p^\infty])^\otimes$$

(defined even over $\mathbb{Z}_{p^r}$ for r sufficiently divisible). These tensors are compatible with the original tensors $\{s_\alpha\}$ in the following way. As in [Kis10, p. 3.2.4], for $x \in \mathscr{S}_K(S)$ with associated Siegel data $(\mathcal{A}_x, \lambda, \eta^p_{K'})$, the section

$$\eta^p_{K'} \in \Gamma(S, \underline{\mathrm{Isom}}(V_{\mathbb{A}^p_f}, \hat{V}^p(\mathcal{A}_x))/K'^p)$$

can be promoted to a section

$$\eta^p_K \in \Gamma(S, \underline{\mathrm{Isom}}(V_{\mathbb{A}^p_f}, \hat{V}^p(\mathcal{A}_x))/K^p),$$

and this isomorphism η^p_K takes s_α to $s_{\alpha,\ell,x}$. At p, there is an isomorphism

$$V^\vee_{\mathbb{Z}_{p^r}} \xrightarrow{\sim} \mathbb{D}(\mathcal{A}_x[p^\infty])(\mathbb{Z}_{p^r})$$

taking s_α to $s_{\alpha,0,x}$. Furthermore, the pointwise tensors $s_{\alpha,0,x} \in \mathbb{D}(\mathcal{A}_x[p^\infty])^\otimes$ can be interpolated to global tensors $s_{\alpha,0} \in \mathbb{D}(\mathcal{A}[p^\infty])^\otimes$ as in [Ham19, p. 2.2] and [HP17, p. 3.1.5].

2.4.3 Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and let v be a prime of E over p determined by this embedding. Denote the residue field by $k(v)$, and let $\overline{\mathscr{S}}_K := \mathscr{S}_K \times_{\mathcal{O}_{E,(p)}} k(v)$. We denote again by $\mathcal{A} \to \overline{\mathscr{S}}_K$ the pullback of the abelian scheme $\mathcal{A} \to \mathscr{S}_K$.

By [Lov17] (cf. [HK19, §4]), the p-divisible group $\mathcal{A}[p^\infty]$ equipped with polarization and tensors on $\mathbb{D}(\mathcal{A}[p^\infty])$ gives rise to an isocrystal with G-structure over $\overline{\mathscr{S}}_K$ in the sense of [RR96]. Restricting to a geometric point $\overline{x} \to \overline{\mathscr{S}}_K$ gives an isocrystal with G-structure $\mathbf{b}_x \in B(G)$ depending only on the point $x \in \overline{\mathscr{S}}_K$ underlying $\overline{x}$. Thereby we obtain a Newton stratification

$$\overline{\mathscr{S}}^{(\mathbf{b})}_K = \{x \in \overline{\mathscr{S}}_K : \mathbf{b}_x = \mathbf{b}\}$$

parametrized by classes $\mathbf{b} \in B(G)$, cf. 2.3.2. (Even if $\overline{x}$ is not an $\overline{\mathbb{F}}_p$-point, $\mathbf{b}_x \in B(G)$ is defined via [RR96, Lem. 1.3].) As in Siegel type, $\overline{\mathscr{S}}^{(\mathbf{b})}_K$ are locally closed subsets [RR96, Thm. 3.6] (cf. [Ham19, pp.726–727]), which we equip with the reduced subscheme structure. The $\mathbf{b}$-stratum is non-empty exactly when $\mathbf{b} \in B(G, \mu^{-1})$, which we assume henceforth, where μ is as in 2.4.1.

2.4.4 Fix a Borel B and maximal torus T in $G_{\mathbb{Z}_{(p)}}$. We choose $\mu \in X_*(T)$ to be the dominant cocharacter representing $\{\mu_X\}$, and let $v = \sigma(\mu^{-1})$.

We will need the following data for the definition of Igusa varieties of Hodge type. Fix a σ-conjugacy class $\mathbf{b} \in B(G, \mu^{-1})$ and let $(\Sigma, \lambda_\Sigma, \{s_{\alpha,\Sigma}\})$ be a p-divisible group with G-structure over $\overline{\mathbb{F}}_p$ of type $\mathbf{b}$, namely

- Σ a p-divisible group over $\overline{\mathbb{F}}_p$,
- λ_Σ a polarization of Σ, and
- $\{s_{\alpha,\Sigma}\} \subset \mathbb{D}(\Sigma)^\otimes$ a collection of tensors,

such that there is an isomorphism $\mathbb{D}(\Sigma) \to V_{\mathcal{O}_L}$ preserving the pairings induced by λ_Σ and ψ, taking $s_{\alpha,\Sigma}$ to s_α, and taking the Frobenius on $D(\Sigma)$ to an endomorphism $b\sigma$ on $V_{\mathcal{O}_L}$ with $b \in \mathbf{b}$; here b is well defined up to σ-conjugation by $G(\mathcal{O}_L)$.

Define the central leaf $C_{\Sigma,K}$ corresponding to $(\Sigma, \lambda_\Sigma, \{s_{\alpha,\Sigma}\})$ to be the set of points $x : \operatorname{Spec} \overline{\mathbb{F}}_p \to \overline{\mathscr{S}}_K^{(\mathbf{b})}$ that admit an isomorphism of p-divisible groups with G-structure

$$(\mathcal{A}_x[p^\infty], \lambda_x, \{s_{\alpha,0,x}\}) \cong (\Sigma, \lambda_\Sigma, \{s_{\alpha,\Sigma}\})$$

is closed in $\overline{\mathscr{S}}_K^{(\mathbf{b})}(\overline{\mathbb{F}}_p)$, cf. [HK19, Cor. 4.12]. This defines a closed subset of $\overline{\mathscr{S}}_K^{(\mathbf{b})}$, which is equipped with the reduced subscheme structure and still denoted by $C_{\Sigma,K}$. By [Ham19, Prop. 2.5] $C_{\Sigma,K}$ is smooth over $\overline{\mathbb{F}}_p$. We know from [KS23, Prop. 5.3.5] that $C_{\Sigma,K}$ is nonempty if and only if the σ-conjugacy orbit of b by $G(\mathcal{O}_L)$ meets the double coset $G(\mathbb{Z}_p^{\mathrm{ur}})v(p)G(\mathbb{Z}_p^{\mathrm{ur}})$. We assume this condition from now on, as Igusa varieties will be empty otherwise. Moreover, following [KS23, §5.3, §6.2] we can change b by an element of $G(\mathcal{O}_L)$ by σ-conjugation and choose a sufficiently divisible $r \in \mathbb{Z}_{\geq 1}$ such that

- $(\Sigma, \lambda_\Sigma, \{s_{\alpha,\Sigma}\})$ and $C_{\Sigma,K}$ are defined over $\mathbb{F}_{p^r}$,
- $b \in G(\mathbb{Z}_{p^r})v(p)G(\mathbb{Z}_{p^r})$,
- $\mathbb{F}_{p^r} \supset k(v)$,
- $r\nu_b$ is a cocharacter of G over $\mathbb{Q}_{p^r}$ (so $fr^s \in J_b(\mathbb{Q}_p)$ is defined whenever $r|s$),
- $b\sigma(b)\cdots\sigma^{r-1}(b) = r\nu_b(p)$ (decency equation).

The condition on $(\Sigma, \lambda_\Sigma, \{s_{\alpha,\Sigma}\})$ is not explicitly stated in *loc. cit.*, but it follows from Dieudonné theory and the transportation of structures via the isomorphism $\mathbb{D}(\Sigma) \to V_{\mathcal{O}_L}$ which descends to an isomorphism of $\mathbb{Z}_{p^r}$-modules. For the remainder of the paper, we fix these choices of $\mathbf{b}$, b, $(\Sigma, \lambda_\Sigma, \{s_{\alpha,\Sigma}\})$, and r.

2.4.5 Following [Ham19, p. 4.1], we define the perfect infinite level Igusa variety of Hodge type

$$\mathscr{J}_{\Sigma,K} \subset (\mathrm{Ig}'_{\Sigma,K'} \times_{C'_{\Sigma,K'}} C_{\Sigma,K})^{(p^{-\infty})}$$

to be the locus of points where $j^*(s_{\alpha,0,x}) = s_{\alpha,\Sigma}$. That is, $\mathscr{J}_{\Sigma,K}$ parametrizes isomorphisms $\Sigma \otimes_{\overline{\mathbb{F}}_p} C_{\Sigma,K} \xrightarrow{\sim} \mathcal{A}[p^\infty]$ preserving polarizations (up to scaling) and tensors. The projective limit $\mathscr{J}_\Sigma := \varprojlim_K \mathscr{J}_{\Sigma,K}$ is equipped with a commuting action of $G(\mathbb{A}_f^p)$ and $J_b(\mathbb{Q}_p)$. The former action is inherited from the prime-to-p Hecke action on $\overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p)$. The action of $J_b(\mathbb{Q}_p)$ is restricted from the action of $J_b^{\mathrm{GSp}}(\mathbb{Q}_p)$ on $(\mathrm{Ig}'_{\Sigma,K'})^{(p^{-\infty})}$ in §2.3.3; see [Ham19, Prop. 4.10].

Define (un-perfected) infinite level and finite level Igusa varieties of Hodge type by

$$\mathrm{Ig}_{\Sigma,K} := \mathrm{im}(\mathscr{J}_{\Sigma,K} \to \mathrm{Ig}'_{\Sigma,K'} \times_{C'_{\Sigma,K'}} C_{\Sigma,K}),$$
$$\mathrm{Ig}_{\Sigma,K,m} := \mathrm{im}(\mathscr{J}_{\Sigma,K} \to \mathrm{Ig}'_{\Sigma,K',m} \times_{C'_{\Sigma,K'}} C_{\Sigma,K}).$$

We will work primarily with the Igusa variety with infinite level at m and infinite level away from p

$$\mathrm{Ig}_\Sigma := \varprojlim_{K^p} \mathrm{Ig}_{\Sigma, K_p K^p},$$

and analogously Ig'_Σ for the Siegel version. The Igusa variety Ig_Σ (or system $\mathrm{Ig}_{\Sigma,K,m}$) receives an action of $G(\mathbb{A}_f^p)$ inherited from the Shimura variety, and a commuting action of the submonoid $S_b := S_b^{\mathrm{GSp}} \cap J_b(\mathbb{Q}_p)$ of $J_b(\mathbb{Q}_p)$.

The perfection of Ig_Σ is naturally isomorphic to $\mathscr{J}_\Sigma$ equivariantly for the $G(\mathbb{A}_f^p) \times S_b$-action, by the argument of [CS17, Prop. 4.3.8] (which works in the case of Hodge type, cf. [KS23, Prop. 6.2.1 (3)]. Since perfection does not change $\overline{\mathbb{F}}_p$-points nor étale cohomology, for these purposes it is essentially similar to work with the perfect Igusa variety. An important example is that the natural map $\mathscr{J}_{\Sigma,K} \to \mathscr{J}'_{\Sigma,K'}$ is a closed embedding [Ham19, Prop. 4.10], so we can regard $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ as a subset of $\mathrm{Ig}'_\Sigma(\overline{\mathbb{F}}_p)$. Since $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ acts on $\mathscr{J}_\Sigma$ and its compact support cohomology (defined as a direct limit over K), we see that the S_b-action on the compact support cohomology of Ig_Σ extends to an action of $J_b(\mathbb{Q}_p)$ (necessarily in a unique way), which commutes with the $G(\mathbb{A}_f^p)$-action.

The Igusa variety Ig_Σ is not an honest moduli space, but we can nonetheless attach useful data to its points, which we will refer to as partial moduli data. A point $x \in \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ parametrizes the equivalence class of data

$$(\mathcal{A}_x, \lambda_x, \eta^p, \{s_{\alpha,0,x}\}, j),$$

where $(\mathcal{A}_x, \lambda_x, \eta^p, \{s_{\alpha,0,x}\})$ is the data associated with the image of x in $\overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p)$, and

$$j : \Sigma \xrightarrow{\sim} \mathcal{A}_x[p^\infty]$$

is the Igusa level structure attached to the image of x in the Siegel Igusa variety $\mathrm{Ig}'_\Sigma(\overline{\mathbb{F}}_p)$, an isomorphism of p-divisible groups over $\overline{\mathbb{F}}_p$ respecting polarizations up to scaling and sending $s_{\alpha,\Sigma}$ to $s_{\alpha,0,x}$.

Consider two sets of data to be equivalent if there is a prime-to-p isogeny between the abelian varieties sending one set of data to the other. With this equivalence, points are distinguished by their partial moduli data.

2.4.6 Let ξ be a finite-dimensional representation of G, and $\mathscr{L}_\xi$ the system of sheaves (omitting K, m by abuse of notation) on $\mathrm{Ig}_{\Sigma,K,m}$ defined by ξ. These sheaves are pullbacks of the sheaves on $\overline{\mathscr{S}}_K$ arising from ξ as in [Kot92, §6] or [KSZ21, §1.5.2]. Denote by H_c^i the étale cohomology with compact supports. Define

$$\begin{aligned} H_c^i(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi) &:= \varinjlim_{K^p,m} H_c^i(\mathrm{Ig}_{\Sigma,K,m}, \mathscr{L}_\xi), \\ H_c(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi) &:= \sum_i (-1)^i H_c^i(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi), \end{aligned} \tag{2.4.7}$$

the former as a $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$-module and the latter as an element of $\mathrm{Groth}(G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p))$. Our eventual goal is to a counting point formula for $H_c(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)$.

We can describe the action of $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ on $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ as follows. Let $x \in \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ be a point with associated partial moduli data $(\mathcal{A}_x, \lambda_x, \eta^p, \{s_{\alpha,0,x}\}, j)$. Note that we can regard $J_b(\mathbb{Q}_p)$ as the group of self-quasi-isogenies of Σ.

The action of $G(\mathbb{A}_f^p)$ is inherited from the Shimura variety, and as there, it acts on the level structure η^p: for $g^p \in G(\mathbb{A}_f^p)$, the data associated to $x \cdot g^p$ is

$$(\mathcal{A}_x, \lambda_x, \eta^p \circ g^p, \{s_{\alpha,0,x}\}, j).$$

To describe the action of $g_p \in J_b(\mathbb{Q}_p)$, regard it as a quasi-isogeny $g_p : \Sigma \to \Sigma$, and choose $m \geq 0$ such that $p^m g_p^{-1} : \Sigma \to \Sigma$ is an isogeny. The Igusa level structure $j : \Sigma \xrightarrow{\sim} \mathcal{A}_x[p^\infty]$ allows us to transfer this to $\mathcal{A}_x$. The data associated to $x \cdot g_p$ is

$$(\mathcal{A}_x / j(\ker p^m g_p^{-1}), g_p^* \lambda_x, \eta^p, \{s_{\alpha,0,x}\}, g_p^* j)$$

where $g_p^*\lambda_x$ is the induced polarization; we can take the same level structure η^p because $\mathcal{A}_x$ is unchanged away from p, and the same tensors $\{s_{\alpha,0,x}\}$ because $J_b(\mathbb{Q}_p)$, being a subgroup of $G(L)$, preserves tensors; and

$$g_p^* j : \Sigma \xrightarrow{\sim} \mathcal{A}_{x \cdot g_p}[p^\infty] = \mathcal{A}_x[p^\infty]/j(\ker p^m g_p^{-1})$$

is the unique map making the following diagram commute.

$$\begin{array}{ccc} \Sigma & \xrightarrow{\quad j \quad} & \mathcal{A}_x[p^\infty] \\ {\scriptstyle p^m g_p^{-1}} \downarrow & & \downarrow \\ \Sigma & \xrightarrow[g_p^* j]{} & \mathcal{A}_x[p^\infty]/j(\ker p^m g_p^{-1}) \end{array}$$

Note that the choice of m does not matter because multiplication by p^k induces an isomorphism $A/\ker p^k \to A$, and this gives an equivalence between moduli data for different choices of m.

2.4.8 There is an alternative partial moduli description which admits a simpler description of the group actions, but has the downside that it makes the map to $\overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p)$ more opaque. By [Shi09, Lem. 7.1], we have the following moduli description of the Siegel Igusa variety:

$$\mathrm{Ig}'_\Sigma(\overline{\mathbb{F}}_p) = \{(A, \lambda, \eta^p, j)\}/\sim$$

where

- A is an abelian variety over $\overline{\mathbb{F}}_p$,
- λ is a polarization of A,
- $\eta^p : V_{\mathbb{A}_f^p} \xrightarrow{\sim} \hat{V}^p(A)$ is an isomorphism preserving the pairings induced by ψ and λ up to scaling,
- $j : \Sigma \to A[p^\infty]$ is a quasi-isogeny preserving polarizations up to scaling, and
- two tuples are equivalent if there is an isogeny $A_1 \to A_2$ sending λ_1 to a scalar multiple of λ_2, and sending η_1^p to η_2^p and j_1 to j_2. (It is equivalent to replace "isogeny" here with "quasi-isogeny").

Note the difference that we allow j to be a quasi-isogeny rather than an isomorphism, and equivalence requires only an isogeny $A_1 \to A_2$, rather than a prime-to-p isogeny.

Under this moduli description, $\mathrm{Ig}'_\Sigma(\overline{\mathbb{F}}_p)$ has a right action of $\mathrm{GSp}(\mathbb{A}_f^p) \times J_b^{\mathrm{GSp}}(\mathbb{Q}_p)$ (where we write b again for the image of b in $\mathrm{GSp}(L)$) described by

$$(g^p, g_p) : (A, \lambda, \eta^p, j) \mapsto (A, \lambda, \eta^p \circ g^p, j \circ g_p).$$

As noted in 2.4.5, we can regard $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p) \subset \mathrm{Ig}'_\Sigma(\overline{\mathbb{F}}_p)$, and furthermore this is compatible with the actions of $G(\mathbb{A}_f^p) \times J_b^G(\mathbb{Q}_p) \subset \mathrm{GSp}(\mathbb{A}_f^p) \times J_b^{\mathrm{GSp}}(\mathbb{Q}_p)$. Thus each point of $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ can be associated data (A, λ, η^p, j) as above, with distinct points having distinct data, and we can write the action of $G(\mathbb{A}_f^p) \times J_b^G(\mathbb{Q}_p)$ in a precisely similar way.

2.5 Galois Gerbs

In this section we review Galois gerbs. We refer to §3 of [Kis17] and §2 of [KSZ21] for details omitted here. Let k'/k be a Galois extension of characteristic zero fields. Recall $\mathrm{Gal}_k := \mathrm{Gal}(\overline{k}/k)$.

2.5.1 A k'/k-*Galois gerb* is a pair $(G, \mathfrak{G})$ consisting of a connected linear algebraic group G over k' and an extension of topological groups (giving $G(k')$ the discrete topology)

$$1 \to G(k') \to \mathfrak{G} \to \mathrm{Gal}(k'/k) \to 1$$

satisfying certain technical conditions [Kis17, p. 3.1.1]. Often we use $\mathfrak{G}$ to refer to the whole data $(G, \mathfrak{G})$. By the *kernel* of $\mathfrak{G}$, we mean the algebraic group G, and write $\mathfrak{G}^\Delta$ for it. A $\bar{k}/k$-Galois gerb is simply called a Galois gerb over k.

A k'/k-Galois gerb $\mathfrak{G}$ induces a $\bar{k}/k$-Galois gerb via pullback by $\mathrm{Gal}_k \to \mathrm{Gal}(k'/k)$ and pushout by $G(k') \to G(\bar{k})$. Similarly, for each place v of $\mathbb{Q}$, a Galois gerb $\mathfrak{G}$ over $\mathbb{Q}$ and $i_v : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_v$ induce a Galois gerb $\mathfrak{G}(v)$ over $\mathbb{Q}_v$ by pulling back by $\mathrm{Gal}_{\mathbb{Q}_v} \to \mathrm{Gal}_{\mathbb{Q}}$ and pushing out by $G(\overline{\mathbb{Q}}) \to G(\overline{\mathbb{Q}}_v)$.

An important example is the *neutral* k'/k-*Galois gerb* attached to a linear algebraic group G over k, defined to be the semi-direct product $\mathfrak{G}_G = G(k') \rtimes \mathrm{Gal}(k'/k)$, where $\mathrm{Gal}(k'/k)$ acts on $G(k')$ according to the k-structure on G.

A morphism of k'/k-Galois gerbs $f : \mathfrak{G}_1 \to \mathfrak{G}_2$ is a continuous homomorphism of topological group extensions compatible with the identity map on $\mathrm{Gal}(k'/k)$ such that the restricted map $f|_{G_1(k')} : G_1(k') \to G_2(k')$ is induced by a map of algebraic groups $f^\Delta : G_1 \to G_2$. If $\mathfrak{G}_1, \mathfrak{G}_2$ are Galois gerbs over $\mathbb{Q}$, then f induces a morphism $f(v) : \mathfrak{G}_1(v) \to \mathfrak{G}_2(v)$ of Galois gerbs over $\mathbb{Q}_v$. Two morphism $\mathfrak{G}_1 \to \mathfrak{G}_2$ are *conjugate* if they are related by conjugation by an element of $G_2(k')$. We denote the conjugacy class of a morphism f by $[f]$.

Let $f : \mathfrak{G}_1 \to \mathfrak{G}_2$ be a morphism. Then we can define the automorphism group scheme I_f over k such that for k-algebras R,

$$I_f(R) = \{g \in G_2(k' \otimes_k R) : \mathrm{Int}(g) \circ f_R = f_R\},$$

where $f_R : \mathfrak{G}_{1,R} \to \mathfrak{G}_{2,R}$ induced by f, and $\mathfrak{G}_{i,R}$ is the pushout via $G_i(k') \to G_i(k' \otimes_k R)$ for $i = 1, 2$. In the case that $\mathfrak{G}_2 = \mathfrak{G}_G$ is the neutral Galois gerb attached to a linear algebraic group G, we have the following lemma.

Lemma 2.5.2 ([Kis17, Lem. 3.1.2])**.** *Let $f : \mathfrak{G}_1 \to \mathfrak{G}_G$ be a map of k'/k-Galois gerbs.*

- *The map $I_{f,k'} \to G_{k'}$ given by*

$$I_{f,k'}(R) \hookrightarrow G(k' \otimes_k R) \to G(R), \qquad R : k'\text{-algebra},$$

  identifies $I_{f,k'}$ with the centralizer $Z_G(f^\Delta)$ in $G_{k'}$.

