Normalizing Iteration Trees and Comparing Iteration Strategies

John R. Steel

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Abstract

In this book, we shall prove a general comparison lemma for iteration strategies. The comparison method involves iterating into a level of a background construction, one that has been done in a universe that is uniquely iterable in the appropriate sense. The proof that it succeeds relies heavily on an analysis the normalization of a stack of normal iteration trees.

We then use this comparison method to develop the basic theory of hod mice in the least branch hierarchy. Modulo the existence of iteration strategies, our results yield a fine structural analysis of $(\mathrm{HOD}|\theta)^M$, whenever M is a model of $\mathsf{AD}_{\mathbb{R}} + V = L(P(\mathbb{R}))$ that has no iteration strategies for mice with long extenders. In particular, $\mathrm{HOD}^M \models \mathsf{GCH}$, for such M.

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1 Introduction

In this book we shall develop a general comparison process for iteration strategies, and show how the process can be used to analyze ordinal definability in models of the Axiom of Determinacy. In this introduction, we look at the context and motivation for the technical results to come.

We begin with a broad overview of *inner model theory*, the subject to which this book belongs. Eventually we reach an outline of the ideas and results that are new here. The journey is organized so that the technical background needed to follow along increases as we proceed.

1.1 Large cardinals and the consistency strength hierarchy

Strong axioms of infinity, or as they are more often called, large cardinal hypotheses, play a central role in set theory. There are at least two reasons.

First, large cardinal hypotheses can be used to decide in a natural way many questions which cannot be decided on the basis of ZFC (the commonly accepted system of axioms for set theory, and hence all of mathematics). Many such questions come from *descriptive set theory*, the theory of simply definable sets of real numbers. For example, the hypothesis that there are infinitely many Woodin cardinals yields a systematic and detailed theory of the *projective* sets of reals, those that are definable in the language of second order arithmetic from real parameters. ZFC by itself yields such a theory at only the simplest levels of second order definability.

Second, large cardinal hypotheses provide a way of organizing and surveying all possible natural extensions of ZFC. This is due to the following remarkable phenomenon: for any natural extension T of ZFC which set theorists have studied, there seems to be an extension S of ZFC axiomatized by large cardinal hypotheses such that the consistency of T is provably (in ZFC) equivalent to that of S. The consistency strengths of the large cardinal hypotheses are linearly ordered, and usually easy to compare. Thus all natural extensions of ZFC seem to fall in a hierarchy linearly ordered by consistency strength, and calibrated by the large cardinal hypotheses.[1]

These two aspects of large cardinal hypotheses are connected, in that the consistency strength order on natural theories corresponds to the inclusion order on the set of their "sufficiently absolute" consequences. For example, if S and T are natural theories extending ZFC, and S has consistency strength less than or equal to that of T, then the arithmetic consequences of S are included in those of T. If in addition, S

[1] Let $\mathrm{con}(T)$ be some natural formalization of the assertion that T is consistent. The consistency strength order is given by: $S \leq_{\mathrm{con}} T$ iff ZFC proves $\mathrm{con}(T) \rightarrow \mathrm{con}(S)$.

and T have consistency strength at least that of "there are infinitely many Woodin cardinals", then the consequences of S in the language of second order arithmetic are included in those of T. This pattern persists at still higher consistency strengths, with still more logically complicated consequences about reals and sets of reals being brought into a uniform order. This beautiful and suggestive phenomenon has a practical dimension as well: one way to develop the absolute consequences of a strong theory T is to compute a consistency strength lower bound S for T in terms of large cardinal hypotheses, and then work in the theory S. For one of many examples, the Proper Forcing Axiom (PFA) yields a canonical inner model with infinitely many Woodin cardinals that is correct for statements in the language of second order arithmetic, and therefore PFA implies all consequences of the existence of infinitely many Woodin cardinals that can be stated in the language of second order arithmetic.

One can think of the consistency strength of a theory as the degree to which it is committed to the existence of the higher infinite. Large cardinal hypotheses make their commitments explicitly: they simply say outright that the infinities in question exist. It is therefore usually easy to compare their consistency strengths. Other natural theories often have their commitments to the existence of the infinite well hidden. Nevertheless, set theorists have developed methods whereby these commitments can be brought to the surface, and compared. These methods have revealed the remarkable phenomenon described in the last paragraph, that natural theories appear to be wellordered by the degrees to which they are committed to the infinite, and that this degree of commitment corresponds exactly to the power of the theory to decide questions about concrete objects, like natural numbers, real numbers, or sets of real numbers.

We should emphasize that the paragraphs above describe a general pattern of existing theorems. There are many examples of natural theories whose consistency strengths have not yet been computed, and perhaps they, or some natural theory yet to be found, will provide counterexamples to the pattern described above. The pervasiveness of the pattern where we know how to compare consistency strengths is evidence that this will not happen.[2] The two methods whereby set theorists compare consistency strengths, forcing and inner model theory, seem to lead inevitably to the pattern. In particular, the wellorder of natural consistency strengths seems to correspond to the inclusion order on canonical minimal inner models for large cardinal hypotheses. Forcing and inner model theory seem sufficiently general to compare all natural consistency strengths, but at the moment, this is just informed

[2]The pattern extends to weak subtheories of ZFC as well. This book is concerned only with theories having very strong commitments to infinity, and so we shall ignore subtheories of ZFC, but the linearity of the consistency strengths below that of ZFC is evidence of linearity higher up.

speculation. So one reasonable approach to understanding the general pattern of consistency strengths is to develop our comparison methods further. In particular, inner model theory is in great need of further development, as there are quite important consistency strengths that it does not yet reach.

1.2 Inner model theory

The inner model program attempts to associate to each large cardinal hypothesis H a canonical minimal universe of sets M_H (an *inner model*) in which H is true. The stronger H is, the larger M_H will be; that is, $G \leq_{\text{con}} H$ if and only if $M_G \subseteq M_H$. Some of our deepest understanding of large cardinal hypotheses comes from the inner model program.

The inner models we have so far constructed have an internal structure which admits a systematic, detailed analysis, a *fine structure theory* of the sort pioneered by Ronald Jensen around 1970 ([11]). Thus being able to construct M_H gives us a very good idea as to what a universe satisfying H might look like. Inner model theory thereby provides evidence of the consistency of the large cardinal hypotheses to which it applies. (The author believes that this will some day include all the large cardinal hypotheses currently studied.) Since forcing seems to reduce any consistency question to the consistency question for some large cardinal hypothesis, it is important to have evidence that the large cardinal hypotheses themselves are consistent! No evidence is more convincing than an inner model theory for the hypothesis in question.

The smallest of the canonical inner models is the universe L of constructible sets, isolated by Kurt Gödel ([7]) in his 1937 proof that CH is consistent with ZFC. It was not until the mid 1960's that J. Silver and K. Kunen ([47],[16]) developed the theory of a canonical inner model going properly beyond L, by constructing M_H for $H =$ "there is a measurable cardinal". Since then, progressively larger M_H for progressively stronger H have been constructed and studied in detail. (See for example [8],[20], and [21].) At the moment, we have a good theory of canonical inner models satisfying "there is a Woodin cardinal", and even slightly stronger hypotheses. (See [19],[23], and [49], for example.) One of the most important open problems in set theory is to extend this theory significantly further, with perhaps the most well-known target being models satisfying "there is a supercompact cardinal".

Inner model theory is a crucial tool in calibrating consistency strengths: in order to prove that $H \leq_{\text{con}} T$, where H is a large cardinal hypothesis, one generally constructs a canonical inner model of H inside an arbitrary model of T. Because we do not have a full inner model theory very far past Woodin cardinals, we lack the means to prove many well-known conjectures of the form $H \leq_{\text{con}} T$, where H

is significantly stronger than "there is a Woodin cardinal". Broadly speaking, there are great defects in our understanding of the consistency strength hierarchy beyond Woodin cardinals.

Inner model theory is also a crucial tool in developing the consequences for real numbers of large cardinal hypotheses. Indeed, the basics of inner model theory for Woodin cardinals were discovered in 1985-86 by D. A. Martin and the author, at roughly the same time they discovered their proof of Projective Determinacy, or PD. (Martin, Moschovakis, and others had shown in the 1960's and 70's that PD decides in a natural way all the classical questions about projective sets left undecided by ZFC alone.) This simultaneous discovery was not an accident, as the fundamental new tool in both contexts was the same: *iteration trees*, and the *iteration strategies* which produce them. Since then, progress in inner model theory has given us a deeper understanding of pure descriptive set theory, and the means to solve some old problems in that field.

The fundamental open problem of inner model theory is to extend the theory to models satisfying "There is a supercompact cardinal". One very well known test question here is whether (ZFC+"there is a supercompact cardinal") $\leq_{\text{con}}$ $\mathsf{ZFC}+\mathsf{PFA}$. The answer is almost certainly yes, and the proof almost certainly involves an inner model theory that is firing on all cylinders. That kind of inner model theory we have now only at the level of many Woodin cardinals, but significant parts of the theory do exist already at much higher levels.

1.3 Mice and iteration strategies

The canonical inner models we seek are often called *mice*. There are two principal varieties, the pure extender mice and the strategy mice.[3]

A pure extender premouse is a model of the form $\mathrm{L}_\alpha[\vec{E}]$ where $\vec{E}$ is a coherent sequence of extenders. Here an extender is a system of ultrafilters coding an elementary embedding, and coherence means roughly that the extenders appear in order of strength, without leaving gaps. These notions were introduced by Mitchell in the 1970s[4], and they have been a foundation for work in inner model theory since then.

In this book, we shall assume that our premice have no long extenders on their coherent sequences.[5] Such premice can model superstrong, and even subcompact,

[3] Strategy mice are sometimes called *hod mice*, because of their role in analyzing the hereditarily ordinal definable sets in models of the Axiom of Determinacy.

[4] See [20] and [21].

[5] An extender is short if all its component ultrafilters concentrate on the critical point. Otherwise, it is long.

cardinals. They cannot model κ^+-supercompactness. Long extenders lead to an additional set of difficulties.

An *iteration strategy* is a winning strategy for player II in the iteration game. For any premouse M, the iteration game on M is a two player game of length $\omega_1 + 1$.[6] In this game, the players construct a tree of models such that each successive node on the tree is obtained by an ultrapower of a model that already exists in the tree. I is the player that describes how to construct this ultrapower. He takes the last model that appeared in the tree and chooses an extender E from the extender sequence of that model. He then chooses another model in the tree and takes the ultrapower by E of this model. If the ultrapower is ill-founded then player I wins; otherwise the resulting ultrapower is the next node on the tree. Player II moves at limit stages λ by choosing a branch of the tree that has been visited cofinally often below λ, and is such that the direct limit of the embeddings along the branch is well-founded. If he fails to do so, he loses. If II manages to stay in the category of wellfounded models through all $\omega_1 + 1$ moves, then he wins. A winning strategy for II in this game is called an *iteration strategy* for M, and M is said to be *iterable* just in case there is an iteration strategy for it. Iterable pure extender premice are called *pure extender mice*.

Pure extender mice are canonical objects; for example, any real number belonging to such a mouse is ordinal definable. Let us say that a premouse M is *pointwise definable* if every element of M is definable over M. For any axiomatizable theory T, the minimal mouse satisfying T is pointwise definable. The canonicity of pure extender mice is due to their iterability, which, via the fundamental *Comparison Lemma*, implies that the pointwise definable pure extender mice are wellordered by inclusion. This is the *mouse order* on pointwise definable pure extender mice. The consistency strength of T is determined by the minimal mouse M having a generic extension satisfying T, and thus the consistency strength order on natural T is mirrored in the mouse order. However, in the case of the mouse order, we have *proved* that we have a wellorder; what we cannot yet do is tie natural T at high consistency strengths to it. As we climb the mouse order, the mice become correct (reflect what is true in the full universe of sets) at higher and higher levels of logical complexity.

Iteration strategies for pointwise definable pure extender mice are also canonical objects; for example, a pointwise definable mouse has exactly one iteration strategy.[7]

[6] Iteration games of other lengths are also important, but this length is crucial, so we shall focus on it.

[7] This follows from Theorem 4.11 of [53], and the fact that any iteration strategy for a pointwise definable M has the weak Dodd-Jensen property with respect to all enumerations of M.

The existence of iteration strategies is at the heart of the fundamental problem of inner model theory, and for a pointwise definable M, to prove the existence of an iteration strategy is to define it. In practice, it seems necessary to give a definition in the simplest possible logical form. As we go higher in the mouse order, the logical complexity of iteration strategies must increase, in a way that keeps pace with the correctness of the mice they identify.

Our most powerful, all-purpose method for constructing iteration strategies is the *core model induction method*. Because iteration strategies must act on trees of length ω_1, they are not coded by sets of reals. Nevertheless, the fragment of the iteration strategy for a countable mouse that acts on countable iteration trees *is* coded by a set of reals. If this set happens to be absolutely definable (that is, Universally Baire) then the strategy can be extended to act on uncountable iteration trees in a unique way. There is no other way known to construct iteration strategies acting on uncountable trees. Thus, having an absolutely definable iteration strategy for countable trees is tantamount to having a full iteration strategy. The key idea in the core model induction is to use the concepts of descriptive set theory, under determinacy hypotheses, to identify a next relevant level of correctness and definability for sets of reals, a target level at which the next iteration strategy should be definable.

Absolute definability leads to determinacy. Thus at reasonably closed limit steps in a core model induction, one has a model M of $\mathsf{AD} + V = L(P(\mathbb{R}))$ that contains the restrictions to countable trees of the iteration strategies already constructed. Understanding the structure of HOD^M is important for going further.

1.4 HOD in models of determinacy

HOD is the class of all hereditarily ordinal definable sets. It is a model of ZFC[8], but beyond that, ZFC does not decide its basic theory, and the same is true of ZFC augmented by any of the known large cardinal hypotheses. The problem is that the definitions one has allowed are not sufficiently absolute. In contrast, the theory of HOD in determinacy models is well-determined, not subject to the vagaries of forcing.[9]

The study of HOD in models of AD has a long history. The reader should see [59] for a survey of this history. HOD was studied by purely descriptive set theoretic

[8]See [24].

[9]We mean here determinacy models of the form $M = L(\Gamma, \mathbb{R})$, where Γ is a proper initial segment of the universally Baire sets. If there are arbitrarily large Woodin cardinals, then for any sentence φ, whether φ is true in all such HOD^M is absolute under set forcing. This follows easily from Woodin's theorem on the generic absoluteness of $(\Sigma^2_1)^{\mathrm{UB}}$ statements. See [52, Theorem 5.1].

methods in the late 70s and 80s, and partial results on basic questions such as whether HOD $\models$ GCH were obtained then. It was known then that inner model theory, if only one could develop it in sufficient generality, would be relevant to characterizing the reals in HOD. It was known that HOD^M is close to M in various ways; for example, if $M \models \mathsf{AD}^+ + V = L(P(\mathbb{R}))$[10], then M can be realized as a symmetric forcing extension of HOD^M, so that the first order theory of M is part of the first order theory of its HOD. [11]

Just how relevant inner model theory is to the study of HOD in models of AD became clear in 1994, when the author showed that if there are ω Woodin cardinals with a measurable above them all, then $\mathrm{HOD}^{L(\mathbb{R})}$ up to $\theta^{L(\mathbb{R})}$ is a pure extender mouse.[12](See [50].) Shortly afterward, this result was improved by Hugh Woodin, who reduced its hypothesis to $\mathsf{AD}^{L(\mathbb{R})}$, and identified the full $\mathrm{HOD}^{L(\mathbb{R})}$ as a model of the form $L[M, \Sigma]$, where M is a pure extender premouse, and Σ is a partial iteration strategy for M. $\mathrm{HOD}^{L(\mathbb{R})}$ is thus a new type of mouse, sometimes called a *strategy mouse*, sometimes called a *hod mouse*. See [66] for an account of this work.

Since the mid-1990s, there has been a great deal of work devoted to extending these results to models of determinacy beyond $L(\mathbb{R})$. Woodin analyzed HOD in models of AD^+ below the minimal model of $\mathsf{AD}_{\mathbb{R}}$ fine structurally, and Sargsyan pushed the analysis further, first to determinacy models below $\mathsf{AD}_{\mathbb{R}}$ + "θ is regular" (see [30] and [31]), and more recently, to models of still stronger forms of determinacy. [13] Part of the motivation for this work is that it seems to be essential in the core model induction: in general, the next iteration strategy seems to be a strategy for a hod mouse, not for a pure extender mouse. This idea comes from work of Woodin and Ketchersid around 2000. (See [14] and [40].)

1.5 Least branch hod pairs

The strategy mice used in the work just described have the form $M = L[\vec{E}, \Sigma]$, where $\vec{E}$ is a coherent sequence of extenders, and Σ is an iteration strategy for M. The strategy information is fed into the model M slowly, in a way that is dictated in part by the determinacy model whose HOD is being analyzed. One says that the

[10] AD^+ is a technical strengthening of AD. It is not known whether $\mathsf{AD} \Rightarrow \mathsf{AD}^+$, but in every model of AD constructed so far, AD^+ also holds. In particular, the models of AD that are relevant in the core model induction satisfy AD^+.

[11] This is a theorem of Woodin from the early 1980s. Cf. [59].

[12] In a determinacy context, θ denotes the least ordinal that is not the surjective image of the reals.

[13] See [32]. Part of this work was done in collaboration with the author; see [55],[56], and [57]. The determinacy principles dealt with here are all weaker than a Woodin limit of Woodin cardinals.

hierarchy of M is *rigidly layered*, or *extender-biased*. The object (M, Σ) is called a rigidly layered (extender biased) *hod pair*.

Perhaps the main motivation for the extender-biased hierarchy is that it makes it possible to prove a comparison theorem. There is no inner model theory without such a theorem. Comparing strategy mice necessarily involves comparing iteration strategies, and comparing iteration strategies is significantly more difficult than comparing extender sequences. Rigid layering lets one avoid the difficulties inherent in the general strategy comparison problem, while proving comparison for a class of strategy mice adequate to analyze HOD in the minimal model of $\mathsf{AD}_{\mathbb{R}}$ + "θ is regular", and somewhat beyond. The key is that in this region, HOD does not have cardinals that are strong past a Woodin cardinal.

Unfortunately, rigid layering does not seem to help in comparing strategy mice that have cardinals that are strong past a Woodin. Moreover, it has serious costs. The definition of "hod premouse" becomes very complicated, and indeed it is not clear how to extend the definition of rigidly layered hod pairs much past that given in [32]. The definition of "rigidly layered hod premouse" is not uniform, in that the extent of extender bias depends on the determinacy model whose HOD is being analyzed. Fine structure, and in particular condensation, become more awkward. For example, it is not true in general that the pointwise definable hull of a level of M is a level of M. (The problem is that the hull will not generally be sufficiently extender biased.)

The more naive notion of hod premouse would abandon extender bias, and simply add the least missing piece of strategy information at essentially every stage. This was originally suggested by Woodin.[14] The focus of this book is a general comparison theorem for iteration strategies that makes it possible to use this approach, at least in the realm of short extenders. The resulting premice are called *least branch premice* (lpm's), and the pairs (M, Σ) are called *least branch hod pairs* (lbr hod pairs). Combining results of this book and [63], one has

Theorem 1.1 ([63]) *Assume* $\mathsf{AD}^+ +$ *"there is an* (ω_1, ω_1) *iteration strategy for a pure extender premouse with a long extender on its sequence"; then*

(1) for any $\Gamma \subseteq P(\mathbb{R})$ *such that* $L(\Gamma, \mathbb{R}) \models \mathsf{AD}_{\mathbb{R}} +$ *"there is no* (ω_1, ω_1) *iteration strategy for a pure extender premouse with a long extender on its sequence",* $HOD^{L(\Gamma,\mathbb{R})}$ *is a least branch premouse, and*

[14]There are some fine-structural problems with the precise method for inserting strategy information originally suggested by Woodin. The method for strategy insertion that is correct in detail is due to Schlutzenberg and Trang. Cf. [46].

(2) there is a $\Gamma \subseteq P(\mathbb{R})$ *such that* $L(\Gamma, \mathbb{R}) \models \mathsf{AD}_{\mathbb{R}}+$ *"there is no* (ω_1, ω_1) *iteration strategy for a pure extender premouse with a long extender on its sequence", and* $HOD^{L(\Gamma,\mathbb{R})} \models$ *"there is a subcompact cardinal".*

Of course, one would like to remove the mouse existence hypothesis of 1.1, and prove its conclusion under AD^+ alone. Finding a way to do this is one manifestation of the long standing iterability problem we have discussed above. Although we do not yet know how to do this, the theorem does make it highly likely that in models of $\mathsf{AD}_{\mathbb{R}}$ that have not reached an iteration strategy for a pure extender premouse with a long extender, HOD is a least branch premouse. It also makes it very likely that there are such HOD's with subcompact cardinals. Subcompactness is one of the strongest large cardinal properties that can be represented with short extenders.[15]

Although we shall not prove Theorem 1.1 here, we shall prove an approximation to it that makes the same points. That approximation is Theorem 8.11 below.

Least branch premice have a fine structure much closer to that of pure extender models than that of rigidly layered hod premice. In this book we develop the basics, including the solidity and universality of standard parameters, and a form of condensation. In [65], the author and N. Trang have proved a sharper condensation theorem, whose pure extender version was used heavily in the Schimmerling-Zeman work ([37]) on $\square$ in pure extender mice. It seems likely that the rest of the Schimmerling-Zeman work extends as well.

Thus least branch hod pairs give us a good theory of HOD in the short extender realm, provided there are enough such pairs. Below, we formulate a conjecture that we call *Hod Pair Capturing*, or HPC, that makes precise the statement that there are enough least branch hod pairs. HPC is the main open problem in the theory to which this book contributes.

1.6 Comparison and the mouse pair order

Let us first say more about the nature of least branch hod pairs (M, Σ). There are some important requirements on Σ in the definition.

Recall that an iteration tree on a premouse M is *normal* iff the extenders $E_\alpha^{\mathcal{W}}$ used in $\mathcal{W}$ have lengths increasing with α, and each $E_\alpha^{\mathcal{W}}$ is applied to the longest possible initial segment of the earliest possible model in $\mathcal{W}$. Suppose now $\vec{\mathcal{T}}$ is a finite stack of iteration trees, with $\mathcal{T}_0$ being a normal tree on M, and $\mathcal{T}_{i+1}$ being a normal tree on the last model of $\mathcal{T}_i$. Let N be the last model of the last tree. There

[15]Until now, there was no very strong evidence that the HOD of a determinacy model could satisfy that there are cardinals that are strong past a Woodin cardinal.

is a natural attempt to construct a "minimal" normal iteration tree $\mathcal{W}$ on M having last model N. This attempt may break down by reaching an illfounded model. If it does not break down, it will in the end produce a model R and an elementary $\pi\colon N \to R$. We call $\mathcal{W}$ the *embedding normalization of* $\vec{\mathcal{T}}$.

The strategies in least branch hod pairs are defined on finite stacks of normal trees.

Definition 1.2 *Suppose Σ is an iteration strategy for a premouse P.*

(1) (Tail strategy) If s is a stack by Σ with last model Q, then Σ_s is the strategy for Q given by: $\Sigma_s(t) = \Sigma(s^\frown t)$.

(2) (Pullback strategy) If $\pi\colon N \to P$ is elementary, then Σ^π is the strategy for N given by: $\Sigma^\pi(s) = \Sigma(\pi s)$, where πs is the lift of s by π to a stack on P.

(3) (Normalizes well) Σ normalizes well *iff whenever s is a stack by Σ with last model Q, and $\mathcal{W}$ is the embedding normalization of s, with associated map $\pi\colon Q \to R$, then*

(i) $\mathcal{W}$ is by Σ, and

(ii) $\Sigma_s = (\Sigma_{\langle \mathcal{W} \rangle})^\pi$.

(4) (Strong hull condensation) Σ has strong hull condensation *iff whenever $\mathcal{T}$ is a normal tree by Σ, and $\mathcal{U}$ is a psuedo-hull of $\mathcal{T}$, then $\mathcal{U}$ is by Σ.*

Here elementarity must be understood fine structurally; our convention is that every premouse P has a degree of soundness attached to it, and elementarity means elementarity at that quantifier level. The notion of psuedo-hull is defined in Definition 3.29 below. Strong hull condensation is a stronger version of the hull condensation property isolated by Sargsyan in [30].

If M is a pure extender premouse, and Σ is a strategy for M that normalizes well and has strong hull condensation, then we call (M, Σ) a *pure extender pair*. If M is a least branch premouse, and Σ is a strategy for M that normalizes well, has strong hull condensation, and whose internal strategy predicate is consistent with Σ, then we call (M, Σ) a *least branch hod pair*. A pair of one of the two types is a *mouse pair*.

If (M, Σ) is a mouse pair, and s is a stack by Σ with last model N, then we call (N, Σ_s) an *iterate of* (M, Σ). If the branch M-to-N of s does not drop, we call it a non-dropping iterate. In that case, we have an iteration map $i_s\colon M \to N$.

We have no hope of showing anything about mouse pairs (M, Σ) unless we assume absolute definabilty for the iteration strategy. Here we assume Σ has scope HC, i.e. that M is countable and Σ is defined on countable stacks of countable trees, and we assume that we are in a model of AD^+. The following is the main new result of the book.

Theorem 1.3 (Comparison theorem, 6.21) *Assume AD^+, and let (P, Σ) and (Q, Ψ) be mouse pairs with scope HC that are of the same type; then they have a common iterate (R, Ω) such that on at least one of the two sides, the iteration does not drop.*

Even for pure extender pairs, this theorem is new, because of the agreement between tail strategies it requires. In fact, it is no easier to prove the theorem for pure extender pairs than it is to prove it for least branch hod pairs. The proof in both cases is the same, and it makes use of the properties of the iteration strategies we have isolated in the definition of mouse pair.

Working in the category of mouse pairs enables us to state a general Dodd-Jensen lemma. Let us say $\pi\colon (P, \Sigma) \to (Q, \Psi)$ is elementary iff π is elementary from P to Q, and $\Sigma = \Psi^\pi$. The iteration maps associated to non-dropping iterations of a mouse pair are elementary.[16]

Theorem 1.4 (Dodd-Jensen lemma) *Let (P, Σ) be a mouse pair, and (Q, Ψ) be an iterate of (P, Σ) via the stack s. Suppose $\pi\colon (P, \Sigma) \to (Q, \Psi)$ is elementary; then s does not drop, and for all ordinals $\eta \in P$, $i_s(\eta) \leq \pi(\eta)$.*

The proof is just the usual Dodd-Jensen proof; the point is just that the language of mouse pairs enables us to formulate the theorem in its proper generality. There is no need to restrict to mice with unique iteration strategies, as is usually done.

Similarly, we can define the mouse order in its proper generality, without restricting to mice with unique iteration strategies. If (P, Σ) and (Q, Ψ) are pairs of the same type, then $(P, \Sigma) \leq^* (Q, \Psi)$ iff (P, Σ) can be elementarily embedded into an iterate of (Q, Ψ). The Comparison and Dodd-Jensen theorems imply that $\leq^*$ is a prewellorder on each type.

1.7 Hod pair capturing

Least branch hod pairs can be used to analyze HOD in models of AD^+, provided that there are enough such pairs.

[16]This is actually not obvious; it is a property of the iteration strategy known as pullback consistency. It follows from strong hull condensation.

Definition 1.5 *(*AD^+*)*

(a) Hod Pair Capturing *(*HPC*) is the assertion: for every Suslin-co-Suslin set A, there is a least branch hod pair (P, Σ) such that A is definable from parameters over $(HC, \in, \Sigma)$.*

(b) $L[E]$ capturing *(*LEC*) is the assertion: for every Suslin-co-Suslin set A, there is a pure extender pair (P, Σ) such that A is definable from parameters over $(HC, \in, \Sigma)$.*

An equivalent (under AD^+) formulation would be that the sets of reals coding strategies of the type in question, under some natural map of the reals onto HC, are Wadge cofinal in the Suslin-co-Suslin sets of reals. The restriction to Suslin-co-Suslin sets A is necessary, for AD^+ implies that if (P, Σ) is a pair of one of the two types, then the codeset of Σ is Suslin and co-Suslin. This is the main result of [63], where it is also shown that the Suslin representation constructed is of optimal logical complexity.

Remark 1.6 HPC is a cousin of Sargsyan's *Generation of Full Pointclasses*. See [30] and [31], §6.1.

Assuming AD^+, LEC is equivalent to the well known Mouse Capturing: for reals x and y, x is ordinal definable from y iff x is in a pure extender mouse over y. This equivalence is shown in [54]. (See especially Theorem 16.6.) We show in Theorem 6.71 below that under AD^+, LEC implies HPC. We do not know whether HPC implies LEC.

Granted $\mathsf{AD}_{\mathbb{R}}$ and HPC, we have enough hod pairs to analyze HOD.

Theorem 1.7 ([63]) *Assume $\mathsf{AD}_{\mathbb{R}}$ and HPC; then $V_\theta \cap HOD$ is the universe of a least branch premouse.*

Some techniques developed in [48] and [63] are needed to prove the theorem, so we shall not prove it here.

The natural conjecture is that LEC and HPC hold in all models of AD^+ that have not reached an iteration strategy for a premouse with a long extender. Because our capturing mice have only short extenders on their sequences, LEC and HPC cannot hold in larger models of AD^+.

Definition 1.8 NLE *("No long extenders") is the assertion: there is no countable, $\omega_1 + 1$-iterable pure extender premouse M such that there is a long extender on the M-sequence.*

Conjecture 1.9 *Assume* AD^+ *and* NLE*; then* LEC*.*

Conjecture 1.10 *Assume* AD^+ *and* NLE*; then* HPC*.*

As we remarked above, 1.9 implies 1.10. Conjecture 1.9 is equivalent to a slight strengthening of the usual Mouse Set Conjecture MSC. (The hypothesis of MSC is that there is no iteration strategy for a pure extender premouse with a superstrong, which is slightly stronger than NLE.) MSC has been a central target for inner model theorists for a long time.

1.8 Constructing mouse pairs

The basic source for mouse pairs is a background construction. In the simplest case, such a construction $\mathbb{C}$ builds pairs $(M_{\nu,k}, \Omega_{\nu,k})$ inductively, putting extenders on the $M_{\nu,k}$-sequence that are restrictions of nice extenders in V. The iteration strategy $\Omega_{\nu,k}$ is induced by an iteration strategy for V, and if we are constructing strategy premice, the relevant information about $\Omega_{\nu,k}$ is inserted into $M_{\nu,k}$ at the appropriate points. $M_{\nu,k+1}$ is the core of $M_{\nu,k}$. The construction breaks down if the standard parameter of $M_{\nu,k}$ behaves poorly, so that there is no core.

There is of course more to say here, and we shall do so later in the book. For now, let us note that the background universe for such a construction should be a model of ZFC that has lots of extenders, and yet knows how to iterate itself. In the AD^+ context, the following theorem of Woodin applies.[17]

Theorem 1.11 (Woodin) *Assume* AD^+*, and let* Γ *be a good pointclass such that all sets in* Γ *are Suslin and co-Suslin; then for any real* x *there is a coarse* Γ*-Woodin pair* (N, Σ) *such that* $x \in N$*.*

Here, roughly speaking, N is a countable transitive model of ZFC with a Woodin cardinal and a term for a universal Γ set, and Σ is an iteration strategy for N that moves this term correctly, and is such that $\Sigma \cap N$ is definable over N. See Definition 4.14.

The following is essentially Theorem 6.70 to follow. It too is one of the main new results of the book.

Theorem 1.12 *Assume* AD^+*, and let* (N, Σ) *be a coarse* Γ*-Woodin pair. Let* $\mathbb{C}$ *be a least branch construction in* N*; then* $\mathbb{C}$ *does not break down. Moreover, each of its levels* $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$ *is a least branch hod pair in* N*, and extends canonically to a least branch hod pair in* V*.*

[17] See [15], and [58, Lemma 3.13].

Background constructions of the sort described in this theorem have an important role to play in our comparison process. Assume AD^+, and let (M, Ω) and (N, Σ) be mouse pairs of the same type. We compare (M, Ω) with (N, Σ) by putting M and N into a common Γ-Woodin universe N^*, where Σ and Ω are in $\Gamma \cap \check{\Gamma}$. We then iterate (M, Σ) and (N, Ω) into levels of a full background construction (of the appropriate type) of N^*. Here are some definitions encapsulating the method.

Definition 1.13 *Let (M, Σ) and (N, Ω) be mouse pairs of the same type; then*

(a) (M, Σ) iterates past (N, Ω) *iff there is a normal iteration tree $\mathcal{T}$ by Σ on M whose last pair is (N, Ω).*

(b) (M, Σ) iterates to (N, Ω) *iff there is a normal $\mathcal{T}$ as in (a) such that the branch M-to-N of $\mathcal{T}$ does not drop.*

(c) (M, Σ) iterates strictly past (N, Ω) *iff it iterates past (N, Ω), but not to (N, Ω).*

Definition 1.14 (AD^+) *Let (P, Σ) be a mouse pair; then $(*)(P, \Sigma)$ is the following assertion: Let (N, Ψ) be any coarse Γ-Woodin pair such that $P \in HC^{N^*}$, and $\Sigma \in \Gamma \cap \check{\Gamma}$. Let $\mathbb{C}$ be a background construction done in N^* of the appropriate type, and let (R, Φ) be a level of $\mathbb{C}$. Suppose that (P, Σ) iterates strictly past all levels of $\mathbb{C}$ that are strictly earlier than (R, Φ); then (P, Σ) iterates past (R, Φ).*

If (M, Ω) is a mouse pair, and N is an initial segment of M, then we write Ω_N for the iteration strategy for trees on N that is induced by Ω. We can unpack the conclusion of 1.14 as follows: suppose the comparison of P with R has produced a normal tree $\mathcal{T}$ on P with last model Q, with $\mathcal{T}$ by Σ, and S is an initial segment of bot Q and R; then $(\Sigma_{\langle \mathcal{T} \rangle})_S = \Phi_S$. Thus the least disagreement between Q and R is an extender disagreement. Moreover, if E on Q and F on R are the extenders involved in it, then $F = \varnothing$.

We shall show (cf. Theorem 5.11 below)

Theorem 1.15 *Assume AD^+; then $(*)(P, \Sigma)$ holds, for all mouse pairs (P, Σ).*

This theorem lets us compare two (or more) mouse pairs of the same type indirectly, by comparing them to the levels of an appropriate construction, done in a Γ-Woodin model, where both strategies are in $\Gamma \cap \check{\Gamma}$. One can show using the Woodinness that $\mathbb{C}$ reaches non-dropping iterates of both pairs[18]. This gives us a stage (M, Ω) of $\mathbb{C}$ such that one of the pairs iterates to it, while the other iterates past it.

[18] See 2.53.

1.9 The comparison argument

In what follows, we shall give fairly complete proofs of the theorems above. The book is long, partly because we wanted to make it as accessible as possible, and partly because we are looking more closely at the construction of iteration strategies in [23], and there are many details there. However, the main new idea behind our strategy-comparison theorem is quite simple. We describe it now.

The first step is to focus on proving $(*)(P, \Sigma)$. That is, rather than directly comparing two strategies, we iterate them both into a common background construction and its strategy. In the comparison-of-mice context, this method goes back to Kunen ([16]), and was further developed by Mitchell and Baldwin ([2]). The first proof of comparison for pure extender mice with Woodin cardinals had this form, and Woodin and Sargsyan had used the method for strategy comparison in the hod mouse context. All these comparisons could be replaced by direct comparisons of the two mice or strategies involved, but in the general case of comparison of strategies, there are serious advantages to the indirect approach. There is no need to decide what to do if one encounters a strategy disagreement, because one is proving that that never happens. The comparison process is just the usual one of comparing least extender disagreements. Instead of the dual problems of designing a process and proving it terminates, one has a given process, and knows why it should terminate: no strategy disagreements show up. The problem is just to show this. These advantages led the author to focus, since 2009, on trying to prove $(*)(P, \Sigma)$.

The main new idea that makes this possible is motivated by Sargsyan's proof in [30] that if Σ has branch condensation, then $(*)(P, \Sigma)$ holds. Branch condensation is too strong to hold once P has extenders overlapping Woodin cardinals; we cannot conclude that $\Sigma(\mathcal{T}) = b$ from having merely realized $\mathcal{M}_b^{\mathcal{T}}$ into a Σ-iterate of P. We need some kind of realization of the entire phalanx $\Phi(\mathcal{T}^\frown b)$ in order to conclude that $\Sigma(\mathcal{T}) = b$. This leads to a weakening of branch condensation that one might call "phalanx condensation", in which one asks for a family of branch-condensation-like realizations having some natural agreement with one another. Phalanx condensation is still strong enough to imply $(*)(P, \Sigma)$, and might well be true in general for background-induced strategies. Unfortunately, Sargsyan's construction of strategies with branch condensation does not seem to yield phalanx condensation in the more general case. For one thing, it involves comparison arguments, and in the general case, this looks like a vicious circle. It was during one of the author's many attempts to break into this circle that he realized that certain properties related to phalanx condensation, namely normalizing well and strong hull condensation, could be obtained directly for background-induced strategies, and that these properties suffice for $(*)(P, \Sigma)$.

Let us explain this last part briefly. Suppose that we are in the context of Theorem 1.15. We have a premouse P with iteration strategy Σ that normalizes well and has strong hull condensation. We have N a premouse occuring in the fully backgrounded construction of N^*, where $P \in \mathrm{HC}^{N^*}$ and N^* captures Σ. We compare P with N by iterating away the least extender disagreement. It has been known since 1985 that only P will move. We must prove that no strategy disagreement shows up.

Suppose we have produced an iteration tree $\mathcal{T}$ on P with last model Q, and that $Q|\alpha = N|\alpha$, and that $\mathcal{U}$ is a tree on $R = Q|\alpha = N|\alpha$ played by both $\Sigma_{\mathcal{T},Q|\alpha}$ (the tail of Σ) and Ω, the N^*-induced strategy for N. Let $\mathcal{U}$ have limit length, and let $b = \Omega(\mathcal{U})$. We must see $b = \Sigma(\langle \mathcal{T}, \mathcal{U} \rangle)$. For this, we look at the embedding normalization $W(\mathcal{T}, \mathcal{U})$ of $\langle \mathcal{T}, \mathcal{U} \rangle$, which also has limit length. We shall see:

(1) b generates (modulo $\mathcal{T}$) a unique cofinal branch a of $W(\mathcal{T}, \mathcal{U})$ (see §3.7).

(2) Letting $i_b^* : N^* \to N_b^*$ come from lifting $i_b^{\mathcal{U}}$ to N^* via the iteration-strategy construction of [23], we have that $W(\mathcal{T}, \mathcal{U})^\frown\langle a \rangle$ is a pseudo-hull of $i_b^*(\mathcal{T})$. This is the key step in the proof. It is carried out in section 4.3.

(3) $i_b^*(\Sigma) \subseteq \Sigma$ because Σ was Suslin-co-Suslin captured by N^*, so $i_b^*(\mathcal{T})$ is by Σ.

(4) Thus $W(\mathcal{T}, \mathcal{U})^\frown\langle a \rangle$ is by Σ, because Σ has strong hull condensation.

(5) Since a determines b (see §3.7), and Σ normalizes well, we must then have $\Sigma(\langle \mathcal{T}, \mathcal{U} \rangle) = b$, as desired.

Here is a diagram of the situation:

$$
\begin{tikzcd}
M_b^{\mathcal{U}} \arrow[r] & M_\infty^{\mathcal{W}_b} \arrow[r] & i_b^*(R) & N_b^* \\
R \arrow[u, "\mathcal{U}"] & & R \arrow[u, "i_b^*"'] & N^* \arrow[u, "i_b^*"'] \\
P \arrow[u, "\mathcal{T}"] \arrow[uur, "\mathcal{W}_b"] \arrow[uurr, "i_b^*(\mathcal{T})"'] & & &
\end{tikzcd}
$$

Figure 1.1: Proof of $(*)(P, \Sigma)$. $\mathcal{W}_b$ is a psuedo-hull of $i_b^*(\mathcal{T})$.

Historical note. The author proved the main comparison theorem of this book in Spring 2015. Its proof was circulated as a handwritten manuscript in July 2015. A preliminary form of the present book was circulated in April 2016, and has been revised and expanded since then, with the last major expansion taking place in March-October 2019. The papers [60], [61],[62], [65],[48], [63], and [64], written in 2016-2018, have extended the work reported here in various directions.

2 Preliminaries

Inner model theory deals with canonical objects, but inner model theorists have presented them in various ways. The conventions we use here are all fairly common. For basic fine structural notions such as projecta, cores, standard parameters, fine ultrapowers, and degrees of elementarity, we shall follow the paper [41] by Schindler and Zeman. We shall use Jensen indexing for the sequences of extenders from which premice are constructed; see for example Zeman's book [69]. The construction of premice using background extenders comes ultimately from Mitchell-Steel [23], but the precise definitions and notation we use come from Neeman-Steel [29]. Here is some further detail.

2.1 Extenders and ultrapowers

Our notation for extenders is standard.

Definition 2.1 *Let M be transitive and rudimentarily closed; then $E = \langle E_a \mid a \in [\theta]^{<\omega}\rangle$ is a (κ, θ)-extender over M with spaces $\langle \mu_a \mid a \in [\theta]^{<\omega}\rangle$ if and only if*

(1) Each E_a is an (M, κ)-complete ultrafilter over $P([\mu_a]^{|a|}) \cap M$, with μ_a being the least μ such that $[\mu]^{|a|} \in E_a$.

(2) (Compatibility) For $a \subseteq b$ and $X \in M$, $X \in E_a \Leftrightarrow X^{ab} \in E_b$.

(3) (Uniformity) $\mu_{\{\kappa\}} = \kappa$.

(4) (Normality) If $f \in M$ and $f(u) < \max(u)$ for E_a a.e. u, then there is a $\beta < \max(u)$ such that for $E_{a\cup\{\beta\}}$ a.e. u, $f^{a,a\cup\{\beta\}}(u) = u^{a,a\cup\{\beta\}}$.

The unexplained notation here can be found in [41, §8]. We shall often identify E with the binary relation $(a, X) \in E$ iff $X \in E_a$. One can also identify it with the other section-function of this binary relation, which is essentially the function $X \mapsto i^M_E(X) \cap \theta$. We call θ the *length* of E, and write $\theta = \mathrm{lh}(E)$. The *space of* E is

$$\mathrm{sp}(E) = \sup\{\mu_a \mid a \in [\mathrm{lh}(E)]^{<\omega}\}.$$

The *domain of* E is the family of sets it measures, that is, $\mathrm{dom}(E) = \{Y \mid \exists (a, X) \in E(Y = X \vee Y = [\mu_a]^{|a|} - X)\}$. If M is a premouse of some kind, we also write $M|\eta = \mathrm{dom}(E)$, where η is least such that $\forall (a, X) \in E(X \in M|\eta)$. By acceptability, $\eta = \sup(\{\mu_a^{+,M} \mid a \in [\theta]^{<\omega}\})$. The critical point of a (κ, θ) extender is κ, and we use either $\mathrm{crit}(E)$ or κ_E to denote it. Given an extender E over M, we form the Σ_0 ultrapower

$$\mathrm{Ult}_0(M,E) = \{[a,f]^M_E \mid a \in [\mathrm{lh}(E)]^{<\omega} \text{ and } f \in M\},$$

as in [41, 8.4]. Our M will always be rudimentarily closed and satisfy the Axiom of Choice, so we have Los' theorem for Σ_0 formulae, and the canonical embedding

$$i^M_E \colon M \to \mathrm{Ult}_0(M,E)$$

is cofinal and Σ_0-elementary, and hence Σ_1-elementary. By normality, $a = [a,\mathrm{id}]^M_E$, so $\mathrm{lh}(E)$ is included in the (always transitivized) wellfounded part of $\mathrm{Ult}_0(M,E)$. More generally,

$$[a,f]^M_E = i^M_E(f)(a).$$

If $X \subseteq \mathrm{lh}(E)$, then $E \restriction X = \{(a,Y) \in E \mid a \subseteq X\}$. $E \restriction X$ has the properties of an extender, except possibly normality, so we can form $\mathrm{Ult}_0(M, E{\restriction}X)$, and there is a natural factor embedding $\tau\colon\ \mathrm{Ult}_0(M,E\restriction X) \to \mathrm{Ult}_0(M,E)$ given by

$$\tau([a,f]^M_{E\restriction X}) = [a,f]^M_E.$$

In the case that $X = \nu > \kappa_E$ is an ordinal, $E \restriction \nu$ is an extender, and $\tau \restriction \nu$ is the identity. We say ν is a *generator of E* iff ν is the critical point of τ, that is, $\nu \neq [a,f]^M_E$ whenever $f \in M$ and $a \subseteq \nu$. Let

$$\nu(E) = \sup(\{\nu + 1 \mid \nu \text{ is a generator of } E\ \}).$$

So $\nu(E) \leq \mathrm{lh}(E)$, and E is equivalent to $E{\restriction}\nu(E)$, in that the two produce the same ultrapower.

We write $\lambda(E)$ or λ_E for $i^M_E(\kappa_E)$. Note that although E may be an extender over more than one M, $\mathrm{sp}(E), \kappa_E, \mathrm{lh}(E), \mathrm{dom}(E), \nu(E)$, and $\lambda(E)$ depend only on E itself. If N is another transitive, rudimentarily closed set, and $P(\mu_a) \cap N = P(\mu_a) \cap M$ for all $a \in [\mathrm{lh}(E)]^{<\omega}$, then E is also an extender over N; moreover i^M_E agrees with i^N_E on $\mathrm{dom}(E)$. However, i^M_E and i^N_E may disagree beyond that. We say E is *short* iff $\nu(E) \leq \lambda(E)$. It is easy to see that E is short iff $\mathrm{lh}(E) \leq \sup(i^M_E``((\kappa^+_E)^M))$. If E is short, then all its interesting measures concentrate on the critical point. When E is short, i^M_E is continuous at $\kappa^{+,M}$, and if M is a premouse, then $\mathrm{dom}(E) = M|\kappa^{+,M}_E$. In this paper, we shall deal almost exclusively with short extenders. If we start with $j\colon M \to N$ with critical point κ, and an ordinal ν such that $\kappa < \nu \leq o(N)$, then for $a \in [\nu]^{<\omega}$ we let μ_a be the least μ such that $a \subseteq j(\mu)$, and for $X \subseteq [\mu_a]^{|a|}$ in M, we put

$$(a,X) \in E_j \Leftrightarrow a \in j(X).$$

E_j is an extender over M, called the (κ, ν) *extender derived from* j. We have the diagram

$$\begin{tikzcd} M \arrow[rr, "j"] \arrow[drr, "i_E^M"'] & & N \\ & & \mathrm{Ult}(M, E) \arrow[u, "k"'] \end{tikzcd}$$

where $i = i^M_{E_j}$, and

$$k(i(f)(a)) = j(f)(a).$$

$k \restriction \nu$ is the identity. If E is an extender over M, then E is derived from i^M_E.

The *Jensen completion* of a short extender E over some M is the $(\kappa_E, i^M_E((\kappa_E^+)^M))$ extender derived from i^M_E. E and its Jensen completion E^* are equivalent, in that $\nu(E) = \nu(E^*)$, and $E = E^* \restriction \mathrm{lh}(E)$.

2.2 Pure extender premice

Our main results apply to premice of various kinds, both strategy premice and pure extender premice, with λ-indexing or ms-indexing for their extender sequences. The comparison theorem for iteration strategies that is our first main goal holds in all these contexts. Although the proof of this theorem requires a detailed fine-structural analysis, the particulars of the fine structure don't affect anything important. We shall prove it first in the case of iteration strategies for pure extender premice with λ-indexing. The essential equivalence of λ-indexing with ms-indexing has been carefully demonstrated by Fuchs in [4] and [5].

The reader should see [1, Def. 2.4] for further details on the following definition. A *Jensen premouse* is a pair

$$M = \langle \hat{M}, k \rangle,$$

where

$$\hat{M} = \langle J^{\vec{E}}_\alpha, \in, \vec{E}, \gamma, F \rangle$$

is an acceptable structure with various properties, and $k < \omega$. The language $\mathcal{L}_0$ of $\hat{M}$ has $\in$, predicate symbols $\dot{E}$ and $\dot{F}$, and a constant symbol $\dot{\gamma}$. We call $\mathcal{L}_0$ the *language of (pure extender) premice*. We write $k = k(M)$; it marks the level of the Levy hierarchy over $\hat{M}$ at which we are considering this structure, and we demand that $\hat{M}$ be $k(M)$-sound. So what we are calling a premouse is just a premouse in the usual sense, paired with a degree of soundness that it has. We usually abuse notation by identifying M with $\hat{M}$.

Abusing notation this way, we set $o(M) = \mathrm{ORD} \cap M$, so that $o(M) = \omega\alpha$ for M as displayed. (The [41] convention differs slightly here.) We write $\hat{o}(M)$ for α itself. The index of M is

$$l(M) = \langle \hat{o}(M), k(M) \rangle.$$

If $\langle \nu, l \rangle \leq_{\mathrm{lex}} l(M)$, then $M|\langle \nu, l \rangle$ is the initial segment N of M with index $l(N) = \langle \nu, l \rangle$. (So $\dot{E}^N = \dot{E}^M \cap N$, and $\dot{F}^N = \dot{E}^M_{\omega\nu}$.) If $\nu \leq \hat{o}(M)$, then we write $M|\nu$ for $M|\langle \nu, 0 \rangle$. We write $M||\nu$, or sometimes $M|\langle \nu, -1 \rangle$, for the structure that agrees with $M|\nu$ except possibly on the interpretation of $\dot{F}$, and satisfies $\dot{F}^{M||\nu} = \emptyset$. By convention, $k(M||\nu) = 0$.

Remark 2.2 We may occasionally consider pairs of the form $M = \langle \hat{M}, \omega \rangle$, call them premice if for all $k < \omega$, $\langle \hat{M}, k \rangle$ is a premouse, and write $k(M) = \omega$. But in most contexts, we are considering premice M such that $k(M) < \omega$.

Definition 2.3 *If P and Q are Jensen premice, then $P \trianglelefteq Q$ iff there are μ and l such that $P = Q|\langle \mu, l \rangle$. Also, $P \triangleleft Q$ iff $P \trianglelefteq Q$ and $P \neq Q$.*

Thus if P and Q have the same universe, but $k(P) < k(Q)$, then $P \triangleleft Q$. Also, if P is passive and Q is active at $o(P)$, then it is not the case that $P \trianglelefteq Q$. So for example, if Q is active, iit is not the case that $Q||o(Q) \trianglelefteq Q$, where $Q||o(Q)$ is Q with its last extender predicate removed. Other conventions would be possible, but this one works best here.

If M is a Jensen premouse, then $\dot{E}^M$ is a sequence of extenders, and $\dot{F}^M$ is either empty, or codes a new extender being added to our model by M. The main requirements are

(1) (λ-indexing) If $F = \dot{F}^M$ is nonempty (i.e., M is *active*), then $M \models \mathrm{crit}(F)^+$ exists, and for $\mu = \mathrm{crit}(F)^{+M}$, $o(M) = i^M_F(\mu) = \mathrm{lh}(F)$. $\dot{F}^M$ is just the graph of $i^M_F \restriction (M|\mu)$.

(2) (Coherence) $i^M_F(\dot{E}^M) \restriction o(M) + 1 = \dot{E}^M {}^\frown \langle \emptyset \rangle$.

(3) (Initial segment condition, J-ISC) If G is a whole proper initial segment of F, then the Jensen completion of G must appear in $\dot{E}^M$. If there is a largest whole proper initial segment, then $\dot{\gamma}^M$ is the index of its Jensen completion in $\dot{E}^M$. Otherwise, $\dot{\gamma}^M = 0$.

(4) If N is an initial segment of M, then N is $k(N)$ sound.

Here an initial segment $G = F\restriction\eta$ of F is *whole* iff $\eta = \lambda_G$. Since Jensen premice are acceptable J-structures, the basic fine structural notions apply to them, so clause (4) above makes sense.

Figure 2.1 illustrates a common situation, one that occurs at successor steps in an iteration tree, for example.

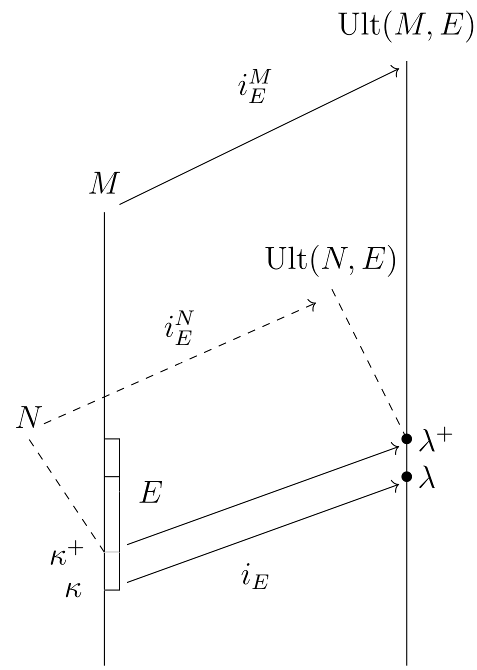


Figure 2.1: E is on the coherent sequence of M, $\kappa = \text{crit}(E)$, and $\lambda = \lambda(E)$. $P(\kappa)^M = P(\kappa)^N = \text{dom}(E)$, so $\text{Ult}(M, E)$ and $\text{Ult}(N, E)$ make sense. The ultrapowers agree with M below $\text{lh}(E)$, and with each other below $\text{lh}(E) + 1$.

There is a significant strengthening of the Jensen initial segment condition (3) above. If M is an active premouse, then we set

$$\nu(M) = \max(\nu(\dot{F}^M), \text{crit}(\dot{F}^M)^{+,M}).$$

$\dot{F}^M\restriction\nu(M)$ is equivalent to $\dot{F}^M$, and so it is not in M. But

Definition 2.4 *Let M be an active premouse with last extender F; then M satisfies the* ms-ISC *(or is* ms-solid*) iff for any $\eta < \nu(M)$, $F\restriction\eta \in M$.*

Clearly the ms-ISC implies the weakening of J-ISC in which we only demand that the whole proper initial segments of $\dot{F}^M$ belong to M. But for iterable M, this then implies the full J-ISC. (See [42].)

Theorem 2.5 *(ms-ISC) Let M be an active premouse with last extender F, and suppose M is 1-sound and $(1, \omega, \omega_1 + 1)$-iterable; then M is ms-solid.*

This is essentially the initial segment condition of [23], but stated for Jensen premice. [23] goes on to say that the trivial completion of $F \restriction \eta$ is either on the M-sequence, or an ultrapower away. This is correct unless $F \restriction \eta$ is type Z. If $F \restriction \eta$ is type Z, then it is the extender of $F \restriction \xi$-then-U, where ξ is its largest generator, and U is an ultrafilter on ξ, and we still get $F \restriction \eta \in M$. (See [42]. Theorem 2.7 of [42] is essentially 2.5 above.)

If M is active, we let its *initial segment ordinal* be

$$\iota(M) = \sup(\{\eta + 1 \mid \dot{F}^M \restriction \eta \in M\}).$$

So M is ms-solid iff $\iota(M) = \nu(M)$. Theorem 2.5 becomes false when its soundness hypothesis is removed, since if $N = Ult_0(M, E)$ where $\nu(M) \leq \text{crit}(E) < \lambda_F$, then $\iota(N) = \iota(M) = \nu(M)$, but $\text{crit}(E) < \nu(N)$.

We shall not use ms-premice, so henceforth we shall refer to Jensen premice as premice, or later, when we need to distinguish them from hod premice, as *pure extender premice*.

2.3 Projecta and cores

If $M = (N, k)$ is a premouse, then N is a k-sound acceptable J-structure. Thus the projecta $\rho_i(N)$ and standard parameters $p_i(N)$ exist for all $i \leq k + 1$, as do the reducts ("Σ_i mastercodes") $N^i = N^{i, p_i(N)}$. As in [41], if $i \leq k$, then

$$\rho_{i+1}(N) = \rho_1(N^i),$$

and

$$p_{i+1}(N) = p_i(N)^\frown \langle r \rangle,$$

where r is the lexicographically least descending sequence of ordinals from which a new subset of $\rho_1(N^i)$ can be Σ_1 defined over N^i. Clearly, $ORD \cap N^i = \rho_i(N)$, and $r \subseteq [\rho_{i+1}(N), \rho_i(N))$. If $i < k$, then r is solid, so each $\alpha \in r$ has a standard solidity witness $W_{N^i}^{\alpha, p_i(N)}$ that belongs to N^i.

Definition 2.6 *(a) If Q is an amenable J-structure, then h^1_Q is its canonical Σ_1 Skolem function.*

(b) If M is a premouse and $n \leq k(M)$, then h^{n+1}_M is the $r\Sigma_{n+1}$ Skolem function obtained by iteratively composing Σ_1 Skolem functions of reducts. (Cf. [41], 5.4.)

(c) Let $M = (N, k)$ be a premouse and $\alpha < \rho_k(N)$ and $r \in [\rho_k(M)]^{<\omega}$; then

$$W^{\alpha,r}_M = \text{ transitive collapse of } h^{k+1}_N``(\alpha \cup r \cup p_k(M)).$$

When $\alpha \in p_{k+1}(M)$ and $r = p_{k+1}(M) - (\alpha + 1)$, we call $W^{\alpha,r}_M$ the standard solidity witness *for α.*

Abusing notation, we speak of $\rho_i(M), M^i$, etc., instead of $\rho_i(N), N^i$, etc. Finally, if $k < \omega$, we set

$$\rho(M) = \rho_{k+1}(M),\ p(M) = p_{k+1}(M),\ \text{and } h_M = h^{k+1}_M,$$

where $k = k(M)$, and call them *the* projectum, parameter, and Skolem function of M. Let

$$\mathcal{C}(M) = \mathcal{C}_{k(M)+1}(M) = \text{ transitive collapse of } h_M``(\rho(M) \cup p(M)),$$

considered as an $\mathcal{L}_0$-structure. Let $\pi\colon \mathcal{C}(M) \to M$ be the anticollapse, and $t = \pi^{-1}(p(M))$. We say that M is $k+1$ *solid*, or M *has a core*, iff $p_{k+1}(M)$ is $k+1$ universal over M, and t is $k+1$ solid over $\mathcal{C}(M)$. This implies that t is $k+1$ universal over $\mathcal{C}(M)$, that $p_{k+1}(M)$ is $k+1$-solid over M, and that $t = p_{k+1}(\mathcal{C}(M))$. If M is $k(M)+1$ solid, then $\mathcal{C}(M)$ is the core of M. We say that M is *sound* iff $M = \mathcal{C}(M)$. When we wish to consider $\mathcal{C}(M)$ as a premouse with degree of soundness attached, we set

$$k(\mathcal{C}(M)) = k(M) + 1.$$

We may occasionally say that M is $k+1$ solid for some $k > k(M)$. This just means that M^{k+1} exists, that is, that the process of starting with M and iteratively taking cores, setting $\mathcal{C}_{k(M)}(M) = M$ and $\mathcal{C}_{i+1}(M) = \mathcal{C}(\mathcal{C}_i(M))$, does not break down by reaching some non-solid $\mathcal{C}_i(M)$ with $i \leq k$. M^{k+1} is the reduct which codes $\mathcal{C}_{k+1}(M)$. We say that M is $k+1$ *sound* if M is $k+1$ solid, and $M = \mathcal{C}_{k+1}(M)$. (If we ignore the distinguished soundness degrees, that is.)

For the notion of *generalized solidity witness*, see [41]. Roughly speaking, a generalized solidity witness for $\alpha \in p_1(M)$ is a transitive structure whose theory includes

$\mathrm{Th}_1^M(\alpha \cup p_1(M) - (\alpha + 1))$. Being a generalized witness for an $\alpha \in p_k(M)$ is a $r\Pi_k$ condition, hence preserved by $r\Sigma_k$ embeddings. Such embeddings may not preserve being a standard witness.

The extension-of-embeddings lemmas relate reducts to the structures they code. The downward extension of embeddings lemma tells us that if S is amenable and $\pi\colon S \to N^n$ is Σ_0, then there is a (unique) M such that $S = M^n$. The upward extension lemma tells us that if $\pi\colon M^n \to S$ is Σ_1 and preserves the wellfoundedness of certain relations (the important one being $\in^M$ as it is described in the predicate of M^n), then there is a unique N such that $S = N^n$. See 5.10 and 5.11 of [41].

Remark 2.7 We have defined cores here as they are defined in [41]. In [23] they are defined in slightly different fashion. First, [23] works directly with the $\mathcal{C}_{k+1}(M)$, rather than with the reducts which code them. The translations indicated above show that is not a real difference; see [23], page 40. Second, if $k \geq 1$, then [23] puts the standard solidity witnesses for $p_k(M)$ into the hull collapsing to $\mathcal{C}_{k+1}(M)$, and if $k \geq 2$, it also puts $\rho_{k-1}(M)$ into this hull if $\rho_{k-1}(M) < o(M)$. The definition from [41] used above does not do this directly. We are grateful to Schindler and Zeman for pointing out that nevertheless these objects do get into the cores as defined in [41], and therefore the two definitions of $\mathcal{C}_{k+1}(M)$ are equivalent. [For example, let $k = 2$ and let M be 1-sound, with $\alpha \in p_1(M)$. Let $r = p_1(M) \setminus (\alpha + 1)$. Let $\pi\colon \mathcal{C}_2(M) \to M$ be the anticore map, and $\pi(\beta) = \alpha$ and $\pi(s) = r$. The relation "W is a generalized solidity witness for α, r" is Π_1 over M. (It is important to add *generalized* here. Being a standard witness is only Π_2.) Since π is Σ_2 elementary, there is a generalized solidity witness for β, s over $\mathcal{C}_2(M)$ in $\mathcal{C}_2(M)$. But any generalized witness generates the standard one ([41], 7.4), so the standard solidity witness U for β, s is in $\mathcal{C}_2(M)$. Being the standard witness is Π_2, so $\pi(U)$ is the standard witness for α, r, and this witness is in $\mathrm{ran}(\pi)$, as desired.]

2.4 Elementarity of maps

Given n-sound acceptable J-structures M and N, and $\pi\colon M^n \to N^n$ a Σ_0 elementary embedding on their n-th reducts, then by decoding the reducts we get a unique $\hat{\pi}\colon M \to N$ that is Σ_n elementary and is such that $\pi \subseteq \hat{\pi}$. If π is Σ_1 elementary, then $\hat{\pi}$ is Σ_{n+1} elementary. The decoding is done iteratively, and yields that for $k < n$, $\hat{\pi}\colon M^k \to N^k$ is Σ_{n-k} or Σ_{n-k+1}, respectively. $\hat{\pi}$ is called the *n-completion* of π. See lemmas 5.8 and 5.9 of [41]. These lemmas record additional elementarity properties of $\hat{\pi}$, codified in definition 5.12 as *$r\Sigma_{n+1}$-elementarity* if π is Σ_1, and *weak $r\Sigma_{n+1}$-elementarity* if π is only Σ_0. Such maps are *cardinal preserving* , in that $M \models$ "γ is a cardinal" iff $N \models$ "$\pi(\gamma)$ is a cardinal", except possibly the weakly $r\Sigma_0$ maps. In this case, we shall always just add cardinal preservation as an additional hypothesis. This leads us to:

Definition 2.8 *Let M and N be Jensen premice with $n = k(M) = k(N)$, and $\pi\colon M \to N$; then*

(a) π is weakly elementary *iff π is the n-completion of $\pi \restriction M^n$, and $\pi \restriction M^n\colon M^n \to N^n$ is Σ_0 and cardinal preserving.*

(b) π is elementary *iff π is the n-completion of $\pi \restriction M^n$, and $\pi \restriction M^n\colon M^n \to N^n$ is Σ_1.*

(c) π is cofinal *iff $\sup \pi``\rho_n(M) = \rho_n(N)$.*

*(d) π is an n-*embedding *iff π is cofinal and elementary.*

The elementary maps correspond to those which are near n-embeddings in the sense of [36]. The cofinal elementary maps correspond to the n-embeddings of [23]. When $n \geq 1$, the weakly elementary embeddings correspond to those that are n-apt in the sense of [36], $\Sigma_0^{(n)}$ in the sense of [69], or n-lifting in the sense of [43]. There are many other levels of elementarity isolated in these references, but for our purposes this is enough.

In particular, we shall not use the notion of *weak n-embedding* defined in [23]. In the end, that notion is not very natural, and in a number of places it does not do the work that the authors of [23] thought that it did. In particular, there are problems with how it was used in the Shift Lemma, the copying construction, and the Weak Dodd-Jensen Lemma. These problems are discussed in [43], and a variety of ways to repair the earlier proofs are given. The simplest of these is to use weakly elementary maps instead of of weak n-embeddings at the appropriate places.

The following is clear from the definition:

Proposition 2.9 *Let M and N be Jensen premice with $n = k(M) = k(N)$, and $\pi\colon M \to N$ be weakly elementary; then*

(1) π is Σ_n elementary,

(2) $\pi(p_k(M)) = p_k(N)$ for all $k \leq n$, and

(3) $\pi(\rho_k(M)) = \rho_k(N)$ for $k < n-1$, and $\sup \pi``\rho_n(M) \leq \rho_n(N)$, and

(4) for any $\alpha < \rho_n(M)$, $\pi(Th_n^M(\alpha \cup p_n(M))) = Th_n^N(\pi(\alpha) \cup p_n(N))$.

Note that we do not necessarily have that $\rho_n(N) \leq \pi(\rho_n(M))$, or that π is Σ_{n+1}-elementary on a set cofinal in $\rho_n(M)$, which are the additional requirements from [23] on weak n-embeddings.

It is easy to see that if π is (weakly) elementary as a map from (M, n) to (N, n), and $k < n$, then π is (weakly) elementary as a map from $(M, k) \to (N, k)$. Indeed, $\pi \upharpoonright M^k$ is a stage in the decoding of $\pi \upharpoonright M^n$. If $k(M) \neq k(N)$, then we say $\pi\colon M \to N$ is (weakly) elementary iff it is (weakly) elementary as a map from (M, n) to (N, n), where $n = \inf(k(M), k(N))$.

Note that if $\pi\colon M \to N$ is weakly elementary, and $k = \inf k(M), k(N)$, then π moves generalized solidity witnesses for $p_k(M)$ to generalized solidity witnesses for $p_k(N)$. For example, being a generalized witness for $p_1(M)$ is a Π_1 fact, so preserved by Σ_1 embeddings. Even cofinal elementary maps may fail to move standard solidity witnesses to standard solidity witnesses.

Here are some natural contexts in which the levels of elementarity play a role.

(i) The natural map from the core of M to M is elementary and cofinal, that is, a full n-embedding for $n = k(M)$.

(ii) The maps $\hat{i}^{\mathcal{T}}_{\alpha,\beta}$ along branches of iteration trees are elementary and cofinal (see below).