- *The set of morphisms $f' : \mathfrak{G}_1 \to \mathfrak{G}_G$ with $f'^\Delta = f^\Delta$ is in bijection with $Z^1(\mathrm{Gal}(k'/k), I_f(k'))$, via the map sending $e \in Z^1(\mathrm{Gal}(k'/k), I_f(k'))$ to the morphism ef defined such that, if $f(q) = g \rtimes \rho$, we have $ef(q) = e_\rho g \rtimes \rho$. Furthermore, ef is conjugate to $e'f$ exactly when e is cohomologous to e'.*

2.5.3 Following [KSZ21, Def. 2.1.11] we can define the category of pro-k'/k-Galois gerbs, e.g., when $k' = \bar{k}$ or when $k'/k = \mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p$. An object, namely a pro-k'/k-Galois gerb, is a projective limit of k'/k-Galois gerbs over a direct set. We are particularly interested in a morphism from a pro-Galois gerb to a Galois gerb. The preceding discussion extends to this generality, e.g., Lemma 2.5.2 is still valid when $\mathfrak{G}_1$ is a pro-Galois gerb, cf. [KSZ21, p. 2.1.14].

2.5.4 There is a distinguished pro-Galois gerb over $\mathbb{Q}$ called the *quasi-motivic Galois gerb*, and denoted $\mathfrak{Q}$, which plays a central role in point counting. Here we review the essential properties we will need, leaving full details to [Kis17, p. 3.1].

For $L/\mathbb{Q}$ a finite Galois extension, define

$$Q^L = (Res_{L(\infty)/\mathbb{Q}}\mathbb{G}_m \times \mathrm{Res}_{L(p)/\mathbb{Q}}\mathbb{G}_m)/\mathbb{G}_m,$$

where the action of $\mathbb{G}_m$ is the diagonal action, and $L(\infty) = L \cap \mathbb{R}$ and $L(p) = L \cap \mathbb{Q}_p$. This group is equipped with cocharacters $\nu(p)^L$ over $\mathbb{Q}_p$ and $\nu(\infty)^L$ over $\mathbb{R}$ defined by

$$\nu(v)^L : \mathbb{G}_m \to \mathrm{Res}_{L(v)/\mathbb{Q}}\mathbb{G}_m \to Q^L.$$

For L'/L Galois there is a natural map $Q^{L'} \to Q^L$, and the limit is a pro-torus $Q = \varprojlim_L Q^L$ over $\mathbb{Q}$ equipped with a fractional cocharacter $\nu(p) : \mathbb{D} \to Q$ over $\mathbb{Q}_p$ and cocharacter $\nu(\infty) : \mathbb{G}_m \to Q$ over $\mathbb{R}$. The kernel of the quasi-motivic Galois gerb is this pro-torus $\mathfrak{Q}^\Delta = Q$.

The quasi-motivic Galois gerb $\mathfrak{Q}$ comes with a morphism

$$\zeta_v : \mathfrak{G}_v \to \mathfrak{Q}(v)$$

from a distinguished (pro-)Galois gerb over $\mathbb{Q}_v$ for each place v of $\mathbb{Q}$. Let us recall the definition of $\mathbb{Q}_v$. For $\ell \neq p, \infty$, $\mathfrak{G}_\ell = \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ is the trivial Galois gerb, and $\mathfrak{G}_\infty$ is isomorphic to the real Weil group as an extension of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ by $\mathbb{C}^\times$. At p, we have a pro-Galois gerb

$$\mathfrak{G}_p = \varprojlim_L \mathfrak{G}_p^L$$

where L runs over finite Galois extensions of $\mathbb{Q}_p$, and $\mathfrak{G}_p^L$ is the $L/\mathbb{Q}_p$-gerb (induced to $\overline{\mathbb{Q}}_p$) with kernel $\mathfrak{G}_p^{L,\Delta} = \mathbb{G}_m$ given by the fundamental class in $H^2(\mathrm{Gal}(L/\mathbb{Q}_p), L^\times)$. The kernel $\mathfrak{G}_p^\Delta = \mathbb{D}$ is a pro-torus with character group $\mathbb{Q}$. At p and ∞ we have $\zeta_p^\Delta = \nu(p)$ and $\zeta_\infty^\Delta = \nu(\infty)$; these cocharacters will play a role in some later arguments.

The quasi-motivic Galois gerb is also equipped with a distinguished morphism $\psi : \mathfrak{Q} \to \mathfrak{G}_{\mathrm{Res}_{\overline{\mathbb{Q}}/\mathbb{Q}} \mathbb{G}_m}$, which allows us to construct a new morphism as follows. Let T be a torus over $\mathbb{Q}$, and μ a cocharacter of T defined over a finite Galois extension $L/\mathbb{Q}$. Then μ induces a map $\mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_m \to T$, thus also a morphism of Galois gerbs $\mathfrak{G}_{\mathrm{Res}_{L/\mathbb{Q}}} \to \mathfrak{G}_T$. We define a morphism $\psi_{T,\mu} : \mathfrak{Q} \to \mathfrak{G}_T$ by the composition

$$\psi_{T,\mu} : \mathfrak{Q} \xrightarrow{\psi} \mathfrak{G}_{\mathrm{Res}_{\overline{\mathbb{Q}}/\mathbb{Q}}} \longrightarrow \mathfrak{G}_{\mathrm{Res}_{L/\mathbb{Q}}} \longrightarrow \mathfrak{G}_T. \tag{2.5.5}$$

2.5.6 In the definition of the pro-Galois gerb $\mathfrak{G}_p$ above, we can restrict L to run over only finite unramified extensions of $\mathbb{Q}_p$ to define a $\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p$-Galois gerb $\mathfrak{D}$ with kernel $\mathfrak{D}^\Delta = \mathbb{D}$, which becomes $\mathfrak{G}_p$ when induced to $\overline{\mathbb{Q}}_p$. Writing $\sigma \in \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$ for the Frobenius, there is a distinguished element $d_\sigma \in \mathfrak{D}$ lying over σ and such that d_σ^n maps to $p^{-1} \in \mathbb{G}_m = \mathfrak{G}_p^{\mathbb{Q}_{p^n},\Delta}$ under the projection to $\mathfrak{G}_p^{\mathbb{Q}_{p^n}}$. Write $\mathfrak{G}_G^{\mathrm{ur}}$ for the neutral $\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p$-gerb attached to a connected linear algebraic group G over $\mathbb{Q}_p$.

Definition 2.5.7. A morphism $\theta : \mathfrak{G}_p \to \mathfrak{G}_G$ is *unramified* if it is induced by a morphism $\theta^{\mathrm{ur}} : \mathfrak{D} \to \mathfrak{G}_G^{\mathrm{ur}}$. To such a θ we assign an element $b_\theta \in G(\mathbb{Q}_p^{\mathrm{ur}})$ by $\theta^{\mathrm{ur}}(d_\sigma) = b_\theta \rtimes \sigma$.

2.5.8 In the setting of 2.5.6, every morphism $f : \mathfrak{G}_p \to \mathfrak{G}_G$ is conjugate to an unramified morphism [KSZ21, Lem. 2.2.4 (i)]. If θ and θ' are unramified morphisms conjugate to f, then b_θ and $b_{\theta'}$ are σ-conjugate in $G(\mathbb{Q}_p^{\mathrm{ur}})$, so we can associate to f a well-defined class $[b_\theta] \in B(G)$.

Lemma 2.5.9 ([KSZ21, Prop. 2.2.6]). *Let G be a connected linear algebraic $\mathbb{Q}_p$-group, $\theta : \mathfrak{G}_p \to \mathfrak{G}_G$ an unramified morphism, and ν the fractional cocharacter $\theta^{\mathrm{ur},\Delta} : \mathbb{D}_{\mathbb{Q}_p^{\mathrm{ur}}} \to G_{\mathbb{Q}_p^{\mathrm{ur}}}$. Then*

- *$\nu = -\nu_{b_\theta}$, where ν_{b_θ} is the slope homomorphism of b_θ, and*
- *there are natural $\mathbb{Q}_p$-isomorphisms $J_{b_\theta} \cong I_{\theta^{\mathrm{ur}}} \cong I_\theta$.*

2.5.10 Now let (G, X) be a Shimura datum, which determines a conjugacy class $\{\mu_X\}$ of cocharacters of $G_\mathbb{C}$ defined over the reflex field E. Since the reflex field E is unramified over $\mathbb{Q}$ at p, we can choose a cocharacter $\mu : \mathbb{G}_m \to G_{\mathbb{Z}_p^{\mathrm{ur}}}$ whose base change to $\mathbb{C}$ (via $\mathbb{C} \cong \overline{\mathbb{Q}}_p$) belongs to $\{\mu_X\}$, cf. [KSZ21, p. 2.4.1].

The morphisms $\mathfrak{Q} \to \mathfrak{G}_G$ that will be used in our point-counting are required to satisfy an admissibility condition. For $\ell \neq p, \infty$, let $\xi_\ell : \mathfrak{G}_\ell \to \mathfrak{G}_G(\ell)$ be the map sending $\rho \mapsto 1 \rtimes \rho$. At ∞, write $\xi_\infty : \mathfrak{G}_\infty \to \mathfrak{G}_G(\infty)$ for the morphism constructed in [Kis17, p. 3.3.5].

Definition 2.5.11. A morphism $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ is *admissible* if

- for $v \neq p$ (including $v = \infty$), the morphism $\phi(v) \circ \zeta_v : \mathfrak{G}_v \to \mathfrak{G}_G(v)$ is conjugate to ξ_v;

- at p, the morphism $\phi(p) \circ \zeta_p : \mathfrak{G}_p \to \mathfrak{G}_G(p)$ is conjugate to an unramified morphism θ such that $b_\theta \in G(\mathbb{Z}_p^{\text{ur}})\mu(p)^{-1}G(\mathbb{Z}_p^{\text{ur}})$;

as well as satisfying a global condition [KSZ21, Def. 2.4.2 (i)]. (The latter is a correction of the definition in [Kis17, p. 3.3.6]. See [KSZ21, Rem. 2.4.3].)

2.5.12 Let S be a set of places of $\mathbb{Q}$ containing ∞. As in Lemma 2.5.2 we can twist a morphism $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ by a cocycle $e \in Z^1(\mathbb{Q}, I_\phi)$. Define

$$\text{III}_G^S(\mathbb{Q}, I_\phi) \subset H^1(\mathbb{Q}, I_\phi)$$

to be the subset of classes which become trivial in $H^1(\mathbb{Q}_v, I_\phi)$ under the localization maps at $v \in S$ and also trivial under the composite map

$$H^1(\mathbb{Q}, I_\phi) \to H^1_{\text{ab}}(\mathbb{Q}, I_\phi) \to H^1_{\text{ab}}(\mathbb{Q}, G),$$

where H^1_{ab} is the abelianized cohomology; the first map is the abelianization map, cf. [KSZ21, p. 1.1.6], and the second is [KSZ21, (2.6.10.1)]. For our purposes S will be $\{\infty\}$ or $\{p, \infty\}$ or $\{\text{all places of } \mathbb{Q}\}$. In the last case, we also write $\text{III}_G(\mathbb{Q}, I_\phi)$ for $\text{III}_G^S(\mathbb{Q}, I_\phi)$. See [KSZ21, p. 1.2.5] for more details.

Proposition 2.5.13 ([KSZ21, Prop. 2.6.11]). *If ϕ is an admissible morphism and $e \in Z^1(\mathbb{Q}, I_\phi)$, then $e\phi$ is admissible exactly when e lies in $\text{III}_G^\infty(\mathbb{Q}, I_\phi)$.*

2.5.14 For an admissible morphism ϕ we can define a set

$$X^p(\phi) = \{x = (x_\ell) \in G(\overline{\mathbb{A}}_f^p) : \text{Int}(x_\ell) \circ \xi_\ell = \phi(\ell) \circ \zeta_\ell\}.$$

It is non-empty by the admissible condition for $\ell \neq p, \infty$, and furthermore is a $G(\mathbb{A}_f^p)$-torsor under the natural right action. Note that $I_{\xi_\ell}(\mathbb{Q}_\ell) = G(\mathbb{Q}_\ell)$. The set $X^p(\phi)$ is equipped with a natural action of $I_\phi(\mathbb{A}_f^p)$ by left multiplication via $I_\phi(\mathbb{A}_f^p) \subset G(\overline{\mathbb{A}}_f^p)$.

Define a cocycle $\zeta_\phi^{p,\infty} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to G(\overline{\mathbb{A}}_f^p)$ by $\rho \mapsto x\rho(x)^{-1}$ for any choice of $x \in X^p(\phi)$. This does not depend on the choice because any other choice x' is related by $x' = xg$ for some $g \in G(\mathbb{A}_f^p)$. For each $\ell \neq p, \infty$, define the cocycle $\zeta_{\phi,\ell}$ to be the projection of $\zeta_\phi^{p,\infty}$ to the ℓ-component. From the definitions we see that $(\phi(\ell) \circ \zeta_\ell)(\rho) = \zeta_{\phi,\ell}(\rho) \rtimes \rho$ for all $\rho \in \text{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$.

2.5.15 For $x_p \in G(\overline{\mathbb{Q}}_p)$, write $\theta_{x_p} := \text{Int}(x_p^{-1}) \circ \phi(p) \circ \zeta_p$. If θ_{x_p} is unramified then it gives rise to $b_{\theta_{x_p}} \in G(\mathbb{Q}_p^{\text{ur}})$ as in Definition 2.5.7. Define

$$X_p(\phi) = \{x_p \in G(\overline{\mathbb{Q}}_p) : \theta_{x_p} \text{ is unramified and } b_{\theta_{x_p}} \in G(\mathbb{Z}_p^{\text{ur}})\mu^{-1}(p)G(\mathbb{Z}_p^{\text{ur}})\},$$

which is a left $I_\phi(\mathbb{Q}_p)$-set via left multiplication and non-empty by definition if ϕ is admissible.

3 Langlands–Rapoport Conjecture for Igusa Varieties of Hodge Type

3.1 Isogeny Classes on Shimura Varieties of Hodge Type

3.1.1 Following [Kis17, p. 1.4.1] we define a cocharacter v of G and an element $b \in G(L)$. These are needed to define the affine Deligne-Lusztig variety $X_v(b)$, which records the p-part of an isogeny class on the Shimura variety. Recall that we fixed a maximal torus T contained in a Borel subgroup of $G_{\mathbb{Z}_{(p)}}$, a cocharacter $\mu \in X_*(T)$ defined over $\mathbb{Z}_{p^r}$ coming from $\{\mu_X\}$, and $v := \sigma(\mu^{-1})$.

Let $x \in \overline{\mathscr{S}}_K(\overline{\mathbb{F}}_p)$. As in 2.4.2, there is an isomorphism

$$V^\vee_{\mathcal{O}_L} \xrightarrow{\sim} \mathbb{D}(\mathcal{A}_x[p^\infty])(\mathcal{O}_L) \tag{3.1.2}$$

taking s_α to $s_{\alpha,0,x}$, under which the Frobenius action on the Dieudonné module is transported to $b_x\sigma$ with $b_x \in G(L)$. We know from [Kis17, p. 1.4.1] that $b_x \in G(\mathcal{O}_L)v(p)G(\mathcal{O}_L)$. The element b_x is canonical up to σ-conjugacy by $G(\mathcal{O}_L)$.

3.1.3 For $b_0 \in G(L)$, define the *affine Deligne-Lusztig variety*

$$X_v(b_0) := \{g \in G(L)/G(\mathcal{O}_L) : g^{-1}b_0\sigma(g) \in G(\mathcal{O}_L)v(p)G(\mathcal{O}_L)\}$$

as a left $J_{b_0}(\mathbb{Q}_p)$-set via left multiplication, equipped with a Frobenius operator

$$\Phi(g) = (b_0\sigma)^r g = b_0 \cdot \sigma(b_0) \cdots \sigma^{r-1}(b_0) \cdot \sigma^r(g).$$

Following [Kis17, p. 1.4.2] we define a map $X_v(b_x) \to \overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p)$ as follows. Choose a base point $x \in \overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p)$, with associated p-divisible group $\mathcal{A}_x[p^\infty]$. For $g \in X_v(b_x)$, the lattice $g \cdot \mathbb{D}(\mathcal{A}_x[p^\infty]) \subset \mathbb{V}(\mathcal{A}_x[p^\infty])$ is again a Dieudonné module, and corresponds to a p-divisible group $\mathscr{G}_{gx}$ equipped with a quasi-isogeny $\mathcal{A}_x[p^\infty] \to \mathscr{G}_{gx}$.

Let $\mathcal{A}_{gx}$ be the corresponding abelian variety equipped with the polarization and level structure induced from $\mathcal{A}_x$. Sending $g \mapsto \mathcal{A}_{gx}$ with polarization and level structure defines a map $X_v(b_x) \to \overline{\mathscr{S}}_{K'_p}(\mathrm{GSp}, S^{\pm})(\overline{\mathbb{F}}_p)$. By [Kis17, Prop. 1.4.4] this map has a unique lift to a map $i_x : X_v(b_x) \to \overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p)$ satisfying $s_{\alpha,0,x} = s_{\alpha,0,i_x(g)} \in \mathbb{D}(\mathcal{A}_{gx}[p^\infty])$. Extending by the action of $G(\mathbb{A}_f^p)$, we get a map

$$i_x : G(\mathbb{A}_f^p) \times X_v(b_x) \longrightarrow \overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p) \tag{3.1.4}$$

which is equivariant for the action of $G(\mathbb{A}_f^p)$ and intertwines the action of Φ on $X_v(b_x)$ with the action of geometric p^r-Frobenius on $\overline{\mathscr{S}}_{K_p}$.

Definition 3.1.5. For $x \in \overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p)$, the *isogeny class of* x, denoted $\mathscr{I}_x^{\mathrm{Sh}}$, is the image of the map (3.1.4).

3.1.6 Define a $\mathbb{Q}$-group $\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x)$ by the rule $\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x)(R) = (\mathrm{End}_{\mathbb{Q}}(\mathcal{A}_x) \otimes_{\mathbb{Q}} R)^\times$ for $\mathbb{Q}$-algebras R. Let $I_x \subset \mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x)$ denote the subgroup preserving the polarization of $\mathcal{A}_x$ up to scaling and fixing the tensors $s_{\alpha,\ell,x}$ ($\ell \neq p$) and $s_{\alpha,0,x}$.

The level structure $\eta^p : V_{\mathbb{A}_f^p} \xrightarrow{\sim} \hat{V}^p(\mathcal{A}_x)$ away from p identifies the tensors s_α and $(s_{\alpha,\ell,x})_{\ell\neq p}$, and therefore identifies $G(\mathbb{A}_f^p)$ with the subgroup of $\mathrm{GL}(\hat{V}^p(\mathcal{A}_x))$ fixing $(s_{\alpha,\ell,x})_{\ell\neq p}$. Thus the embedding $\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x)(\mathbb{Q}) \hookrightarrow \mathrm{GL}(\hat{V}^p(\mathcal{A}_x))$ induces an embedding $I_x(\mathbb{Q}) \hookrightarrow G(\mathbb{A}_f^p)$, canonical up to conjugation by $G(\mathbb{A}_f^p)$. Similarly, (3.1.2) allows us to identify $J_{b_x}(\mathbb{Q}_p)$ with the subgroup of $\mathrm{GL}(\mathbb{V}(\mathcal{A}_x[p^\infty]))$ fixing the tensors $s_{\alpha,0,x}$ and commuting with the Frobenius. Thus the embedding $\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x)(\mathbb{Q}) \hookrightarrow \mathrm{GL}(\mathbb{V}(\mathcal{A}_x[p^\infty]))$ induces an embedding $I_x(\mathbb{Q}) \hookrightarrow J_{b_x}(\mathbb{Q}_p)$, canonical up to conjugation by $J_{b_x}(\mathbb{Q}_p)$. Thus we have a group embedding

$$I_x(\mathbb{Q}) \hookrightarrow G(\mathbb{A}_f^p) \times J_{b_x}(\mathbb{Q}_p),$$

canonical up to conjugation. We fix such a choice of embedding, through which we take a quotient. By [Kis17, Prop. 2.1.3], the map (3.1.4) induces an injective map

$$i_x : I_x(\mathbb{Q})\backslash\big(G(\mathbb{A}_f^p) \times X_v(b_x)\big) \hookrightarrow \overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p). \tag{3.1.7}$$

Thus the points in the isogeny class $\mathscr{I}_x^{\mathrm{Sh}}$ are parametrized by the set $I_x(\mathbb{Q})\backslash(G(\mathbb{A}_f^p) \times X_v(b_x))$. We can also give a description of isogeny classes in terms of the partial moduli structure, which (in addition to being useful) gives a plain relation to isogenies of the moduli data.

Proposition 3.1.8 ([Kis17, Prop. 1.4.15])**.** *Two points $x, x' \in \overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p)$ lie in the same isogeny class exactly when there is a quasi-isogeny $\mathcal{A}_x \to \mathcal{A}_{x'}$ preserving polarizations up to scaling and such that the induced maps $\mathbb{D}(\mathcal{A}_{x'}[p^\infty]) \to \mathbb{D}(\mathcal{A}_x[p^\infty])$ and $\hat{V}^p(A_x) \to \hat{V}^p(\mathcal{A}_{x'})$ send $s_{\alpha,0,x'}$ to $s_{\alpha,0,x}$ and $s_{\alpha,\ell,x}$ to $s_{\alpha,\ell,x'}$.*

3.2 Isogeny Classes on Igusa Varieties of Hodge Type

Let $x \in \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$. Write x' for the image of x in $\overline{\mathscr{S}}_{K_p}^{(\mathbf{b})}(\overline{\mathbb{F}}_p)$, and often write I_x, b_x for $I_{x'}, b_{x'}$. Since x determines a tensor-preserving quasi-isogeny $j : \Sigma \cong \mathcal{A}_{x'}[p^\infty]$ (see 2.4.8), it induces an isomorphism $J_b \cong J_{b_{x'}}$. So we obtain a canonical embedding

$$I_x(\mathbb{Q}) \hookrightarrow G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$$

from the embedding $I_x(\mathbb{Q}) \hookrightarrow G(\mathbb{A}_f^p) \times J_{b_x}(\mathbb{Q}_p)$ of 3.1.6. We begin by giving the definition and parametrization of an isogeny class in $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$.