(iii) If $\pi\colon M \to N$ is weakly elementary, and $\mathcal{T}$ is a weakly normal tree on M, then $\pi\mathcal{T}$ is weakly normal, and the copy maps $\pi_\alpha\colon \mathcal{M}^{\mathcal{T}}_\alpha \to \mathcal{M}^{\pi\mathcal{T}}_\alpha$ are weakly elementary. The Dodd-Jensen and weak Dodd-Jensen lemmas hold in the category of weakly elementary maps.

(iv) If π, M, N, and $\mathcal{T}$ are as in (iii), and in addition, $\rho_k(N) \leq \pi(\rho_k(M))$ for $k = k(M)$, then all the π_α satisfy the corresponding condition, and if $\mathcal{T}$ is normal, then so is $\pi\mathcal{T}$. (See Remark 2.26 below.)

(v) By Lemma 1.3 of [36]), if $\pi\colon M \to N$ is elementary, and $\mathcal{T}$ is a weakly normal tree on M, then the copy maps $\pi_\alpha\colon \mathcal{M}^{\mathcal{T}}_\alpha \to \mathcal{M}^{\pi\mathcal{T}}_\alpha$ are elementary. (They are not necessarily cofinal.)

(vi) The maps $\pi^{\nu,\gamma}_\tau$ occuring in an embedding normalization are elementary. The maps σ_γ are weakly elementary, but may not be elementary, so far as we can see. See Chapter 3.

(vii) The lifting maps that occur in the proof of iterability are only weakly elementary. They are not in general elementary. (See below.)

2.5 Iteration trees

If M is a premouse with $n = k(M)$, and E is a short extender over M with $\kappa_E < \rho_n(M)$ and $P(\kappa_E)^M \subseteq \mathrm{dom}(E)$, then we set

$$\begin{aligned}\mathrm{Ult}(M, E) &= \mathrm{Ult}_n(M, E) \\ &= \text{ decoding of } \mathrm{Ult}_0(M^n, E).\end{aligned}$$

The canonical embedding of M^n into $\mathrm{Ult}(M^n, E)$ is Σ_1 and cofinal. Its n-completion $i_E^M \colon M \to \mathrm{Ult}_n(M, E)$ is therefore an n-embedding. (We assume here that $\mathrm{Ult}_n(M, E)$ is wellfounded, though one could make sense of these statements even if it is not.) By convention,

$$k(M) = k(\mathrm{Ult}(M, E)).$$

Rather than coding and decoding, one can define $\mathrm{Ult}(M, E)$ directly, as in [23]:

$$\mathrm{Ult}(M, E) = \{[a, f_{\tau,q}]_E^M \mid a \in [\lambda]^{<\omega} \wedge q \in M \wedge \tau \in \mathrm{SK}_n\},$$

where $n = k(M)$ and SK_n is the set of $r\Sigma_n$ Skolem terms.

If in addition $\rho(M) \leq \kappa_E$, $p(M)$ is solid, and E is close to M, then $\rho(M) = \rho(\mathrm{Ult}(M, E))$, and $i_E^M(p(M)) = p(\mathrm{Ult}(M, E))$, and $p(\mathrm{Ult}(M, E))$ is also solid.

Our notation and terminology regarding iteration trees is essentially that of [53]. If $\mathcal{T}$ is a tree on M, then $\mathcal{M}_\alpha^\mathcal{T}$ is its α-th model, and $E_\alpha^\mathcal{T}$ is the *exit extender* taken from the sequence of $\mathcal{M}_\alpha^\mathcal{T}$ and used to form

$$\mathcal{M}_{\alpha+1}^\mathcal{T} = \mathrm{Ult}(\mathcal{M}_{\alpha+1}^{*,\mathcal{T}}, E_\alpha^\mathcal{T}).$$

where

$$\mathcal{M}_{\alpha+1}^{*,\mathcal{T}} = \mathcal{M}_\beta^\mathcal{T} | \langle \xi, k \rangle$$

for some $\beta = T\text{-pred}(\alpha + 1)$, and some $\langle \xi, k \rangle \leq l(\mathcal{M}_\beta^\mathcal{T})$ such that $\mathrm{crit}(E_\alpha^\mathcal{T}) < \rho_k(\mathcal{M}_\beta^\mathcal{T} | \xi)$. We put $\alpha + 1 \in D^\mathcal{T}$ iff $\mathcal{M}_{\alpha+1}^{*,\mathcal{T}} \lhd \mathcal{M}_\beta^\mathcal{T}$ iff $l(\mathcal{M}_{\alpha+1}^{*,\mathcal{T}}) < l(\mathcal{M}_\beta^\mathcal{T})$, and we say $\mathcal{T}$ drops at $\alpha + 1$ in this case. So unlike [53], drops in degree yield elements of $D^\mathcal{T}$ too. If $\alpha \leq_T \beta$ and $(\alpha, \beta]_T \cap D^\mathcal{T} = \emptyset$, then the canonical embedding

$$i_{\alpha,\beta}^\mathcal{T} \colon \mathcal{M}_\alpha^\mathcal{T} \to \mathcal{M}_\beta^\mathcal{T}$$

is cofinal and elementary; that is, it is an n-embedding, where $n = k(\mathcal{M}_\alpha^\mathcal{T}) = k(\mathcal{M}_\beta^\mathcal{T})$. All extenders in $\mathcal{T}$ are close to the models to which they are applied, so if $\mathrm{crit}(i_{\alpha,\beta}^\mathcal{T}) \geq \rho(\mathcal{M}_\alpha^\mathcal{T})$, then $\rho(\mathcal{M}_\alpha^\mathcal{T}) = \rho(\mathcal{M}_\beta^\mathcal{T})$ and $i_{\alpha,\beta}^\mathcal{T}(p(\mathcal{M}_\alpha^\mathcal{T})) = p(\mathcal{M}_\beta^\mathcal{T})$.

We shall also have a use for the natural partial embeddings that exist along branches that have dropped.

Definition 2.10 *Let $\mathcal{U}$ be an iteration tree, and $\alpha <_U \beta$. Then $\hat{\imath}^{\mathcal{U}}_{\alpha,\beta}$ is the natural map from a (perhaps proper!) initial segment of $\mathcal{M}^{\mathcal{U}}_\alpha$ into $\mathcal{M}^{\mathcal{U}}_\beta$. More precisely,*

$$i^{*,\mathcal{U}}_{\beta+1} \colon \mathcal{M}^{*,\mathcal{U}}_{\beta+1} \to \mathrm{Ult}(\mathcal{M}^{*,\mathcal{U}}_{\beta+1}, E^{\mathcal{U}}_\beta)$$

is the canonical embedding,

$$\hat{\imath}^{\mathcal{U}}_{\alpha,\beta+1} = i^{*\mathcal{U}}_{\beta+1} \circ \hat{\imath}^{\mathcal{U}}_{\alpha,\gamma}$$

if $\gamma = U\text{-pred}(\beta+1)$, and

$$\hat{\imath}^{\mathcal{U}}_{\alpha,\lambda}(x) = i^{\mathcal{U}}_{\beta,\lambda}(\hat{\imath}_{\alpha,\beta}(x))$$

if λ is a limit ordinal, and β is past the last drop in $[0,\lambda)_U$.

It would have been more natural to have originally defined $i^{\mathcal{U}}_{\alpha,\beta}$ the way we just defined $\hat{\imath}^{\mathcal{U}}_{\alpha,\beta}$, but it is too late for that now. The difference between "$\hat{\imath}$" and "i" is barely visible anyway.

If $\mathcal{T}$ is an iteration tree, then $\mathrm{lh}(\mathcal{T})$ is the domain of its tree order, that is, $\mathrm{lh}(\mathcal{T}) = \{\alpha \mid \mathcal{M}^{\mathcal{T}}_\alpha \text{ exists }\}$. So if $\mathrm{lh}(\mathcal{T}) = \alpha+1$, then $\mathcal{M}^{\mathcal{T}}_\alpha$ exists, but $E^{\mathcal{T}}_\alpha$ does not. $\mathcal{T}{\restriction}\beta$ is the initial segment $\mathcal{U}$ of $\mathcal{T}$ such that $\mathrm{lh}(\mathcal{U}) = \beta$. So $\mathcal{M}^{\mathcal{T}\restriction\alpha+1}_\alpha$ exists, but there is no exit extender $E^{\mathcal{T}\restriction\alpha+1}_\alpha$.

Remark 2.11 We allow iteration trees of length 1. Such a degenerate tree has no extenders, and thus consists of only its base model. This convention plays some role in the definitions of tree embeddings and strong hull condensation.

By *normal* we shall mean "Jensen normal".

Definition 2.12 *Let $\mathcal{T}$ be an iteration tree on a premouse M; then $\mathcal{T}$ is* normal *iff*

(1) if $\beta+1 < \mathrm{lh}(\mathcal{T})$ and $\alpha < \beta$, then $\mathrm{lh}(E^{\mathcal{T}}_\alpha) < \mathrm{lh}(E^{\mathcal{T}}_\beta)$, and

(2) if $\alpha+1 < \mathrm{lh}(\mathcal{T})$, then $T\text{-pred}(\alpha+1)$ is the least β such that $\mathrm{crit}(E^{\mathcal{T}}_\alpha) < \lambda(E^{\mathcal{T}}_\beta)$, and

(3) $\mathcal{M}^{*,\mathcal{T}}_{\alpha+1} = \mathcal{M}^{\mathcal{T}}_\beta|\langle\eta,k\rangle$, where $\langle\eta,k\rangle \le l(\mathcal{M}^{\mathcal{T}}_\beta)$ is largest so that $\mathrm{crit}(E^{\mathcal{T}}_\alpha) < \rho_k(\mathcal{M}^{\mathcal{T}}_\beta|\eta)$.

Definition 2.13 *Let $\mathcal{T}$ be a normal iteration tree on a Jensen premouse; then for any $\beta < \mathrm{lh}(\mathcal{T})$,*

$$\begin{aligned}\lambda^{\mathcal{T}}_\beta &= \sup\{\lambda_F \mid \exists\eta < \beta(F = E^{\mathcal{T}}_\eta)\}\\ &= \sup\{\lambda_F \mid \exists\eta(\eta+1 \le_T \beta \wedge F = E^{\mathcal{T}}_\eta)\}\end{aligned}$$

So $\lambda^{\mathcal{T}}_\beta$ is the sup of the "Jensen generators" of extenders used to produce $\mathcal{M}^{\mathcal{T}}_\beta$. For $k = k(\mathcal{M}^{\mathcal{T}}_\beta)$, $\mathcal{M}^{\mathcal{T}}_\beta = h^{k+1}``(\mathrm{ran}(\hat{\imath}_{0,\beta}) \cup \lambda^{\mathcal{T}}_\beta)$.

If $\mathcal{T}$ is normal, then T-pred$(\beta + 1)$ is the largest α such that $\lambda^{\mathcal{T}}_\alpha \leq \mathrm{crit}(E^{\mathcal{T}}_\beta)$. Another useful characterization is the following. Let θ be $\mathrm{crit}(E^{\mathcal{T}}_\beta)^+$, as computed in $\mathcal{M}^{\mathcal{T}}_\beta | \mathrm{lh}(E^{\mathcal{T}}_\beta)$. Then

$$T\text{-pred}(\beta + 1) = \text{ least } \alpha \text{ such that } \mathcal{M}^{\mathcal{T}}_\alpha | \theta = \mathcal{M}^{\mathcal{T}}_\beta | \theta.$$

Note here that θ is passive in $\mathcal{M}^{\mathcal{T}}_\beta$, so for α as on the right, θ is passive in $\mathcal{M}^{\mathcal{T}}_\alpha$. The formula may fail if we replace the $|$ by $||$, for when $\lambda_{E^{\mathcal{T}}_\alpha} = \mathrm{crit}(E^{\mathcal{T}}_\beta)$, T-pred$(\beta + 1)$ is $\alpha + 1$, not α.

Figure 2.2 shows how the agreement of models in a normal iteration tree is propagated when the tree is augmented by one new extender. (Figures like this were first drawn by Itay Neeman.)

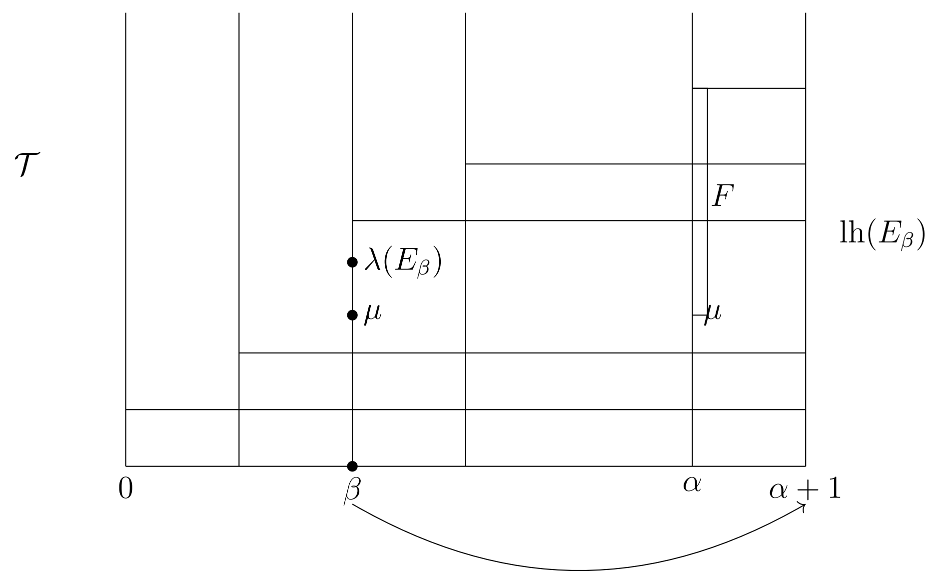


Figure 2.2: A normal tree $\mathcal{T}$, extended normally by F. The vertical lines represent the models, and the horizontal ones represent their levels of agreement. $\mathrm{crit}(F) = \mu$, and β is least such that $\mu < \lambda(E^{\mathcal{T}}_\beta)$. The arrow at the bottom represents the ultrapower embedding generated by F.

If one replaces the condition $\mathrm{crit}(E^{\mathcal{T}}_\alpha) < \lambda(E^{\mathcal{T}}_\beta)$ by the condition $\mathrm{crit}(E^{\mathcal{T}}_\alpha) < \nu(E^{\mathcal{T}}_\beta)$ in the definition of (Jensen) normality, one obtains a definition of *ms-normality*.

(This is called s-normality in [5, §5].) In fact, there are some advantages to working with ms-normal trees, even in the context of Jensen premice. One is that full background constructions of Jensen-normally iterable M seem to require superstrong extenders in V (but see [29]). On the other hand, one can show granted only a Woodin with a measurable above that there is a ms-normally iterable Jensen mouse with a Woodin cardinal, granted that there is in V a Woodin with a measurable above it. ([23] yields an ms-iterable ms-mouse with a Woodin, and [4] and [5] then translates it to an ms-normally iterable Jensen mouse with a Woodin.) Nevertheless, 2.12 is the more common notion of normality in the setting of Jensen premice, and it will serve our purposes. We believe that there are elementary simulations of Jensen normal trees by ms-normal trees, and vice-versa, but we have not verified this carefully.

Remark 2.14 ms-normal iterations preserve ms-solidity. As we remarked earlier, Jensen normal iterations may not.

We also need stacks of normal trees.

Definition 2.15 *Let M be a premouse; then s is a* normal M-stack *iff* $s = \langle (\nu_\alpha, k_\alpha, \mathcal{T}_\alpha) \mid \alpha < \beta \rangle$, *and there are premice M_α for $\alpha < \beta$ such that*

(1) $\mathcal{T}_\alpha$ is a normal tree on $M_\alpha | \langle \nu_\alpha, k_\alpha \rangle$,

(2) $M_0 = M$,

(3) if $\alpha < \beta$ and α is a limit ordinal, then M_α is the direct limit of the M_β for $\beta < \alpha$, and

(4) if $\gamma + 1 = \alpha < \beta$, then M_α is the last model of $\mathcal{T}_\gamma$

The definition allows a gratuitous drop at the beginning of each normal tree $\mathcal{T}_\alpha$. If $\langle \nu_\alpha, k_\alpha \rangle = l(M_\alpha)$ for all α, then we say s is *maximal*. We allow $k_\alpha = -1$, with the convention that $P|\langle \nu, -1 \rangle = P||\nu$ as above.

In (3), the direct limit is under the obvious partial maps $\hat{\imath}^s_{\xi,\gamma} \colon M_\xi \to M_\gamma$, for $\xi < \gamma < \alpha$. We demand that for $\alpha < \beta$ a limit, there are only finitely many drops along the branches producing these maps, and that the direct limit is wellfounded. We write $M_\xi(s)$ and $\mathcal{T}_\xi(s)$ for M_ξ and $\mathcal{T}_\xi$. If $\mathrm{dom}(s) = \alpha + 1$, then we write $\mathcal{U}(s) = \mathcal{T}_\alpha(s)$ for the last tree in the stack. $\mathcal{U}(s)$ could have no last model.

2.6 Jensen normal genericity iterations

Jensen normal genericity iterations must be allowed to drop, unless our identities are generated by superstrong extenders. However, this dropping will not occur along the main branch, so it is harmless. We explain this briefly now. The reader should see [53, §7] for more detail on the extender algebra and genericity iterations.

Let M be a premouse, and $\mu < \delta$ cardinals of M. We let $\mathbb{B} = \mathbb{B}^M_{\mu,\delta}$ be the ω-generator *extender algebra* determined by the extenders on the $M|\delta$-sequence with critical point $> \mu$. . $\mathbb{B}$ is the Lindenbaum algebra of a certain infinitary theory T in the propositional language $\mathcal{L}_{\delta,0}$ generated by the sentence symbols A_n, for $n < \omega$. For $x \subset \omega$, $x \models A_n$ iff $n \in x$, and then $x \models \varphi$ for φ an arbitrary sentence of $\mathcal{L}_0$ has the natural meaning. The axioms of T are those sentences of the form

$$\bigvee_{\alpha<\kappa} \varphi_\alpha \longleftrightarrow \bigvee_{\alpha<\lambda} i_E(\langle \varphi_\xi \colon \xi < \kappa\rangle) \restriction \lambda,$$

whenever E is on the $M|\delta$-sequence, $\mathrm{crit}(E) = \kappa > \mu$, $i_E(\langle \varphi_\xi \colon \xi < \kappa\rangle) \restriction \lambda \in M|\eta$, for some cardinal η of M such that $\eta < \lambda_E$. Let us write $T = T(M|\delta, \mu)$.

The usual argument shows that if δ is Woodin in M, then $M \models$ "$\mathbb{B}$ is δ-c.c.". It is also clear that if M comes from a background construction in V, then every $x \in V$ satisfies all axioms of T. This is because if E generates an axiom as above, and E^* is its background extender, then $E \restriction \eta = E^* \restriction \eta \cap M$, for all M-cardinals η.

Given an iterable M as above, and an $x \subset \omega$, we form a Jensen normal tree $\mathcal{T}$ on M as follows: $E^{\mathcal{T}}_\alpha$ is the first extender on the sequence of $\mathcal{M}^{\mathcal{T}}_\alpha$ with critical point above μ that induces an axiom of $T(\mathcal{M}^{\mathcal{T}}_\alpha | \sup \hat{\imath}^{\mathcal{T}}_{0,\alpha}``\delta, \mu)$ not satisfied by x. The rest is determined by the rules of Jensen normal trees. Note the hat above the i in the formula! $[0, \alpha)_T$ may have dropped. $\hat{\imath}_{0,\alpha}(\mu) = \mu$, but it may happen that $\hat{\imath}_{0,\alpha}(\delta)$ is undefined.

As usual, the construction of $\mathcal{T}$ terminates with a last model $\mathcal{M}^{\mathcal{T}}_\alpha$ such that x satisfies all the axioms of $T(\mathcal{M}^{\mathcal{T}}_\alpha | \sup \hat{\imath}^{\mathcal{T}}_{0,\alpha}``\delta, \mu)$. We must see that in this case, $[0, \alpha)_T$ has not dropped. Suppose that it has, and let $\xi + 1 \leq_T \alpha$ be the site of the last drop, and T-pred$(\xi + 1) = \gamma$. Let $E = E^{\mathcal{T}}_\gamma$, and let

$$\psi = \bigvee_{\alpha<\kappa} \varphi_\alpha \longleftrightarrow \bigvee_{\alpha<\lambda} i_E(\langle \varphi_\nu \colon \nu < \kappa\rangle) \restriction \lambda$$

be the bad axiom induced by E, and η a cardinal of $\mathcal{M}^{\mathcal{T}}_\gamma$ such that $\psi \in \mathcal{M}^{\mathcal{T}}_\gamma|\eta$. Since we dropped when applying $E^{\mathcal{T}}_\xi$, $\eta \leq \mathrm{crit}(E^{\mathcal{T}}_\xi)$, so $\hat{\imath}^{\mathcal{T}}_{\gamma,\alpha} \restriction \eta$ is the identity. But also, $\mathcal{M}^{\mathcal{T}}_\gamma | \mathrm{lh}(E) \trianglelefteq \mathcal{M}^*_{\xi+1}$, so $\hat{\imath}^{\mathcal{T}}_{\gamma,\alpha}(E)$ exists. Clearly, $\hat{\imath}^{\mathcal{T}}_{\gamma,\alpha}(E)$ still induces ψ as an axiom of $T(\mathcal{M}^{\mathcal{T}}_\alpha | \sup \hat{\imath}^{\mathcal{T}}_{0,\alpha}``\delta, \mu)$. Since x does not satisfy ψ, the genericity iteration did not terminate at α, contradiction.

2.7 Iteration strategies

Let M be a premouse. $G(M, \theta)$ is the game of length θ in which I and II cooperate to produce a normal tree on M, with II picking branches at limit steps, and being obliged to stay in the category of wellfounded models. See [53], where the game is called $G_k(M, \theta)$, for $k = k(M)$. A *θ-iteration strategy* for M is a winning strategy for II in $G(M, \theta)$.

If λ is a limit ordinal, then $G(M, \lambda, \theta)$ is the game in which the players play λ rounds, the α-th round being a play of $G(N, \theta)$, where N has been produced by the prior rounds. Thus a postion in $G(M, \lambda, \theta)$ that is not yet a loss for II is a normal M-stack of length $< \lambda$ whose component normal trees each have length $< \theta$. I extends the stack at successor stages, including starting a new normal tree if he wishes. II picks branches at limit stages, and his obligation is just to insure all models are wellfounded, including the direct limit of the base models in the final stack. A *(λ, θ)-iteration strategy* for M is a winning strategy for II in $G(M, \lambda, \theta)$. See [53].

In order to unify the notation, let us set $G(M, 1, \theta) = G(M, \theta)$.

It is natural to generalize these standard iteration games so that player I has the freedom to "drop gratuitously" on any of his moves. For example, if M is a premouse, we let $G^+(M, \theta)$ be the variant of $G(M, \theta)$ in which player II must pick cofinal wellfounded branches at limit steps as before, and given that $\mathcal{T}$ with $\mathrm{lh}(\mathcal{T}) = \alpha + 1$ is the play so far, I must pick E_α from the $\mathcal{M}_\alpha = \mathcal{M}_\alpha^\mathcal{T}$ sequence such that $\mathrm{lh}(E_\beta) < \mathrm{lh}(E_\alpha)$ for all $\beta < \alpha$. (Here $\mathcal{M}_0 = M$.) As before, we set

$$\xi = T\text{-pred}(\alpha + 1) = \text{ least } \beta \text{ s.t. } \mathrm{crit}(E_\alpha) < \lambda(E_\beta).$$

Let $\langle \nu, k \rangle$ be least such that $\rho(\mathcal{M}_\xi^\mathcal{T} | \langle \nu, k \rangle) \le \mathrm{crit}(E_\alpha)$, or $\langle \nu, k \rangle = l(\mathcal{M}_\xi)$. Let $\gamma = \mathrm{crit}(E_\alpha)^+$ in the sense of $\mathcal{M}_\alpha | \mathrm{lh}(E_\alpha)$, or equivalently, in the sense of $\mathcal{M}_\xi | \langle \nu, k \rangle$. We now allow I to pick *any* $\langle \eta, l \rangle$ such that

$$\langle \gamma, 0 \rangle \le \langle \eta, l \rangle \le \langle \nu, k \rangle,$$

and we set

$$\mathcal{M}_{\alpha+1} = \mathrm{Ult}(\mathcal{M}_\xi | \langle \eta, l \rangle, E_\alpha).$$

We write $\mathcal{M}_\xi | \langle \eta, l \rangle = T\text{-pred}(\alpha + 1)$.

Definition 2.16 *A* weakly normal *tree on an lpm M is a play of some $G^+(M, \theta)$ in which player* II *has not yet lost.*

In older terminology, a weakly normal tree is just one that is length-increasing and nonoverlapping.

Remark 2.17 Again, we allow degenerate weakly normal trees that use no extenders. If M is the base model of some such $\mathcal{T}$, then $\mathcal{T}$ may drop to some $N \trianglelefteq M$, then end.

For λ a limit ordinal or $\lambda = 1$, we let $G^+(M, \lambda, \theta)$ be the variant of $G(M, \lambda, \theta)$ in which I is allowed gratuitous dropping within each of the λ rounds. (So $G^+(M, 1, \theta) = G^+(M, \theta)$.) For notational reasons, we'll allow I to drop in the base model for the beginning of a round as well, though this is no extra generality in fact. II wins iff all models reached are wellfounded, and if $\lambda > 1$, there are finitely many drops along the sequence of base models, and their direct limit is wellfounded. We call a position in some $G^+(M, \lambda, \theta)$ in which II has not yet lost an *M-stack*.

Definition 2.18 *An M-*stack *is a sequence $s = \langle(\nu_\alpha, k_\alpha, \mathcal{T}_\alpha) \mid \alpha < \beta\rangle$ with all the properties of normal M-stacks, save that the $\mathcal{T}_\alpha$ may be only weakly normal.*

We allow some or all of the weakly normal trees in our M-stack to be empty. Given an an M-stack s as above, we write $(\nu_i(s), k_i(s), \mathcal{T}_i(s))$ for $s(i)$, $M_0(s) = M$, and $M_{i+1}(s)$ for the last model of $\mathcal{T}_i(s)$, when $i < \mathrm{dom}(s) - 1$. We write $\mathcal{U}(s)$ for $\mathcal{T}_{\mathrm{dom}(s)-1}(s)$, the last normal tree in s. We write $M_\infty(s)$ for the last model of $\mathcal{U}(s)$, if it has one. If s is a maximal M-stack, then we identify s with its sequence of trees $\mathcal{T}_i(s)$, the $\nu_i(s)$ and $k_i(s)$ being determined by maximality. If s is merely normal, we must specify the base models of the $\mathcal{T}_i(s)$ as well.

Tail strategies are defined by

Definition 2.19 *Let Ω be a winning strategy for* II *in $G^+(M, \lambda, \theta)$, and let s be an M-stack according to Ω with* $\mathrm{lh}(s) < \lambda$ *such that $M_\infty(s)$ exists; then Ω_s is the strategy for $G^+(M_\infty(s), \lambda - \mathrm{lh}(s), \theta)$ given by:*

$$\Omega_s(t) = \Omega(s^\frown t),$$

for all $M_\infty(s)$-stacks t.

The following notation will be useful:

Definition 2.20 *Let Ω be a winning strategy for* II *in $G^+(M, \lambda, \theta)$, and let s be an M-stack according to Ω such that* $\mathrm{lh}(s) < \lambda$ *and $M_\infty(s)$ exists, and let $N = M_\infty(s)|\langle\nu, k\rangle$; then $\Omega_{s,N} = \Omega_{s^\frown\langle\nu,k,\emptyset\rangle}$. We also write $\Omega_{s,\langle\nu,k\rangle}$ for $\Omega_{s,N}$.*

When $N = M|\langle \nu, k\rangle$, we write Ω_N or $\Omega_{\langle \nu,k\rangle}$ for $\Omega_{\emptyset,N}$. Clearly, Ω_N is a complete strategy for N. Our definitions so far allow the tails of a iteration strategy to be inconsistent with the strategy itself; for example, one could have a strategy Ω for $G^+(M, \lambda, \theta)$, and $P \trianglelefteq N \trianglelefteq M$ such that $(\Omega_N)_P \neq \Omega_P$. One could even have $\Omega_M \neq \Omega$. We shall eventually completely rule out such internal inconsistencies for the iteration strategies we care about. The following definitions make a start on that.

Definition 2.21 *Let Ω be a winning strategy for* II *in $G^+(M, \lambda, \theta)$; then Ω is* positional *iff whenever s and t are M-stacks by Ω of length $< \lambda$, and $N \trianglelefteq M_\infty(s)$ and $N \trianglelefteq M_\infty(t)$, then $\Omega_{s,N} = \Omega_{t,N}$.*

The iteration strategies that are our focus are positional, but it is beyond the scope of this book to show that. We shall instead use some approximations to positionality here.

Let $\mathcal{T}$ be a weakly normal tree on Q, and $Q \trianglelefteq R$. We can think of $\mathcal{T}$ as a weakly normal tree $\mathcal{T}_0$ on R that always drops at least as far as Q when it applies an extender to the base model. $\mathcal{T}_0$ uses the same extenders and has the same models as $\mathcal{T}$, except that the base model of $\mathcal{T}_0$ is R. Let us say that $\mathcal{T}_0$ is the *R-equivalent of $\mathcal{T}$*.

Definition 2.22 *Let Ω be a winning strategy for* II *in $G^+(M, \lambda, \theta)$, where $\lambda > 1$. We say that Ω is* mildly positional *iff whenever s is a M-stack by Ω with* $\mathrm{lh}(s) < \lambda$, *then*

(a) $\Omega_{s,M_\infty(s)} = \Omega_s$,

(b) whenever $P \trianglelefteq N \trianglelefteq M_\infty(s)$, then $(\Omega_{s,N})_P = \Omega_{s,P}$, and

(c) whenever $P \trianglelefteq N \trianglelefteq M_\infty(s)$, $\mathcal{T}$ is a weakly normal tree by $\Omega_{s,P}$, and $\mathcal{T}_0$ is the N-equivalent of $\mathcal{T}$, then $\mathcal{T}_0$ is by $\Omega_{s,N}$.

The iteration strategies that are our focus in this book have two much stronger internal consistency properties: they normalize well, and have strong hull condensation.

Definition 2.23 *Let λ be a limit ordinal, or $\lambda = 1$. An* complete (λ, θ)-iteration strategy for M *is a mildly positional winning strategy for player* II *in $G^+(M, \lambda, \theta)$. We say Σ is a* complete strategy for M *iff it is a complete (λ, θ)-iteration strategy, for some ordinal λ and some θ.*

In Lemma 4.59 we show that complete strategies that normalize well and have strong hull condensation have a property that we call *strategy coherence*. One clause in strategy coherence is positionality, but restricted to stacks s and t consisting of a single normal tree. In [48], we show that complete strategies that normalize well and have strong hull condensation are fully positional.

We shall be especially interested in strategies defined on M-stacks of finite length.

Definition 2.24 *Let θ be regular; then Σ is a* complete iteration strategy with scope H_θ *iff Σ is a complete (ω, θ)-iteration strategy.*

Remark 2.25 Here we have isolated (ω, θ)-strategies, rather than (θ, θ)-strategies, because we wish to avoid the theory of normalizing infinite stacks. In order to compare complete (ω_1, ω_1)-strategies one would have to normalize stacks of arbitrary countable length. This can be done, (see [44] and [48]), but we have chosen not to go into the process here. Complete (ω, ω_1)-strategies are adequate for the theory of HOD in models of $\mathsf{AD}_{\mathbb{R}}$ that we develop in Chapters 5-7.

The background constructions in Chapter 3 do produce complete (ω_1, ω_1)-strategies that normalize well for countable stacks. These strategies are determined by their action on normal trees. Once that has been shown, our strategy comparison process becomes applicable.

Although a complete strategy with scope H_θ is only be required to act on finite stacks, it is part of player II's winning condition that whenever s is a run of $G^+(M, \omega, \theta)$ by Σ, then the direct limit $M_\omega(s)$ of the $M_i(s)$ for $i < \omega$ sufficiently large exists, and is wellfounded. This requirement on Σ is crucial in the proof of the Dodd-Jensen Lemma, 6.19.

The complete iteration strategies for premice that we consider in this book are entirely determined by their action on normal trees (see 4.60), but we do need to consider how the strategies act on finite, non-maximal stacks of normal trees. We have allowed them to act on non-maximal stacks of weakly normal trees because it seemed natural to do so.

Given $\pi \colon M \to N$ weakly elementary, we can copy an M-stack s to an N-stack πs, until we reach an illfounded model on the πs side. Thus if Ω is a complete strategy for N, we have the complete pullback strategy Ω^π for M.

Remark 2.26 It is possible that $\pi \colon M \to N$ is weakly elementary, $\mathcal{T}$ is normal on M, and $\pi\mathcal{T}$ is not normal. For example, we might have $k(M) = 1$, and E on the M-sequence such that $\rho_1(M) \leq \text{crit}(E)$, but $\pi(\text{crit}(E))) < \rho_1(N)$. If $\mathcal{T}$ starts normally with E, will drop to M^-, that is, to M with its degree reduced by one, and form $\mathrm{Ult}(M^-, E)$. Our copying process then requires $\pi\mathcal{T}$ to start by forming $\mathrm{Ult}(N^-, \pi(E))$, which is for it a gratuitous drop.

Nevertheless, if $\mathcal{T}$ is weakly normal and π is weakly elementary, then $\pi\mathcal{T}$ will be weakly normal. In §5.2 we describe the natural normal tree on N into which $\pi\mathcal{T}$ embeds; this tree is called $(\pi\mathcal{T})^+$.

Definition 2.27 *[Pullback strategies] If Σ is a strategy for N, and $\pi\colon M \to N$ is weakly elementary, then Ω^π is the pullback strategy for M, given by*

$$\Omega^\pi(s) = \Omega(\pi s),$$

for all s such that $\pi s \in \mathrm{dom}(\Omega)$.

The copy maps are all weakly elementary, and if π is fully elementary, then the copy maps are all fully elementary. (Cf. 1.3 of [36].)

It is also useful to have a notation for a join of strategies:

Definition 2.28 *Let Ω be a complete strategy for M, and s an M-stack by Ω; then $\Omega_{s,<\nu} = \langle \Omega_{s,\langle \eta,k\rangle} \mid \eta < \nu \wedge k \le \omega\rangle$.*

Note that in general, $\Omega_{s,<\nu}$ is strictly weaker than $\Omega_{s,\langle \nu,0\rangle}$.

We shall often be working with a countable premouse M, and an iteration strategy Σ for M that is defined on countable trees of some sort, with AD^+ as our background assumption. We can then extend Σ so that it acts on trees of length ω_1, because under AD^+, ω_1 is measurable. Here is a simple proposition along these lines.

Proposition 2.29 . *Assume AD, and let Σ be an ω_1-iteration strategy for a countable premouse M; then Σ can be extended to an $\omega_1 + 1$ strategy for M.*

Proof. Let $\mathcal{T}$ be a normal tree of length ω_1 on M that is played by Σ. It will suffice to show $\mathcal{T}$ has a cofinal, wellfounded branch. But let $j\colon V \to N$ with $\mathrm{crit}(j) = \omega_1$ witness the measurability of ω_1. The pair $\langle \mathcal{T}, M\rangle$ can be coded by a set of ordinals A, and Los's Theorem holds for ultrapowers of wellordered structures, so $j\colon L[A] \to L[j(A)]$ is elementary. It follows that $j(\mathcal{T})$ is an iteration tree on M, $\mathcal{T} = j(\mathcal{T}) \restriction \omega_1$, and $\omega_1 < \mathrm{lh}(j(\mathcal{T}))$. But this implies that $[0, \omega_1)_{j(T)}$ is a cofinal, wellfounded branch of $\mathcal{T}$. □

Although it is quite easy to prove, this proposition stands at a key junction in inner model theory. The direct proofs of iterability only produce branches for countable iteration trees, even in the realm of linear iterations. Yet $\omega_1 + 1$-iterability is the minimal useful kind of iterability; for example, it is the kind needed to compare countable premice. All known proofs of $\omega_1 + 1$-iterability involve at some point producing an ω_1-strategy Σ, and showing that Σ is sufficiently absolutely definable that one can extend it to an $\omega_1 + 1$ strategy. In the proposition above, the absolute definability of Σ is evidenced by its membership in a model of AD. In contexts where one's goal is more ambitious than analyzing HOD in models of AD, the absolute definability of Σ has to be more finely calibrated, and a model of some fragment of AD that contains Σ constructed along with Σ. This leads into the core model induction method, our most all-purpose method for constructing iteration strategies.

Proposition 2.29, simple as it is, is one important reason that inner model theory and descriptive set theory have become so entangled in recent years.

When calibrating definablity in terms of pointclasses, the standard procedure is to code elements of HC (e.g. premice) by reals, and subsets of HC (e.g. ω_1-iteration strategies) by sets of reals. Of course, any reasonable way of doing this is fine, but we may as well spell one out. For $x \in \mathbb{R} = \omega^\omega$, we say Cd(x) iff $E_x =_{\text{df}} \{\langle n, m\rangle \mid x(2^n 3^m) = 0\}$ is a wellfounded, extensional relation on ω. If Cd(x), then

$$\pi_x \colon (\omega, E_x) \cong (M, \in)$$

is the transitive collapse map, and

$$\text{set}(x) = M \text{ and } \text{set}_0(x) = \pi_x(0).$$

So Cd is Π^1_1, and set_0 maps Cd onto HC. For $A \subseteq \text{HC}$, we let

$$\text{Code}(A) = \{x \in \mathbb{R} \mid \text{Cd(x)} \wedge \text{set}_0(\text{x}) \in \text{A}\}.$$

If Σ is an iteration strategy with scope HC for a countable M, and Γ is a pointclass, then we sometimes say "$\Sigma \in \Gamma$" when we mean $\text{Code}(\Sigma) \in \Gamma$.

Recall that a set $A \subseteq \mathbb{R}$ is *κ-Universally Baire* (κ-UB) iff there are trees T and U on some $\omega \times Z$ such that $p[T] = \mathbb{R} \setminus p[U]$ holds in $V[g]$ whenever g is V-generic for a poset of size $< \kappa$, and $p[T] = A$ holds in V. We call such a pair (T, U) a *κ-UB code* of A.[19] If κ is a limit of Woodin cardinals, then the κ-UB are the same as the $< \kappa$-homogeneously Suslin sets; moreover, if A is κ-UB, as witnessed by the pair of trees (T, U), then the theory of $(\text{HC}, \in, p[T])$ is absolute for forcing of size $< \kappa$ (cf. [52]). This enables us to extend ω_1-iteration strategies that are κ-UB to κ-iteration strategies. As is well known, the extension is independent of the partucular UB code chosen. In fact, with a little care, we do not need the Woodin cardinals to make it.

[19]The concept was first isolated and studied for its own sake by Q. Feng, M. Magidor, and W.H. Woodin. See [3]. There are earlier related results due to K. Schilling and R. Vaught in [33].

Proposition 2.30 *Let $A \subseteq HC$, and suppose (T, U) is a κ-UB code of* $\mathrm{Code}(A)$. *For $b \in H_\kappa$, put*

$$b \in B \text{ iff } \mathrm{Col}(\omega, < \kappa) \Vdash \exists x \in p[T](\mathrm{set}_0(x) = b).$$

Then $(HC, \in, A) \prec_{\Sigma_1} (H_\kappa, \in, B)$.

Proof.(Sketch.) Note that $p[T]$ and $p[U]$ remain invariant in $V^{\mathrm{Col}(\omega,\kappa)}$, in that if $\mathrm{set}_0(x) = \mathrm{set}_0(y)$, then $x \in p[T]$ iff $y \in p[T]$, and similarly for U. Also, whether $x \in p[T]$ for any and all x such that $\mathrm{set}_0(x) = b$ is decided by the empty condition. Suppose $(H_\kappa, \in, B) \models \varphi[a]$, where φ is Σ_1 and $a \in \mathrm{HC}$. Let $\pi \colon N \to V_\theta$ with N countable and transitive, and $\pi(\langle \bar{T}, \bar{U} \rangle) = \langle T, U \rangle$. Let $\pi(M) = H_\kappa$ and $\pi(\bar{B}) = B$. We have $\pi(a) = a$, and $(M, \in, \bar{B}) \models \varphi[a]$. Using $\bar{T}$ and $\bar{U}$ and a simple absoluteness argument, we see that $\bar{B} = A \cap M$. So $(M, \in, A \cap M) \models \varphi[a]$. But φ is Σ_1, so $(\mathrm{HC}, \in, A) \models \varphi[a]$, as desired. □

In order to apply the proposition to iteration strategies, we have to be careful about how we present them. Given an ω_1 strategy Σ, let A_Σ be the set of all pairs $(\mathcal{T}, \alpha)$ such that $\mathcal{T}$ is a tree of limit length by Σ, and $\alpha \in \Sigma(\mathcal{T})$.

Corollary 2.31 *Let Σ be an ω_1-iteration strategy for a countable premouse P, and suppose that* $\mathrm{Code}(A_\Sigma)$ *is κ-UB; then there is a κ-iteration strategy Ψ extending Σ.*

Proof. Let $B \subseteq H_\kappa$ be such that $(\mathrm{HC}, \in, A_\Sigma) \prec_{\Sigma_1} (H_\kappa, \in, B)$. It is not hard to see that $B = A_\Psi$, where Ψ is the desired extension of Σ. □

Clearly, the extension Ψ to H_κ is independent of the particular κ-UB code of A_Σ chosen. We call Ψ the *canonical κ-extension of* Σ. Abusing language somewhat, we may say that a κ-iteration strategy is κ-UB when it is the canonical κ-extension of an ω_1-strategy. The extension process works equally well for (λ, ω_1)-strategies.

The following simple fact about such strategies is useful.

Proposition 2.32 *Let Σ be a κ-UB κ-iteration strategy for some countable P, and $j \colon V \to M$ with M transitive; then $j(\Sigma) \cap H_\kappa \subseteq \Sigma$.*

Proof. Let (T, U) be a κ-UB code for $\mathrm{Code}(A_\Sigma)$. Suppose $\mathcal{T} \in H_\kappa$ is by both Σ and $j(\Sigma)$, and has limit length λ. If $\alpha < \lambda$, and $\alpha \in j(\Sigma)(\mathcal{T})$, then letting $\mathrm{set}_0(x) = \langle \mathcal{T}, \alpha \rangle$ with x in $V^{\mathrm{Col}(\omega,<\kappa)}$, we get $x \in p[j(T)]$. As usual, this implies $x \notin p[U]$, and hence $x \in p[T]$. Thus $\alpha \in \Sigma(\mathcal{T})$, as desired. □

We shall show in 4.55 below that the conclusion $j(\Sigma) \cap \Sigma$ also follows from strong hull condensation for Σ.

2.8 Coarse structure

One must consider also iteration trees on transitive models M that are not equipped with any distinguished fine structural hierarchy. In that case, we shall always assume $M \models \mathsf{ZFC}$, for simplicity. In general, V_α^M plays the role that $M|\alpha$ would in the fine structural case. All extenders are total on the models to which they are applied, and all embeddings are fully elementary in the $\in$-language. We shall sometimes call such M, and associated objects like iteration trees or embeddings acting on them, *coarse*, in order to distinguish them from their fine-structural cousins.

Definition 2.33 *Let E be an extender over V; then E is* nice *iff*

(a) E is strictly short, that is, $\mathrm{lh}(E) < \lambda(E)$,

(b) $\mathrm{lh}(E)$ *is strongly inaccessible, but not a measurable cardinal,*

(c) $V_{\mathrm{lh}(E)} \subseteq \mathrm{Ult}(V, E)$.

Nice E can be used to background extenders in a Jensen premouse, even though $\mathrm{lh}(E) < \lambda(E)$. In practice, our background extenders will be such that $\mathrm{lh}(E)$ is the least strongly inaccessible strictly above η, for some η, so that (b) holds. The requirements of (b) enable us to avoid a counterexample to UBH for stacks of normal trees due to Woodin. See 4.40 below.

Definition 2.34 *Let $\mathcal{T}$ be an iteration tree on a coarse M; then*

(a) $\mathcal{T}$ is nice *iff whenever $\alpha + 1 < \mathrm{lh}(\mathcal{T})$, then $\mathcal{M}_\alpha^{\mathcal{T}} \models$ "$E_\alpha^{\mathcal{T}}$ is nice".*

(b) $\mathcal{T}$ is normal *iff*

(i) if $\alpha < \beta$ and $\beta + 1 < \mathrm{lh}(\mathcal{T})$, then $\mathrm{lh}(E_\alpha^{\mathcal{T}}) < \mathrm{lh}(E_\beta^{\mathcal{T}})$, and

(ii) if $\alpha + 1 < \mathrm{lh}(\mathcal{T})$, then $T\text{-pred}(\alpha + 1)$ is the least β such that $\mathrm{crit}(E_\alpha^{\mathcal{T}}) < \mathrm{lh}(E_\beta^{\mathcal{T}})$.

This definition of normality is only appropriate for nice trees, but all our coarse iteration trees will be nice, so that is ok. It would be possible to allow gratuitous dropping, but we shall not do that. Nice iteration trees do not drop anywhere. Moreover, we shall often restrict the choice of extenders in $\mathcal{T}$ even further.

Definition 2.35 *Let $\mathcal{T}$ be an iteration tree on a coarse M, and $\mathcal{F}$ a set or class of M; then*

(a) $\mathcal{T}$ *is an* $\mathcal{F}$-tree *iff whenever* $\alpha + 1 < \mathrm{lh}(\mathcal{T})$, *then* $E_\alpha^{\mathcal{T}} \in i_{0,\alpha}^{\mathcal{T}}(\mathcal{F})$.

(b) $\mathcal{T}$ *is* above κ *iff* $\mathcal{T}$ *is an* $\mathcal{F}$*-tree, where* $\mathcal{F} = \{E \mid \mathrm{crit}(E) > \kappa\}$.

(c) $\mathcal{T}$ *is* based on V_δ^M *iff* $\mathcal{T}$ *is an* $\mathcal{F}$*-tree, where* $\mathcal{F} = V_\delta^M$.

(d) *A* putative $\mathcal{F}$-tree *on* M *is a system having all the properties of an* $\mathcal{F}$*-tree on* M*, except that its last model may be illfounded.*

In Definition 2.35 we are not assuming that $\mathcal{T}$ is normal. It may be a stack of normal trees, in which case we may call it an $\mathcal{F}$-stack, or a putative $\mathcal{F}$-stack. The non-normal iteration trees that we consider will always be stacks of normal trees. One could venture further into the wilds, but we shall not do that.

Definition 2.36 *Let* $M \models \mathsf{ZFC}$ *be transitive, and* $\mathcal{F}$ *be a set or class of* M*; then*

(a) $G(M, \eta, \theta, \mathcal{F})$ *is the variant of* $G(M, \eta, \theta)$ *in which* I *must choose his exit extenders from the current image of* $\mathcal{F}$*, and*

(b) *an* $(\eta, \theta, \mathcal{F})$*-iteration strategy for* M *is a winning strategy for* II *in* $G(M, \eta, \theta, \mathcal{F})$.

By convention, these strategies are complete.

In general, the iteration strategies for coarse M that we consider choose branches that, when allowed to act on the largest possible base model, become the unique cofinal wellfounded branch.

Definition 2.37 *Let* $M \models \mathsf{ZFC}$ *be transitive, let* $\mathcal{F}$ *be a set or class of* M*, and let* $\lambda, \theta \in OR$*; then*

(a) M *is* strongly uniquely $(\lambda, \theta, \mathcal{F})$-iterable *iff there is a* $(\lambda, \theta, \mathcal{F})$*-iteration strategy* Σ *for* M *such that whenever* $\mathcal{T}$ *is a tree by* Σ *of limit length, then* $\Sigma(\mathcal{T})$ *is the unique cofinal, wellfounded branch of* $\mathcal{T}$.

(b) M *is* strongly uniquely $\theta, \mathcal{F}$-iterable for normal trees *iff* M *is strongly uniquely* $(1, \theta, \mathcal{F})$*-iterable.*

We say that M is strongly uniquely (λ, θ)-iterable above κ, or for trees based on V_δ^M, iff M is strongly uniquely $(\lambda, \theta, \mathcal{F})$-iterable for the associated $\mathcal{F}$. Notice that strong unique iterability is more than just having a unique iteration strategy; that strategy must be to choose the unique cofinal, wellfounded branch.

Often, our $\mathcal{F}$ will be the class of extenders occuring in some coarsely coherent sequence.

Definition 2.38 *A sequence* $\vec{F} = \langle F_\alpha \mid \alpha < \mu \rangle$ *is* coarsely coherent *iff each* F_α *is a nice extender over* V*, and*

(1) $\alpha < \beta \Rightarrow \mathrm{lh}(F_\alpha) < \mathrm{lh}(F_\beta)$*, and*

(2) *if* $i \colon V \to \mathrm{Ult}(V, F_\alpha)$ *is the canonical embedding, and* $\vec{E} = i(\vec{F})$*, then* $\vec{E}{\restriction}\alpha = \vec{F}{\restriction}\alpha$*, and* $\mathrm{lh}(F_\alpha) < \mathrm{lh}(E_\alpha)$*.*

Definition 2.39 *A* coarse extender premouse *is a structure* $M = (|M|, \in, \vec{F})$ *such that* $|M|$ *is transitive,* $\vec{F} \in |M|$*, and* $(|M|, \in) \models \mathsf{ZFC} +$ "$\vec{F}$ *is coarsely coherent*"*.*

We sometimes identify M with its universe $|M|$. We write $\vec{F}^M$ for the distinguished coarsely coherent sequence of M.

Given a coarsely coherent $\vec{F}$, an $\vec{F}$-iteration tree is a $\{F_\alpha \mid \alpha < \mathrm{lh}(\vec{F})\}$ -iteration tree. That is, all extenders used must be taken from $\vec{F}$ and its images. Similarly for $\vec{F}$-stacks of normal trees. So the trees in an $\vec{F}$-stack are nice. In the coarse case, iteration trees do not have any necessary drops, and we prohibit gratuitous dropping just to keep things simple. Thus all $\vec{F}$-stacks are maximal, by convention, and a complete $(\lambda, \theta, \vec{F})$-strategy is just a (λ, θ) strategy for $\vec{F}$-trees in the usual sense.

The following simple lemma uses only clause (1) of coarse coherence.

Lemma 2.40 *Let* $(M, \vec{F})$ *be a coarse premouse, and let* Σ *be an* $\vec{F}$*-iteration strategy for* M*; then for any* N*, there is at most one normal* $\vec{F}$*-iteration tree played according to* Σ *whose last model is* N*.*

Proof. Let $\mathcal{T}$ and $\mathcal{U}$ be distinct such trees. Because both are played by Σ and normal, there must be a β such that $\mathcal{T}{\restriction}\beta + 1 = \mathcal{U}{\restriction}\beta + 1$, but $G \neq H$, where $G = E^{\mathcal{T}}_\beta$ and $H = E^{\mathcal{U}}_\beta$. Both G and H are taken from $i(\vec{F})$, where $i = i^{\mathcal{T}}_{0,\beta} = i^{\mathcal{U}}_{0,\beta}$. Say G occurs before H in $i(\vec{F})$. Then $G \in N$ because $\mathcal{U}$ is normal. But $G \notin N$ because $\mathcal{T}$ is normal. □

Assuming AD^+, we get coarse extender premice $(M, \vec{F})$ via the Γ-Woodin construction. (See [58][§3] and [54][§10].) These M can have a Woodin cardinal δ, and yet be correct for predicates in some complicated pointclass Γ. We shall have that δ is countable in V, and M is strongly uniquely (ω_1, ω_1)-iterable for trees based on V^M_δ. The same construction also produces coarse strategy premice, although these do not have Woodin cardinals. We say more about this in section 3.2.

Woodin has shown that if κ is supercompact, $\vec{F}$ is coarsely coherent and such that $\kappa < \mathrm{crit}(E)$ for all E on $\vec{F}$, and UBH holds in $V^{\mathrm{Col}(\omega, <\kappa)}$ for normal $\vec{F}$-trees on V, then V is strongly uniquely $\vec{F}$-iterable for normal trees. See Theorem 4.31. We show in 4.40 below that this implies that V is strongly uniquely $(\omega, \theta, \vec{F})$-iterable, for all θ.

2.9 Full background extender constructions

In this book, we shall be looking very carefully at full background extender constructions, and in particular at how an iteration strategy Σ^* for the background universe induces iteration strategies for the premice occurring in such a construction. In our applications, the background universes will satisfy "I am strongly uniquely $\vec{F}$-iterable", where $\vec{F}$ is the sequence of background extenders used in the construction, and Σ^* will be the corresponding $\vec{F}$-iteration strategy. In this section we look at the well known construction of pure extender premice. Section 5.5 lays out the obvious generalization to strategy mice.

We shall use the notation of [29] in this context. The reader should look at [29], and at [1] on which it relies, for full definitions.

Definition 2.41 *Let w be a wellorder of V_δ, and $\kappa < \delta$. A w-*construction above κ *is a full background construction in which the background extenders are nice, have critical points $> \kappa$, cohere with w, have strictly increasing strengths, and are minimal (first in Mitchell order, then in w).*

More precisely, such a construction $\mathbb{C}$ consists of premice $M^{\mathbb{C}}_{\nu,k}$, with $k(M_{\nu,k}) = k$, and extenders $F^{\mathbb{C}}_\nu$ obtained as follows. (In the notation of [23], $M_{\nu,k} = \mathbb{C}_k(\mathcal{N}_\nu)$, and $F^{\mathbb{C}}_\nu$ is a choice of background extender for the last extender of $M_{\nu,0} = \mathcal{N}_\nu$.) We let $M_{0,0}$ be the passive premouse with universe V_ω. For any k, ν,

$$M_{\nu,k+1} = \mathrm{core}(M_{\nu,k}) =_{\mathsf{def}} \mathcal{C}(M_{\nu,k}).$$

We have an anti-core embedding $\pi : M_{\nu,k+1} \to M_{\nu,k}$ with $\mathrm{crit}(\pi) \geq \rho(M_{\nu,k})$. For $k < \omega$ sufficiently large, $M_{\nu,k} = M_{\nu,k+1}$, except of course that its associated k has changed. That is, $\hat{M}_{\nu,k}$ is eventually constant as $k \to \omega$. We set

$$\hat{M}_{\nu,\omega} = \text{eventual value of } \hat{M}_{\nu,k} \text{ as } k \to \omega,$$

and

$$\begin{aligned}\hat{M}_{\nu+1,0} = &\text{ rud closure of } \hat{M}_{\nu,\omega} \cup \{\hat{M}_{\nu,\omega}\},\\ &\text{ arranged as a passive premouse,}\end{aligned}$$

and

$$M_{\nu+1,0} = (\hat{M}_{\nu+1,0}, 0).$$

Finally, if ν is a limit, put

$$\begin{aligned}M^{<\nu} = &\text{ unique passive } P \text{ such that for all premice } N,\\ &\ N \lhd P \text{ iff } N \lhd M_{\alpha,0} \text{ for all sufficiently large } \alpha < \nu.\end{aligned}$$

There are two possibilities now: we may add a new extender to the sequence, or not.

Case 1. For some F such that $(M^{<\nu}, F)$ is a Jensen premouse, and F is certifiable, in the sense of Definition 2.1 of [29], we set

$$M_{\nu,0} = (M^{<\nu}, F).$$

A bicephalus argument shows that, under a natural iterability hypothesis, there is at most one certifiable F such that $(M^{<\nu}, F)$ is a premouse.

Case 2. Otherwise.

Then we set

$$M_{\nu,0} = M^{<\nu}.$$

(Again, our convention is that in case 1, $M^{<\nu}$ is not an initial segment of $M_{\nu,0}$.) A w-construction need not add an extender whenever possible. We say $\mathbb{C}$ is *maximal* iff it does so, that is, iff case 1 occurs whenever there is an F meeting its requirements.
.

A certificate for F in the sense of 2.1 of [29] is a short extender F^*. Let us write $\kappa_F = \mathrm{crit}(F)$ and $\lambda_F = i_F(\kappa_F)$. F^* must have strength some inaccessible cardinal $\eta > \lambda_F$, and satisfy

$$F^* \restriction \lambda_F \cap M^{<\nu} = F \restriction \lambda_F.$$

Since F^* is short, $i_{F^*}(\kappa_F) \geq \eta > \lambda_F$, so we cannot replace λ_F by $\lambda_F + 1$ in this equation. We add here to the demands on certificates that

(*i*) F^* is nice (so $\mathrm{lh}\, F^* = \eta$),

(*ii*) $\forall \tau < \nu\ (\mathrm{lh}\, F^{\mathbb{C}}_\tau < \eta)$,

(*iii*) $i_{F^*}(w) \cap V_\eta = w \cap V_\eta$,

(*iv*) $F^* \in V_\delta$, and $\mathrm{crit}(F^*) > \kappa$.

We then choose $F^{\mathbb{C}}_\nu$ to be the unique certificate for F such that

($*$) $F^{\mathbb{C}}_\nu$ is a certificate for F, minimal in the Mitchell order among all certificates for F, and w-least among all Mitchell order minimal certificates for F.

This has the consequence that $\mathrm{lh}(F_\nu^{\mathbb{C}})$ is the *least* strongly inaccessible η such that $\lambda_F < \eta$ and $\forall \tau < \nu$ $(\mathrm{lh}\, F_\tau^{\mathbb{C}} < \eta)$. We also get that $F_\nu^{\mathbb{C}}$ "coheres with $\mathbb{C}$". That is, letting $\mathbb{C}{\restriction}\gamma = \langle (M_{\tau,k}, F_\tau) \mid \tau < \gamma \wedge k \leq \omega \rangle$ and $F^* = F_\nu^{\mathbb{C}}$,

1. $i_{F^*}(\mathbb{C}){\restriction}\nu = \mathbb{C}{\restriction}\nu$,
2. $M_{\nu,0}^{i_{F^*}(\mathbb{C})}$ is passive.

Thus the sequence $\vec{F}^{\mathbb{C}}$ of all $F_\nu^{\mathbb{C}}$ is coarsely coherent, and $(V, \vec{F}^{\mathbb{C}})$ is a coarse extender premouse.

We may want to start with some coarsely coherent $\vec{F}$ given in advance. An $\vec{F}$-*construction* is then a $\mathbb{C}$ with the properties above, except that in case 1 we require that F have a certificate in $\vec{F}$, and we let $F_\nu^{\mathbb{C}}$ be the first such certificate in $\vec{F}$. Of course, if $\mathbb{C}$ is a w-construction, then it is a $\vec{F}^{\mathbb{C}}$-construction. It is an easy exercise to show that if $\mathbb{C}$ is an $\vec{F}$-construction, then it is a w-construction, for some wellorder w.

Definition 2.42 *A* background construction *(for pure extender mice) is a sequence* $\mathbb{C} = \langle M_{\nu,k}^{\mathbb{C}}, F_\nu^{\mathbb{C}} \rangle$ *with the properties listed above. We say that* $\mathbb{C}$ *is* maximal *iff it adds an extender whenever there is one that meets the requirements of Case 1.*

Of course, maximality is relative to the requirements for adding extenders. Any construction $\mathbb{C}$ is maximal as an $\vec{F}^{\mathbb{C}}$-construction.

The background constructions described above extract pure extender premice from coarse extender premice. In Chapter 5 we shall describe background constructions that extract fine-structural strategy mice from coarse strategy mice.

Let $\mathbb{C}$ be a background construction. By a $\mathbb{C}$-iteration, we mean a $\vec{F}^{\mathbb{C}}$-iteration in the sense explained above. The length of $\mathbb{C}$ is the lexicographically least $\langle \mu, l \rangle$ such that $M_{\mu,l}^{\mathbb{C}}$ does not exist.

Lemma 2.43 *Let $\mathbb{C}$ be a background construction; then for any premouse N, there is at most one $\langle \nu, k \rangle$ such that $N = M_{\nu,k}^{\mathbb{C}}$.*

The proof is easy and well known. Notice that $N = M_{\nu,k}$ implies by convention that $k = k(N)$. Without this convention, the lemma would fail.

Associated to a construction $\mathbb{C}$ we have *resurrection maps* $\mathrm{Res}_{\nu,k}$ acting on initial segments N of $M_{\nu,k}$, with $\mathrm{Res}_{\nu,k}[N] = \langle \eta, l \rangle$ for some $\langle \eta, l \rangle \leq_{\mathrm{lex}} \langle \nu, k \rangle$. The idea is that N traces back to $M_{\eta,l}$ by following anti-core maps. $\sigma_{\nu,k}[N]$ is the associated elementary (at level l) embedding of N into $M_{\eta,l}$. For example, suppose $\mathrm{Res}_{\nu,k}$ and $\sigma_{\nu,k}$ are defined. We define $\mathrm{Res}_{\nu,k+1}$, $\sigma_{\nu,k+1}$ by

A. If $N = M_{\nu,k+1}$, then $\mathrm{Res}_{\nu,k+1}[N] = \langle \nu, k+1 \rangle$ and $\sigma_{\nu,k+1}[N] = \text{identity}$.

B. If $N \lhd M_{\nu,k+1} | (\rho^+)^{M_{\nu,k+1}}$, where $\rho = \rho(M_{\nu,k})$, then $\mathrm{Res}_{\nu,k+1}[N] = \mathrm{Res}_{\nu,k}[N]$ and $\sigma_{\nu,k+1}[N] = \sigma_{\nu,k}[N]$.

C. Otherwise, letting $\pi : M_{\nu,k+1} \to M_{\nu,k}$ be the anti-core map, $\mathrm{Res}_{\nu,k+1}[N] = \mathrm{Res}_{\nu,k}[\pi(N)]$ and $\sigma_{\nu,k+1}[N] = \sigma_{\nu,k}[\pi(N)] \circ \pi$.

The reader should see [1] for the remainder of the definition. Two points on agreeement of resurrection maps:

1. if $N \lhd M_{\nu,k}$ and $\forall N' \ (N \trianglelefteq N' \trianglelefteq M_{\nu,k} \Rightarrow \rho(N') \geq \gamma)$, then $\sigma_{\nu,k}[N] \restriction \gamma = \text{identity}$.

2. if $N \trianglelefteq N^* \trianglelefteq M_{\nu,k}$, and $\forall N' \ (N \trianglelefteq N' \trianglelefteq N^* \Rightarrow \rho(N') \geq \gamma)$, then $\sigma_{\nu,k}[N] \restriction \gamma = \sigma_{\nu,k}[N^*] \restriction \gamma$.

These of course just come from the fact that the anti-core map $\pi : \mathcal{C}(M) \to M$ is the identity on $\rho(M)$.

Now let $\mathbb{C} = \langle (M_{\nu,k}, F^*_\nu) \mid \langle \nu, k \rangle <_{\text{lex}} \langle \mu, l \rangle \rangle$ be a construction above κ. Take $\kappa = 0$ to save notation. Let Σ^* be a (λ, θ)-iteration strategy for $(V, \vec{F}^{\mathbb{C}})$. We wish to describe the induced complete strategy Σ for $M_{\nu,k}$. For $\mathcal{T}$ a weakly normal iteration tree played by Σ, we shall have a *conversion system for* $\mathcal{T}$ in the sense of Definition 2.2 of [29]. Such a conversion system converts trees on $M_{\nu,k}$ to trees on V. The particular conversion system we construct we call $\mathrm{lift}(\mathcal{T}, M_{\nu,k}, \mathbb{C})$. In general, a $\mathbb{C}$-conversion system for a weakly normal tree $\mathcal{T}$ consists of

(i) an iteration tree $\mathcal{T}^*$ on V,

(ii) indices $\langle \eta_\xi, l_\xi \rangle$ for $\xi < \mathrm{lh}\,\mathcal{T}$,

(iii) maps π_ξ for $\xi < \mathrm{lh}\,\mathcal{T}$,

so that, using P_ξ, $i_{\xi,\nu}$, F_ξ, P^*_ξ, $i^*_{\xi,\nu}$, F^*_ξ for the models, embeddings, and exit extenders of $\mathcal{T}$ and $\mathcal{T}^*$

1. $\pi_\xi : P_\xi \to M^{P^*_\xi}_{\eta_\xi, l_\xi}$ is weakly elementary (where $M^{P^*_\xi}_{\eta_\xi, l_\xi}$ is M_{η_ξ, l_ξ} in $i^*_{0,\xi}(\mathbb{C})$),

2. $\mathcal{T}$ and $\mathcal{T}^*$ have the same tree order,

3. if $\xi <_T \nu$ and $(\xi, \nu]_T$ does not drop in model or degree, then $\langle \eta_\nu, l_\nu \rangle = i^*_{\xi,\nu}(\langle \eta_\xi, l_\xi \rangle)$ and $\pi_\nu \circ i_{\xi,\nu} = i^*_{\xi,\nu} \circ \pi_\xi$.

4. if $\xi = T$-pred$(\nu+1)$ and this is a drop in model or degree to $\bar{P} \lhd P_\xi$, then $\langle \eta_{\nu+1}, l_{\nu+1}\rangle = i^*_{\xi,\nu+1}(\mathrm{Res}^{P^*_\xi}_{\eta_\xi,l_\xi}[\pi_\xi(\bar{P})])$.

5. Let $\lambda_\xi = i_{F_\xi}(\mathrm{crit}(F_\xi))$, and $\alpha_\xi = \mathrm{lh}\, F_\xi$ be the index of F_ξ in P_ξ, and σ_ξ be the resurrection map $\sigma^{i^*_{0,\xi}(\mathbb{C})}_{\eta_\xi,l_\xi}[\pi_\xi(P_\xi \parallel \alpha_\xi)]$. Then for $\xi < \nu$,

$$\pi_\nu \restriction \lambda_\xi = \sigma_\xi \circ \pi_\xi \restriction \lambda_\xi$$

and

$$P^*_\xi | \sup \sigma_\xi \circ \pi_\xi \text{“} \lambda_\xi = P^*_\nu | \sup \sigma_\xi \circ \pi_\xi \text{“} \lambda_\xi.$$

The particular conversion system $\mathrm{lift}(\mathcal{T}, M_{\nu,k}, \mathbb{C})$ is determined by these conditions and the fact that

(a) $\langle \eta_0, l_0\rangle = \langle \nu, k\rangle$ and $\pi_0 =$ identity,

(b) let $\xi = T$-pred$(\nu+1)$, and $\alpha_\nu = \mathrm{lh}\, F_\nu$, so that F_ν is the last extender of $P_\nu|\alpha_\nu$. Let

$$G = \text{last extender of } \mathrm{Res}^{P^*_\nu}_{\eta_\nu,l_\nu}[\pi_\nu(P_\nu|\langle\alpha_\nu, 0\rangle)];$$

then

$$F^*_\nu = \text{background extender for } G \text{ provided by } i^*_{0,\nu}(\mathbb{C}).$$

(c) let ξ, ν etc. be as in (b). If $(\xi, \nu+1]_T$ is not a drop in model or degree, then

$$\pi_{\nu+1}([a,f]^{P_\xi}_{F_\nu}) = [\sigma \circ \pi_\nu(a), \pi_\xi(f)]^{P^*_\xi}_{F^*_\nu},$$

where $\sigma = \sigma_{\eta_\nu,l_\nu}[\pi_\nu(P_\nu|\alpha_\nu)]$. If it is a drop, to $\bar{P} \lhd P_\xi$, then

$$\pi_{\nu+1}([a,f]^{\bar{P}}_{F_\nu}) = [\sigma \circ \pi_\nu(a), \tau \circ \pi_\xi(f)]^{P^*_\xi}_{F^*_\nu},$$

where σ is as above, and $\tau = \sigma_{\eta_\xi,l_\xi}[\pi_\xi(\bar{P})]^{P^*_\xi}$.