Lemma 3.2.1. *For $x \in \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$, the map*

$$\begin{aligned} i_x : I_x(\mathbb{Q})\backslash\big(G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)\big) &\to \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p) \\ (g^p, g_p) &\mapsto x \cdot (g^p, g_p) \end{aligned}$$

is well-defined and injective.

Proof. We need to show that $I_x(\mathbb{Q})$ is the stabilizer of x under the action of $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$. Let (A, λ, η^p, j) be the data associated with x by 2.4.8. Then the data corresponding to $x \cdot (g^p, g_p)$ is $(A, \lambda, \eta^p \circ g^p, j \circ g_p)$. Again in light of 2.4.8, $x = x \cdot (g^p, g_p)$ if and only if there is a quasi-isogeny $\theta : A \to A$ preserving λ up to $\mathbb{Q}^\times$-scaling and sending η^p to $\eta^p \circ g^p$ and sending j to $j \circ g_p$. This means that θ acts as $\eta^p g^p (\eta^p)^{-1}$ on $\hat{V}^p(A)$, and therefore preserves the tensors $s_{\alpha,\ell,x}$. Likewise the θ-action on $\mathbb{V}(\mathcal{A}[p^\infty])$ preserves the tensors $s_{\alpha,0,x}$. Thus we can regard θ as an element of $I_x(\mathbb{Q})$, which is identified with (g^p, g_p) under our embedding, i.e., $(g^p, g_p) = \theta \in I_x(\mathbb{Q})$. Hence the stabilizer of x is $I_x(\mathbb{Q})$ as desired. □

Definition 3.2.2. Let $x \in \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$. The *isogeny class of x*, denoted $\mathscr{I}_x^{\mathrm{Ig}}$, is the image of the map of 3.2.1, i.e., the $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$-orbit of x. By an *isogeny class* $\mathscr{I}^{\mathrm{Ig}}$ we mean a $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$-orbit in $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$.

Hence $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ is partitioned into isogeny classes. The next two lemmas relate isogeny classes on the Igusa variety and the Shimura variety. Since $b_x \in G(\mathcal{O}_L)v(p)G(\mathcal{O}_L)$, we have $1 \in X_v(b_x)$. Hence there is a natural composite map $J_b(\mathbb{Q}_p) \cong J_{b_x}(\mathbb{Q}_p) \to X_v(b_x)$, where the first map is given by x as above, and the second map is induced by the identity map on the ambient group $G(L)$. We extend this map to $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p) \to G(\mathbb{A}_f^p) \times X_v(b_x)$ by the identity map on $G(\mathbb{A}_f^p)$.

Lemma 3.2.3. *Let $x \in \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$, and $x' \in \overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p)$ the image of x under the natural map. The isogeny class maps for x and x' fit into a pullback diagram as below, where the left vertical map is as above.*

$$\begin{array}{ccc} I_x(\mathbb{Q})\backslash\big(G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)\big) & \xrightarrow{i_x} & \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p) \\ \downarrow & & \downarrow \\ I_x(\mathbb{Q})\backslash\big(G(\mathbb{A}_f^p) \times X_v(b_x)\big) & \xrightarrow[i_{x'}]{} & \overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p) \end{array}$$

In particular, an isogeny class on the Igusa variety is the preimage of an isogeny class on the Shimura variety.

Proof. First we show that this diagram commutes. For this we can ignore the quotients by $I_x(\mathbb{Q})$. The whole diagram is $G(\mathbb{A}_f^p)$-equivariant, so it suffices to check commutativity for elements of $J_{b_x}(\mathbb{Q}_p)$. Let $(1, g_p) \in G(\mathbb{A}_f^p) \times J_{b_x}(\mathbb{Q}_p)$. Write $(A_x, \lambda, \eta^p, \{s_{\alpha,0,x}\}, j)$ for the data at x described in 2.4.5. Then the data at $x \cdot g_p$, namely $i_x(1, g_p) \in \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$, is

$$(A_x / j(\ker p^m g_p^{-1}), g_p^* \lambda, \{s_{\alpha,0,x}\}, \eta^p, g_p^* j),$$

and the data at the image in $\overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p)$ (following the right vertical arrow of the diagram) is the same, simply forgetting $g_p^* j$.

Going the other way around the diagram, $i_{x'}(1, g_p) \in \overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p)$ is defined as in 3.1.3 by taking the p-divisible group $\mathscr{G}_{g_p x}$ associated to the Dieudonné module $g_p \cdot \mathbb{D}(\mathcal{A}_x[p^\infty])$, with quasi-isogeny $\mathcal{A}_x[p^\infty] \to \mathscr{G}_{g_p x}$ induced by the isomorphism $g_p^{-1} : g_p \cdot \mathbb{V}(\mathcal{A}_x[p^\infty]) \xrightarrow{\sim} \mathbb{V}(\mathcal{A}_x[p^\infty])$, then the corresponding abelian variety $A_{g_p x}$ with induced polarization and level structure, and the same tensors $s_{\alpha,0,x}$ (as usual note g_p preserves tensors).

Since g_p is (the image of) an element of $J_{b_x}(\mathbb{Q}_p)$, which consists of *self*-quasi-isogenies, we see that $\mathscr{G}_{g_p x}$ is isomorphic to $\mathcal{A}_x[p^\infty]$, and the quasi-isogeny $\mathcal{A}_x[p^\infty] \to \mathscr{G}_{g_p x}$ corresponds via j to g_p^{-1}. Thus, taking m large enough that $p^m g_p^{-1}$ is an isogeny, we can identify $\mathscr{G}_{g_p x}$ with $\mathcal{A}_x[p^\infty]/j(\ker p^m g_p^{-1})$ and $\mathcal{A}_{g_p x}$ with $\mathcal{A}_x/j(\ker p^m g_p^{-1})$, with the induced polarization and away-from-p level structure, the same tensors, and the Igusa level structure $g_p^* j$. This matches the data produced by traversing the diagram the other way, and we see that the diagram commutes.

To show the diagram is a pullback, let $x_1 \in \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ be any point whose image x_1' in $\overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p)$ is in the isogeny class of x'. We want to show that x_1 is in the isogeny class of x.

Since x_1' and x' lie in the same isogeny class, they are related by a pair $(g^p, g_0) \in G(\mathbb{A}_f^p) \times X_v(b_x)$. In particular, the p-divisible groups are related by a quasi-isogeny $\mathcal{A}_x[p^\infty] \to \mathcal{A}_{x_1}[p^\infty]$ corresponding to the isomorphism

$$\mathbb{V}(\mathcal{A}_{x_1}[p^\infty]) \cong g_0 \cdot \mathbb{D}(\mathcal{A}_x[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} \mathbb{V}(\mathcal{A}_x[p^\infty]).$$

Using the Igusa level structures j at x and j_1 at x_1, we can translate this to a quasi-isogeny $\Sigma \to \Sigma$ given by an element $g_p^{-1} \in J_{b_x}(\mathbb{Q}_p)$ (i.e., we define g_p to be the inverse of this quasi-isogeny). We claim that x_1 is related to x by $(g^p, g_p) \in G(\mathbb{A}_f^p) \times J_{b_x}(\mathbb{Q}_p)$.

Indeed, g_p maps to $g_0 \in X_v(b_x)$ because by construction they send $\mathbb{D}(\mathcal{A}_x[p^\infty])$ to the same lattice $g_0 \cdot \mathbb{D}(\mathcal{A}_x[p^\infty]) = g_p \cdot \mathbb{D}(\mathcal{A}_x[p^\infty])$ in $\mathbb{V}(\mathcal{A}_x[p^\infty])$. Thus x_1 and $x \cdot (g^p, g_p)$ have the same image in $\overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p)$, so it only remains to show they have the same Igusa level structure. This is also essentially by construction: g_p was defined to make the left-hand diagram commute, and the Igusa level structure $g_p^* j$ at $x \cdot (g^p, g_p)$ is defined to make the right-hand diagram commute.

$$\begin{array}{ccc} \Sigma & \xrightarrow{j} & \mathcal{A}_x[p^\infty] \\ {\scriptstyle g_p^{-1}}\downarrow & & \downarrow \\ \Sigma & \xrightarrow[j_1]{} & \mathcal{A}_{x_1}[p^\infty] \end{array} \qquad\qquad \begin{array}{ccc} \Sigma & \xrightarrow{j} & \mathcal{A}_x[p^\infty] \\ {\scriptstyle p^m g_p^{-1}}\downarrow & & \downarrow \\ \Sigma & \xrightarrow[g_p^* j]{} & \mathcal{A}_x[p^\infty]/j(\ker p^m g_p^{-1}) \end{array}$$

But $\mathcal{A}_{x_1}[p^\infty] = \mathcal{A}_x[p^\infty]/j(\ker p^m g_p^{-1})$, because x_1 and $x \cdot (g^p, g_p)$ have the same associated abelian variety. Thus (after adjusting the vertical arrows by p^m in the first diagram to make them isogenies), we see that j_1 and $g_p^* j$ both make the same diagram commute. Since there is a unique isomorphism making the diagram commute, we conclude $j_1 = g_p^* j$ as desired. ☐

Recall that in 2.4.4 we have fixed a class $\mathbf{b} \in B(G)$, which specifies the isogeny class of our fixed p-divisible group Σ with G-structure, and therefore the Newton stratum over which our Igusa variety Ig_Σ lies. We also fixed $b \in G(\mathbb{Z}_{p^r}) v(p) G(\mathbb{Z}_{p^r})$ representing $\mathbf{b}$ and specifying the isomorphism class of Σ.

Lemma 3.2.4. *Each isogeny class in $\overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p)$ is contained in a single Newton stratum. The isogeny classes in $\overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p)$ which give rise to a non-empty isogeny class in $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ are precisely those contained in the $\mathbf{b}$-stratum $\overline{\mathscr{S}}_{K_p}^{(\mathbf{b})}(\overline{\mathbb{F}}_p)$.*

Proof. The first assertion follows from the definitions of isogeny classes and Newton strata.

By Lemma 3.2.3, the isogeny classes in $\overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p)$ which give rise to non-empty isogeny classes in $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ are precisely those which intersect the central leaf C_Σ. Thus to prove the second assertion, it suffices to show that every isogeny class in $\overline{\mathscr{S}}_{K_p}^{(\mathbf{b})}(\overline{\mathbb{F}}_p)$ intersects the central leaf. Let $x \in \overline{\mathscr{S}}_{K_p}^{(\mathbf{b})}(\overline{\mathbb{F}}_p)$. Then $[b_x] = \mathbf{b}$, so there exists $g_p \in G(L)$ such that $b = g_p^{-1} b_x \sigma(g_p)$. Since $b \in G(\mathcal{O}_L) v(p) G(\mathcal{O}_L)$, $g_p \in X_v(b_x)$. Then $x' := i_x(1, g_p)$ lies in the isogeny class of x and $b_{x'} = b$ (more precisely their $G(\mathcal{O}_L)$-orbits under σ-conjugation are equal), so $x' \in C_\Sigma(\overline{\mathbb{F}}_p)$. The second assertion is proved. ☐

We summarize the results of this section as follows.

Corollary 3.2.5. *There is a canonical bijection between isogeny classes on $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ and isogeny classes on $\overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p)$ contained in the $\mathbf{b}$-stratum, given by taking preimage under the map $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p) \to \overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p)$.*

Proof. This follows from Lemma 3.2.3 and Lemma 3.2.4. ☐

3.3 b-admissible Morphisms of Galois Gerbs

Recall from 2.4.4 that we fixed a representative $b \in G(L)$ of $\mathbf{b}$ such that $b \in G(\mathbb{Q}_{p^r})$ and satisfying the decency equation $b\sigma(b)\cdots\sigma^{r-1}(b) = r\nu_b(p)$. This allows us to define a morphism $\theta_b^{\mathrm{ur}} : \mathfrak{D} \to \mathfrak{G}^{\mathrm{ur}}$ whose algebraic part $\theta_b^{\mathrm{ur},\triangle} : \mathbb{D}_{\mathbb{Q}_p^{\mathrm{ur}}} \to G_{\mathbb{Q}_p^{\mathrm{ur}}}$ is ν_b and maps $d_\sigma \in \mathfrak{D}$ to $b \in G(\mathbb{Q}_{p^r})$. (Clearly there is at most one such θ_b^{ur}. The decency equation implies that there is indeed such a morphism θ_b^{ur}, and it factors through $\mathfrak{D}_r$ in the notation of [KSZ21, p. 2.2.1].) By pulling back $\theta_b^{\mathrm{ur},\triangle}$ we obtain a morphism of (pro-)Galois gerbs over $\mathbb{Q}_p$:

$$\theta_b : \mathfrak{G}_p \to \mathfrak{G}_G.$$

By construction $b_{\theta_b} = b$.

Definition 3.3.1. A morphism $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ is $\mathbf{b}$-*admissible* if it is admissible (Definition 2.5.11) and if $\phi(p) \circ \zeta_p : \mathfrak{G}_p \to \mathfrak{G}_G(p)$ is conjugate to an unramified morphism θ with $[b_\theta] = \mathbf{b}$.

The notion of $\mathbf{b}$-admissibility is visibly well defined for a conjugacy class $[\phi]$ of admissible morphisms. The condition at p in the definition is equivalent to the condition that $\phi(p) \circ \zeta_p$ is conjugate to θ_b. Define

$$X_p^{\mathbf{b}}(\phi) := \{x_p \in G(\overline{\mathbb{Q}}_p) : \mathrm{Int}(x_p) \circ \theta_b = \phi(p) \circ \zeta_p\},$$

which is non-empty if ϕ is $\mathbf{b}$-admissible. If so, $X_p^{\mathbf{b}}(\phi)$ is a right $J_b(\mathbb{Q}_p)$-torsor under right multiplication. (A priori $J_b(\mathbb{Q}_p)$ is only a subgroup of $G(L)$, and the latter is not contained in $G(\overline{\mathbb{Q}}_p)$, but since b is decent, the proof of [RZ96, Cor. 1.14] shows that $J_b(\mathbb{Q}_p) \subset G(\mathbb{Q}_p^{\mathrm{ur}})$.) Moreover $I_\phi(\mathbb{Q}_p)$ has a natural left multiplication action on $X_p^{\mathbf{b}}(\phi)$ via the inclusion $I_\phi(\mathbb{Q}_p) \subset G(\overline{\mathbb{Q}}_p)$. The identity map on $G(\overline{\mathbb{Q}}_p)$ induces a natural $I_\phi(\mathbb{Q}_p)$-equivariant map $X_p^{\mathbf{b}}(\phi) \to X_p(\phi)$.

If $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ is $\mathbf{b}$-admissible then we define a $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$-set

$$S^{\mathrm{Ig}}(\phi) = I_\phi(\mathbb{Q}) \backslash \big(X^p(\phi) \times X_p^{\mathbf{b}}(\phi)\big), \tag{3.3.2}$$

by letting $I_\phi(\mathbb{Q})$ act through $I_\phi(\mathbb{A}_f^p)$ and $I_\phi(\mathbb{Q}_p)$ on $X^p(\phi)$ and $X_p^{\mathbf{b}}(\phi)$, respectively. The set $S^{\mathrm{Ig}}(\phi)$ will be used to parametrize the isogeny class $\mathscr{I}^{\mathrm{Ig}}$ corresponding to $[\phi]$, up to a twist to be described in §3.5.

3.4 Kottwitz Triples

To make the connection between isogeny classes and admissible morphisms, we import the technique of [Kis17].

3.4.1 A *Kottwitz triple of level* $m \in \mathbb{Z}_{\geq 1}$ is a triple $\mathfrak{k} = (\gamma_0, \gamma, \delta)$ consisting of

- $\gamma_0 \in G(\mathbb{Q})$ a semi-simple element which is elliptic in $G(\mathbb{R})$,
- $\gamma = (\gamma_\ell)_{\ell \neq p} \in G(\mathbb{A}_f^p)$ conjugate to γ_0 in $G(\overline{\mathbb{A}}_f^p)$, and
- $\delta \in G(\mathbb{Q}_{p^m})$ such that γ_0 is conjugate to $\gamma_p = \delta\sigma(\delta)\cdots\sigma^{m-1}(\delta)$ in $G(\overline{\mathbb{Q}}_p)$;

this data is required to satisfy the further condition that

(∗) there is an inner twist I of I_0 over $\mathbb{Q}$ with $I \otimes_{\mathbb{Q}} \mathbb{R}$ anisotropic mod center, and $I \otimes_{\mathbb{Q}} \mathbb{Q}_v$ is isomorphic to I_v as inner twists of I_0 for all finite places v of $\mathbb{Q}$,

where I_0 is the centralizer of γ_0^n in G, and I_ℓ for $\ell \neq p$ is the centralizer of γ_ℓ^n in $G_{\mathbb{Q}_\ell}$, and I_p is a $\mathbb{Q}_p$-group defined on points by

$$I_p = \{g \in G(W(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} R) : g^{-1}\delta\sigma(g) = \delta\}$$

for sufficiently divisible $n \in \mathbb{Z}_{\geq 1}$. This makes sense as all of these groups stabilize for n sufficiently divisible.

A *Kottwitz triple* is an equivalence class of Kottwitz triples of various level, where we take the smallest equivalence relation such that a triple $(\gamma_0, \gamma, \delta)$ of level m is equivalent to the triple $(\gamma_0^n, \gamma^n, \delta)$ of level mn, where $m, n \in \mathbb{Z}_{\geq 1}$. Two Kottwitz triples $\mathfrak{k}, \mathfrak{k}'$ are equivalent if there exist representatives $(\gamma_0, \gamma, \delta), (\gamma_0', \gamma', \delta')$ of the same level m for $\mathfrak{k}, \mathfrak{k}'$ such that (i) γ, γ' are conjugate in $G(\mathbb{A}_f^p)$ and (ii) δ, δ' are σ-conjugate in $G(\mathbb{Q}_{p^m})$. (If so, γ_0, γ_0' are conjugate in $G(\overline{\mathbb{Q}})$ by (i).) Define a Kottwitz triple $(\gamma_0, \gamma, \delta)$ to be $\mathbf{b}$-*admissible* if the σ-conjugacy class of δ is $\mathbf{b}$. This is clearly seen to be preserved under the equivalences above.

3.4.2 Let $x \in \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ and write $\overline{x} \in \overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p)$ for the image of x. We will attach a Kottwitz triple to x on the Igusa variety, following [Kis17, p. 4.4.6]. The same construction assigns a Kottwitz triple to x' on the Shimura variety. Since it is the same construction the triples for x and $\overline{x}$ match by Corollary 3.2.5.

We may assume that the data $(\mathcal{A}_x, \lambda, \{s_{\alpha,\ell,x}\}, \{s_{\alpha,0,x}\})$ are defined over some finite field $\mathbb{F}_{p^m}$. The level structure η^p at x identifies the group $G_{\mathbb{Q}_\ell}$ with the subgroup of $\mathrm{GL}(H^1_{\text{ét}}(\mathcal{A}_x, \mathbb{Q}_\ell))$ fixing the tensors $\{s_{\alpha,\ell,x}\} \subset H^1_{\text{ét}}(\mathcal{A}_x, \mathbb{Q}_\ell)^{\otimes}$, and this allows us to write the geometric p^m Frobenius on $H^1_{\text{ét}}(\mathcal{A}_x, \mathbb{Q}_\ell)$ as an element $\gamma_\ell \in G(\mathbb{Q}_\ell)$. Let $\gamma = (\gamma_\ell) \in G(\mathbb{A}_f^p)$. At p, we similarly have an isomorphism

$$V^\vee_{\mathbb{Z}_{p^m}} \xrightarrow{\sim} \mathbb{D}(\mathcal{A}_x[p^\infty])(\mathbb{Z}_{p^m})$$

which identifies $G_{\mathbb{Z}_{p^m}}$ with the subgroup of $\mathrm{GL}(\mathbb{D}(\mathcal{A}_x[p^\infty])(\mathbb{Z}_{p^m}))$ fixing the tensors $\{s_{\alpha,0,x}\}$, and allows us to write the Frobenius on $\mathbb{D}(\mathcal{A}_x[p^\infty])$ as $\delta\sigma$ for some $\delta \in G(\mathbb{Q}_{p^m})$. With this choice of γ and δ, [Kis17, Cor. 2.3.1, 2.3.5] states that there is an element $\gamma_0 \in G(\mathbb{Q})$ that makes $(\gamma_0, \gamma, \delta)$ a Kottwitz triple, which we denote $\mathfrak{k}(x)$. Since the equivalence class of $\mathfrak{k}(x)$ depends only on the isogeny class $\mathscr{I}^{\mathrm{Ig}}_x$ containing x, we also write $\mathfrak{k}(\mathscr{I}^{\mathrm{Ig}}_x)$ for it. Similarly $\mathfrak{k}(\mathscr{I}^{\mathrm{Sh}})$ denotes the Kottwitz triple (up to equivalence) attached to an isogeny class in $\overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p)$.

We establish a simple compatibility between isogeny classes and their associated Kottwitz triples.