In clauses (4) and (c), we treat a gratuitous drop to $\bar{P}$ in exactly the same way as a necessary one, by resurrecting $\bar{P}$.

Definition 2.44 *Let $\mathbb{C}$ be a background construction, let $M = M^{\mathbb{C}}_{\nu,k}$ be a model of $\mathbb{C}$, and let $\mathcal{T}$ be a weakly normal iteration tree on $M_{\nu,k}$; then*

$$\mathrm{lift}(\mathcal{T}, M, \mathbb{C}) = \langle \mathcal{T}^*, \langle(\eta_\xi, l_\xi) \mid \xi < \mathrm{lh}(\mathcal{T})\rangle, \langle \pi_\xi \mid \xi < \mathrm{lh}(\mathcal{T})\rangle\rangle,$$

is the unique conversion system satisfying (1)-(5) and (a),(b) above. We write

$$\mathcal{T}^* = \text{lift}(\mathcal{T}, M, \mathbb{C})_0$$

for its tree component.

We can take $M_{\nu,k}$ to be the input for the lift function, rather than $\langle \nu, k \rangle$, because of 2.43. $\text{lift}(\mathcal{T}, M, \mathbb{C})$ is defined so long as its tree component $\mathcal{T}^*$ is a *putative iteration tree*, that is, all models of $\mathcal{T}^*$ except possibly the last are wellfounded. We are most interested in the case that the background universe is iterable. If Σ^* is a strategy for the background universe, or even just a partial strategy defined on all trees of the form $\text{lift}(\mathcal{T}, M, \mathbb{C})$, then Σ^* induces a strategy Σ for M: for $\mathcal{T}$ weakly normal on M,

$$\mathcal{T} \text{ is by } \Sigma \Leftrightarrow \text{lift}(\mathcal{T}, M, \mathbb{C})_0 \text{ is by } \Sigma^*.$$

We write

$$\Sigma = \Omega(\mathbb{C}, M, \Sigma^*)$$

for this induced strategy. We may occasionally use the notation $\text{lift}(\mathcal{T}, M, \mathbb{C}, \Sigma^*)$ for the largest initial segment of $\text{lift}(\mathcal{T}, M, \mathbb{C})$ that is by Σ^*. So $\mathcal{T}$ is by $\Omega(\mathbb{C}, M, \Sigma^*)$ iff $\text{lift}(\mathcal{T}, M, \mathbb{C}) = \text{lift}(\mathcal{T}, M, \mathbb{C}, \Sigma^*)$.

We need to see that the lifted tree $\mathcal{T}^*$ is normal. (This is true even if $\mathcal{T}$ itself is only weakly normal.)

Lemma 2.45 *Let $\mathcal{T}$ be weakly normal, and let* $\text{lift}(\mathcal{T}, M_{\nu,k}, \mathbb{C}, \Sigma^*) = \langle \mathcal{T}^*, \langle (\eta_\xi, l_\xi) \mid \xi < \text{lh}\,\mathcal{T} \rangle, \langle \pi_\xi \mid \xi < \text{lh}\,\mathcal{T} \rangle \rangle$*; then $\mathcal{T}^*$ is normal.*

Proof. Let P_ξ, $i_{\xi,\nu}$, F_ξ, P^*_ξ, $i^*_{\xi,\nu}$, F^*_ξ be the models, embeddings, and extenders of $\mathcal{T}$ and $\mathcal{T}^*$. Set

$$\begin{aligned} \kappa_\xi &= \text{crit}\, F_\xi, & \lambda_\xi &= i_{F_\xi}(\kappa_\xi), \\ \kappa^*_\xi &= \text{crit}\, F^*_\xi, & \lambda^*_\xi &= i_{F^*_\xi}(\kappa^*_\xi). \end{aligned}$$

Normality for $\mathcal{T}^*$ is determined by its agreement ordinals, which are the $\text{lh}\, F^*_\alpha$, not the λ^*_α. So what we want to show is that for all α, β, $\alpha < \beta$ implies $\text{lh}\, F^*_\alpha < \text{lh}\, F^*_\beta$, and for all β, T^*-pred$(\beta + 1)$ is the least ξ such that $\kappa^*_\beta < \text{lh}\, F^*_\xi$. Let

$$\sigma_\xi = \sigma^{i^*_{0,\xi}(\mathbb{C})}_{\eta_\xi, l_\xi}[\pi_\xi(P_\xi \parallel \langle \alpha_\xi, 0 \rangle)]$$

be the resurrection embedding, so that

$$F^*_\xi = \text{background extender for } \sigma_\xi \circ \pi_\xi(F_\xi) \text{ provided by } i^*_{0,\xi}(\mathbb{C}).$$

Recall that in Jensen indexing, F is indexed at $\text{lh}\, F = (\lambda^+_F)^{\text{Ult}(M,F)}$.

Sublemma 2.45.1 *Let $\xi + 1 < \text{lh}\,\mathcal{T}$; then*

(a) $\sigma_\xi \circ \pi_\xi(\lambda_\xi) < \lambda^*_\xi = \pi_{\xi+1}(\lambda_\xi)$,

(b) $\sigma_\beta \restriction \pi_{\xi+1}(\text{lh}\,F_\xi) =$ *identity, for all* $\beta \geq \xi + 1$,

(c) $\pi_\beta \restriction (\text{lh}\,F_\xi + 1) = \pi_{\xi+1} \restriction (\text{lh}\,F_\xi + 1)$, *for all* $\beta \geq \xi + 1$.

Proof. For (a): let $G = \sigma_\xi \circ \pi_\xi(F_\xi)$. Since F^*_ξ is the background in $i^*_{0,\xi}(\mathbb{C})$ for G, $\lambda^*_\xi > \lambda_G = \sigma_\xi \circ \pi_\xi(\lambda_\xi)$. But

$$\begin{aligned}\pi_{\xi+1}(\lambda_\xi) &= \pi_{\xi+1}([\varnothing, \text{constant } \kappa_\xi \text{ function}]^{\bar{P}_\tau}_{F_\xi}) \\ &= [\varnothing, \text{constant } \kappa^*_\xi \text{ function}]^{\bar{P}^*_\tau}_{F_\xi} \\ &= \lambda^*_\xi,\end{aligned}$$

where $\tau = T\text{-pred}(\xi + 1)$ and $\bar{P}_\tau \trianglelefteq P_\tau$ is appropriate.

For (b), we have since $\mathcal{T}$ is weakly normal that for all $\beta \geq \xi+1$, $\text{lh}\,F_\xi$ is a cardinal in P_β, and $\rho_{k(P_\beta)}(P_\beta) \geq \text{lh}\,F_\xi$. We then get by induction on β that $\rho_{l_\beta}(M^{i^*_{0,\beta}(\mathbb{C})}_{\eta_\beta, l_\beta}) \geq \pi_{\xi+1}(\text{lh}\,F_\xi)$, and $\pi_{\xi+1}(\text{lh}\,F_\xi)$ is a cardinal in $M^{i^*_{0,\beta}(\mathbb{C})}_{\eta_\beta, l_\beta}$, for all $\beta \geq \xi+1$. This gives (b).

For (c), we have $\lambda_{\xi+1} > \text{lh}\,F_\xi$, so

$$\begin{aligned}\pi_\beta \restriction (\text{lh}\,F_\xi + 1) &= \sigma_{\xi+1} \circ \pi_{\xi+1} \restriction (\text{lh}\,F_\xi + 1), \\ &= \pi_{\xi+1} \restriction (\text{lh}\,F_\xi + 1),\end{aligned}$$

for all $\beta > \xi + 1$. ☐

Now we show $\mathcal{T}^*$ is normal. First, let $\alpha < \beta$, with $\beta + 1 < \text{lh}\,\mathcal{T}^*$. Then

$$\begin{aligned}\text{lh}\,F^*_\alpha < \lambda^*_\alpha &= \pi_{\alpha+1}(\lambda_\alpha) = \pi_\beta(\lambda_\alpha) \\ &= \sigma_\beta \circ \pi_\beta(\lambda_\alpha) < \sigma_\beta \circ \pi_\beta(\lambda_\beta) < \text{lh}\,F^*_\beta,\end{aligned}$$

as desired.

For the rest, it is enough to show that whenever $\alpha < \beta$, then

$$\kappa_\beta < \lambda_\alpha \quad \text{iff} \quad \kappa^*_\beta < \text{lh}\,F^*_\alpha.$$

Suppose first $\kappa_\beta < \lambda_\alpha$. Then

$$\begin{aligned}\kappa^*_\beta = \sigma_\beta \circ \pi_\beta(\kappa_\beta) = \pi_\beta(\kappa_\beta) &= \sigma_\alpha \circ \pi_\alpha(\kappa_\beta) \\ &< \sup \sigma_\alpha \circ \pi_\alpha``\lambda_\alpha < \text{lh}\,F^*_\alpha.\end{aligned}$$

Suppose next $\kappa_\beta \geq \lambda_\alpha$. Then

$$\kappa^*_\beta = \sigma_\beta \circ \pi_\beta(\kappa_\beta) \geq \sigma_\beta \circ \pi_\beta(\lambda_\alpha) = \pi_\beta(\lambda_\alpha) = \lambda^*_\alpha.$$

But $\lambda^*_\alpha > \mathrm{lh}\, F^*_\alpha$, so $\kappa^*_\beta > \mathrm{lh}\, F^*_\alpha$. $\square$

If Σ^* is defined on stacks of normal trees, of any length, then we can extend the lifting process and the induced strategy Σ for $M_{\nu,k}$ so that it is defined on stacks of weakly normal trees if the same length. For example, if $\langle \mathcal{T}, \mathcal{U} \rangle$ is a stack on $M = M^{\mathbb{C}}_{\nu,k}$, and $P_\xi = \mathcal{M}^{\mathcal{T}}_\xi$ is the last model of $\mathcal{T}$, and $\mathrm{lift}(\mathcal{T}, M, \mathbb{C})_0 = \mathcal{T}^*$ by Σ^* with last model P^*_ξ, then we have

$$\pi_\xi : P_\xi \to M^{P^*_\xi}_{\eta_\xi, l_\xi}$$

from this lift. But $\Sigma^*_{\mathcal{T}^*, P^*_\xi}$ is a strategy for P^*_ξ on normal trees and by what we just said, it induces a strategy Ω on $N = M^{i^{\mathcal{T}^*}_{0,\xi}(\mathbb{C})}_{\eta_\xi, l_\xi}$. (We did not need that the background universe was V.) We let

$$\begin{aligned} \Sigma_{\mathcal{T}, P_\xi} &= \Omega^{\pi_\xi} \\ &= \pi_\xi\text{-pullback of } \Omega. \end{aligned}$$

Similarly, we let

$$\mathrm{lift}(\langle \mathcal{T}, \mathcal{U} \rangle, M, \mathbb{C}\rangle = \langle \mathrm{lift}(\mathcal{T}, M_{\nu,k}, \mathbb{C}), \mathrm{lift}(\pi_\xi \mathcal{U}, N, i^{\mathcal{T}^*}_{0,\xi}(\mathbb{C}))\rangle.$$

In this way we can define $\mathrm{lift}(s, M, \mathbb{C})$ for any M-stack s. We let $\mathrm{lift}(s, M, \mathbb{C})_0$ be the stack of normal trees in $\mathrm{lift}(s, M, \mathbb{C})$. The trees in s may be only weakly normal, but those in $\mathrm{lift}(s, M, \mathbb{C})_0$ have no drops.

Definition 2.46 *Let $\mathbb{C}$ be a background construction, and let Σ^* be a (λ, θ)-iteration strategy for $(V, \vec{F}^{\mathbb{C}})$; then for any $M = M^{\mathbb{C}}_{\nu,k}$, $\Omega(\mathbb{C}, M, \Sigma^*)$ is the complete (λ, θ)-strategy for M given by*

$$s \text{ is by } \Omega(\mathbb{C}, M, \Sigma^*) \Leftrightarrow \mathrm{lift}(\mathcal{T}, M, \mathbb{C})_0 \text{ is by } \Sigma^*.$$

.

If we fix a construction $\mathbb{C}$ and a strategy Σ^* that witnesses the strong unique $\vec{F}^{\mathbb{C}}$-iterability of its background universe, then the induced strategies $\Omega(\mathbb{C}, M, \Sigma^*)$ are all strategy coherent. We prove this in Lemma 4.59 below, but it should be plausible. The strategies and their tails are all derived from the same strategy Σ^*, and Σ^* is itself coherent because it picks unique wellfounded branches. Here we show the induced strategies are mildly positional.

Lemma 2.47 *Let $\mathbb{C}$ be a background construction, let Σ^* be a (λ, θ)-iteration strategy for $(V, \vec{F}^{\mathbb{C}})$, and let $M = M^{\mathbb{C}}_{\nu,k}$. Let $N \trianglelefteq M$, and let $\langle \eta, l \rangle = \mathrm{Res}_{\nu,k}[N]$ and $\sigma = \sigma_{\nu,k}[N]$; then*

$$\Omega(\mathbb{C}, M, \Sigma^*)_N = \Omega(\mathbb{C}, M^{\mathbb{C}}_{\eta,l}, \Sigma^*)^{\sigma}.$$

Proof. This is immediate from the definitions. Letting $\Sigma = \Omega(\mathbb{C}, M, \Sigma^*)$, $\Sigma_N = \Sigma_{\langle \emptyset, N \rangle}$ is the tail of Σ after the empty normal tree followed by a gratuitous drop to N. But then if $\mathcal{T}$ is the first normal tree in a stack on N and $\mathrm{lift}(\mathcal{T}, M, \mathbb{C}) = \langle \mathcal{T}^*, \langle (\nu_\xi, k_\xi) \mid \xi < \mathrm{lh}(\mathcal{T}) \rangle, \langle \varphi_\xi \mid \xi < \mathrm{lh}(\mathcal{T}) \rangle$, we see from the way dropping is handled in conversion systems that $\langle \nu_0, k_0 \rangle = \langle \eta, l \rangle$ and $\varphi_0 = \sigma$. This is what we need. □

Corollary 2.48 *Let $\mathbb{C}$ be a background construction, let Σ^* be a (λ, θ)-iteration strategy for $(V, \vec{F}^{\mathbb{C}})$, and let $M = M^{\mathbb{C}}_{\nu,k}$; then $\Omega(\mathbb{C}, M, \Sigma^*)$ is a complete strategy.*

Proof. Let $\Sigma = \Omega(\mathbb{C}, M, \Sigma^*)$. We must show that Σ is mildly positional.

Σ_M comes from lifting the empty tree on M to the empty tree on V, then resurrecting M to itself. So $\Sigma_M = \Sigma$. Similarly, $\Sigma_{s, M_\infty(s)} = \Sigma_s$ for all s by Σ.

Let $P \trianglelefteq N \trianglelefteq M$; we show that $(\Sigma_N)_P = \Sigma_P$. For let $\langle \eta, l \rangle = \mathrm{Res}_{\nu,k}[N]$ and $\sigma = \sigma_{\nu,k}[N]$. Let $\langle \theta, j \rangle = \mathrm{Res}_{\eta,l}[\sigma(P)]$ and $\tau = \sigma_{\eta,l}[\sigma(P)]$. It is not hard to see that $\langle \theta, j \rangle = \mathrm{Res}_{\nu,k}[P]$ and $\tau \circ \sigma = \sigma_{\nu,k}[P]$. We have then that both Σ_P and $(\Sigma_N)_P$ (the tail of Σ after two empty trees and two drops) are both equal to $\Omega(\mathbb{C}, M_{\theta,j}, \Sigma^*)^{\tau \circ \sigma}$.

The proof of the last paragraph applies also to tails Σ_s of Σ, so we have clause (b) of mild positionality.

For clause (c), let $Q \trianglelefteq M$, let $\mathcal{T}$ be weakly normal on Q and by Σ_Q, and let $\mathcal{U}$ be the M-equivalent of $\mathcal{T}$. Using Lemma 2.47, it is easy to see that $\mathrm{lift}(\langle \langle \emptyset, Q \rangle, \mathcal{T} \rangle, M, \mathbb{C})_0$ is the same as $\mathrm{lift}(\mathcal{U}, M, \mathbb{C})_0$, except that the first of these normal trees has one step of padding at the beginning. When $\mathcal{T}$ applies an extender to some initial segment of its base model Q, $\mathrm{lift}(\langle \langle \emptyset, Q \rangle, \mathcal{T} \rangle, M, \mathbb{C})$ resurrects in $\mathbb{C}$ the image of P from the resurrection of Q. At the corresponding step in $\mathcal{U}$, $\mathrm{lift}(\mathcal{U}, M, \mathbb{C})$ will resurrect P in $\mathbb{C}$. Lemma 2.47 tells us we get the same lifting map both ways. □

Another elementary fact we need later is that lifting to a background universe commutes with the copying construction. The proof is completely routine, but it has the structure of somewhat less routine inductions we shall do later, so we run through it here.

Lemma 2.49 *Let R and S be transitive models of* ZFC*, $R \models$ "$\mathbb{C}$ is a background construction", and let $\pi \colon R \to S$ be elementary with $\pi(\mathbb{C}) = \mathbb{D}$. Let M be a model of $\mathbb{C}$, $N = \pi(M)$, and let s be an M-stack; then $\pi \, \mathrm{lift}(s, M, \mathbb{C})_0 = \mathrm{lift}(\pi s, N, \mathbb{D})_0$.*

Proof. We assume for simplicity that s consists of one weakly normal tree $\mathcal{T}$ on M. The general case is quite similar. Let $\mathcal{U} = \pi\mathcal{T}$, $\mathcal{T}^* = \mathrm{lift}(\mathcal{T}, M, \mathbb{C})_0$ and $\mathcal{U}^* = \mathrm{lift}(\mathcal{U}, N, \mathbb{D})_0$. We must see that $\mathcal{U}^* = \pi\mathcal{T}^*$.

Let $\pi_\xi \colon \mathcal{M}_\xi^{\mathcal{T}} \to \mathcal{M}_\xi^{\mathcal{U}}$ be the copy map. Let

$$\mathrm{lift}(\mathcal{T}, M, \mathbb{C}) = \langle \mathcal{T}^*, \langle (\nu_\xi, k_\xi) \mid \xi < \mathrm{lh}(\mathcal{T})\rangle, \langle \varphi_\xi \mid \xi < \mathrm{lh}(\mathcal{T})\rangle$$

and

$$\mathrm{lift}(\mathcal{U}, N, \mathbb{D}) = \langle \mathcal{U}^*, \langle (\eta_\xi, l_\xi) \mid \xi < \mathrm{lh}(\mathcal{U})\rangle, \langle \psi_\xi \mid \xi < \mathrm{lh}(\mathcal{U})\rangle\rangle.$$

Let us write $\mathbb{C}_\xi = i_{0,\xi}^{\mathcal{T}^*}(\mathbb{C})$ and $\mathbb{D}_\xi = i_{0,\xi}^{\mathcal{U}^*}(\mathbb{D})$. Set $P_\xi = M_{\nu_\xi,k_\xi}^{\mathbb{C}_\xi}$ and $Q_\xi = M_{\eta_\xi,l_\xi}^{\mathbb{D}_\xi}$. The map that resurrects $\varphi_\xi(E_\xi^{\mathcal{T}})$ inside $\mathbb{C}_\xi$ is

$$\rho_\xi = \sigma_{\nu_\xi,l_\xi}[M_{\nu_\xi,l_\xi}^{\mathbb{C}_\xi}|\langle \mathrm{lh}(\varphi_\xi(E_\xi^{\mathcal{T}})), 0\rangle].$$

Similarly, the resurrection map for $\psi_\alpha(E_\alpha^{\mathcal{U}})$ is

$$\tau_\xi = \sigma_{\eta_\xi,l_\xi}[M_{\eta_\xi,l_\xi}^{\mathbb{D}_\xi}|\langle \mathrm{lh}(\psi_\xi(E_\xi^{\mathcal{U}})), 0\rangle].$$

For any construction $\mathbb{D}$ and G the last extender of $M_{\nu,0}^{\mathbb{D}}$, we write $B^{\mathbb{D}}(G) = F_\nu^{\mathbb{D}}$ for the background extender of G given by $\mathbb{D}$. Thus

$$E_\xi^{\mathcal{T}^*} = B^{\mathbb{C}_\xi} \circ \rho_\xi \circ \varphi_\xi(E_\xi^{\mathcal{T}}),$$

and

$$E_\xi^{\mathcal{U}^*} = B^{\mathbb{D}_\xi} \circ \tau_\xi \circ \psi_\xi(E_\xi^{\mathcal{U}}).$$

We define $\sigma_\xi \colon \mathcal{M}_\xi^{\mathcal{T}^*} \to \mathcal{M}_\xi^{\mathcal{U}^*}$ by induction on ξ, maintaining by induction on ξ

(a) $\mathcal{U}^*{\restriction}\xi+1 = \pi\mathcal{T}^*{\restriction}\xi+1$, and for all $\alpha \le \xi$, σ_α is the associated copy map,

(b) $\sigma_\xi(P_\xi) = Q_\xi$, and

(c) $\sigma_\xi \circ \varphi_\xi = \psi_\xi \circ \pi_\xi$.

Let $(\dagger)_\xi$ be the conjunction of (a) and (b), and assume that it holds. Let

$$E = E_\xi^{\mathcal{T}} \text{ and } F = E_\xi^{\mathcal{U}}.$$

Thus $\pi_\xi(E) = F$. Let $E^* = E_\xi^{\mathcal{T}^*}$ and $F^* = E_\xi^{\mathcal{U}^*}$. So

$$\begin{aligned}\sigma_\xi(E^*) &= \sigma_\xi(B^{\mathbb{C}_\xi} \circ \rho_\xi \circ \varphi_\xi(E))\\ &= B^{\mathbb{D}_\xi} \circ \tau_\xi(\sigma_\xi(\varphi_\xi(E)))\\ &= B^{\mathbb{D}_\xi} \circ \tau_\xi \circ \psi_\xi \circ \pi_\xi(F)\\ &= F^*.\end{aligned}$$

Line 2 comes from the fact that $\sigma_\xi(\rho_\xi) = \tau_\xi$ by (b), and line 3 comes from (c). Since $\sigma_\xi(E^*) = F^*$, we get that F^* is the next extender used in $\pi\mathcal{T}^*$, and thus $\pi\mathcal{T}^*\restriction\xi+2 = \mathcal{U}^*\restriction\xi+2$. We let $\sigma_{\xi+1}$ be the copy map,

$$\sigma_{\xi+1}([a,f]_{E^*}^{\mathcal{M}_\beta^{\mathcal{T}^*}}) = [\sigma_\xi(a), \sigma_\beta(f)]_{F^*}^{\mathcal{M}_\beta^{\mathcal{U}^*}},$$

where $\beta = T\text{-pred}(\xi+1)$ is the predeccessor of $\xi+1$ in all our trees.

We must verify (b) and (c) of $(\dagger)_{\xi+1}$. Suppose first that $\xi+1$ is not a drop in T (gratuitous or otherwise). It is then not a drop in $\mathcal{U}$ either, so $P_{\xi+1} = i^{\mathcal{T}^*}_{\beta,\xi+1}(P_\beta)$ and $Q_{\xi+1} = i^{\mathcal{U}^*}_{\beta,\xi+1}(Q_\beta)$. But then

$$\begin{aligned}\sigma_{\xi+1}(P_{\xi+1}) &= \sigma_{\xi+1} \circ i^{\mathcal{T}^*}_{\beta,\xi+1}(P_\beta)\\ &= i^{\mathcal{U}^*}_{\beta,\xi+1} \circ \sigma_\beta(P_\beta)\\ &= i^{\mathcal{U}^*}_{\beta,\xi+1}(Q_\beta)\\ &= Q_{\xi+1},\end{aligned}$$

so we have (b). For (c), let us consider the diagram

$$\begin{tikzcd}
 & \mathcal{M}^{\mathcal{U}}_{\xi+1} \arrow[rr, "\psi_{\xi+1}"] & & Q_{\xi+1} & & & \mathcal{M}^{\mathcal{U}^*}_{\xi+1} \\
\mathcal{M}_{\xi+1}\mathcal{T} \arrow[ur, "\pi_\xi"] \arrow[rr, "\varphi_{\xi+1}"] & & P_{\xi+1} \arrow[ur, "\sigma_{\xi+1}"] & & & \mathcal{M}^{\mathcal{T}^*}_{\xi+1} \arrow[ur, "\sigma_{\xi+1}"] & \\
 & \mathcal{M}^{\mathcal{U}}_\beta \arrow[uu, "F"] \arrow[rr, "\psi_\beta"'] & & Q_\beta \arrow[uu, "F^*"'] & & & \mathcal{M}^{\mathcal{U}^*}_\beta \arrow[uu, "F^*"'] \\
\mathcal{M}^{\mathcal{T}}_\beta \arrow[uu, "E"] \arrow[ur, "\pi_\beta"] \arrow[rr, "\varphi_\beta"'] & & P_\beta \arrow[uu, "E^*"'] \arrow[ur, "\sigma_\beta"'] & & & \mathcal{M}^{\mathcal{T}^*}_\beta \arrow[uu, "E^*"] \arrow[ur, "\sigma_\beta"] &
\end{tikzcd}$$

We are asked to show that $\sigma_{\xi+1} \circ \varphi_{\xi+1} = \psi_{\xi+1} \circ \pi_{\xi+1}$, that is, that the rectangle on the top face of the cube commutes. We are given that all other faces of the cube commute, so we have that $\sigma_{\xi+1} \circ \varphi_{\xi+1}$ agrees with $\psi_{\xi+1} \circ \pi_{\xi+1}$ on $\operatorname{ran}(i^{\mathcal{T}}_{\beta,\xi+1})$. Since $\mathcal{M}^{\mathcal{T}}_{\xi+1}$ is generated by $\operatorname{ran}(i^{\mathcal{T}}_{\beta,\xi+1}) \cup \lambda(E)$, it is enough to show that $\sigma_{\xi+1} \circ \varphi_{\xi+1}$ agrees with $\psi_{\xi+1} \circ \pi_{\xi+1}$ on $\lambda(E)$. But on $\lambda(E)$, $\sigma_{\xi+1} \circ \varphi_{\xi+1}$ agrees with $\sigma_\xi \circ \varphi_\xi$ and $\psi_{\xi+1} \circ \pi_{\xi+1}$

agrees with $\psi_\xi \circ \pi_\xi$, by the Shift Lemma. Hence our induction hypothesis $(\dagger)_\xi$ (c) gives us what we want.

The case that $\mathcal{T}$ drops at $\xi+1$ is similar. Suppose the drop is to $J \lhd \mathcal{M}^{\mathcal{T}}_\beta$. Let $K = \pi_\beta(J)$ be what $\mathcal{U}$ drops to at $\xi+1$, and let $L = \varphi_\beta(J)$ and $N = \psi_\beta(K)$. To get to $P_{\xi+1}$ and $Q_{\xi+1}$ we must resurrect our drop. Let $\langle \nu, k\rangle = \mathrm{Res}^{\mathbb{C}_\xi}_{\nu_\beta, l_\beta}[L])$ and $Y = M^{\mathbb{C}_\beta}_{\nu,k}$. Let $t \colon L \to Y$ be the resurrection map, that is, $t = \sigma_{\nu_\beta,k_\beta}[L]$. Similarly, let Z be the resurrection of N is $\mathbb{D}_\beta$ from stage $\langle \eta_\beta, l_\beta\rangle$, and let $u \colon N \to Z$ be the resurrection map. From the definition of a conversion system, we see that

$$P_{\xi+1} = i^{\mathcal{T}^*}_{\beta,\xi+1}(Y)$$

and

$$Q_{\xi+1} = i^{\mathcal{U}^*}_{\beta,\xi+1}(Z).$$

But $\sigma_\beta(L) = N$ by $(\dagger)_\beta$, so $\sigma_\beta(Y) = Z$ by elementarity, so $\sigma_{\xi+1}(P_{\xi+1}) = Q_{\xi+1}$. This gives us (b) of $(\dagger)_{\xi+1}$. The reader can easily check (c) using a diagram like the one above. Note here that $\sigma_\beta(t) = u$. $\square$

We get at once

Corollary 2.50 *Let R and S be transitive models of ZFC, $R \models$ "$\mathbb{C}$ is a background construction", and let $\pi \colon R \to S$ be elementary with $\pi(\mathbb{C}) = \mathbb{D}$. Let M be a model of $\mathbb{C}$, $N = \pi(M)$, and let Σ be a complete strategy for S; then*

$$\Omega(\mathbb{C}, M, \Sigma^\pi) = \Omega(\mathbb{D}, N, \Sigma)^\pi.$$

2.10 Iterating into a background construction

The idea that if one compares a countable mouse P with some level $M^{\mathbb{C}}_{\nu,k}$ of a background construction, then only the P side moves, goes back to Baldwin and Mitchell, and in some sense even to Kunen. The proof is very much like the proof one learns now that least disagreement comparisons terminate. The Skolem-hull-of-V embedding is replaced by by some background extender embedding, and one gets thereby that no backgrounded extender ever particpates in a disagreement.

The argument has been used many times at the level of Woodin cardinals (cf. [35, Theorem 2.5] for example), but we know of no exposition in print of the very simple form we need in this book. So we give one here.

Definition 2.51 *Let M and P be premice, and let Σ be an iteration strategy for P; then* *(a)* (P, Σ) iterates past M *iff there is a normal iteration tree* $\mathcal{T}$ *by* Σ *on* P *with last model* Q *such that* $M \trianglelefteq Q$*,*

(b) (P, Σ) iterates to M *iff there are* $\mathcal{T}$ *and* Q *as in (a), and moreover,* $M = Q$*, and the branch* P*-to-*Q *of* $\mathcal{T}$ *does not drop.*

(c) (P, Σ) iterates strictly past M *iff it iterates past* M*, but not to* M*.*

Lemma 2.52 *(Only the mouse moves.) Let* $\mathbb{C}$ *be a background construction above* κ *such that each* $F_\nu^{\mathbb{C}}$ *is in* V_δ*, where* δ *is inaccessible. Let* P *be a premouse such that* $|P| < \kappa$*, and let* Σ *be a* δ*-iteration strategy for* P*. Suppose that whenever* $E^* = F_\nu^{\mathbb{C}}$ *for some* ν*, we have*

$$i_{E^*}(\Sigma) \subseteq \Sigma.$$

Let $M = M_{\nu,k}^{\mathbb{C}}$*, and suppose that* (P, Σ) *iterates strictly past* $M_{\eta,j}^{\mathbb{C}}$ *for all* $\langle \eta, j \rangle <_{\text{lex}} \langle \nu, k \rangle$*; then* (P, Σ) *iterates past* $M_{\nu,k}^{\mathbb{C}}$*.*

Proof. Suppose not. Let E be the first extender used on the $M_{\nu,k}$-side in its comparison with P.

Claim 1. If (P, Σ) iterates strictly past $M_{\nu,n}$, then (P, Σ) iterates past $M_{\nu,n+1}$.

Proof. Let $\mathcal{T}$ with last model Q witness that (P, Σ) iterates strictly past $M_{\nu,n}$. If $M_{\nu,n}$ and $M_{\nu,n+1}$ are the same except for their distinguished soundness degrees n and $n+1$, then $\mathcal{T}$ witnesses that (P, Σ) iterates past $M_{\nu,n+1}$ (perhaps not strictly), as desired. Otherwise $M_{\nu,n}$ is not sound, so it must be equal to Q. But then $M_{\nu,n+1} = \text{core}(M_{\nu,n}) = \mathcal{M}_\xi^{*,\mathcal{T}}$ for some ξ on the main branch of $\mathcal{T}$. This implies that $\mathcal{T} \restriction \xi + 1$ witnesses that (P, Σ) iterates past $M_{\nu,n+1}$.

□

Claim 2. Suppose (P, Σ) iterates strictly past $M_{\nu,k}$ for all $k < \omega$; then (P, Σ) iterates past $M_{\nu+1,0}$.

Proof. The literal premouse $\hat{M}_{\nu,k}$ is eventually constant as $k \to \omega$. Thus there is a fixed normal tree $\mathcal{T}$ of minimal length witnessing that (P, Σ) iterates strictly past $M_{\nu,k}$ for all $k < \omega$. Letting Q be the last model of $\mathcal{T}$, we have $M_{\nu,k} \triangleleft Q$ for all sufficiently large k, and thus $M_{\nu+1,0} \trianglelefteq Q$. □

Claim 3. Let ν be a limit ordinal, and suppose that (P, Σ) iterates strictly past $M_{\eta,j}$ for all $\eta < \nu$, and that $M_{\nu,0}$ is passive; then (P, Σ) iterates past $M_{\nu,0}$.

Proof. This is immediate. □

Now suppose the lemma fails at ν, k. By the claims, $k = 0$, $M_{\nu,0}$ is active, and (P, Σ) iterates past $M^{<\nu}$. Let E be the last extender of $M_{\nu,0}$, and let $E^* = F_\nu^{\mathbb{C}}$ be the background extender for E, and let $\mathcal{T}$ be the tree by Σ on P of minimal length iterating it past $M_{\nu,0}||\,\mathrm{lh}(E) = M^{<\nu}$. Since the lemma is failing, E gets used in the comparison of P with $M_{\nu,0}$. So setting $\alpha + 1 = \mathrm{lh}(\mathcal{T})$, we have that

(i) $M||\,\mathrm{lh}(E) = \mathcal{M}_\alpha^{\mathcal{T}}||\,\mathrm{lh}(E)$,

(ii) $M|\,\mathrm{lh}(E) \neq \mathcal{M}_\alpha^{\mathcal{T}}|\,\mathrm{lh}(E)$, and

(iii) for all $\xi < \alpha$, $\mathrm{lh}(E_\xi^{\mathcal{T}}) < \mathrm{lh}(E)$.

Let $\kappa = \mathrm{crit}(E)$, let $i_{E^*} \colon V \to N$ be the canonical embedding, and let $\mathcal{U} = i_{E^*}(\mathcal{T})$. Note that because P is countable and κ is a (measurable) cardinal, $\kappa \leq \alpha$. Let $\lambda = i_{E^*}(\kappa)$.

Claim 4. $\kappa <_U \lambda$, $[\kappa, \lambda)_U$ does not drop, and $i_{E^*}\restriction\mathcal{M}_\kappa^{\mathcal{T}} = i_{\kappa,\lambda}^{\mathcal{U}}$.

Proof. If $\beta <_T \kappa$, then $\beta = i_{E^*}(\beta) <_U \lambda$. Since $[0, \lambda)_U$ is a closed set of ordinals, $\kappa \leq_U \lambda$. Since $[0, \kappa)_T$ has only finitely many drops, these are the same as the drops of $[0, \lambda)_U$, so $[\kappa, \lambda)_U$ does not drop. Finally, if $x \in \mathcal{M}_\kappa^{\mathcal{T}}$, then we have $\beta <_T \kappa$ and $\bar{x}$ such that $i_{\beta,\kappa}^{\mathcal{T}}(\bar{x}) = x$. But then

$$\begin{aligned} i_{E^*}(x) &= i_{E^*}(i_{\beta,\kappa}^{\mathcal{T}}(\bar{x})) \\ &= i_{E^*}(i_{\beta,\kappa}^{\mathcal{T}})(\bar{x}) \\ &= i_{\beta,\lambda}^{\mathcal{U}}(\bar{x}) \\ &= i_{\kappa,\lambda}^{\mathcal{U}}(i_{\beta,\kappa}^{\mathcal{U}}(\bar{x})) \\ &= i_{\kappa,\lambda}^{\mathcal{U}}(x), \end{aligned}$$

as desired. □

Claim 5. $\mathcal{U}$ is by Σ, and $\mathcal{U}\restriction\alpha + 1 = \mathcal{T}$.

Proof. $\mathcal{U}$ is by $i_{E^*}(\Sigma)$. But $i_{E^*}(\Sigma) \subseteq \Sigma$, so $\mathcal{U}$ is by Σ. So in N, $\mathcal{U}$ is obtained by iterating P, using Σ, so as to remove least disagreements with $i_{E^*}(M)$. Since E^* ceritifies E, we have $i_{E^*}(M)||\,\mathrm{lh}(E) = \mathrm{Ult}(M|\,\mathrm{lh}(E), E)||\,\mathrm{lh}(E) = M||\,\mathrm{lh}(E)$. Thus the process that produces $\mathcal{U}$ is the same as the process that produced $\mathcal{T}$, until extenders with length $\geq \mathrm{lh}(E)$ are used, so $\mathcal{T} = \mathcal{U}\restriction\alpha + 1$. □

Now let $G = E_\xi^{\mathcal{U}}$, where $\kappa = U\text{-pred}(\xi+1)$ and $\xi+1 <_U \lambda$. G is an initial segment of the extender of $i_{\kappa,\lambda}^{\mathcal{U}}$, so by Claim 4, G is compatible with E. If G is a proper initial segment of E, then G is on the $M||\operatorname{lh}(E)$-sequence, so G is on the $\mathcal{M}^{\mathcal{U}}_\lambda$-sequence because $\mathcal{M}^{\mathcal{U}}_\alpha||\operatorname{lh}(E) = i_{E^*}(M)||\operatorname{lh}(E)$. But then $\operatorname{lh}(G)$ is not a cardinal in $\mathcal{M}^{\mathcal{U}}_\lambda$, contrary to its having been used in $\mathcal{U}$. If E is an initial segment of G, we get that E is on the sequence of $\mathcal{M}^{\mathcal{U}}_\xi$, and hence on the sequence of $\mathcal{M}^{\mathcal{U}}_\alpha = \mathcal{M}^{\mathcal{T}}_\alpha$. But this means that E was not part of the least disagreement between $\mathcal{M}^{\mathcal{T}}_\alpha$ and M, contradiction.

□

We can use this to show that the output of a maximal construction done below a Woodin cardinal is universal for mice of size strictly less than its additivity. This argument has probably been known since the late 1980s, but we can find no appropriate reference. A stronger version involving partial background extenders and universality with respect to weasels traces back to the paper [22] by Mitchell and Schindler. The author adapted the stronger version to full background constructions, where the Woodin cardinal becomes necessary. See [54, Lemma 11.1] and [29].

Theorem 2.53 *(Universality at a Woodin cardinal) Suppose that $\vec{F}$ be coarsely coherent, $\vec{F} \subseteq V_\delta$, and δ is Woodin as witnessed by extenders on $\vec{F}$. Let $\mathbb{C}$ be a maximal $\vec{F}$-construction. Let P be a premouse, $|P| < \operatorname{crit}(F_\nu)$ for all ν, and let Σ be a $\delta + 1$-iteration strategy for P. Suppose that whenever E^* is on $\vec{F}$, we have*

$$i_{E^*}(\Sigma) \subseteq \Sigma.$$

Then there is a $\nu < \delta$ and $k < \omega$ such that $M^{\mathbb{C}}_{\nu,k}$ exists, and (P, Σ) iterates to $M^{\mathbb{C}}_{\nu,k}$.

Proof. The proof would be a bit easier if we assumed that $\mathbb{C}$ never breaks down, but we do not need to do that. Here we say that $\mathbb{C}$ *breaks down at* $\langle \nu, k\rangle$ iff $M^{\mathbb{C}}_{\nu,k}$ exists, and either

(i) the standard parameter of $M_{\nu,k}$ is either not solid, or not universal, or

(ii) ν is a limit ordinal, $k = 0$, and the Bicephalus Lemma fails, in that there are background certified F and G such that $(M^{<\nu}, F)$ and $M^{<\nu}, G)$ are premice, and $F \neq G$.

Claim 1. Suppose $\mathbb{C}$ breaks down at $\langle \nu, k\rangle$; then there is an $\langle \eta, j\rangle <_{\text{lex}} \langle \nu, k\rangle$ such that (P, Σ) iterates to $M^{\mathbb{C}}_{\eta,j}$.

Proof. Suppose first that there is an $\langle \eta, j\rangle <_{\text{lex}} \langle \nu, k\rangle$ such that (P, Σ) does not iterate strictly past $M_{\eta,j}$. Then for the lexicographically least such $\langle \eta, j\rangle$, (P, Σ) iterates to $M_{\eta,j}$, by 2.52, so we are done. Thus we may assume (P, Σ) iterates strictly past $M_{\eta,j}$ for all $\langle \eta, j\rangle <_{\text{lex}} \langle \nu, k\rangle$. By Lemma 2.52, we get that (P, Σ) iterates past $M_{\nu,k}$.

P is iterable, so the standard parameters of its iterates are solid and universal. So (i) does not hold at $\langle \nu, k\rangle$, and it must be (ii) that holds. Let F and G witness this, and let F^* and G^* be background certificates for them. Let $\mathcal{T}$ be the shortest tree by which (P, Σ) iterates past $M_{\nu,0}||\operatorname{lh}(F) = M_{\nu,0}||\operatorname{lh}(G)$, and let $\alpha + 1 = \operatorname{lh}(\mathcal{T})$. We now simply apply the proof of Lemma 2.52 to both F and G, and it shows that both of them are on the sequence of $\mathcal{M}_\alpha^{\mathcal{T}}$. Thus $F = G$, contradiction. $\square$

Thus we may assume that $\mathbb{C}$ never breaks down, and that (P, Σ) iterates past $M_{\nu,k}$ for all $\nu < \delta$ and $k \leq \omega$. Let

$$M = (M^{<\delta})^{\mathbb{C}}$$

be the unique passive premouse such that $o(M) = \delta$ and for all $\xi < \delta$, $M|\xi \trianglelefteq M_{\alpha,0}^{\mathbb{C}}$ for all sufficiently large $\alpha < \delta$. Clearly, (P, Σ) iterates past M. Let $\mathcal{T}$ on P be the normal tree by Σ that witnesses this. We have that $\operatorname{lh}(\mathcal{T}) = \delta + 1$, $\delta(\mathcal{T}) = \delta$, and

$$M \triangleleft M_\delta^{\mathcal{T}},$$

because δ is inaccessible. Let $b = [0, \delta)_T$, and for $\beta < \delta$, let $f(\beta) = \min(b - (\beta + 1))$. Since δ is Woodin, we can find a nice extender F^* with critical point α and length η such that for $j = i_{F^*}$

(1) $f``\alpha \subseteq \alpha$, and $j(f)(\alpha) < \eta$,

(2) $M||\eta = j(M)||\eta$, and

(3) $j(b) \cap \eta = b \cap \eta$.

Let $\tau + 1 <_T \delta$ be such that $\alpha = T\text{-pred}(\tau + 1)$, and let $F = E_\tau^{\mathcal{T}}$. By (1) and (3), $\tau + 1 = j(f)(\alpha)$ is the first point in $j(b)$ above α. Letting $\mathcal{U} = j(\mathcal{T})$ and $\lambda = j(\alpha)$, we have as usual that $\mathcal{M}_\alpha^{\mathcal{T}} = \mathcal{M}_\alpha^{\mathcal{U}}$, and

$$j \restriction \mathcal{M}_\alpha^{\mathcal{T}} = i_{\alpha,\lambda}^{\mathcal{U}}.$$

But in fact $\mathcal{T} \restriction \eta = \mathcal{U} \restriction \eta$ by (2) and the fact that $j(\Sigma) \subseteq \Sigma$. So $F = E_\tau^{\mathcal{U}}$, where $\alpha <_U \tau + 1 <_U \lambda \in j(b)$, which implies that F^* is a background certificate for F.

Let ν be the least stage of $\mathbb{C}$ such that $M||\operatorname{lh}(F) \trianglelefteq M^{<\nu}$. Because $\operatorname{lh}(F)$ is a cardinal of M, we must have $M^{<\nu} = M||\operatorname{lh}(F)$. But then $M_{\nu,0} = (M^{<\nu}, F)$, because our construction is maximal. After $\langle \nu, 0\rangle$ the levels of $\mathbb{C}$ do not project strictly below λ_F, because $M \trianglelefteq \mathcal{M}_\delta^{\mathcal{T}}$. This implies that F is on the M-sequence, contrary to its being used in $\mathcal{T}$. $\square$

3 Normalizing stacks of iteration trees

In this chapter, we shall show how one can re-order the use of extenders in a finite stack s of normal iteration trees, so as to produce a single normal tree $W(s)$ such that the last model of s embeds into the last model of $W(s)$. We call this process *embedding normalization*. Our goal here is to give some basic definitions and prove some elementary theorems that help one deal with the complexities of this process. In Chapter 4 we shall apply the resulting theory to the comparison of iteration strategies.

We shall focus throughout the chapter on finite, maximal M-stacks, that is, on finite stacks of normal trees on M that involve no gratuitous dropping at the beginning of a round. Everything we prove generalizes easily to arbitrary finite M-stacks, but rather than complicate the notation further, we shall just make some occasional remarks on how the generalization goes.

The results of this chapter have the pleasant feature that one need only understand the basic facts about iteration trees and premice in order to follow their proofs. Indeed, it seems to us that this is a place where someone with minimal background knowledge could get a feel for iteration tree combinatorics. With that in mind, we have gone more slowly, including more examples and variant proofs than a more advanced reader would require.

In that spirit, we begin in §2.1 by considering the simplest possible case, normalizing a stack of length two in which each component tree uses only one extender. The results of this section are not used later, but they do help give a feel for what's going on. We also show in §2.1 that these simple stacks can be fully normalized, in that, granted an iterability assumption, one can find a normal tree $X(s)$ whose last model is equal to the last model of s.

In §2.2 we consider the special case of stacks $\langle \mathcal{T}, \mathcal{U} \rangle$ in which $\mathcal{U}$ uses only one extender, and in §2.5 we define $W(\langle \mathcal{T}, \mathcal{U} \rangle) = W(\mathcal{T}, \mathcal{U})$ for the general maximal stack of length two. We do use some of the definitions of §2.2 in §2.5.

In §2.3 we introduce *extender trees*, which are simple re-packagings of iteration trees that are sometimes helpful. In §2.4 we introduce something much more important, the notion of a *tree embedding*.[20] This notion is absolutely central to our work here. What makes an iteration strategy Σ comparable with other strategies is that if $\mathcal{U}$ is by Σ, and $\mathcal{T}$ is tree-embeddable into $\mathcal{U}$, then $\mathcal{T}$ is by Σ. We call this property of Σ *strong hull condensation*. Tree embeddings play an important role in the definition of $W(\mathcal{T}, \mathcal{U})$, as we shall see.

§2.6 and §2.7 are devoted to elementary facts about $W(\mathcal{T}, \mathcal{U})$. The most sub-

[20] Tree embeddings were isolated independently by Schlutzenberg and the author. See [44].

stantial result here concerns the way branches of $W(\mathcal{T},\mathcal{U})$ correspond in one-one fashion with pairs consisting of a branch of $\mathcal{T}$ and a branch of $\mathcal{U}$. Finally, in §2.8 we describe the normalization of stacks of arbitrary finite length, and say a few words about normalizing stacks of infinite length.

In general, there are two sorts of base models M for the iteration trees we deal with in this book: coarse premice, and fine-structural premice. Both sorts divide further into pure extender and strategy premice. The definition of $W(\mathcal{T},\mathcal{U})$ will make sense in both cases. In this chapter we shall focus on the case that M is a fine structural, pure extender premouse, with Jensen indexing for its extender sequence. Until we get to Chapter 5, this is what we shall mean by the unqualified *premouse*. We do also need to define $W(\mathcal{T},\mathcal{U})$ in the coarse structural case as well, and we shall indicate how to do so as we proceed. But then we are just talking about ultrapowers of models of ZFC by nice extenders, so various things simplify.

One useful consequence of our definitions is:

Lemma 3.0 *Let M and N be premice, coarse or fine, and let Σ an iteration strategy for M; then there is at most one normal iteration tree $\mathcal{T}$ according to Σ having last model N.*

In the coarse structural case, this is not clear, even if Σ chooses unique cofinal wellfounded branches, unless we restrict our iterations to $\vec{F}$-trees, for some fixed coarsely coherent $\vec{F}$. In that case, we have proved in in Lemma 2.40. That proof works also in the fine structural case. We shall use Lemma 3.0 in an important way in the proof of Lemma 3.67 below.

The construction of $W(\mathcal{T},\mathcal{U})$ does not require that any iteration strategy for M be fixed; however, it may break down by reaching illfounded models, even if the models of $\mathcal{T}^\frown\mathcal{U}$ are wellfounded. In the case we care about, M has an iteration strategy Σ , $\langle\mathcal{T},\mathcal{U}\rangle$ is played according to Σ, and the initial segment of $W(\mathcal{T},\mathcal{U})$ up to our point of interest is also played by Σ. We can then invoke Fact 3.0, relative to Σ, for the models in $W(\mathcal{T},\mathcal{U})$ up to our point of interest. We shall eventually show that if Σ has strong hull condensation, then $W(\mathcal{T},\mathcal{U})$ is also by Σ, and hence the construction of $W(\mathcal{T},\mathcal{U})$ does not break down.

3.1 Normalizing trees of length 2

We begin by looking closely at stacks of the form $\langle\langle E\rangle,\langle F\rangle\rangle$.

Let M be a premouse, E on the sequence of M, $\mathrm{crit}(E) < \rho_{k(M)}(M)$, and $N = \mathrm{Ult}(M,E)$. Let F be on the sequence of N, and $\mathrm{crit}(F) < \lambda(E)$. It follows that

Ult(N, F) makes sense; let $Q = \mathrm{Ult}(N, F)$. So $k(M) = k(N)$, and both ultrapowers are $k(M)$-ultrapowers.

Let

$$\kappa = \mathrm{crit}(E), \quad \mu = \mathrm{crit}(F).$$

Let $\mathcal{T}$ be the iteration tree such that $E_0^{\mathcal{T}} = E$, $E_1^{\mathcal{T}} = F$, $\mathcal{M}_0^{\mathcal{T}} = M$, $\mathcal{M}_1^{\mathcal{T}} = N$, and $\mathcal{M}_2^{\mathcal{T}} = Q$. Since $\mu < \lambda(E)$, $\mathcal{T}$ is not normal. We show how to normalize it. There are two cases.

Case 1. $\mathrm{crit}(F) \leq \mathrm{crit}(E)$.

Since $\mu \leq \kappa$ and E is an extender over M (that is, over the reduct M^n, for $n = k(M)$), F is also an extender over M. Let $P = \mathrm{Ult}(M, F)$, and $i_F^M : M \to P$ be the canonical embedding. We have the diagram

$$\begin{tikzcd}
N \arrow[r, "F"] & Q \arrow[r, "\tau"] & i_F^M(N) = \mathrm{Ult}(P, i_F^M(E)) \\
M \arrow[u, "E"] \arrow[r, "F"] & P \arrow[ur, "i_F^M(E)"'] &
\end{tikzcd}$$

Suppose first that $M \models \mathsf{ZFC}$; then N is definable over M from E, and i_F^M moves the fact that $N = \mathrm{Ult}(M, E)$ over to the fact that $i_F^M(N) = \mathrm{Ult}(P, i_F^M(E))$. τ is the natural embedding from $i_F^N(N)$ to $i_F^M(N)$. That is,

$$\tau([a, g]_F^N) = [a, g]_F^M$$

for $g : [\mu]^{|a|} \to N$, with $g \in N$. The tree $\mathcal{U}$ with models

$$\mathcal{M}_0^{\mathcal{U}} = M, \ \mathcal{M}_1^{\mathcal{U}} = N, \ \mathcal{M}_2^{\mathcal{U}} = P, \ \mathcal{M}_3^{\mathcal{U}} = \mathrm{Ult}_0(\mathcal{P}, i_F^M(E))$$

and extenders

$$E_0^{\mathcal{U}} = E, \ E_1^{\mathcal{U}} = F, \ E_2^{\mathcal{U}} = i_F^M(E),$$

is normal. We call $\mathcal{U}$ the *embedding normalization* of $\mathcal{T}$.

Remark 3.1 This implicitly assumes $\mathrm{lh}\, E < \mathrm{lh}\, F$. If $\mathrm{lh}\, F < \mathrm{lh}\, E$, then F is already on the M-sequence, and the extenders of $\mathcal{U}$ would be $E_0^{\mathcal{U}} = F$, $E_1^{\mathcal{U}} = i_F^M(E)$. The diagrams and calculations above don't change, however.

The proof just given was based on N being definable over M as its E-ultrapower and i_F^M acting elementarily on this definition. But of course, $\mathrm{OR}^N > \mathrm{OR}^M$ is possible, and anyway, we need to know i_F^M has enough elementarity. If $M \models \mathsf{ZFC}$, all is fine. We now give a more careful proof that works in general.

We assume $k(M) = k(N) = 0$ so that we can avoid the details of ultrapowers of reducts and their decodings. The general case is similar. So every $x \in Q$ has the form $i_F^N(g)(b)$ for $g \in N$ and $b \in [\nu(F)]^{<\omega}$. We can write $g = i_E^M(h)(a)$, where $h \in M$ and $a \in [\nu(E)]^{<\omega}$. So

$$\begin{aligned} x &= i_F^N(i_E^M(h)(a))(b) \\ &= i_F^N \circ i_E^M(h)(i_F^N(a))(b), \end{aligned}$$

with b, $i_F^N(a) \in [\sup i_F^N\text{``}\,\nu(E)]^{<\omega}$. Let

$$G = (\text{extender of } i_F^N \circ i_E^M){\restriction} \sup i_F^N\text{``}\,\nu(E),$$

so that

$$Q = \mathrm{Ult}(M, G).$$

The space of G is κ, and its critical point is μ. Let us write

$$\begin{aligned} R &= \mathrm{Ult}_0(P, i_F^M(E)) \\ H &= (\text{extender of } i^P_{i_F^M(E)} \circ i_F^M){\restriction} \sup i_F^M\text{``}\nu(E). \end{aligned}$$

It is easy to see that

$$R = \mathrm{Ult}(M, H).$$

But then we can calculate that G is a subextender of H. For let $b \in [\nu(F)]^{<\omega}$ and $g : [\mu]^{|b|} \to [\nu(E)]^l$ with $g \in N$. Let $A \subseteq [\mathrm{crit}(E)]^l$ with $A \in N$. (Equivalently, $A \in M$.) We have

$$\begin{aligned} ([b,g]_F^N, A) \in G \quad &\text{iff} \quad [b,g]_F^N \in i_F^N \circ i_E^M(A) \\ &\text{iff} \quad \text{for } F_b \text{ a.e. } \bar{\mu},\ g(\bar{\mu}) \in i_E^M(A) \\ &\text{iff} \quad \text{for } F_b \text{ a.e. } \bar{\mu},\ (g(\bar{\mu}), A) \in E \\ &\text{iff} \quad ([b,g]_F^M, i_F^M(A)) \in i_F^M(E) \\ &\text{iff} \quad [b,g]_F^M \in i^P_{i_F^M(E)} \circ i_F^M(A) \\ &\text{iff} \quad ([b,g]_F^M, A) \in H. \end{aligned}$$

So letting $\sigma : \mathrm{lh}\, G \to \mathrm{lh}\, H$ be given by

$$\sigma([b, g]^N_F) = [b, g]^M_F,$$

we have

$$(a, A) \in G \quad \text{iff} \quad (\sigma(a), A) \in H,$$

so G is a subextender of H under σ. We can therefore define τ from Q into R by

$$\tau([a, f]^M_G) = [\sigma(a), f]^M_H.$$

Note $\tau \restriction \mathrm{lh}(F) = \sigma \restriction \mathrm{lh}(F) =$ identity. One can easily show that in the case $M \models \mathsf{ZFC}$, our current definition of τ coincides with the earlier one.

Here is another way to obtain τ, one that is closer to the way we shall handle the general case below. Let $\psi\colon \mathrm{Ult}(M, E) \to \mathrm{Ult}(P, E^*)$ be the Shift Lemma map, where $E^* = i^M_F(E)$. That is,

$$\psi([a, f]^M_E) = [i^M_F(a), i^M_F(f)]^P_{E^*}.$$

By the Shift Lemma, ψ agrees with i^M_F on $\lambda(E)$. It follows that F is an initial segment of E_ψ, the extender of ψ. Let θ be the factor embedding from $\mathrm{Ult}(N, F)$ to $\mathrm{Ult}(N, E_\psi)$, given by

$$\theta([a, g]^N_F) = [a, g]^N_{E_\psi} = \psi(g)(a),$$

for all $a \in [\nu(F)]^{<\omega}$. We claim that $\theta = \tau$.

To see this, note that θ is the unique map π from Q to $\mathrm{Ult}(P, E^*)$ such that $\psi = \pi \circ i^N_F$ and $\pi \restriction \nu(F)$ is the identity. Clearly $\tau \restriction \nu(F) = \sigma \restriction \nu(F) =$ identity, so we must see that $\psi = \tau \circ i^N_F$. Now both θ and τ make the diagram

$$\begin{tikzcd}
N \arrow[r, "F"] & Q \arrow[r] & i^M_F(N) = \mathrm{Ult}(P, i^M_F(E)) \\
M \arrow[u, "E"] \arrow[r, "F"] & P \arrow[ur, "i^M_F(E)"'] &
\end{tikzcd}$$

commute, so ψ agrees with $\tau \circ i^N_F$ on $\mathrm{ran}(i^M_E)$. Thus it is enough to see that ψ agrees with $\tau \circ i^N_F$ on the generators of E, that is, on $\nu(E)$. But for $a \in [\nu(E)]^{<\omega}$,

$$\psi(a) = i^M_F(a) = \tau(i^N_F(a)),$$

by the definitions of ψ and τ. This completes our proof that $\tau = \theta$.

Before we move on to the case that $\text{crit}(E) < \text{crit}(F)$, let us look at the problem of full normalization when $\text{crit}(F) \leq \text{crit}(E)$. That is, we seek a normal tree on M whose last model is literally equal to Q. Full normalization is not important in this monograph, but it is very useful in its sequels, for example [63] and [61]. The paper [48] proves a general theorem on the existence of full normalizations for stacks of normal trees on premice. The argument we are about to give contains one of the main ideas in that proof.

Clearly, a full normalization of $\mathcal{T}$ must start with E and then F. We are now at the model P, and to get to Q, we must replace $i_F^M(E)$ by a subextender of itself. One can see from the analysis above that the appropriate subextender is $i_F^M(E) \restriction \sigma``i_F^N(\nu(E))$. What we need to see is that the transitive collapse of this subextender is on the P-sequence. Here we must use the condensation properties of mice, and hence we are assuming that P has these condensation properties. Of course our true interest is in iterable P, which do have them.

We shall apply condensation iteratively. Let $\langle(\beta_i, k_i) \mid 0 \leq i < n\rangle$ be the $\text{lh}(E)$-dropdown sequence of M. That is

$$(\beta_0, k_0) = (\text{lh}\, E, 0)$$

and

$$\begin{aligned}(\beta_{i+1}, k_{i+1}) = {} & \text{lexicographically least } (\alpha, l) \text{ such that} \\ & \langle \alpha, l\rangle <_{\text{lex}} l(M) \text{ and } \rho(M|\langle\alpha, l\rangle) < \rho(M|\langle\beta_i, k_i\rangle).\end{aligned}$$

So long as they are defined, the ordinals

$$\rho_i^* = \rho(M|\langle\beta_i, k_i\rangle)$$

are strictly decreasing as i increases. The $\langle\beta_i, k_i\rangle$ increase, lexicographically. Note that ρ_i^* is a cardinal of $M|\beta_{i+1}$ with respect to $r\Sigma_{k_{i+1}}$ functions, and $\langle\beta_{i+1}, k_{i+1}\rangle$ is lex-largest such that this is true.

Let n be least such that (β_n, k_n) cannot be defined this way, and set

$$(\beta_n, k_n) = l(M) = \langle\hat{o}(M), k(M)\rangle.$$

Notice that E was total on the reduct $M^{k(M)}$, so that $\text{crit}(E) < \rho(M|\langle\beta_i, k_i\rangle)$ for all $i < n$, so by our case hypothesis, $\text{crit}(F) < \rho(M|\langle\beta_i, k_i\rangle)$ for all $i < n$. Thus we have

$$\pi_i : M|\langle\beta_i, k_i\rangle \to \text{Ult}(M|\langle\beta_i, k_i\rangle, F)$$

for all $i \leq n$. We have

$$\pi_n = i_F^M$$

and $\mathrm{Ult}(M|\langle\beta_n, k_n\rangle, F) = \mathrm{Ult}_0(M, F) = P$. So $R = \mathrm{Ult}(P, \pi_n(E))$ was the last model of our embedding normalization.

Claim 3.2 *$Q = \mathrm{Ult}(P, \pi_0(E))$.*

Proof. $\mathrm{lh}(E)$ is a regular cardinal in N. So

$$\pi_0 = i_F^{M\|\mathrm{lh}(E)} = i_F^N \restriction N\|\,\mathrm{lh}(E),$$

and thus

$$\pi_0(\nu(E)) = i_F^N(\nu(E)).$$

Let

$$L = (\text{extender of } i^P_{\pi_0(E)} \circ i^M_F)\restriction i^N_F(\nu(E)),$$

then it is easy to see that

$$\mathrm{Ult}(P, \pi_0(E)) = \mathrm{Ult}(M, L).$$

Recall that G was the extender of length $i_F^N(\nu(E))$ given by $i_F^N \circ i_E^M$. As before, we get $\bar{\sigma} : \mathrm{lh}(G) \to \mathrm{lh}(L)$ by

$$\bar{\sigma}([b, g]^N_F) = [b, g]_F^{M\|\,\mathrm{lh}(E)},$$

defined for $b \in [\nu(E)]^{<\omega}$ and $g : [\mu]^{|b|} \to \nu(E)$ with $g \in N$. (We assume here $k(M) = k(N) = 0$; otherwise replace M and N by their $k(M)$-reducts.) But all such g are in $M\|\,\mathrm{lh}(E)$, so

$$\bar{\sigma} = \text{identity}.$$

As before, we get that G is a subextender of L under $\bar{\sigma}$, but this just means that $G = L$, proving Claim 3.2. $\square$

Claim 3.3 *For $0 \le i \le n$, $\mathrm{Ult}(M|\langle\beta_i, k_i\rangle, F)$ is an initial segment of P.*

Proof. $\mathrm{Ult}(M|\langle\beta_n, k_n\rangle, F) = P$. Now suppose $\mathrm{Ult}(M|\langle\beta_{i+1}, k_{i+1}\rangle, F)$ is an initial segment of P. So then $\pi_{i+1}(M|\langle\beta_i, k_i\rangle)$ is an initial segment of P. It will suffice to show $\mathrm{Ult}(M|\langle\beta_i, k_i\rangle, F) \trianglelefteq \pi_{i+1}(M|\langle\beta_i, k_i\rangle)$. But consider the factor map

$$\psi : \mathrm{Ult}(M|\langle\beta_i, k_i\rangle, F) \to \pi_{i+1}(M|\langle\beta_i, k_i\rangle)$$

given by

$$\psi([a, f]_F^{M|\beta_i}) = [a, f]_F^{M|\beta_{i+1}}$$

for f a function given by a $r\Sigma_{k_i}$-Skolem term interpreted over $M|\beta_i$. For simplicity, let us assume $k_i = k_{i+1} = 0$, so this just amounts to $f \in M|\beta_i$. Let $\rho = \rho_i^*$; that is, assuming $k_i = 0$, let

$$\begin{aligned}\rho &= \rho_1(M|\beta_i),\\ p &= p_1(M|\beta_i),\\ S &= \mathrm{Ult}(M|\beta_i, F).\end{aligned}$$

So $\psi : S \to \pi_{i+1}(M|\beta_i)$. Now ρ is still a cardinal in $M|\beta_{i+1}$. So $({}^{\mu}\alpha)^{M|\beta_i} = ({}^{\mu}\alpha)^{M|\beta_{i+1}}$ for all $\alpha < \rho$. So

$$\mathrm{crit}(\psi) \geq \sup \pi_i``\rho.$$

Also,

$$S = \mathrm{Hull}_1^S(\sup \pi_i``\rho \cup \{\pi_i(p)\}),$$

as is easily checked. So $\rho_1(S) \leq \sup \pi_i``\rho$. Using the solidity witnesses, it is easy to see that

$$\rho_1(S) = \sup \pi_i``\rho \quad \text{and} \quad p_1(S) = \pi_i(p).$$

We can apply condensation to ψ to see that $S \trianglelefteq \pi_{i+1}(M|\beta_i)$ once we show that $\sup \pi_i``\rho$ is not an index of an extender on the $\pi_{i+1}(M|\beta_i)$-sequence.

Suppose it were. Then $\sup \pi_i``\rho$ is not a cardinal of $\pi_{i+1}(M|\beta_i)$, so $\mathrm{crit}(\psi) = \sup \pi_i``\rho$. This implies that π_{i+1} is discontinuous at ρ and that

$$M|\beta_{i+1} \models \mathrm{cof}(\rho) = \mu.$$

But then

$$\mathrm{Ult}(M|\beta_{i+1}, F) \models \mathrm{cof}(\sup \pi_i``\rho) = \mu.$$

But indices of extenders have successor cardinal cofinalities, and μ is a limit cardinal in $\mathrm{Ult}(M|\beta_{i+1}, F)$, so $\sup \pi_i``\rho$ is not an index in $\mathrm{Ult}(M|\beta_{i+1}, F)$-sequence. Therefore it is not an index in the $\pi_{i+1}(M|\beta_i)$-sequence. $\square$

By Claim 3.3, $\pi_0(E)$ is on the sequence of P. Thus our full normalization of $\mathcal{T}$ is the tree $\mathcal{S}$, where

$$\mathcal{M}_0^{\mathcal{S}} = M,\ \mathcal{M}_1^{\mathcal{S}} = N,\ \mathcal{M}_2^{\mathcal{S}} = P,\ \mathcal{M}_3^{\mathcal{S}} = Q,$$

and

$$E_0^{\mathcal{S}} = E,\ E_1^{\mathcal{S}} = F,\ E_2^{\mathcal{S}} = \pi_0(E).$$

Again, this assumes $\mathrm{lh}(F) \geq \mathrm{lh}(E)$. Otherwise it is $E_0^{\mathcal{S}} = F$ and $E_1^{\mathcal{S}} = \pi_0(E)$.

The following diagram summarizes Case 1.

$$\begin{tikzcd} N \arrow[r, "i_F^N"] & Q \arrow[r, "\tau"] & R \\ M \arrow[u, "E"] \arrow[r, "i_F^M"] & P \arrow[u, "i_F^N(E)"] \arrow[ur, "i_F^M(E)"'] & \end{tikzcd}$$

Here $i_F^N(E) = \pi_0(E)$. The notation is justified because $(N|\operatorname{lh}(E), E) = M|\operatorname{lh}(E)$, so i_F^N moves E as an amenable predicate, and produces thereby what we called $\pi_0(E)$. The construction in Claim 3.3 shows that in fact $i_F^N(E)$ is a subextender of $i_F^M(E)$ under the map $\sigma : i_F^N(\nu(E)) \to i_F^M(\nu(E))$ we identified earlier, $\sigma([b, g]_F^N) = [b, g]_F^M$ for $g : [\mu]^{|b|} \to \nu(E)$ with g in N.

Remark 3.4 All embeddings in the diagram above are all elementary and cofinal. All but τ are ultrapower embeddings. τ is easily seen to be weakly elementary, and it is cofinal because all the other embeddings are cofinal.

Remark 3.5 If G is the extender of $i_F^N \circ i_E^M$, then in fact $\nu(G) = \sup i_F^N``\nu(E)$, as shown by our earlier calculation. So $\nu(i_F^N(E)) = \sup i_F^N``\nu(E)$.

Remark 3.6 Let us consider the case that $\nu(E)$ is a cardinal in M. Then $({}^{\mu}\alpha)^M = ({}^{\mu}\alpha)^N$ for all $\alpha < \nu(E)$, so for σ as above, $\sigma \restriction \sup i_F^N``\nu(E) = \text{identity}$. Thus $i_F^N(E)$ is the trivial completion of $i_F^M(E) \restriction \sup i_F^M``\nu(E)$. If i_F^M is continuous at $\nu(E)$ (i.e. $\operatorname{cof}^M(\nu(E)) \neq \mu$), then $i_F^N(E) = i_F^M(E)$ and $Q = R$. If i_F^M is discontinuous at $\nu(E)$ (i.e. $\operatorname{cof}^M(\nu(E)) = \mu$), then $Q \neq R$, and in fact $\operatorname{crit}(\tau) = \sup i_F^M``\nu(E)$.

So in this case, the embedding normalization of $\mathcal{T}$ uses $i_F^M(E)$ to continue from P, while the full normalization may use a proper initial segment of $i_F^M(E)$ to continue from P.

Case 2. $\operatorname{crit}(E) < \operatorname{crit}(F)$.

Let $\mu = \operatorname{crit}(F)$ and $\kappa = \operatorname{crit}(E)$. We have assumed $\mu < \lambda(E)$, as otherwise $\mathcal{T}$ is already normal. Let

$$P = \operatorname{Ult}(M|\langle \xi, k\rangle, F)$$

where $\langle \xi, k\rangle$ is lexicographically least such that $\rho(M|\langle \xi, k\rangle) \leq \mu$. Let

$$i : M|\langle \xi, k\rangle \to P$$

be the canonical embedding, $i = i_F^{M|\langle \xi, k\rangle}$. As in Case 1, $N = \operatorname{Ult}(M, E)$ and $Q = \operatorname{Ult}(N, F)$. The embedding normalization of $\mathcal{T}$ continues from M, N (assuming

$\mathrm{lh}(E) < \mathrm{lh}(F)$), and then P by using $i(E)$ now. Note $i(E)$ should be applied to M, not P, in a normal tree. So let

$$R = \mathrm{Ult}(M, i(E)).$$

Let G be the extender of $i_F^N \circ i_E^M$, and notice that G is short, with $\lambda(G) = i_F^N(\lambda(E)) = \sup i_F^N``\lambda(E)$. Let

$$\sigma : i_F^N(\lambda(E)) \to i_F^{M|\langle\xi,k\rangle}(\lambda(E))$$

be given by

$$\sigma([b,g]_F^N) = [b,g]_F^{M|\langle\xi,k\rangle},$$

for $g : [\mu]^{|b|} \to \lambda(E)$ with $g \in N$. (Note that for $n = k(M) = k(N)$, we have $\kappa < \rho_n(M)$, so $\lambda(E) < \rho_n(N)$, so every $r\Sigma_n^N$ such function g belongs to N.) We claim that

Claim 3.7 *G is a subextender of $i(E)$ under σ.*

Remark 3.8 *In this case, G and $i(E)$are short, and σ is the identity on their common domain.*

Proof. Let $a \subseteq \sup i_F^N``\lambda(E)$ be finite, and let $A \subseteq [\kappa]^{|a|}$ be in M. Let $a = [b,g]_F^N$, where $g \in N$ and $g : [\mu]^{|b|} \to [\nu(E)]^{|a|}$. Then

$$\begin{aligned}
(a, A) \in G \quad &\text{iff} \quad ([b,g]_F^N, A) \in G \\
&\text{iff} \quad [b,g]_F^N \in i_F^N \circ i_E^M(A) \\
&\text{iff} \quad \text{for } F_b \text{ a.e. } \bar{\mu},\ g(\bar{\mu}) \in i_E^M(A) \\
&\text{iff} \quad \text{for } F_b \text{ a.e. } \bar{\mu},\ (g(\bar{\mu}), A) \in E \\
&\text{iff} \quad ([b,g]_F^{M|\xi}, A) \in i(E) \\
&\text{iff} \quad (\sigma(a), A) \in i(E).
\end{aligned}$$

□

Thus we have a factor map $\tau : Q \to R$ from $Q = \mathrm{Ult}(M, G)$ to $\mathrm{Ult}(M, i(E))$ given by

$$\tau([a,f]_G^M) = [\sigma(a), f]_{i(E)}^M.$$

Assuming $\mathrm{lh}(E) < \mathrm{lh}(F)$, the embedding normalization of $\mathcal{T}$ is then $\mathcal{U}$, where

$$E_0^{\mathcal{U}} = E,\ E_1^{\mathcal{U}} = F,\ E_2^{\mathcal{U}} = i(E).$$

If $\mathrm{lh}(F) < \mathrm{lh}(E)$, it is $E_0^{\mathcal{U}} = F$, $E_1^{\mathcal{U}} = i(E)$.

The full normalization is obtained as in Case 1. Let

$$\pi_0 : M||\,\mathrm{lh}(E) \to \mathrm{Ult}(M||\,\mathrm{lh}(E), F)$$

be the canonical embedding. Letting $\bar{\sigma}([b, g]_F^N) = [b, g]_F^{M||\,\mathrm{lh}(E)}$ for b, g as above, we have $\bar{\sigma} =$ identity, which yields $G = \pi_0(E)$. One can show that $\pi_0(E)$ is on the P-sequence by considering the $\mathrm{lh}(E)$-dropdown sequence of $M|\xi$ and using condensation, as in Case 1.

The situation in Case 2 is summarized by the diagram

$$\begin{tikzcd}
N \arrow[r, "i_F^N"] & Q \arrow[r, "\tau"] & R \\
M \arrow[u, "E"] \arrow[ur, "i_F^N(E)"] \arrow[urr, "i_F^{M|\xi}(E)"'] & & \\
M|\xi \arrow[u, "\trianglelefteq"] \arrow[r, "i_F^{M|\xi}"'] & P &
\end{tikzcd}$$

We have assumed here $k = 0$ to remove some clutter. Again, all the embeddings in the diagram are cofinal and elementary. In the case of τ, this is because it is weakly elementary, and it is cofinal because all the other embeddings are cofinal.

Remark 3.9 If $\langle \xi, k \rangle = \langle \mathrm{lh}(E), 0 \rangle$, then $i_F^{M|\xi} = i_F^N \restriction N|\,\mathrm{lh}(E)$, so $i_F^N(E) = i_F^{M|\xi}(E)$, and $Q = R$. This is what happens if $\nu(E) \leq \mathrm{crit}(F) < \lambda(E)$. The original $\mathcal{T}$ is ms-normal but not Jensen normal. Its embedding normalization is Jensen normal, and has the same last model as $\mathcal{T}$.

If $\langle \xi, k \rangle = l(M)$, then the diagram simplifies to

$$\begin{tikzcd}
N \arrow[r, "i_F^N"] & Q \arrow[r, "\tau"] & R \\
M \arrow[u, "E"] \arrow[ur, "i_F^N(E)"] \arrow[urr, "i_F^M(E)"'] \arrow[r, "i_F^M"'] & P &
\end{tikzcd}$$

If $\mu < \nu(E)$ and $\nu(E)$ is a cardinal of M and $\langle \xi, k \rangle = l(M)$, then $i_F^N(E)$ is the trivial completion of $i_F^M(E) \restriction \sup i_F^N``\nu(E)$. In this case, $Q = R$ iff $\mathrm{cof}^M(\nu(E)) \neq \mu$, and if $Q \neq R$, then $\mathrm{crit}(\tau) = \sup i_F^N``\nu(E)$.

Remark 3.10 In both cases, the embedding normalization of $\langle \langle E \rangle, \langle F \rangle \rangle$ may break down by reaching an illfounded model. Similarly for full normalization. (There we also used condensation, hence indirectly iterability.)

Again we are interested in the case M has an iteration strategy Σ. In that case, the models are all wellfounded, and things work out as above. It doesn't yet matter what Σ is, since the trees are finite.

3.2 Normalizing $\mathcal{T}^\frown\langle F \rangle$

Let M be a premouse, let $\mathcal{T}$ a normal tree on M having last model N, and let F be on the N-sequence. Let Q be the longest initial segment of N such that $\mathrm{Ult}(Q, F)$ makes sense, that is, such that F is total on Q and $\mathrm{crit}(F) < \rho_{k(Q)}(Q)$. We construct a normal tree $\mathcal{W}$ on M such that $\mathrm{Ult}(Q, F)$ embeds into the last model of $\mathcal{W}$ via a weakly elementary map. We call $\mathcal{W}$ the *embedding normalization of* $\mathcal{T}^\frown\langle F \rangle$, and write

$$\mathcal{W} = W(\mathcal{T}, F).$$

The reader can find some diagrams which may help visualize the construction of $\mathcal{W}$ at the end of this section.

Let α be least such that F is on the $\mathcal{M}_\alpha^\mathcal{T}$-sequence. Then $\mathcal{M}_\alpha^\mathcal{T}$ agrees with Q up to $\mathrm{lh}(F) + 1$, and Q agrees with $\mathrm{Ult}(Q, F)$ up to $\mathrm{lh}(F)$, but not $\mathrm{lh}(F) + 1$. By Fact 3.0, $\mathcal{W}$ must start out with $\mathcal{T} \restriction (\alpha + 1)$, if it is being played by some iteration strategy Σ for M such that $\mathcal{T} \restriction (\alpha + 1)$ is played by Σ. This is the context that is motivating our definition of $\mathcal{W}$, so we set

$$\mathcal{W} \restriction (\alpha + 1) = \mathcal{T} \restriction (\alpha + 1).$$

(This does *not* imply $E_\alpha^\mathcal{W} = E_\alpha^\mathcal{T}$, just $\mathcal{M}_\alpha^\mathcal{W} = \mathcal{M}_\alpha^\mathcal{T}$.)

Now let $\mu = \mathrm{crit}(F)$, and let $\beta \leq \alpha$ be least such that either $\mu < \lambda(E_\beta^\mathcal{T})$, or $\beta = \alpha$. F must be applied to an initial segment of $\mathcal{M}_\beta^\mathcal{W} = \mathcal{M}_\beta^\mathcal{T}$ in $\mathcal{W}$. That is

$$E_\alpha^\mathcal{W} = F,$$

and the rest is dictated by normality:

$$W\text{-pred}(\alpha + 1) = \beta,$$

and

$$\mathcal{M}^{*,\mathcal{W}}_{\alpha+1} = \mathcal{M}^{\mathcal{T}}_{\beta}|\langle \xi_0, k_0\rangle$$

where $\langle \xi_0, k_0\rangle$ is least such that $\rho(\mathcal{M}_\beta|\langle \xi_0, k_0\rangle) \leq \mu$ or $\langle \xi_0, k_0\rangle = l(\mathcal{M}^{\mathcal{W}}_{\beta})$, and

$$\mathcal{M}^{\mathcal{W}}_{\alpha+1} = \mathrm{Ult}(\mathcal{M}^{*,\mathcal{W}}_{\alpha+1}, F).$$

This gives us $\mathcal{W}{\restriction}(\alpha+2)$.

Case 1. $Q \neq N$.

If $\beta+1 < \mathrm{lh}(\mathcal{T})$, then Q is a proper initial segment of $\mathcal{M}^{\mathcal{T}}_{\beta}|\,\mathrm{lh}(E^{\mathcal{T}}_{\beta})$, by the following claim.

Claim 3.11 *Let $\mathcal{T}$ be a normal iteration tree, $\beta+1 < \mathrm{lh}(\mathcal{T})$, and $\mathcal{M}^{\mathcal{T}}_{\beta}|\,\mathrm{lh}(E^{\mathcal{T}}_{\beta}) \trianglelefteq R \trianglelefteq \mathcal{M}^{\mathcal{T}}_{\theta}$ for some $\theta \geq \beta+1$; then $\mathrm{lh}(E^{\mathcal{T}}_{\beta}) \leq \rho_{k(R)}(R)$.*

Proof. Let $S = \mathcal{M}^{\mathcal{T}}_{\theta}$. It is easy to see that $\rho_{k(S)}(S) \geq \mathrm{lh}(G)$ for all extenders G used in the branch $[0,\theta)_T$. Since some G with $\mathrm{lh}(G) \geq \mathrm{lh}(E^{\mathcal{T}}_{\beta})$ was used in $[0,\theta)_T$, we are done if $R = S$. If $\hat{o}(R) = \hat{o}(S)$ but $k(R) < k(S)$, then $\rho_{k(S)}(S) \leq \rho_{k(R)}(R)$, so again we are done. Finally, if $\hat{o}(R) < \hat{o}(S)$, then $R \in S$, so $\rho_{k(R)}(R) < \mathrm{lh}(E^{\mathcal{T}}_{\beta}) \leq o(R)$ implies that $\mathrm{lh}(E^{\mathcal{T}}_{\beta})$ is not a cardinal in S. This is a contradiction. □

Let $N = \mathcal{M}^{\mathcal{T}}_{\theta}$ and $Q = N|\langle \xi, k\rangle$. If $\beta < \theta$, then we apply the claim to $R = N|\langle \xi, k+1\rangle$. We have $Q \lhd N$, so this makes sense. We have $\rho(Q) = \rho_{k(R)}(R) \leq \mu < \mathrm{lh}(E^{\mathcal{T}}_{\beta})$. It follows from the claim that $R \lhd \mathcal{M}^{\mathcal{T}}_{\beta}|\,\mathrm{lh}(E^{\mathcal{T}}_{\beta})$. But $Q \lhd R$. Thus Q is a proper initial segment of $\mathcal{M}^{\mathcal{T}}_{\beta}|\,\mathrm{lh}(E^{\mathcal{T}}_{\beta})$.

So if $\beta+1 < \mathrm{lh}(\mathcal{T})$, then $Q = \mathcal{M}_\beta|\langle \xi_0, k_0\rangle \trianglelefteq \mathcal{M}^{\mathcal{T}}_{\beta}$, $\alpha = \beta$, and $\mathcal{M}^{\mathcal{W}}_{\alpha+1} = \mathrm{Ult}(Q, F)$. These conclusions hold trivially if $\beta+1 = \mathrm{lh}(\mathcal{T})$, so in either case we set

$$\begin{aligned} W(\mathcal{T}, F) &= \mathcal{W}{\restriction}(\alpha+2) \\ &= \mathcal{T}{\restriction}(\beta+1)^\frown\langle F\rangle. \end{aligned}$$

We call this the *dropping case* in the definition of $W(\mathcal{T}, F)$. In this case, $\mathrm{Ult}(Q, F)$ is actually equal to the last model of $W(\mathcal{T}, F)$.

Case 2. $Q = N$, and $\mathrm{lh}(\mathcal{T}) = \beta+1$.

Since $\mathrm{lh}(\mathcal{T}) = \beta+1$, then $Q = N = \mathcal{M}^{\mathcal{T}}_{\beta}$. Thus $\alpha = \beta$, and again

$$\begin{aligned} W(\mathcal{T}, F) &= \mathcal{W}{\restriction}(\alpha+2) \\ &= \mathcal{T}{\restriction}(\beta+1)^\frown\langle F\rangle. \end{aligned}$$

Again, $\mathrm{Ult}(Q, F)$ is actually equal to the last model of $W(\mathcal{T}, F)$. The difference between this and the previous case is just that we did not drop when we applied F to $\mathcal{T}$.

Case 3. $Q = N$, and $\mathrm{lh}(\mathcal{T}) > \beta + 1$.

In this case, $\mathrm{Ult}(N, F)$ makes sense, so $\langle \mathrm{lh}(E^{\mathcal{T}}_\beta), 0\rangle \leq \langle \xi_0, k_0\rangle$, and in fact $\mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_\eta, F)$ makes sense for all η such that $\beta < \eta < \mathrm{lh}(\mathcal{T})$.

For $\eta < \mathrm{lh}(\mathcal{T})$, set

$$\phi(\eta) = \begin{cases} \eta, & \text{if } \eta < \beta; \\ (\alpha+1) + (\eta - \beta), & \text{if } \eta \geq \beta. \end{cases}$$

So $\phi : [0, \mathrm{lh}(\mathcal{T})) \cong [0, \beta) \cup [\alpha+1, (\alpha+1) + (\mathrm{lh}(\mathcal{T}) - \beta))$ order-preservingly. We define $\mathcal{M}^{\mathcal{W}}_{\phi(\eta)}$, and

$$\pi_\eta : \mathcal{M}^{\mathcal{T}}_\eta \to \mathcal{M}^{\mathcal{W}}_{\phi(\eta)}.$$

For $\eta < \beta$, $\phi(\eta) = \eta$ and $\mathcal{M}^{\mathcal{T}}_\eta = \mathcal{M}^{\mathcal{W}}_\eta$ and $\pi_\eta = \text{identity}$. We let

$$\pi_\beta = \text{canonical embedding of } \mathcal{M}^{\mathcal{T}}_\beta | \langle \xi_0, k_0\rangle \text{ into } \mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_\beta | \langle \xi_0, k_0\rangle, F).$$

(So the display above is a bit off; for $\eta = \beta$, π_η may not act on all of $\mathcal{M}^{\mathcal{T}}_\eta$. For $\eta \neq \beta$, π_η will act on all of $\mathcal{M}^{\mathcal{T}}_\eta$.) Note that F is close to $\mathcal{M}^{\mathcal{T}}_\beta | \langle \xi_0, k_0\rangle$ because it arose in a later model of $\mathcal{T}$, so that π_β is cofinal and elementary.

We define π_η and $\mathcal{M}^{\mathcal{W}}_{\phi(\eta)}$ for $\eta \geq \beta + 1$ by induction.

For $\eta = \beta + 1$, we let

$$E^{\mathcal{W}}_{\phi(\beta)} = \pi_\beta(E^{\mathcal{T}}_\beta),$$

and let $\tau \leq \beta$ be least such that $\mathrm{crit}(E^{\mathcal{W}}_{\phi(\beta)}) < \lambda(E^{\mathcal{W}}_\tau)$, and $\langle \gamma, k\rangle$ be least such that $\mathrm{crit}(E^{\mathcal{W}}_{\phi(\beta)}) \geq \rho_{k+1}(\mathcal{M}^{\mathcal{W}}_\tau | \gamma)$, and set

$$\mathcal{M}^{\mathcal{W}}_{\phi(\beta+1)} = \mathrm{Ult}(\mathcal{M}^{\mathcal{W}}_\tau | \langle \gamma, k\rangle, E^{\mathcal{W}}_{\phi(\beta)}),$$

as required by normality. We get $\pi_{\beta+1}$ from the Shift Lemma. There are two cases.

Case A. $\mathrm{crit}(E^{\mathcal{T}}_\beta) \geq \mu$.

Since $\pi_\beta = i^{M_\beta | \langle \xi_0, k_0\rangle}_F$, $\mathrm{crit}(\pi_\beta(E^{\mathcal{T}}_\beta)) > \mathrm{lh}(F)$. But $F = E^{\mathcal{W}}_\alpha$. Thus $\pi_\beta(E^{\mathcal{T}}_\beta) = E^{\mathcal{W}}_{\phi(\beta)}$ is applied to $\mathcal{M}^{\mathcal{W}}_{\alpha+1} = \mathcal{M}^{\mathcal{W}}_{\phi(\beta)}$, or an initial segment of it. That is

$$\tau = \phi(\beta) = \alpha + 1$$

in this case. In $\mathcal{T}$, we must have

$$T\text{-pred}(\beta+1)=\beta,$$

because β was least such that $\mu<\nu(E^{\mathcal{T}}_\beta)$. Similarily, the case hypothesis implies that

$$\mathcal{M}^{\mathcal{T}}_{\beta+1}=\mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_\beta|\langle\xi_1,k_1\rangle,E^{\mathcal{T}}_\beta)$$

where $\langle\xi_1,k_1\rangle\leq_{\mathrm{lex}}\langle\xi_0,k_0\rangle$. We have that $\pi_\beta:\mathcal{M}^{\mathcal{T}}_\beta|\langle\xi_1,k_1\rangle\to\pi_\beta(\mathcal{M}^{\mathcal{T}}_\beta|\langle\xi_1,k_1\rangle)$ is elementary, so when $k_1=0$ we can set

$$\pi_{\beta+1}([a,f]^{M^{\mathcal{T}}_\beta|\xi_1}_{E^{\mathcal{T}}_\beta})=[\pi_\beta(a),\pi_\beta(f)]^{\mathcal{M}^{\mathcal{W}}_{\phi(\beta)}|\pi_\beta(\xi_1)}_{E^{\mathcal{W}}_{\phi(\beta)}}$$

as in the Shift Lemma. (If $k_1>0$, the ultrapowers are decoded from ultrapowers of reducts, but the Shift Lemma still applies. In the notation of [23], $\pi_\beta(f^{\mathcal{M}^{\mathcal{T}}_\beta|\xi_1}_{\tau,q})=f^{\mathcal{M}^{\mathcal{W}}_{\phi(\beta)}|\pi_\beta(\xi_1)}_{\tau,\pi_\beta(q)}$.) We have that $\pi_{\beta+1}$ is elementary (a near k_1-embedding) by [36], and $\pi_{\beta+1}\restriction\mathrm{lh}(E^{\mathcal{T}}_\beta)+1=\pi_\beta\restriction\mathrm{lh}(E^{\mathcal{T}}_\beta)$.

Case B. $\mathrm{crit}(E^{\mathcal{T}}_\beta)<\mu$.

Then $\mathrm{crit}(\pi_\beta(E^{\mathcal{T}}_\beta))=\mathrm{crit}(E^{\mathcal{T}}_\beta)$, so $\tau=T\text{-pred}(\beta+1)=W\text{-pred}(\phi(\beta+1))$. It is clear that $E^{\mathcal{T}}_\beta$ and $\pi_\beta(E^{\mathcal{T}}_\beta)$ are applied to the same initial segment of $\mathcal{M}^{\mathcal{T}}_\tau=\mathcal{M}^{\mathcal{W}}_\tau$. Letting this be $\mathcal{M}^{\mathcal{T}}_\tau|\langle\gamma,k\rangle$, we get

$$\pi_{\beta+1}:\mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_\tau|\langle\gamma,k\rangle,E^{\mathcal{T}}_\beta)\to\mathrm{Ult}(\mathcal{M}^{\mathcal{W}}_\tau|\langle\gamma,k\rangle,\pi_\beta(E^{\mathcal{T}}_\beta))$$

from

$$\pi_{\beta+1}([a,f]^{\mathcal{M}^{\mathcal{T}}_\tau|\gamma}_{E^{\mathcal{T}}_\beta})=[\pi_\beta(a),f]^{\mathcal{M}^{\mathcal{T}}_\tau|\gamma}_{E^{\mathcal{W}}_{\phi(\beta)}}.$$

Again, $\pi_{\beta+1}$ is elementary, and $\pi_{\beta+1}$ agrees with π_β on $\mathrm{lh}(E^{\mathcal{T}}_\beta)+1$.

Remark 3.12 In Case A, $\phi(T\text{-pred}(\beta+1))=W\text{-pred}(\phi(\beta+1))$, while in Case B, this fails, and in fact $T\text{-pred}(\beta+1)=W\text{-pred}(\beta+1)$. It is because ϕ may not preserve point-of-application for extenders that $\mathcal{T}$ may not be a hull of $\mathcal{W}$, under ϕ and the π_η's, in the sense of Sargsyan's thesis [30]. In fact, $\mathcal{T}$ will be such a hull iff $\mathrm{crit}(E^{\mathcal{T}}_\eta)\geq\mu$ for all $\eta\geq_T\beta$. For example, this happens when $\mathcal{T}$ factors as $\mathcal{T}\restriction(\beta+1)^\frown\mathcal{S}$, where $\mathcal{S}$ is a tree on $\mathcal{M}^{\mathcal{T}}_\beta$ with all critical points $\geq\mu$.

The successor case when $\eta>\beta$ is similar. Suppose by induction that whenever $\xi,\delta\leq\eta$:

(1) $E^{\mathcal{W}}_{\phi(\delta)} = \pi_\delta(E^{\mathcal{T}}_\delta)$.

(2) if $\delta \neq \beta$, then π_δ is an elementary embedding from $\mathcal{M}^{\mathcal{T}}_\delta$ to $\mathcal{M}^{\mathcal{W}}_{\phi(\delta)}$. ($\pi_\beta$ is cofinal elementary from $\mathcal{M}^{\mathcal{T}}_\beta|\langle \xi_0, k_0\rangle$ to $\mathcal{M}^{\mathcal{W}}_{\phi(\beta)}$.)

(3) if $\xi < \delta$, then π_δ agrees with π_ξ on $\mathrm{lh}(E^{\mathcal{T}}_\xi) + 1$.

(4) (a) if T-pred$(\delta) \neq \beta$ then $\phi(T\text{-pred}(\delta)) = W\text{-pred}(\phi(\delta))$

(b) if T-pred$(\delta) = \beta$, then

i. $\mathrm{crit}(E^{\mathcal{T}}_{\delta-1}) \geq \mu \Rightarrow \phi(T\text{-pred}(\delta)) = W\text{-pred}(\phi(\delta))$

ii. $\mathrm{crit}(E^{\mathcal{T}}_{\delta-1}) < \mu \Rightarrow W\text{-pred}(\phi(\delta)) = \beta$

(c) i. if $\delta \neq \beta$, then ($\delta \, T \, \xi$ iff $\phi(\delta) \, W \, \phi(\xi)$)

ii. $\beta \, T \, \xi \Rightarrow \phi(\beta) \, W \, \phi(\xi)$ iff the first extender used in $(\beta, \xi]_T$ has critical point $\geq \mu$.

(5) (a) if $\delta \neq \beta$, then $\delta \in D^{\mathcal{T}}$ iff $\phi(\delta) \in D^{\mathcal{W}}$, and $\deg^{\mathcal{T}}(\delta) = \deg^{\mathcal{W}}(\phi(\delta))$

(b) if $\delta \neq \beta$, $\delta \, T \, \xi$, and $D^{\mathcal{T}} \cap (\xi, \delta]_T = \varnothing$, then $\pi_\xi \circ i^{\mathcal{T}}_{\delta,\xi} = i^{\mathcal{W}}_{\phi(\delta),\phi(\xi)} \circ \pi_\delta$

we then define $\pi_{\eta+1} : \mathcal{M}^{\mathcal{T}}_{\eta+1} \to \mathcal{M}^{\mathcal{W}}_{\phi(\eta+1)}$ so as to maintain those conditions. Namely,

$$E^{\mathcal{W}}_{\phi(\eta)} = \pi_\eta(E^{\mathcal{T}}_\eta),$$

and letting τ be least such that $\mathrm{crit}(E^{\mathcal{W}}_{\phi(\eta)}) < \lambda(E^{\mathcal{W}}_\tau)$, and $\langle \gamma, k\rangle$ be appropriate for normal trees,

$$\mathcal{M}^{\mathcal{W}}_{\phi(\eta+1)} = \mathrm{Ult}(\mathcal{M}^{\mathcal{W}}_\tau|\langle \gamma, k\rangle, E^{\mathcal{W}}_{\phi(\eta)}).$$

We get $\pi_{\eta+1}$ from the Shift Lemma, with two cases, as before.

Case A. $\mathrm{crit}(E^{\mathcal{T}}_\eta) \geq \mu$.

Let $\sigma = T\text{-pred}(\eta + 1)$, i.e. σ is least such that $\mathrm{crit}(E^{\mathcal{T}}_\eta) < \lambda(E^{\mathcal{T}}_\sigma)$. Clauses (1) and (3) above tell us that $\phi(\sigma)$ is the least θ in $\mathrm{ran}(\phi)$ such that $\mathrm{crit}(E^{\mathcal{W}}_{\phi(\eta)}) < \lambda(E^{\mathcal{W}}_\theta)$. But $\tau \geq \phi(\beta)$ by our case hypotheses, so $\tau \in \mathrm{ran}(\phi)$, so $\tau = \phi(\sigma)$. We leave it to the reader to show that if

$$\mathcal{M}^{\mathcal{T}}_{\eta+1} = \mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_\sigma|\langle \lambda, i\rangle, E^{\mathcal{T}}_\eta),$$

then in fact $i = k$, and $\pi_\sigma(\lambda) = \gamma$. Thus we set

$$\pi_{\eta+1}([a, f]^{\mathcal{M}^{\mathcal{T}}_\sigma|\lambda}_{E^{\mathcal{T}}_\eta}) = [\pi_\eta(a), \pi_\sigma(f)]^{\mathcal{M}^{\mathcal{W}}_\tau|\gamma}_{E^{\mathcal{W}}_{\phi(\eta)}},$$

and everything works out so that (1)-(5) still hold.

Case B. $\mathrm{crit}(E^{\mathcal{T}}_\eta) < \mu$.

Again, let $\sigma = T\text{-pred}(\eta+1)$. So $\sigma \leq \beta$. Since $\pi_\eta \restriction \mathrm{lh}(E^{\mathcal{T}}_\beta) = \pi_\beta \restriction \mathrm{lh}(E^{\mathcal{T}}_\beta)$, $\pi_\eta \restriction \mu =$ identity, so $\mathrm{crit}(E^{\mathcal{T}}_\eta) = \mathrm{crit}(E^{\mathcal{W}}_{\phi(\eta)})$. Thus $\sigma = \tau$. One can show that $E^{\mathcal{T}}_\eta$ and $E^{\mathcal{W}}_{\phi(\eta)}$ are applied to the same initial segment of $\mathcal{M}^{\mathcal{T}}_\tau = \mathcal{M}^{\mathcal{W}}_\tau$, via ultrapowers of the same degree. So we have

$$\pi_{\eta+1} : \mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_\tau | \langle \gamma, k \rangle, E^{\mathcal{T}}_\eta) \to \mathrm{Ult}(\mathcal{M}^{\mathcal{W}}_\tau | \langle \gamma, k \rangle, E^{\mathcal{W}}_{\phi(\eta)})$$

given by

$$\pi_{\eta+1}([a,f]^{\mathcal{M}^{\mathcal{T}}_\tau | \gamma}_{E^{\mathcal{T}}_\eta}) = [\pi_\eta(a), f]^{\mathcal{M}^{\mathcal{W}}_\tau | \gamma}_{E^{\mathcal{W}}_{\phi(\eta)}}.$$

The reader can check (1)-(5) still hold.

This finishes the definition of $\pi_{\eta+1}$. For λ a limit, $\mathcal{M}^{\mathcal{W}}_{\phi(\lambda)}$ and $\pi_\lambda : \mathcal{M}^{\mathcal{T}}_\lambda \to \mathcal{M}^{\mathcal{W}}_{\phi(\lambda)}$ are defined by

$$\mathcal{M}^{\mathcal{W}}_{\phi(\lambda)} = \mathrm{dir\,lim\ of\ } \mathcal{M}^{\mathcal{W}}_{\phi(\alpha)} \text{ for } \alpha \, T \, \lambda \text{ sufficiently large,}$$
$$\pi_\lambda(i^{\mathcal{T}}_{\alpha\lambda}(x)) = i^{\mathcal{W}}_{\phi(\alpha),\phi(\lambda)}(\pi_\alpha(x)), \text{ for } \alpha \, T \, \lambda \text{ sufficiently large.}$$

(1)-(5) imply this makes sense, and that (1)-(5) continue to hold. This completes our description of the embedding-normalization of $\mathcal{T}^\frown\langle F \rangle$.

We must see that for N the last model of $\mathcal{T}$ and R the last model of $\mathcal{W}$, $\mathrm{Ult}(N,F)$ embeds elementarily into R. But

Lemma 3.13 *For any $\gamma \geq \beta$, F is an initial segment of the extender of π_γ.*

Proof. F is the extender of π_β. Since $\pi_\beta \restriction (\mu^+)^{\mathcal{M}^{\mathcal{T}}_\beta | \xi} = \pi_\gamma \restriction (\mu^+)^{\mathcal{M}^{\mathcal{T}}_\beta | \xi}$ (because $(\mu^+)^{\mathcal{M}^{\mathcal{T}}_\beta | \xi} < \mathrm{lh}(E^{\mathcal{T}}_\beta)$), we are done. $\square$

Thus there is a natural factor embedding τ from $\mathrm{Ult}(N,F)$ into R, given by $\tau([a,f]^N_F) = \pi_\gamma(f)(a)$, where $N = \mathcal{M}^{\mathcal{T}}_\gamma$.

Lemma 3.14 *τ is weakly elementary.*

Proof. Let $n = k(N)$. Let G be the shortest initial segment of the extender of π_γ such that $\pi_\gamma(N^n) = \mathrm{Ult}_0(N^n, G)$. Then F is an initial segment of G, and $\tau \restriction \mathrm{Ult}_0(N^n, F)$ is Σ_0 elementary from $\mathrm{Ult}_0(N^n, F)$ to $\mathrm{Ult}_0(N^n, G)$, and Σ_1 elementary on $\mathrm{ran}(i_F^{N^n})$, which is cofinal in $\mathrm{Ult}_0(N^n, F)$. This implies that τ is $r\Sigma_n$ elementary, and $r\Sigma_{n+1}$ elementary on a set cofinal in $\rho_n(\mathrm{Ult}(N, F))$.

The remaining clauses in definition 2.8, concerning the preservation of parameters and projecta, follow from the fact that i_F^N and π_γ are weakly elementary, and $\tau \circ i_F^N = \pi_\gamma$.

□

Remark 3.15 We do not know whether τ must be fully elementary. The problem is that π_γ"$\rho_n(N)$ may not be cofinal in $\rho_n(R)$. If M-to-N does not drop in $\mathcal{T}$, then M-to-R does not drop in $\mathcal{W}$, and therefore π_γ is cofinal and elementary, so τ is cofinal and elementary. When M-to-N drops, τ may fail to be elementary, so far as we can see.

Remark 3.16 The definition of $W(\mathcal{T}, F)$ needs no change at all in the case that $\mathcal{T}$ is only weakly normal. In this case, $W(\mathcal{T}, F)$ will only be weakly normal itself, in general.

In a sufficiently coarse case, $\mathcal{W}$ is also the full normalization of $\langle \mathcal{T}, F\rangle$.

Remark 3.17 There is an analogous construction that starts with an ms-normal tree $\mathcal{T}$ on M, and an extender F on the sequence of its last model N, and produces an ms-normal tree $\mathcal{W}^{\mathrm{ms}}(\mathcal{T}, F)$ such that $\mathrm{Ult}(N, F)$ embeds into its last model.

We shall write $X(\mathcal{T}, F)$ for the full normalization of $\langle \mathcal{T}, F\rangle$. In a sufficiently coarse case, $X(\mathcal{T}, F) = W(\mathcal{T}, F)$.

Proposition 3.18 *Let M, $\mathcal{T}$, F, and β be as above. Suppose also that $\mathcal{T}$ is ms-normal, and that $k(M) = \omega$ and $\rho_\omega(M) = o(M)$. Let $\mu = \mathrm{crit}(F)$, and suppose that for all $\gamma + 1 < \mathrm{lh}(\mathcal{T})$,*

$$\mathcal{M}_\gamma^{\mathcal{T}} \models \nu(E_\gamma^{\mathcal{T}}) \textit{ is a cardinal of } \mathrm{cof} \neq \mu.$$

(So $\mathcal{T}$ does not drop anywhere, and all models have degree ω.) Then for all $\gamma < \mathrm{lh}\,\mathcal{T}$ such that $\gamma \geq \beta$

$$\mathcal{M}_{\phi(\gamma)}^{\mathcal{W}} = \mathrm{Ult}_\omega(\mathcal{M}_\gamma^{\mathcal{T}}, F),$$

and the embedding normalization map π_γ is the same as the F-ultrapower map.

Remark 3.19 A Jensen-normal tree that does not drop is ms-normal. We have stated the proposition using the weaker hypothesis of ms-normality because its greater generality may be useful, and anyway is natural in the coarse case.

Proof. We show this by induction on γ. For $\gamma = \beta$, this is the definition of $\mathcal{M}^{\mathcal{W}}_{\phi(\beta)}$ and π_β. Suppose it holds for all $\gamma \leq \eta$, we must show it holds at $\eta + 1$. Let $E = E^{\mathcal{T}}_\eta$ and $E^* = \pi_\eta(E) = E^{\mathcal{W}}_{\phi(\eta)}$. Let $\sigma = T\text{-pred}(\eta + 1)$.

Case 1. $\mu \leq \mathrm{crit}(E)$.

Then $\sigma \geq \beta$, and $\phi(\sigma) = W\text{-pred}(\phi(\eta + 1))$. Let $S = \mathrm{Ult}_\omega(\mathcal{M}^{\mathcal{T}}_{\eta+1}, F)$, and let $i_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}}$ be the canonical embedding. We have the diagram

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{T}}_{\eta+1} \arrow[r, "i_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}}"] & S \arrow[r, "\tau"] & \mathcal{M}^{\mathcal{W}}_{\phi(\eta+1)} \\
\mathcal{M}^{\mathcal{T}}_\sigma \arrow[u, "E"] \arrow[r, "i_F^{\mathcal{M}^{\mathcal{T}}_\sigma} = \pi_\sigma"'] & \mathcal{M}^{\mathcal{W}}_{\phi(\sigma)} \arrow[ur, "E^*"'] &
\end{tikzcd}$$

Here τ comes from the argument in Case 1 of two-step normalization. Namely, let G be the extender of $i_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}} \circ i_E^{\mathcal{M}^{\mathcal{T}}_\sigma}$, and H be the extender of $i_{E^*}^{\mathcal{M}^{\mathcal{W}}_{\phi(\sigma)}} \circ i_F^{\mathcal{M}^{\mathcal{T}}_\sigma}$. Note $\nu(G) = \sup i_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}} “\nu(E)$ and $\nu(H) = \sup i_F^{\mathcal{M}^{\mathcal{T}}_\eta} “\nu(E)$, by our cofinality assumption.

Claim 3.20 *G is a subextender of H under the map ψ, where*

$$\psi([b, g]_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}}) = [b, g]_F^{\mathcal{M}^{\mathcal{T}}_\eta},$$

for $b \in [\nu(F)]^{<\omega}$ and $g : [\mu]^{|b|} \to \nu(E)$, $g \in \mathcal{M}^{\mathcal{T}}_{\eta+1}$.

Proof. We calculate as before: for b, g as above and $A \subseteq [\mathrm{crit}(E)]^{<\omega}$ with $A \in \mathcal{M}^{\mathcal{T}}_\sigma$,

$$\begin{aligned}
([b, g]_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}}, A) \in G \quad &\text{iff} \quad [b, g]_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}} \in i_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}} \circ i_E^{\mathcal{M}^{\mathcal{T}}_\sigma}(A) \\
&\text{iff} \quad \text{for } F_b \text{ a.e. } u,\ g(u) \in i_E^{\mathcal{M}^{\mathcal{T}}_\sigma}(A)
\end{aligned}$$

(by Łos for $\mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_{\eta+1}, F)$)

$$\begin{aligned}
&\text{iff} \quad \text{for } F_b \text{ a.e. } u,\ (g(u), A) \in E \\
&\text{iff} \quad ([b, g]_F^{\mathcal{M}^{\mathcal{T}}_\eta}, i_F^{\mathcal{M}^{\mathcal{T}}_\eta}(A)) \in E^*
\end{aligned}$$

(by Łos for $\mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_\eta, F)$)

$$\text{iff} \quad [b, g]_F^{\mathcal{M}^{\mathcal{T}}_\eta} \in i_{E^*}^{\mathcal{M}^{\mathcal{T}}_{\phi(\sigma)}}(i_F^{\mathcal{M}^{\mathcal{T}}_\eta}(A))$$

(since $i_{E^*}^{\mathcal{M}^{\mathcal{T}}_{\phi(\sigma)}}$ and $i_{E^*}^{\mathcal{M}^{\mathcal{T}}_{\eta}}$ agree on subsets of $\mathrm{crit}(E^*)$)

$$\text{iff} \quad [b,g]_F^{\mathcal{M}^{\mathcal{T}}_{\eta}} \in i_{E^*}^{\mathcal{M}^{\mathcal{T}}_{\phi(\sigma)}}(i_F^{\mathcal{M}^{\mathcal{T}}_{\sigma}}(A))$$

(since $i_F^{\mathcal{M}^{\mathcal{T}}_{\eta}}$ agrees with π_η, hence π_γ, hence $i_F^{\mathcal{M}^{\mathcal{T}}_{\sigma}}$ on subsets of $\mathrm{crit}(E)$)

$$\text{iff} \quad ([b,g]_F^{\mathcal{M}^{\mathcal{T}}_{\eta}}, A) \in H.$$

□

But now $\mathcal{M}^{\mathcal{T}}_{\eta}$ and $\mathcal{M}^{\mathcal{T}}_{\eta+1}$ have the same functions $g : [\mu]^{<\omega} \to \nu(E)$, by our "coarseness" assumptions. So $\psi = \text{identity}$, and $G = H$, and $S = \mathcal{M}^{\mathcal{W}}_{\phi(\eta+1)}$. So our diagram is

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{T}}_{\eta+1} \arrow[r, "i_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}}"] \arrow[r, bend right, "\pi_{\eta+1}"'] & \mathcal{M}^{\mathcal{W}}_{\phi(\eta+1)} \\
\mathcal{M}^{\mathcal{T}}_{\sigma} \arrow[u, "E"] \arrow[r, "\pi_\sigma = i_F^{\mathcal{M}^{\mathcal{T}}_{\sigma}}"'] & \mathcal{M}^{\mathcal{W}}_{\phi(\sigma)} \arrow[u, "E^*"']
\end{tikzcd}$$

It remains to show $i_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}} = \pi_{\eta+1}$. Since both maps make the diagram commute, it is enough to show $i_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}} \restriction \nu(E) = \pi_{\eta+1} \restriction \nu(E)$. But $\pi_{\eta+1} \restriction \nu(E) = \pi_\eta \restriction \nu(E)$ by the Shift Lemma, and $\pi_\eta \restriction \nu(E) = i_F^{\mathcal{M}^{\mathcal{T}}_{\eta}} \restriction \nu(E)$ by induction, and $i_F^{\mathcal{M}^{\mathcal{T}}_{\eta}} \restriction \nu(E) = i_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}} \restriction \nu(E)$ because $\mathcal{M}^{\mathcal{T}}_{\eta}$ and $\mathcal{M}^{\mathcal{T}}_{\eta+1}$ have the same functions $g : [\mu]^{<\omega} \to \nu(E)$.

Case 2. $\mathrm{crit}(E) < \mu$.

Let $\sigma = T\text{-pred}(\eta+1)$. Then in this case, $\sigma = W\text{-pred}(\eta+1)$. Let $S = \mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_{\eta+1}, F)$. We have the diagram

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{T}}_{\eta+1} \arrow[r, "i_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}}"] & S \arrow[r, "\tau"] & \mathcal{M}^{\mathcal{W}}_{\phi(\eta+1)} \\
\mathcal{M}^{\mathcal{T}}_{\sigma} = \mathcal{M}^{\mathcal{W}}_{\sigma} \arrow[u, "E"] \arrow[urr, "E^*"'] & &
\end{tikzcd}$$

We show that $S = \mathcal{M}^{\mathcal{W}}_{\phi(\eta+1)}$ and $i_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}} = \pi_{\eta+1}$ by the calculations in Case 2 of two-step normalization. □

Definition 3.21 *For $\mathcal{U}$ a normal iteration tree on M, let*

$$\mathcal{U}^{<\gamma} = \mathcal{U}\restriction(\alpha+1), \text{ where } \alpha \text{ is least such that } \mathrm{lh}\, E^{\mathcal{U}}_{\alpha} \geq \gamma,$$

and $\mathcal{U}^{<\gamma} = \mathcal{U}$ if there is no such α. Let

$$\mathcal{U}^{>\gamma} = \langle \mathcal{M}^{\mathcal{U}}_{\eta} \mid E^{\mathcal{U}}_{\eta} \text{ exists} \wedge \gamma < \lambda(E^{\mathcal{U}}_{\eta})\rangle.$$

Definition 3.22 *Let M, $\mathcal{T}$, F and $\mathcal{W}$ be as above, then we write*

$$W(\mathcal{T}, F) = \mathcal{T}^{<\mathrm{lh}\, F}{}^\frown\langle F\rangle^\frown i_F \text{``}\mathcal{T}^{>\mathrm{crit}(F)}$$

for the embedding normalization of $\mathcal{T}^\frown\langle F\rangle$ just defined. We write $\alpha^{\mathcal{T},F}$, $\beta^{\mathcal{T},F}$, $\phi^{\mathcal{T},F}$, and $\pi^{\mathcal{T},F}_{\xi}$ for the auxiliary objects $\alpha, \beta, \phi, \pi_{\xi}$ that we defined above.

The full normalization $X(\mathcal{T}, F)$ of $\mathcal{T}^\frown\langle F\rangle$ can be obtained as follows. We assume that $\mathcal{T}$ is normal on M, N is the last model of $\mathcal{T}$, F is on the N sequence, and $\mathrm{crit}(F) < \rho_n(N)$, for $n = k(N)$. Let

$$\mathcal{W} = \mathcal{T}^{<\mathrm{lh}\, F}{}^\frown\langle F\rangle^\frown i_F \text{``}\mathcal{T}^{>\mathrm{crit}(F)}$$

be the embedding normalization. Let $\mathcal{T}^{<\mathrm{lh}\, F} = \mathcal{T}\restriction(\alpha+1)$, $\beta = W\text{-pred}(\alpha+1)$, and $\phi : \mathrm{lh}\, \mathcal{T} \to \mathrm{lh}\, W$ be as above. The full normalization is $\mathcal{X}$, where

$$\mathcal{X}\restriction(\alpha+2) = \mathcal{W}\restriction(\alpha+2)$$

and

$$\mathcal{M}^{\mathcal{X}}_{\phi(\eta)} = \mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_{\eta}, F) \text{ for } \eta > \beta.$$

(Note that if $\eta > \beta$, then some G such that $\mathrm{crit}(F) = \mu < \lambda(G)$ was used on the branch to $\mathcal{M}^{\mathcal{T}}_{\eta}$, so for $k = k(\mathcal{M}^{\mathcal{T}}_{\eta})$, $\mu < \rho_k(\mathcal{M}^{\mathcal{T}}_{\eta})$.) The tree order of $\mathcal{X}$ is the same as that of $\mathcal{W}$. We have

$$\begin{tikzcd} \mathcal{M}^{\mathcal{T}}_{\eta} \arrow[r, "i_F^{\mathcal{M}^{\mathcal{T}}_{\eta}}"] \arrow[rr, bend right, "\pi_{\eta}"'] & \mathcal{M}^{\mathcal{X}}_{\phi(\eta)} \arrow[r, "\tau"] & \mathcal{M}^{\mathcal{W}}_{\phi(\eta)} \end{tikzcd}$$

where τ is the natural factor map. What remains is to find the extenders $E^{\mathcal{X}}_{\phi(\eta)}$ that make $\mathcal{X}$ into a normal iteration tree. For this, let $E = E^{\mathcal{T}}_{\eta}$, and

$$\pi : \mathcal{M}^{\mathcal{T}}_{\eta}|\langle \mathrm{lh}(E), 0\rangle \to \mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_{\eta}|\langle \mathrm{lh}(E), 0\rangle, F)$$

be the canonical embedding. One can show using condensation that $\pi(E)$ is on the sequence of $\mathcal{M}^{\mathcal{X}}_{\phi(\eta)}$. Moreover, for $\sigma = W\text{-pred}(\eta + 1)$,

$$\mathcal{M}^{\mathcal{X}}_{\phi(\eta+1)} = \mathrm{Ult}(\mathcal{M}^{\mathcal{W}}_{\sigma}|\langle \xi, n\rangle, \pi(E)),$$

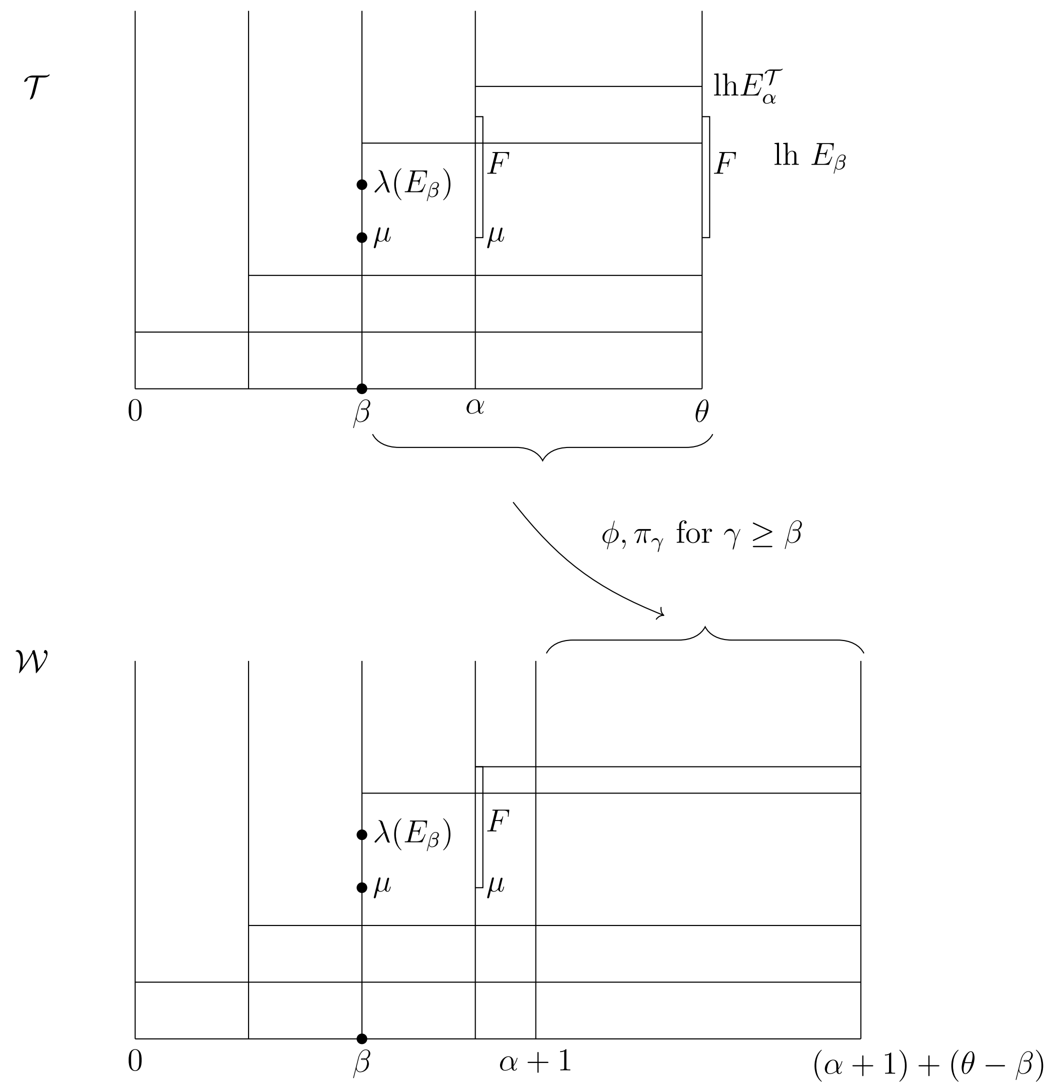
where $n = k(\mathcal{M}^{\mathcal{W}}_{\eta+1}) = k(\mathcal{M}^{\mathcal{T}}_{\eta+1})$ and ξ is appropriate. The details here are like those in the two-step case. Since we don't actually need full normalization in comparing iteration strategies, we give no further detail here. There is a much more careful discussion in [48]. Here is a diagram of the situation.

$$\begin{tikzcd}
N \arrow[r, "i^N_F"] & \mathrm{Ult}(N,F) \arrow[r, "\tau"] & R \\
M \arrow[u, "\hat{\imath}^{\mathcal{T}}"] \arrow[ur, "\hat{\imath}^{\mathcal{X}}"'] \arrow[urr, "\hat{\imath}^{\mathcal{W}}"'] & &
\end{tikzcd}$$

Each $\mathcal{M}^{\mathcal{T}}_{\eta}$ is mapped into $\mathcal{M}^{\mathcal{X}}_{\phi(\eta)}$, and that in turn is mapped into $\mathcal{M}^{\mathcal{W}}_{\phi(\eta)}$.

Returning to $W(\mathcal{T}, F)$, here are a few illustrations that the reader may or may not find helpful. Let $\mathcal{T}$ be normal on M of length $\theta + 1$, F on the sequence of $\mathcal{M}^{\mathcal{T}}_{\theta}$, $\mu = \mathrm{crit}(F)$, β least such that $\mu < \lambda(E^{\mathcal{T}}_{\beta})$, and α least such that F is on the sequence of $\mathcal{M}^{\mathcal{T}}_{\alpha}$, as above. We assume in the diagram that $\beta < \theta$, and that $\mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_{\theta}, F)$ makes sense. Let $\phi : \theta \cong [0, \beta) \cup [\alpha + 1, (\alpha + 1) + (\theta - \beta)]$ be the order-isomorphism as above.

We illustrate first the embedding of $\mathcal{T}$ into $\mathcal{W}(\mathcal{T}, F)$, as it appears in the agreement diagrams. We draw them as if $\beta < \alpha$, although $\beta = \alpha$ is possible.



We have

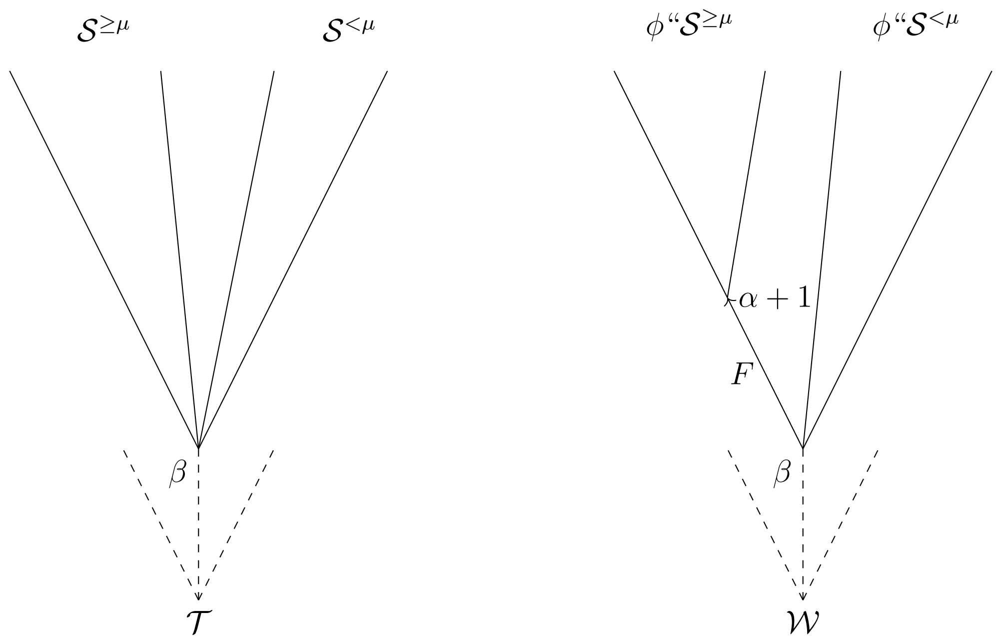
$$\mathcal{T}\restriction(\alpha+1) = \mathcal{W}\restriction(\alpha+1),$$
$$F = E_\alpha^{\mathcal{W}},$$

and

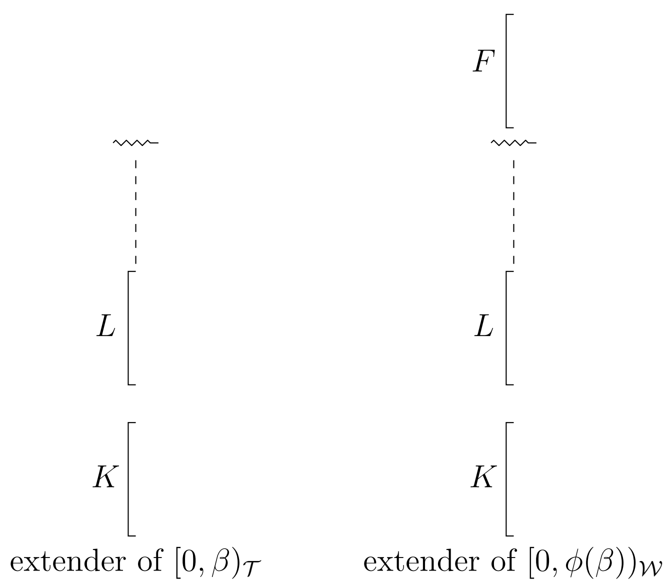
$$i_F\text{``}\mathcal{T}^{>\mu} = \text{ remainder of } \mathcal{W}.$$

The next diagram shows how ϕ may fail to preserve tree order. By (4)(c) above, we can have $\delta \leq_T \xi$ but $\phi(\delta) \not\leq_W \phi(\xi)$ iff $\delta = \beta$, and the first extender G used in

$(0,\xi)_T$ such that G is applied to an initial segment of $\mathcal{M}_\beta^\mathcal{T}$ satisfies $\mathrm{crit}(G) < \mu$. Let $S^{<\mu}$ be the set of such $\xi >_\mathcal{T} \beta$, and $S^{\geq\mu}$ the remaining $\xi >_\mathcal{T} \beta$. The picture is

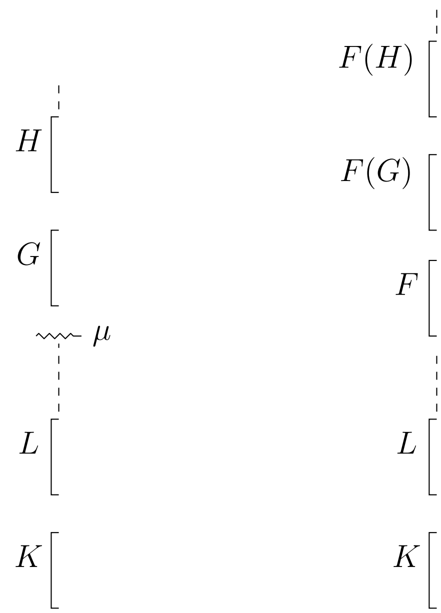


Finally, we illustrate the relationship between the branch extenders of $[0,\xi)_T$ and $[0,\phi(\xi))_W$. If $\xi < \beta$, they are equal. For $\xi = \beta$, the picture is



because $[0,\beta)_T \subseteq [0,\phi(\beta))_W$, and just the one additional extender F is used.

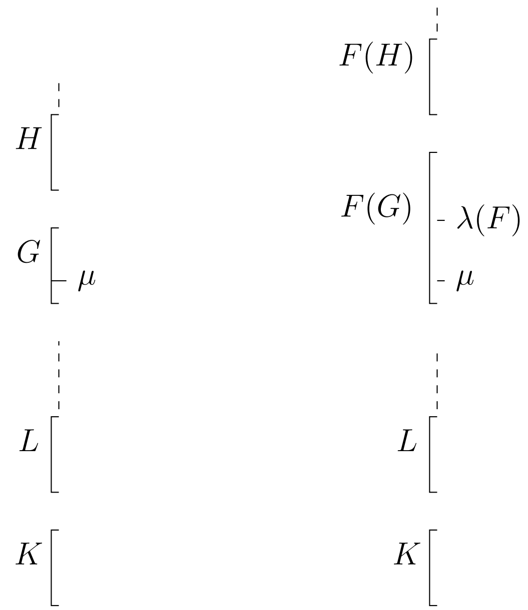
For $\xi > \beta$, let G be the first extender used in $[0,\xi)_T$ such that $\lambda(G) \geq \lambda(E_\beta^{\mathcal{T}})$. The picture depends on whether $\mu \leq \text{crit}(G)$. If $\mu \leq \text{crit}(G)$, it is



extender of $[0,\beta)_{\mathcal{T}}$ extender of $[0,\phi(\beta))_{\mathcal{W}}$

In this case, F is used on $[0,\phi(\xi))_W$, and the remaining extender used are the images of old ones under copy maps.

If $\text{crit}(G) < \mu < \lambda(G)$, the picture is



extender of $[0, \beta)_{\mathcal{T}}$ extender of $[0, \phi(\beta))_{\mathcal{W}}$

In this case, the two branches use the same extenders until G is used on $[0, \xi)_T$. At that point and after, $[0, \phi(\xi))_W$ uses the images of extenders under the copy maps.

Notice that in either case, there is an L used in $[0, \phi(\xi))_W$ such that $\mathrm{crit}(L) \leq \mathrm{crit}(F) < \lambda(F) \leq \lambda(L)$. This will be important later.

Remark 3.23 There is nothing guaranteeing that the models of $W(\mathcal{T}, F)$ are well-founded. In our context of interest, $\mathcal{T}$ is played according to an iteration strategy Σ. Part of "normalizing well" for Σ will then be that $W(\mathcal{T}, F)$ is according to Σ.

3.3 The extender tree $\mathcal{V}^{\mathrm{ext}}$

The fact that $\phi^{\mathcal{T},F}$ does not fully preserve tree order or tree predecessor is awkward. Here is another way to visualize our embedding of $\mathcal{T}$ into $W(\mathcal{T}, F)$ given by $\phi^{\mathcal{T},F}$ and the $\pi_\xi^{\mathcal{T},F}$'s.

For $\mathcal{V}$ a normal tree, let

$$\mathrm{Ext}(\mathcal{V}) = \{E_\alpha^{\mathcal{V}} \mid \alpha + 1 < \mathrm{lh}\,\mathcal{V}\}$$

be the set of extenders used. Note $\mathrm{Ext}(\mathcal{V})$ determines $\mathcal{V}$ modulo a strategy Σ for the base model of $\mathcal{V}$, by normality. For $\gamma < \mathrm{lh}(\mathcal{V})$,

$$e_\gamma^{\mathcal{V}} = \text{increasing enumeration of } \{E_\alpha^{\mathcal{V}} \mid \alpha + 1 \leq_V \gamma\},$$

increasing in order of use (index, length).

Note that each of $e^{\mathcal{V}}_\gamma$, $\mathcal{M}^{\mathcal{V}}_\gamma$ and $\mathcal{V}\restriction(\gamma+1)$ determines the others, by normality. Set

$$\mathcal{V}^{\text{ext}} = \{e^{\mathcal{V}}_\gamma \mid \gamma < \operatorname{lh}\mathcal{V}\}.$$

$\mathcal{V}^{\text{ext}}$ determines $\mathcal{V}$. The structure $(\mathcal{V}^{\text{ext}}, \subseteq)$ is the *extender-tree* of $\mathcal{V}$.

If F and G are extenders, then F and G *overlap* iff $[\operatorname{crit}(F), \lambda(F)) \cap [\operatorname{crit}(G), \lambda(G)) \neq \varnothing$. We say F and G are *compatible* iff $\exists\alpha(F = G\restriction\alpha$ or $G = F\restriction\alpha)$. Here are two elementary facts:

Proposition 3.24 *Let $\mathcal{V}$ be a normal iteration tree; then*

(1) if $s^\frown\langle F\rangle \in \mathcal{V}^{\text{ext}}$ and $s^\frown\langle G\rangle \in \mathcal{V}^{\text{ext}}$, then F and G overlap, and

(2) if $s, t \in \mathcal{V}^{\text{ext}}$ and $s(i)$ is compatible with $t(k)$, then $i = k$ and $s\restriction(i+1) = t\restriction(i+1)$.

Now let $\mathcal{T}$ be normal on M, and $\mathcal{W} = W(\mathcal{T}, F)$. Let $\phi = \phi^{\mathcal{T},F}$, $\pi_\xi = \pi^{\mathcal{T},F}_\xi$, etc. We define a partial map

$$p_{\mathcal{T},F} : \operatorname{Ext}(\mathcal{T}) \to \operatorname{Ext}(\mathcal{W})$$

by

$$p_{\mathcal{T},F}(E^{\mathcal{T}}_\xi) = \pi_\xi(E^{\mathcal{T}}_\xi) = E^{\mathcal{W}}_{\phi(\xi)}.$$

So $p_{\mathcal{T},F}(E^{\mathcal{T}}_\xi)\downarrow$ iff $\xi \in \operatorname{dom}\phi$, and either $\xi \neq \beta$, or $\xi = \beta$ and $\mathcal{M}^{\mathcal{T}}_\beta | \operatorname{lh}(E^{\mathcal{T}}_\beta) \trianglelefteq \mathcal{M}^{*,\mathcal{W}}_{\alpha+1}$.

We can view p as acting on branch extenders. For $s \in \mathcal{T}^{\text{ext}}$, let

$$i^F_s = i_s = \begin{cases} \text{least } i \text{ such that } \operatorname{crit}(F) < \lambda(s(i)), & \text{if this exists;} \\ \text{undefined,} & \text{otherwise.} \end{cases}$$

Let $\xi \in \operatorname{dom}\phi$ and $s = e^{\mathcal{T}}_\xi$. Then if $\operatorname{dom}(\phi) = \beta + 1$, we have

$$e^{\mathcal{W}}_{\phi(\xi)} = \begin{cases} s, & \text{if } \xi < \beta; \\ s^\frown\langle F\rangle, & \text{if } \xi = \beta. \end{cases}$$

If $\operatorname{dom}(\phi) > \beta + 1$, then i_s exists precisely when $s = e^{\mathcal{T}}_\xi$ for some $\xi \geq \beta + 1$, and

$$e^{\mathcal{W}}_{\phi(\xi)} = \begin{cases} s, & \text{if } \xi < \beta; \\ s^\frown\langle F\rangle, & \text{if } \xi = \beta; \\ s\restriction i_s{}^\frown\langle F\rangle^\frown\langle \psi^{\mathcal{T},F}(s(i)) \mid i \geq i_s\rangle, & \text{if } \operatorname{crit}(F) \leq \operatorname{crit}(s(i_s)); \\ s\restriction i_s{}^\frown\langle p^{\mathcal{T},F}(s(i)) \mid i \geq i_s\rangle, & \text{if } \operatorname{crit}(s(i_s)) < \operatorname{crit}(F). \end{cases}$$

So if E is used before H in $e^{\mathcal{T}}_\xi$, then $p_{\mathcal{T},F}(E)$ is used before $p_{\mathcal{T},F}(H)$ in $e^{\mathcal{W}}_{\phi(\xi)}$.