Lemma 3.4.3. *Let $\mathscr{I}^{\mathrm{Sh}}$ be an isogeny class in $\overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p)$. The associated Kottwitz triple $\mathfrak{k}(\mathscr{I}^{\mathrm{Sh}})$ is $\mathbf{b}$-admissible if and only if $\mathscr{I}^{\mathrm{Sh}}$ is contained in the $\mathbf{b}$-stratum.*

Proof. The Kottwitz triple $\mathfrak{k}(\mathscr{I}^{\mathrm{Sh}}) = (\gamma_0, \gamma, \delta)$ is $\mathbf{b}$-admissible if the σ-conjugacy class of δ is $\mathbf{b}$. Since δ arises from the Frobenius on the Dieudonné module at some point in $\mathscr{I}^{\mathrm{Sh}}$, the σ-conjugacy class of δ records the Newton stratum in which the isogeny class lies. ☐

3.4.4 Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be an admissible morphism and pick $y = (y^p, y_p) \in X^p(\phi) \times X_p^{\mathbf{b}}(\phi)$. Write $\overline{y} = (y^p, \overline{y}_p) \in X^p(\phi) \times X_p(\phi)$ for the image under the left $I_\phi(\mathbb{Q}_p)$-equivariant map $X_p^{\mathbf{b}}(\phi) \to X_p(\phi)$. We refer to [Kis17, p. 4.5.1] for the construction of a Kottwitz triple $\mathfrak{k}(\phi, y)$, which depends only on the the conjugacy class $[\phi]$ and the image of $\overline{y}$ in the quotient $I_\phi(\mathbb{Q})\backslash(X^p(\phi) \times X_p(\phi))$. We record two facts from *loc. cit.* (where our $y^p, \overline{y}_p$ are g^p, g_0): (i) the element δ appearing in the Kottwitz triple $\mathfrak{k}(\phi, y)$ is σ-conjugate to the element b_θ produced by an unramified morphism θ conjugate to $\phi(p) \circ \zeta_p$ as in Definition 2.5.11; (ii) the Kottwitz triple $\mathfrak{k}(\phi, y) = (\gamma_0, \gamma, \delta)$ has a natural refinement $\tilde{\mathfrak{k}}(\phi, y)$ taking $I = I_\phi$.

We observe below that $\mathbf{b}$-admissibility is passed down from admissible morphisms to Kottwitz triples, as it was the case for isogeny classes.

Lemma 3.4.5. *An admissible morphism $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ is $\mathbf{b}$-admissible if and only if the associated Kottwitz triple $\mathfrak{k}(\phi) = (\gamma_0, \gamma, \delta)$ is $\mathbf{b}$-admissible.*

Proof. The $\mathbf{b}$-admissibility of an admissible morphism ϕ depends on the σ-conjugacy class $[b_\theta]$ produced by an unramified morphism θ conjugate to $\phi(p) \circ \zeta_p$, while for a Kottwitz triple $(\gamma_0, \gamma, \delta)$ it depends on the σ-conjugacy class of δ. But δ is a σ-conjugate of b_θ, so these conditions are equivalent. ☐

3.5 τ-twists

Our goal is to parametrize each isogeny class $\mathscr{I}^{\mathrm{Ig}} \subset \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ by a set $S^{\mathrm{Ig}}(\phi)$. However, we will only identify the action of $I_x(\mathbb{Q})$ with $I_\phi(\mathbb{Q})$ up to twisting by an element $\tau \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$. We develop the necessary theory in this section, following [KSZ21, §2.6].

3.5.1 Let ϕ be an admissible morphism, and define

$$\mathcal{H}(\phi) = I_\phi(\mathbb{A}_f)\backslash I_\phi^{\mathrm{ad}}(\mathbb{A}_f)/I_\phi^{\mathrm{ad}}(\mathbb{Q}),$$
$$\mathfrak{E}^p(\phi) = I_\phi(\mathbb{A}_f^p)\backslash I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p).$$

These sets can be given the structure of abelian groups (by comparison with certain abelianized cohomology groups, see [KSZ21, 2.6.13, Lemma 2.6.14]).

Let $\mathcal{AM}$ be the set of admissible morphisms. Since we want to consider assignments of an element of $\mathcal{H}(\phi)$ for all ϕ simultaneously, it is convenient to consider $\mathcal{H}(\phi)$ as the stalks of a sheaf $\mathcal{H}$ on $\mathcal{AM}$ (regarded as a discrete topological space), and similarly $\mathfrak{E}^p(\phi)$ the stalks of a sheaf $\mathfrak{E}^p$. Let $\Gamma(\mathcal{H})$ and $\Gamma(\mathfrak{E}^p)$ be the global sections of these sheaves, so an element $\tau \in \Gamma(\mathcal{H})$ assigns to each admissible morphism ϕ an element $\tau(\phi) \in \mathcal{H}(\phi)$, and similarly for $\mathfrak{E}^p$.

By weak approximation the natural inclusion $I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p) \to I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$ induces a surjection $\mathfrak{E}^p(\phi) \to \mathcal{H}(\phi)$, which allows us to lift an element of $\mathcal{H}(\phi)$ to $I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p)$, or further $I_\phi(\overline{\mathbb{A}}_f^p)$. We will often do this implicitly so that $\tau(\phi)$ also denotes a lift in $I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p)$ for each $\tau \in \Gamma(\mathcal{H})$, when the ambiguity in the lift is harmless. The surjections $\mathfrak{E}^p(\phi) \to \mathcal{H}(\phi)$ produce surjections $\mathfrak{E}^p \to \mathcal{H}$ and $\Gamma(\mathfrak{E}^p) \to \Gamma(\mathcal{H})$.

Define an equivalence relation $\phi_1 \approx \phi_2$ if ϕ_1^Δ is conjugate to ϕ_2^Δ by $G(\overline{\mathbb{Q}})$. If $\phi_1 \approx \phi_2$, then there are canonical isomorphisms

$$\mathrm{Comp}_{\phi_1,\phi_2} : \mathcal{H}(\phi_1) \to \mathcal{H}(\phi_2)$$
$$\mathrm{Comp}^{\mathfrak{E}^p}_{\phi_1,\phi_2} : \mathfrak{E}^p(\phi_1) \to \mathfrak{E}^p(\phi_2)$$

satisfying the relations $\mathrm{Comp}_{\phi_2,\phi_3} \circ \mathrm{Comp}_{\phi_1,\phi_2} = \mathrm{Comp}_{\phi_1,\phi_3}$ and $\mathrm{Comp}_{\phi_1,\phi_1} = \mathrm{id}_{\mathcal{H}(\phi_1)}$, and similarly for $\mathfrak{E}^p$. These isomorphisms show that $\mathcal{H}$ and $\mathfrak{E}^p$ are pulled back from sheaves $\mathcal{H}/\approx$ and $\mathfrak{E}^p/\approx$ on $\mathcal{AM}/\approx$, under the natural quotients

$$\mathcal{AM} \to \mathcal{AM}/\mathrm{conj} \to \mathcal{AM}/\approx .$$

Write $\mathcal{H}/\mathrm{conj}$ and $\mathfrak{E}^p/\mathrm{conj}$ for the intermediate pullbacks to $\mathcal{AM}/\mathrm{conj}$, the set of admissible morphisms up to conjugacy.

3.5.2 Let $\Gamma(\mathcal{H})_0$ be the set of global sections of $\mathcal{H}$ that descend to $\mathcal{AM}/\approx$, and $\Gamma(\mathcal{H})_1$ those that descend to $\mathcal{AM}/\mathrm{conj}$, so we have $\Gamma(\mathcal{H})_0 \subset \Gamma(\mathcal{H})_1 \subset \Gamma(\mathcal{H})$. Define $\Gamma(\mathfrak{E}^p)_0 \subset \Gamma(\mathfrak{E}^p)_1 \subset \Gamma(\mathfrak{E}^p)$ similarly. The surjection $\Gamma(\mathfrak{E}^p) \to \Gamma(\mathcal{H})$ induces a surjection $\Gamma(\mathfrak{E}^p)_0 \to \Gamma(\mathcal{H})_0$.

There is one further technical definition we will need, namely the notion of *tori-rationality* of an element of $\Gamma(\mathcal{H})$ or $\Gamma(\mathfrak{E}^p)$. For this we refer to [KSZ21, Def. 2.6.19]. We will also need the fact [KSZ21, Lem. 2.6.20] that an element of $\Gamma(\mathcal{H})$ is tori-rational exactly when one (equivalently, every) lift to $\Gamma(\mathfrak{E}^p)$ is tori-rational.

Let $\tau \in \Gamma(\mathcal{H})_1$. We define a $\tau(\phi)$-twisted analogue of (3.3.2):

$$S_\tau^{\mathrm{Ig}}(\phi) := I_\phi(\mathbb{Q})_{\tau(\phi)} \backslash \big(X^p(\phi) \times X_p^{\mathbf{b}}(\phi)\big), \tag{3.5.3}$$

where the quotient is taken with respect to the $\tau(\phi)$-twisted embedding

$$I_\phi(\mathbb{Q}) \hookrightarrow I_\phi(\mathbb{A}_f) \overset{\tau(\phi)}{\cong} I_\phi(\mathbb{A}_f)$$

followed by the natural action of $I_\phi(\mathbb{A}_f)$ on $X^p(\phi) \times X_p^{\mathbf{b}}(\phi)$.

3.6 Langlands–Rapoport-τ Conjecture for Igusa Varieties of Hodge Type

A crucial ingredient for us is the analogue of

Theorem 3.6.1. *There exists a tori-rational element $\tau \in \Gamma(\mathcal{H})_0$ admitting a $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$-equivariant bijection*

$$\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p) \xrightarrow{\sim} \coprod_{[\phi]} S_\tau^{\mathrm{Ig}}(\phi),$$

where the disjoint union ranges over conjugacy classes of $\mathbf{b}$-admissible morphisms $\phi : \mathfrak{Q} \to \mathfrak{G}_G$.

Proof. Let us start by recalling relevant results for Shimura varieties. Following [KSZ21, Thm. 5.13.9] (summarized in §0.4 and elaborated in §6 therein) we have a bijection (inverse of the bijection $\mathscr{B}$ therein)

$$\mathscr{B}^{-1} : \left\{ \begin{array}{c} \text{isogeny classes} \\ \text{in } \overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{conjugacy classes of} \\ \text{admissible morphisms} \end{array} \right\}$$

such that if an isogeny class $\mathscr{I}$ corresponds to a conjugacy class $[\phi]$, then $(\mathscr{I},[\phi])$ is an amicable pair [KSZ21, Def. 5.10.1]. In particular, the bijection is compatible with the maps from both sides to the equivalence classes of Kottwitz triples. To the above bijection they assign $\tau \in \Gamma(\mathcal{H})_0$, which is tori-rational by [KSZ21, Thm. 5.12.2, Lem. 6.1.7]. (In their notation, our $[\phi], \tau$ are $\mathscr{J}, \underline{\tau}_{\mathscr{B}}$. Roughly speaking, τ measures the difference between refined Kottwitz triples arising from the two sides of the bijection.)

By Corollary 3.2.5 we can consider the set of isogeny classes in $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ as the set of isogeny classes in $\overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p)$ contained in the $\mathbf{b}$-stratum. By Lemma 3.4.3, the latter set is characterized as the set of isogeny classes whose Kottwitz triples are $\mathbf{b}$-admissible. On the other side, we can regard the set of conjugacy classes of $\mathbf{b}$-admissible morphisms as a subset of the set of conjugacy classes of admissible morphisms. By Lemma 3.4.5, this subset is characterized as those conjugacy classes whose associated Kottwitz triple is $\mathbf{b}$-admissible. Because the bijection $\mathscr{B}^{-1}$ is compatible with Kottwitz triples, it induces a bijection, again denoted by $\mathscr{B}^{-1}$, between the subsets on each side corresponding to $\mathbf{b}$-admissible Kottwitz triples:

$$\mathscr{B}^{-1} : \left\{ \begin{array}{c} \text{isogeny classes} \\ \text{in } \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{conjugacy classes of} \\ \mathbf{b}\text{-admissible morphisms} \end{array} \right\}.$$

As $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ can be partitioned into isogeny classes, it suffices to show that, for the tori-rational $\tau \in \Gamma(\mathcal{H})_0$ above, there is a $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$-equivariant bijection

$$\mathscr{I}^{\mathrm{Ig}} \xrightarrow{\sim} S_\tau^{\mathrm{Ig}}(\phi)$$

for each isogeny class $\mathscr{I}^{\mathrm{Ig}}$ on the left hand side and the corresponding conjugacy class of $\mathbf{b}$-admissible ϕ. Let $\tilde{x} \in \mathscr{I}^{Ig}$. Write $\mathscr{I}$ for the isogeny class in $\overline{\mathscr{S}}_{K_p}(\overline{\mathbb{F}}_p)$ determined by $\mathscr{I}^{\mathrm{Ig}}$. Denote by $x \in \mathscr{I}$ the image of $\tilde{x}$. To find the desired bijection, we follow the ideas of [KSZ21, 5.10.3, Prop. 5.10.4] and use the notation from there. So we fix a point $x \in \mathscr{I}$ and $\tilde{y} \in X^p(\phi) \times X^{\mathbf{b}}(\phi)$. Note that there is a tautological map from $X^p(\phi) \times X^{\mathbf{b}}(\phi)$ to $Y(\phi)$ and further to its quotient $\overline{Y}(\phi)$. We choose a marking $(\overline{y}, \overline{y}')$ for the amicable pair $(\mathscr{I},[\phi])$ such that $\overline{y}' \in \overline{Y}(\phi)$ equals the image of $\tilde{y}$. Further we choose lifts

$$y = (y^p, y_p) \in Y(x) = Y^p(x) \times Y_p(x) \quad \text{and} \quad y' = (y'^{,p}, y'_p) \in Y(\phi) = X^p(\phi) \times Y_p(\phi)$$

of $\overline{y}$ and $\overline{y}'$ as in *loc. cit.* Here $Y^p(x)$ is the $G(\mathbb{A}_f^p)$-torsor of tensor-preserving trivializations $V_{\mathbb{A}_f^p} \cong \hat{V}^p(\mathcal{A}_x)$, and $Y_p(x)$ is the $G(\mathbb{Q}_p^{\mathrm{ur}})$-torsor of tensor-preserving trivializations $V_{\mathbb{Z}_p^{\mathrm{ur}}} \cong \mathbb{V}(\mathcal{A}_x[p^\infty])$ (as $\mathbb{Z}_p^{\mathrm{ur}}$-modules). The $G(\mathbb{Q}_p^{\mathrm{ur}})$-torsor $Y_p(\phi)$ is the subset of $y''_p \in G(\overline{\mathbb{Q}}_p)$ such that $\theta_{y''_p}$ is unramified, cf. 2.5.15.

The element $\tau_{\overline{y},\overline{y}'} \in I_\phi(\mathbb{Q})\backslash I_\phi^{\mathrm{ad}}(\mathbb{A}_f)/I_\phi^{\mathrm{ad}}(\mathbb{Q})$ in [KSZ21, p. 5.10.3] maps to $\tau(\phi) \in \mathcal{H}(\phi)$, cf. [KSZ21, 5.10.6, Def. 5.10.9]. Put $G(\mathbb{A}_f^*) := G(\mathbb{A}_f^p) \times G(\mathbb{Q}_p^{\mathrm{ur}})$. The base points y, y' induce trivializations of the torsors $Y(x) \cong G(\mathbb{A}_f^*)$ and $Y(\phi) \cong G(\mathbb{A}_f^*)$ as well as embeddings

$$\iota_y : I_x(\mathbb{Q}) \hookrightarrow I_x(\mathbb{A}_f) \hookrightarrow G(\mathbb{A}_f^*), \qquad \iota_{y'} : I_\phi(\mathbb{Q}) \hookrightarrow I_\phi(\mathbb{A}_f) \hookrightarrow G(\mathbb{A}_f^*).$$

As in the first four lines in the proof of [KSZ21, Prop. 5.10.4], we have (without taking the right quotients) right $G(\mathbb{A}_f^*)$-equivariant bijections

$$\overline{\xi}_y : I_x(\mathbb{Q})\backslash Y(x) \cong \iota_y(I_x(\mathbb{Q}))\backslash G(\mathbb{A}_f^*),$$
$$\overline{\xi}_{y'} : I_\phi(\mathbb{Q})_{\tau(\phi)}\backslash Y(\phi) \cong \iota_y(I_x(\mathbb{Q}))\backslash G(\mathbb{A}_f^*).$$

The $J_b(\mathbb{Q}_p)$-torsor $X_p^{\mathbf{b}}(\phi)$ is a subset of the $G(\mathbb{Q}_p^{\mathrm{ur}})$-torsor $Y_p(\phi)$ by definition, compatibly with $J_b(\mathbb{Q}_p) \subset G(\mathbb{Q}_p^{\mathrm{ur}})$. (We remarked on the latter embedding in §3.3.) So $\overline{\xi}_{y'}$ restricts to a $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$-equivariant bijection

$$S_\tau^{\mathrm{Ig}}(\phi) \stackrel{\mathrm{def}}{=} I_\phi(\mathbb{Q})_{\tau(\phi)}\backslash\big(X^p(\phi) \times X_p^{\mathbf{b}}(\phi)\big) \cong \iota_y(I_x(\mathbb{Q}))\backslash\big(G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)\big).$$

Write $Y_p^{\mathrm{Ig}}(x) \subset Y_p(x)$ for the subset of isomorphisms $V_{\mathbb{Z}_p^{\mathrm{ur}}} \cong \mathbb{V}(\mathcal{A}_x[p^\infty])$ which are compatible with the Frobenius actions: $b\sigma$ on $V_{\mathbb{Z}_p^{\mathrm{ur}}}$ and the usual one on $\mathbb{V}(\mathcal{A}_x[p^\infty])$. Then $Y_p^{\mathrm{Ig}}(x)$ is nonempty since x lies in the $\mathbf{b}$-stratum, and it is a right $J_b(\mathbb{Q}_p)$-torsor with a left $I_x(\mathbb{Q})$-action. Although y_p chosen above may not lie in $Y_p^{\mathrm{Ig}}(x)$, there exists $g_p \in G(\mathbb{Q}_p^{\mathrm{ur}})$ such that $y_p \circ g_p \in Y_p^{\mathrm{Ig}}(x)$. Thus the right translation by g_p^{-1} on $I_x(\mathbb{Q})\backslash Y(x)$ followed by $\overline{\xi}_y$ restricts to a $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$-equivariant bijection

$$I_x(\mathbb{Q})\backslash\big(Y^p(x) \times Y_p^{\mathrm{Ig}}(x)\big) \cong \iota_y(I_x(\mathbb{Q}))\backslash\big(G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)\big).$$

The left hand side is exactly $\mathscr{I}^{\mathrm{Ig}}$, cf. §3.2, so we conclude by combining the last two displayed bijections. □

3.6.2 Only in this paragraph we consider general Shimura data (G, X), where $G_{\mathbb{Q}_p}$ need not be unramified. Let $\mathbf{b} \in B(G, \mu^{-1})$, where μ belongs to the conjugacy class $\{\mu_X\}$. Given $b \in \mathbf{b}$, we expect to be able to define a perfect Igusa variety Ig_b over $\overline{\mathbb{F}}_p$ with a $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$-action; see the relevant discussion in [BMS, pp. 3.3.8–3.3.10]. Notice that $\mathbf{b}$-admissible morphisms still make sense as well as the right $G(\mathbb{A}_f) \times J_b(\mathbb{Q}_p)$-set $S^{\mathrm{Ig}}(\phi)$ for each $\mathbf{b}$-admissible morphism ϕ. Assuming Ig_b is non-empty, we may still contemplate a Langlands–Rapoport conjecture for Igusa varieties as a conjectural $G(\mathbb{A}_f) \times J_b(\mathbb{Q}_p)$-equivariant bijection

$$\mathrm{Ig}_b(\overline{\mathbb{F}}_p) \overset{?}{\cong} \coprod_{[\phi]} S^{\mathrm{Ig}}(\phi),$$

where the disjoint union is over conjugacy classes of $\mathbf{b}$-admissible morphisms. The analogue of the Langlands–Rapoport-τ conjecture for Igusa varieties can also be formulated, and this weaker version should suffice for the purpose of deriving a point-counting formula.

4 Point-Counting Formula for Igusa Varieties of Hodge Type

We continue to use the notation of the previous sections. In particular, we have an Igusa variety from a Shimura datum (G, X) of Hodge type, a class $\mathbf{b} \in B(G, \mu^{-1})$, and a representative $b \in G(L)$ of this class satisfying the list of conditions in 2.4.4 for some $r \in \mathbb{Z}_{\geq 1}$. We fixed these data.

4.1 Acceptable Functions and Fujiwara–Varshavsky's Trace Formula

4.1.1 The (right) action of $G(\mathbb{A}_f^p) \times S_b$ on Ig_Σ extends to an (left) action of $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ on $H_c^i(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)$ as in (2.4.7). Let $f \in C_c^\infty(G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p))$, a smooth compactly supported function on $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$. We write

$$\mathrm{tr}(f \mid H_c(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)) = \sum_i (-1)^i \,\mathrm{tr}(f \mid H_c^i(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)).$$

Any $f \in C_c^\infty(G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p))$ is a finite linear combination of indicator functions $\mathbb{1}_{UgU}$ for varying $U \subset G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ compact open and $g \in G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$, so by linearity of trace it suffices to consider $f = \mathbb{1}_{UgU}$. Writing $UgU = \coprod_i g_i g U$ for some finite collection $g_i \in U$, the action of $\mathbb{1}_{UgU}$ on $H_c^i(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)$ is given by

$$\int_{G(\mathbb{A}_f^p)\times J_b(\mathbb{Q}_p)} \mathbb{1}_{UgU}(x) x \cdot v \, dx = \sum_i g_i g \int_U x \cdot v \, dx.$$

Now $\mathrm{vol}(U)^{-1} \int_U x \cdot v \, dx$ is a projection onto $H_c^i(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)^U$. Thus $\mathrm{vol}(U)^{-1}\mathbb{1}_{UgU}$ acts on $H_c^i(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)^U$ as the following *double coset operator*, called $[UgU]$:

$$[UgU] : v \mapsto \sum_i g_i g \cdot v \qquad \text{on} \qquad H_c^i(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)^U.$$

A neighborhood basis of the identity in $J_b(\mathbb{Q}_p)$ is given by the following open compact subgroups for $m \geq 1$:

$$U_p(m) := \ker(\mathrm{Aut}(\Sigma, \lambda_\Sigma, \{s_{\alpha,\Sigma}\}) \to \mathrm{Aut}(\Sigma[p^m], \lambda_\sigma, \{s_{\alpha,\Sigma}\})) \subset J_b(\mathbb{Q}_p).$$

So we can assume $U = U^p \times U_p(m)$ for $U^p \subset G(\mathbb{A}_f^p)$ compact open. Then

$$H_c^i(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)^U = H_c^i(\mathrm{Ig}_{\Sigma,U^p,m}, \mathscr{L}_\xi).$$

Taking the alternating sum over i, we have

$$\mathrm{vol}(U)^{-1}\,\mathrm{tr}(\mathbb{1}_{UgU} \mid H_c(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)) = \mathrm{tr}([UgU] \mid H_c(\mathrm{Ig}_{\Sigma,U^p,m}, \mathscr{L}_\xi)). \tag{4.1.2}$$

4.1.3 Now assume $g = g^p \times g_p \in G(\mathbb{A}_f^p) \times S_b$ (recall S_b from 2.3.4), so that we can consider the action of g on finite-level Igusa varieties. Then the double coset operator $[UgU]$ is induced by the following correspondence, considered on the sets of $\overline{\mathbb{F}}_p$-points as this is enough for our purpose.

$$\begin{array}{ccc} & \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)/(U \cap gUg^{-1}) & \\ {\scriptstyle [\cdot 1]}\swarrow & & \searrow{\scriptstyle [\cdot g]} \\ \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)/U & & \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)/U \end{array}$$

Definition 4.1.4. Define the fixed point set of the above correspondence by

$$\mathrm{Fix}(UgU) = \{x \in \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)/(U \cap gUg^{-1}) : x = xg \text{ in } \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)/U\}.$$

Recall from 2.4.1 that a $\mathbb{Q}$-embedding $\varrho : G \hookrightarrow \mathrm{GL}(V)$ was fixed.

Definition 4.1.5. A function $f \in C_c^\infty(G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p))$ is *ϱ-acceptable* if

1. for all $(g, \delta) \in \mathrm{supp}\, f$, we have $\delta \in S_b$ and δ is ϱ-acceptable (Definition 2.2.4);
2. there is a sufficiently small compact open subgroup $U = U^p \times U_p(m) \subset G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ and a finite subset $I \subset G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ such that $f = \sum_{g \in I} c_g \mathbb{1}_{UgU}$ with $c_g \in \overline{\mathbb{Q}}_\ell$; and for each term in this sum, we have
   (a) $\mathrm{Fix}(UgU)$ is finite, and
   (b) the trace of the correspondence on cohomology is given by Fujiwara's formula:

$$\mathrm{tr}([UgU] \mid H_c(\mathrm{Ig}_{\Sigma,U^p,m}, \mathscr{L}_\xi)) = \sum_{x \in \mathrm{Fix}(UgU)} \mathrm{tr}([UgU] \mid (\mathscr{L}_\xi)_x). \tag{4.1.6}$$

Lemma 4.1.7. *Let $f \in C_c^\infty(G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p))$ be a ϱ-acceptable function, written as $f = \sum_{g \in I} c_g \mathbb{1}_{UgU}$ as above. Then*

$$\mathrm{tr}(f \mid H_c(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)) = \sum_{g \in I} c_g\, \mathrm{vol}(U) \sum_{x \in \mathrm{Fix}(UgU)} \mathrm{tr}([UgU] \mid (\mathscr{L}_\xi)_x). \tag{4.1.8}$$

Proof. Combine (4.1.2) and (4.1.6). ☐

We would like to know that ϱ-acceptable functions are a sufficient class of test functions to determine a representation such as $H_c(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)$ in the Grothendieck group. This is addressed by the next two lemmas, slightly adjusted from Lemmas 6.3 and 6.4 of [Shi09]. Recall from 2.2.8 that p and $fr^r = r\nu_b(p)$ denote central elements of $J_b(\mathbb{Q}_p)$. Write $\mathrm{Groth}(G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p))$ for the Grothendieck group of admissible representations of $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$.

Lemma 4.1.9. *For any $f \in C_c^\infty(G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p))$, the function $f^{(m,n)}$ defined by $f^{(m,n)}(x) := f(x \cdot p^m (fr^r)^n)$ is a ϱ-acceptable function for sufficiently large m, n.*

Proof. The exact analogue in the setting of some PEL Shimura varieties was shown in [Shi09, Lem. 6.3]. The proof there carries over to our case without change, as long as we verify the following nontrivial point: we need a model $\mathcal{J}$ of our Igusa variety over a finite field $\mathbb{F}_{p^s}$ such that the s-th power of the absolute Frobenius on $\mathcal{J}$ is transported to the action of $fr^s \in J_b(\mathbb{Q}_p)$ on our Igusa variety. This point is justified by part (2) of [KS23, Lem. 6.2.1] when s equals our fixed integer r. ☐

Lemma 4.1.10. *If* $\Pi_1, \Pi_2 \in \mathrm{Groth}(G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p))$ *satisfy* $\mathrm{tr}(f \mid \Pi_1) = \mathrm{tr}(f \mid \Pi_2)$ *for all* ϱ*-acceptable functions* f*, then* $\Pi_1 = \Pi_2$ *in the Grothendieck group.*

Proof. The purely representation-theoretic proof of [Shi09, Lem. 6.4] works in our setting verbatim. The only change is that Lemma 6.3 used in that paper is justified by Lemma 4.1.9 in our case. □

Thus it is enough to work in the setting of Lemma 4.1.7 and analyze the right hand side, in particular the set $\mathrm{Fix}(UgU)$ (Definition 4.1.4). This is where the Langlands–Rapoport conjecture for Igusa varieties (Theorem 3.6.1) comes in. After some preparation, we will return to (4.1.8) in §4.

4.2 Langlands–Rapoport Pairs and Kottwitz Parameters

We introduce Langlands–Rapoport pairs and Kottwitz parameters to parametrize fixed points in the trace formula. We will define a notion of acceptability for them. The definition depends on the embedding $\varrho : G \hookrightarrow \mathrm{GL}(V)$, but with this understanding, we will simply call it acceptability (omitting the reference to ϱ) since ϱ is fixed throughout.

Definition 4.2.1. A *Langlands–Rapoport (LR) pair* is a pair (ϕ, ε) where $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ is a morphism of Galois gerbs and $\varepsilon \in I_\phi(\mathbb{Q})$. The element ε can also be regarded as an element of $G(\overline{\mathbb{Q}})$ via $I_\phi(\mathbb{Q}) \subset G(\overline{\mathbb{Q}})$. Two LR pairs (ϕ_1, ε_1) and (ϕ_2, ε_2) are *conjugate* if there is an element $g \in G(\overline{\mathbb{Q}})$ which conjugates ϕ_1 to ϕ_2 and ε_1 to ε_2. An LR pair (ϕ, ε) is *semi-admissible* if ϕ is admissible, and **b**-*admissible* if ϕ is **b**-admissible, cf. Definitions 2.5.11 and 3.3.1.

Definition 4.2.2. An LR pair (ϕ, ε) is *gg* (abbreviated from *günstig gelegen*, German for "well-positioned") if

- ϕ^Δ is defined over $\mathbb{Q}$;
- ε lies in $G(\mathbb{Q})$ (not just in $G(\overline{\mathbb{Q}})$) and is semi-simple and elliptic in $G(\mathbb{R})$;
- for any $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, letting $q_\rho \in \mathfrak{Q}$ a lift of ρ and $\phi(q_\rho) = g_\rho \rtimes \rho$, we have $g_\rho \in G_\varepsilon^\circ$.

If (ϕ, ε) is a semi-admissible LR pair then ε is semi-simple, as I_ϕ / Z_G is compact over $\mathbb{R}$; see [KSZ21, p. 3.1.2]. The notion of gg LR pairs is independent of the choice of q_ρ by [KSZ21, Rem. 3.2.2].

4.2.3 Let (ϕ, ε) be an LR pair. As in 2.5.8, the morphism $\phi(p) \circ \zeta_p : \mathfrak{G}_p \to \mathfrak{G}_G(p)$ is conjugate by some $g \in G(\overline{\mathbb{Q}}_p)$ to an unramified morphism $\theta_g : \mathfrak{G}_p \to \mathfrak{G}_G(p)$, which defines an element $b_{\theta_g} \in G(\mathbb{Q}_p^{\mathrm{ur}})$. Since ε commutes with ϕ, its conjugate $\varepsilon_g := g\varepsilon g^{-1} \in G(\overline{\mathbb{Q}}_p)$ commutes with θ_g. Then for any $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{ur}})$ we have

$$1 \rtimes \rho = \theta(\rho) = \mathrm{Int}(\varepsilon_g) \circ \theta_g(\rho) = \mathrm{Int}(\varepsilon_g)(1 \rtimes \rho) = \varepsilon_g \rho(\varepsilon_g)^{-1} \rtimes \rho.$$

Hence $\varepsilon_g \in G(\mathbb{Q}_p^{\mathrm{ur}})$. Furthermore, since ε_g commutes with θ_g^{ur}, it must σ-centralize b_{θ_g}, and we can regard ε_g as an element of $J_{b_{\theta_g}}(\mathbb{Q}_p)$.

If $\phi(p) \circ \zeta_p$ is conjugated to an unramified morphism by another $g' \in G(\overline{\mathbb{Q}}_p)$ then $g'g^{-1} \in G(\mathbb{Q}_p^{\mathrm{ur}})$, and we see that the resulting pair $(b_{\theta_{g'}}, \varepsilon_{g'})$ is conjugate to $(b_{\theta_g}, \varepsilon_g)$ in that

$$b_{\theta_{g'}} = (g'g^{-1}) b_{\theta_g} \sigma(g'g^{-1})^{-1} \tag{4.2.4}$$

and conjugation by $g'g^{-1}$ gives an isomorphism $J_{b_{\theta_g}} \to J_{b_{\theta_{g'}}}$ sending ε_g to $\varepsilon_{g'}$. To simplify notation, we will often write (b_θ, ε') for $(b_{\theta_g}, \varepsilon_g)$, which is well defined up to conjugation. By (4.2.4), $[b_\theta]$ is a well-defined element of $B(G)$.

Definition 4.2.5. Let (ϕ, ε) be an LR pair with $(b_{\theta_g}, \varepsilon_g)$ as above. Define (ϕ, ε) to be *acceptable* if ε_g is ϱ-acceptable as an element of $J_{b_{\theta_g}}(\mathbb{Q}_p)$. (This does not depend on the choice of g by the paragraph just above.)

We denote the set of LR pairs by $\mathcal{LRP}$. The subset of semi-admissible pairs, $\mathbf{b}$-admissible pairs, acceptable, and gg pairs are respectively denoted by

$$\mathcal{LRP}_{\mathrm{sa}}, \quad \mathcal{LRP}_{\mathbf{b}}, \quad \mathcal{LRP}_{\mathrm{acc}}, \quad \mathcal{LRP}^{\mathrm{gg}}.$$

The first three subsets are closed under conjugation by elements of $G(\overline{\mathbb{Q}})$, cf. Definition 4.2.1, but the last one is not. Nevertheless we can define the equivalence relation by conjugacy on all four subsets. We write $\mathcal{LRP}_{\mathrm{sa}}/\mathrm{conj.}$, etc. for the quotient sets. We will impose more than one conditions by using multiple superscripts or subscripts, e.g., $\mathcal{LRP}_{\mathbf{b},\mathrm{acc}}$ stands for the set of $\mathbf{b}$-admissible and acceptable LR pairs.

4.2.6 A *special point datum* is a triple (T, h_T, i) where

- T is a torus,
- $h_T : \mathbb{S} \to T_{\mathbb{R}}$ is a morphism from the Deligne torus, and
- $i : T \to G$ is an embedding realizing T as a maximal torus of G defined over $\mathbb{Q}$, and sending h_T into X.

We simply write (T, h_T) if it is clear what i is, e.g., if T is a subtorus of G.

A special point datum induces an admissible morphism as follows. We obtain $\mu_T \in X_*(T)$ from h_T by restricting the base change $h_T \times_{\mathbb{R}} \mathbb{C}$ to the copy of $\mathbb{G}_m$ in $\mathbb{S}_{\mathbb{C}} \cong \mathbb{G}_m \times \mathbb{G}_m$ corresponding to the identity map of $\mathbb{C}$. From ψ_{T,μ_T} constructed in (2.5.5), we obtain a morphism

$$\phi := i \circ \psi_{T,\mu_T} : \mathfrak{Q} \xrightarrow{\psi_{T,\mu_T}} \mathfrak{G}_T \xrightarrow{i} \mathfrak{G}_G,$$

which is admissible by [Kis17, Lem. 3.5.8].

Furthermore, in this setup $T(\mathbb{Q})$ as a subgroup of $G(\overline{\mathbb{Q}})$ lies inside $I_\phi(\mathbb{Q})$, so any $\varepsilon \in T(\mathbb{Q})$ makes a semi-admissible LR pair (ϕ, ε). Moreover this LR pair is gg as the conditions in Definition 4.2.2 are readily checked.

An LR pair is said to be *special* if it is conjugate to the LR pair arising from a special point datum (T, h_T, i) and $\varepsilon \in T(\mathbb{Q})$ as above. It is a fact [KSZ21, p. 3.3.9] that every semi-admissible pair is special. In particular, every semi-admissible pair is conjugate to a gg LR pair.

The following definition is inspired by [Shi09, Def. 10.1] and [KSZ21, Def. 1.6.2].

Definition 4.2.7. A *classical Kottwitz parameter* of *type* $\mathbf{b}$ is a triple $(\gamma_0, \gamma, \delta)$ where

- $\gamma_0 \in G(\mathbb{Q})$ is semi-simple and elliptic in $G(\mathbb{R})$,
- $\gamma = (\gamma_\ell)_\ell \in G(\mathbb{A}_f^p)$ such that γ_ℓ is stably conjugate to γ_0 in $G(\mathbb{Q}_\ell)$, and
- $\delta \in J_b(\mathbb{Q}_p)$ is conjugate to γ_0 in $G(\overline{\mathbb{Q}}_p)$ under the embedding $J_b(\mathbb{Q}_p) \to G(\overline{\mathbb{Q}}_p)$. If δ is ϱ-acceptable, the parameter $(\gamma_0, \gamma, \delta)$ is said to be *acceptable*.

We say that $(\gamma_0, \gamma, \delta)$ and $(\gamma_0', \gamma', \delta')$ are *equivalent* if γ_0 is stably conjugate to γ_0' in $G(\mathbb{Q})$, γ is conjugate to γ' in $G(\mathbb{A}_f^p)$, and δ is conjugate to δ' in $J_b(\mathbb{Q}_p)$. The set of classical Kottwitz parameters of type $\mathbf{b}$ is denoted by $\mathcal{CKP}_{\mathbf{b}}$. By $\mathcal{CKP}_{\mathbf{b},\mathrm{acc}}$ we mean the subset of acceptable parameters.

In order to handle the case that G_{der} is not simply connected, we need a closely related but more Galois-cohomological treatment.

Definition 4.2.8 (cf. [KSZ21, Def. 1.6.4])**.** A *Kottwitz parameter* is a triple $\mathfrak{c} = (\gamma_0, a, [b_0])$ where

1. $\gamma_0 \in G(\mathbb{Q})$ is semi-simple and elliptic in $G(\mathbb{R})$, and we write $I_0 = G_{\gamma_0}^\circ$;
2. a is an element of
$$\mathfrak{D}(I_0, G; \mathbb{A}_f^p) = \ker\left(H^1(I_0, \mathbb{A}_f^p) \to H^1(G, \mathbb{A}_f^p)\right);$$
3. $[b_0] \in B(I_0)$; and
4. the image of $[b_0]$ under the Kottwitz map $B(I_0) \to B(G) \xrightarrow{\kappa} \pi_1(G)_{\Gamma_p}$ is equal to the image of μ^{-1}, where μ is the cocharacter induced by an(y) element of X.

If $b_0 \in I_0(L)$ is a representative of $[b_0]$ then b_0 commutes with γ_0 so $\gamma^{-1} b \sigma(\gamma) = \gamma^{-1} b \gamma = b$, telling us that $\gamma_0 \in J_{b_0}(\mathbb{Q}_p)$.

4.2.9 An isomorphism of Kottwitz parameters is essentially given by conjugation by $G(\overline{\mathbb{Q}})$. We make this precise as follows. Let $(\gamma_0, a, [b_0])$ be a Kottwitz parameter, $I_0 = G^\circ_{\gamma_0}$, and $u \in G(\overline{\mathbb{Q}})$ an element such that $\mathrm{Int}(u)\gamma_0 = \gamma_0'$ is again in $G(\mathbb{Q})$ and $u^{-1}\rho(u) \in I_0$ for all $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Write $I_0' = G^\circ_{\gamma_0'}$.

To relate the away-from-p parts, we consider the bijection

$$\begin{aligned} u_* : \mathfrak{D}(I_0, G; \mathbb{A}_f^p) &\to \mathfrak{D}(I_0', G; \mathbb{A}_f^p) \\ e_\rho &\mapsto u e_\rho \rho(u)^{-1} \end{aligned}$$

induced by the element u.

To relate the p-parts, we construct a bijection $u_* : B(I_0) \to B(I_0')$. The cocycle $\rho \mapsto u^{-1}\rho(u) \in Z^1(\mathbb{Q}_p, I_0)$ is trivial in $H^1(\breve{\mathbb{Q}}_p, I_0)$ by the Steinberg vanishing theorem. That is, we can find $d \in I_0(\overline{\breve{\mathbb{Q}}}_p)$ so that $u^{-1}\rho(u) = d^{-1}\rho(d)$ for all $\rho \in \mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/\breve{\mathbb{Q}}_p)$. Then we have $ud^{-1} = \rho(ud^{-1})$ for all such ρ, so $u_0 := ud^{-1}$ lies in $G(\breve{\mathbb{Q}}_p)$.

Since d commutes with γ_0, we have $u_0\gamma_0u_0^{-1} = \gamma_0'$, and thus u_0 induces a bijection

$$\begin{aligned} u_* : B(I_0) &\to B(I_0') \\ [b] &\mapsto [u_0 b \sigma(u_0)^{-1}]. \end{aligned}$$

This bijection is independent of d (and therefore deserves the name u_*) because any other choice of d is related by an element of $I_0(\breve{\mathbb{Q}}_p)$ and therefore its σ-conjugation of b does not change the class in $B(I_0)$.

Now with the above setup, an *isomorphism* between Kottwitz parameters $(\gamma_0, a, [b_0])$ and $(\gamma_0', a', [b_0'])$ is an element $u \in G(\overline{\mathbb{Q}})$ such that

- $\mathrm{Int}(u)\gamma_0 = \gamma_0'$ and $u^{-1}\rho(u) \in I_0$ for all $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (i.e., u stably conjugates γ_0 to γ_0'),
- the bijection $u_* : \mathfrak{D}(I_0, G; \mathbb{A}_f^p) \to \mathfrak{D}(I_0', G; \mathbb{A}_f^p)$ sends a to a', and
- the bijection $u_* : B(I_0) \to B(I_0')$ sends $[b_0]$ to $[b_0']$.

Definition 4.2.10. A Kottwitz parameter $(\gamma_0, a, [b_0])$ is $\mathbf{b}$-*admissible* if the map $B(I_0) \to B(G)$ sends $[b_0]$ to $\mathbf{b}$. Note that we have fixed $\mathbf{b}$ in $B(G, \mu^{-1})$, so a $\mathbf{b}$-admissible Kottwitz parameter automatically satisfies item 4 of Definition 4.2.8. Say that $(\gamma_0, a, [b_0])$ is *acceptable* if γ_0 is ϱ-acceptable as an element of $J_{b_0}(\mathbb{Q}_p)$ for some representative b_0 of $[b_0]$.

The acceptability is insensitive to the choice of b_0: if $b_0' = ub_0\sigma(u)^{-1}$ for $u \in I_0(L)$ then conjugation by u induces an isomorphism $J_{b_0} \xrightarrow{\sim} J_{b_0'}$ sending γ_0 to itself, and ϱ-acceptability of γ_0 does not change under the isomorphism by Lemma 2.2.5.

Write $\mathcal{KP}$ for the set of Kottwitz parameters. When decorated with subscripts $\mathbf{b}$ and acc, they designate the subsets of $\mathbf{b}$-admissible and acceptable parameters, respectively.

Lemma 4.2.11. *If $(\gamma_0, a, [b_0])$ is acceptable then $[b_0]$ is basic in $B(I_0)$.*

Proof. It follows from Lemma 2.2.10 that I_0 we have $I_0 \subset M_{b_0}$. Since ν_{b_0} is central in M_{b_0}, it is also central in I_0. ☐

4.2.12 Let $\mathfrak{c} = (\gamma_0, a, [b_0])$ be a Kottwitz parameter, $I_0 = G^\circ_{\gamma_0}$, and consider the group

$$\mathfrak{E}(I_0, G; \mathbb{A}/\mathbb{Q}) = \mathrm{coker}\left(H^0_{\mathrm{ab}}(\mathbb{A}, G) \to H^0_{\mathrm{ab}}(\mathbb{A}/\mathbb{Q}, I_0 \to G)\right)$$

where H^0_{ab} is the abelianized Galois cohomology of [Col99]. We write $\alpha(\mathfrak{c}) \in \mathfrak{E}(I_0, G; \mathbb{A}/\mathbb{Q})$ for the *Kottwitz invariant* of $\mathfrak{c}$ as defined in [KSZ21, p. 1.7].

4.2.13 We now define a map from semi-admissible LR pairs to Kottwitz parameters, following [KSZ21, p. 3.5.1].

Let (ϕ, ε) be a semi-admissible LR pair, and $\tau(\phi) \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p)$. After conjugation by an element of $G(\overline{\mathbb{A}}_f^p)$ we may assume that (ϕ, ε) is gg; the construction will be shown to be insensitive to this conjugation in Lemma 4.2.15 below. We will define a Kottwitz parameter $\mathbf{t}(\phi, \varepsilon, \tau(\phi)) = (\gamma_0, a, [b_0])$ associated to (ϕ, ε) and $\tau(\phi)$.

Define $\gamma_0 = \varepsilon$. By the gg condition, ε is contained in $G(\mathbb{Q})$ and is semi-simple and elliptic in $G(\mathbb{R})$, verifying the requirements for γ_0 in Definition 4.2.8. We write $I_0 = G_{\gamma_0}^\circ = G_\varepsilon^\circ$.