Definition 3.25 *Let* $\mathcal{W} = W(\mathcal{T}, F)$*, and suppose* $s \in \mathcal{T}^{\text{ext}}$ *is such that* $\forall \mu \in \text{dom}(s)$*,* $p_{\mathcal{T},F}(s(\mu))\downarrow$*; then*

$$\hat{p}_{\mathcal{T},F}(s) = \textit{unique shortest } t \in \mathcal{W}^{\text{ext}} \textit{ such that}$$
$$\forall \mu \in \text{dom}(s),\ p_{\mathcal{T},F}(s(\mu)) \in \text{ran}(t).$$

For $\hat{p} = \hat{p}_{\mathcal{T},F}$, we have that $\hat{p}(e^{\mathcal{T}}_{\xi}) = e^{\mathcal{W}}_{\phi(\xi)}$, except when $\xi = \beta$. At β, we have $e^{\mathcal{W}}_{\phi(\beta)} = \hat{p}(e^{\mathcal{T}}_{\beta})^\frown\langle F\rangle$. The map $\hat{p}\colon \mathcal{T}^{\text{ext}} \to W(\mathcal{T}, F)^{\text{ext}}$ does preserve $\subseteq$.

Proposition 3.26 *Let* $s, t \in \text{dom}(\hat{p}^{\mathcal{T},F})$*; then*

(1) $s \subseteq t \Rightarrow \hat{p}(s) \subseteq \hat{p}(t)$*, and*

(2) $s \perp t \Rightarrow \hat{p}(s) \perp \hat{p}(t)$*.*

3.4 Tree embeddings

An iteration strategy Σ for M *condenses well* iff whenever $\mathcal{U}$ is by Σ, and π is a sufficiently elementary embedding from $\mathcal{T}$ into $\mathcal{U}$ such that $\pi\restriction(M \cup \{M\})$ is the identity, then $\mathcal{T}$ is by Σ. By weakening the elementarity required of π, we obtain stronger condensation properties.

In the Hull Condensation property of [30], one is given an embedding $\sigma : \text{lh}(\mathcal{T}) \to \text{lh}(\mathcal{U})$ and embeddings $\tau_\alpha : \mathcal{M}^{\mathcal{T}}_\alpha \to \mathcal{M}^{\mathcal{U}}_{\sigma(\alpha)}$. σ preserves tree order and tree-predecessor. The τ_α's have the agreement one would get from a copying construction, and they commute with the branch embeddings of $\mathcal{T}$ and $\mathcal{U}$. Moreover, $\tau_\alpha(E^{\mathcal{T}}_\alpha) = E^{\mathcal{U}}_{\sigma(\alpha)}$. A simple example in the way $\mathcal{T} = \pi\mathcal{W}$ sits inside $\mathcal{U} = \pi(\mathcal{W})$, in the case $\pi\colon H \to V$ is elementary and $\pi\restriction(M \cup \{M\}) =$ id .

A hull embedding $(\sigma, \vec{\tau})$ as above induces a map $p : \text{Ext}(\mathcal{T}) \to \text{Ext}(\mathcal{U})$ by

$$p(E^{\mathcal{T}}_\alpha) = \tau_\alpha(E^{\mathcal{T}}_\alpha).$$

We then get $\hat{p}\colon \mathcal{T}^{\text{ext}} \to \mathcal{U}^{\text{ext}}$ from p as in 3.25.. $\hat{p}$ preserves $\subseteq$ and incompatibility in the extender trees. $\hat{p}$ is related to σ by

$$\hat{p}(e^{\mathcal{T}}_{\alpha+1}) = e^{\mathcal{U}}_{\sigma(\alpha+1)}.$$

But for λ a limit, $\hat{p}(e^{\mathcal{T}}_\lambda)$ may be a proper initial segment of $e^{\mathcal{U}}_{\sigma(\lambda)}$.

We now define the notion of a tree embedding from $\mathcal{T}$ into $\mathcal{U}$. This will be a tuple with most of the properties of σ, $\vec{\tau}$, ψ above. The pair $(\sigma, \vec{\tau})$ is resolved into two pairs: the pair $(v, \vec{s})$, which embeds the models of $\mathcal{T}$ into models of $\mathcal{U}$ in a minimal way, and the pair $(u, \vec{t})$, which connects the exit extenders of $\mathcal{T}$ to exit extenders in $\mathcal{U}$. The the requirement that σ preserves tree predecessors is relaxed to the requirement that if $\beta = T\text{-pred}(\gamma+1)$, then $U\text{-pred}(u(\gamma)+1) \in [v(\beta), u(\beta)]_U$. We shall also allow the t_α's to be partial, in a controlled way. Recall here the partial branch embeddings $\hat{\imath}^{\mathcal{U}}_{\alpha,\beta}$. (Cf. 2.10.)

Definition 3.27 *Let $\mathcal{T}$ and $\mathcal{U}$ be normal iteration trees on a premouse M, with $\text{lh}(\mathcal{T}) > 1$. A* tree embedding *of $\mathcal{T}$ into $\mathcal{U}$ is a system*

$$\langle u, \langle s_\beta \mid \beta < \text{lh}(\mathcal{T})\rangle, \langle t_\beta \mid \beta + 1 < \text{lh}(\mathcal{T})\rangle, p\rangle$$

such that

(a) $u : \{\alpha \mid \alpha + 1 < \text{lh}(\mathcal{T})\} \to \{\alpha \mid \alpha + 1 < \text{lh}(\mathcal{U})\}$*, and* $\alpha < \beta \Rightarrow u(\alpha) < u(\beta)$.

(b) $p : \text{Ext}(\mathcal{T}) \to \text{Ext}(\mathcal{U})$ *is such that E is used before F on the same branch of $\mathcal{T}$ iff $p(E)$ is used before $p(F)$ on the same branch of $\mathcal{U}$. Thus p induces $\hat{p} : \mathcal{T}^{\text{ext}} \to \mathcal{U}^{\text{ext}}$ as in Definition 3.25.*

(c) *Let $v : \text{lh}(\mathcal{T}) \to \text{lh}(\mathcal{U})$ be given by*

$$e^{\mathcal{U}}_{v(\beta)} = \hat{p}(e^{\mathcal{T}}_\beta)$$

Then

$$s_\beta : \mathcal{M}^{\mathcal{T}}_\beta \to \mathcal{M}^{\mathcal{U}}_{v(\beta)}$$

is total and elementary. Moreover, for $\alpha <_T \beta$,

$$s_\beta \circ \hat{\imath}^{\mathcal{T}}_{\alpha,\beta} = \hat{\imath}^{\mathcal{U}}_{v(\alpha),v(\beta)} \circ s_\alpha.$$

In particular, the two sides have the same domain.

(d) *For $\alpha + 1 < \text{lh}(\mathcal{T})$, $v(\alpha) \leq_U u(\alpha)$, and*

$$t_\alpha = \hat{\imath}^{\mathcal{U}}_{v(\alpha),u(\alpha)} \circ s_\alpha.$$

Moreover,

$$\begin{aligned} p(E^{\mathcal{T}}_\alpha) &= t_\alpha(E^{\mathcal{T}}_\alpha) \\ &= E^{\mathcal{U}}_{u(\alpha)}. \end{aligned}$$

Moreover, for $\alpha < \beta < \text{lh}(\mathcal{T})$,

$$s_\beta \restriction \text{lh}(E^{\mathcal{T}}_\alpha) + 1 = t_\alpha \restriction \text{lh}(E^{\mathcal{T}}_\alpha) + 1.$$

(*e*) *If* $\beta = T\text{-pred}(\alpha+1)$, *then* $U\text{-pred}(u(\alpha)+1) \in [v(\beta), u(\beta)]_U$, *and setting* $\beta^* = U\text{-pred}(u(\alpha)+1)$,

$$s_{\alpha+1}([a,f]^P_{E^{\mathcal{T}}_\alpha}) = [t_\alpha(a), \hat{\imath}^{\mathcal{U}}_{v(\beta),\beta^*} \circ s_\beta(f)]^{P^*}_{E^{\mathcal{U}}_{u(\alpha)}},$$

where $P \trianglelefteq \mathcal{M}^{\mathcal{T}}_\beta$ *is what* $E^{\mathcal{T}}_\alpha$ *is applied to, and* $P^* \trianglelefteq \mathcal{M}^{\mathcal{U}}_{\beta^*}$ *is what* $E^{\mathcal{U}}_{u(\alpha)}$ *is applied to.*

The appropriate diagram to go with (e) of Definition 3.27 (for the non-dropping case is)

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{T}}_{\alpha+1} \arrow[rr, "s_{\alpha+1}"] & & \mathcal{M}^{\mathcal{U}}_{v(\alpha+1)} \\
& \mathcal{M}^{\mathcal{U}}_{u(\beta)} & \\
& & \mathcal{M}^{\mathcal{U}}_{\beta^*} \arrow[uu, "E^{\mathcal{U}}_{u(\alpha)}"'] \arrow[ul] \\
\mathcal{M}^{\mathcal{T}}_{\beta} \arrow[uuu, "E^{\mathcal{T}}_{\alpha}"] \arrow[uur, "t_\beta"] \arrow[urr, "\rho"] \arrow[rr, "s_\beta"] & & \mathcal{M}^{\mathcal{U}}_{v(\beta)} \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{T}}_{\alpha} \arrow[rr, "t_\alpha"] & & \mathcal{M}^{\mathcal{U}}_{u(\alpha)}
\end{tikzcd}$$

Here $\hat{\imath}^{\mathcal{U}}_{v(\beta),\beta^*} \circ s_\beta = \rho$ is a possibly partial map, defined and elementary on P.

Definition 3.27 is not quite right in the case that $\mathcal{T}$ is only weakly normal.

Definition 3.28 *If* $\mathcal{T}$ *and* $\mathcal{U}$ *are weakly normal trees on* M, *with* $\mathrm{lh}(\mathcal{T}) > 1$, *then a* tree embedding *from* $\mathcal{T}$ *to* $\mathcal{U}$ *is a system*

$$\langle u, \langle s_\beta \mid \beta < \mathrm{lh}(\mathcal{T})\rangle, \langle t_\beta \mid \beta + 1 < \mathrm{lh}(\mathcal{T})\rangle, p\rangle$$

satisfying all the clauses of 3.27, except that in clause (c), we demand that

$$s_\beta \colon \mathcal{M}^{\mathcal{T}}_\beta \to N_\beta$$

elementarily, where $N_\beta \trianglelefteq \mathcal{M}^{\mathcal{U}}_{v(\beta)}$. *The* N_ξ *must be given by* *(i)* $N_0 = M$,

(ii) $N_{\alpha+1} = i^{\mathcal{U}}_{\beta^*,\alpha+1} \circ \hat{\imath}^{\mathcal{U}}_{v(\beta),\beta^*} \circ s_\beta(P)$, *where* $P \trianglelefteq N_\beta$ *is such that* $\mathcal{M}^{\mathcal{T}}_{\alpha+1} = \mathrm{Ult}(P, E^{\mathcal{T}}_\alpha)$, *and*

(iii) if λ *is a limit, then* $N_\lambda = i^{\mathcal{U}}_{\xi,\lambda}(N_\xi)$, *for all sufficiently large* $\xi <_U \lambda$.

We need the N_β's because we want to allow $\mathcal{T}$ to drop gratuitously along the branch to β, while at the corresponding step along the branch of $\mathcal{U}$ to $v(\beta)$, $\mathcal{U}$ has not dropped, or not dropped as far. The Shift Lemma formula in clause (e) needs no change, but now what it may be giving us is an elementary embedding of $\mathrm{Ult}(P, E^{\mathcal{T}}_\alpha)$ into a proper initial segment $N_{\alpha+1}$ of $\mathrm{Ult}(P^*, E^{\mathcal{U}}_{u(\alpha)})$. Note that Φ completely determines the N_β, so we can write $N_\beta = N^\Phi_\beta$.

One can easily see that if $\mathcal{T}$ is normal, then $\mathcal{U}$ must also be normal, and $N_\beta = \mathcal{M}^{\mathcal{U}}_{v(\beta)}$ for all β. So the two definitions of tree embedding are consistent with each other. Only occasionally do we actually need to consider the case that $\mathcal{T}$ is weakly normal, but not normal. We don't need to consider the case that $\mathcal{U}$ is not normal at all, but we have allowed it for the sake of completeness. The notion of tree embedding we have defined doesn't seem to make much sense if $\mathcal{T}$ fails to be weakly normal.

Definition 3.29 *For weakly normal iteration trees* $\mathcal{T}$ *and* $\mathcal{U}$,

(a) $\Phi\colon \mathcal{T} \to \mathcal{U}$ *iff* Φ *is a tree embedding of* $\mathcal{T}$ *into* $\mathcal{U}$,

(b) if $\Phi\colon \mathcal{T} \to \mathcal{U}$, *then* $u^\Phi, v^\Phi, s^\Phi_\alpha, t^\Phi_\alpha$, *and* p^Φ *are the component maps of* Φ, *and*

(c) $\mathcal{T}$ *is a* psuedo-hull *of* $\mathcal{U}$ *iff there is a tree embedding of* $\mathcal{T}$ *into* $\mathcal{U}$.

Remark 3.30 It is easy to see that $\Phi\colon \mathcal{T} \to \mathcal{U}$ if and only if $\Phi\colon \mathcal{T} \to \mathcal{U} \restriction \gamma$, where $\gamma = \sup(\{v^\Phi(\alpha) + 1 \mid \alpha < \mathrm{lh}(\mathcal{T})\})$.

Definition 3.31 *A tree embedding* $\Phi\colon \mathcal{T} \to \mathcal{U}$ *is* cofinal *iff* $\mathrm{lh}(\mathcal{U}) = \sup(\{v^\Phi(\alpha)+1 \mid \alpha < \mathrm{lh}(\mathcal{T})\})$.

Remark 3.32 By clause (c), $v(0) = 0$ and $s_0 = \mathrm{id}$. It is possible that $u(0) > 0$. By clause (d), $v(\alpha + 1) = u(\alpha) + 1$. Clause (b) implies that $\alpha + 1 \leq_T \beta + 1$ iff $v(\alpha+1) \leq_U v(\beta+1)$. If $\lambda < \mathrm{lh}(\mathcal{T})$ is a limit ordinal, then $v(\lambda) = \sup\{v(\xi) \mid \xi <_T \lambda\}$. So v preserves tree order, and is continuous at limits. The map u may not preserve tree order.

Remark 3.33 Given $u(\alpha)$ and t_α, we can characterize $v(\alpha)$ as the least $\xi \leq_U u(\alpha)$ such that $\mathrm{ran}(t_\alpha) \subseteq \mathrm{ran}(\hat{\imath}^{\mathcal{U}}_{\xi,u(\alpha)})$.

If $\Phi\colon \mathcal{T} \to \mathcal{U}$ is a tree embedding, then $\mathcal{T}$ and $\mathcal{U}$ have the same base model, and s_0^Φ is the identity map. One might ask whether there is a natural more general concept, one that allows $\mathcal{M}_0^{\mathcal{T}} \neq \mathcal{M}_0^{\mathcal{U}}$. Indeed there is, but it reduces to the notion above. Namely, one can have an elementary $\pi\colon \mathcal{M}_0^{\mathcal{T}} \to \mathcal{M}_0^{\mathcal{U}}$, together with a tree embedding from the copied tree $\pi\mathcal{T}$ into $\mathcal{U}$. This seems to be the natural way to relate trees on different base models.

Any tree embedding induces an embedding of extender trees:

Proposition 3.34 *Let $\Phi\colon \mathcal{T} \to \mathcal{U}$ be a tree embedding, let $p = p^\Phi$, and let $\hat{p}\colon \mathcal{T}^{\mathrm{ext}} \to \mathcal{U}^{\mathrm{ext}}$ be the induced map on extender trees; then Let $s, t \in \mathrm{dom}(\hat{p}^{\mathcal{T},F})$; then*

(1) *$s \subseteq t \Rightarrow \hat{p}(s) \subseteq \hat{p}(t)$, and*

(2) *$s \perp t \Rightarrow \hat{p}(s) \perp \hat{p}(t)$.*

Let us record the agreement properties of the maps in a tree embedding. In the context of Jensen premice, embeddings that agree on $\mathrm{lh}(E)$ will generally be forced to agree on $\mathrm{lh}(E) + 1$. For example, in clause (e) of 3.27, $s_{\alpha+1}$ agrees with t_α on $\mathrm{lh}(E_\alpha^{\mathcal{T}}) + 1$, because the Shift Lemma produces this kind of agreement. One does encounter embeddings that agree on λ_E, but not on $\lambda_E + 1$. With this in mind, we see that

Lemma 3.35 *Let $\langle u, \langle s_\beta \mid \beta < \mathrm{lh}\,\mathcal{T}\rangle, \langle t_\beta \mid \beta + 1 < \mathrm{lh}\,\mathcal{T}\rangle, p\rangle$ be a tree embedding of $\mathcal{T}$ into $\mathcal{U}$; then*

(a) if $\alpha + 1 < \mathrm{lh}(\mathcal{T})$, then t_α agrees with s_α on $\lambda_\alpha^{\mathcal{T}}$,

(b) if $\beta < \alpha < \mathrm{lh}(\mathcal{T})$, then s_α agrees with t_β on $\mathrm{lh}(E_\beta^{\mathcal{T}}) + 1$, and

(b) if $\beta < \alpha < \mathrm{lh}(\mathcal{T})$, then s_α agrees with s_β on $\lambda_\beta^{\mathcal{T}}$

Proof. For (a), notice that if F is used in $e_\alpha^{\mathcal{T}}$, then $p(F)$ is used in $e^{\mathcal{U}}_{v(\alpha)}$, and so $\lambda_{p(F)} \leq \mathrm{crit}(\hat{\imath}^{\mathcal{U}}_{v(\alpha),u(\alpha)})$. Thus $\sup s_\alpha``\lambda_\alpha^{\mathcal{T}} \leq \mathrm{crit}(\hat{\imath}^{\mathcal{U}}_{v(\alpha),u(\alpha)})$. But $t_\alpha = \hat{\imath}^{\mathcal{U}}_{v(\alpha),u(\alpha)} \circ s_\alpha$, so we have (a).

Part (b) is just a clause in the definition. Part (c) follows at once from (a) and (b). $\square$

One could not replace $\lambda_\alpha^{\mathcal{T}}$ by $\sup\{\mathrm{lh}(F) \mid F \in \mathrm{ran}(e_\alpha^{\mathcal{T}})\}$ in the lemma above. The reason is that there could be a last extender F used in $e_\alpha^{\mathcal{T}}$. (So $F = E_\beta^{\mathcal{T}}$

where $\alpha = \beta + 1$.) Then $p(F)$ is the last extender used in $e^{\mathcal{U}}_{v(\alpha)}$. It could be that $\text{crit}(\hat{\imath}^{\mathcal{U}}_{v(\alpha),u(\alpha)}) = \lambda_{p(F)}$, and thus t_α and $s_{\alpha+1}$ both disagree with s_α at λ_F. (This is the only way the stronger agreement lemma can fail.)

Remark 3.36 The proof of 5.3 in Chapter 4 gives a formula for the point of application of $E^{\mathcal{U}}_{u(\alpha)}$ under a tree embedding of $\mathcal{T}$ into $\mathcal{U}$, namely

$$U\text{-pred}(u(\alpha)+1) = \text{least } \eta \in [v(\beta), u(\beta)]_U \text{ such that crit } \hat{\imath}^{\mathcal{U}}_{\eta,u(\beta)} > \hat{\imath}^{\mathcal{U}}_{v(\beta),\eta} \circ s_\beta(\mu),$$

where

$$\beta = T\text{-pred}(\alpha + 1) \text{ and } \mu = \text{crit}(E^{\mathcal{T}}_\alpha).$$

Remark 3.37 It is easy to see that $\mathcal{T}$,$\mathcal{U}$, and u determine the rest of the tree embedding. For p is given by $p(E^{\mathcal{T}}_\alpha) = E^{\mathcal{U}}_{u(\alpha)}$, and p determines $\hat{p}$ and v. We then determine the copy maps s_α and t_α by induction on α. t_α is determined by s_α by $t_\alpha = \hat{\imath}^{\mathcal{U}}_{v(\alpha),u(\alpha)} \circ s_\alpha$. If α is a limit, we easily get s_α from $v(\alpha)$ and the fact that $s_\alpha \circ \hat{\imath}^{\mathcal{T}}_{\beta,\alpha} = \hat{\imath}^{\mathcal{U}}_{v(\beta),v(\alpha)} \circ s_\beta$ holds whenever $\beta <_T \alpha$. Clause (e) determines $s_{\alpha+1}$ from earlier s and t values.

p determines u, hence p determines the whole of the tree embedding as well.

Remark 3.38 Suppose that $\text{lh}(\mathcal{T}) = \alpha + 1$ and $\Phi \colon \mathcal{T} \to \mathcal{U}$ is a tree embedding. Let $s = s^\Phi$, $u = u^\Phi$, etc., so that $s_\alpha \colon \mathcal{M}^{\mathcal{T}}_\alpha \to \mathcal{M}^{\mathcal{U}}_{v(\alpha)}$ is our enlargement of the last model of $\mathcal{T}$. Then for all $\beta < \alpha$,

$$s_\alpha(\text{lh}(E^{\mathcal{T}}_\beta)) = \text{lh}(E^{\mathcal{U}}_{u(\beta)}),$$

by 3.35. Thus $s_\alpha, \mathcal{T}$, and $\mathcal{U} \restriction v(\alpha) + 1$ determine u, and hence the whole of Φ. As far as Φ is concerned, $M^{\mathcal{U}}_{v(\alpha)}$ is the last relevant model of $\mathcal{U}$. So we can say that if $\mathcal{T}$ has successor length, then a tree embedding from $\mathcal{T}$ to $\mathcal{U}$ is just a map from the last model of $\mathcal{T}$ into some model of $\mathcal{U}$ that is elementary in a certain strong sense.

The reader might wonder why the u-map and t-maps of $\Phi \colon \mathcal{T} \to \mathcal{U}$ are undefined at α, where $\alpha + 1 = \text{lh}(\mathcal{T})$. In general, forcing Φ to include a value for $u(\alpha)$ is wrong, because u is being used to connect exit extenders, and $\mathcal{T}$ has not yet chosen an exit extender at α. If we demand Φ include a value for $u(\alpha)$, then what we would like to call extensions of Φ may have to revise this value. That is awkward. (See Lemma 5.3 for a characterization of when it is possible to extend $\Phi \colon \mathcal{T} \to \mathcal{U}$ to $\Psi \colon (\mathcal{T}^\frown \langle F \rangle) \to \mathcal{U}$.)

In the case $\mathcal{U} = W(\mathcal{T}, F)$, there is a natural way to define u and $\vec{t}$ at $\alpha = \text{lh}(\mathcal{T}) - 1$, namely, $u(\alpha) = \text{lh}(\mathcal{U}) - 1$, and $t_\alpha = \hat{\imath}^{\mathcal{U}}_{v(\alpha),u(\alpha)} \circ s_\alpha$. It helps to make a definition here.

Definition 3.39 *Let $\mathcal{T}$ and $\mathcal{U}$ be weakly normal iteration trees of lengths $\alpha+1>1$ and $\beta+1$, and let $\Phi\colon \mathcal{T}\to\mathcal{U}$ be a tree embedding, with $\Phi=\langle u,\langle s_\xi \mid \xi\leq\alpha\rangle,\langle t_\xi\mid \xi<\alpha\rangle,p\rangle$. Suppose that $v(\alpha)\leq\beta$; then we define*

$$\Psi(\Phi,\mathcal{U})=\langle u\cup\{\langle\alpha,\beta\rangle\},\langle s_\xi\mid\xi\leq\alpha\rangle,\langle t_\xi\mid\xi<\alpha\rangle^\frown\langle\hat{i}^{\mathcal{U}}_{v(\alpha),u(\alpha)}\circ s_\alpha\rangle,p\rangle.$$

We say that Ψ is an extended tree embedding *iff $\Psi=\Psi(\Phi,\mathcal{U})$ for some Φ and $\mathcal{U}$, and write $\Phi=c(\Psi)$ and $\mathcal{U}=r(\Psi)$ for the unique such Φ and $\mathcal{U}$.*

Extended tree embeddings are not tree embeddings, they are tree embeddings that have been extended in a small way. If $\Phi\colon\mathcal{T}\to\mathcal{U}$ is a cofinal tree embedding, then its extension $\Psi(\Phi,\mathcal{U})$ is completely trivial. In general, an extended tree embedding from $\mathcal{T}$ into $\mathcal{U}$ is completely determined by $\mathcal{T}$, $\mathcal{U}$, and its last s-map.

Remark 3.40 $\mathcal{T}$ is a pseudo-hull of $W(\mathcal{T},F)$, and in fact, there is an extended tree embedding $\Psi=\langle u,\vec{s},\vec{t},p\rangle$ from $\mathcal{T}$ into $W(\mathcal{T},F)$. In our embedding normalization notation, $u=\phi^{\mathcal{T},F}$, $t_\beta=\pi^{\mathcal{T},F}_\beta$, and $p(E^{\mathcal{T}}_\xi)=E^{W(\mathcal{T},F)}_{u(\xi)}$ for $\xi+1<\mathrm{lh}(\mathcal{T})$. This determines $\hat{p}$ and v. u agrees with v except at $\beta=\beta^{\mathcal{T},F}$, where we have $v(\beta)=\beta$ and $u(\beta)=\alpha^{\mathcal{T},F}+1$.

Letting $\Phi=c(\Psi)$ be the associated tree embedding, it is easy to see that Φ is cofinal iff $\mathcal{T}^\frown\langle F\rangle$ is not normal.

Definition 3.41 *Let Φ be a tree embedding from $\mathcal{T}$ into $\mathcal{U}$, and Ψ be a tree embedding from $\mathcal{U}$ into $\mathcal{V}$; then $\Psi\circ\Phi$ is the tree embedding from $\mathcal{T}$ into $\mathcal{V}$ obtained by composing the corresponding component maps of Φ and Ψ. Similarly, if Φ and Ψ are extended tree embeddings, then $\Psi\circ\Phi$ is the extended tree embedding obtained by composing corresponding maps.*

It is easy to check that composing corresponding maps does indeed produce a tree embedding or extended tree embedding, as the case may be.

3.5 Normalizing $\mathcal{T}^\frown\mathcal{U}$

In this section we define the embedding normalization $W(\mathcal{T},\mathcal{U})$ of a maximal M-stack $\langle\mathcal{T},\mathcal{U}\rangle$ of length 2. It is not hard to extend our definitions so that they apply to arbitrary M-stacks of length 2, but the additional notation introduced by gratuitous dropping would be a burden. So rather than deal with arbitrary finite stacks here, we shall show later that in our context of interest, they can be reduced to maximal stacks. (See 4.60.)

To begin with, note that $W(\mathcal{T}, F)$ makes sense in somewhat greater generality. Let $\mathcal{T}$ be a normal tree on the premouse M. Let $\mathcal{S}$ be another normal tree on M, and F be on the sequence of the last model of $\mathcal{S}$. Let α be least such that F is on the sequence of $\mathcal{M}_\alpha^\mathcal{S}$, so that $\mathcal{S}\restriction(\alpha+1) = \mathcal{S}^{<\mathrm{lh}(F)}$. Let β be such that $\beta = \mathcal{S}\text{-pred}(\alpha+1)$ would hold in any normal $\mathcal{S}'$ extending $\mathcal{S}\restriction(\alpha+1)$ such that $F = E_\alpha^{\mathcal{S}'}$. That is, $\mathcal{S}\restriction\beta+1 = \mathcal{S}^{<\mathrm{crit}(F)}$. Suppose that

$$\mathcal{T}\restriction\beta+1 = \mathcal{S}\restriction\beta+1.$$

Suppose also that if $\beta+1 < \mathrm{lh}(\mathcal{T})$, then $\mathrm{dom}(F) = \mathcal{M}_\beta^\mathcal{T}|\eta$ for some $\eta < \lambda(E_\beta^\mathcal{T})$, that is, assume that

$$\mathcal{T}\restriction\beta+1 = \mathcal{T}^{<\mathrm{crit}(F)}.$$

We define a normal tree $W(\mathcal{T}, \mathcal{S}, F)$.

Remark 3.42 The last supposition holds if either $\alpha = \beta$ and $\mathrm{lh}(F) < \mathrm{lh}(E_\beta^\mathcal{T})$, or $\alpha > \beta$, and $\mathrm{lh}(E_\beta^\mathcal{S}) \leq \mathrm{lh}(E_\beta^\mathcal{T})$. This will be the case when we use $W(\mathcal{T}, \mathcal{S}, F)$ to define $W(\mathcal{T}, \mathcal{U})$.

Let $Q \trianglelefteq N = \mathcal{M}_\theta^\mathcal{T}$, where $\theta+1 = \mathrm{lh}(\mathcal{T})$, and let

$$\mu = \mathrm{crit}(F).$$

Suppose that $\mathrm{Ult}(Q, F)$ makes sense, that is, $\mathrm{dom}(F) \leq \rho_{k(Q)}(Q)$. Suppose also that Q is the longest initial segment of N to which F applies, that is, either $Q = N$, or $\rho(Q) \leq \mu < \rho_{k(Q)}(Q)$. We want to define $W(\mathcal{T}, \mathcal{S}, F)$ so that $\mathrm{Ult}(Q, F)$ embeds weakly elementarily into the last model of $W(\mathcal{T}, \mathcal{S}, F)$.

There are three cases.

Case 1. $Q \neq N$.

In this case Q is a proper initial segment of $\mathcal{M}_\beta^\mathcal{T}|\mathrm{lh}(E_\beta^\mathcal{T})$, by the argument given in the dropping case of the definition of $W(\mathcal{T}, F)$.

$$W(\mathcal{T}, \mathcal{S}, F) = \mathcal{S}\restriction(\alpha+1)^\frown\langle F\rangle$$

is the unique normal continuation $\mathcal{W}$ of $\mathcal{S}\restriction(\alpha+1)$ of length $\alpha+2$ such that $E_\alpha^\mathcal{W} = F$. Note here that $\mathcal{M}_\beta^\mathcal{T} = \mathcal{M}_\beta^\mathcal{S}$, and Q is what F would be applied to in a normal continuation of $\mathcal{S}\restriction\alpha+1$. (Unlike the case $\mathcal{T} = \mathcal{S}$ we discussed before, it is possible that $Q \neq N$ and $\alpha > \beta$.) In this dropping case, the last model of $W(\mathcal{T}, \mathcal{S}, F)$ is equal to $\mathrm{Ult}(Q, F)$, and doesn't just embed it.

Case 2. $Q = N$, and $\mathrm{lh}(\mathcal{T}) = \beta + 1$.

Again

$$W(\mathcal{T}, \mathcal{S}, F) = \mathcal{S}\restriction(\alpha+1)^\frown\langle F\rangle$$

is the unique normal $\mathcal{S}'$ of length $\alpha + 2$ extending $\mathcal{S}$ such that $E_\alpha^{\mathcal{S}'} = F$. $Q = N = \mathcal{M}_\beta^{\mathcal{T}}$, and so $\mathrm{Ult}(Q, F)$ is equal to the last model of $W(\mathcal{T}, \mathcal{S}, F)$.

Case 3. $\mathrm{lh}\,\mathcal{T} > \beta + 1$, and $Q = N$.

In this case, we construct $\mathcal{W} = W(\mathcal{T}, \mathcal{S}, F)$ just as before. We set

$$\mathcal{W}\restriction(\alpha+1) = \mathcal{S}\restriction(\alpha+1),$$

and

$$\mathcal{M}_{\alpha+1}^{\mathcal{W}} = \mathrm{Ult}(\mathcal{M}_\beta^{\mathcal{T}}|\langle\gamma, k\rangle, F),$$

where k, γ are appropriate for normality. (Note $\mathcal{M}_\beta^{\mathcal{T}} = \mathcal{M}_\beta^{\mathcal{S}} = \mathcal{M}_\beta^{\mathcal{W}}$.) Let $\phi(\xi) = \xi$ for $\xi < \beta$, and $\phi(\xi) = (\alpha + 1) + (\xi - \beta)$ for $\xi \geq \beta$. Let $\pi_\xi = \mathrm{id}$ for $\xi < \beta$, and $\pi_\beta : \mathcal{M}_\beta^{\mathcal{T}}|\langle\gamma, k\rangle \to \mathcal{M}_{\alpha+1}^{\mathcal{W}}$ be the canonical embedding. Note that by our case hypothesis, F applies to $\mathcal{M}_\theta^{\mathcal{T}}$, and hence to $\mathcal{M}_\beta^{\mathcal{T}}|\,\mathrm{lh}(E_\beta^{\mathcal{T}})$, so $\langle \mathrm{lh}(E_\beta^{\mathcal{T}}), 0\rangle \leq \langle\gamma, k\rangle$. Thus π_β moves $E_\beta^{\mathcal{T}}$. So we can use the Shift lemma to lift the rest of $\mathcal{T}$, defining an elementary

$$\pi_\xi : \mathcal{M}_\xi^{\mathcal{T}} \to \mathcal{M}_{\phi(\xi)}^{\mathcal{W}}$$

for $\xi > \beta$, by induction on ξ. If $\sigma = T\text{-pred}(\xi)$, then $\phi(\sigma) = W\text{-pred}(\phi(\xi))$, unless $\sigma = \beta$ and $\mathrm{crit}(E_{\xi-1}^{\mathcal{T}}) < \mu$. In this case, $\mathrm{crit}(E_{\phi(\xi)-1}^{W}) = \mathrm{crit}(E_{\xi-1}^{\mathcal{T}}) < \mu$, so $W\text{-pred}(\phi(\xi)) = \beta$, rather than $\phi(\beta)$. We write

$$W(\mathcal{T}, \mathcal{S}, F) = \mathcal{S}^{<\mathrm{lh}\,F\frown}\langle F\rangle^\frown i_F\text{``}\mathcal{T}^{>\mathrm{crit}(F)}$$

in this case.

Remark 3.43 Recall that $\mathcal{T}$ and $\mathcal{S}$ were normal on M. Let Σ be an iteration strategy according to which both $\mathcal{T}$ and $\mathcal{S}$ are played. F and Σ determine $\mathcal{S}\restriction(\alpha+1)$, because F determines $\mathcal{M}_\alpha^{\mathcal{S}}|\,\mathrm{lh}\,F$, and thus $\mathcal{S}\restriction(\alpha+1)$ as the unique normal tree on M by Σ leading to a model having F on its sequence, and using only extenders of length less than $\mathrm{lh}\,F$. $\mathcal{S}\restriction(\alpha+1)$ is all we need of $\mathcal{S}$ to determine $W(\mathcal{T}, \mathcal{S}, F)$. So we could write $W(\mathcal{T}, \Sigma, F)$ for $W(\mathcal{T}, \mathcal{S}, F)$, or if Σ is understood, write $W(\mathcal{T}, F) = W(\mathcal{T}, \mathcal{S}, F)$.

Notation 3.43.1 Let $\alpha^{\mathcal{T},\mathcal{S},F}$ and $\beta^{\mathcal{T},\mathcal{S},F}$ be the α and β described above. In Case 3, let $\phi^{\mathcal{T},\mathcal{S},F}$ and $\pi_\xi^{\mathcal{T},\mathcal{S},F}$ for $\xi < \operatorname{lh}\mathcal{T}$ be the maps ϕ and π_ξ described there. In Cases 1 and 2, let $\operatorname{dom}(\phi^{\mathcal{T},\mathcal{S},F}) = \beta+1$, with $\phi^{\mathcal{T},\mathcal{S},F}(\xi) = \xi$ if $\xi < \beta$, and $\phi^{\mathcal{T},\mathcal{S},F}(\beta) = \alpha+1$. (Where $\alpha = \alpha^{\mathcal{T},\mathcal{S},F}$ and $\beta = \beta^{\mathcal{T},\mathcal{S},F}$.) Let $\pi_\xi^{\mathcal{T},\mathcal{S},F} = \mathrm{id}$ if $\xi < \beta$, and $\pi_\beta^{\mathcal{T},\mathcal{S},F} : \mathcal{M}_{\alpha+1}^{*,\mathcal{W}} = \mathcal{M}_\beta^{\mathcal{T}}|\xi \to \mathcal{M}_{\alpha+1}^{\mathcal{W}}$ be the canonical embedding in those cases.

In cases 2 and 3, we have an extended tree embedding

$$\Phi_{\mathcal{T},\mathcal{S},F} = \langle u, \langle s_\xi \mid \xi < \operatorname{lh}\mathcal{T}\rangle, \langle t_\xi \mid \xi+1 < \operatorname{lh}(\mathcal{T})\rangle, p\rangle$$

from $\mathcal{T}$ into $W(\mathcal{T},\mathcal{S},F)$. It is determined by setting

$$u = \phi^{\mathcal{T},\mathcal{S},F}.$$

Some of its other maps are given by

$$t_\xi = \pi_\xi^{\mathcal{T},\mathcal{S},F}$$

and

$$p(E_\xi^{\mathcal{T}}) = \pi_\xi^{\mathcal{T},\mathcal{S},F}(E_\xi^{\mathcal{T}}).$$

In case 1, these objects determine a partial extended tree embedding from $\mathcal{T}{\restriction}\beta{+}1$ into $\mathcal{W}(\mathcal{T},\mathcal{S},F)$. This is a system with all the properties of an extended tree embedding, except that its last map t_β may only be defined on some $Q \trianglelefteq \mathcal{M}_\beta^{\mathcal{T}}$. We call it $\Phi_{\mathcal{T},\mathcal{S},F}$ as well.

The illustrations associated to $W(\mathcal{T},\mathcal{S},F)$ are pretty much the same as before, allowing for the possibility that $\mathcal{S} \neq \mathcal{T}$. In particular, if $\xi \geq \beta^{\mathcal{T},\mathcal{S},F}$, then F either appears directly as one of the extenders used in $[0,\phi(\xi))_W$, or appears indirectly via some extender $F(G)$ used in $[0,\phi(\xi))_W$, where $\operatorname{crit}(G) < \mu < \lambda(G)$ and G is used in $[0,\xi)_T$.

Now let $\mathcal{T}$ be a normal tree on a premouse M, with last model Q, and let $\mathcal{U}$ be a normal tree on Q. We do not assume that $\mathcal{U}$ has a last model. We shall define $W(\mathcal{T},\mathcal{U}) = \mathcal{W}$, the embedding normalization of $\mathcal{T}^\frown\mathcal{U}$. For this, we define

$$\mathcal{W}_\gamma = W(\mathcal{T},\mathcal{U}|(\gamma+1)),$$

the embedding normalization of $\mathcal{T}^\frown\mathcal{U}|(\gamma+1)$, by induction on γ. Let us write

$$Q_\gamma = \mathcal{M}_\gamma^{\mathcal{U}} = \text{last model of } \mathcal{U}|(\gamma+1).$$

We shall maintain that each $\mathcal{W}_\gamma$ successor length, with last model

$$
\begin{aligned}
R_\gamma &= \text{last model of } \mathcal{W}_\gamma \\
&= \mathcal{M}^{\mathcal{W}_\gamma}_{z(\gamma)},
\end{aligned}
$$

and that there is an elementary embedding

$$\sigma_\gamma : Q_\gamma \to R_\gamma.$$

As we go we construct extended tree embeddings $\Phi_{\eta,\gamma}$, for $\eta <_U \gamma$, from an appropriate initial segment of $\mathcal{W}_\eta$ to $\mathcal{W}_\gamma$. $\Phi_{\eta,\gamma}$ is determined by its u-map $\phi_{\eta,\gamma}$ acting on an initial segment of $\mathrm{lh}(\mathcal{W}_\eta)$, and its t-maps we call

$$\pi^{\eta,\gamma}_\tau \colon \mathcal{M}^{\mathcal{W}_\eta}_\tau \to \mathcal{M}^{\mathcal{W}_\gamma}_{\phi_{\eta,\gamma}(\tau)},$$

defined when $\tau \in \mathrm{dom}(\phi_{\eta,\gamma})$. (There is the possibility that $\pi^{\eta,\gamma}_\tau$ acts only on some proper initial segment of $\mathcal{M}^{\mathcal{W}_\eta}_\tau$. That happens iff $(\eta,\gamma]_U$ has a drop.) Roughly, the system $(\langle \mathcal{W}_\gamma \mid \gamma < \mathrm{lh}(\mathcal{U})\rangle, \langle \Phi_{\eta,\gamma} \mid \eta <_U \gamma\rangle)$ is an iteration tree of iteration trees, whose base node is $\mathcal{W}_0 = \mathcal{T}$, and whose overall structure is induced by $\mathcal{U}$. The $\Phi_{\eta,\gamma}$ are the branch embeddings of this tree.

We set

$$\mathcal{W}_0 = \mathcal{T},$$

and let σ_0 be the identity. Now suppose everything is given up to γ. We let

$$F_\gamma = \sigma_\gamma(E^{\mathcal{U}}_\gamma).$$

Let α_γ be the least ξ such that F_γ is on the sequence of $\mathcal{M}^{\mathcal{W}_\gamma}_\xi$. So F_γ is on the sequence of $\mathcal{M}^{\mathcal{W}_\gamma}_\xi$ for all ξ such that $\alpha_\gamma \le \xi \le z(\gamma)$. We assume the following agreement hypotheses:

$(*)_\gamma$

(i) For $\eta \le \xi \le \gamma$, $\sigma_\eta \restriction (\mathrm{lh}(E^{\mathcal{U}}_\eta)+1) = \sigma_\xi \restriction (\mathrm{lh}(E^{\mathcal{U}}_\eta)+1)$.

(ii) For $\eta < \xi < \gamma$, $\alpha_\eta < \alpha_\xi$ and $\mathrm{lh}(F_\eta) < \mathrm{lh}(F_\xi)$.

(iii) For $\eta < \xi \le \gamma$, R_η agrees with R_ξ up to $\mathrm{lh}(F_\eta)$, but $\mathrm{lh}(F_\eta)$ is a cardinal of R_ξ, so they disagree at $\mathrm{lh}(F_\eta)$.

(iv) For $\eta < \xi \le \gamma$, $\mathcal{W}_\eta \restriction (\alpha_\eta+1) = \mathcal{W}_\xi \restriction (\alpha_\eta+1)$, and $E^{\mathcal{W}_\xi}_{\alpha_\eta} = F_\eta$.

(v) For $\eta < \gamma$,

(a) for all $\xi < \alpha_\eta$, $\mathrm{lh}(E_\xi^{\mathcal{W}_\eta}) < \mathrm{lh}(F_\eta)$, and

(b) if $\alpha_\eta < z(\eta)$, then $\mathrm{lh}(F_\eta) < \mathrm{lh}(E_{\alpha_\eta}^{\mathcal{W}_\eta})$.

Claim 3.44 *(ii) and (v) of* $(*)_{\gamma+1}$ *hold.*

Proof. For (ii), if $\eta < \gamma$, then $\mathrm{lh}(E_\eta^{\mathcal{U}}) < \mathrm{lh}(E_\gamma^{\mathcal{U}})$, so $\mathrm{lh}(F_\eta) < \mathrm{lh}(F_\gamma)$ by (i) at γ. Moreover, if $\alpha_\gamma \leq \alpha_\eta$, then by (iv), F_γ is on the sequence of $\mathcal{M}_{\alpha_\eta}^{\mathcal{W}_\gamma} = \mathcal{M}_{\alpha_\eta}^{\mathcal{W}_\eta}$. But F_η is also on the $\mathcal{M}_{\alpha_\eta}^{\mathcal{W}_\gamma}$ sequence, by (iv). Since $\mathrm{lh}(F_\eta) < \mathrm{lh}(F_\gamma)$ and F_γ is on the R_γ sequence, we get that F_η is on the R_γ sequence. However, F_η is used in $\mathcal{W}_\gamma$ by (iv) at γ, and thus F_η is not on the R_γ sequence.

(v)(a) holds because otherwise F_γ would be on the sequence of some $\mathcal{M}_\xi^{\mathcal{W}_\gamma}$ for $\xi < \alpha_\gamma$. For (v)(b), suppose $\alpha_\gamma < z(\gamma)$. Since F_γ is on the sequences of $\mathcal{M}_{\alpha_\gamma}^{\mathcal{W}_\gamma}$ and of $\mathcal{M}_{\alpha_\gamma+1}^{\mathcal{W}_\gamma}$, we must have $\mathrm{lh}(F_\gamma) < \mathrm{lh}(E_{\alpha_\gamma}^{\mathcal{W}_\gamma})$. □

Now suppose $\eta = U\text{-pred}(\gamma+1)$. We set

$$\mathcal{W}_{\gamma+1} = W(\mathcal{W}_\eta, \mathcal{W}_\gamma, F_\gamma).$$

Let us check that this makes sense. Let us write $F = F_\gamma$ and $\alpha = \alpha_\gamma$. Clearly $\alpha = \alpha^{\mathcal{W}_\eta, \mathcal{W}_\gamma, F}$. Let

$$\bar{\mu} = \mathrm{crit}(E_\gamma^{\mathcal{U}}),$$

and

$$\mu = \sigma_\gamma(\bar{\mu}) = \mathrm{crit}(F).$$

Let

$$\begin{aligned} \beta &= \beta^{\mathcal{W}_\eta, \mathcal{W}_\gamma, F} \\ &= \text{ least } \xi \text{ such that } \mu < \lambda(E_\xi^{\mathcal{W}_\gamma}) \text{ or } \xi = z(\gamma) \end{aligned}$$

be the tree predecessor of $\alpha+1$ in any normal continuation $\mathcal{S}$ of $\mathcal{W}_\gamma \restriction (\alpha+1)$ that uses F. Since η is the least ξ such that $\bar{\mu} < \lambda(E_\xi^{\mathcal{U}})$, we have by (i) of $(*)_\gamma$ that

$$\eta = \text{ the least } \xi \text{ such that } \mu < \lambda(F_\eta).$$

But $\mathcal{W}_\eta \restriction (\alpha_\eta+1) = \mathcal{W}_\gamma \restriction (\alpha_\eta+1)$, and $E_{\alpha_\eta}^{\mathcal{W}_\gamma} = F_\eta$ or else $\eta = \gamma$. In either case, $\beta \leq \alpha_\eta$, so

$$\mathcal{W}_\eta \restriction (\beta+1) = \mathcal{W}_\gamma \restriction (\beta+1).$$

Moreover, since $\beta \leq \alpha_\eta$, if $\beta < z(\eta)$ then

$$\mathrm{lh}(E_\beta^{\mathcal{W}_\gamma}) \leq \mathrm{lh}(E_\beta^{\mathcal{W}_\eta}),$$

with equality holding iff $\beta < \alpha_\eta$. These are the conditions we needed to check, so $W(\mathcal{W}_\eta, \mathcal{W}_\gamma, F)$ makes sense.

Let $\Phi_{\eta,\gamma+1}$ be the (possibly partial) extended tree embedding $\Phi_{\mathcal{W}_\eta,\mathcal{W}_\gamma,F}$. Its u-map is

$$\phi_{\eta,\gamma+1} = \phi^{\mathcal{W}_\eta,\mathcal{W}_\gamma,F},$$

and its t maps are

$$\pi_\tau^{\eta,\gamma+1} = \pi_\tau^{\mathcal{W}_\eta,\mathcal{W}_\gamma,F}.$$

For $\delta <_U \eta$,

$$\Phi_{\delta,\gamma+1} = \Phi_{\eta,\gamma+1} \circ \Phi_{\delta,\eta}.$$

This of course means that $\phi_{\delta,\gamma+1} = \phi_{\eta,\gamma+1} \circ \phi_{\delta,\eta}$, and $\pi_\tau^{\delta,\gamma+1} = \pi_{\phi_{\delta,\eta}(\tau)}^{\eta,\gamma+1} \circ \pi_\tau^{\delta,\eta}$. Here the compositions are considered as defined wherever they make sense.

Note that $\Phi_{\eta,\gamma+1}$ is partial iff $\gamma+1 \in D^{\mathcal{U}}$. If $\gamma+1 \in D^{\mathcal{U}}$, then $\mathrm{dom}(\phi_{\eta,\gamma+1}) = \beta+1$, and $\pi_\beta^{\eta,\gamma+1}$ acts on a proper initial segment of $\mathcal{M}_\beta^{\mathcal{W}_\eta}$.

$\sigma_{\gamma+1}$ is determined as follows. Let

$$Q_{\gamma+1} = \mathrm{Ult}(Q^*, E_\gamma^{\mathcal{U}}),$$

where $Q^* \trianglelefteq Q_\eta$.

Let $R^* = R_\eta$ if $Q^* = Q_\eta$, and $R^* = \sigma_\eta(Q^*)$ otherwise. $\sigma_\eta \restriction Q^*$ is elementary from Q^* to R^*.

Suppose first that we drop in $\mathcal{U}$, i.e. $Q^* \neq Q_\eta$. Then $\rho(Q^*) \leq \bar{\mu}$, and σ_η is a near $k(Q^*)+1$ embedding, so

$$\mu = \sigma_\gamma(\bar{\mu}) = \sigma_\eta(\bar{\mu}) \leq \rho(R^*),$$

while $\rho_{k(R^*)}(R^*) = \sigma_\eta(\rho_{k(Q)}(Q)) > \mu$. So R^* is what we would apply F to in a normal continuation of $\mathcal{W}_\gamma \restriction (\alpha+1)$. Moreover,

$$\mathcal{W}_{\gamma+1} = \mathcal{W}_\gamma^{<\mathrm{lh}\, F} {}^\frown \langle F \rangle {}^\frown \mathrm{Ult}(R^*, F)$$

because we are in case 1 of the definition of $W(\mathcal{W}_\eta, \mathcal{W}_\gamma, F)$. So $R_{\gamma+1} = \mathrm{Ult}(R^*, F)$, and we can take $\sigma_{\gamma+1}$ to be the Shift Lemma map.

Suppose next that $Q^* = Q_\eta$, so that we are in case 2 or case 3, and

$$\mathcal{W}_{\gamma+1} = \mathcal{W}_\gamma^{<\mathrm{lh}\, F} {}^\frown \langle F \rangle {}^\frown i_F \text{“} \mathcal{W}_\eta^{>\mathrm{crit}(F)}.$$

For $\tau \leq z(\eta)$, we have an elementary $\pi^{\eta,\gamma+1}_{\tau} : \mathcal{M}^{\mathcal{W}_\eta}_{\tau} \to \mathcal{M}^{W_{\gamma+1}}_{\phi_{\eta,\gamma+1}(\tau)}$. Since we are not dropping in $\mathcal{U}$,

$$Q^{\mathcal{U}}_{\gamma+1} = \mathrm{Ult}(Q^{\mathcal{U}}_{\eta}, E^{\mathcal{U}}_{\gamma}).$$

and

$$\phi_{\eta,\gamma+1}(z(\eta)) = z(\gamma+1).$$

We have then the diagram

$$\begin{tikzcd}
Q_{\gamma+1} \arrow[r, "\theta"] & \mathrm{Ult}(R_\eta, F) \arrow[r, "\psi"] & R_{\gamma+1} = \mathcal{M}^{\mathcal{W}_{\gamma+1}}_{z(\gamma+1)} \\
Q_\eta \arrow[u, "i^{\mathcal{U}}_{\eta,\gamma+1}"] \arrow[r, "\sigma_\eta"'] & R_\eta = \mathcal{M}^{\mathcal{W}_\eta}_{z(\eta)} \arrow[u] \arrow[ur, "\pi^{\eta,\gamma+1}_{z(\eta)}"']
\end{tikzcd}$$

Here θ is given by the Shift Lemma, and ψ comes from the fact that F is an initial segment of the extender of $\pi^{\eta,\gamma+1}_{z(\eta)}$, as we remarked before. (So $\psi \restriction \mathrm{lh}\, F = \mathrm{id}$.) We then set

$$\sigma_{\gamma+1} = \psi \circ \theta.$$

So when $\gamma + 1 \notin D^{\mathcal{U}}$, we have the diagram

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{U}}_{\gamma+1} \arrow[r, "\sigma_{\gamma+1}"] & R_{\gamma+1} \\
\mathcal{M}^{\mathcal{U}}_{\eta} \arrow[u, "i^{\mathcal{U}}_{\eta,\gamma+1}"] \arrow[r, "\sigma_\eta"'] & R_\eta \arrow[u, "\pi^{\eta,\gamma+1}_{z(\eta)}"']
\end{tikzcd}$$

When $\gamma + 1 \in D^{\mathcal{U}}$, we have the diagram

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{U}}_{\gamma+1} \arrow[r, "\sigma_{\gamma+1}"] & R_{\gamma+1} \\
\mathcal{M}^{*,\mathcal{U}}_{\gamma+1} \arrow[u, "i^{*,\mathcal{U}}_{\gamma+1}"] \arrow[r, "\sigma_\eta"'] & \sigma_\eta(\mathcal{M}^{*,\mathcal{U}}_{\gamma+1}) \arrow[u, "\pi^{\eta,\gamma+1}_{\beta}"']
\end{tikzcd}$$

where $\beta = \beta^{W_\eta, \mathcal{W}_\gamma, F}$.

Claim 3.45 $(*)_{\gamma+1}$ *holds.*

Proof. Left to the reader. □

We have completed the definition of $\mathcal{W}_{\gamma+1}$.

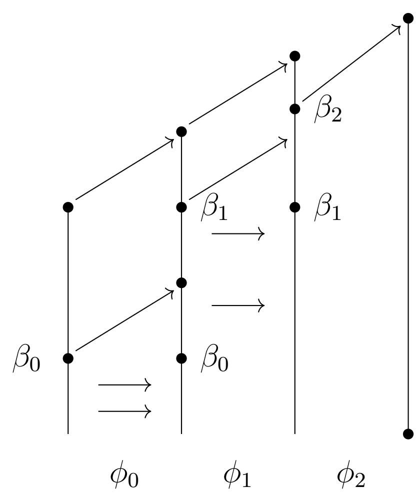
If $\lambda < \mathrm{lh}(\mathcal{U})$ is a limit ordinal, then

$$\mathcal{W}_\lambda = \lim_{\alpha <_U \lambda} \mathcal{W}_\alpha,$$

where we make sense of the direct limit using the tree embeddings $\Phi_{\eta,\gamma}$ for $\eta <_U \gamma <_U \lambda$. We give a little more detail on this below.

In our context of interest, $\langle \mathcal{T}, \mathcal{U} \rangle$ is played by a background-induced iteration strategy Σ for M, and we shall show that all $\mathcal{W}_\alpha$ are by Σ. So in our context of interest, all models above are wellfounded.

Here are a couple illustrations that the reader may or may not find helpful. Let $\gamma_0 U \gamma_1 U \gamma_2 U \gamma_3$ be successive elements of a branch of U. Write $\phi_i = \phi_{\gamma_i,\gamma_{i+1}}$. Let $\beta_i = \beta^{W_{\gamma_i}, \mathcal{W}_{\tau_i}, F_i}$, where $\tau_i = \gamma_{i+1} - 1$ and $F_i = \sigma_{\tau_i}(E^{\mathcal{U}}_{\tau_i})$. Thus $\mathcal{W}_{\gamma_{i+1}} = W(\mathcal{W}_{\gamma_i}, \mathcal{W}_{\tau_i}, F_i)$, and $\beta_i = \mathrm{crit}(\phi_i)$. The ϕ_i might look like:

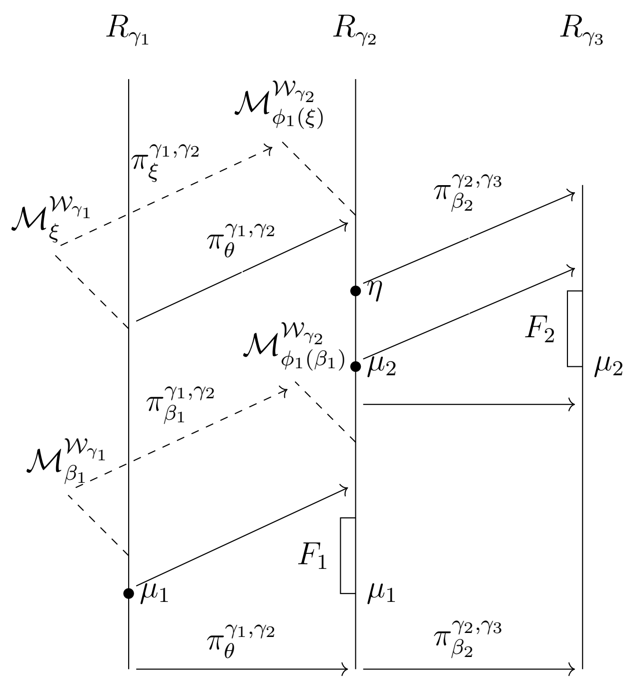


The last step pictured involves a drop. Notice that $\beta_{i+1} \geq \phi_i(\beta_i)$. (equality is possible.) This is because $\mathcal{U}$ is normal. In $\mathcal{W}_{\gamma_{i+1}}$, $\mathcal{M}^{\mathcal{W}_{\gamma_{i+1}}}_{\phi_i(\beta_i)}$ is immediately above $\mathcal{M}^{\mathcal{W}_{\gamma_{i+1}}}_{\beta_i}$ via an F_i-ultrapower. Moreover, $\mathcal{W}_{\gamma_{i+1}} \restriction (\alpha+1) = \mathcal{W}_{\tau_i} \restriction (\alpha+1)$, where $\alpha+1 = \phi_i(\beta_i)$. By our choice of α, $\lambda(E^{\mathcal{W}_{\tau_i}}_\xi) \leq \lambda(F_i)$ for all $\xi < \alpha$. But $\lambda(F_i) \leq \mathrm{crit}(F_{i+1})$, since $\mathcal{U}$ is normal, so F_{i+1} cannot be applied to any $\mathcal{M}^{\mathcal{W}_{\gamma_{i+1}}}_\xi$ for $\xi < \phi_i(\beta_i)$.

Because $\beta_{i+1} \geq \phi_i(\beta_i)$, and above $\phi_i(\beta_i)$, $\mathrm{ran}(\phi_i)$ is an initial segment of $\mathrm{ORD} - \phi(\beta_i)$, we see that along any branch b of $\mathcal{U}$, the direct limit of the $\phi_{\gamma,\eta}$ for $\gamma, \eta \in b$ is wellfounded.

In fact, the direct limit has order type $\lambda + \theta$, where $\lambda = \sup_{\eta \in b} \operatorname{crit}(\phi_{\eta,b})$, and $\theta = \operatorname{lh} \mathcal{T} - \beta$, where β is least such that $\phi_{0,b}(\beta) \geq \lambda$.

In addition to the ϕ-maps on indices of models, we have the π-maps on the models. Let $\mu_i = \operatorname{crit}(F_i)$, and let $\operatorname{lh}(\mathcal{W}_{\gamma_1}) = \theta + 1$. Let η be the level of R_{γ_2}, or equivalently $\mathcal{M}_{\beta_2}^{\mathcal{W}_{\gamma_2}}$, that we drop to when we apply F_2. The picture is



One can look at $\Phi_{\eta,\gamma}$, for $\eta <_U \gamma$, as a map on the extender trees. Let $p_{\eta,\gamma}$ be the p-map of $\Phi_{\eta,\gamma}$, that is

$$p_{\eta,\gamma} : \operatorname{Ext}(\mathcal{W}_\eta) \to \operatorname{Ext}(\mathcal{W}_\gamma)$$

and

$$p_{\eta,\gamma}(E_\xi^{\mathcal{W}_\eta}) = \pi_\xi^{\eta,\gamma}(E_\xi^{\mathcal{W}_\eta}) = E_{\phi_{\eta,\gamma}(\xi)}^{\mathcal{W}_\gamma}.$$

So $p_{\eta,\gamma}(E_\xi^{\mathcal{W}_\eta})\downarrow$ iff $\xi \in \operatorname{dom} \phi_{\eta,\gamma}$. Let

$$\hat{p}(s) = \text{ least } t \in \mathcal{W}_\gamma^{\text{ext}} \text{ such that } p\text{``}\operatorname{ran}(s) \subseteq \operatorname{ran}(t).$$

By Proposition 3.34, $\hat{p}_{\eta,\gamma}$ preserves extender tree order and incompatibility; that is $s \subseteq t \Rightarrow \hat{p}_{\eta,\gamma}(s) \subseteq \hat{p}_{\eta,\gamma}(t)$, and $s \perp t \Rightarrow \hat{p}_{\eta,\gamma}(s) \perp \hat{p}_{\eta,\gamma}(t)$. Moreover

Proposition 3.46 *Let $\eta <_U \gamma$ and $\phi_{\eta,\gamma}(\alpha)\downarrow$, and suppose whenever $\eta \leq_U \xi <_U \gamma$, then $\phi_{\eta,\xi}(\alpha) \geq \mathrm{crit}(\phi_{\xi,\gamma})$. Then for $s = e_\alpha^{\mathcal{W}_\eta}$,*

$$e^{\mathcal{W}_\gamma}_{\phi_{\eta,\gamma}(\alpha)} = \hat{p}_{\eta,\gamma}(s)^\frown\langle F_\tau \mid \tau + 1 \leq_U \gamma \text{ and for all } i \in \mathrm{dom}\, \hat{p}_{\eta,\gamma}(s), \\ \lambda(\hat{p}_{\eta,\gamma}(s)(i)) \leq \mathrm{crit}(F_\tau)\rangle$$

We omit the simple proof. The proposition says that the branch extender to $\mathcal{M}^{\mathcal{W}_\gamma}_{\phi_{\eta,\gamma}(\alpha)}$ consists of blow-ups by $p_{\eta,\gamma}$ of extenders used in the branch to $\mathcal{M}^{\mathcal{W}_\eta}_\alpha$, together with certain F_τ's used in $\mathcal{U}$ from η to γ. It generalizes our pictures on page 86 and before.

Suppose now that $\lambda \leq \mathrm{lh}(\mathcal{U})$ is a limit ordinal, and we have defined $\mathcal{W}_\gamma, \sigma_\gamma$, and the $\Phi_{\eta,\gamma}$ for $\eta, \gamma < \lambda$. We let $W(\mathcal{T}, \mathcal{U}{\restriction}\lambda)$ be the lim inf of the $\mathcal{W}_\gamma$ for $\gamma < \mathrm{lh}\,\mathcal{U}$. More precisely, let

$$F_\gamma = \sigma_\gamma(E^{\mathcal{U}}_\gamma)$$

and

$$\begin{aligned} \alpha_\gamma = &\text{ least } \alpha \text{ such that } F_\gamma \text{ is on the sequence of } \mathcal{M}^{\mathcal{W}_\gamma}_\alpha \\ = &\text{ largest } \alpha \text{ such that } \mathcal{W}_{\gamma+1}{\restriction}(\alpha+1) = \mathcal{W}_\gamma{\restriction}(\alpha+1). \end{aligned}$$

We put

$$W(\mathcal{T}, \mathcal{U}{\restriction}\lambda) = \bigcup_{\gamma < \mathrm{lh}\,\mathcal{U}} \mathcal{W}_\gamma{\restriction}(\alpha_\gamma + 1).$$

Since $\gamma < \eta \Rightarrow \alpha_\gamma < \alpha_\eta$, $W(\mathcal{T}, \mathcal{U} \restriction \lambda)$ has limit length. There are no new σ's or Φ's to be defined at this stage.

Now let b be a cofinal branch of $\mathcal{U}{\restriction}\lambda$ (not necessarily a wellfounded one). We define the embedding normalization

$$\mathcal{W}_b = W(\mathcal{T}, \mathcal{U}^\frown b)$$

by forming the direct limit of the $\mathcal{W}_\gamma$, for $\gamma \in b$, under the $\Phi_{\eta,\gamma}$ for $\eta <_U \gamma$ in b.

We begin with $\mathrm{lh}(\mathcal{W}_b)$. Let us put

$$\langle \eta, \xi \rangle \in I \quad \text{iff} \quad \eta \in b, \text{ and for all sufficiently large } \gamma \in b,\ \phi^{\eta,\gamma}(\xi)\downarrow.$$

Put

$$\langle \eta, \xi \rangle \leq_I \langle \delta, \theta \rangle \quad \text{iff} \quad \text{for all sufficiently large } \gamma \in b,\ \phi_{\eta,\gamma}(\xi) \leq \phi_{\delta,\gamma}(\theta).$$

It is easy to see that $\leq_I$ is a prewellorder (even if b is illfounded, or drops infinitely often). We set

$$\mathrm{lh}(\mathcal{W}_b) = \mathrm{otp}(I, \leq_I).$$

For $\eta \in b$, we let $\phi_{\eta,b}(\xi)\downarrow$ iff $\langle \eta, \xi \rangle \in I$, and in that case, set

$$\phi_{\eta,b}(\xi) = \text{rank of } \langle \eta, \xi \rangle \text{ in } (I, \leq_I).$$

We define the tree order $\leq_{W_b}$ by: given $\langle \eta, \xi \rangle$ and $\langle \delta, \theta \rangle \in I$

$$\phi_{\eta,b}(\xi) \leq_{W_b} \phi_{\delta,b}(\theta) \quad \text{iff} \quad \text{for all sufficiently large } \gamma \in b,\ \phi_{\eta,\gamma}(\xi) \leq_{W_\gamma} \phi_{\delta,\gamma}(\theta).$$

Although the $\phi_{\eta,\gamma}$ do not completely preserve tree order, they almost do so. See clause (4) in the list following Remark 3.12, and the illustration on p.85. Using this, we can show $\leq_{W_b}$ is a tree order. $\phi_{\eta,b}$ may fail to preserve tree order, but again, this can only happen in a way similar to the possible failure described after 3.12. We record this in a proposition.

Proposition 3.47 *Let $\langle \eta, \xi \rangle$, $\langle \eta, \delta \rangle \in I$, and suppose $\xi \leq_{W_\eta} \delta$ but $\phi_{\eta,b}(\xi) \not\leq_{W_b} \phi_{\eta,b}(\delta)$. Then there is a unique $\gamma \geq \eta$ in b such that letting U-pred$(\theta + 1) = \gamma$ with $\theta + 1 \in b$, $F = F_\theta$, and $\beta = \beta^{\mathcal{W}_\gamma, \mathcal{W}_\theta, F}$, we have*

1. *$\beta = \phi_{\eta,\gamma}(\xi) \leq_{W_\gamma} \phi_{\eta,\gamma}(\delta)$, and*
2. *letting G be the first extender used in $[0, \phi_{\eta,\gamma}(\delta))$ such that $\lambda(G) \geq \lambda(E_\beta^{\mathcal{W}_\gamma})$, we have $\mathrm{crit}(G) < \mathrm{crit}(F) < \lambda(G)$.*

Moreover, in this case, if $\xi = W_\eta$-pred(δ), $\beta = \phi_{\eta,\gamma}(\xi) = W_\gamma$-pred$(\phi_{\eta,\gamma}(\delta))$, and

$$W_{\theta+1}\text{-pred}(\phi_{\eta,\theta+1}(\delta)) = \beta = W_{\theta+1}\text{-pred}(\phi_{\eta,\theta+1}(\xi)).$$

We omit the easy proof. Using such arguments, we can show $\leq_{W_b}$ is a tree order, and

Proposition 3.48 *Let $\langle \eta, \xi \rangle$ and $\langle \delta, \theta \rangle \in I$. Then $\phi_{\eta,b}(\xi) = W_b$-pred$(\phi_{\delta,b}(\theta))$ iff for all sufficiently large $\gamma \in b$, $\phi_{\eta,\gamma}(\xi) = W_\gamma$-pred$(\phi_{\delta,\gamma}(\theta))$.*

Here is a more concrete description of $\mathrm{lh}(\mathcal{W}_b)$ and $\phi_{\eta,b}$. Let

$$\begin{aligned}\delta &= \mathrm{lh}\, W(\mathcal{T}, \mathcal{U}\restriction\lambda) \\ &= \sup_{\gamma<\lambda} \alpha_\gamma \\ &= \sup\{\mathrm{crit}\, \phi_{\eta,\gamma} \mid \eta <_U \gamma \wedge \gamma \in b\}.\end{aligned}$$

(The last equality holds because if $\eta = U\text{-pred}(\gamma+1)$ and $\gamma + 1 \leq_U \tau$ where $\tau \in b$, then $\text{crit}(\phi_{\eta,\gamma+1}) \leq \alpha_\gamma < \text{crit}(\phi_{\gamma+1,\tau})$.)

Case 1. b drops somewhere.

Let $\gamma+1$ be least in $b \cap D^{\mathcal{U}}$, and $\eta = U\text{-pred}(\gamma+1)$, and $\beta = \beta^{\mathcal{W}_\eta, \mathcal{W}_\gamma, F_\gamma} = \text{crit}(\phi_{\eta,\gamma+1})$. Let $\beta = \phi_{0,\eta}(\tau)$. Then for all $\gamma + 1 \leq_U \theta <_U \rho$, with $\rho \in b$,

$$\begin{aligned} \text{crit}(\phi_{\theta,\rho}) &= \phi_{\eta,\theta}(\beta) \\ &= \text{lh}(\mathcal{W}_\theta) - 1. \end{aligned}$$

(Further dropping cuts down on the domains of the π-maps, not on that of the ϕ-maps.) Thus

$$\begin{aligned} \text{lh}(\mathcal{W}_b) &= \delta + 1 \\ &= \phi_{\eta,b}(\beta) + 1 = \phi_{0,b}(\tau) + 1. \end{aligned}$$

Case 2. b does not drop.

Let

$$\begin{aligned} \tau = \tau_b = \text{least } \alpha < \text{lh}\,\mathcal{T} \text{ such that for all } \gamma <_U \xi \\ \text{with } \xi \in b,\ \phi_{0,\gamma}(\alpha) \geq \text{crit}(\phi_{\gamma,\xi}). \end{aligned}$$

Then

$$\begin{aligned} \phi_{0,b}(\tau) &= \delta, \\ \text{lh}(\mathcal{W}_b) &= \delta + (\text{lh}\,\mathcal{T} - \tau), \end{aligned}$$

and for $\xi \geq \tau$ with $\xi < \text{lh}(\mathcal{T})$,

$$\phi_{0,b}(\xi) = \delta + (\xi - \tau).$$

This case can happen in two ways: it can be that $\phi_{0,\eta}(\tau) = \text{crit}(\phi_{\eta,\gamma})$ for some $\eta <_U \gamma$ with $\gamma \in b$, in which case that is true for all sufficiently large such η, γ. Or it can happen that $\phi_{0,\eta}(\tau) > \text{crit}(\phi_{\eta,\gamma})$, for all $\eta <_U \gamma$ with $\gamma \in b$. In that case, τ is a limit ordinal, and the extenders in b are being inserted cofinally into the branch extender of $[0, \tau)_T$.

It can happen in Case 2 that τ is a limit ordinal, but some $\phi_{0,\eta}(\tau)$ and its images are in the "eventual critical points" along b. In that case, some tail of the extenders used in b are being inserted after the blow-ups of all those in $[0, \tau)_T$.

Now we define the models and extenders of $\mathcal{W}_b$. Suppose $\alpha = \phi_{\eta,b}(\gamma) < \mathrm{lh}(\mathcal{W}_b)$. Suppose $\eta \le \xi < \delta \in b$. Then we have the map $\pi^{\xi,\delta}_{\phi_{\eta,\xi}(\gamma)}$ acting on either $\mathcal{M}^{\mathcal{W}_\xi}_{\phi_{\eta,\xi}(\gamma)}$ or an initial segment thereof. We let

$$\mathcal{M}^{\mathcal{W}_\gamma}_\alpha = \text{dirlim of the } \mathcal{M}^{\mathcal{W}_\xi}_{\phi_{\eta,\xi}(\gamma)} \text{ under the } \pi^{\xi,\delta}_{\phi_{\eta,\xi}(\gamma)}\text{'s.}$$

If b does not drop after η, then we have

$$\pi^{\eta,b}_\gamma : \mathcal{M}^{\mathcal{W}_\eta}_\gamma \to \mathcal{M}^{\mathcal{W}_b}_{\phi_{\eta,b}(\gamma)}$$

as the direct limit map. Otherwise $\pi^{\eta,b}_\gamma$ may (or may not) act on a proper initial segment of $\mathcal{M}^{\mathcal{W}_\eta}_\gamma$.

Finally, if $\alpha = \phi_{\eta,b}(\gamma) < \mathrm{lh}(\mathcal{W}_b)$ and $\alpha + 1 < \mathrm{lh}(\mathcal{W}_\gamma)$, then

$$E^{\mathcal{W}_b}_\alpha = \pi^{\eta,b}_\gamma(E^{\mathcal{W}_\eta}_\gamma).$$

One can check that with this choice of extenders, $\mathcal{W}_b$ is a normal iteration tree on M. For example, suppose that $\eta \in b$ and that for all $\xi \ge \eta$ in b, W_ξ-pred$(\phi_{\eta,\xi}(\gamma+1)) = \phi_{\eta,\xi}(\theta)$, and we aren't dropping, so

$$\mathcal{M}^{\mathcal{W}_\xi}_{\phi_{\eta,\xi}(\gamma+1)} = \mathrm{Ult}(\mathcal{M}^{\mathcal{W}_\xi}_{\phi_{\eta,\xi}(\theta)}, E^{\mathcal{W}_\xi}_{\phi_{\eta,\xi}(\gamma)}).$$

Then

$$\mathcal{M}^{\mathcal{W}_b}_{\phi_{\eta,b}(\gamma+1)} = \mathrm{Ult}(\mathcal{M}^{\mathcal{W}_b}_{\phi_{\eta,b}(\theta)}, E^{\mathcal{W}_b}_{\phi_{\eta,b}(\gamma)}).$$

because each of the three objects in this equation is a direct limit of its ξ-approximations, for $\xi \in b$, and the maps commute appropriately. We omit further detail.

Now we also have the natural map

$$\sigma_b \colon \mathcal{M}^{\mathcal{U}}_b \to R_b,$$

where R_b is the last model of $\mathcal{W}_b$, given by

$$\sigma_b(i^{\mathcal{U}}_{\gamma,b}(x)) = \pi^{\gamma,b}_{z(\gamma)}(\sigma_\gamma(x)).$$

In the abstract, it may happen that not all models of $\mathcal{W}_b$ are wellfounded. In our context of interest, $\langle \mathcal{T}, \mathcal{U}^\frown b\rangle$ is played according to an iteration strategy Σ for M, and we show that Σ is sufficiently good that $\mathcal{W}_b$ is also played by Σ.

Now suppose $\lambda < \mathrm{lh}\,\mathcal{U}$ and $b = [0, \lambda)_U$, and all models of $\mathcal{W}_b$ are wellfounded. Then we set

$$\begin{aligned}\mathcal{W}_\lambda &= \mathcal{W}_b,\\ \phi_{\eta,\lambda} &= \phi_{\eta,b},\\ \pi_\gamma^{\eta,\lambda} &= \pi_\gamma^{\eta,b},\\ \sigma_\lambda &= \sigma_b,\end{aligned}$$

and continue with the inductive construction of $W(\mathcal{T},\mathcal{U})$. If some model of $\mathcal{W}_b$ is illfounded, we stop the construction, and say that $W(\mathcal{T},\mathcal{U})$ is undefined.

Finally, if $\mathcal{U}$ has a last model, we set $W(\mathcal{T},\mathcal{U}) = \mathcal{W}_\gamma$, where $\mathrm{lh}\,\mathcal{U} = \gamma + 1$. If $\mathcal{U}$ has limit length λ, then $W(\mathcal{T},\mathcal{U}) = W(\mathcal{T},\mathcal{U}{\restriction}\lambda)$ has already been defined.

To summarize our notation associated to $W(\mathcal{T},\mathcal{U})$: for $\gamma < \mathrm{lh}\,\mathcal{U}$,

$$F_\gamma = \sigma_\gamma(E_\gamma^{\mathcal{U}})$$

where $\sigma_\gamma : \mathcal{M}_\gamma^{\mathcal{U}} \to R_\gamma =$ last model of $\mathcal{W}_\gamma$, and

$$\mathcal{W}_{\gamma+1} = W(\mathcal{W}_\eta, \mathcal{W}_\gamma, F_\gamma)$$

where $\eta = U\text{-pred}(\gamma + 1)$. By normality, modulo an iteration strategy according to which all $\mathcal{W}_\gamma$ are played, R_γ and $\mathcal{W}_\gamma$ determine each other, while F_γ and $\mathcal{W}_\gamma{\restriction}(\alpha_\gamma+1)$ determine each other. The R_γ's are not the models of a single iteration tree; they constitute and enlargement of $\mathcal{U}$, with accompanying maps $\sigma_\gamma\colon \mathcal{M}_\gamma^{\mathcal{U}} \to R_\gamma$. We proved the basic facts about agreement of models and maps in this enlargement in $(*)_\gamma$ above; we list some of them again here for reference.

Proposition 3.49 *Let $\gamma < \eta < \mathrm{lh}\,\mathcal{U}$. Then*

(*a*) *R_γ agrees with R_η below $\mathrm{lh}\,F_\gamma$,*

(*b*) *$\sigma_\eta \restriction (\mathrm{lh}(E_\eta^{\mathcal{U}}) + 1) = \sigma_\gamma \restriction (\mathrm{lh}(E_\eta^{\mathcal{U}}) + 1)$, and*

(*c*) *F_γ is on the sequence of R_γ, but not that of R_η. In fact, $\mathrm{lh}(F_\gamma)$ is a cardinal of R_η.*

The following diagram summarizes the situation. We draw the diagram as if the maps in question exist, although sometimes they may not, because of dropping. Let $z(\eta) + 1 = \mathrm{lh}(\mathcal{W}_\eta)$, and let $i^{\mathcal{W}_\eta} : M \to R_\eta$ be the canonical embedding (assuming M-to-R_η does not drop).

$$
\begin{tikzcd}
R_0 = \mathcal{M}_0^{\mathcal{U}} \arrow[rr] \arrow[drr] & & \mathcal{M}_\eta^{\mathcal{U}} \arrow[r, "i^{\mathcal{U}}_{\eta,\gamma}"] \arrow[d, "\sigma_\eta"] & \mathcal{M}_\gamma^{\mathcal{U}} \arrow[dd, "\sigma_\gamma"] \\
& & R_\eta \arrow[dr, "\pi^{\eta,\gamma}_{z(\eta)}"] & \\
& \mathcal{M}_\sigma^{\mathcal{W}_\eta} \arrow[dr, "\pi_\sigma^{\eta,\gamma}"] & & R_\gamma \\
& & \mathcal{M}^{\mathcal{W}_\gamma}_{\phi_{\eta,\gamma}(\sigma)} & \\
M \arrow[uuuu, "i^{\mathcal{W}_0}"] \arrow[uur, "i^{\mathcal{W}_\eta}_{0\sigma}"] \arrow[uuurr, "i^{\mathcal{W}_\eta}"] \arrow[urr] \arrow[uurrr, bend right, "i^{\mathcal{W}_\gamma}"'] & & &
\end{tikzcd}
$$

The various embeddings all commute:

(i) $i^{\mathcal{W}_\gamma} = \pi^{\eta,\gamma}_{z(\eta)} \circ i^{\mathcal{W}_\eta}$

(ii) $\pi_\sigma^{\eta,\gamma} \circ i^{\mathcal{W}_\eta}_{\xi,\sigma} = i^{\mathcal{W}_\gamma}_{\phi_{\eta,\gamma}(\xi),\phi_{\eta,\gamma}(\sigma)} \circ \pi_\xi^{\eta,\gamma}$ (general version of (i))

(iii) $\sigma_\gamma \circ i^{\mathcal{U}}_{\eta,\gamma} = \pi^{\eta,\gamma}_{z(\eta)} \circ \sigma_\eta$.

In a sufficiently coarse case, the upper triangle in the diagram above collapses.

Proposition 3.50 *Let $\mathcal{T}$ be normal on M, and $\mathcal{U}$ normal on the last model $\mathcal{T}$. Suppose also that $\mathcal{T}$ and $\mathcal{U}$ are ms-normal Suppose that whenever $\alpha + 1 < \mathrm{lh}\,\mathcal{T}$,*

$$\mathcal{M}_\alpha^{\mathcal{T}} \models \nu(E_\alpha^{\mathcal{T}}) \textit{ is strongly inaccessible.}$$

Let $\mathcal{W}_\eta$, $\sigma_\eta : \mathcal{M}_\eta^{\mathcal{U}} \to R_\eta$, $R_\eta = \mathcal{M}^{\mathcal{W}_\eta}_{z(\eta)}$ etc., be as above. Then

(1) *$R_\eta = \mathcal{M}_\eta^{\mathcal{U}}$, and $\sigma_\eta = \mathrm{id}$, for all $\eta < \mathrm{lh}(\mathcal{U})$;*

(2) *if $\eta <_U \gamma$, then $i^{\mathcal{U}}_{\eta,\gamma} = \pi^{\eta,\gamma}_{z(\eta)}$.*

Proof. Proposition 3.18 generalizes to $W(\mathcal{W}_\eta, \mathcal{W}_\gamma, F)$, where F comes from $\mathcal{W}_\gamma$. We use that repeatedly. $\square$

Remark 3.51 There is a tacit hypothesis in 3.50 that all models in W_γ are well-founded. The ms-normality hypothesis is there because if we replace $\nu(E_\alpha^\mathcal{T})$ by $\lambda(E_\alpha^\mathcal{T})$ above, then the hypothesis implies that $M \models$ " there is a superstrong cardinal".

Remark 3.52 We shall need also to consider $W(\mathcal{T},\mathcal{U})$ when $\langle \mathcal{T},\mathcal{U}\rangle$ is a stack on some M that is not a premouse of any kind. In that case we shall assume that $M \models \mathsf{ZFC}$, and M is the background universe for some construction for a fine-structural object. The background extenders used in this construction will constitute a coarsely coherent sequence $\vec{F} \in M$. (See 2.38.) We shall only be interested in $\vec{F}$-trees on M. Normality for such trees $\mathcal{S}$ means

1. $\alpha < \beta \Rightarrow \mathrm{lh}\, E_\alpha^\mathcal{S} < \mathrm{lh}\, E_\beta^\mathcal{S}$, and
2. S-pred$(\gamma + 1) =$ least β such that $\mathrm{crit}(E_\alpha^\mathcal{T}) < \mathrm{lh}(E_\beta^\mathcal{T})$.

Given $\langle \mathcal{T},\mathcal{U}\rangle$ a normal $\vec{F}$-stack on M, we can define $W(\mathcal{T},\mathcal{U})$ as above. In this coarse case we shall have $\sigma_\gamma =$ id for all γ, and hence $F_\gamma = E_\gamma^\mathcal{U}$ for all γ. Having defined $\mathcal{W}_\eta$ for $\eta \le \gamma$, and with $R_\gamma = \mathcal{M}_\gamma^\mathcal{U}$, we let

$$\alpha = \text{least } \tau \text{ such that for } \eta = \mathrm{lh}(E_\gamma^\mathcal{U}), V_\eta^{\mathcal{M}_\tau^\mathcal{U}} = V_\eta^{\mathcal{M}_\gamma^\mathcal{U}}.$$

It is easy to see that α is the least τ such that $E_\gamma^\mathcal{U} \in i_{0,\tau}^\mathcal{U} \circ i^\mathcal{T}(\vec{F})$. We define

$$\begin{aligned}\mathcal{W}_{\gamma+1} &= W(\mathcal{W}_\eta, \mathcal{W}_\gamma, E_\gamma^\mathcal{U}) \\ &= \mathcal{W}_\gamma {\restriction} (\alpha+1)^\frown \langle E_\gamma^\mathcal{U}\rangle^\frown i_{E_\gamma^\mathcal{U}} \text{``} \mathcal{W}_\eta^{> \mathrm{crit}(E_\gamma^\mathcal{U})}.\end{aligned}$$

The coherence of $\vec{F}$ implies that if $\sigma < \alpha$, then $\mathrm{lh}(E_\sigma^{\mathcal{W}_\gamma}) < \mathrm{lh}(E_\gamma^\mathcal{U})$, so that $\mathcal{W}_\gamma {\restriction} (\alpha + 1)^\frown \langle E_\gamma^\mathcal{U}\rangle$ is normal, so $\mathcal{W}_{\gamma+1}$ is normal.

This completes our definition of embedding normalization. Since we do not need full normalization in this paper, we shall not discuss it further here.

Remark 3.53 One can regard the sequence of iteration trees $\langle \mathcal{W}_\gamma \mid \gamma < \mathrm{lh}(\mathcal{U})\rangle$ that occurs in the definition of $W(\mathcal{T},\mathcal{U})$ as an iteration tree of iteration trees. One might call such a system a *meta-iteration tree*, or *meta-tree*. The nodes in the meta-tree are iteration trees, with $\mathcal{T}$ being the base node. The F_γ are used to extend the meta-tree at successor steps, via the W-operation. We have tree embeddings from one node to the later ones along branches of our meta-tree.

The meta-tree associated to $W(\mathcal{T},\mathcal{U})$ is not the general case, however, because there is in general no need to require that the F_γ be obtained by lifting extenders used in some tree $\mathcal{U}$ on the last model of $\mathcal{T}$. This was first realized by Schlutzenberg, who defined the general notion of "meta-iterate of $\mathcal{T}$". (Schlutzenberg's term is "inflation of $\mathcal{T}$".) Schlutzenberg also showed that if $\mathcal{T}$ is played by a strategy Σ with the weak Dodd-Jensen property, then Σ induces a meta-iteration strategy for $\mathcal{T}$. See [44]. Schlutzenberg's work was streamlined and re-written by Jensen, who introduced the general notion of meta-tree. See [12]. Further general results on meta-iteration trees and strategies can be found in [48], along with a more detailed discussion of the evolution of the idea.

3.6 Normalization commutes with copying

We prove that normalization commutes with copying. The proof is completely straightforward, but takes a while to put on paper, because of the many embeddings involved. We shall use this fact to show that the pullback of a strategy that normalizes well also normalizes well. The proof also serves as an introduction to our proof that normalization commutes with lifting to a background universe. That in turn is used in the proof that if a strategy for the background universe normalizes well, then so do the strategies on premice that it induces. (See 4.41.)

Theorem 3.54 *Let $\langle \mathcal{T},\mathcal{U}\rangle$ be a maximal M-stack, and let $\psi\colon M \to N$ be elementary. Let $\langle \mathcal{T}^*,\mathcal{U}^*\rangle = \psi\langle \mathcal{T},\mathcal{U}\rangle$ be the stack on N obtained by copying. Suppose that $W(\mathcal{T}^*,\mathcal{U}^*)$ exists; then*

(1) $W(\mathcal{T},\mathcal{U})$ exists, and $\psi W(\mathcal{T},\mathcal{U}) = W(\mathcal{T}^*,\mathcal{U}^*)$, and

(2) let $\mathcal{U}$ and $\mathcal{U}^*$ have last models Q and Q^* respectively, and $W(\mathcal{T},\mathcal{U})$ and $W(\mathcal{T}^*,\mathcal{U}^*)$ have last model R and R^* respectively, and let

(i) $\rho\colon Q \to Q^*$ be the map from copying $\langle \mathcal{T},\mathcal{U}\rangle$ to $\langle \mathcal{T}^*,\mathcal{U}^*\rangle$,

(ii) $\sigma\colon Q \to R$ be the normalization map associated to $W(\mathcal{T},\mathcal{U})$,

(iii) $\theta\colon R \to R^*$ be the map from copying $W(\mathcal{T},\mathcal{U})$ to $W(\mathcal{T}^*,\mathcal{U}^*)$, and

(iv) $\sigma^*\colon Q^* \to R^*$ be the normalization map associated to $W(\mathcal{T}^*,\mathcal{U}^*)$;

then $\theta \circ \sigma = \sigma^* \circ \rho$.

The following diagram depicts our situation:

$$
\begin{tikzcd}
& & R^* \\
N \arrow[urr, "\mathcal{W}^*"] \arrow[r, "\mathcal{T}^*"'] & P^* \arrow[r, "\mathcal{U}^*"'] & Q^* \arrow[u, "\sigma^*"] \\
& & R \arrow[uu, bend right, "\theta"'] \\
M \arrow[uu, "\psi"] \arrow[urr, "\mathcal{W}"] \arrow[r, "\mathcal{T}"'] & P \arrow[r, "\mathcal{U}"'] & Q \arrow[u, "\sigma"'] \arrow[uu, bend left, "\rho"]
\end{tikzcd}
$$

Proof.

The embedding normalization $W(\mathcal{T},\mathcal{U})$ has associated to it normal trees

$$\mathcal{W}_\gamma = W(\mathcal{T},\mathcal{U} \restriction \gamma + 1)$$

on M, for $\gamma < \mathrm{lh}\,\mathcal{U}$. We also have extended tree embeddings

$$\Phi_{\eta,\gamma} \colon W_\eta \to W_\gamma,$$

defined for $\eta \leq_U \gamma$. For $\eta \leq_U \gamma$, we set

$$\phi_{\eta,\gamma} = u^{\Phi_{\eta,\gamma}},$$

so that $\phi_{\eta,\gamma} \colon \mathrm{lh}\,\mathcal{W}_\eta \to \mathrm{lh}\,\mathcal{W}_\gamma$, and for $\tau \in \mathrm{dom}\,\phi_{\eta,\gamma}$,

$$\pi_\tau^{\eta,\gamma} = t_\tau^{\Phi_{\eta,\gamma}},$$

so that $\pi_\tau^{\eta,\gamma} : \mathcal{M}_\tau^{\mathcal{W}_\eta} \to \mathcal{M}_{\phi_{\eta,\gamma}(\tau)}^{\mathcal{W}_\gamma}$. Let R_γ be the last model of $\mathcal{W}_\gamma$, $\sigma_\gamma : \mathcal{M}_\gamma^{\mathcal{U}} \to R_\gamma$ as before, and $F_\gamma = \sigma_\gamma(E_\gamma^{\mathcal{U}})$. So

$$\mathcal{W}_{\gamma+1} = W(\mathcal{W}_\eta, F_\gamma)$$

when $\eta = U\text{-pred}(\gamma + 1)$.

Similarly, $W(\mathcal{T}^*,\mathcal{U}^*)$ has associated trees

$$\mathcal{W}^*_\gamma = W(\mathcal{T}^*,\mathcal{U}^* \restriction \gamma + 1)$$

on N for $\gamma < \mathrm{lh}\,\mathcal{U}^* = \mathrm{lh}\,\mathcal{U}$, together tree embeddings

$$\Phi^*_{\eta,\gamma} \colon \mathcal{W}^*_\eta \to \mathcal{W}^*_\gamma$$

defined when $\eta \leq_U \gamma$. We call the u maps of these tree embeddings $\phi^*_{\eta,\gamma} : \mathrm{lh}\,\mathcal{W}^*_\eta \to \mathrm{lh}\,\mathcal{W}^*_\gamma$, and for $\tau \in \mathrm{dom}\,\phi^*_{\eta,\gamma}$, the t map is $\overset{*}{\pi}{}^{\eta,\gamma}_\tau$. We let $R^*_\gamma =$ be the last model of $\mathcal{W}^*_\gamma$, $\sigma^*_\gamma : \mathcal{M}^{\mathcal{U}^*}_\gamma \to R^*_\gamma$, and $F^*_\gamma = \sigma^*_\gamma(E^{\mathcal{U}^*}_\gamma)$. We have that $\mathcal{W}^*_{\gamma+1} = W(\mathcal{W}^*_\eta, F^*_\gamma)$ when $\eta = U^*\text{-pred}(\gamma+1)$ (equivalently, $\eta = U\text{-pred}(\gamma+1)$).

We shall prove that for all γ,

$$\psi\mathcal{W}_\gamma = \mathcal{W}^*_\gamma.$$

The proof is by induction on γ, with a subinduction on initial segments of $\mathcal{W}_\gamma$. Given that we know this holds for $\mathcal{W}_\gamma\restriction\eta$, we have copy maps

$$\psi^\gamma_\tau \colon \mathcal{M}^{\mathcal{W}_\gamma}_\tau \to \mathcal{M}^{\mathcal{W}^*_\gamma}_\tau$$

defined for all $\tau < \eta$. $\psi^\gamma_0 = \psi$ for all γ.

For $\gamma < \mathrm{lh}\,\mathcal{U}$, let

$$\psi^{\mathcal{U}}_\gamma \colon \mathcal{M}^{\mathcal{U}}_\gamma \to \mathcal{M}^{\mathcal{U}^*}_\gamma$$

be the copy map. So $\psi^{\mathcal{U}}_0$ is the copy map given by the fact that $\mathcal{T}^* = \psi\mathcal{T}$, and the remaining $\psi^{\mathcal{U}}_\gamma$ come from the fact that $\mathcal{U}^* = (\psi^{\mathcal{U}}_0)\mathcal{U}$.

We write $z(\nu)$ for $\mathrm{lh}\,\mathcal{W}_\nu - 1$ and $z^*(\nu)$ for $\mathrm{lh}\,\mathcal{W}^*_\nu - 1$. (Once we have shown that $\psi\mathcal{W}_\nu = \mathcal{W}^*_\nu$, we get $z(\nu) = z^*(\nu)$, of course.) We may use ∞ for $z(\nu)$ or $z^*(\nu)$ when context permits. So $R_\nu = \mathcal{M}^{\mathcal{W}_\nu}_{z(\nu)} = \mathcal{M}^{\mathcal{W}_\nu}_\infty$. If $(\nu,\gamma]_U$ does not drop, then $\phi_{\nu,\gamma}(z(\nu)) = z(\gamma)$, and $\pi^{\nu,\gamma}_{z(\nu)} = \pi^{\nu,\gamma}_\infty \colon R_\nu \to R_\gamma$.

Lemma 3.55 *Let $\gamma < \mathrm{lh}\,\mathcal{U}$. Then*

(1) $\mathcal{W}^*_\gamma = \psi\mathcal{W}_\gamma$.

(2) $\phi_{\eta,\nu} = \phi^*_{\eta,\nu}$, *if* $\eta,\nu \leq \gamma$ *and* $\eta \leq_U \nu$.

(3) *Whenever $\nu <_U \gamma$ and $(\nu,\gamma]_U$ does not drop in model or degree, then for all $\tau < \mathrm{lh}\,\mathcal{W}_\nu$, $\psi^\gamma_{\phi_{\nu,\gamma}(\tau)} \circ \pi^{\nu,\gamma}_\tau = \overset{*}{\pi}{}^{\nu,\gamma}_\tau \circ \psi^\nu_\tau$.*

(4) $\psi^\gamma_{z(\gamma)} \circ \sigma_\gamma = \sigma^*_\gamma \circ \psi^{\mathcal{U}}_\gamma$.

Letting Ω_η be the system of all copy maps from W_η to W^*_η, item (3) is keeping track of the sense in which $\Omega_\gamma \circ \Phi_{\nu,\gamma} = \Phi^*_{\nu,\gamma} \circ \Omega_\nu$. Here is a diagram of (3):

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{W}_\gamma}_{\phi_{\nu,\gamma}(\tau)} \arrow[r, "\psi^\gamma_{\phi_{\nu,\gamma}(\tau)}"] & \mathcal{M}^{\mathcal{W}^*_\gamma}_{\phi^*_{\nu,\gamma}(\tau)} \\
\mathcal{M}^{\mathcal{W}_\nu}_\tau \arrow[u, "\pi^{\nu,\gamma}_\tau"] \arrow[r, "\psi^\nu_\tau"] & \mathcal{M}^{\mathcal{W}^*_\nu}_\tau \arrow[u, "{}^*\pi^{\nu,\gamma}_\tau"']
\end{tikzcd}$$

There is a diagram related to (4) and the case $\tau = z(\nu)$ of (3) near the end of the proof.

Proof. We prove 3.55 by induction. Suppose that it is true at all $\nu \leq \gamma$. We show it at $\gamma + 1$. Let $\nu = U\text{-pred}(\gamma + 1)$, and

$$F = F_\gamma = \sigma_\gamma(E^{\mathcal{U}}_\gamma),$$

and

$$\begin{aligned}
\alpha &= \alpha^{\mathcal{T},\mathcal{U}}_\gamma \\
&= \alpha(\mathcal{W}_\nu, \mathcal{W}_\gamma, F) \\
&= \text{least } \tau \text{ such that } F \text{ is on the } M^{\mathcal{W}_\gamma}_\tau\text{-sequence.}
\end{aligned}$$

So

$$\begin{aligned}
\mathcal{W}_{\gamma+1} &= W(\mathcal{W}_\nu, \mathcal{W}_\gamma, F) \\
&= \mathcal{W}_\gamma \restriction (\alpha + 1)^\frown \langle F \rangle^\frown i_F \text{``} \mathcal{W}^{>\text{crit}(F)}_\nu .
\end{aligned}$$

Let also

$$F^* = F^*_\gamma = \sigma^*_\gamma(E^{\mathcal{U}^*}_\gamma).$$

Since $\mathcal{U}^*$ is a copy of $\mathcal{U}$, $\nu = U^*\text{-pred}(\gamma + 1)$, so

$$\mathcal{W}^*_{\gamma+1} = W(\mathcal{W}^*_\nu, \mathcal{W}^*_\gamma, F^*).$$

Claim 3.56 *(1)* $\psi^\gamma_{z(\gamma)}(F) = F^*$,

(2) $\alpha = \alpha(\mathcal{W}^*_\gamma, \mathcal{W}^*_\nu, F^*)$, *and*

(3) $\beta(\mathcal{W}_\nu, \mathcal{W}_\gamma, F) = \beta(\mathcal{W}^*_\nu, \mathcal{W}^*_\gamma, F^*)$.

Proof. For (1), we have

$$\begin{aligned}\psi^{\gamma}_{z(\gamma)}(F) &= \psi^{\gamma}_{z(\gamma)} \circ \sigma_\gamma(E^{\mathcal{U}}_\gamma)\\ &= \sigma^*_\gamma \circ \psi^{\mathcal{U}}_\gamma(E^{\mathcal{U}}_\gamma)\\ &= \sigma^*_\gamma(E^{\mathcal{U}^*}_\gamma)\\ &= F^*.\end{aligned}$$

For (2), it is enough to show that $\mathrm{lh}(F) < \mathrm{lh}(E^{\mathcal{W}_\gamma}_\tau)$ if and only if $\mathrm{lh}(F^*) < \mathrm{lh}(E^{\mathcal{W}^*_\gamma}_\tau)$. But if $\mathrm{lh}(F) < \mathrm{lh}(E^{\mathcal{W}_\gamma}_\tau)$, then applying the copy maps ψ^γ, we have

$$\begin{aligned}\mathrm{lh}(F^*) = \mathrm{lh}(\psi^{\gamma}_{z(\gamma)}(F)) &= \mathrm{lh}(\psi^{\gamma}_{\tau}(F))\\ &< \mathrm{lh}(\psi^{\gamma}_{\tau}(E^{\mathcal{W}_\gamma}_\tau))\\ &= \mathrm{lh}(E^{\mathcal{W}^*_\gamma}_\tau).\end{aligned}$$

The first line holds because $\psi^{\gamma}_{z(\gamma)}$ agrees with ψ^γ_τ on $\mathrm{lh}(E^{\mathcal{W}_\gamma}_\tau)$. Conversely, if $\mathrm{lh}(F) > \mathrm{lh}(E^{\mathcal{W}_\gamma}_\tau)$, then if $\mathrm{lh}(F^*) > \mathrm{lh}(E^{\mathcal{W}^*_\gamma}_\tau)$ by the same calculation.

For (3), we must show that $\mathrm{crit}(F) < \lambda(E^{\mathcal{W}_\gamma}_\tau)$ if and only if $\mathrm{crit}(F^*) < \lambda(E^{\mathcal{W}^*_\gamma}_\tau)$. But this follows from the agreement of the copy maps ψ^γ in exactly the same way. □

The claim easily implies that $\phi_{\nu,\gamma+1} = \phi^*_{\nu,\gamma+1}$, which then gives us (2) of 3.55 at $\gamma + 1$.

We now define the copy maps $\psi^{\gamma+1}_\tau \colon \mathcal{M}^{\mathcal{W}_{\gamma+1}}_\tau \to \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_\tau$ that witness $\mathcal{W}^*_{\gamma+1} = \psi\mathcal{W}_{\gamma+1}$. As we do so, we show that (3) of 3.55 holds, that is, the ψ^ν and $\psi^{\gamma+1}$ maps commute with the embedding normalization maps of models of $\mathcal{W}_\nu$ into models of $\mathcal{W}_{\gamma+1}$ and models of $\mathcal{W}^*_\nu$ into models of $\mathcal{W}^*_{\gamma+1}$.

We have $\mathcal{W}_{\gamma+1}\restriction(\alpha+1) = \mathcal{W}_\gamma\restriction(\alpha+1)$ and $\mathcal{W}^*_{\gamma+1}\restriction(\alpha+1) = \mathcal{W}^*_\gamma\restriction(\alpha+1)$, so we can set

$$\psi^{\gamma+1}_\tau = \psi^\gamma_\tau, \text{ for all } \tau \leq \alpha.$$

Now $F = E^{\mathcal{W}_{\gamma+1}}_\alpha$ and $F^* = E^{\mathcal{W}^*_{\gamma+1}}_\alpha$, moreover $\psi^\gamma_\alpha(F) = \psi^\gamma_{z(\gamma)}(F) = F^*$ because $\mathrm{lh}(F) < \mathrm{lh}(E^{\mathcal{W}_\gamma}_\alpha)$ if $\alpha < z(\gamma)$. Letting $P = \mathcal{M}^{\mathcal{W}_\nu}_\beta|\langle \eta, k\rangle$ be such that

$$\mathcal{M}^{\mathcal{W}_{\gamma+1}}_{\alpha+1} = \mathrm{Ult}(P, F),$$

we have

$$\mathcal{M}_{\alpha+1}^{\mathcal{W}^*_{\gamma+1}} = \mathrm{Ult}(P^*, F^*),$$

where $P^* = \mathcal{M}_\beta^{\mathcal{W}^*_\nu}|\langle \psi^\nu_\beta(\eta), k\rangle$. (Here we make the usual convention if $\eta = o(\mathcal{M}_\beta^{\mathcal{W}_\nu})$.) This is because $\mathcal{W}_\nu\restriction(\beta+1) = \mathcal{W}_\gamma\restriction(\beta+1)$, and similarly at the (*) level, by the properties of embedding normalization. So $\psi^\nu_\beta = \psi^\gamma_\beta$, and thus agrees with $\psi^\gamma_{z(\gamma)}$ up to $\lambda(E_\beta^{\mathcal{W}_\gamma})$, hence past $\mathrm{crit}(F)$. So we can let

$$\psi^{\gamma+1}_{\alpha+1}([a,f]^P_F) = [\psi^{\gamma+1}_\alpha(a), \psi^{\gamma+1}_\beta(f)]^{P^*}_{F^*},$$

by the Shift lemma, and we have $\psi\mathcal{W}_{\gamma+1}\restriction(\alpha+2) = \mathcal{W}^*_{\gamma+1}\restriction(\alpha+2)$. Note that $\alpha+1 = \phi_{\nu,\gamma+1}(\beta)$, so $\psi^{\gamma+1}_{\phi_{\nu,\gamma+1}(\beta)} \circ \pi^{\nu,\gamma+1}_\beta = \overset{*}{\pi}{}^{\nu,\gamma+1}_\beta \circ \psi^\nu_\beta$ by the Shift lemma, and this gives us the new instance of (3) of 3.55.

The general successor case above $\alpha+1$ is similar. Suppose we have $\psi\mathcal{W}_{\gamma+1}\restriction(\eta+1) = \mathcal{W}^*_{\gamma+1}\restriction(\eta+1)$ as witnessed by $\psi^{\gamma+1}_\tau$ for $\tau \le \eta$. Suppose $\eta > \alpha$. Let

$$\eta = \phi_{\nu,\gamma+1}(\xi) = \phi^*_{\nu,\gamma+1}(\xi),$$

$$G = E_\eta^{\mathcal{W}_{\gamma+1}},$$

and

$$G^* = E_\eta^{\mathcal{W}^*_{\gamma+1}}.$$

Then

$$\begin{aligned}
\psi^{\gamma+1}_\eta(G) &= \psi^{\gamma+1}_{\phi_{\nu,\gamma+1}(\xi)}(\pi^{\nu,\gamma+1}_\xi(E_\xi^{\mathcal{W}_\nu}))\\
&= \overset{*}{\pi}{}^{\nu,\gamma+1}_\xi(\psi^\nu_\xi(E_\xi^{\mathcal{W}_\nu}))\\
&= \overset{*}{\pi}{}^{\nu,\gamma+1}_\xi(E_\xi^{\mathcal{W}^*_\nu})\\
&= E_\eta^{\mathcal{W}^*_{\gamma+1}} = G^*.
\end{aligned}$$

The Shift lemma now gives us $\psi^{\gamma+1}_{\eta+1}$ as above, and we have $\psi\mathcal{W}_{\gamma+1}\restriction(\eta+2) = \mathcal{W}^*_{\gamma+1}\restriction(\eta+2)$.

We leave the limit case of the subinduction to the reader. This finishes the subinduction proving (1), (2), and (3) of 3.55 at step $\gamma+1$. For (4), let us set $\tau = \gamma+1$. To simplify things, let us assume that $(\nu, \gamma+1]_U$ is not a drop. Consider the diagram

$$
\begin{tikzcd}
& R_\tau \arrow[rr, "\psi^\tau_\infty"] & & R^*_\tau \\
\mathcal{M}^{\mathcal{U}}_\tau \arrow[ur, "\sigma_\tau"] \arrow[rr] & & \mathcal{M}^{\mathcal{U}^*}_\tau \arrow[ur, "\sigma^*_\tau"'] & \\
& R_\nu \arrow[uu, "\pi^{\nu,\tau}_\infty"'] \arrow[rr, "\psi^\nu_\infty"] & & R^*_\nu \arrow[uu, "\overset{*}{\pi}^{\nu,\tau}_\infty"'] \\
\mathcal{M}^{\mathcal{U}}_\nu \arrow[uu, "E^{\mathcal{U}}_\gamma"] \arrow[ur, "\sigma_\nu"] \arrow[rr, "\psi^{\mathcal{U}}_\nu"'] & & \mathcal{M}^{\mathcal{U}^*}_\nu \arrow[uu] \arrow[ur, "\sigma^*_\nu"'] &
\end{tikzcd}
$$

We are asked to show that $\sigma^*_\tau \circ \psi^{\mathcal{U}}_\tau = \psi^\tau_\infty \circ \sigma_\tau$, in other words, that the square on the top face of the cube commutes. The square on the bottom commutes by our induction hypothesis. The square in front commutes because $\mathcal{U}^*$ is a copy of $\mathcal{U}$. That the square in back commutes is clause (3) of our lemma at $\gamma + 1$, which we just proved. The squares on the left and right faces commute by the properties of embedding normalization.

It is clear from these facts that the top square commutes on $\mathrm{ran}(i^{\mathcal{U}}_{\nu,\tau})$. Since $\mathcal{M}^{\mathcal{U}}_\tau$ is generated by $\mathrm{ran}(i^{\mathcal{U}}_{\nu,\tau}) \cup \lambda(E^{\mathcal{U}}_\gamma)$, it is enough to see that the top square commutes on $\lambda(E^{\mathcal{U}}_\gamma)$.

Let $a \in [\lambda(E^{\mathcal{U}}_\gamma)]^{<\omega}$. So $\sigma_\gamma(a) \in [\lambda(F)]^{<\omega}$, and $\sigma_\tau(a) = \sigma_\gamma(a)$ by Proposition 3.49 on the agreement properties of embedding normalization maps. Thus

$$
\begin{aligned}
\psi^\tau_\infty(\sigma_\tau(a)) &= \psi^\tau_\infty(\sigma_\gamma(a)) \\
&= \psi^\gamma_\infty(\sigma_\gamma(a)),
\end{aligned}
$$

using that the copy maps ψ^τ_∞ and ψ^γ_∞ both agree with ψ^γ_γ on $\lambda(F)$. On the other hand, $\psi^{\mathcal{U}}_\tau(a) \in [\lambda(E^{\mathcal{U}^*}_\tau]^{<\omega}$, so

$$
\begin{aligned}
\sigma^*_\tau(\psi^{\mathcal{U}}_\tau(a)) &= \sigma^*_\gamma(\psi^{\mathcal{U}}_\tau(a)) \\
&= \sigma^*_\gamma(\psi^{\mathcal{U}}_\gamma(a),
\end{aligned}
$$

by the agreement in normalization maps on the $\mathcal{W}^*$ side. But $\psi^\gamma_\infty \circ \sigma_\gamma = \sigma^*_\gamma \circ \psi^{\mathcal{U}}_\gamma$ by induction, so

$$\begin{aligned}\psi^{\tau}_{\infty} \circ \sigma_{\tau}(a) &= \psi^{\gamma}_{\infty} \circ \sigma_{\gamma}(a)\\ &= \sigma^{*}_{\gamma} \circ \psi^{\mathcal{U}}_{\gamma}(a)\\ &= \sigma^{*}_{\tau} \circ \psi^{\mathcal{U}}_{\tau}(a),\end{aligned}$$

as desired.

This finishes the step from γ to $\gamma + 1$ in the inductive proof of 3.55. We leave the limit step to the reader. □

It is easy to see that Theorem 3.54 follows from Lemma 3.55.

□

3.7 The branches of $W(\mathcal{T},\mathcal{U})$

Let $\mathcal{T}$ be normal on M, and $\mathcal{U}$ be normal on the last model of $\mathcal{T}$. Let us adopt the notation of the last section, so that we have $\mathcal{W}_{\gamma}$, F_{γ}, α_{γ}, β_{γ}, $\phi_{\eta,\gamma}$, $\pi^{\eta,\gamma}_{\tau}$, and so on. Suppose $\mathrm{lh}\,\mathcal{U}$ is a limit ordinal θ, and let

$$\lambda = \mathrm{lh}\, W(\mathcal{T},\mathcal{U}) = \sup_{\gamma<\theta} \alpha_{\gamma}.$$

Here we assume $W(\mathcal{T},\mathcal{U})$ exists, i.e. embedding normalization has so far produced only wellfounded models. Let b be a cofinal branch of $\mathcal{U}$. We do not assume $\mathcal{M}^{\mathcal{U}}_{b}$ is wellfounded. Note that $\mathcal{W}_{b}$ still makes sense, as defined above.

Proposition 3.57 *$\lambda = \phi_{0,b}(\tau)$, where τ is least such that whenever $\eta, \gamma \in b$ and $\eta <_U \gamma$, then $\mathrm{crit}\,\phi_{\eta,\gamma} \leq \phi_{0,\eta}(\tau)$.*

Proof. Let $\eta + 1 \in b$, and $\sigma \in U\text{-pred}(\eta + 1)$. Then $\phi_{\sigma,\eta+1}(\mathrm{crit}(\phi_{\sigma,\eta+1})) = \alpha_{\eta} + 1$, so $\alpha_{\eta} + 1 \leq \mathrm{crit}(\phi_{\eta+1,\xi})$ for all $\xi \in b$. It follows that $\phi_{0,b}(\tau) \geq \lambda$. But if $\sigma < \tau$, we can find $\gamma + 1 \in b$ with $\eta = U\text{-pred}(\gamma + 1)$ such that $\phi_{0,\eta}(\sigma) < \mathrm{crit}(\phi_{\eta,\gamma+1})$. Then $\phi_{0,b}(\sigma) = \phi_{0,\eta}(\sigma) < \alpha_{\gamma} < \lambda$. Finally, $\lambda \in \mathrm{ran}\,\phi_{0,b}$ (because any $\xi < \mathrm{lh}(\mathcal{W}_{\gamma})$ not in $\mathrm{ran}\,\phi_{0,\gamma}$ is fixed by $\phi_{\gamma,b}$), so $\lambda = \phi_{0,b}(\tau)$. □

Proposition 3.58 *Let $a = [0, \lambda)_{W_b}$ and $\lambda = \phi_{0,b}(\tau)$; then*

$$\xi \in a \quad \text{iff} \quad \exists \eta \in b\, (\xi \leq \mathrm{crit}(\phi_{\eta,b}) \wedge \xi \leq_{W_\eta} \phi_{0,\eta}(\tau)).$$

We omit the easy proof.

Remark 3.59 We don't get a "continuously" from b. If τ is fixed in advance, then continuously in those b such that $\tau = \tau_b$, we can produce the corresponding a's.

Definition 3.60 *In the situation above, we write*

$$a = \mathrm{br}(b, \mathcal{T}, \mathcal{U})$$

and

$$\tau = m(b, \mathcal{T}, \mathcal{U})$$

for the branch of $W(\mathcal{T}, \mathcal{U})$ and model of $\mathcal{T}$ determined by b.

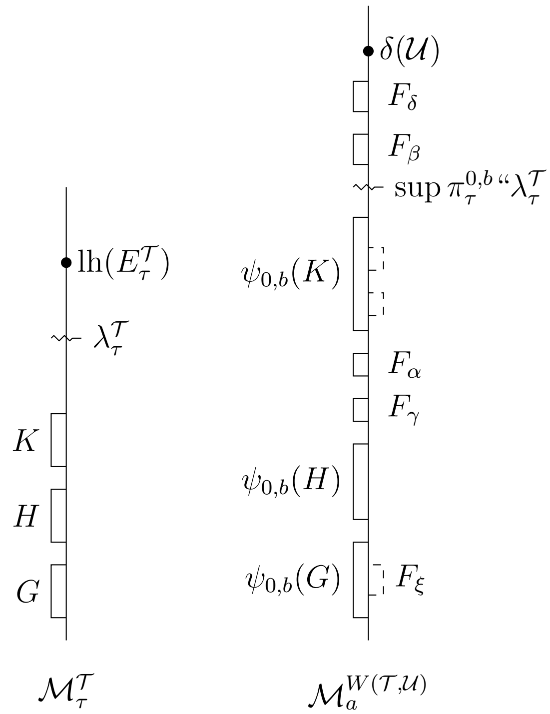
Remark 3.61 Let E_b be the extender of $i_b^{\mathcal{U}}$. It is an extender over the model $\mathcal{M}_\xi^{\mathcal{T}}$, where $\xi + 1 = \mathrm{lh}\,\mathcal{T}$. One can show that τ is the least α such that either E_b is an extender over $\mathcal{M}_\alpha^{\mathcal{T}} | \mathrm{lh}\, E_\alpha^{\mathcal{T}}$ (that is, $\mathrm{dom}(E_b) \subseteq \mathcal{M}_\alpha^{\mathcal{T}} | \mathrm{lh}(E_\alpha^{\mathcal{T}})$) , or $\alpha = \xi$.

The branch extender of a is given by

Proposition 3.62 *Let $a = \mathrm{br}(b, \mathcal{T}, \mathcal{U})$ and $\tau = m(b, \mathcal{T}, \mathcal{U})$ be as above; then*

$$\begin{aligned} e_a^{W(\mathcal{T},\mathcal{U})} = \hat{p}_{0,b}(e_\tau^{\mathcal{T}})^\frown\langle F_\sigma \mid \sigma + 1 \in b \wedge \forall i \in \mathrm{dom}(\hat{p}_{0,b}(e_\tau^{\mathcal{T}})) \\ \lambda(\hat{p}_{0,b}(e_\tau^{\mathcal{T}})(i)) \leq \mathrm{crit}(F_\sigma)\rangle. \end{aligned}$$

Here we are writing $e_a^{W(\mathcal{T},\mathcal{U})}$ for $e_\lambda^{\mathcal{W}_b}$, because $e_a^{W(\mathcal{T},\mathcal{U})}$ really only depends on a and $W(\mathcal{T}, \mathcal{U})$. We omit the proof of 3.62. For what it's worth, here is a picture



Note $\delta(\mathcal{U}) = \delta(W(\mathcal{T},\mathcal{U}))$. The F's in the picture were all used in b. Some got put directly into $e^{W(\mathcal{T},\mathcal{U})}_a$, others indirectly via some $p_{0,b}(G)$. $\lambda^{\mathcal{T}}_\tau$ is the sup of the Jensen generators of extenders used to get to $\mathcal{M}^{\mathcal{T}}_\tau$. (In general, $\lambda^{\mathcal{T}}_\tau < \lambda(E^{\mathcal{T}}_\tau)$.) The extenders in $e^{W(\mathcal{T},\mathcal{U})}_a$ with generators beyond $\sup \pi^{0,b}_\tau$"$\lambda^{\mathcal{T}}_\tau$ are all directly inserted F's.

Branches of $W(\mathcal{T},\mathcal{U})$ of the form $\mathrm{br}(b,\mathcal{T},\mathcal{U})$ come from cofinal branches of $\mathcal{U}$ and *models* of $\mathcal{T}$. There may also be cofinal branches of $W(\mathcal{T},\mathcal{U})$ coming from cofinal branches of $\mathcal{U}$ and *maximal* (perhaps not cofinal) *branches* of $\mathcal{T}$. So we extend our definitions.

Definition 3.63 *Let $\mathcal{W} = W(\mathcal{T},\mathcal{U})$, where $\mathcal{T}$ is normal on M and $\mathcal{U}$ is normal on the last model of $\mathcal{T}$. For $\xi < \mathrm{lh}\,\mathcal{T}$,*

(a) *for $\gamma+1 < \mathrm{lh}\,\mathcal{U}$, letting $\eta = U\text{-pred}(\gamma+1)$, we set*

$$\mathrm{nd}_{\mathcal{W}}(\xi,\gamma+1) = \begin{cases} \phi_{0,\eta}(\xi), & \textit{if } \phi_{0,\eta}(\xi)\!\downarrow \textit{ and } \phi_{0,\eta}(\xi) \leq_{W_\eta} \mathrm{crit}(\phi_{\eta,\gamma+1}); \\ \textit{undefined}, & \textit{otherwise.} \end{cases}$$

(b) *For any $\gamma < \mathrm{lh}\,\mathcal{U}$,*

$$\tau \in \mathrm{br}_{\mathcal{W}}(\xi,\gamma) \quad \text{iff} \quad \tau = \mathrm{nd}_{\mathcal{W}}(\xi_0,\gamma_0+1), \quad \textit{for some } \xi_0 \leq_T \xi \textit{ and } \gamma_0+1 \leq_U \gamma.$$

"nd" stands for "node". We shall drop the subscript and write $\mathrm{nd}(\xi, \gamma+1)$ and $\mathrm{br}(\xi, \gamma)$ when context permits. Notice that if $\tau = \mathrm{nd}(\xi, \gamma+1)$, then whenever $\gamma + 1 \leq_U \delta$, then $\phi_{0,\delta}(\xi)\downarrow$, and $\tau \leq_{W_\delta} \phi_{0,\delta}(\xi)$. This is true even if $\tau = \mathrm{crit}(\phi_{\eta,\gamma+1})$ holds in the definition of nd_W, because $\mathrm{crit}(\phi_{\eta,\gamma+1}) \leq_{W_\delta} \phi_{\eta,\delta}(\mathrm{crit}(\phi_{\eta,\gamma+1}))$. This gives

Proposition 3.64 *1. Let $\xi_0 \leq_T \xi_1$ and $\gamma_0 + 1 \leq_U \gamma_1 + 1$. Then*

$$\mathrm{nd}(\xi_0, \gamma_0 + 1) \leq_{W(\mathcal{T},\mathcal{U})} \mathrm{nd}(\xi_1, \gamma_1 + 1)$$

if both are defined,

2. $\mathrm{br}(\xi, \gamma)$ is a branch of $W(\mathcal{T}, \mathcal{U})$ (not cofinal),

3. $\xi_0 \leq_T \xi_1$ and $\gamma_0 \leq_U \gamma_1 \Rightarrow \mathrm{br}(\xi_0, \gamma_0)$ is an initial segment of $\mathrm{br}(\xi_1, \gamma_1)$.

Proof. Routine. □

Definition 3.65 *Let c be a branch of $\mathcal{T}$ and b be a branch of $\mathcal{U}$. Then*

1. $\mathrm{br}_{\mathcal{W}}(c, b) = \bigcup_{\xi \in c, \gamma \in b} \mathrm{br}_{\mathcal{W}}(\xi, \gamma)$,

*2. c is b-*minimal *iff for any $\xi \in c$, $\mathrm{br}_{\mathcal{W}}(c \cap \xi, b) \neq \mathrm{br}_{\mathcal{W}}(c, b)$.*

Again we omit the subscript $\mathcal{W}$ when possible.

Remark 3.66 1. If b is cofinal in $\mathrm{lh}(\mathcal{U})$, then $\mathrm{br}(c, b)$ is the $\leq_{W(\mathcal{T},\mathcal{U})}$-downward closure of $\phi_{0,b}$"$\, c \cap \mathrm{lh}(W(\mathcal{T}, \mathcal{U}))$.

2. Equivalent are: (1) c is b-minimal, (2) for cofinally many $\xi \in c$, $\exists \gamma + 1 \in b$ such that $\mathrm{nd}(\xi, \gamma + 1)\downarrow$, (3) for all $\xi \in c$, $\exists \gamma + 1 \in b$, $\mathrm{nd}(\xi, \gamma + 1)\downarrow$.

We do not assume in Definition 3.65 that b and c are maximal branches. So for example $\mathrm{br}([0, \xi]_T, [0, \gamma]_U) = \mathrm{br}(\xi, \gamma)$.

We shall show that if a is a cofinal branch of $W(\mathcal{T}, \mathcal{U})$, then $a = \mathrm{br}(c, b)$ for some cofinal branch b of $\mathcal{U}$ and some c; moreover, there is a unique such b, and a unique such b-minimal c. For this, we must assume that all $\mathcal{W}_\gamma$ are played according to a common iteration strategy. The following is the key lemma.

Lemma 3.67 *Let $\mathcal{T}, \mathcal{U}$ be as above, and suppose there is an iteration strategy Σ for M such that all $\mathcal{W}_\gamma$, $\gamma < \mathrm{lh}\,\mathcal{U}$, are according to Σ. Let γ and δ be $\leq_U$-incomparable, and let η be largest such that $\eta <_U \gamma$ and $\eta <_U \delta$. Let $\alpha = \phi_{\eta,\gamma}(\bar{\alpha})$ and $\varepsilon = \phi_{\eta,\delta}(\bar{\varepsilon})$, where $\bar{\alpha} \geq \mathrm{crit}(\phi_{\eta,\gamma})$ and $\bar{\varepsilon} \geq \mathrm{crit}(\phi_{\eta,\delta})$; then $e_\alpha^{\mathcal{W}_\gamma}$ is incompatible with $e_\varepsilon^{\mathcal{W}_\delta}$.*

Proof. Let $u = e_{\alpha}^{\mathcal{W}_\gamma}$, $\bar{u} = e_{\bar{\alpha}}^{\mathcal{W}_\eta}$, $v = e_{\varepsilon}^{\mathcal{W}_\delta}$ and $\bar{v} = e_{\bar{\varepsilon}}^{\mathcal{W}_\eta}$. Assume toward contradiction that either $u \subseteq v$, or $v \subseteq u$.

Let

$$\gamma_0 + 1 = \text{least } \xi \in (\eta, \gamma]_U,$$
$$\delta_0 + 1 = \text{least } \xi \in (\eta, \delta]_U,$$

so that $E_{\gamma_0}^{\mathcal{U}}$ and $E_{\delta_0}^{\mathcal{U}}$ are the extenders used in $\mathcal{U}$ along the two branches of $\mathcal{U}$ at the point where they diverge, and F_{γ_0} and F_{δ_0} stretch $\mathcal{W}_\eta$ into $\mathcal{W}_{\gamma_0+1}$ and $\mathcal{W}_{\delta_0+1}$. Let

$$k(\bar{u}) = \begin{cases} \text{least } i \text{ such that } \mathrm{crit}(F_{\gamma_0}) < \lambda(\bar{u}(i)), & \text{if this exists;} \\ \mathrm{dom}(\bar{u}), & \text{otherwise,} \end{cases}$$

and

$$k(\bar{v}) = \begin{cases} \text{least } i \text{ such that } \mathrm{crit}(F_{\delta_0}) < \lambda(\bar{v}(i)), & \text{if this exists;} \\ \mathrm{dom}(\bar{v}), & \text{otherwise.} \end{cases}$$

Claim 3.68 *$k(\bar{u}) = k(\bar{v})$, and for $k = k(\bar{u})$, $\bar{u}\restriction k = \bar{v}\restriction k = u\restriction k = v\restriction k$.*

Proof. Let $k = k(\bar{u})$. If $k < k(\bar{v})$, then $v(k) = \bar{v}(k)$, so $\lambda(v(k)) \leq \mathrm{crit}(F_{\delta_0})$. But $\lambda(u(k)) \geq \lambda(F_{\gamma_0})$. $[e_{\phi_{\eta,\gamma_0+1}(\bar{\alpha})}^{W_{\gamma_0+1}}(k) = H$ is defined because $\bar{\alpha} \geq \mathrm{crit}(\phi_{\eta,\gamma_0+1})$. H is either F_{γ_0} or the stretch by F_{γ_0} of some G such that $\mathrm{crit}(G) < \mathrm{crit}(F_{\gamma_0})$. In either case, $\lambda(H) \geq \lambda(F_{\gamma_0})$. $u(k) = \psi_{\gamma_0+1,\gamma}(H)$, so $\lambda(u(k)) \geq \lambda(H)$.$]$ Since $u(k) = v(k)$, we have $\lambda(F_{\gamma_0}) \leq \mathrm{crit}(F_{\delta_0})$, so F_{γ_0} and F_{δ_0} do not overlap, contradiction. $k(\bar{v}) < k(\bar{u})$ leads to a parallel contradiction. So we have $k(\bar{u}) = k(\bar{v}) = k$.

For $i < k$, $u(i) = \bar{u}(i)$ and $v(i) = \bar{v}(i)$. So $\bar{u}\restriction k = \bar{v}\restriction k = u\restriction k = v\restriction k$. $\square$

Fix $k = k(\bar{u})$. We may assume by symmetry that $\gamma_0 < \delta_0$.

Claim 3.69 *$k \in \mathrm{dom}(\bar{u})$, and moreover, $\mathrm{crit}(\bar{u}(k)) < \mathrm{crit}(F_{\gamma_0})$.*

Proof. If either statement fails, then

$$e_{\phi_{\eta,\gamma_0+1}(\bar{\alpha})}^{\mathcal{W}_{\gamma_0+1}}(k) = F_{\gamma_0}.$$

Since the extenders used in $(\gamma_0 + 1, \gamma]_U$ have critical point at least $\lambda(E_{\gamma_0}^{\mathcal{U}})$, we get

$$p_{\gamma_0+1,\gamma}(F_{\gamma_0}) = F_{\gamma_0}.$$

(In fact, $\phi_{\gamma_0+1,\gamma}\restriction(\gamma_0 + 1) = \text{identity}$, and $\pi_{\gamma_0}^{\gamma_0+1,\gamma} = \text{identity}$.) So

$$u(k) = F_{\gamma_0}.$$

But $k = k(\bar{v})$, and from this we get

$$\lambda(F_{\delta_0}) \leq \lambda(v(k))$$

as in Claim 3.68. Since $\lambda(F_{\gamma_0}) < \lambda(F_{\delta_0})$, we have a contradiction. $\square$

Let $G = \bar{u}(k)$ and $H = u(k)$. By Claim 3.69, along the branch from η to γ, G is being stretched above its critical point into H, by the copy maps corresponding to the F_τ for $\tau + 1 \leq_U \gamma$ and $\eta \leq \tau$. Let $\gamma_1 \leq \gamma$ be least such that the stretching is finished at γ_1. That is, setting

$$\begin{aligned} G &= E_\xi^{\mathcal{W}_\eta} \\ \gamma_1 &= \text{least } \tau \leq \gamma \text{ such that } \mathrm{crit}(\phi_{\tau,\gamma}) > \phi_{\eta,\tau}(\xi) \\ &= \text{least } \tau \leq \gamma \text{ such that } \pi_\xi^{\eta,\tau}(G) = H. \end{aligned}$$

If $\eta <_U \tau + 1 \leq_U \gamma_1$, so that F_τ was used in producing $\mathcal{W}_{\gamma_1}$ from $\mathcal{W}_\eta$, then F_τ is an initial segment of all the extenders of copy maps $\pi_\rho^{\mu,\tau+1}$, where $\mu = U\text{-pred}(\tau + 1)$, and $\rho \geq \mathrm{crit}(\phi_{\mu,\tau+1})$. From this we get

Claim 3.70 *For $\eta <_U \tau + 1 \leq_U \gamma_1$, $\lambda(F_\tau) < \lambda(H)$.*

Proof. Just given. $\square$

Claim 3.71 *$H \neq F_{\delta_0}$.*

Proof. Suppose $H = F_{\delta_0}$. We claim that $\gamma_1 \leq \delta_0$. If γ_1 is a limit ordinal, then $\gamma_1 = \sup\{\tau + 1 \mid \eta <_U \tau + 1 <_U \gamma_1\}$, so by Claim 3.70, $\lambda(F_{\delta_0}) > \lambda(F_\tau)$ for cofinally many τ in γ_1, which implies $\delta_0 \geq \gamma_1$. If γ_1 is not a limit ordinal, we have $\gamma_1 = \tau + 1$ where F_τ is used, so that $\lambda(F_\tau) < \lambda(H) = \lambda(F_{\delta_0})$. Thus $\tau < \delta_0$, so $\gamma_1 = \tau + 1 \leq \delta_0$.

On the other hand, H is used in $\mathcal{W}_{\gamma_1}$ on the way to R_{γ_1}. Thus R_{γ_1} and R_{δ_0} agree below $\mathrm{lh}(H)$, while $H = F_{\delta_0}$ is on the R_{δ_0}-sequence, but not on the R_{γ_1}-sequence. This implies $\delta_0 < \gamma_1$, a contradiction. $\square$

By Claim 3.71, $k \in \mathrm{dom}(\bar{v})$, and letting $L = \bar{v}(k)$, $\mathrm{crit}(L) < \mathrm{crit}(F_{\delta_0})$. So L is being stretched above its critical point into H along the branch from η to δ. Let $\delta_1 \leq \delta$ be least such that the stretching is over with at δ_1; that is, setting

$$\begin{aligned} L &= E_\mu^{\mathcal{W}_\eta} \\ \delta_1 &= \text{least } \tau \leq_U \delta \text{ such that } \mathrm{crit}(\phi_{\tau,\delta}) > \phi_{\eta,\tau}(\mu) \\ &= \text{least } \tau \leq_U \delta \text{ such that } \pi_\mu^{\eta,\tau}(L) = H. \end{aligned}$$

Since $\gamma_1 \neq \delta_1$, we have $\lambda_{\gamma_1}^{\mathcal{U}} \neq \lambda_{\delta_1}^{\mathcal{U}}$. Assume $\lambda_{\gamma_1}^{\mathcal{U}} < \lambda_{\delta_1}^{\mathcal{U}}$. (It no longer matters whether $\gamma_0 < \delta_0$, so this is not a loss of generality.) That is, we have a $\tau + 1 \leq_U \delta_1$ such that for all $\sigma + 1 \leq_U \gamma_1$, $\lambda(E_\sigma^{\mathcal{U}}) < \lambda(E_\tau^{\mathcal{U}})$. This yields:

(∗) $\tau \leq \delta_1$, and whenever $\sigma + 1 \leq_U \gamma_1$, then $\lambda(F_\sigma) < \lambda(F_\tau)$.

Thus $\tau > \sigma$, whenever $\sigma + 1 \leq_U \gamma_1$. So $\tau \geq \gamma_1$. We have that H is used in both $\mathcal{W}_{\gamma_1}$ and $\mathcal{W}_{\delta_1}$, so R_{γ_1} agrees with R_{δ_1} below $\mathrm{lh}(H)$, which is a cardinal in both models. But F_τ is used in $\mathcal{W}_{\delta_1}$, before H, so $\mathrm{lh}(F_\tau)$ is a cardinal in both R_{γ_1} and R_{δ_1}.