Next we consider a. Recall the cocycles $\zeta_\phi^{p,\infty}$ and $\zeta_{\phi,\ell}$ of 2.5.14. The gg condition

$$\phi(q_\rho) = g_\rho \rtimes \rho \text{ has } g_\rho \in G_\varepsilon^\circ = I_0 \text{ for } q_\rho \text{ any lift of } \rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \tag{4.2.14}$$

implies that $\zeta_\phi^{p,\infty}$ is valued in $I_0(\overline{\mathbb{A}}_f^p)$.

Choose a lift $\tilde{\tau} \in I_\phi(\overline{\mathbb{A}}_f^p)$ of $\tau(\phi)$, and define a cocycle $A : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to I_0(\overline{\mathbb{A}}_f^p)$ by

$$A(\rho) = t_\rho \zeta_\phi^{p,\infty}(\rho)$$

where $t_\rho = \tilde{\tau}^{-1}\rho(\tilde{\tau}) \in Z_{I_\phi}(\overline{\mathbb{A}}_f^p)$, acting by ρ via the $\mathbb{Q}$-structure of I_ϕ. We can regard t_ρ as an element of $I_0(\overline{\mathbb{A}}_f^p)$ because the natural embedding $Z_{I_\phi} \to G$ factors through I_0.

The cocycle A splits over $G(\overline{\mathbb{A}}_f^p)$. To see this, we write $\tilde{\tau}_G$ for the image of $\tilde{\tau}$ in G. We distinguish these because $\tilde{\tau}$ is subject to the Galois action given by the $\mathbb{Q}$-structure on I_0, while $\tilde{\tau}_G$ is subject to that given by the $\mathbb{Q}$-structure on G. With respect to the Galois action on G, the element $\rho(\tilde{\tau})$ becomes $\zeta_\phi^{p,\infty}(\rho)\rho(\tilde{\tau}_G)\zeta_\phi^{p,\infty}(\rho)^{-1}$, and so

$$\begin{aligned} A(\rho) &= \tilde{\tau}^{-1}\rho(\tilde{\tau})\zeta_\phi^{p,\infty}(\rho) \\ &= \tilde{\tau}_G^{-1}\zeta_\phi^{p,\infty}(\rho)\rho(\tilde{\tau}_G)\zeta_\phi^{p,\infty}(\rho)^{-1}\zeta_\phi^{p,\infty}(\rho) \\ &= \tilde{\tau}_G^{-1}\zeta_\phi^{p,\infty}(\rho)\rho(\tilde{\tau}_G). \end{aligned}$$

Combined with the fact that $\zeta_\phi^{p,\infty}$ splits in $G(\overline{\mathbb{A}}_f^p)$ (realized as $\rho \mapsto x\rho(x)^{-1}$ for $x \in X^p(\phi)$), this shows that A splits in $G(\overline{\mathbb{A}}_f^p)$ as well.

We define $a \in \mathfrak{D}(I_0, G; \mathbb{A}_f^p)$ in our Kottwitz parameter to be the class defined by the image of A. This does not depend on the choice of lift $\tilde{\tau}$, because two choices differ by an element of $Z_{I_\phi}(\overline{\mathbb{A}}_f^p)$ which commutes with $\zeta_\phi^{p,\infty}$.

Finally we construct $[b_0]$. The same gg condition (4.2.14) above implies that ϕ factors

$$\phi : \mathfrak{Q} \xrightarrow{\phi_0} \mathfrak{G}_{I_0} \to \mathfrak{G}_G.$$

Then (ϕ_0, ε) is again an LR pair for I_0, giving rise to a well-defined class $[b_0] := [b_\theta] \in B(I_0)$ by 4.2.3. This finishes the construction of $\mathbf{t}(\phi, \varepsilon, \tau(\phi)) = (\gamma_0, a, [b_0])$.

Note that we have taken $\tau(\phi) \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p)$, but by [KSZ21, Prop. 3.5.2] this construction only depends on its image in $\mathfrak{E}^p(\phi) = I_\phi(\mathbb{A}_f^p)\backslash I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p)$, so we have a well-defined Kottwitz parameter $\mathbf{t}(\phi, \varepsilon, \tau(\phi))$ associated to a semi-admissible pair (ϕ, ε) and an element $\tau \in \Gamma(\mathfrak{E}^p)$.

We want to show that this construction only depends on the conjugacy class of the LR pair. In particular, after conjugating we have worked in the case that our LR pair is gg, so we want to show that if two gg pairs are conjugate then the resulting Kottwitz parameters are isomorphic. For this we need to assume that the τ-twists are well-behaved under conjugation as well.

Lemma 4.2.15. *Let (ϕ, ε) and (ϕ', ε') be gg LR pairs, and $\tau \in \Gamma(\mathfrak{E}^p)_1$. Write $\mathbf{t}(\phi, \varepsilon, \tau(\phi)) = (\gamma_0, a, [b_0])$ and $\mathbf{t}(\phi', \varepsilon', \tau(\phi')) = (\gamma_0', a', [b_0'])$ for the associated Kottwitz parameters. If $u \in G(\overline{\mathbb{Q}})$ conjugates (ϕ, ε) to (ϕ', ε'), then*

1. *$u\rho(u)^{-1} \in G_{\varepsilon'}^\circ$ for all $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and*
2. *u gives an isomorphism $(\gamma_0, a, [b_0]) \cong (\gamma_0', a', [b_0'])$.*

Proof. The proof is the same as in [KSZ21, Prop. 3.5.3]. ☐

Thus for any $\tau \in \Gamma(\mathfrak{E}^p)_1$ we have a well-defined map

$$\begin{aligned} \mathbf{t}_\tau : \mathcal{LRP}_{\mathrm{sa}}/\mathrm{conj.} &\to \mathcal{KP}/\mathrm{isom.} \\ (\phi, \varepsilon) &\mapsto \mathbf{t}(\phi, \varepsilon, \tau(\phi)). \end{aligned} \tag{4.2.16}$$

Lemma 4.2.17. *Let (ϕ, ε) be a semi-admissible pair, and $\tau \in \Gamma(\mathfrak{E}^p)_1$. Consider the associated Kottwitz parameter $\mathbf{t}_\tau(\phi, \varepsilon) = (\gamma_0, a, [b_0])$.*

1. *$(\gamma_0, a, [b_0])$ is $\mathbf{b}$-admissible if and only if (ϕ, ε) is $\mathbf{b}$-admissible.*
2. *$(\gamma_0, a, [b_0])$ is acceptable if and only if (ϕ, ε) is acceptable.*

Proof. For the first claim, recall that a semi-admissible pair (ϕ, ε) is $\mathbf{b}$-admissible if $\phi(p) \circ \zeta_p : \mathfrak{G}_p \to \mathfrak{G}_G(p)$ is conjugate to an unramified morphism θ with $b_\theta \in \mathbf{b}$. On the other hand, $[b_0]$ is the class (in $B(I_0)$) defined by precisely such a b_θ, and $(\gamma_0, a, [b_0])$ is defined to be $\mathbf{b}$-admissible if $[b_0]$ maps to the class $\mathbf{b}$ under $B(I_0) \to B(G)$. These conditions are manifestly equivalent.

For the second claim, by conjugation we may assume that (ϕ, ε) is gg (as acceptability only depends on the conjugacy or isomorphism class). In view of Definition 2.2.4 as well as 4.2.10, it is enough to check that (b_0, γ_0) is a pair of the form $(b_{\theta_g}, \varepsilon_g)$ arising from (ϕ, ε) by the construction of 4.2.3. This follows from the fact that b_0 is defined by an unramified morphism that is conjugate to $\phi(p) \circ \zeta_p$ by $g \in I_0(\overline{\mathbb{Q}}_p)$. Indeed $b_{\theta_g} = b_0$ by construction, and $\varepsilon_g = g\varepsilon g^{-1} = \gamma_0$ since $\varepsilon = \gamma_0$ is in the center of I_0. ☐

4.2.18 Next we define a map cl : $\mathcal{KP}_{\mathbf{b}}/\mathrm{isom.} \to \mathcal{CKP}_{\mathbf{b}}/\mathrm{equiv.}$ Let $(\gamma_0, a, [b_0])$ be a $\mathbf{b}$-admissible Kottwitz parameter. The element γ_0 of our classical Kottwitz parameter is chosen to be the same element γ_0 of our Kottwitz parameter. The class a determines a conjugacy class in $G(\mathbb{A}_f^p)$ stably conjugate to γ_0, and we choose γ to be an arbitrary element of this class. Finally, let $b_0 \in I_0(L)$ be a representative of $[b_0]$. Then $\gamma_0 \in J_{b_0}(\mathbb{Q}_p)$. Since b_0 becomes σ-conjugate to the fixed element b in $G(L)$, there is a $\mathbb{Q}_p$-isomorphism $J_{b_0} \cong J_b$ canonical up to $J_b(\mathbb{Q}_p)$-conjugacy. We obtain $\delta \in J_b(\mathbb{Q}_p)$ by transporting γ_0 via this isomorphism. It is readily checked that $(\gamma_0, \gamma, \delta)$ is a classical Kottwitz parameter whose equivalence class depends only on the isomorphism class of $(\gamma_0, a, [b_0])$. Moreover $(\gamma_0, a, [b_0])$ is acceptable if and only if $(\gamma_0, \gamma, \delta)$ is so.

Finally, recall that $[b_0]$ is basic if $(\gamma_0, a, [b_0])$ is acceptable; in this case, $J^\circ_{b,\delta}$ (the connected centralizer of δ in J_{b_0}) is isomorphic to the inner form $J^{I_0}_{b_0}$ of I_0 defined by $[b_0]$. Indeed, $J_{b,\delta} \cong J_{b_0,\gamma_0}$ by construction, and $J_{b_0,\gamma_0} = J^{I_0}_{b_0}$ as $\mathbb{Q}_p$-subgroups of J_b since centralizing γ_0 is equivalent to being contained in I_0 (inside of G).

4.2.19 In light of (4.2.16), 4.2.18, and Lemma 4.2.17, we have constructed the following maps

$$\mathcal{LRP}_{\mathbf{b}}/\mathrm{conj.} \xrightarrow{\mathbf{t}_\tau} \mathcal{KP}_{\mathbf{b}}/\mathrm{isom.} \xrightarrow{\mathrm{cl}} \mathcal{CKP}_{\mathbf{b}}/\mathrm{equiv.}$$

restricting to

$$\mathcal{LRP}_{\mathbf{b},\mathrm{acc}}/\mathrm{conj.} \xrightarrow{\mathbf{t}_\tau} \mathcal{KP}_{\mathbf{b},\mathrm{acc}}/\mathrm{isom.} \xrightarrow{\mathrm{cl}} \mathcal{CKP}_{\mathbf{b},\mathrm{acc}}/\mathrm{equiv.}$$

4.3 Kottwitz parameters arising from LR pairs

Recall from §4.2 we have defined a map $\mathbf{t}_\tau : \mathcal{LRP}_{\mathrm{sa}}/\mathrm{conj.} \to \mathcal{KP}/\mathrm{isom.}$ for any $\tau \in \Gamma(\mathfrak{E}^p)_1$. We have seen in 4.2.19 that the image of acceptable $\mathbf{b}$-admissible LR pairs lies in the set of acceptable $\mathbf{b}$-admissible Kottwitz parameters. Its image also has trivial Kottwitz invariant by the following lemma. The goal of this section is a surjectivity result, namely that every acceptable $\mathbf{b}$-admissible Kottwitz parameter with trivial Kottwitz invariant is in the image of $\mathbf{t}_\tau$ for $\tau \in \Gamma(\mathfrak{E}^p)_0$ tori-rational.

Lemma 4.3.1 ([KSZ21, Prop. 3.6.3])**.** *Let (ϕ, ε) be a semi-admissible LR pair, $\tau \in \Gamma(\mathfrak{E}^p)_1$ a tori-rational element, and $\mathfrak{c} = \mathbf{t}(\phi, \varepsilon, \tau(\phi))$ the associated Kottwitz parameter. Then the Kottwitz invariant $\alpha(\mathfrak{c})$ is zero.*

4.3.2 Given a connected reductive group H over $\mathbb{Q}_p$ and a cocharacter $\mu : \mathbb{G}_m \to H_{\overline{\mathbb{Q}}_p}$, we define $[b_{\mathrm{bas}}(\mu)] \in B(H)$ to be the unique basic class in $B(H, \mu)$. For each $x \in X$, write $h_x : \mathbb{S} \to G_{\mathbb{R}}$ for the corresponding morphism.

Lemma 4.3.3. *Let $(\gamma_0, a, [b_0]) \in \mathcal{KP}_{\mathbf{b},\mathrm{acc}}$. Write $I_0 := G^{\circ}_{\gamma_0}$. Possibly after changing $(\gamma_0, a, [b_0])$ within its isomorphism class (cf. 4.2.9), there exist a maximal torus $T \subset I_0$ over $\mathbb{Q}$ and $x \in X$ such that*

- *h_x factors through $T_{\mathbb{R}}$ (so $\mu_x \in X_*(T)$; the notation is as in 2.4.1) and*
- *$[b_x] \in B(T)$ maps to $\mathbf{b}$ in $B(G)$, where $[b_x] := [b_{\mathrm{bas}}(\mu_x^{-1})]$,*
- *$\nu_{b_x} = \nu_{b_0}$, where $b_x \in T(L)$ and $b_0 \in I_0(L)$ are representatives of $[b_x]$ and $[b_0]$. (It is implicit here that ν_{b_x}, ν_{b_0} are independent of choice of b_x, b_0.)*

Proof. This lemma and its proof are inspired from [KMPS22, Prop. 1.2.5]. The main difference lies in the third bullet point and the requirement that $T \subset I_0$, which are not found in *loc. cit.* (In their setting, there is simply no I_0.) The basic idea is to find suitable $T_p \subset I_{0,\mathbb{Q}_p}$ and $T_\infty \subset I_{0,\mathbb{R}}$, and then to globalize them to a maximal torus in I_0 over $\mathbb{Q}$. We will sketch the argument while explaining in some detail how to ensure the additional properties.

At p, we need some preparation to apply [KMPS22, Cor. 1.1.17]. Since $[b_0]$ is basic in $B(I_0)$ (Lemma 4.2.11), the morphism $\nu_{b_0} : \mathbb{D} \to G$ is defined over $\mathbb{Q}_p$ and factors through the center of I_0. Thus I_0 is contained in the $\mathbb{Q}_p$-rational Levi subgroup M_{b_0} of $G_{\mathbb{Q}_p}$. By [KMPS22, Cor. 1.1.15], we have $[b_0] \in B(M_{b_0}, \mu_0^{-1})$ for a suitable cocharacter $\mu_0 : \mathbb{G}_m \to M_{b_0}$ which belongs to the G-conjugacy class $\{\mu_X\}$; thus $([b_0], \mu_0)$ is M_{b_0}-admissible in the terminology of *loc. cit.* Moreover $[b_0]$ is basic in $B(M_{b_0})$ as ν_{b_0} remains central in M_{b_0}. On the other hand, the basic class $[b_0]$ determines an inner form $J^{I_0}_{b_0}$ of $I_{0,\mathbb{Q}_p}$. The $\mathbb{Q}_p$-group $J^{I_0}_{b_0}$ is contained in J_{b_0}, where J_{b_0} is defined with $G_{\mathbb{Q}_p}$ (rather than $I_{0,\mathbb{Q}_p}$) as the ambient group, but since $[b_0]$ is basic in $B(M_{b_0})$, the definition of J_{b_0} does not change if the ambient group is M_{b_0}; moreover J_{b_0} is an inner form of M_{b_0}. Choose an elliptic maximal torus T' in $I_{0,\mathbb{Q}_p}$, then it transfers to a maximal torus of $J^{I_0}_{b_0}$, still to be denoted by T'. Thus T' may be viewed as a maximal torus of J_{b_0}, which in turn transfers to a maximal torus T_p of M_{b_0}.

In the setting of the preceding paragraph, the existence result of [KMPS22, Cor. 1.1.17] applies with our b_0, μ_0, M_{b_0} playing the roles of their b, μ, G. As in the second paragraph in the proof of [KMPS22, Prop. 1.2.5], the output is a cocharacter $\mu_p \in X_*(T_p)$ which belongs to the conjugacy class $\{\mu_X\}$ (under $T_p \subset G_{\mathbb{Q}_p}$) such that $[b_{\mathrm{bas}}(\mu_p^{-1})] \in B(T_p)$ maps to $[b_0] \in B(M_{b_0})$. If we write $b_p \in T_p(L)$ for a representative of $[b_{\mathrm{bas}}(\mu_p^{-1})]$, then as an immediate consequence, $[b_{\mathrm{bas}}(\mu_p^{-1})]$ maps to $\mathbf{b} \in B(G)$, and $\nu_{b_p} = \nu_{b_0}$. (A priori ν_{b_p} and ν_{b_0} are only conjugate in M_{b_0} but since ν_{b_0} is central, the two are equal.)

Now we turn to the matter at ∞. Since γ_0 is elliptic in $G(\mathbb{R})$, we can choose a maximal torus $T_\infty \subset G_{\mathbb{R}}$ containing γ_0, that is, $T_\infty \subset I_{0,\mathbb{R}}$. Since every element of X factors through an elliptic maximal torus of $G_{\mathbb{R}}$ and all such tori are $G(\mathbb{R})$-conjugate, we can choose $x \in X$ such that h_x factors through T_∞.

We are ready to globalize. First, there exists a maximal torus $T \subset I_0$ over $\mathbb{Q}$ such that $T_{\mathbb{Q}_p}$ and $T_{\mathbb{R}}$ are conjugate to T_p and T_∞ by elements of $I_0(\mathbb{Q}_p)$ and $I_0(\mathbb{R})$, respectively, by [KMPS22, Lem. 1.2.2]. Next, the rest of the argument in [KMPS22, Prop. 1.2.5] (other than we chose T_p, T_∞, T, μ_p somewhat differently from *loc. cit.*) applies verbatim and shows the existence of $x \in X$ such that T and x satisfy the properties stated in the lemma; the properties for b_x follow from those for b_p since the argument produces $(T, [b_x])$ that is conjugate to $(T_p, [b_p])$ by an element of $I_0(\mathbb{Q}_p)$. □

Proposition 4.3.4. *Let $(\gamma_0, a, [b_0]) \in \mathcal{KP}_{\mathbf{b},\mathrm{acc}}$. Possibly after changing $(\gamma_0, a, [b_0])$ by an isomorphism, there is an admissible morphism ϕ_0 such that $(\phi_0, \gamma_0) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathbf{b},\mathrm{acc}}$.*

Proof. By Lemma 4.3.3, there exist a maximal torus $T \subset I_0$ over $\mathbb{Q}$ and $x \in X$ such that h_x factors through $T_{\mathbb{R}}$ and $[b_{\mathrm{bas}}(\mu_x^{-1})] \in B(T)$ maps to $\mathbf{b} \in B(G)$. In particular, (T, h_x) forms a special point datum. Write $i : T \hookrightarrow G$ for the embedding induced by $I_0 \subset G$.

Let $\phi_0 = i \circ \psi_{\mu_x}$ be the admissible morphism induced from the special point datum (T, h) as in 4.2.6. Observing that $\gamma_0 \in T(\mathbb{Q})$, we can form the gg LR pair (ϕ_0, γ_0) as explained there. Let us show that this LR pair is $\mathbf{b}$-admissible and acceptable.

We check $\mathbf{b}$-admissibility first. Consider $\psi_{\mu_x} : \mathfrak{Q} \to \mathfrak{G}_T$ and its p-part $\psi_{\mu_x}(p) \circ \zeta_p : \mathfrak{G}_p \to \mathfrak{G}_T(p)$. As in 2.5.8, the latter is conjugate by some element $y \in T(\overline{\mathbb{Q}}_p)$ to an unramified morphism $\theta : \mathfrak{G}_p \to \mathfrak{G}_T(p)$ which gives rise to an element $b_\theta \in T(\mathbb{Q}_p^{\mathrm{ur}})$ and a class $[b_\theta] \in B(T)$. By [KSZ21, Lem. 2.2.10], the image $\kappa_T([b_\theta]) \in X_*(T)_{\Gamma_p}$

of $[b_\theta]$ under the Kottwitz map is equal to the image of $\mu_x^{-1} \in X_*(T)$. It follows that $[b_\theta] = [b_{\mathrm{bas}}(\mu_x^{-1})]$. The second condition of Lemma 4.3.3 tells us that $[b_\theta] \in B(T)$ maps to $\mathbf{b}$. Since $i \circ \psi_{\mu_x}(p) \circ \zeta_p$ is conjugate by y to the unramified morphism $i \circ \theta$, and since $[b_{i\circ\theta}] = i([b_\theta]) = \mathbf{b}$, we see that $\phi_0 = i \circ \psi_{\mu_x}$ is $\mathbf{b}$-admissible as desired.

Now we check acceptability. The element $y \in T(\overline{\mathbb{Q}}_p)$ above commutes with γ_0 since $T \subset I_0$. So the recipe of 4.2.3 assigns to (ϕ_0, γ_0) a pair $(b_{i\circ\theta}, y\gamma_0 y^{-1}) = (i(b_\theta), \gamma_0)$. Thus we need to show that γ_0 is ϱ-acceptable with respect to $i(b_\theta)$, cf. Definition 4.2.5. We have assumed that γ_0 is ϱ-acceptable with respect to $b_0 \in I_0(L)$. Since $\gamma_0 \in M_{i(b_\theta)}(L) \cap M_{b_0}(L)$ (following from the fact that $i(b_\theta), b_0$ commute with γ_0), and since the slope morphisms of $i(b_\theta)$ and b_0 coincide by the last condition of Lemma 4.3.3, the paragraph below Definition 2.2.4 tells us that the ϱ-acceptability with respect to $b_0 \in I_0(L)$ implies the same property with respect to $i(b_\theta)$. □

Lemma 4.3.5. *Let (ϕ, ε) be an acceptable LR pair, and suppose that ϕ^Δ is defined over $\mathbb{Q}$ and $\varepsilon \in G(\mathbb{Q})$ (in particular, this applies to any acceptable gg pair). Then the inclusion $I_{\phi,\varepsilon} \hookrightarrow G_\varepsilon$ over $\overline{\mathbb{Q}}$ is an isomorphism.*

Proof. Recall from 2.5.2 that $I_{\phi,\overline{\mathbb{Q}}}$ is the centralizer of (the image of) ϕ^Δ in $G_{\overline{\mathbb{Q}}}$, so our goal is to show that any element commuting with ε must commute with ϕ^Δ.