But then R_{γ_1} and R_τ agree up to $\mathrm{lh}(F_\tau)$, since $R_\tau \parallel \mathrm{lh}(F_\tau) = R_{\delta_1} \parallel \mathrm{lh}(F_\tau)$. F_τ is on the R_τ-sequence, and not the R_{γ_1}-sequence, so $\tau < \gamma_1$. Contradiction. □

Corollary 3.72 *Let $\sigma = \mathrm{nd}(\xi, \gamma_0 + 1)$ and $\tau = \mathrm{nd}(\rho, \gamma_1 + 1)$, where $\gamma_0 + 1$ and $\gamma_1 + 1$ are $\leq_U$-minimal. (I.e. $\gamma_0' + 1 <_U \gamma_0 + 1 \Rightarrow \sigma \neq \mathrm{nd}(\xi, \gamma_0' + 1)$, and similarly for $\gamma_1 + 1$, τ, and ρ.) Suppose that U-pred$(\gamma_0 + 1)$ is $\leq_U$-incomparable with U-pred$(\gamma_1 + 1)$; then σ and τ are $\leq_{W(\mathcal{T},\mathcal{U})}$-incomparable.*

Proof. Let η be largest such that $\eta <_U \gamma_0 + 1$ and $\eta <_U \gamma_1 + 1$. Let $\eta = U\text{-pred}(\eta_0 + 1) = U\text{-pred}(\eta_1 + 1)$, where $\eta_0 + 1 \leq_U \gamma_0 + 1$ and $\eta_1 + 1 \leq_U \gamma_1 + 1$. By the minimality of γ_0 and γ_1,

$$\mathrm{crit}(\phi_{\eta,\eta_0+1}) \leq \phi_{0,\eta}(\xi)$$

and

$$\mathrm{crit}(\phi_{\eta,\eta_1+1}) \leq \phi_{0,\eta}(\rho).$$

[To see this, recall that the ϕ maps along a branch of $\mathcal{U}$ have increasing critical points, so if $\mathrm{crit}(\phi_{\eta,\eta_0+1}) > \phi_{0,\eta}(\xi)$, then $\sigma = \phi_{0,\eta}(\xi)$, so $\sigma = \mathrm{nd}(\xi, \eta_0 + 1)$. Similarly on the η_1 side.] But then

$$\mathrm{crit}(\phi_{\eta,\gamma_0+1}) \leq \phi_{0,\eta}(\xi)$$

and

$$\mathrm{crit}(\phi_{\eta,\gamma_1+1}) \leq \phi_{0,\eta}(\rho).$$

By Lemma 3.67,

$$e_\sigma^{\mathcal{W}_{\gamma_0+1}} \perp e_\tau^{\mathcal{W}_{\gamma_1+1}}.$$

But $\sigma \leq \beta_{\gamma_0}$ by the definition of $\mathrm{nd}(\xi, \gamma_0 + 1)$, so $\sigma \leq \alpha_{\gamma_0}$, so $e_\sigma^{\mathcal{W}_{\gamma_0+1}} = e_\sigma^{W(\mathcal{T},\mathcal{U})}$. Similarly, $e_\tau^{\mathcal{W}_{\gamma_1+1}} = e_\tau^{W(\mathcal{T},\mathcal{U})}$, so we are done. □

Corollary 3.73 *Let a be a cofinal branch of $W(\mathcal{T},\mathcal{U})$, and suppose $a = \mathrm{br}(c_0, b_0) = \mathrm{br}(c_1, b_1)$. Then $b_0 = b_1$, and b_0 is cofinal in $\mathcal{U}$. Moreover, if c_0 and c_1 are b_0-minimal, then $c_0 = c_1$.*

Proof. We show first that b_0 is cofinal. Let $\mu < \text{lh}\,\mathcal{U}$, and let $\tau \in a$ with $\tau > \alpha_\mu$, and

$$\tau = \text{nd}(\xi, \gamma + 1),$$

for $\xi \in c_0$ and $\gamma + 1 \in b_0$. Let $\eta = U\text{-pred}(\gamma + 1)$. Then

$$\tau = \phi_{0,\eta}(\xi) \leq \text{crit}(\phi_{\eta,\gamma+1}) < \alpha_\gamma + 1,$$

so $\alpha_\mu < \alpha_\gamma + 1$, so $\mu \leq \gamma$. Hence b_0 is cofinal. Similarly for b_1.

Remark 3.74 The proof showed that if $\text{nd}(\xi, \gamma + 1)\downarrow$ and $\text{nd}(\xi, \gamma + 1) > \alpha_\mu$, then $\gamma \geq \mu$.

Suppose toward contradiction that $b_0 \neq b_1$. Let $\eta_0 \in b_0$ and $\eta_1 \in b_1$ be $\leq_U$-incomparable. Let $\tau_0, \tau_1 \in a$ with $\tau_0 > \alpha_{\eta_0}$ and $\tau_1 > \alpha_{\eta_1}$ and $\tau_0 = \text{nd}(\xi, \gamma_0 + 1)$, $\tau_1 = \text{nd}(\rho, \gamma_1 + 1)$ for some $\gamma_0 + 1 \in b_0$ and $\gamma_1 + 1 \in b_1$. Then $\eta_0 \leq_U \gamma_0 + 1$ and $\eta_1 \leq_U \gamma_1 + 1$ by the remark above. By Corollary 3.72, τ_0 is $\leq_{W(\mathcal{T},\mathcal{U})}$-incomparable with τ_1. Since $\tau_0, \tau_1 \in a$, this is a contradiction.

Finally, suppose c_0 and c_1 are b_0-minimal. We claim $c_0 = c_1$. For that it suffices to show

Claim 3.74.1 *Suppose* $\text{nd}(\xi, \gamma+1)$ *and* $\text{nd}(\rho, \delta+1)$ *are defined and* $\leq_{W(\mathcal{T},\mathcal{U})}$*-comparable. Suppose* $\gamma + 1$ *and* $\delta + 1$ *are* $\leq_U$*-comparable. Then* ξ *and* ρ *are* $\leq_T$*-comparable.*

Proof. Although the ϕ-maps do not fully preserve tree order, we do have

(*i*) $\phi_{\eta,\gamma}(\xi) \leq_{W_\gamma} \phi_{\eta,\gamma}(\rho) \Rightarrow \xi \leq_{W_\eta} \rho$

(*ii*) ξ, ρ are $\leq_{W_\eta}$-incomparable and $\phi_{\eta,\gamma}(\xi)\downarrow$ and $\phi_{\eta,\gamma}(\rho)\downarrow$ implies $\phi_{\eta,\gamma}(\xi)$ and $\phi_{\eta,\gamma}(\rho)$ are $\leq_{W_\gamma}$-incomparable.

Now let $\xi, \gamma + 1, \rho, \delta + 1$ be as in our hypotheses, and suppose ξ and ρ are $\leq_T$-incomparable. By (*ii*), we cannot have $\gamma + 1 = \delta + 1$. Suppose without loss of generality $\gamma + 1 <_U \delta + 1$. Let

$$\eta = U\text{-pred}(\gamma + 1)$$

and

$$\mu = U\text{-pred}(\delta + 1).$$

Then $\phi_{0,\eta}(\xi)$ is $\leq_{W_\eta}$-incomparable with $\phi_{0,\eta}(\rho)$. Since $\phi_{0,\eta}(\xi) \leq \text{crit}(\phi_{\eta,\gamma+1})$ we see that $\phi_{0,\eta}(\xi)$ is incomparable in $\mathcal{W}_{\gamma+1}$ with $\phi_{0,\gamma+1}(\rho)$. (If $\phi_{0,\eta}(\xi) < \text{crit}(\phi_{\eta,\gamma+1})$, this follows from (ii). If $\phi_{0,\eta}(\xi) = \text{crit}(\phi_{\eta,\gamma+1})$, it follows from the definition of $\mathcal{W}_{\gamma+1}$.) Since $\phi_{0,\eta}(\xi) < \text{crit}(\phi_{\gamma+1,\mu})$, $\phi_{0,\eta}(\xi)$ is $\mathcal{W}_\mu$-incomparable with $\phi_{0,\mu}(\rho)$, contradiction. $\square$

$\square$

Finally, we show (assuming still that all $\mathcal{W}_\gamma$, $\gamma < \text{lh}\,\mathcal{U}$, are by a common Σ.)

Lemma 3.75 *For any cofinal branch a of $W(\mathcal{T},\mathcal{U})$, there is a cofinal branch b of $\mathcal{U}$ and a branch c of $\mathcal{T}$ such that $\text{br}_{\mathcal{W}}(c,b) = a$.*

Proof. We begin by decoding notes of $\mathcal{U}$ from nodes of $W(\mathcal{T},\mathcal{U})$. For $\xi < \text{lh}(W(\mathcal{T},\mathcal{U}))$, set

$$d(\xi) = \text{least } \gamma \text{ such that } \xi \leq \alpha_\gamma.$$

Claim 3.75.1

$$\begin{aligned} d(\xi) &= \textit{least } \gamma \textit{ such that } e_\xi^{\mathcal{W}_\gamma} = e_\xi^{W(\mathcal{T},\mathcal{U})} \\ &= \textit{least } \gamma \textit{ such that } \mathcal{M}_\xi^{\mathcal{W}_\gamma} = \mathcal{M}_\xi^{W(\mathcal{T},\mathcal{U})}. \end{aligned}$$

Proof. The two characterization are clearly equivalent. So it is enough to show that $\xi \leq \alpha_\gamma \Leftrightarrow \mathcal{M}_\xi^{\mathcal{W}_\gamma} = \mathcal{M}_\xi^{W(\mathcal{T},\mathcal{U})}$. The $\Rightarrow$ direction is trivial. But if $\mathcal{M}_\xi^{\mathcal{W}_\gamma} = \mathcal{M}_\xi^{W(\mathcal{T},\mathcal{U})}$, then $\mathcal{W}_\gamma \restriction (\xi{+}1) = W(\mathcal{T},\mathcal{U}) \restriction (\xi{+}1)$ by normality. Since $\mathcal{W}_\gamma \restriction (\alpha_\gamma{+}2) = W(\mathcal{T},\mathcal{U}) \restriction (\alpha_\gamma{+}2)$ (because F_γ was used in the latter, and not the former), $\xi \leq \alpha_\gamma$. $\square$

Claim 3.75.2 $\xi_0 \leq_{W(\mathcal{T},\mathcal{U})} \xi_1 \Rightarrow d(\xi_0) \leq_U d(\xi_1)$.

Proof. Let $\gamma_0 = d(\xi_0)$ and $\gamma_1 = d(\xi_1)$. We claim that $\xi_0 \in \text{ran}\,\phi_{0,\gamma_0}$. For let τ be least such that $\phi_{0,\gamma_0}(\tau) \geq \xi_0$. If $\phi_{0,\gamma_0}(\tau) \neq \xi_0$, then there must be $0 \leq_U \eta <_U \sigma + 1 \leq_U \gamma_0$ such that

$$\text{crit}(\phi_{\eta,\sigma+1}) \leq \xi_0 < \phi_{\eta,\sigma+1}(\text{crit}(\phi_{\eta,\sigma+1}))$$

and $\eta = U\text{-pred}(\sigma + 1)$. (All discontinuities in ϕ_{0,γ_0} arise this way.) But then $\xi_0 < \alpha_\sigma + 1$, so $\xi_0 \leq \alpha_\sigma$, and $\sigma < \gamma_0$, contradiction.

Similarly, $\xi_1 \in \text{ran}\,\phi_{0,\gamma_1}$.

We claim that γ_0 and γ_1 are comparable in $\mathcal{U}$. Suppose not, and let η be largest such that $\eta <_U \gamma_0$ and $\eta <_U \gamma_1$. Let

$$\xi_0 = \phi_{\eta,\gamma_0}(\bar{\xi}_0)$$

and

$$\xi_1 = \phi_{\eta,\gamma_1}(\bar{\xi}_1).$$

The hypotheses of 3.67 are satisfied, noting that $\bar{\xi}_0 \geq \mathrm{crit}(\phi_{\eta,\gamma_0})$ because otherwise $e^{\mathcal{W}_{\gamma_0}}_{\xi_0} = e^{\mathcal{W}_\eta}_{\xi_0}$, whilst γ_0 was least such that $e^{\mathcal{W}_{\gamma_0}}_{\xi_0}$ appears as a branch extender. Similarly, $\bar{\xi}_1 \geq \mathrm{crit}(\phi_{\eta,\gamma_1})$. The other hypotheses of 3.67 hold, so we conclude $e^{\mathcal{W}_{\gamma_0}}_{\xi_0}$ is incompatible with $e^{\mathcal{W}_{\gamma_1}}_{\xi_1}$. This implies ξ_0 and ξ_1 are incomparable in $W(\mathcal{T},\mathcal{U})$. Finally, $\xi_0 \leq_{W(\mathcal{T},\mathcal{U})} \xi_1 \Rightarrow \xi_0 \leq \xi_1$, and trivially $\xi_0 \leq \xi_1 \Rightarrow d(\xi_0) \leq d(\xi_1)$. Since $d(\xi_0)$ and $d(\xi_1)$ are $\leq_U$-comparable, $d(\xi_0) \leq_U d(\xi_1)$, as desired. □

Claim 3.75.3 *$d : \mathrm{lh}(W(\mathcal{T},\mathcal{U})) \to \mathrm{lh}\,\mathcal{U}$ is an order-homomorphism, and $\mathrm{ran}(d)$ is cofinal in $\mathrm{lh}(\mathcal{U})$.*

Proof. As we remarked, $\xi_0 \leq \xi_1 \Rightarrow d(\xi_0) \leq d(\xi_1)$ is trivial. Pick any $\gamma < \mathrm{lh}\,\mathcal{U}$, and $\xi < \mathrm{lh}\, W(\mathcal{T},\mathcal{U})$ with $\xi > \alpha_\gamma$. (The α_γ's are strictly increasing.) Then $d(\xi) > \gamma$. □

It follows that for any branch a of $W(\mathcal{T},\mathcal{U})$, we can set

$$d(a) = \{\gamma \mid \exists \xi \in a\, (\gamma \leq_U d(\xi))\},$$

and $d(a)$ is a branch of $\mathcal{U}$. If a is cofinal in $W(\mathcal{T},\mathcal{U})$, then $d(a)$ is cofinal in $\mathcal{U}$.

Next we decode nodes of $\mathcal{T}$. For any $\xi < \mathrm{lh}(W(\mathcal{T},\mathcal{U}))$, set

$$e(\xi) = \text{unique } \alpha < \mathrm{lh}\,\mathcal{T} \text{ such that } \phi_{0,d(\xi)}(\alpha) = \xi.$$

We showed in the proof of Claim 3.75.2 that $\xi \in \mathrm{ran}(\phi_{0,d(\xi)})$.

Claim 3.75.4 *$\xi_0 \leq_{W(\mathcal{T},\mathcal{U})} \xi_1 \Rightarrow e(\xi_0) \leq_T e(\xi_1)$.*

Proof. Let $\gamma_i = d(\xi_i)$ and $\bar{\xi}_i = e(\xi_i)$. As we noted above, the σ maps do not introduce new tree-order relationships in $\mathrm{ran}\,\phi$.

Subclaim 3.75.1 *If $\phi_{\eta,\gamma}(\mu) \leq_{W_\gamma} \phi_{\eta,\gamma}(\nu)$, then $\mu \leq_{W_\eta} \nu$.*

Proof. Easy induction on γ. □

So if $\bar{\xi}_0 \not\leq_T \bar{\xi}_1$, then $\phi_{0,\gamma_0}(\bar{\xi}_0) \not\leq_{W_{\gamma_0}} \phi_{0,\gamma_0}(\bar{\xi}_1)$. That is, $\xi_0 \not\leq_{W_{\gamma_0}} \phi_{0,\gamma_0}(\bar{\xi}_1)$. If $\mathrm{crit}(\phi_{\gamma_0,\gamma_1}) > \xi_0$, then we get $\xi_0 \not\leq_{W_{\gamma_1}} \xi_1$, and since $\xi_1 \leq \alpha_{\gamma_1}$, $\xi_0 \not\leq_{W(\mathcal{T},\mathcal{U})} \xi_1$, as desired. So assume $\xi_0 \geq \mathrm{crit}(\phi_{\gamma_0,\gamma_1})$.

If $\xi_0 = \mathrm{crit}(\phi_{\gamma_0,\gamma_1})$, then $\xi_0 \leq_{W_{\gamma_1}} \phi_{\gamma_0,\gamma_1}(\sigma)$ iff $\xi_0 \leq_{W_{\gamma_0}} \sigma$ for all σ. Since $\xi_0 \not\leq_{W_{\gamma_0}} \phi_{0,\gamma_0}(\bar{\xi}_1)$, this yields $\xi_0 \not\leq_{W_{\gamma_1}} \xi_1$, so $\xi_0 \not\leq_{W(\mathcal{T},\mathcal{U})} \xi_1$, as desired.

Finally, suppose $\xi_0 > \mathrm{crit}(\phi_{\gamma_0,\gamma_1})$. So letting $\tau + 1 \leq \gamma_1$ be least such that $\gamma_0 < \tau + 1$, and

$$\beta = \beta(\mathcal{W}_{\gamma_0}, \mathcal{W}_\tau, F_\tau),$$

we have

$$\beta < \xi_0 \leq \alpha_{\gamma_0} < \alpha_\tau.$$

No extender in $\operatorname{ran} \psi_{\gamma_0,\gamma_1}$ can have critical point in the interval $[\mathrm{crit}(F_\tau), \lambda(F_\tau)]$. This implies that if $\tau+1 \leq_U \gamma$ and $\beta < \xi \leq \alpha_\tau$, then for all $\sigma \in \operatorname{dom} \phi_{\gamma_0,\gamma}$, $\xi \not\leq_{W_\gamma} \phi_{\gamma_0,\gamma}(\sigma)$. In particular, $\xi_0 \not\leq_{W_{\gamma_1}} \xi_1$, so $\xi_0 \not\leq_{W(\mathcal{T},\mathcal{U})} \xi_1$, as desired. $\square$

For a branch a of $W(\mathcal{T},\mathcal{U})$, we set

$$e(a) = \{\beta \mid \exists \xi \in a\, (\beta \leq_T e(\xi))\}.$$

So $e(a)$ is a branch of $\mathcal{T}$. Even if a is cofinal in $W(\mathcal{T},\mathcal{U})$, $e(a)$ may not be cofinal in $\mathcal{T}$. $e(a)$ may have a largest element, or be a maximal branch of $\mathcal{T}$ not chosen by $\mathcal{T}$.

Claim 3.75.5 *Let a be cofinal in $W(\mathcal{T},\mathcal{U})$. Then $a = \mathrm{br}_{\mathcal{W}}(e(a), d(a))$, and $e(a)$ is $d(a)$-minimal.*

Proof. Let $b = d(a)$ and $c = e(a)$. Let $\xi \in a$, we wish to show $\xi \in \mathrm{br}(c, b)$. Let η be least such that $\xi \leq \alpha_\eta$, so that $\eta \in b$. Let $\phi_{0,\eta}(\bar{\xi}) = \xi$, so that $\bar{\xi} \in c$. Let $\gamma+1 \in b$ be such that $\eta = U\text{-pred}(\gamma+1)$. It will be enough to show that $\xi = \mathrm{nd}(\bar{\xi}, \gamma+1)$. For that, it is enough to show that $\xi \leq \mathrm{crit}(\phi_{\eta,\gamma+1})$.

Let $\rho \in a$ be such that $\alpha_\eta < \rho$. Let σ be least such that $\rho \leq \alpha_\sigma$, so that $\sigma \in b$ and $\gamma + 1 \leq_U \sigma$. Let $\phi_{0,\sigma}(\bar{\rho}) = \rho$. If $\xi > \mathrm{crit}(\phi_{\eta,\gamma+1})$, then $\xi \in (\mathrm{crit}(\phi_{\eta,\gamma+1}), \alpha_\eta]$. But we observed above that ξ is "dead" along branches containing $\gamma+1$ for extensions in $\operatorname{ran} \phi_{\eta,\sigma}$, so since ρ is in $\operatorname{ran} \phi_{\eta,\sigma}$, $\xi \not\leq_{W_\sigma} \rho$. But $\mathcal{W}_\sigma \restriction (\alpha_\sigma + 1) = W(\mathcal{T},\mathcal{U}) \restriction (\alpha_\sigma + 1)$, so $\xi \not\leq_{W(\mathcal{T},\mathcal{U})} \rho$, contrary to $\rho \in a$.

It is easy to see that $e(a)$ is $d(a)$-minimal. $\square$

$\square$

Definition 3.76 *Given $\mathcal{T}$ normal on M, and $\mathcal{U}$ normal on the last model of $\mathcal{T}$, we write $\mathrm{br}_{\mathcal{W}}(\mathcal{T},\mathcal{U})$ for the function $\mathrm{br}_{\mathcal{W}}$ (defined on pairs of nodes and pairs of branches) defined above. We write $\mathrm{br}^{\mathcal{W}}_{\mathcal{U}}$ for the function d and $\mathrm{br}^{\mathcal{W}}_{\mathcal{T}}$ for the function e defined above.*

Notation 3.76.1 To reconcile with our previous notation: if b is cofinal in $\mathcal{U}$, there is exactly one branch c of $\mathcal{T}$ such that

(*i*) $c = [0, \tau]_T$ or $c = [0, \tau)_T$ for some $\tau < \mathrm{lh}\, \mathcal{T}$, and (*ii*) $\mathrm{br}_{\mathcal{W}}(c,b)$ is cofinal in $W(\mathcal{T},\mathcal{U})$.

This uses that $\mathcal{T}$ has a last model. We defined $\mathrm{br}(b,\mathcal{T},\mathcal{U})$ to be $\mathrm{br}_{\mathcal{W}}(c,b)$, for the unique such c. We define $m(b,\mathcal{T},b)$ to be the unique τ as in (*i*). We probably won't use that earlier notation much.

For τ in (*i*) a limit ordinal, the earlier notation does not distinguish between $c=[0,\tau)_T$ and $c=[0,\tau]_T$, whereas the current one does. $c=[0,\tau)_T$ is the case where, roughly speaking, the measures in E_b concentrate on proper initial segments of $\mathcal{M}_c^{\mathcal{T}}|\delta(\mathcal{T}\restriction\sup c)=\mathcal{M}_\tau^{\mathcal{T}}|\lambda_\tau^{\mathcal{T}}$.

Remark 3.77 We assumed $\mathcal{T}$ has a last model, but one could generalize some of this by dropping that, and assuming that $\mathcal{U}$ is on $\mathcal{M}(\mathcal{T})$.

Remark 3.78 There are two special cases worth mentioning.

(*a*) $\mathcal{T}^\frown\mathcal{U}$ is already normal. Then $W(\mathcal{T}^\frown\mathcal{U})=\mathcal{T}^\frown\mathcal{U}$, and $\mathrm{br}_W(c,b)=c^\frown b$.

(*b*) $\mathcal{U}$ is a tree on $M|\kappa$, where $\kappa=\inf\{\mathrm{crit}(E_\eta^{\mathcal{T}})\mid\eta+1<\mathrm{lh}\,\mathcal{T}\}$. Then if $\mathcal{U}$ has limit length, then $W(\mathcal{T},\mathcal{U})=\mathcal{U}$-on-$M$, i.e. $\mathcal{U}$ regarded as a tree on M. For b a cofinal branch of $\mathcal{U}$, $\mathcal{W}_b=W(\mathcal{T},\mathcal{U}^\frown b)=\mathcal{U}^\frown b^\frown(i_b^{\mathcal{U}})\mathcal{T}$, and $\mathrm{br}_W(c,b)=b^\frown\phi``c$, where $\phi(\eta)=\mathrm{lh}\,\mathcal{U}+\eta$.

In our application, however, $\mathcal{T}$ and $\mathcal{U}$ will definitely not be separated this way.

Remark 3.79 $\mathrm{br}_{\mathcal{W}}(\mathcal{T},\mathcal{U})$ makes sense in the coarse structural case. Our proof that it is 1-1 and onto used fine structure (via 3.67), as well as the hypothesis that all $\mathcal{W}_\gamma$ are by some fixed Σ. So that part is limited to the fine structural case. But not much fine structure was used, and we shall adapt the proof to the coarse structural case later.

3.8 Normalizing longer stacks

There seem to be in the abstract many different ways to normalize a stack $\langle\mathcal{U}_1,...,\mathcal{U}_n\rangle$, one for each way of associating the $\mathcal{U}_i$. If we are in the case that embedding normalization coincides with full normalization, and there is a fixed strategy Σ for M according to which all these normalizations are played, such that for any N there is at most one normal Σ-iteration from M, then clearly all these normalizations are the same. They are just the unique normal tree by Σ from M to the last model of $\vec{\mathcal{U}}$. We shall be in that situation below when we deal with coarse iterations of a background universe. But in general, it seems that the various normalizations of $\vec{\mathcal{U}}$ might all be different from one another.

We shall define Σ *normalizes well* by demanding that whenever $\vec{\mathcal{U}}$ is a finite stack by Σ, then all normalizations of $\vec{\mathcal{U}}$ are by Σ. In addition, we demand that Σ pull back to itself under normalization maps.

Definition 3.80 *Let $\vec{\mathcal{U}} = \langle \mathcal{U}_1, \ldots, \mathcal{U}_n \rangle$ be a finite stack of normal trees on M, where $n > 1$. Let $M_0 = M$, and M_i be the last model of $\mathcal{U}_i$ for $1 \leq i \leq n$. A* 1-step normalization *of $\vec{\mathcal{U}}$ is a triple $\langle k, \vec{\mathcal{V}}, \vec{\pi} \rangle$ such that $\vec{\mathcal{V}}$ is a stack of length $n-1$ on $M = M_0$, and*

(1) $1 \leq k < n$,

(2) $\mathcal{V}_m = \mathcal{U}_m$ for all $m < k$, and $\mathcal{V}_k = W(\mathcal{U}_k, \mathcal{U}_{k+1})$,

(3) Letting $N_0 = M$ and N_i be the last model of $\mathcal{V}_i$ for $i < n$, we have that

(a) $\pi_i \colon M_i \to N_i$ is the identity for $i < k$,

(b) $\pi_k \colon M_{k+1} \to N_k$ is the map given by embedding normalization, and

(c) for $k < i < n$, $\mathcal{V}_i = \pi_{i-1}\mathcal{U}_{i+1}$, and $\pi_i \colon M_{i+1} \to N_i$ is the copy map.

Clearly, $\vec{\mathcal{U}}$ and k determine the rest of the normalization.

Definition 3.81 *Let $\vec{\mathcal{U}} = \langle \mathcal{U}_1, \ldots, \mathcal{U}_n \rangle$ be a finite, maximal M-stack, with $n > 1$. Let $1 \leq t < n$; then a* t-step normalization *of $\vec{\mathcal{U}}$ is a sequence s with domain $t+1$ such that $s(0) = (0, \vec{\mathcal{U}}, \emptyset)$, and whenever $0 \leq i < t$, $s(i+1)$ is a 1-step normalization of $\vec{\mathcal{V}}$, where $\vec{\mathcal{V}}$ is the second coordinate of $s(i)$.*

A *complete normalization* of $\langle \mathcal{U}_1, \ldots, \mathcal{U}_n \rangle$ is an $n-1$ step normalization of $\langle \mathcal{U}_1, \ldots, \mathcal{U}_n \rangle$. We shall sometimes identify a t-step normalization s of $\vec{\mathcal{U}}$ with the stack of trees in the second coordinate of $s(t)$. If $t = n-1$, then this is a single normal tree on M.

Remark 3.82 Benjamin Siskind has recently shown that the normalization operation is associative, in that if $\vec{\mathcal{U}}$ is a finite stack of normal trees on a premouse M, then all complete normalizations of s produce the same normal tree on M. This is not at all obvious, even in the case that $\mathrm{lh}(\vec{\mathcal{U}}) = 3$, where there are only two possible ways to normalize $\vec{\mathcal{U}}$.

For $m \geq 1$, and $i \geq 0$, let us write $\mathcal{V}_m^{s(i)}$ for the m-th tree in the second coordinate of $s(i)$ (or in its third coordinate, if $i > 0$), and $N_m^{s(i)}$ for the last model of $\mathcal{V}_m^{s(i)}$. Let $N_0^{s(i)} = M$, for all i. For any $e < i < n$, and any m such that $N_m^{s(i)}$ exists, there is a unique l such that $N_m^{s(i)}$ comes from $N_l^{s(e)}$, in the sense that $s(e)\restriction(l+1)$ is normalized to $s(i)\restriction(m+1)$ by $s\restriction(e,i]$. Let us write

$$l = o^{s,i,e}(m)$$

in this case. Composing normalization maps and copy maps given by $s\restriction(e,i]$ yields a canonical

$$\pi_{l,m}^{s,i,e} \colon N_l^{s(e)} \to N_m^{s(i)},$$

where $l = o^{s,i,e}(m)$. So if s is a normalization of $\langle \mathcal{U}_1, \ldots, \mathcal{U}_n\rangle$ with $\mathrm{dom}(s) = i+1$, then the stack $\vec{\mathcal{V}}^{s(i)}$ has last model $N_m^{s(i)}$, where $m = n - i$, and $n = o^{s,i,0}(m)$, and $\pi_{n,m}^{s,i,0}$ is the natural map from the last model of $\vec{\mathcal{U}}$ to the last model of $\vec{\mathcal{V}}$. Let us write

$$\pi^s = \pi_{n,m}^{s,i,0}$$

in this case. So π^s is the natural map from the last model of $s(0)$ to the last model of the stack in $s(\mathrm{dom}(s)-1)$ that is given by s. All $\pi_{l,m}^{s,i,e}$ have the form π^u, for u obtained from s in a simple way.

Probably the most natural order in which to normalize a stack is bottom-up.

Definition 3.83 *Let $\vec{\mathcal{U}} = \langle \mathcal{U}_1, \ldots, \mathcal{U}_n\rangle$ be a finite, maximal stack of normal trees on M; then the* bottom-up normalization *of $\vec{\mathcal{U}}$ is the complete normalization s of $\vec{\mathcal{U}}$ such that for each $i \geq 1$ in* $\mathrm{dom}(s)$, *$s(i)$ has first coordinate* 1. *We write $W(\vec{\mathcal{U}})$ for the normal tree on M in the second coordinate of* $s(\mathrm{dom}(s)-1)$, *and also call $W(\vec{\mathcal{U}})$ the bottom-up normalization of $\vec{\mathcal{U}}$.*

The definitions above extend to stacks $\vec{\mathcal{U}}$ on M of infinite length. Again, it seems to makes sense to normalize in any order, but the most natural way is bottom-up. Suppose for example that $\vec{\mathcal{U}} = \langle \mathcal{U}_n \mid n < \omega\rangle$. Let $\mathcal{W}_0 = \mathcal{U}_0$, and for $n \geq 1$ let

$$\mathcal{W}_n = W(\langle \mathcal{U}_i \mid i \leq n\rangle).$$

For $n \geq 0$, let

$$\Phi_n \colon \mathcal{W}_n \to \mathcal{W}_{n+1}$$

be the tree embedding given by the fact that $\mathcal{W}_{n+1} = W(\mathcal{W}_n, \pi\mathcal{U}_{n+1})$ for the appropriate π. (Φ_n is partial iff $\mathcal{U}_{n+1}$ drops along its main branch.) Then we set

$$W(\vec{\mathcal{U}}) = \lim_n \mathcal{W}_n,$$

where the limit is taken using the Φ_n. It is clear how to define this limit as an algebraic structure, but not at all clear that it is an iteration tree. Its length may be illfounded, and the models occurring in it may be illfounded. As in the case of finite stacks, what we need is that $\vec{\mathcal{U}}$ has been played according to a sufficiently good iteration strategy. The optimal result in this direction is due to Schlutzenberg; see [44]. We discuss this matter further in the next chapter.

One can continue further into the transfinite. $W(\vec{\mathcal{U}})$ makes sense as an algebraic structure for stacks $\vec{\mathcal{U}}$ of normal trees of any length, and under appropriate iterability hypotheses it is an iteration tree. In fact, one could go beyond linear stacks of normal trees, and consider normalizing arbitrary trees on M. There is as of now no good theory at this level of generality.

In this book we shall not need more than normalization for finite stacks of normal trees.

4 Strategies that condense and normalize well

In this chapter we define what it is for an iteration strategy to normalize well, and to have strong hull condensation. We prove some elementary facts related to these two properties, and we show that they follow from strong unique iterability.

All the good behavior of iteration strategies one could wish for seems to follow from their normalizing well and having strong hull condensation. This good behavior then follows from strong uniqueness, but strong uniquenessis too restrictive. Mice with Woodin cardinals do not in general have strongly unique iteration strategies. On the other hand, we shall see later that every iterable pure extender premouse has an iteration strategy that normalizes well and has strong hull condensation. (See Proposition 6.25.)

Assuming AD^+, one can obtain strongly uniquely iterable coarse premice having Woodin cardinals via the Γ-Woodin construction. We discuss this in section 3.2. In section 3.3, we show that UBH together with the existence of a Woodin cardinal above a supercompact cardinal implies the existence of strongly uniquely iterable coarse premice with Woodin cardinals. These are our main existence theorems for coarse premice with strongly unique iteration strategies.

In section 3.4, we show that if $\mathbb{C}$ is a background construction done inside a coarse premouse N^* with an iteration strategy Σ^* that normalizes well, then for any model M of $\mathbb{C}$, the induced strategy $\Omega(\mathbb{C}, M, \Sigma^*)$ for M normalizes well. In section 3.5 we show that strong hull condensation is similarly preserved. In particular, if Σ^* is a strongly unique strategy for N^*, then the background-induced strategies $\Omega(\mathbb{C}, M, \Sigma^*)$ all normalize well and have strong hull condensation. This (together with its counterpart later for strategy mice) is our main existence theorem for fine structural mice with strategies that normalize well and have strong hull condensation.

4.1 The definitions

The definitions in this section apply to both fine-structural premice and coarse premice.

Definition 4.1 *Let Σ be a complete iteration (λ, θ)-strategy for M, where $\lambda > 1$.*

(1) We say that Σ 2-normalizes well *iff whenever $\langle \mathcal{T}, \mathcal{U} \rangle$ is a maximal 2-stack by Σ such that $\mathcal{U}$) has last model Q, then*

(a) $W(\mathcal{T}, \mathcal{U})$ is by Σ, and

(b) letting $\mathcal{W} = W(\mathcal{T}, \mathcal{U})$ have last model R, and $\pi \colon Q \to R$ be the last t-map of the embedding normalization, we have that $\Sigma_{\vec{\mathcal{U}},Q} = (\Sigma_{\vec{\mathcal{V}},R})^\pi$.

(2) We say that Σ normalizes well *iff all its tails Σ_s 2-normalize well.*

Clearly, if Σ normalizes well, then so do all its tail strategies.

In 4.1(1), we restrict ourselves to maximal stacks $\langle \mathcal{T}, \mathcal{U}\rangle$ because we have not defined $W(\mathcal{T}, \mathcal{U})$ when $\langle \mathcal{T}, \mathcal{U}\rangle$ is not maximal. It is in fact not hard to define $W(s)$ for non-maximal stacks of merely weakly normal trees. The theorems we prove here, for example Theorem 4.21 and Theorem 4.41, hold for the resulting stronger version of normalizing well.

Suppose that Σ normalizes well, and $\mathcal{T}$ is a normal tree on M with last model Q that is according to Σ. Let $\mathcal{U}$ on Q be normal and by $\Sigma_{\mathcal{T},Q}$ and of limit length, and let

$$b = \Sigma_{\mathcal{T},Q}(\mathcal{U}) = \Sigma(\langle \mathcal{T}, \mathcal{U}\rangle),$$

and

$$a = \Sigma(W(\mathcal{T}, \mathcal{U})).$$

Then

$$a = \mathrm{br}_W^{\mathcal{T},\mathcal{U}}(c, b)$$

where c is some branch $[0, \tau)_{\mathcal{T}}$ or $[0, \tau]_{\mathcal{T}}$ of $\mathcal{T}$ that is chosen by Σ. Moreover,

$$b = \mathrm{br}_{\mathcal{U}}^{\mathcal{T},\mathcal{U}}(a).$$

In other words, $\Sigma(\langle \mathcal{T}, \mathcal{U}\rangle)$ and $\Sigma(W(\mathcal{T}, \mathcal{U}))$ determine each other, modulo $\mathcal{T}$. (This "moreover" part applies in the fine-structural case, and to the case of $\vec{F}$-trees with $\vec{F}$ coarsely coherent, with all $\mathcal{W}_\gamma$ by a fixed Σ.) Applying this repeatedly we get

Proposition 4.2 *Let Σ and Ψ be complete strategies for M with scope H_δ that normalize well, and suppose that Σ agrees with Ψ on normal trees; then Σ agrees with Ψ on finite, maximal M-stacks.*

Proof. We just gave the proof for stacks of length 2. Let $\langle \mathcal{U}_1, ..., \mathcal{U}_{n+1}\rangle$ be a maximal stack by Σ and Ψ such that $\mathcal{U}_{n+1}$ has limit length. Let Q be the base model of $\mathcal{U}_{n+1}$, and R the last model of $W(\langle \mathcal{U}_1, ..., \mathcal{U}_n\rangle) = \mathcal{W}$, and $\sigma \colon Q \to R$ the embedding normaliztion map. We have that $\langle \mathcal{W}, \sigma\mathcal{U}_{n+1}\rangle$ is by both strategies because they normalize well. By our result for stacks of length 2, $\Sigma(\langle \mathcal{W}, \sigma\mathcal{U}_{n+1}\rangle) = \Psi(\langle \mathcal{W}, \sigma\mathcal{U}_{n+1}\rangle)$. But $\Sigma(\langle \mathcal{W}, \sigma\mathcal{U}_{n+1}\rangle) = \Sigma(\langle \mathcal{U}_1, ..., \mathcal{U}_{n+1}\rangle)$ because Σ normalizes well, and similarly for Ψ, so we are done. $\square$

We now show that if Σ normalizes well, then in fact it does so for arbitrary finite stacks, not just stacks of length 2.

Proposition 4.3 *Let Σ be an complete (λ, θ)-iteration strategy for M that normalizes well, and let r be a stack of length $< \lambda$ by Σ. Suppose $\vec{\mathcal{U}}$ is a finite maximal stack by Σ_r, and s is a t-step normalization of $\vec{\mathcal{U}}$, and $\vec{\mathcal{V}} = \vec{\mathcal{V}}^{s(t)}$ is the stack in $s(t)$, then*

(1) $\vec{\mathcal{V}}$ is by Σ_r, and

(2) if $\pi = \pi^s$ is the natural map from the last model Q of $\vec{\mathcal{U}}$ to the last model R of $\vec{\mathcal{V}}$, then $\Sigma_{r^\frown \vec{\mathcal{U}}, Q} = (\Sigma_{r^\frown \vec{\mathcal{V}}, R})^\pi$.

Proof. We show by induction on n that Σ normalizes well for stacks of length n. The same proof works for tails Σ_r of Σ.

For $n = 2$ this is true by hypothesis. Let $\vec{\mathcal{T}}^\frown\langle \mathcal{U}_1, \mathcal{U}_2\rangle^\frown\vec{\mathcal{S}}$ be a stack of length $n+1$ by Σ. We want to see that the 1-step normalization obtained by replacing $\langle \mathcal{U}_1, \mathcal{U}_2\rangle$ by $W(\mathcal{U}_1, \mathcal{U}_2)$, and $\vec{\mathcal{S}}$ by $\pi\vec{\mathcal{S}}$ for π the normalization map, behaves well. It is clear that this implies t-step normalizations behave well, for all t.

Let $\mathcal{V}$ be a complete normalization of $\vec{\mathcal{T}}$, with θ the normalization map from $N = \mathcal{M}_\infty^{\vec{\mathcal{T}}}$ to $N^* = \mathcal{M}_\infty^{\mathcal{V}}$. θ lifts $\mathcal{U}_1$ to $\theta\mathcal{U}_1$; let $\rho\colon \mathcal{M}_\infty^{\mathcal{U}_1} \to \mathcal{M}_\infty^{\theta\mathcal{U}_1}$ be the copy map. Note that $\langle \mathcal{V}, \theta\mathcal{U}_1, \rho\mathcal{U}_2\rangle$ is a stack by Σ, because $\Sigma_{\mathcal{V}, N^*}$ pulls back under θ to $\Sigma_{\vec{\mathcal{T}}, N}$ by our induction hypothesis. Let Q^* be its last model. Let

$$\mathcal{W}^* = W(\theta\mathcal{U}_1, \rho\mathcal{U}_2),$$

and let R^* be the last model of $\mathcal{W}^*$, and $\sigma^*\colon Q^* \to R^*$ the normalization map. The hypothesis of our proposition tells us that $\langle \mathcal{V}, \mathcal{W}^*\rangle$ is by Σ, and that

$$\Sigma_{\langle \mathcal{V}, \theta\mathcal{U}_1, \rho\mathcal{U}_2\rangle, Q^*} = (\Sigma_{\langle \mathcal{V}, \mathcal{W}^*\rangle, R^*})^{\sigma^*}.$$

Let Q be the last model of $\vec{\mathcal{T}}^\frown\langle \mathcal{U}_1, \mathcal{U}_2\rangle$, let

$$\mathcal{W} = W(\mathcal{U}_1, \mathcal{U}_2),$$

and let R be the last model of $\mathcal{W}$. Let $\sigma\colon Q \to R$ be the normalization map. The situation can be encapsulated in the following diagram.

$$\begin{tikzcd}
& & & R^* \\
& N^* \arrow[r, "\theta\mathcal{U}_1"'] \arrow[urr, "\mathcal{W}^*"] & P^* \arrow[r, "\rho\mathcal{U}_2"'] & Q^* \arrow[u, "\sigma^*"] \\
& & & R \arrow[uu, bend right, "\phi"'] \\
M \arrow[r, "\vec{\mathcal{T}}"'] \arrow[uur, "\mathcal{V}"] & N \arrow[uu, "\theta"] \arrow[urr, "\mathcal{W}"] \arrow[r, "\mathcal{U}_1"'] & P \arrow[r, "\mathcal{U}_2"'] & Q \arrow[u, "\sigma"'] \arrow[uu, bend left, "\psi"]
\end{tikzcd}$$

Here $P = \mathcal{M}_\infty^{\mathcal{U}_1}$, and $P^* = \mathcal{M}_\infty^{\theta\mathcal{U}_1}$, and $\rho\colon P \to P^*$ is the copy map. The maps $\psi\colon Q \to Q^*$ and $\phi\colon R \to R^*$ are copy maps. We get ϕ from Theorem 3.54; in this case, copying $\langle \mathcal{U}_1, \mathcal{U}_2\rangle$ via θ commutes with normalizing $\langle \mathcal{U}_1, \mathcal{U}_2\rangle$. We have

$$\phi \circ \sigma = \sigma^* \circ \psi$$

from 3.54.

Since $\theta\mathcal{W} = \mathcal{W}^*$, and Σ pulls back to itself under θ by induction, we have that $\vec{\mathcal{T}}^\frown\langle\mathcal{W}\rangle$ is by Σ, and $\Sigma_{\vec{\mathcal{T}}^\frown\langle\mathcal{W}\rangle, R} = (\Sigma_{\langle\mathcal{V},\mathcal{W}^*\rangle, R^*})^\phi$. It follows that

$$\begin{aligned}
(\Sigma_{\vec{\mathcal{T}}^\frown\langle\mathcal{W}\rangle, R})^\sigma &= (\Sigma_{\langle\mathcal{V},\mathcal{W}^*\rangle, R^*})^{\phi\circ\sigma} \\
&= (\Sigma_{\langle\mathcal{V},\mathcal{W}^*\rangle, R^*})^{\sigma^*\circ\psi} \\
&= ((\Sigma_{\langle\mathcal{V},\mathcal{W}^*\rangle, R^*})^{\sigma^*})^{\psi} \\
&= (\Sigma_{\langle\mathcal{V},\theta\mathcal{U}_1,\rho\mathcal{U}_2\rangle, Q^*})^{\psi} \\
&= \Sigma_{\vec{\mathcal{T}}^\frown\langle\mathcal{U}_1,\mathcal{U}_2\rangle, Q}.
\end{aligned}$$

Line 1 holds because Σ normalizes well for $\vec{\mathcal{T}}$, line 2 comes from 3.54, line 4 holds because $\Sigma_{\mathcal{V},N^*}$ 2-normalizes well, and line 5 holds because Σ normalizes well for $\vec{\mathcal{T}}$.

This takes care of the case $\vec{\mathcal{S}} = \emptyset$. The general case follows easily. Since $\Sigma_{\vec{\mathcal{T}}^\frown\langle\mathcal{W}\rangle, R})^\sigma = \Sigma_{\vec{\mathcal{T}}^\frown\langle\mathcal{U}_1,\mathcal{U}_2\rangle, Q}$ and $\vec{\mathcal{S}}$ is by $\Sigma_{\vec{\mathcal{T}}^\frown\langle\mathcal{U}_1,\mathcal{U}_2\rangle, Q}$, we have that $\sigma\vec{\mathcal{S}}$ is by $\Sigma_{\vec{\mathcal{T}}^\frown\langle\mathcal{W}\rangle, R}$, and moreover the $\vec{\mathcal{T}}^\frown\langle\mathcal{W}\rangle^\frown\sigma\vec{\mathcal{S}}$-tail of Σ pulls back under the relevant copy map to the $\vec{\mathcal{T}}^\frown\langle\mathcal{U}_1,\mathcal{U}_2\rangle^\frown\vec{\mathcal{S}}$-tail of Σ. $\square$

A very similar argument shows that the property of normalizing well passes to pullback strategies.

Theorem 4.4 *Let Σ be an iteration strategy for N that normalizes well, and let $\pi \colon M \to N$ be sufficiently elementary that the pullback strategy Σ^π exists; then Σ^π normalizes well.*

Proof. Let $\langle \mathcal{V}, \mathcal{U}_1, \mathcal{U}_2 \rangle$ be a stack by Σ^π, with last model Q. Let $\mathcal{W} = W(\mathcal{U}_1, \mathcal{U}_2)$ have last model R, and $\sigma \colon Q \to R$ be the normalization map. We want to see that $\langle \mathcal{V}, \mathcal{W} \rangle$ is by Σ^π, and that the $\langle \mathcal{V}, \mathcal{W} \rangle$-tail of Σ^π pulls back under σ to the $\langle \mathcal{V}, \mathcal{U}_1, \mathcal{U}_2 \rangle$-tail of Σ^π.

We have the diagram

$$\begin{tikzcd}
 & & & R^* \\
N \arrow[r, "\pi\mathcal{V}"] & K^* \arrow[r, "\theta\mathcal{U}_1"'] \arrow[urr, "\mathcal{W}^*"] & P^* \arrow[r, "\rho\mathcal{U}_2"'] & Q^* \arrow[u, "\sigma^*"] \\
 & & & R \arrow[uu, bend right, "\phi"'] \\
M \arrow[uu, "\pi"] \arrow[r, "\mathcal{V}"'] & K \arrow[uu, "\theta"] \arrow[r, "\mathcal{U}_1"'] \arrow[urr, "\mathcal{W}"] & P \arrow[r, "\mathcal{U}_2"'] & Q \arrow[u, "\sigma"'] \arrow[uu, bend left, "\psi"]
\end{tikzcd}$$

Here $\theta \colon K \to K^*$ and $\rho \colon P \to P^*$ are copy maps generated by π, and $\mathcal{W}^*$ is the normalization of $\langle \theta\mathcal{U}_1, \rho\mathcal{U}_2 \rangle$. σ^* is the associated normalization map. ψ and ϕ are copy maps, which we have because copying commutes with normalization. $\phi \circ \sigma = \sigma^* \circ \psi$ by 3.54.

The copy map ϕ tells us that $\langle \mathcal{V}, \mathcal{W} \rangle$ is by Σ^π. The rest is given by

$$\begin{aligned}
(\Sigma^\pi_{\langle \mathcal{V}, \mathcal{W} \rangle, R})^\sigma &= (\Sigma_{\langle \pi\mathcal{V}, \mathcal{W}^* \rangle, R^*})^{\phi \circ \sigma} \\
&= (\Sigma_{\langle \pi\mathcal{V}, \mathcal{W}^* \rangle, R^*})^{\sigma^* \circ \psi} \\
&= ((\Sigma_{\langle \pi\mathcal{V}, \mathcal{W}^* \rangle, R^*})^{\sigma^*})^{\psi} \\
&= (\Sigma_{\langle \pi\mathcal{V}, \theta\mathcal{U}_1, \rho\mathcal{U}_2 \rangle, Q^*})^{\psi} \\
&= \Sigma^\pi_{\vec{\mathcal{T}}^\frown \langle \mathcal{U}_1, \mathcal{U}_2 \rangle, Q}.
\end{aligned}$$

This is what we want. $\square$

We turn to strong hull condensation. It will be conveneient here to extend the definition of extended tree embeddings (3.39) so that they can act on weakly normal trees $\mathcal{T}$ of length 1.

Definition 4.5 *Let $\mathcal{U}$ be a weakly normal tree on M of length $\beta+1$, and let $N \trianglelefteq M$; then we say that $\hat{\imath}^{\mathcal{U}}_{0,\beta}$ is an* extended tree embedding *from the weakly normal tree $\langle \emptyset, N\rangle$ into $\mathcal{U}$.*

The point of this perhaps strange terminology is to streamline the following definition.

Definition 4.6 *Let Σ be a complete iteration strategy for a premouse M. Then Σ has* strong hull condensation *iff whenever s is a stack of weakly normal trees by Σ with last model N, and $\mathcal{U}$ is a weakly normal tree on N by $\Sigma_{s,N}$, then for any weakly normal $\mathcal{T}$ on N,*

(a) if $\mathcal{T}$ is a psuedo-hull of $\mathcal{U}$, then $\mathcal{T}$ is by $\Sigma_{s,N}$, and

(b) if $\Phi\colon \mathcal{T} \to \mathcal{U}$ is an extended tree embedding, with last t-map $\pi\colon Q \to R \trianglelefteq \mathcal{M}^{\mathcal{U}}_\alpha$ then $\Sigma_{s^\frown\langle\mathcal{T}\rangle,Q} = (\Sigma_{s^\frown\langle\mathcal{U}\restriction(\alpha+1)\rangle,R})^\pi$.

Because less is required of a tree embedding than is required of a hull embedding in [30], the property is stronger than the property called Hull Condensation in [30]. Hence its name.

Clause (b) was not part of our original definition of strong hull condensation. B. Siskind then showed that (b) follows abstractly from (a) and normalizing well (see [48]), via a strategy-comparison argument. We have made clause (b) part of the definition here because it is useful, and one can obtain it directly for background-induced strategies.

Despite the title of this book, it will turn out that strong hull condensation is the fundamental regularity property of iteration strategies. All the other regularity properties are implied strong hull condensation together with normalizing well. We believe that a complete strategy with strong hull condensation need not normalize well, although we have no example at the moment. However, any complete strategy for normal trees that has strong hull condensation can be extended in a unique way to a complete strategy for finite stacks of normal trees that has strong hull condensation and normalizes well. This is a result of Schlutzenberg and the author. Schlutzenberg also proved a stronger version of the theorem in which the extended strategy can act on infinite stacks. See [44] and [48], and Theorem 4.34 in the next section.

Remark 4.7 The papers [60] and [48] introduce a still weaker sort of embedding of iteration trees, and make use of the resulting "very strong hull condensation". It turns out that strategies for premice that have strong hull condensation also have very strong hull condensation, and this implies that they fully normalize well. However, the proof of this requires a strategy-comparison argument. Strong hull condensation has the virtue that we can verify it directly for background-induced strategies, so we can use it in proving a comparison theorem.

Because we have included clause (b) in the definition of strong hull condensation, it implies a property usually known as *pullback consistency*. Indeed, what is usually called pullback consistency is just clause (b), as applied to $\mathcal{T}$ of length one.

Definition 4.8 *Let Ω be a complete iteration strategy for M. We say that Ω is* pullback consistent *iff whenever u is an M-stack by Ω, and s is a finite M-stack by Ω_u, and $\pi\colon \mathcal{M}_\alpha^{\mathcal{T}_m(s)} \to \mathcal{M}_\infty(s)$ is an iteration map of s, then for $t = s\restriction(m-1)^\frown\langle(\nu_m(s), k_m(s), \mathcal{T}_m(s)\restriction(\alpha+1))\rangle$,*

$$\Omega_{u^\frown t} = (\Omega_{u^\frown s})^\pi.$$

A pullback consistent strategy pulls back to itself under its own iteration maps, where by "iteration map" we mean any map of a branch segment generated somewhere in a finite stack s by the strategy, from one model to a later one. This is a strengthening of the pullback consistency condition from [30]. It follows at once from strong hull condensation.

Lemma 4.9 *Let Ω be a complete strategy for M that has strong hull condensation; then Ω is pullback consistent.*

Proof. Suppose first that $\mathcal{T}$ is a weakly normal tree on $M_\infty(u)$ by Ω_u of length $\beta+1$, and that $\alpha \le_T \beta$. Suppose that $Q \trianglelefteq \mathcal{M}_\alpha^{\mathcal{T}}$ and $Q \subseteq \mathrm{dom}(\hat{\imath}^{\mathcal{T}}_{\alpha,\beta})$. Let $\mathcal{U}$ be $\mathcal{T}\restriction(\alpha+1)$, followed by a gratuitous drop to Q, and let $\mathcal{W}$ be $\mathcal{T}$ followed by a gratuitous drop to $\hat{\imath}^{\mathcal{T}}_{\alpha,\beta}(Q)$. Letting $\pi = \hat{\imath}^{\mathcal{T}}_{\alpha,\beta}\restriction Q$, we have that π is the last t-map of an extended tree embedding from $\mathcal{U}$ to $\mathcal{W}$. (If $\alpha > 0$, its associated tree embedding is just the identity on $\mathcal{T}\restriction\alpha+1$, and if $\alpha = 0$, we appeal to definition 4.5.) By part (b) of definition 4.6, $(\Omega_{u^\frown\langle\mathcal{T}\rangle,R})^\pi = \Omega_{u^\frown\langle\mathcal{T}\restriction\alpha+1\rangle,Q}$, which is what we need.

It is routine the extend this argument to finite M-stacks by Ω_u, by pulling back under the branch embeddings of the constituent normal trees, one at a time. □

Strong hull condensation is preserved by pullbacks:

Proposition 4.10 *Let $\pi\colon M \to N$ be weakly elementary, and let Σ be a strategy for N having strong hull condensation; then Σ^π has strong hull condensation.*

Proof.(Sketch.) There is a relevant diagram below. Let s be a stack on M with last model K, and let K^* be the last model of πs, with $\theta\colon K \to K^*$ the copy map. Let $\mathcal{U}$ be on K and by $(\Sigma^\pi)_s$, and let $\mathcal{T}$ be a psuedo-hull of $\mathcal{U}$. It is not hard to see that $\theta\mathcal{T}$ is a psuedo-hull of $\theta\mathcal{U}$. Since $\theta\mathcal{U}$ is by $\Sigma_{\pi s,K^*}$, $\theta\mathcal{T}$ is by $\Sigma_{\pi s,K^*}$, so $\mathcal{T}$ is by $(\Sigma^\pi)_s$, as desired for part (a).

For part (b), let $\Phi\colon \mathcal{T} \to \mathcal{U}$ be an extended tree embedding with last t-map $\sigma\colon Q \to R$. By the (suppressed) construction of the first part, we have an extended tree embedding $\Psi\colon \theta\mathcal{T} \to \theta\mathcal{U}$. Let $\sigma^*\colon Q^* \to R^*$ be the last t-map of Ψ. Let $\psi\colon Q \to Q^*$ come from the copying of $\mathcal{T}$ to $\theta\mathcal{T}$, and $\phi\colon R \to R^*$ come from copying $\mathcal{U}$ to $\theta\mathcal{U}$. We have the diagram

$$\begin{tikzcd}
& & R^* \\
N \arrow[r, "\pi s"] & K^* \arrow[r, "\theta\mathcal{T}"'] \arrow[ur, "\theta\mathcal{U}"] & Q^* \arrow[u, "\sigma^*"] \\
& & R \arrow[uu, bend right, "\phi"'] \\
M \arrow[uu, "\pi"] \arrow[r, "s"'] & K \arrow[uu, "\theta"'] \arrow[r, "\mathcal{T}"'] \arrow[ur, "\mathcal{U}"] & Q \arrow[u, "\sigma"'] \arrow[uu, bend left, "\psi"]
\end{tikzcd}$$

This is quite similar to the diagram in 4.4, because the situations are quite similar. Again, we calculate

$$\begin{aligned}
(\Sigma_{\langle s^\frown\langle\mathcal{U}\rangle,R\rangle})^\sigma &= (\Sigma_{\langle\pi s,\theta\mathcal{U}\rangle,R^*})^{\phi\circ\sigma} \\
&= (\Sigma_{\langle\pi s,\theta\mathcal{U}\rangle,R^*})^{\sigma^*\circ\psi} \\
&= ((\Sigma_{\langle\pi s,\theta\mathcal{U}\rangle,R^*})^{\sigma^*})^{\psi} \\
&= (\Sigma_{\langle\pi s,\theta\mathcal{T}\rangle,Q^*})^{\psi} \\
&= (\Sigma_{s^\frown\langle\mathcal{T}\rangle,Q})^{\psi}.
\end{aligned}$$

□

The following elementary lemma on extending tree embeddings at limit steps will be useful.

Lemma 4.11 *Let Σ be a strategy for the premouse M having strong hull condensation, and let $\mathcal{T}$ and $\mathcal{U}$ be trees of limit length by Σ. Let $\Phi\colon \mathcal{T} \to \mathcal{U}$ be a tree embedding such that*

$$\exists\alpha < \mathrm{lh}(\mathcal{U})\forall\beta(\alpha < \beta < \mathrm{lh}(\mathcal{U}) \Rightarrow \beta \in \mathrm{ran}(u^\Phi)).$$

Let $b = \Sigma(\mathcal{T})$ and $c = \Sigma(\mathcal{U})$; then there is a unique tree embedding $\Psi\colon \mathcal{T}^\frown b \to \mathcal{U}^\frown c$ such that $\Phi \subseteq \Psi$.

Proof. Let $u = u^\Phi$, and $d = u^{-1}$"c. By our hypothesis that $\mathrm{ran}(u)$ contains a final segment of the ordinals below $\mathrm{lh}(\mathcal{U})$, we see that d is cofinal in $\mathrm{lh}(\mathcal{T})$. Moreover, Φ extends to a tree embedding of $\mathcal{T}^\frown d$ into $\mathcal{U}^\frown c$. By strong hull condensation, $d = \Sigma(\mathcal{T}) = b$, so we are done. □

If one weakens the hypothesis of Lemma 4.11 by requiring only that $\mathrm{ran}(u^\Phi)$ be cofinal in $\mathrm{lh}(\mathcal{U})$, then the conclusion may not hold. There is a counterexample in [48], just after definition 1.3.

4.2 Coarse Γ-Woodins and Γ-universality

Of course, one cannot prove that there are any nontrivial iteration strategies without making assumptions that go beyond ZF. Determinacy assumptions are particularly useful in this regard. Under AD^+, every Suslin-co-Suslin set is Wadge reducible to an iteration strategy; in fact, there are countable iterable structures at every Suslin-co-Suslin degree of correctness. More precisely

Definition 4.12 *Let $A \subseteq \mathbb{R}$. We say that $(M, \delta, \tau, \Sigma)$* captures *$A$ iff*

(a) $M \models \mathsf{ZFC}$ + "δ is Woodin",

(b) δ is countable, and Σ is a complete strategy with scope HC for $V^M_{\delta+1}$, and

(c) $\tau \in M$ is a $\mathrm{Col}(\omega, \delta)$-term for a set of reals, and

(d) whenever $i\colon M \to N$ is by Σ and g is $\mathrm{Col}(\omega, i(\delta))$-generic over N, then $i(\tau)_g = A \cap N[g]$.

Notice here that $(M, \delta, \tau, \Sigma)$ does indeed determine A, because for every real x there are N and g as in (d) such that $x \in N[g]$.

The following came out of Woodin's work in the late 1980s on large cardinals in HOD under determinacy hypotheses. See [15] and [58].

Theorem 4.13 *[Woodin] Assume* AD*; then for any Suslin and co-Suslin set* A*, there is a tuple* $(M, \delta, \tau, \Sigma)$ *that captures* A*.*

Unfortunately, the models M produced by the proof of 4.13 are not given as fine structural. They are "HOD-like", but that doesn't help until the analysis of HOD to which this book contributes is done. However, one can use M as a background universe for the construction of some fine structural premouse N, and hope to show that N and its induced strategy capture some set close to A. This is the basic plan behind the proofs we currently have for fragments of LEC and HPC, and it is therefore the main source for the iteration strategies to which the theorems of this book apply.

In this context, it helps to be working with a background universe M having more structure than is recorded in 4.12. We shall call the resulting pairs *coarse* Γ*-Woodin pairs*.

Assume AD^+, and let Γ, Γ_1 be good (i.e. closed under $\exists^{\mathbb{R}}$) lightface pointclasses with the scale property such that $\Gamma \subseteq \Delta_1$. Let A be a universal Γ_1 set, and let $U \subset \mathbb{R}$ code $\{\langle \varphi, x\rangle \mid (V_{\omega+1}, \in, A) \models \varphi[x]\}$. Let S and T be trees on some $\omega \times \kappa$ that project to U and $\neg U$. Using his work in [15], Woodin has shown ([58, Lemma 3.13]) that there is a countable transitive $N^* \in \mathrm{HC}$, a wellorder $\lhd$ of N^*, and an iteration strategy Σ such that for $\delta = o(N^*)$,

(a) (fullness) $N^* = V_\delta^{L(N^* \cup \{S,T,\lhd\})}$,

(b) N^* is f-Woodin, for all $f \colon \delta \to \delta$ such that $f \in C_\Gamma(N^*, \lhd)$,

(c) for all $\eta \leq \delta$, there is an $f \colon \eta \to \eta$ such that $f \in C_{\Gamma_1}(V_\eta^{N^*}, \lhd \cap V_\eta^{N^*})$ and $V_\eta^{N^*}$ is not f-Woodin, and

(d) Σ is an (ω_1, ω_1)-iteration strategy for $L(N^*, S, T, \lhd)$, with respect to nice trees based on N^*.

Concerning item (d), recall that ω_1-iterability implies $\omega_1 + 1$-iterability, granted AD.

Definition 4.14 *Assume* AD^+*, and let* Γ *be a good pointclass with the scale property, and let* $N^*, \delta, S, T, \lhd$*, and* Σ *be as in (a)-(d); then*

(1) we call $\langle N^*, \delta, S, T, \lhd, \Sigma\rangle$ *a* coarse Γ-Woodin tuple*, and*

(2) letting $M = (L[N^*, S, T, \lhd], \in, S, T)$*, we call* (M, Σ) *a* coarse Γ-Woodin pair.

Of course, S and T determine U, and hence A and Γ_1. U is self-dual, so S is only there for notational convenience. We write $A = A_T$. If (M, Σ) is a coarse Γ-Woodin pair, then we write δ^M, $\lhd^M$, T^M, and S^M for the associated objects.

From [15] (see also [58, Lemma 3.13]), we have

Theorem 4.15 (Woodin) *Let* Γ *be a good lightface pointclass with the scale property, and assume that all sets in* $\breve{\Gamma}$ *are Suslin; then for any real* x *there is a coarse* Γ*-Woodin pair* (M, Σ) *such that* $x \in M$*.*

Lemma 4.16 *Let* (P, Σ) *be a coarse* Γ*-Woodin pair,* $\delta = \delta^P$*,* $T = T^P$*, and* $S = S^P$*. Let* s *be a* P*-stack all of whose models are wellfounded, with iteration map* $i \colon P \to Q$ *be an iteration map by* Σ*; then*

(i) $p[i(T)] = p[T]$ *and* $p[i(S)] = p[S]$*,*

(ii) if g *is* $\mathrm{Col}(\omega, i(\delta))$*-generic over* N*, then for* $A = A_T$*,* $(V_{\omega+1}^{N[g]}, \in, A \cap N[g]) \prec (V_{\omega+1}, \in, A)$*, and*

(iii) (Q, Σ_s) *is a coarse* Γ*-Woodin pair.*

Proof. As usual: $p[T] \subseteq p[i(T)]$ and $p[S] \subseteq p[i(S)]$, while $p[i(T)] \cap p[i(S)] = \emptyset$ because Q is wellfounded, and wellfoundedness is absolute to wellfounded models. This gives us (i). For (ii), we use the Tarski-Vaught criterion. Suppose $x \in N[g]$ and $(V_{\omega+1}, \in, A) \models \exists y \in \mathbb{R}\varphi[y, x]$. There is then a branch of T of the form $(\varphi, \langle y, x\rangle, f)$. But then $(\varphi, \langle y, x\rangle, i(f))$ is a branch of $i(T)$, so there is a branch $(\varphi, \langle y, x\rangle, h)$ of $i(T)$ such that $y \in N[g]$, as desired.

(iii) follows easily from (i) and (ii). $\square$

Note that we did not assume in the lemma that s was by Σ. We shall show in a moment that this follows, that is, that Σ witnesses strong unique iterability.

If we drop down from P to $L(N^*, W, \lhd)$, where W is the tree of a Γ-scale on a universal Γ set, then δ becomes Woodin, and Lemma 4.16 yields a pair capturing Γ in the sense of Definition 4.12.

Corollary 4.17 *Let* (P, Σ) *be a coarse* Γ*-Woodin pair, and* $\delta = \delta^P$*. Let* W *be the tree of a scale on a universal* Γ *set, and let* τ *be the natural term for* $p[W]$*; then* $(L[V_\delta^P, \lhd^P, W], \delta, \tau, \Sigma)$ *captures* $p[W]$*.*

Let $M = L[N^*, S, T, \lhd]$, where $(N^*, \delta, S, T, \lhd, \Sigma)$ is a coarse Γ-Woodin tuple. Let $A = A_T$, and let Γ_1 be the good pointclass whose universal set is A. If P is a Σ^*-iterate of M, and g is is P-generic over $\mathrm{Col}(\omega, i(\delta))$, then $P[g]$ is projectively-in-A correct. Thus the C_Γ and C_{Γ_1} operators are correctly defined over $P[g]$. It follows that M and its iterates are C_{Γ_1}-full, and Σ is guided at $\mathcal{T}$ by a Q-structure in $C_{\Gamma_1}(\mathcal{M}(\mathcal{T}))$. More precisely,

Lemma 4.18 *Assume* AD^+, *and let* (M, Σ) *be a coarse* Γ-*Woodin pair. Let* $\vec{\mathcal{T}}, \mathcal{U}$ *be a stack of nice normal trees played by* Σ*; then the following are equivalent*

(1) $\Sigma_{\vec{\mathcal{T}}}(\mathcal{U}) = b$,

(2) $C_{\Gamma_1}(\mathcal{M}(\mathcal{U})) \subseteq \mathcal{M}_b^{\mathcal{U}}$,

(3) $\mathcal{M}_b^{\mathcal{U}}$ *is wellfounded.*

Proof. Just outlined. □

It follows that if (M, Σ) is a coarse Γ-Woodin pair, then Σ is positional, that is, $\Sigma_{s,Q}$ depends only on Q. (Cf. 6.24.) Moreover, if Q is an iterate of M via the stack s, then for $\theta = \omega_1^V$,

(i) Q is strongly uniquely (θ, θ)-iterable, and

(ii) $Q \models$ " I am strongly uniquely (θ, θ)-iterable."

The strategy witnessing (i) is Σ_Q, and the strategy witnessing (ii) is $\Sigma_Q \upharpoonright Q$. Moreover, Σ_Q is definable over $(V_{\omega+1}, \in, A)$ from the parameter $(V^Q_{\delta^Q}, \lhd^Q)$, uniformly in Q, and Q and its generic extensions are correct for the theory of $(V_{\omega+1}, \in, A)$. So we have

Corollary 4.19 *Assume* AD^+, *and let* (M, Σ) *be a coarse* Γ-*Woodin pair; then* M *is strongly uniquely iterable for countable stacks of countable normal trees. Moreover, for* $\kappa = \omega_1^V$,

$$M \models \text{“I am strongly uniquely } (\kappa, \kappa)\text{-iterable”}.$$

If (M, Σ) is a coarse Γ-Woodin pair, and $\mathbb{C}$ is a background construction done in M, then $\mathbb{C}$ never breaks down, because all its levels have iteration strategies induced by Σ. $(M, \vec{F}^{\mathbb{C}})$ is then a coarse premouse,, and Σ is a complete (ω_1, ω_1) iteration strategy for $(M, \vec{F}^{\mathbb{C}})$. If $\mathbb{C}$ is maximal, in that it never passes on the opportunity to add an extender, then $\mathbb{C}$ is universal in the following sense.