We begin by showing that any element commuting with ε must commute with $\phi^\Delta \circ \nu(p)$ and $\phi^\Delta \circ \nu(\infty)$.

At p: let $g \in G(\overline{\mathbb{Q}}_p)$ such that $\theta = \mathrm{Int}(g) \circ \phi(p) \circ \zeta_p$ is unramified, and let $b_\theta \in G(\mathbb{Q}_p^{\mathrm{ur}})$ defined by $\theta(d_\sigma) = b_\theta \rtimes \sigma$. Write $\varepsilon' = \mathrm{Int}(g)\varepsilon$. By Lemma 2.5.9 we have

$$\mathrm{Int}(g) \circ \phi^\Delta \circ \nu(p) = (\mathrm{Int}(g) \circ \phi(p) \circ \zeta_p)^\Delta = -\nu_{b_\theta},$$

so the centralizer of $\mathrm{Int}(g) \circ \phi^\Delta \circ \nu(p)$ is M_{b_θ}. On the other hand, (b_θ, ε') is associated with our acceptable pair (ϕ, ε) by 4.2.3. In particular, we can apply Lemma 2.2.10 to conclude that $G_{\varepsilon'} \subset M_{b_\theta}$.

Since the centralizer of $\mathrm{Int}(g) \circ \phi^\Delta \circ \nu(p)$ is M_{b_θ} we see the centralizer of $\phi^\Delta \circ \nu(p)$ is $\mathrm{Int}(g^{-1})M_{b_\theta}$, and likewise we have $G_\varepsilon = \mathrm{Int}(g^{-1})G_{\varepsilon'}$. The above analysis then shows that $\mathrm{Int}(g^{-1})G_{\varepsilon'} = \mathrm{Int}(g^{-1})M_{b_\theta}$, which is to say that the centralizer of ε is contained in the centralizer of $\phi^\Delta \circ \nu(p)$, as desired.

At ∞: as in the proof of [KSZ21, Lem. 3.1.9], the fact is that $\phi^\Delta \circ \nu(\infty)$ is central in G, and therefore any element commuting with ε trivially commutes with $\phi^\Delta \circ \nu(\infty)$.

Now, suppose that $g \in G(\overline{\mathbb{Q}})$ commutes with ε, and we want to see that g commutes with ϕ^Δ.

Recall that ϕ^Δ is a morphism $Q \to G$ where $Q = \varprojlim_L Q^L$ is the kernel of $\mathfrak{Q}$. For each finite Galois $L/\mathbb{Q}$, the torus Q^L is generated by the $\mathrm{Gal}(L/\mathbb{Q})$-conjugates of the images of $\nu(p)^L$ and $\nu(\infty)^L$. Thus Q is generated by the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugates of the images of $\nu(p)$ and $\nu(\infty)$.

For any $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the conjugate $\rho(g)$ again commutes with ε by our hypothesis that ε is rational, and therefore by the above arguments $\rho(g)$ commutes with $\phi^\Delta \circ \nu(v)$ for $v = p, \infty$. Applying ρ^{-1} and using our hypothesis that ϕ^Δ is defined over $\mathbb{Q}$, we see that g commutes with $\phi^\Delta \circ \rho^{-1}(\nu(v))$ for $v = p, \infty$. Since ρ was arbitrary and the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugates of the images of $\nu(v)$ generate Q, this implies that g commutes with ϕ^Δ, as desired. □

4.3.6 Let (ϕ, ε) be a gg acceptable pair, so that letting $q_\rho \in \mathfrak{Q}$ a lift of $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we have $\phi(q_\rho) = g_\rho \rtimes \rho$ with $g_\rho \in G_\varepsilon^\circ$. The group I_ϕ is defined as an inner form of $Z_G(\phi^\Delta)$ by the cocycle

$$\rho \mapsto \mathrm{Int}(g_\rho) \in \mathrm{Aut}((G_\varepsilon^\circ)_{\overline{\mathbb{Q}}}) \qquad \rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

In view of the isomorphism of Lemma 4.3.5, this produces compatible inner twists

$$(I_{\phi,\varepsilon}^\circ)_{\overline{\mathbb{Q}}} \xrightarrow{\sim} (G_\varepsilon^\circ)_{\overline{\mathbb{Q}}},$$
$$(I_{\phi,\varepsilon})_{\overline{\mathbb{Q}}} \xrightarrow{\sim} (G_\varepsilon)_{\overline{\mathbb{Q}}}$$

defined by the same cocycle.

Lemma 4.3.7. *Suppose that $(\gamma_0, a, [b_0]), (\gamma_0, a_1, [b_1]) \in \mathcal{KP}_{\mathbf{b},\mathrm{acc}}$ share the same element γ_0. Then $\nu_{b_0} = \nu_{b_1}$.*

Proof. Both Kottwitz parameters are assumed to be $\mathbf{b}$-admissible, so $[b_0]$ and $[b_1]$ both map to $\mathbf{b}$ in $B(G)$. In particular, b_0 and b_1 are σ-conjugate in $G(L)$. Furthermore, the semi-simple element γ_0 as an element of $G(L)$ lies in both $J_{b_0}(\mathbb{Q}_p)$ and $J_{b_1}(\mathbb{Q}_p)$, and is acceptable with respect to both. Thus we are in the situation of Lemma 2.2.11, and we conclude that $\nu_{b_0} = \nu_{b_1}$. □

Proposition 4.3.8. *Suppose that $(\gamma_0, a, [b_0]) \in \mathcal{KP}_{\mathbf{b},\mathrm{acc}}$ has trivial Kottwitz invariant and that there is a ϕ_0 such that (ϕ_0, γ_0) belongs to $\mathcal{LRP}^{\mathrm{gg}}_{\mathbf{b},\mathrm{acc}}$. Then there exists $(\phi_1, \varepsilon_1) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathbf{b},\mathrm{acc}}$ such that*

$$\mathbf{t}(\phi_1, \varepsilon_1, 1) \cong (\gamma_0, a, [b_0]).$$

This is the analogue of [KSZ21, Prop. 3.5.9], except that their "p^n-admissible" hypothesis has been replaced by our "acceptable **b**-admissible" hypothesis. We briefly sketch how their proof carries over to our case.

Proof. Write $\mathbf{t}(\phi_0, \gamma_0, 1) = (\gamma_0, a', [b'_0])$, and $I_0 = G^{\circ}_{\gamma_0}$. By Lemma 4.2.17, our hypothesis that (ϕ_0, γ_0) is acceptable and **b**-admissible implies that $(\gamma_0, a', [b'_0])$ is acceptable and **b**-admissible.

Lemma 4.2.11 tells us that $\nu_{b'_0}$ is central in I_0. This is the first ingredient; we also need the fact from Lemma 4.3.7 that $\nu_{b_0} = \nu_{b'_0}$; and the fact from 4.3.6 of a canonical inner twisting $(I^{\circ}_{\phi_0,\gamma_0})_{\overline{\mathbb{Q}}} \xrightarrow{\sim} I_{0,\overline{\mathbb{Q}}}$.

In the presence of these three ingredients, the proof of [KSZ21, Prop. 3.5.9] carries over without modification to show the existence of a gg semi-admissible LR pair (ϕ_1, ε_1) with $\mathbf{t}(\phi_1, \varepsilon_1, 1) = (\gamma_0, a, [b_0])$. By Lemma 4.2.17, (ϕ_1, ε_1) is acceptable and **b**-admissible. ☐

The last step is to incorporate τ-twists. We collect the full result in the following proposition.

Proposition 4.3.9. *Let $\tau \in \Gamma(\mathfrak{E}^p)_0$ be a tori-rational element. If $(\gamma_0, a, [b_0]) \in \mathcal{KP}_{\mathbf{b},\mathrm{acc}}$ has trivial Kottwitz invariant, then there exists $(\phi, \varepsilon) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathbf{b},\mathrm{acc}}$ such that*

$$\mathbf{t}(\phi, \varepsilon, \tau(\phi)) = (\gamma_0, a, [b_0]).$$

Proof. This is our analogue of [KSZ21, Prop. 3.6.5]. The proof in their case applies equally well here, with our Propositions 4.3.4 and 4.3.8 replacing their and Propositions 3.4.8 and 3.5.9 respectively.

The resulting LR pair (ϕ, ε) is furthermore acceptable and **b**-admissible by Lemma 4.2.17. ☐

4.3.10 As before, $\tau \in \Gamma(\mathfrak{E}^p)_0$ is tori-rational. Having completed this result, and combining it with the discussion at the beginning of this section, we conclude that the image of the map (4.2.19)

$$\mathbf{t}_\tau : \mathcal{LRP}_{\mathbf{b},\mathrm{acc}}/\mathrm{conj.} \to \mathcal{KP}_{\mathbf{b},\mathrm{acc}}/\mathrm{isom.}, \qquad (\phi, \varepsilon) \mapsto \mathbf{t}(\phi, \varepsilon, \tau(\phi)),$$

consists of exactly those with trivial Kottwitz invariant.

4.4 LR pairs mapping to the same Kottwitz parameter

Now that we understand the image of the map $\mathbf{t}_\tau : \mathcal{LRP}_{\mathbf{b},\mathrm{acc}}/\mathrm{conj.} \to \mathcal{KP}_{\mathbf{b},\mathrm{acc}}/\mathrm{isom.}$, we now examine the fibers of this map in terms of cohomological twists.

4.4.1 Recall from 2.5.2 that we can twist a morphism $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ by a cocycle $e \in Z^1(\mathbb{Q}, I_\phi)$ to get another morphism $e\phi$. We saw in 2.5.13 that if ϕ is admissible, then $e\phi$ is again admissible exactly when e lies in $Ш^{\infty}_G(\mathbb{Q}, I_\phi)$. We can also twist LR pairs. For an LR pair (ϕ, ε) and cocycle $e \in Z^1(\mathbb{Q}, I_{\phi,\varepsilon}) \subset Z^1(\mathbb{Q}, I_\phi)$, we define the twist to be $(e\phi, \varepsilon)$ by simply twisting the morphism. By [KSZ21, Lem. 3.2.5] this is again an LR pair. As in the case of twisting morphisms, two twists $(e\phi, \varepsilon)$ and $(e'\phi, \varepsilon)$ are conjugate by $G(\overline{\mathbb{Q}})$ exactly when e, e' define the same class in $H^1(\mathbb{Q}, I_{\phi,\varepsilon})$ ([KSZ21, Lem. 3.2.6]).

Now, suppose that (ϕ, ε) is gg. Then [KSZ21, Lem. 3.2.5] also tells us that if $e \in Z^1(\mathbb{Q}, I^{\circ}_{\phi,\varepsilon}) \subset Z^1(\mathbb{Q}, I_{\phi,\varepsilon})$ then the twist $(e\phi, \varepsilon)$ is again gg. This gives us a map $Z^1(\mathbb{Q}, I^{\circ}_{\phi,\varepsilon}) \to \mathcal{LRP}^{\mathrm{gg}}$, which descends to a map

$$H^1(\mathbb{Q}, I^{\circ}_{\phi,\varepsilon}) \to \mathcal{LRP}^{\mathrm{gg}}/\mathrm{conj.}$$

Lemma 4.4.2. *Let (ϕ, ε) a gg acceptable LR pair, and $\mathrm{Int}(g)\varepsilon \in G(\mathbb{Q})$ a rational element stably conjugate to ε, i.e., $g \in G(\overline{\mathbb{Q}})$ and $g^{-1}\rho(g) \in G^{\circ}_{\varepsilon}$ for $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then the conjugate $(\mathrm{Int}(g) \circ \phi, \mathrm{Int}(g)\varepsilon)$ is again gg and acceptable.*

Proof. Acceptability is insensitive to conjugacy, so $(\mathrm{Int}(g) \circ \phi, \mathrm{Int}(g)\varepsilon)$ is acceptable. That it is also gg is proven as in [KSZ21, Lem. 3.6.4], with our 4.3.6 replacing their 3.2.14. ☐

Lemma 4.4.3. *Suppose that $(\phi, \varepsilon), (\phi', \varepsilon) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathbf{b},\mathrm{acc}}$ share the same $\varepsilon \in G(\mathbb{Q})$. Then $\phi^\Delta = \phi'^\Delta$.*

Proof. By the same reasoning as the last paragraphs of the proof of Lemma 4.3.5, it suffices to show that $\phi^\Delta \circ \nu(v) = \phi'^\Delta \circ \nu(v)$ for $v = p, \infty$ (this implies that ϕ^Δ and ϕ'^Δ agree on a generating set for $\mathfrak{Q}^\Delta = Q$, and being morphisms they must then agree everywhere).

At ∞: by [KSZ21, Lem. 3.1.9], simply the fact that both ϕ and ϕ' are admissible implies that $\phi^\Delta \circ \nu(\infty) = \phi'^\Delta \circ \nu(\infty)$.

At p: since (ϕ, ε) is gg, we can factor

$$\phi : \mathfrak{Q} \to \mathfrak{G}_{G^\circ_\varepsilon} \to \mathfrak{G}_G,$$

and therefore we can conjugate $\phi(p) \circ \zeta_p$ to an unramified morphism θ by an element $u \in G^\circ_\varepsilon(\overline{\mathbb{Q}})$. Let $b_\theta \in G^\circ_\varepsilon(\mathbb{Q}_p^{\mathrm{ur}})$ be the corresponding element as in 2.5.7. In the same way we can conjugate $\phi'(p) \circ \zeta_p$ by an element $u' \in G^\circ_\varepsilon(\overline{\mathbb{Q}})$ and produce an element $b'_\theta \in G^\circ_\varepsilon(\mathbb{Q}_p^{\mathrm{ur}})$. Then the pairs (b_θ, ε) and (b'_θ, ε) are associated with the LR pairs (ϕ, ε) and (ϕ', ε) as in 4.2.3, respectively.

We have assumed that our LR pairs are acceptable and $\mathbf{b}$-admissible, which implies that b_θ, b'_θ and ε satisfy the hypotheses of Lemma 2.2.11, and we conclude that $\nu_{b_\theta} = \nu_{b'_\theta}$.

On the other hand, we have

$$\begin{aligned}\mathrm{Int}(u) \circ \phi^\Delta \circ \nu(p) &= (\mathrm{Int}(u) \circ \phi(p) \circ \zeta_p)^\Delta \overset{*}{=} -\nu_{b_\theta} \\ &= -\nu_{b'_\theta} \overset{*}{=} (\mathrm{Int}(u') \circ \phi'(p) \circ \zeta_p)^\Delta = \mathrm{Int}(u') \circ \phi'^\Delta \circ \nu(p)\end{aligned}$$

where the starred equalities are given by Lemma 2.5.9. By our acceptable hypothesis, we can apply Lemma 2.2.10 to see that G_ε commutes with $\nu_{b_\theta} = \nu_{b'_\theta}$. But the above equation demonstrates that $\phi^\Delta \circ \nu(p)$ and $-\nu_{b_\theta}$ and $-\nu_{b'_\theta}$ and $\phi'^\Delta \circ \nu(p)$ are all conjugate by $G^\circ_\varepsilon(\overline{\mathbb{Q}})$, so they must all be equal, and in particular $\phi^\Delta \circ \nu(p) = \phi'^\Delta \circ \nu(p)$. □

Now we are prepared to show that points in the same fiber of $\mathbf{t}$ are related by an H^1-twist.

Lemma 4.4.4. *Suppose that $(\phi, \varepsilon), (\phi', \varepsilon) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathbf{b},\mathrm{acc}}$ give rise to isomorphic Kottwitz parameters under $\mathbf{t}_\tau$. Then the conjugacy classes of (ϕ, ε) and (ϕ', ε') are related by twisting by an element of $H^1(\mathbb{Q}, I^\circ_{\phi,\varepsilon})$.*

Proof. In fact we only need to make a weaker assumption that the rational elements, say γ_0 and γ'_0, appearing in the two Kottwitz parameters are stably conjugate. (This assumption is insensitive to τ-twists.) So our hypothesis implies that ε and ε' are stably conjugate, and by Lemma 4.4.2 we can conjugate (ϕ', ε') to a gg pair (ϕ_0, ε) which is again acceptable and $\mathbf{b}$-admissible. By Lemma 4.4.3 we have $\phi^\Delta = \phi_0^\Delta$. As in Lemma 2.5.2 we can choose $e \in Z^1(\mathbb{Q}, I_\phi)$ so that $\phi_0 = e\phi$. Now write

$$\phi(q_\rho) = g_\rho \rtimes \rho, \qquad \phi_0(q_\rho) = e\phi(q_\rho) = e_\rho g_\rho \rtimes \rho$$

where as usual $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $q_\rho \in \mathfrak{Q}$ is a lift of ρ. Since our LR pairs are gg, we have $g_\rho \in G^\circ_\varepsilon$ and $e_\rho g_\rho \in G^\circ_\varepsilon$, so we conclude that $e_\rho \in G^\circ_\varepsilon$. By 4.3.5 this shows that in fact $e \in Z^1(\mathbb{Q}, I^\circ_{\phi,\varepsilon})$, demonstrating that the conjugacy classes of (ϕ, ε) and (ϕ', ε') are related by twisting by $H^1(\mathbb{Q}, I^\circ_{\phi,\varepsilon})$. □

We also have an analogue of Proposition 2.5.13 characterizing which twists of a semi-admissible LR pair are semi-admissible.

Proposition 4.4.5. *If $(\phi, \varepsilon) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathrm{sa}}$ and $e \in Z^1(\mathbb{Q}, I^\circ_{\phi,\varepsilon})$, then the twist $(e\phi, \varepsilon)$ is gg and semi-admissible exactly when e lies in $Ш^\infty_G(\mathbb{Q}, I^\circ_{\phi,\varepsilon})$. For such an e, if (ϕ, ε) is $\mathbf{b}$-admissible then so is $(e\phi, \varepsilon)$; if (ϕ, ε) is acceptable then so is $(e\phi, \varepsilon)$.*

Proof. The first assertion is [KSZ21, Prop. 3.2.19]. For the second assertion, we may write $\mathbf{t}_\tau(\phi, \varepsilon) = (\gamma_0, a, [b_0])$ and $\mathbf{t}_\tau(e\phi, \varepsilon) = (\gamma_0, a', [b'_0])$ with $\gamma_0 = \varepsilon \in G(\mathbb{Q})$. By [KSZ21, Prop. 3.5.5], we have that ν_{b_0} and $\nu_{b'_0}$ are conjugate in I_0 and that the difference under the Kottwitz morphism $\kappa_{I_0}(b_0) - \kappa_{I_0}(b'_0) \in H^1_{\mathrm{ab}}(\mathbb{Q}_p, I_0)$ is equal to the image of e under

$$Ш^\infty_G(\mathbb{Q}, I^\circ_{\phi,\varepsilon}) \to \ker(H^1_{\mathrm{ab}}(\mathbb{Q}, I_0) \to H^1_{\mathrm{ab}}(\mathbb{Q}, G)) \to \ker(H^1_{\mathrm{ab}}(\mathbb{Q}_p, I_0) \to H^1_{\mathrm{ab}}(\mathbb{Q}_p, G)),$$

in their notation. The fact that $\kappa_{I_0}(b_0) - \kappa_{I_0}(b_0')$ maps trivially into $H^1_{\mathrm{ab}}(\mathbb{Q}_p, G)$ tells us that the images of $[b_0], [b_0'] \in B(I_0)$ in $B(G)$ have equal image under the Kottwitz morphism for G. Since ν_{b_0} and $\nu_{b_0'}$ are conjugate in G, we deduce that the images of $[b_0], [b_0'] \in B(I_0)$ in $B(G)$ are equal by [Kot97, p. 4.13]. Therefore $(\gamma_0, a, [b_0])$ is $\mathbf{b}$-admissible if and only if $(\gamma_0, a', [b_0'])$ is. It follows that (ϕ, ε) is $\mathbf{b}$-admissible if and only if $(e\phi, \varepsilon)$ is.

Finally, suppose that (ϕ, ε) is acceptable. Then $(\gamma_0, a, [b_0])$ is also acceptable so ν_{b_0} is central in I_0 by Lemma 4.2.11. Since ν_{b_0} and $\nu_{b'}$ are conjugate in I_0 again by [KSZ21, Prop. 3.5.5], it follows that $\nu_{b_0} = \nu_{b_0'}$. Now $\gamma_0 \in J_{b_0}(\mathbb{Q}_p) \cap J_{b_0'}(\mathbb{Q}_p)$ (intersected in $G(L)$), we see from the discussion below Definition 2.2.4 that if γ_0 is acceptable with respect to b_0 then the same is true with respect to b_0'; that is, $\mathbf{t}_\tau(e\phi, \varepsilon)$ is acceptable. Hence $(e\phi, \varepsilon)$ is acceptable. □

Combining Proposition 4.4.5 with the discussion of 4.4.1, we have for each $(\phi, \varepsilon) \in \mathcal{LRP}^{\mathrm{gg}}_{\mathbf{b},\mathrm{acc}}$ a map

$$\eta_{\phi,\varepsilon} : Ш^\infty_G(\mathbb{Q}, I^\circ_{\phi,\varepsilon}) \to \mathcal{LRP}^{\mathrm{gg}}_{\mathbf{b},\mathrm{acc}}/\mathrm{conj}.$$

By Lemma 4.4.4 and Proposition 4.4.5, two LR pairs in $\mathcal{LRP}^{\mathrm{gg}}_{\mathbf{b},\mathrm{acc}}$ giving rise to isomorphic Kottwitz parameters must both lie in the image of one such map $\eta_{\phi,\varepsilon}$.

4.5 Point-Counting Formula

We return to the task of analyzing the right hand side of (4.1.8), putting ourselves in that setting. To understand the fixed point set $\mathrm{Fix}(UgU)$, a crucial ingredient is a description of $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ as a $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$-set by Theorem 3.6.1. Thereby we have a bijection

$$\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)/U \xrightarrow{\sim} \coprod_{[\phi]} I_\phi(\mathbb{Q})_{\tau(\phi)} \backslash \big(X^p(\phi) \times X_p^{\mathbf{b}}(\phi)\big)/U, \tag{4.5.1}$$

where ϕ ranges over a set of representatives for conjugacy classes of $\mathbf{b}$-admissible morphisms, and with the $I_\phi(\mathbb{Q})$-action twisted by a tori-rational element $\tau \in \Gamma(\mathcal{H})_0$. Recall from 3.5 that we can lift $\tau(\phi) \in \mathcal{H}(\phi)$ to an element of $I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p)$, which we will also call $\tau(\phi)$ by abuse of notation. Before going further, we prove a group-theoretic lemma that will be needed.

Lemma 4.5.2. *Let U be a sufficiently small compact open subgroup of $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$. Then*

1. *$Z_G(\mathbb{Q}) \cap U = \{1\}$, and*
2. *for each $\mathbf{b}$-admissible morphism ϕ and $\tau(\phi) \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p)$, the stabilizer in $I_\phi(\mathbb{Q})$ of $(x^p, x_p) \in G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)/U$ for the $\tau(\phi)$-twisted action is $Z_G(\mathbb{Q}) \cap U$ for every (x^p, x_p).*

Proof. (1) Since G is part of a Shimura datum of Hodge type, Z_G° satisfies the Serre condition—this is (equivalent to) the condition that Z_G° is isogenous over $\mathbb{Q}$ to a torus $T^+ \times T^-$ where T^+ is split over $\mathbb{Q}$ and T^- is compact over $\mathbb{R}$. This implies that $Z_G^\circ(\mathbb{Q})$ is discrete in $Z_G^\circ(\mathbb{A}_f)$ (e.g., [KSZ21, Lem. 1.5.5]), and via

$$Z_G^\circ(\mathbb{A}_f) \hookrightarrow G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$$

we see $Z_G^\circ(\mathbb{Q})$ is discrete in $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ (the embedding $Z_G \hookrightarrow J_b$ coming from the fact that J_b is an inner form of a Levi subgroup of G). Since $[Z_G(\mathbb{Q}) : Z_G^\circ(\mathbb{Q})]$ is finite, $Z_G(\mathbb{Q})$ is also discrete in $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$. Thus any sufficiently small compact open subgroup U will intersect $Z_G(\mathbb{Q})$ trivially.

(2) The proof of [KSZ21, Lem. 3.7.2(i)] carries over to our case, the only difference being the component at p. In our case $\varepsilon \in I_\phi(\mathbb{Q})$ stabilizing an element $\varepsilon x_p = x_p \bmod U_p$ implies that ε is contained in the compact subgroup $x_p U_p x_p^{-1}$ of $J_b(\mathbb{Q}_p)$ (or of $G(\overline{\mathbb{Q}}_p)$ via the embedding $J_b \to G$ over $\overline{\mathbb{Q}}_p$), providing the necessary ingredient for the proof. □

Returning to the matter of $\mathrm{Fix}(UgU)$ and (4.5.1), we consider left $I_\phi(\mathbb{Q})$-sets

$$X^\phi := \big(X^p(\phi)\times X_p^{\mathbf{b}}(\phi)\big)/U \quad\text{and}\quad Y^\phi := \big(X^p(\phi)\times X_p^{\mathbf{b}}(\phi)\big)/(U\cap gUg^{-1}),$$

and maps $a,b : Y^\phi\to X^\phi$ given by $a : x\mapsto x \bmod U$ and $b : x\mapsto xg \bmod U$. Write $(I^\phi\backslash Y^\phi)^{a=b}$ for the subset of $I^\phi\backslash Y^\phi$ on which $a=b$. It follows from Definition 4.1.4 and (4.5.1) that

$$\mathrm{Fix}(UgU) = \coprod_\phi (I^\phi\backslash Y^\phi)^{a=b},$$

where the disjoint union runs over the same set of ϕ as above. On the other hand, we can apply Milne's combinatorial lemma [Mil92, Lem. 5.3] in this setting (taking $C=\{1\}$ in the notation there) as the necessary hypotheses are satisfied by Lemma 4.5.2. The outcome is that

$$(I^\phi\backslash Y^\phi)^{a=b} = \coprod_\varepsilon \mathcal{O}(\phi,\varepsilon,g,\tau), \qquad \text{where}$$
$$\mathcal{O}(\phi,\varepsilon,g,\tau) := I_{\phi,\varepsilon}(\mathbb{Q})_{\tau(\phi)}\backslash\{x\in \big(X^p(\phi)\times X_p^{\mathbf{b}}(\phi)\big)/(U\cap gUg^{-1}) : \varepsilon x = xg \bmod U\},$$

and the disjoint union runs over a set of representatives for conjugacy classes in $I_\phi(\mathbb{Q})$; as usual, $I_{\phi,\varepsilon}$ denotes the centralizer of ε in I_ϕ. Arguing as in the proof of [KSZ21, Lem. 3.7.3], we compute the local term $\mathrm{tr}([UgU] \mid (\mathscr{L}_\xi)_x) = \mathrm{tr}\,\xi(\varepsilon)$ (computed by viewing ε as an element of $G(\overline{\mathbb{Q}})$. All in all, using the same index sets for ϕ and ε as above, (4.1.8) for an acceptable function of the form $\mathbb{1}_{UgU}$ can be rewritten as

$$\mathrm{tr}(\mathbb{1}_{UgU}\mid H_c(\mathrm{Ig}_\Sigma,\mathscr{L}_\xi)) = \mathrm{vol}(U)\sum_\phi\sum_\varepsilon \#\mathcal{O}(\phi,\varepsilon,g,\tau)\cdot \mathrm{tr}\,\xi(\varepsilon). \tag{4.5.3}$$

The formula for a general acceptable function is an obvious linear combination thereof.

4.5.4 Suppose that $\mathfrak{c} = (\gamma_0, a, [b_0]) \in \mathcal{KP}_{\mathbf{b},\mathrm{acc}}$ has trivial Kottwitz invariant. (This is the case when $\mathfrak{c}$ arises from a $\mathbf{b}$-admissible acceptable LR pair.) BY Lemma 4.2.11, $[b_0]$ is basic in $B(I_0)$. We can define an inner form $I_\mathfrak{c}$ of $I_0 = G^\circ_{\gamma_0}$ as follows. Writing $a = (a_\ell)_{\ell\neq p,\infty}$, let I_ℓ be the inner form of I_0 over $\mathbb{Q}_\ell$ defined by a_ℓ (or to be precise, the image of a_ℓ in $H^1(\mathbb{Q}_\ell, I_0^{\mathrm{ad}})$). At p, let $I_p = J^{I_0}_{b_0}$ be the inner form of I_0 over $\mathbb{Q}_p$ defined by the basic class $[b_0]\in B(I_0)$. At ∞, let I_∞ be the inner form of I_0 over $\mathbb{R}$ which is compact modulo Z_G. By [KSZ21, Prop. 1.7.12], these local components determine a unique inner form I_c of I_0 over $\mathbb{Q}$ such that $I_\mathfrak{c}\otimes\mathbb{Q}_v \cong I_v$ for all places v of $\mathbb{Q}$.

Write $(\gamma_0,\gamma,\delta) = \mathrm{cl}(\gamma_0,a,[b_0]) \in \mathcal{CKP}_{\mathbf{b},\mathrm{acc}}$ for the corresponding classical parameter, well-defined up to equivalence. By construction $I_\mathfrak{c}\cong G^\circ_\gamma$ over $\mathbb{A}_f^p$. At p we have $I_{\mathfrak{c},\mathbb{Q}_p}\cong J^\circ_{b,\delta}$ as observed in §4.2.18.

Since both sides of (4.5.3) are proportional to the Haar measures on $G(\mathbb{A}_f^p)$ and $J_b(\mathbb{Q}_p)$, it is harmless to fix the Haar measures giving each of U^p and U_p volume 1. Every group of $\mathbb{Q}$-points will be endowed with the counting measure. We choose Haar measures on $I_\mathfrak{c}(\mathbb{A}_f^p)$ and $I_\mathfrak{c}(\mathbb{Q}_p)$, inducing Haar measures on isomorphic groups G°_γ and $J^\circ_{b,\delta}$ as well as a quotient measure on $I_\mathfrak{c}(\mathbb{Q})\backslash I_\mathfrak{c}(\mathbb{A}_f)$. For $f\in C_c^\infty(G(\mathbb{A}_f^p)\times J_b(\mathbb{Q}_p))$ define

$$T_\xi^f(\mathfrak{c}) = T_\xi^f(\gamma_0,a,[b_0]) := \mathrm{vol}\left(I^\circ_\mathfrak{c}(\mathbb{Q})\backslash I^\circ_\mathfrak{c}(\mathbb{A}_f)\right) O_{\gamma\times\delta}^{G(\mathbb{A}_f^p)\times J_b(\mathbb{Q}_p)}(f)\,\mathrm{tr}\,\xi(\gamma_0),$$

which is independent of the choice of Haar measures on $I_\mathfrak{c}(\mathbb{A}_f^p)$ and $I_\mathfrak{c}(\mathbb{Q}_p)$. The definition of $T_\xi^f(\mathfrak{c})$ depends on $\mathfrak{c}$ only up to isomorphism.

So far $\mathfrak{c}$ has been assumed to be acceptable. If $\mathfrak{c}$ is not acceptable (but still $\mathbf{b}$-admissible), simply set $T_\xi^f(\mathfrak{c}) := 0$.

4.5.5 Now let ϕ,ε be as in (4.5.3). Define $\iota_{I_\phi}(\varepsilon) := [I_{\phi,\varepsilon}(\mathbb{Q}) : I^\circ_{\phi,\varepsilon}(\mathbb{Q})]$. From the $\mathbf{b}$-admissible LR pair (ϕ,ε) we obtain $\mathbf{t}_\tau(\phi,\varepsilon) = (\gamma_0,[a],b)\in\mathcal{KP}_\mathbf{b}$ and $\mathrm{cl}(\gamma_0,[a],b) = (\gamma_0,\gamma,\delta)\in\mathcal{CKP}_\mathbf{b}$.

Lemma 4.5.6. *In the setting above,* $\#\mathcal{O}(\phi,\varepsilon,g,\tau) = \iota_{I_\phi}(\varepsilon)^{-1}T_\xi^f(\mathfrak{c})$.

Proof. We can mimic the proof of the analogous assertion [KSZ21, Lem. 3.7.4] for Shimura varieties. The proof there shows that

$$\mathrm{vol}\left(I^\circ_{\mathfrak{c}}(\mathbb{Q})\backslash I^\circ_{\mathfrak{c}}(\mathbb{A}_f)\right) = \mathrm{vol}\left(I^\circ_{\phi,\varepsilon}(\mathbb{Q})\backslash I^\circ_{\phi,\varepsilon}(\mathbb{A}_f)\right).$$

We can rewrite $\iota_{I_\phi}(\varepsilon)\#\mathcal{O}(\phi,\varepsilon,g,\tau)$ as

$$\begin{aligned}
&\# I^\circ_{\phi,\varepsilon}(\mathbb{Q})_{\tau(\phi)}\backslash\{x \in \left(X^p(\phi)\times X^{\mathbf{b}}_p(\phi)\right)/(U\cap gUg^{-1}) : \varepsilon x = xg \bmod U\}\cdot \mathrm{tr}\,\xi(\varepsilon)\\
&= \mathrm{vol}\left(I^\circ_{\phi,\varepsilon}(\mathbb{Q})\backslash I^\circ_{\phi,\varepsilon}(\mathbb{A}_f)\right)^{-1} C^p C_p\,\mathrm{tr}\,\xi(\varepsilon) = \mathrm{vol}\left(I^\circ_{\mathfrak{c}}(\mathbb{Q})\backslash I^\circ_{\mathfrak{c}}(\mathbb{A}_f)\right)^{-1} C^p C_p\,\mathrm{tr}\,\xi(\gamma_0),
\end{aligned}$$

where we used that γ_0 and ε are conjugate in $G(\overline{\mathbb{Q}})$ and we put

$$\begin{aligned}
C^p &:= \# I^\circ_{\phi,\varepsilon}(\mathbb{A}^p_f)_{\tau(\phi)}\backslash\{x^p \in \left(X^p(\phi)/(U^p\cap g^pU(g^p)^{-1}\right) : \varepsilon x^p = x^pg^p \bmod U^p\},\\
C_p &:= \# I^\circ_{\phi,\varepsilon}(\mathbb{Q}_p)\backslash\{x_p \in \left(X_p(\phi)/(U_p\cap g_pUg_p^{-1}\right) : \varepsilon x_p = x_pg_p \bmod U_p\}.
\end{aligned}$$

The formula for C_p has no twisting since $\tau(\phi)\in I_\phi(\mathbb{A}^p_f)$ is away from p. The proof of [KSZ21, Lem. 3.7.4] also shows that $C^p = O^{G(\mathbb{A}^p_f)}_\gamma(\mathbb{1}_{U^pg^pU^p})$, the situation being the same away from p (except that $(g^p)^{-1}$ is used in place of g^p in that source). It follows from a similar argument, which is only easier thanks to the absence of twisting, that $C_p = O^{J_b(\mathbb{Q}_p)}_\delta(\mathbb{1}_{U_pg_pU_p})$. If (ϕ,ε) is not acceptable then δ is not acceptable, but $\mathbb{1}_{U_pg_pU_p}$ is supported on acceptable elements so $C_p = 0$. The proof of the lemma is complete by putting everything together. □

Proposition 4.5.7. *Let $\tau\in\Gamma(\mathcal{H})_0$ a tori-rational element satisfying Theorem 3.6.1, lifted to a tori-rational element of $\Gamma(\mathfrak{E}^p)_0$ (still called τ by abuse of notation). For every ϱ-acceptable function $f\in C^\infty_c(G(\mathbb{A}^p_f)\times J_b(\mathbb{Q}_p))$, we have*

$$\mathrm{tr}(f\mid H_c(\mathrm{Ig}_\Sigma,\mathscr{L}_\xi)) = \sum_{(\phi,\varepsilon)\in\mathcal{LRP}_{\mathbf{b},\mathrm{acc}}/conj.} \iota_{I_\phi}(\varepsilon)^{-1}T^f_\xi(\mathfrak{c}),$$

where (ϕ,ε) ranges over a set of representatives for conjugacy classes of $\mathbf{b}$-admissible acceptable LR pairs, and $\mathfrak{c} = \mathfrak{t}_\tau(\phi,\varepsilon)$ is the corresponding Kottwitz parameter (well-defined up to isomorphism).

Proof. Write $f = \sum_{g\in I} c_g\mathbb{1}_{UgU}$ as in Definition 4.1.5. Then a finite linear combination of (4.5.3) holds over $g\in I$, in which we can plug in Lemma 4.5.6. Then we obtain the equation of the proposition except that the sum a priori runs over $\mathbf{b}$-admissible LR pairs. Since $T^f_\xi(\mathfrak{c}) = 0$ if $\mathfrak{c}$ is not acceptable, or equivalently if (ϕ,ε) is not acceptable, so we can restrict the sum to acceptable pairs. □

It remains to rewrite the sum in Proposition 4.5.7 in terms of Kottwitz parameters.

4.5.8 Let $\tau\in\Gamma(\mathfrak{E}^p)_0$ tori-rational, and $(\gamma_0,a,[b_0])\in\mathcal{KP}_{\mathbf{b},\mathrm{acc}}$ a $\mathbf{b}$-admissible acceptable Kottwitz parameter with trivial Kottwitz invariant. By Proposition 4.3.9, there exists $(\phi_0,\gamma_0)\in\mathcal{LRP}^{\mathrm{gg}}_{\mathbf{b},\mathrm{acc}}$ such that $\mathbf{t}_\tau(\phi_0,\gamma_0) = (\gamma_0,a,[b_0])$. Furthermore Lemma 4.4.4 and Proposition 4.4.5 imply that every $(\phi,\varepsilon)\in\mathcal{LRP}^{\mathrm{gg}}_{\mathbf{b},\mathrm{acc}}$ satisfying $\mathbf{t}_\tau(\phi,\varepsilon)\cong(\gamma_0,a,[b_0])$ is contained in the image of η_{ϕ_0,γ_0}. Let $D(\phi_0,\gamma_0)\subset Ш^\infty_G(\mathbb{Q},I^\circ_{\phi_0,\gamma_0})$ be the subset of classes e such that

$$\mathbf{t}_\tau(e\phi_0,\gamma_0)\cong\mathbf{t}_\tau(\phi_0,\gamma_0) = (\gamma_0,a,[b_0]),$$

i.e., twists that preserve the Kottwitz parameter up to isomorphism. So the composite map

$$D(\phi_0,\gamma_0)\hookrightarrow Ш^\infty_G(\mathbb{Q},I^\circ_{\phi_0,\gamma_0})\xrightarrow{\eta_{\phi_0,\gamma_0}}\mathcal{LRP}^{\mathrm{gg}}_{\mathbf{b},\mathrm{acc}}/\mathrm{conj}.$$

is a surjection onto the set of conjugacy classes of LR pairs in the fiber $\mathbf{t}^{-1}(\gamma_0,a,[b_0])$. To account for failure for this map to be a bijection, we compute:

Lemma 4.5.9. *Let $e\in D(\phi_0,\gamma_0)$. In the setting of 4.5.8, we have*

$$\#\{\textit{fiber of } \eta_{\phi_0,\gamma_0} \textit{ containing } e\} = \frac{|(I_{e\phi_0,\gamma_0}/I^\circ_{e\phi_0,\gamma_0})(\mathbb{Q})|}{[I_{e\phi_0,\gamma_0}(\mathbb{Q}) : I^\circ_{e\phi_0,\gamma_0}(\mathbb{Q})]}.$$

Proof. This is proven in the last paragraph of the proof of Lemma 3.7.6 in [KSZ21]. (In their notation, $\iota_H(\varepsilon) = [H_\varepsilon(\mathbb{Q}) : H_\varepsilon^\circ(\mathbb{Q})]$ and $\bar{\iota}_H(\varepsilon) = |(H_\varepsilon/H_\varepsilon^\circ)(\mathbb{Q})|$.) □

Lemma 4.5.10. *In the setting of 4.5.8, we have*

$$|D(\phi_0,\gamma_0)| = \sum_{(a',[b'_0])} |\text{Ш}_G(\mathbb{Q}, I^\circ_{\phi_0,\gamma_0})|,$$

where the sum runs over pairs $(a',[b'_0])$ such that $(\gamma_0,a',[b'_0])$ is an acceptable **b***-admissible Kottwitz parameter with trivial Kottwitz invariant, and Ш$_G$ is defined as in 2.5.12.*

Proof. Having fixed an LR pair (ϕ_0,γ_0) (and *not* simply a conjugacy class), [KSZ21, Prop. 3.6.2(iii)] tells us that twisting by an element $e \in Z^1(\mathbb{Q}, I^\circ_{\phi_0,\gamma_0})$ representing a class in $\text{Ш}^\infty_G(\mathbb{Q}, I^\circ_{\phi_0,\gamma_0})$ results in a well-defined Kottwitz parameter $\mathbf{t}_\tau(e\phi_0,\gamma_0)$ (*not* simply an isomorphism class). Thus we can write

$$D(\phi_0,\gamma_0) = \coprod_{(a',[b'_0])} D_{(a',[b'_0])},$$

where $(a',[b'_0])$ runs over all pairs for which $(\gamma_0,a',[b'_0])$ forms a Kottwitz parameter isomorphic to $(\gamma_0,a,[b_0])$, and $D_{(a',[b'_0])} \subset D(\phi_0,\gamma_0)$ is the subset of twists giving rise to the Kottwitz parameter $(\gamma_0,a',[b'_0])$.

If $D_{(a',[b'_0])}$ is non-empty, then by [KSZ21, Prop. 3.6.2] it must be a coset of $\text{Ш}_G(\mathbb{Q}, I^\circ_{\phi_0,\gamma_0})$ inside $\text{Ш}^\infty_G(\mathbb{Q}, I^\circ_{\phi_0,\gamma_0})$ (note that that proposition only uses their "p^n-admissible" hypothesis to show that $[b_0]$ is basic, which we know by our "acceptable" hypothesis). The proof that $D_{(a',[b'_0])}$ is indeed non-empty proceeds precisely as in the proof of [KSZ21, Lem. 3.7.6] (where they call this set D_i), replacing their Lemma 3.6.2 with our Lemma 4.4.2 and their Proposition 3.6.1 with our Lemma 4.2.15. □

Since $(e\phi_0,\gamma_0)$ is a gg acceptable pair in the last two lemmas, we are in the situation of 4.3.6 and we can equally well replace $I_{e\phi_0,\gamma_0}$ and $I^\circ_{e\phi_0,\gamma_0}$ by G_{γ_0} and $G^\circ_{\gamma_0}$, respectively. With Lemmas 4.5.9 and 4.5.10 applied to Proposition 4.5.7, our point-counting formula is transformed into the following final form.

Theorem 4.5.11. *For every ϱ-acceptable function $f \in C_c^\infty(G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p))$, we have*

$$\operatorname{tr}(f \mid H_c(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)) = \sum_{\gamma_0 \in \Sigma_{\mathbb{R}-\mathrm{ell}}(G)} \sum_{(a,[b_0]) \in \mathcal{KP}(\gamma_0)} \frac{|\text{Ш}_G(\mathbb{Q}, G^\circ_{\gamma_0})|}{|(G_{\gamma_0}/G^\circ_{\gamma_0})(\mathbb{Q})|} \operatorname{vol}\left(I^\circ_{\mathfrak{c}}(\mathbb{Q})\backslash I^\circ_{\mathfrak{c}}(\mathbb{A}_f)\right) O^{G(\mathbb{A}_f^p)\times J_b(\mathbb{Q}_p)}_{\gamma\times\delta}(f) \operatorname{tr}\xi(\gamma_0)$$

where $I_{\mathfrak{c}}$ is the inner form of $G^\circ_{\gamma_0}$ associated with the Kottwitz parameter $\mathfrak{c} = (\gamma_0,a,[b_0])$ as in 4.5.4, and γ,δ are coming from the classical Kottwitz parameter $(\gamma_0,\gamma,\delta) = \mathrm{cl}(\mathfrak{c})$ as in 4.2.18.

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