Theorem 4.20 (Γ-universality) *Assume* AD^+, *and let* $(N^*, S, T, \lhd, \Lambda)$ *be a coarse* Γ*-Woodin tuple. Let* P *be a premouse in* N^* *that is countable in* N^*, *and let* Σ *be an* ω_1*-iteration strategy for* P *such that* $\mathrm{Code}(\Sigma) \in \Gamma(\mathrm{Code}(P))$. *Let* $\mathbb{C}$ *be the maximal* $\lhd$*-construction of* N^**; then there is some* ν, k *such that* $\nu < o(N^*)$ *and* (P, Σ) *iterates to* $M^{\mathbb{C}}_{\nu,k}$.

Proof. Let $\delta = o(N^*)$, and let $N = L[N^*, \lhd, W]$, where W is the tree of a Γ-scale on a universal Γ-set. We have that $N \models$ "δ is Woodin", moreover, for $\kappa = \omega_1^V$, $\Sigma \restriction H^N_\kappa$ can be computed from W, and is therefore in N. (Note that H^N_κ is closed under Σ, because if $\mathcal{T}$ is by Σ, then $\Sigma(\mathcal{T})$ is a $\Gamma(\mathcal{T}, P)$-singleton.)

Letting $\Sigma_0 = \Sigma \restriction H^N_\kappa$, we have that whenever $F = F^{\mathbb{C}}_\nu$ for some ν, then $i_F(\Sigma_0) \subseteq \Sigma_0$. For suppose $\mathcal{T} \in H^N_\kappa$ is a tree of limit length by both Σ_0 and $i_F(\Sigma_0)$, and let $b = \Sigma_0(\mathcal{T})$ and $c = i_F(\Sigma_0)(\mathcal{T})$. Let g be N-generic for $\mathrm{Col}(\omega, |\mathcal{T}|)$. Let W_0 be the tree projecting to $\mathrm{Code}(\Sigma)$ we get out of W, and let $\mathrm{Code}(\mathcal{T}) = t$, $\mathrm{Code}(b) = u$, and $\mathrm{Code}(c) = v$. We have f such that $(t, u, f) \in [W_0]$, so that $(t, u, i_F \circ f) \in [i_F(W_0)]$. We also have g such that $(t, v, g) \in [i_F(W_0)]$. But $i_F(W_0)$ projects in V to the codeset of a single-valued partial function, by absoluteness of wellfoundedness. Thus $b = c$, as desired.

We can now apply Theorem 2.53 in N. □

It is easy to see that a strongly unique strategy has strong hull condensation and normalizes well.

Theorem 4.21 *Let* M *be a coarse premouse, and let* Σ *witness that* M *is is strongly uniquely* (η, θ)*-iterable; then* Σ *has strong hull condensation, and the complete strategy determined by* Σ *normalizes well.*

Proof. Strong hull condensation is immediate. For if $\mathcal{U}$ is by Σ_s and $\mathcal{T}$ is a psuedo-hull of $\mathcal{U}$, then all models of $\mathcal{T}$ are wellfounded, so $\mathcal{T}$ is by Σ_s. Further, if π is the map on last models, then $\Sigma^\pi_{s,\mathcal{U}} = \Sigma_{s,\mathcal{T}}$ because $\Sigma^\pi_{s,\mathcal{U}}$ chooses wellfounded branches, and $\Sigma_{s,\mathcal{T}}$ chooses unique wellfounded branches.

We show now that the complete strategy induced by Σ normalizes well. So let s be by Σ and $\langle \mathcal{T}, \mathcal{U} \rangle$ by Σ_s; we must see that $W(\mathcal{T}, \mathcal{U})$ is by Σ_s. Let $\mathrm{lh}(\mathcal{U}) = \mu + 1$, and for $\gamma \le \mu$ set

$$\mathcal{W}_\gamma = W(\mathcal{T}, \mathcal{U} \restriction \gamma + 1).$$

We show by induction on γ that $\mathcal{W}_\gamma$ is by Σ_s.

$\mathcal{W}_0 = \mathcal{T}$ is by Σ_s. Suppose now that $\mathcal{W}_\gamma$ is by Σ_s, and let

$$\mathcal{W}_{\gamma+1} = W(\mathcal{W}_\nu, \mathcal{W}_\gamma, F),$$

where $F = \sigma_\gamma(E_\gamma^{\mathcal{U}})$. Since we are in the coarse case, full normalization coincides with embedding normalization, and σ_γ is the identity, but we don't need this. Let $\alpha = \alpha(\mathcal{W}_\nu, \mathcal{W}_\gamma, F)$ an $\beta = \beta(\mathcal{W}_\nu, \mathcal{W}_\gamma, F)$. We have that $\mathcal{W}_{\gamma+1} \restriction \alpha + 1 = \mathcal{W}_\gamma \restriction \alpha + 1$ is by Σ_s. So it is enough to show by induction that $\mathcal{W}_{\gamma+1} \restriction \alpha + \lambda + 1$ is by Σ_s for all $\lambda < \mathrm{lh}(\mathcal{W}_\eta)$. Clearly, we may assume that λ is a limit ordinal.

The construction of $W(\mathcal{W}_\nu, \mathcal{W}_\gamma, F)$ gives us a tree embedding Φ from $\mathcal{W}_\eta \restriction \beta + \lambda$ into $\mathcal{W}_{\gamma+1} \restriction \alpha + \lambda$ whose u-map satisfies $u(\beta + \xi) = \alpha + 1 + \xi$ for all $\xi < \lambda$. We can use 4.11 to extend Φ. If

$$c = \Sigma_s(\mathcal{W}_{\gamma+1} \restriction \alpha + \lambda),$$

then letting $b = u^{-1}``c$, we can extend Φ to a tree embedding of $(\mathcal{W}_\nu \restriction \beta + \lambda)^\frown b$ to $(\mathcal{W}_{\gamma+1} \restriction \alpha + \lambda)^\frown c$, and since psuedo-hulls of normal trees by Σ are by Σ,

$$b = \Sigma(\mathcal{W}_\nu \restriction \beta + \lambda).$$

So $b = [0, \beta + \lambda]_{W_\nu}$, so $c = [0, \alpha + \lambda]_{W_{\gamma+1}}$, as desired.

Now suppose λ is a limit ordinal. We want to see $\mathcal{W}_\lambda$ is by Σ_s. Let $\mathcal{W} = W(\mathcal{T}, \mathcal{U}{\restriction}\lambda)$ and let $a = \Sigma(\mathcal{W})$. The results of section 2.7 go through for $\vec{F}^M$-iteration trees on M, because of 2.40. Adopting the notation of 2.7, let

$$b = \mathrm{br}_{\mathcal{U}}^{\mathcal{W}}(a)$$

be the cofinal branch of $\mathcal{U}$ determined by a. So $W(\mathcal{T}, \mathcal{U})^\frown a$ is an initial segment of $\mathcal{W}_b$, and is by Σ_s.

We show by induction on ξ that $\mathcal{W}_b \restriction \xi + 1$ is by Σ_s, the proof being like the one in the successor case above. Let $\eta = \mathrm{lh}(W(\mathcal{T}, \mathcal{U}{\restriction}\lambda))$. Let

$$\Phi = \Phi_{0,b} \colon \mathcal{T} \to \mathcal{W}_b$$

be the "putative tree embedding" we get from the construction of $\mathcal{W}_b$. (We don't know yet that the models of $\mathcal{W}_b$ are wellfounded, so Φ may not be a true tree embedding.) Let $u = u^\Phi$, and let τ be such that

$$\eta = \sup_{\gamma < \lambda} \alpha_\gamma = u(\tau),$$

so that $\tau < \mathrm{lh}(\mathcal{T})$, and $\tau = m(b, \mathcal{T}, \mathcal{U}{\restriction}\lambda)$. We show by induction on ξ that if $\eta \le \xi < \mathrm{lh}(\mathcal{W}_b)$, then $\mathcal{W}_b{\restriction}(\xi + 1)$ is by Σ_s. This is trivial if ξ is a successor ordinal, because Σ_s cannot lose at a successor step. But if ξ is a limit, then we have

$$\xi = u(\bar{\xi})$$

for some limit ordinal $\bar{\xi} < \mathrm{lh}(\mathcal{T})$. Moreover, $\xi - \eta$ is contained in $\mathrm{ran}(u)$. Thus by 4.11, letting $c = \Sigma(\mathcal{W}_b \upharpoonright \xi)$ and $b = [0, \bar{\xi})_T$, we have $u``b \subseteq c$. It follows that $c = [0, \xi)_{W_b}$, so that $\mathcal{W}_b \upharpoonright \xi + 1$ is by Σ_s, as desired.

So $\mathcal{W}_b$ is by Σ_s. But there is an embedding of $\mathcal{M}_b^{\mathcal{U}}$ into the last model of $\mathcal{W}_b$, so $\mathcal{M}_b^{\mathcal{U}}$ is wellfounded, so $b = \Sigma_s(\langle \mathcal{T}, \mathcal{U} \upharpoonright \lambda \rangle)$, that is $b = [0, \lambda)_U$, and $\mathcal{W}_\lambda = \mathcal{W}_b$ is by Σ_s, as desired.

This shows that $W(\mathcal{T}, \mathcal{U})$ is by Σ_s. Let π be the embedding normalization map from the last model of $\mathcal{U}$ to the last model of $W(\mathcal{T}, \mathcal{U})$. ($\pi$ is the identity in this coarse case, but we don't need that.) Then $\Sigma^{\pi}_{s^\frown \langle W(\mathcal{T},\mathcal{U}) \rangle} = \Sigma_{s^\frown \langle \mathcal{T}, \mathcal{U} \rangle}$ because the π-pullback strategy picks wellfounded branches, and these are unique. $\square$

Let us assume AD^+ for a while. Let (M, Σ) be a coarse Γ-Woodin pair. M is uncountable, because it incorporates the trees S and T. Σ acts on countable iteration trees based on V_δ^M, which is countable, but if we think of Σ as moving only V_α^M for some $\alpha < \omega_1^V$, then there will no longer be unique wellfounded branches, just unique C_{Γ_1}-full branches. To get equivalent (3) of Lemma 4.18, we really needed to let $i_b^{\mathcal{U}}$ act on S and T. This showed up in the proof of 4.16.

In the AD^+ context it is natural to be working with countable base models. This leads us to

Definition 4.22 *A* coarse extender pair *is a pair* $((N, \vec{F}), \Sigma)$ *such that* $(N, \vec{F})$ *is a coarse extender premouse, and* Σ *is a complete* (ω_1, ω_1)*-iteration strategy for* $(N, \vec{F})$ *that normalizes well and has strong hull condensation.*

From 4.21 we get at once

Corollary 4.23 *Let* $(M, \vec{F})$ *be a coarse premouse, with* $\vec{F} \subseteq V_\delta^M$*, and let* Σ *witness that* $(M, \vec{F})$ *is strongly uniquely* (ω_1, ω_1) *iterable. Let* $\delta < \alpha$*, with* $V_\alpha^M \models \mathsf{ZFC}$*, and* $N = V_\alpha^M$ *being countable; then* $((N, \vec{F}), \Sigma)$ *is a coarse extender pair.*

In particular, if (M, Σ) is a coarse Γ-Woodin pair, $\delta = \delta^M$, and $M \models$ "$\vec{F}$ is coarsely coherent and $\vec{F} \subseteq V_\delta$", then whenever $\delta < \alpha < \omega_1$ is such that $V_\alpha \models \mathsf{ZFC}$, we have that $((V_\alpha^M, \vec{F}), \Sigma)$ is a coarse extender pair.

4.3 Strong unique iterability from UBH

We now look at consequences of the Unique Branches Hypothesis for for the existence of iteration strategies. The value of these iterability proofs that assume UBH is an open question. Perhaps they will play an important role in the ultimate construction of iteration strategies for mice with very large cardinals, perhaps not. Perhaps in the end UBH will be simply be a corollary of strategy-existence theorems that are proved without assuming it. This is closer to the way inner model theory has developed so far. In any case, we devote this section to describing some consequences of UBH for iterability.

Definition 4.24 *Let $\mathcal{F}$ be a set or class of extenders; then $\mathcal{F} - \mathsf{UBH}$ holds iff whenever $\mathcal{T}$ is a normal $\mathcal{F}$-tree on V, then $\mathcal{T}$ has at most one cofinal, wellfounded branch.*

In particular, nice-UBH is the restriction of UBH to nice trees. Woodin has observed that a Löwenheim-Skolem argument shows that $\mathcal{F}$-UBH follows from $\mathcal{F}$-UBH for countable trees.

Although $\mathcal{F}$-UBH involves only normal trees, we can show

Lemma 4.25 *Let $\vec{F}$ be coarsely coherent, and suppose that$\vec{F}$-UBH holds; then whenever s is a stack of $\vec{F}$-trees with last tree $\mathcal{U}$, then $\mathcal{U}$ has at most one cofinal, wellfounded branch.*

Proof. Suppose first that we have a stack $s = \langle \vec{\mathcal{T}}, \mathcal{U} \rangle$ of length two. Let b and c be a cofinal, wellfounded branches of $\mathcal{U}$. Let $\mathcal{W} = W(\mathcal{T}, \mathcal{U})$, and let

$$a = \mathrm{br}(b, \mathcal{T}, \mathcal{U})$$

and

$$d = \mathrm{br}(c, \mathcal{T}, \mathcal{U}).$$

It will be enough to show that $a = d$, for then $b = c$ by the results of Chapter 2. We have assumed $\vec{F}$-UBH for normal trees, so it is enough to show that $\mathcal{M}_a^{\mathcal{W}}$ and $\mathcal{M}_d^{\mathcal{W}}$ are wellfounded. The situation is symmetric, so it is enough to show $\mathcal{M}_a^{\mathcal{W}}$ is wellfounded. So suppose toward contradiction that

$$\mathcal{M}_a^{\mathcal{W}} \text{ is illfounded.}$$

Let $\phi_{0,b}(\tau) = \mathrm{lh}(W(\mathcal{T}, \mathcal{U})$. We see then from the normalization construction that

$$\mathcal{M}_a^{\mathcal{W}} = \mathrm{Ult}(\mathcal{M}_\tau^{\mathcal{T}}, E_b),$$

where E_b is the extender of b.

We need some elementary covering properties of the models in $\mathcal{T}$. For $\eta < \mathrm{lh}(\mathcal{T})$, let

$$\nu_\eta = \sup(\{\mathrm{lh}(G) \mid G \text{ is used in } [0, \eta)_T\}).$$

It is clear that ν_η is either inaccessible or a limit of inaccessibles in $\mathcal{M}_\eta^{\mathcal{T}}$. It is clear that ν_η is either inaccessible or a limit of inaccessibles in $\mathcal{M}_\eta^{\mathcal{T}}$.

Claim 4.26 *Let $X \subseteq \mathcal{M}^{\mathcal{T}}_\eta$ be countable in V; then there is a $Y \supseteq X$ such that $Y \in \mathcal{M}^{\mathcal{T}}_\eta$ and $\mathcal{M}^{\mathcal{T}}_\eta \models |Y| \leq \nu_\eta$.*

Proof. There are $f_n \in V$, for $n < \omega$, such that every $x \in X$ is of the form $i_{0,\eta}(f_n)(a)$, for some $a \in [\nu_\eta]^{<\omega}$. So we can take $Y = \{i_{0,\eta}(f_n)(a) \mid n < \omega$ and $a \in [\nu_\eta]^{<\omega}\}$. □

Claim 4.27 *Suppose $\mathcal{M}_\eta \models$ "θ is regular but not measurable"; then θ has uncountable cofinality in V.*

Proof. We prove this by induction on η. It is trivial for $\eta = 0$. Suppose we have it for $\eta < \lambda$, where λ is a limit ordinal. Let θ be regular but not measurable in $\mathcal{M}_\lambda$, and let $\theta = i_{\alpha,\lambda}(\beta)$. By induction, $\mathrm{cof}^V(\beta) > \omega$. But $i_{\alpha,\lambda}$ is continuous at β, because β is regular but not measurable in $\mathcal{M}_\alpha$. Thus $\mathrm{cof}^V(\theta) > \omega$.

Finally, suppose the claim holds at η, and let θ be regular but not measurable in $\mathcal{M}_{\eta+1}$. Let $\nu = lh(E^{\mathcal{T}}_\eta) = \nu_{\eta+1}$. If $\theta < \nu$, then the agreement between $\mathcal{M}_\eta$ and $\mathcal{M}_{\eta+1}$ implies θ is regular but not measurable in $\mathcal{M}_\eta$, so $\mathrm{cof}^V(\theta) > \omega$ by induction. If $\theta = \nu$, then θ is regular but not measurable in $\mathcal{M}_\eta$ by our hypothesis on the extenders in $\vec{F}$, so again $\mathrm{cof}^V(\theta) > \omega$. Finally, if $\theta > \nu$ and $\mathrm{cof}^V(\theta) = \omega$, then θ is singular in $\mathcal{M}_{\eta+1}$ by claim 4.26, contradiction. □

Now let $\nu = \nu_{\tau+1} = \mathrm{lh}(E^{\mathcal{T}}_\tau)$. We have that $i^{\mathcal{U}}_b(\nu) \geq \delta(\mathcal{U})$, for if not, then $\phi_{0,b}(\tau) < \lambda$. (See 3.57, and the discussion near it.) But ν is regular and not measurable in $\mathcal{M}^{\mathcal{U}}_0 = \mathcal{M}^{\mathcal{T}}_\infty$, so $i^{\mathcal{U}}_b$ is continuous at ν. Moreover, $\mathrm{cof}^V(\nu) > \omega$, while $\mathrm{cof}^V(\delta(U)) = \omega$ because b is not the only cofinal branch of $\mathcal{U}$. Thus we can fix ρ such that

$$\rho < \nu \text{ and } i^{\mathcal{U}}_b(\rho) > \delta(\mathcal{U}).$$

Since the measures in E_b all concentrate on bounded subsets of ρ, we also have

$$\nu_\tau \leq \rho.$$

Let us fix a witness to the illfoundedness of $\mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_\tau, E_b)$, namely $f_n \in \mathcal{M}_\tau$ and $a_n \in [\delta(\mathcal{U})]^{<\omega}$ such that $\pi(f_{n+1})(a_{n+1}) \in \pi(f_n)(a_n)$ for all n, where

$$\pi \colon \mathcal{M}_\tau \to \mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_\tau, E_b)$$

is the canonical embedding. By 4.26, we can cover $\{f_n \mid n < \omega\}$ by a set $Y \in \mathcal{M}^{\mathcal{T}}_\tau$ such that $|Y| \leq \rho$ in $\mathcal{M}^{\mathcal{T}}_\tau$. Let $Y \subseteq N$, where N is a rank initial segment of $\mathcal{M}^{\mathcal{T}}_\tau$, and let P be the transitive collapse of $\mathrm{Hull}^N(Y \cup \rho)$. Letting g_n be the collapse of f_n, we see that

$$\mathrm{Ult}(P, E_b) \text{ is illfounded,}$$

as witnessed by the g_n's and a_n's. But $\mathcal{M}_0^{\mathcal{U}}$ agrees with $\mathcal{M}_\tau^{\mathcal{T}}$ up to ν, so

$$P \in \mathcal{M}_0^{\mathcal{U}}.$$

Further, $\mathrm{Ult}(P, E_b)$ embeds into $i_b^{\mathcal{U}}(P)$, so $i_b^{\mathcal{U}}(P)$ is illfounded. But $i_b^{\mathcal{U}}(P)$ is wellfounded in $\mathcal{M}_b^{\mathcal{U}}$, so $\mathcal{M}_b^{\mathcal{U}}$ is illfounded, contradiction.

This takes care of the case that s has length two. Given an arbitrary finite stack $s = t^\frown \mathcal{U}$, with t having last model N, set $\mathcal{T} = W(t)$. Because we are in the coarse case, $\mathcal{T}$ has last model N. But $\mathcal{T}$ is normal, so the proof above shows that $\mathcal{U}$ has at most one cofinal, wellfounded branch.

One can prove the full lemma for arbtrary stacks using the normalizability of such stacks. This is shown in [44].

□

We do not know whether the coarse coherence hypothesis in the lemma is necessary, but we would guess that nice-UBH implies nice-UBH for stacks. We shall see below that one cannot drop the niceness hypothesis completely.

We turn to branch existence. The main results here come from [19]. That paper shows that nice-UBH implies that every countable, normal tree on V has a cofinal wellfounded branch. Combining it with Lemma 4.25, we get

Lemma 4.28 *Let $\vec{F}$ be coarsely coherent, and suppose that $\vec{F}$-UBH holds; then V is strongly uniquely $(\omega_1, \omega_1, \vec{F})$- iterable.*

For iterations of uncountable length, we need UBH in the appropriate collapse extension.

Theorem 4.29 (Folk.) *Let $\vec{F}$ be coarsely coherent, $\theta < \mathrm{crit}(F_\nu)$ for all ν, and suppose that $\vec{F}$-UBH holds in $V[G]$, where G is $\mathrm{Col}(\omega, \theta)$ generic over V; then V is strongly uniquely $(\theta^+, \theta^+, \vec{F})$-iterable.*

Proof. [*Sketch.*] Given $\mathcal{T}$ in V of limit length $< \theta^+$, we can regard $\mathcal{T}$ as a tree on $V[G]$ because $\theta < \kappa$. In $V[G]$, $\mathcal{T}$ is countable, so by UBH in $V[G]$ and [19] in $V[G]$, it has a unique cofinal, wellfounded branch. Because the collapse is homogeneous, this branch is in V. □

In one situation, UBH in V implies instances of UBH in $V[G]$:

Theorem 4.30 (Woodin) *Let δ be Woodin, and assume that $\mathcal{F}$-UBH holds, where $\mathcal{F}$ is a set or class of extenders with all critical points $> \delta$. Let $\mathcal{T}$ be a normal $\mathcal{F}$-tree, with $|\mathcal{T}| < \delta$, and let G be V-generic for a poset of size $< \delta$; then $V[G] \models$ "$\mathcal{T}$ has at most one cofinal, wellfounded branch".*

Proof. [*Sketch.*] We may assume G is countable in $V[H]$, where H is V-generic for the countable stationary tower $\mathbb{Q}_{<\delta}$. Suppose toward contradiction that b and c are distinct cofinal branches of $\mathcal{T}$ in $V[G]$. $\mathcal{T}$ can be regarded as a tree on $V[H]$, and b and c are still wellfounded when it is regarded this way.

But let $\pi\colon V \to M = \mathrm{Ult}(V, H)$ be the generic elementary embedding. Since M is closed under countable sequences in $V[H]$, $\pi\mathcal{T} \in M$, and one can check that b and c are wellfounded as branches of $\pi\mathcal{T}$. (Essentially the same functions into the ordinals are used in forming $\mathcal{M}_b^{\mathcal{T}}$ and $\mathcal{M}_b^{\pi\mathcal{T}}$, for example.) One can also check that in M, $\pi\mathcal{T}$ is a $\pi(\mathcal{F})$-tree. Thus $\pi(\mathcal{F})$-UBH fails in M, contrary to the elementarity of π. $\square$

At supercompacts, we catch our tail:

Theorem 4.31 (Woodin) *Suppose that κ is supercompact, $\vec{F}$ is coarsely coherent, $\mathrm{crit}(F_\nu) > \kappa$ for all ν, and $\vec{F}$-UBH holds; then for all θ, V is strongly uniquely $(\theta, \theta, \vec{F})$-iterable.*

Proof. Given s an $\vec{F}$-stack on V with last normal tree $\mathcal{T}$, with $s \in V_\theta$, let $j : V \to M$, $\mathrm{crit}(j) = \kappa$, $j{\restriction}V_\theta \in M$. In M, the lifted stack js has size $< j(\kappa)$, and all its critical points are above $j(\kappa)$. So by 4.29 and 4.30, $j\mathcal{T}$ has a cofinal wellfounded branch b in M. (Note $j(\kappa)$ is a limit of Woodin cardinals in M.) The copy map $\sigma : M_b^{\mathcal{T}} \to M_b^{j\mathcal{T}}$ witnesses that b is wellfounded branch of $\mathcal{T}$. $\square$

In the theory of hod mice, it is important that strategies be moved to themselves by their own iteration maps. More precisely, we would like to know that if $i\colon M \to N$ comes from a stack of trees $\vec{\mathcal{T}}$ by Σ, then $i(\Sigma \cap M) = \Sigma_{\vec{\mathcal{T}},N} \cap N$. We shall obtain this from the corresponding property of coarse strategies Σ such that Σ witnesses that V is strongly uniquely $(\lambda, \theta, \vec{F})$-iterable.

Lemma 4.32 *Let $\vec{F}$ be coarsely coherent, and let Σ witness that V is strongly uniquely $(\lambda, \theta, \vec{F})$-iterable. Suppose that $i\colon V \to N$ comes from a stack of trees $\vec{\mathcal{T}}$ by Σ; then $i(\Sigma) = \Sigma_{\vec{\mathcal{T}},N} \cap N$.*

Proof. Both $i(\Sigma)$ and $\Sigma_{\vec{\mathcal{T}},N}$ choose wellfounded branches. Since these are unique (in V!), the two strategies cannot disagree. $\square$

The remainder of this section contains some examples and results related to unique iterability that are somewhat removed from the main line of this book.

First, there are some counterexamples to forms of UBH to keep in mind when considering strong unique iterability for stacks on V. The counterexamples involve extenders overlapping Woodin cardinals, and thus do not apply to the Γ-Woodin models of 4.14, which have no such extenders. They involve stacks of trees that are not nice.

If we allow our trees to use extenders that do not have ω-closed ultrapowers in the models where they apppear, then as we said above, Woodin has shown in [67] that there are in fact normal trees of length ω on V having distinct wellfounded branches. (His construction requires a supercompact cardinal.) The construction relies heavily on the non-ω-closure, and it is quite plausible to the author that normal trees on V using only extenders that are ω-closed in the models they are taken from can have at most one cofinal wellfounded branch.

When one moves to stacks of normal trees, ω-closure is no longer enough to avoid counterexamples, as Woodin has shown. His example builds on one due to Neeman and the author. In [28], they construct a stack $\vec{\mathcal{U}} = \langle \mathcal{U}_0, \mathcal{U}_1 \rangle$ of normal iteration trees on V such that for some strong limit cardinal δ of cofinality ω,

(i) $\mathcal{U}_0 = \langle F \rangle$, where $\mathrm{lh}\, F = \text{ strength } (F) = \delta$,

(ii) $\mathcal{U}_1$ is an alternating chain on $V_\delta = V_\delta^{\mathrm{Ult}(V,F)}$, with distinct branches b and c, and

(iii) both $\mathcal{M}_b^{\mathcal{U}_1}$ and $\mathcal{M}_c^{\mathcal{U}_1}$ are wellfounded.

The key here is that because $V_\delta = V_\delta^{\mathrm{Ult}(V,F)}$, both $i_b^{\mathcal{U}_1}$ and $i_c^{\mathcal{U}_1}$ can be extended so as to act on V, and the construction arranges that $i_b(F) = i_c(F)$. But then $\mathcal{M}_b^{\mathcal{U}_1} = \mathrm{Ult}(V, i_b(F)) = \mathrm{Ult}(V, i_c(F)) = \mathcal{M}_c^{\mathcal{U}_1}$. So not only are b and c both wellfounded as branches of $\vec{\mathcal{U}}$, in fact $\mathcal{M}_b^{\mathcal{U}_1} = \mathcal{M}_c^{\mathcal{U}_1}$!

In the example above, $\mathrm{Ult}(V, F)$ is not closed under ω-sequences. However, Woodin showed that under stronger large cardinal assumptions, we can modify the example so as to get a stack of length 2 of "almost nice" trees on V. Namely, suppose we start with μ a normal measure on δ_0, where δ_0 is Woodin, and F_0 an extender with length = strength equal to δ_0. Let $\mathcal{I}$ be a linear iteration of μ of length ω, with direct limit model N. Let F and δ be the images in N of F_0 and δ_0. Then let $\mathcal{U}_0$ be the normal tree determined by $\mathcal{I}^\frown\langle F \rangle$, so that the last model of $\mathcal{U}_0$ is $M = \mathrm{Ult}(V, F)$. and let $\mathcal{U}_1$ be an alternating chain on M with branches b and c which, when acting on N, satisfy $i_b(F) = i_c(F)$. The construction of [28] gives us this $\mathcal{U}_2$; we only need $\mathrm{cof}(\delta) = \omega$ to hold in V, it need not hold in N. Again we have $\mathcal{M}_b^{\vec{\mathcal{U}}} = \mathcal{M}_c^{\vec{\mathcal{U}}}$, so both branches are wellfounded. But now $\vec{\mathcal{U}}$ is satisfies all the requirements of niceness, with the exception that $\mathrm{lh}(F_0)$ is measurable in M.

Remark 4.33 We shall see in 4.40 that this apparently small departure from niceness is essential.

In both examples, the branches b and c are not equally good. For example, consider the first example. Let E_b and E_c be the two branch extenders. Since our chain was constructed by the one-step method, exactly one of $\mathrm{Ult}(V, E_b)$ and $\mathrm{Ult}(V, E_c)$ is wellfounded. But in $\langle \mathcal{U}_0, \mathcal{U}_1{}^\frown b \rangle$ and $\langle \mathcal{U}_0, \mathcal{U}_1{}^\frown c \rangle$, these branch extenders are applied to $\mathrm{Ult}(V, F)$ rather than V. We have taken advantage of non-normality to hide the difference between b and c. If we normalize, the difference shows up:

$$W(\mathcal{U}_0, \mathcal{U}_1{}^\frown b) = \mathcal{U}_1{}^\frown b^\frown i_b^{\mathcal{U}_1}(F)$$

and

$$W(\mathcal{U}_0, \mathcal{U}_1{}^\frown c) = \mathcal{U}_1{}^\frown c^\frown i_c^{\mathcal{U}_1}(F).$$

Here $\mathcal{U}_1{}^\frown b$ and $\mathcal{U}_1{}^\frown c$ are acting on V, where only one of the two is actually an iteration tree, in that all its models are wellfounded.

This suggests that we might iterate V for finite stacks by simply choosing branches that are consistent with the iteration tree we get by normalizing. We shall show now that in fact *any* iteration strategy with strong hull condensation that acts on normal trees can be extended in this way.

In the fine-structural context, this was first proved independently by Schlutzenberg and the author. Schlutzenberg went on to prove a stronger form of the theorem, in which the extended strategy acts on infinite stacks. (See [44].) The proof of Schlutzenberg's stronger form requires significant new ideas. The construction in the finite-stack case is at bottom the same as the one we are about to give in a coarse setting. The details are simpler in the coarse case, however, because our assumptions will imply embedding normalization coincides with full normalization, and hence various maps are the identity that would not otherwise be.

Theorem 4.34 *Let $M \models \mathsf{ZFC}$ + "$\vec{F}$ is coarsely coherent", and let Σ be a complete $(1, \theta, \vec{F})$ iteration strategy for M. Suppose that Σ has strong hull condensation; then there is a unique $(\omega, \theta, \vec{F})$ strategy Σ^* such that*

(a) $\Sigma \subseteq \Sigma^*$, and

(b) Σ^* normalizes well, and has strong hull condensation.

Remark 4.35 Let s be a stack of length ω all of whose finite initial segments are by Σ^*. We do not demand that the direct limit along s be wellfounded, as would be required if Σ^* were to be a complete strategy. Adding this demand would take us into the difficulties that Schlutzenberg overcame in the fine-structural case.

Remark 4.36 We do not assume in 4.34 that Σ witnesses strong unique iterability.

Proof. Because $\vec{F}$ is coarsely coherent, $\vec{F}$-trees on M are nice, and thus embedding normalization coincides with full normalization. In particular, if $\langle \mathcal{T}, \mathcal{U} \rangle$ is an $\vec{F}$-stack on M, with Q being the last model of $\mathcal{T}$ and N the last model of $\mathcal{U}$, and $W(\mathcal{T}, \mathcal{U})$ exists, then $W(\mathcal{T}, \mathcal{U})$ also has last model N. The embedding normalization map $\sigma \colon N \to N$ is the identity, and the last t-map of the extended tree embedding from $\mathcal{T}$ into $\mathcal{U}$ is equal to the main branch embedding $i^{\mathcal{U}} \colon Q \to N$.

We begin by extending Σ to Σ_2, acting on stacks of length ≤ 2. Let $\langle \mathcal{T}, \mathcal{U} \rangle$ be a 2-stack of $\vec{F}$-trees, with $\mathcal{T}$ by Σ. We define $\Sigma_2(\langle \mathcal{T}, \mathcal{U} \rangle)$ by induction on $\mathrm{lh}(\mathcal{U})$, maintaining by induction that $W(\mathcal{T}, \mathcal{U})$ is by Σ. Let us write

$$\mathcal{W}_\gamma = W(\mathcal{T}, \mathcal{U} \restriction \gamma + 1)$$

as before.

Suppose that $\mathcal{W}_\gamma$ is by Σ; we wish to show that $\mathcal{W}_{\gamma+1}$ is by Σ. For let η be such that

$$\mathcal{W}_{\gamma+1} = W(\mathcal{W}_\eta, F),$$

where $F = E^{\mathcal{U}}_\gamma$. Let $\alpha = \alpha(\mathcal{W}_\eta, \mathcal{W}_\gamma, F)$ an $\beta = \beta(\mathcal{W}_\eta, \mathcal{W}_\gamma, F)$. We have that $\mathcal{W}_{\gamma+1} \restriction \alpha + 1 = \mathcal{W}_\gamma \restriction \alpha + 1$ is by Σ. So it is enough to show by induction that $\mathcal{W}_{\gamma+1} \restriction \alpha + \lambda + 1$ is by Σ for all $\lambda < \mathrm{lh}(\mathcal{W}_\eta)$. Clearly, we may assume that λ is a limit ordinal.

But now the construction of $W(\mathcal{W}_\eta, \mathcal{W}_\gamma, F)$ gives us a tree embedding Φ from $\mathcal{W}_\eta \restriction \beta + \lambda$ into $\mathcal{W}_{\gamma+1} \restriction \alpha + \lambda$ whose u-map satisfies $u(\beta + \xi) = \alpha + 1 + \xi$. We can use 4.11 to extend Φ. To repeat its proof: if

$$c = \Sigma(\mathcal{W}_{\gamma+1} \restriction \alpha + \lambda),$$

then letting $b = u^{-1}``c$, we can extend Φ to a tree embedding of $(\mathcal{W}_\eta \restriction \beta + \lambda)^\frown b$ to $(\mathcal{W}_{\gamma+1} \restriction \alpha + \lambda)^\frown c$, and since psuedo-hulls of normal trees by Σ are by Σ,

$$b = \Sigma(\mathcal{W}_\eta \restriction \beta + \lambda).$$

So $b = [0, \beta + \lambda]_{W_\eta}$, so $c = [0, \alpha + \lambda]_{W_{\gamma+1}}$, as desired.

Now suppose $\mathcal{U}$ of limit length λ. It is enough show that there is a unique cofinal branch b of $\mathcal{U}$ such that setting

$$\mathcal{W}_b = W(\mathcal{T}, \mathcal{U}^\frown b),$$

$\mathcal{W}_b$ is by Σ. For then we can set

$$\Sigma_2(\langle \mathcal{T}, \mathcal{U} \rangle) = b,$$

and our induction hypothesis remains true at $\lambda + 1$. To show this, let $\mathcal{W} = W(\mathcal{T}, \mathcal{U})$ and let $a = \Sigma(\mathcal{W})$. The results of section 2.7 go through for $\vec{F}$-iteration trees on M, because of 2.40. Adopting the notation of 2.7, let

$$b = \mathrm{br}^{\mathcal{W}}_{\mathcal{U}}(a)$$

be the cofinal branch of $\mathcal{U}$ determined by a. So $W(\mathcal{T}, \mathcal{U})^\frown a$ is an initial segment of $\mathcal{W}_b$, and is by Σ. We show by induction on ξ that $\mathcal{W}_b \restriction \xi + 1$ is by Σ, the proof being like the one in the successor case above. Let $\eta = \mathrm{lh}(W(\mathcal{T}, \mathcal{U}))$. Let

$$\Phi = \Phi_{0,b} \colon \mathcal{T} \to \mathcal{W}_b$$

be the "putative tree embedding" we get from the construction of $\mathcal{W}_b$. (We don't know yet that the models of $\mathcal{W}_b$ are wellfounded, so Φ may not be a true tree embedding.) Let $u = u^\Phi$, and let τ be such that

$$\eta = \sup_{\gamma < \lambda} \alpha_\gamma = u(\tau),$$

so that $\tau < \mathrm{lh}(\mathcal{T})$, and $\tau = m(b, \mathcal{T}, \mathcal{U})$. We show by induction on ξ that if $\eta \leq \xi < \mathrm{lh}(\mathcal{W}_b)$, then $\mathcal{W}_b \restriction (\xi + 1)$ is by Σ. This is trivial if ξ is a successor ordinal, because Σ cannot lose at a successor step. But if ξ is a limit, then we have

$$\xi = u(\bar{\xi})$$

for some limit ordinal $\bar{\xi} < \mathrm{lh}(\mathcal{T})$. Moreover, $\xi - \eta$ is contained in $\mathrm{ran}(u)$. Thus by 4.11, letting $c = \Sigma(\mathcal{W}_b \restriction \xi$ and $b = [0, \bar{\xi})_T$, we have $u``b \subseteq c$. It follows that $c = [0, \xi)_{W_b}$, so that $\mathcal{W}_b \restriction \xi + 1$ is by Σ, as desired.

This completes the definition of Σ_2 on stacks of length ≤ 2. Clearly, normalizations of stacks by Σ_2 are by Σ. Suppose now we have Σ_n where $n \geq 2$, and

$(*)_n$ whenever $\vec{\mathcal{T}}$ is an $\vec{F}$-stack of length $\leq n$ played by Σ_n, and having last model R, then there is a normal $\vec{F}$-iteration tree on V with last model R.

There is then exactly one such $\mathcal{T}$ by 2.40, and we write

$$\mathcal{T} = W(\vec{\mathcal{T}}).$$

We define Σ_{n+1} as follows: if $\vec{\mathcal{T}}^\frown\langle \mathcal{U} \rangle$ is a stack of length $\leq n + 1$ played by Σ_{n+1},

$$\Sigma_{n+1}(\vec{\mathcal{T}}^\frown\langle \mathcal{U} \rangle) = \Sigma_2(\langle W(\vec{\mathcal{T}}), \mathcal{U} \rangle).$$

Clearly, Σ_{n+1} is an $\vec{F}$-iteration strategy defined on stacks of length at most $n+1$, extending Σ_n. If $\vec{\mathcal{T}}^\frown\langle U\rangle$ is a stack on V by Σ_{n+1} with last model R, then $\langle W(\vec{\mathcal{T}}),\mathcal{U}\rangle$ is a 2-stack by Σ_2 with last model R, so $W(W(\vec{\mathcal{T}}),\mathcal{U})$ is a normal tree with last model R. Thus $(*)_{n+1}$ holds, and we can go on.

Let

$$\Sigma^* = \bigcup_n \Sigma_n.$$

We now show that Σ normalizes well. For this, the following definition is useful.

Definition 4.37 *(1) Let $\mathcal{W}$ be a normal iteration tree, and δ a limit ordinal. We say that b is a δ-*branch *of $\mathcal{W}$ iff $\delta = \sup\{lh(E_\alpha^{\mathcal{W}}) \mid \alpha+1 \in b\}$.*

(2) Let $\mathcal{W}$ and $\mathcal{U}$ be normal iteration trees, let b be a branch of $\mathcal{U}$ of limit order type (perhaps maximal), and let c be a branch of $\mathcal{W}$ (perhaps maximal). We say that b fits into *c iff for any extender F used in b, there is an extender G used in c such that $\mathrm{crit}(G) \leq \mathrm{crit}(F) \leq \mathrm{lh}(F) \leq \mathrm{lh}(G)$.*

Lemma 4.38 *Let $\mathcal{W}$ and $\mathcal{U}$ be normal iteration trees, and let δ be a limit ordinal; then for any δ-branch c of $\mathcal{W}$, there is at most one δ-branch b of $\mathcal{U}$ such that b fits into c.*

Proof. Suppose a and b fit into c, where $a \neq b$. We get the zipper pattern, that is F_n's used in a and G_n's used in b such that $\mathrm{crit}(F_n) \leq \mathrm{crit}(G_n) < \nu(F_n) < \mathrm{crit}(F_{n+1}) < \nu(G_n)$. If H is used in c and F_0 fits into H, then G_0 must also fit into H, since it doesn't fit anywhere else in c. By induction, all the F_n and G_n fit into H. But then $\delta \leq \nu(H)$, contradiction. □

Lemma 4.39 *Let $\langle\mathcal{T},\mathcal{U}\rangle$ be a stack of nice iteration trees on M, and b a cofinal branch of $\mathcal{U}$; then b fits into $\mathrm{br}(b,\mathcal{T},\mathcal{U})$.*

Proof. This is clear from the construction, and the fact that the σ-maps of embedding normalization are the identity in this coarse case. See the earlier diagrams of the extender tree of $W(\mathcal{T},\mathcal{U})$. □

We show now that all tails of Σ 2-normalize well. So let $\vec{\mathcal{S}}$ be a stack by Σ with last model Q, and let $\langle\mathcal{T},\mathcal{U}\rangle$ be by $\Sigma_{\vec{\mathcal{S}},Q}$ with last model R. We must see that $W(\mathcal{T},\mathcal{U})$ is by $\Sigma_{\vec{\mathcal{S}},Q}$, and that $\Sigma_{\vec{\mathcal{S}}^\frown\langle\mathcal{T},\mathcal{U}\rangle,R} = \Sigma_{\vec{\mathcal{S}}^\frown\langle W(\mathcal{T},\mathcal{U})\rangle,R}$. Here we are making use of the fact that the σ-maps in this coarse case are all the identity.

The proof is by induction on $\text{lh}(\mathcal{U})$, and the harder case is $\text{lh}(\mathcal{U}) = \lambda + 1$ for some limit ordinal λ, so let us just handle that case. Let $b = [0, \lambda)_U$, and $\delta = \delta(\mathcal{U})$. Since $\vec{\mathcal{S}}^\frown\langle \mathcal{T}, \mathcal{U}\rangle$ is by Σ, we see from the definition of Σ that

$$\mathcal{W}_0 = W(W(\vec{\mathcal{S}}^\frown\langle \mathcal{T}\rangle), \mathcal{U})$$

is the unique normal $\vec{F}$-tree on V with last model $R = \mathcal{M}^{\mathcal{U}}_\lambda$. Moreover $\mathcal{W}_0$ chooses the δ-branch

$$a = \text{br}(b, \mathcal{V}, \mathcal{U}) = \Sigma(\mathcal{W}_0 \restriction \eta)),$$

where we have set $W(\vec{\mathcal{S}}^\frown\langle \mathcal{T}\rangle) = \mathcal{V}$. Letting

$$c = \text{br}(b, \mathcal{T}, \mathcal{U}),$$

and

$$c_1 = \Sigma_2(\langle W(\vec{\mathcal{S}}), W(\mathcal{T}, \mathcal{U}\restriction\lambda)\rangle),$$

we must show that $c = c_1$. Setting

$$\mathcal{W}_1 = W(W(\vec{\mathcal{S}}), W(\mathcal{T}, \mathcal{U}\restriction\lambda)),$$

we have by our induction hypothesis that $\mathcal{W}_1$ is according to Σ. Because the embedding normalization σ-maps are the identity, the common part model $\mathcal{M}(\mathcal{W}_1) = V_\delta^{W(\mathcal{T}, \mathcal{U}\restriction\lambda)} = V_\delta^R$. By our uniqueness lemma for normal $\vec{F}$-iterations,

$$\mathcal{W}_1 = \mathcal{W}_0\restriction\eta,$$

so c_1 fits into $\Sigma(\mathcal{W}_1) = a$. Thus it is enough to see that c also fits into a.

Let $\tau = m(b, \mathcal{T}, \mathcal{U})$, and

$$p\colon \text{Ext}(\mathcal{T}) \to \text{Ext}(W(\mathcal{T}, \mathcal{U}))$$

be the map on extenders induced by the tree embedding Φ of $\mathcal{T}$ into $W(\mathcal{T}, \mathcal{U})$. Suppose F is used in c; we must see that F fits into some H used in a. This is true if F is used in b, since b fits into a. The other possibility is that $F = p(G)$, where $G \in \text{ran}(s^{\mathcal{T}}_\tau)$, so assume that. Let

$$q\colon \text{Ext}(\mathcal{V}) \to \text{Ext}(\mathcal{W}_0)$$

be induced by the tree embedding Ψ of $\mathcal{V}$ into $W(\mathcal{V}, \mathcal{U})$, and let $\rho = m(b, \mathcal{V}, \mathcal{U})$. Letting E_b be the extender of $i^{\mathcal{U}}_b$, we have that τ is least such that E_b is an extender over $\mathcal{M}^{\mathcal{T}}_\tau$, and ρ is least such that E_b is an extender over $\mathcal{M}^{\mathcal{V}}_\rho$, so that ρ is least such that $\mathcal{M}^{\mathcal{V}}_\rho$ agrees with $\mathcal{M}^{\mathcal{T}}_\tau$ through $\mathrm{dom}(E_b)$. It follows that $br([0,\tau)_T, W(\vec{\mathcal{S}}), \mathcal{T}\restriction\tau+1) = [0,\rho)_V$, and thus G fits into some K that is used in $[0,\rho)_V$. But then $F = p(G)$ fits into $q(K)$, because t^Φ_τ and t^Ψ_ρ are both E_b-ultrapower maps, so agree with one another on $\mathrm{lh}(K)+1$. (Letting N be the last model of $\mathcal{T}$ and $i^{\mathcal{U}}\colon N \to R$ the canonical embedding, t^Φ_τ and t^Ψ_ρ agree with the common last t-map $i^{\mathcal{U}}$ of Φ and Ψ this far.) Since $q(K)$ is used in a, we are done.

We shall not give a full proof that Σ has strong hull condensation. To see how it goes, suppose $\Phi\colon \mathcal{T} \to \mathcal{U}$ is an extended tree embedding, where $\mathcal{U}$ is by Σ. Let $\pi\colon N \to P$ be its last t-map, where these are the last models of $\mathcal{T}$ and $\mathcal{U}$. We must see that $\Sigma_{\mathcal{T},N} = \Sigma^\pi_{\mathcal{U},P}$. Let $\mathcal{V}$ be of limit length and by both strategies. Now $\Sigma_{\mathcal{T},N}(\mathcal{V})$ is determined by $\Sigma(W(\mathcal{T},\mathcal{V}))$, and $\Sigma_{\mathcal{U},P}(\pi\mathcal{V})$ is determined by $\Sigma(W(\mathcal{U},\pi\mathcal{V}))$. Using Φ, we can obtain a tree embedding from $W(\mathcal{T},\mathcal{V})$ into $W(\mathcal{U},\pi\mathcal{V})$. We can then use the fact that Σ condenses well on normal trees to show that $\Sigma_{\mathcal{T},N}(\mathcal{V}) = \Sigma_{\mathcal{U},P}(\pi\mathcal{V})$. □

This gives us a result on strong unique iterability that does not require a supercompact.

Theorem 4.40 *Let $\vec{F}$ be coarsely coherent, and suppose that V is strongly uniquely $(1,\theta,\vec{F})$- iterable; then V is strongly uniquely $(\omega,\theta,\vec{F})$-iterable. Moreover, letting Σ be the complete strategy that witnesses this,*

- *(a) Σ normalizes well and has strong hull condensation, and*
- *(b) if s is a stack of length ω of countable normal trees on V with last models $M_i(s)$, then the direct limit of the $M_i(s)$ under the iteration maps of s is wellfounded.*

Proof. By the first part of the proof of 4.34, we have a strategy Σ witnessing that V is $(\omega,\theta,\vec{F})$-iterable. Our hypothesis implies $\vec{F}$-UBH, so by 4.25, Σ witnesses strong uniqueness.

That Σ normalizes well and has strong hull condensation follows from 4.21. Item (b) in the conclusion comes from the branch existence arguments of [19]. Note for example that each $\mathcal{T}_i(s)$ is continuously illfounded off the branches it chooses. □

4.4 Fine strategies that normalize well

Next, we show that if Σ^* is an iteration strategy for a coarse N^* that normalizes well, then the strategies for premice induced by Σ^* via a full background extender construction also normalize well.

The reader should see the preliminaries section for our definitions and notation related to background constructions, and to the conversion of iteration strategies they mediate.

Theorem 4.41 *Let $\mathbb{C}$ be a background construction done in some universe $N^* \models$ ZFC, and let Σ^* be a complete (λ, θ)-iteration strategy for $(N^*, \vec{F}^{\mathbb{C}})$. Suppose that Σ^* normalizes well. Let M be a model of $\mathbb{C}$, and $\Sigma = \Omega(\mathbb{C}, M, \Sigma^*)$ be its induced strategy; then Σ normalizes well.*

Remark 4.42 We believe that the proof of 4.41 works even if the construction $\mathbb{C}$ is allowed to use extenders that are not nice, so that embedding normalization does not coincide with full normalization at the background level. This just means that certain embeddings are no longer the identity, and hence must be given names in the proof to follow.

Proof. We must show that all tails of Σ 2-normalize well. We consider first a 2-stack on $M^{\mathbb{C}}_{\nu_0,k_0}$ itself.

Let $\mathcal{T}$ be normal on $M^{\mathbb{C}}_{\nu_0,k_0}$, and $\mathcal{U}$ normal on the last model of $\mathcal{T}$, with $\langle \mathcal{T}, \mathcal{U} \rangle$ by Σ. Let $\langle \mathcal{T}^*, \mathcal{U}^* \rangle$ come from lifting $\langle \mathcal{T}, \mathcal{U} \rangle$ as above. We shall show that $W(\mathcal{T}, \mathcal{U})$ lifts to an initial segment of $W(\mathcal{T}^*, \mathcal{U}^*)$. (If $\mathcal{U}$ has limit length, $W(\mathcal{T}, \mathcal{U})$ lifts to $W(\mathcal{T}^*, \mathcal{U}^*)$. If it has successor length, then dropping along the main branch of $\mathcal{U}$ can cause $W(\mathcal{T}, \mathcal{U})$ to lift to a proper initial segment of $W(\mathcal{T}^*, \mathcal{U}^*)$.) Since $W(\mathcal{T}^*, \mathcal{U}^*)$ is by Σ^*, we get that $W(\mathcal{T}, \mathcal{U})$ is by Σ.

More precisely, let

$$\mathrm{lift}(\mathcal{T}, M_{\nu_0,k_0}, \mathbb{C}) = \langle \mathcal{T}^*, \langle \eta^{\mathcal{T}}_\xi, l^{\mathcal{T}}_\xi \mid \xi \le \xi_0 \rangle, \langle \psi^{\mathcal{T}}_\xi \mid \xi \le \xi_0 \rangle \rangle.$$

We are using "ψ" rather than "π" for the maps so as not to clash with our notation for embedding normalization.

Let

$$\mathrm{lift}(\psi^{\mathcal{T}}_{\xi_0} \mathcal{U}, M^{i^{\mathcal{T}^*}_{0,\xi_0}(\mathbb{C})}_{\eta^{\mathcal{T}}_{\xi_0}, l^{\mathcal{T}}_{\xi_0}}, i^{\mathcal{T}^*}_{0,\xi_0}(\mathbb{C})) = \langle \mathcal{U}^*, \langle \langle \eta^{\mathcal{U}}_\xi, l^{\mathcal{U}}_\xi \rangle \mid \xi < \mathrm{lh}\,\mathcal{U} \rangle, \langle \rho_\xi \mid \xi < \mathrm{lh}\,\mathcal{U} \rangle \rangle.$$

Let $\tau_\xi : \mathcal{M}^{\mathcal{U}}_\xi \to \mathcal{M}^{(\psi^{\mathcal{T}}_{\xi_0})\mathcal{U}}_\xi$ be the copy map, and

$$\psi^{\mathcal{U}}_\xi = \rho_\xi \circ \tau_\xi,$$

so that

$$\psi^{\mathcal{U}}_\xi : \mathcal{M}^{\mathcal{U}}_\xi \to Q_\xi,$$

where

$$Q_\xi = M^{i^{\mathcal{U}^*}_{0,\xi}\circ i^{\mathcal{T}^*}_{0,\xi_0}(\mathbb{C})}_{\eta^{\mathcal{U}}_\xi, l^{\mathcal{U}}_\xi}.$$

So $\psi^{\mathcal{U}}_\xi$ is the lifting map on $\mathcal{M}^{\mathcal{U}}_\xi$ given by our conversion of $\langle \mathcal{T}, \mathcal{U}\rangle$ to $\langle \mathcal{T}^*, \mathcal{U}^*\rangle$.

The embedding normalization $W(\mathcal{T}, \mathcal{U})$ has associated to it normal trees $\mathcal{W}_\gamma$ on $M^{\mathbb{C}}_{\nu_0,k_0}$, for $\gamma < \operatorname{lh}\mathcal{U}$, and tree embeddings

$$\Phi_{\eta,\gamma} \colon \mathcal{W}_\eta \to \mathcal{W}_\gamma$$

defined when $\eta <_U \gamma$. $\Phi_{\eta,\gamma}$ consists of its u-map $\phi_{\eta,\gamma} : \operatorname{lh}\mathcal{W}_\eta \to \operatorname{lh}\mathcal{W}_\gamma$ for $\eta <_{\mathcal{U}} \gamma$, and for $\tau \in \operatorname{dom}\phi_{\eta,\gamma}$, a its t-map $\pi^{\eta,\gamma}_\tau : \mathcal{M}^{\mathcal{W}_\eta}_\tau \to \mathcal{M}^{\mathcal{W}_\gamma}_{\phi_{\eta,\gamma}(\tau)}$. We have also

$$R_\gamma = \text{last model of } \mathcal{W}_\gamma,$$

and $\sigma_\gamma : \mathcal{M}^{\mathcal{U}}_\gamma \to R_\gamma$, and $F_\gamma = \sigma_\gamma(E^{\mathcal{U}}_\gamma)$.

$$\mathcal{W}_{\gamma+1} = W(\mathcal{W}_\eta, F_\gamma)$$

when $\eta = U\text{-pred}(\gamma+1)$.

Similarly, $W(\mathcal{T}^*, \mathcal{U}^*)$ has associated trees $\mathcal{W}^*_\gamma$ on N^* for $\gamma < \operatorname{lh}\mathcal{U}^* = \operatorname{lh}\mathcal{U}$, together with tree embeddings

$$\Phi^*_{\eta,\gamma} \colon \mathcal{W}^*_\eta \to \mathcal{W}^*_\gamma$$

defined when $\eta <_{U^*} \gamma$, or equivalently, $\eta <_U \gamma$. $\Phi^*_{\eta,\gamma}$ determines a u-map $\phi^*_{\eta,\gamma} : \operatorname{lh}\mathcal{W}^*_\eta \to \operatorname{lh}\mathcal{W}^*_\gamma$, and for $\tau \in \operatorname{dom}\phi^*_{\eta,\gamma}$, a t-map $\pi^{*\eta,\gamma}_\gamma$. Since Σ^* normalizes well, the $\mathcal{W}^*_\gamma$ are by Σ^*; moreover, by 3.50, the last model of $\mathcal{W}^*_\gamma$ is $\mathcal{M}^{\mathcal{U}^*}_\gamma$. We have that

$$\mathcal{W}^*_{\gamma+1} = W(\mathcal{W}^*_\eta, E^{\mathcal{U}^*}_\gamma)$$

when $\eta = U^*\text{-pred}(\gamma+1)$ (equivalently, $\eta = U\text{-pred}(\gamma+1)$).

We shall prove that each $\mathcal{W}_\gamma$ lifts into $\mathcal{W}^*_\gamma \restriction \operatorname{lh}\mathcal{W}_\gamma$, and hence is by Σ. The proof is by induction on γ, with a subinduction on initial segments of $\mathcal{W}_\gamma$. Basically, we are just showing that embedding normalization commutes with our conversion method. The proof is like the proof that embedding normalization commutes with copying given in 3.54, but there is more to it because in addition to copying, we are passing to resurrected background extenders. Nevertheless, the main quality required to put such a proof on paper is sufficient patience.

For $\gamma < \operatorname{lh}\mathcal{U}$, set

$$\operatorname{lift}(\mathcal{W}_\gamma, M_{\nu_0,k_0}, \mathbb{C}) = \langle \mathcal{S}^*_\gamma, \langle\langle \eta^\gamma_\xi, l^\gamma_\xi\rangle \mid \xi < \operatorname{lh}\mathcal{W}_\gamma\rangle, \langle \psi^\gamma_\xi \mid \xi < \operatorname{lh}\mathcal{W}_\gamma\rangle\rangle.$$

We shall show, among other things, that $\mathcal{S}^*_\gamma = \mathcal{W}^*_\gamma \restriction \operatorname{lh}\mathcal{W}_\gamma$, so that $\mathcal{W}_\gamma$ is by Σ. Our overall plan is summarized in the diagram:

$$\begin{tikzcd}
\mathcal{W}_\gamma \arrow[r, "\text{lift}"] & \mathcal{S}^*_\gamma \trianglelefteq \mathcal{W}^*_\gamma \\
\mathcal{W}_\nu \arrow[u, "\Phi_{\nu,\gamma}"] \arrow[r, "\text{lift}"] & \mathcal{S}^*_\nu \trianglelefteq \mathcal{W}^*_\nu \arrow[u, "\Phi^*_{\nu,\gamma}"']
\end{tikzcd}$$

As before, we write $z(\nu)$ for $\mathrm{lh}\,\mathcal{W}_\nu - 1$ and $z^*(\nu)$ for $\mathrm{lh}\,\mathcal{W}^*_\nu - 1$. We write ∞ for $z(\nu)$ or $z^*(\nu)$ when context permits. So $R_\nu = \mathcal{M}^{\mathcal{W}_\nu}_{z(\nu)} = \mathcal{M}^{\mathcal{W}_\nu}_\infty$, and if $(\nu,\gamma]_U$ does not drop, then $\phi_{\nu,\gamma}(z(\nu)) = z(\gamma)$, and $\pi^{\nu,\gamma}_{z(\nu)} = \pi^{\nu,\gamma}_\infty \colon R_\nu \to R_\gamma$. Let us also write

$$\mathbb{C}^\gamma_\xi = i^{\mathcal{W}^*_\gamma}_{0,\xi}(\mathbb{C})$$

for the construction of $\mathcal{M}^{\mathcal{W}^*_\gamma}_\xi$. In this notation,

$$Q_\gamma = (M_{\eta^{\mathcal{U}}_\gamma, l^{\mathcal{U}}_\gamma})^{\mathbb{C}^\gamma_{z^*(\gamma)}},$$

because $\mathcal{M}^{\mathcal{W}^*_\gamma}_{z^*(\gamma)} = \mathcal{M}^{\mathcal{U}^*}_\gamma$, and $i^{\mathcal{W}^*_\gamma}_{0,z^*(\gamma)} = i^{\mathcal{U}^*}_{0,\gamma} \circ i^{\mathcal{T}^*}_{0,\xi_0}$.

Lemma 4.43 *Let $\gamma < \mathrm{lh}\,\mathcal{U}$. Then*

(1) $\mathcal{S}^*_\gamma = \mathcal{W}^*_\gamma \restriction \mathrm{lh}\,\mathcal{W}_\gamma$.

(2) *Whenever $\nu <_U \gamma$ and $(\nu,\gamma]_U$ does not drop in model or degree, then for all $\tau < \mathrm{lh}\,\mathcal{W}_\nu$,*

  (i) $\langle \eta^\gamma_{\phi_{\nu,\gamma}(\tau)}, l^\gamma_{\phi_{\nu,\gamma}(\tau)} \rangle = \overset{*}{\pi}{}^{\nu,\gamma}_\tau(\langle \eta^\nu_\tau, l^\nu_\tau \rangle)$, *and*

  (ii) $\psi^\gamma_{\phi_{\nu,\gamma}(\tau)} \circ \pi^{\nu,\gamma}_\tau = \overset{*}{\pi}{}^{\nu,\gamma}_\tau \circ \psi^\nu_\tau$.

(3) $\phi_{\eta,\nu} \subseteq \phi^*_{\eta,\nu}$, *if $\eta, \nu \leq \gamma$ and $\eta \leq_U \nu$.*

(4) (i) $\langle \eta^\gamma_{z(\gamma)}, l^\gamma_{z(\gamma)} \rangle = \langle \eta^{\mathcal{U}}_\gamma, l^{\mathcal{U}}_\gamma \rangle$, *and $\mathbb{C}^\gamma_{z(\gamma)}$ agrees with $\mathbb{C}^\gamma_{z^*(\gamma)}$ at and below this point,*

  (ii) $\psi^\gamma_{z(\gamma)} \circ \sigma_\gamma = \psi^{\mathcal{U}}_\gamma$.

Proof.
Here is a diagram related to 4.43:

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{U}}_{\gamma} \arrow[r, "\sigma_\gamma"] \arrow[rr, bend left, "\psi^{\mathcal{U}}_{\gamma}"] & R_\gamma \arrow[r, "\psi^{\gamma}_{\infty}"] & Q_\gamma \in \mathcal{M}^{\mathcal{S}^*_\gamma}_{\infty} \\
\mathcal{M}^{\mathcal{U}}_{\nu} \arrow[u, "i^{\mathcal{U}}_{\nu,\gamma}"] \arrow[r, "\sigma_\nu"] \arrow[rr, bend right, "\psi^{\mathcal{U}}_{\nu}"'] & R_\nu \arrow[u, "\pi^{\nu,\gamma}_{\infty}"] \arrow[r, "\psi^{\nu}_{\infty}"] & Q_\nu \in \mathcal{M}^{\mathcal{S}^*_\nu}_{\infty} \arrow[u, "{}^*\pi^{\nu,\gamma}_{\infty}"]
\end{tikzcd}$$

The fact that ψ^{γ}_{∞} maps to Q_γ is (4)(i). The fact that the triangle on the top commutes is (4)(ii). That the square on the right commutes is (2), in the case $\tau = z(\nu)$. We of course need (2) at other τ as well. That square on the left commutes is a basic fact about embedding normalization.

The reader might look back at the diagram near the end of the proof of 3.55. $\mathcal{M}^{\mathcal{U}^*}_{\nu}$ in that diagram corresponds to Q_ν in the present one. We can take R^*_ν of that diagram to also be Q_ν in the present one, because our tree on the background universe is nice. We don't actually need that; if the background extenders were not nice, then in the present case we would be introducing some $\sigma^*_\nu \colon Q_\nu \to R^*_\nu$ via the embedding normalization of $\langle \mathcal{T}^*, \mathcal{U}^* \rangle$. ψ^{ν}_{∞} would map into R^*_ν, rather than Q_ν, and the present diagram would transform into the previous one. (See remark 4.42 above.)

We prove 4.43 by induction on γ. For $\gamma = 0$, $\mathcal{W}_0 = \mathcal{T}$ and $\mathcal{W}^*_0 = \mathcal{T}^*$, so (1) holds; moreover, $\langle \eta^0_\xi, l^0_\xi \rangle = \langle \eta^{\mathcal{T}}_\xi, l^{\mathcal{T}}_\xi \rangle$ and $\psi^0_\xi = \psi^{\mathcal{T}}_\xi$. (2) and (3) are vacuous. (4) holds: in this case, $z(0) = z^*(0) = \mathrm{lh}(\mathcal{T}) - 1$, and $\langle \eta^0_{z(0)} l^0_{z(0)} \rangle = \langle \eta^{\mathcal{U}}_0, l^{\mathcal{U}}_0 \rangle$ because $\mathcal{U}$ is on the last model of $\mathcal{T}$. That gives (i). For (ii), $\psi^{\mathcal{U}}_0 = \rho_0 \circ \tau_0 = \psi^{\mathcal{T}}_{\xi_0}$, since $\rho_0 =$ identity and $\tau_0 = \psi^{\mathcal{T}}_{\xi_0}$. But $\sigma_0 =$ identity, so $\psi^{\mathcal{U}}_0 = \psi^0_{\xi_0} \circ \sigma_0$, as desired.

Now suppose Lemma 4.43 is true at all $\nu \leq \gamma$. We show it at $\gamma + 1$. Let $\nu = U\text{-pred}(\gamma + 1)$, and

$$\begin{aligned} \alpha &= \alpha^{\mathcal{T},\mathcal{U}}_{\gamma} \\ &= \text{least } \tau \text{ such that } F_\gamma \text{ is on the } \mathcal{M}^{\mathcal{W}_\gamma}_{\tau}\text{-sequence.} \end{aligned}$$

Set $F = F_\gamma$. So

$$\begin{aligned} \mathcal{W}_{\gamma+1} &= W(\mathcal{W}_\nu, F) \\ &= \mathcal{W}_\gamma \restriction (\alpha + 1)^\frown \langle F \rangle^\frown i_F``\mathcal{W}^{>\mathrm{crit}(F)}_{\nu}. \end{aligned}$$

Then $\nu = U^*\text{-pred}(\gamma + 1)$, and

$$\mathcal{W}^*_{\gamma+1} = W(\mathcal{W}^*_\nu, E^{\mathcal{U}^*}_{\gamma}).$$

$E_\gamma^{\mathcal{U}^*}$ came from lifting $E_\gamma^{\mathcal{U}}$ by $\psi_\gamma^{\mathcal{U}}$, then resurrecting it inside the construction of $\mathcal{M}_\gamma^{\mathcal{U}^*}$, then using the background extender provided by this construction. $\mathcal{M}_\gamma^{\mathcal{U}^*}$ is the last model of $\mathcal{W}_\gamma^*$, so the construction in question is $\mathbb{C}^\gamma_{z^*(\gamma)}$. More precisely, let $\psi_\gamma^{\mathcal{U}}(E_\gamma^{\mathcal{U}})$ be the last extender of

$$Q_\gamma|\langle\theta, 0\rangle =_{\mathsf{def}} \bar{P}$$

and

$$G = \sigma_{\langle \eta_\gamma^{\mathcal{U}}, l_\gamma^{\mathcal{U}}\rangle}[\bar{P}](\psi_\gamma^{\mathcal{U}}(E_\gamma^{\mathcal{U}})),$$

where the resurrection is computed in $\mathbb{C}^\gamma_{z^*(\gamma)}$. Set

$$G^* = \text{background extender for } G \text{ provided by } \mathbb{C}^\gamma_{z^*(\gamma)}.$$

Then $E_\gamma^{\mathcal{U}^*} = G^*$, and

$$\mathcal{W}^*_{\gamma+1} = W(\mathcal{W}^*_\nu, G^*).$$

Recall that $\alpha = \alpha(\mathcal{W}_\gamma, F)$.

Claim 4.44 *$\alpha = \alpha(\mathcal{W}_\gamma^*, G^*)$, and G^* is the background extender for $\sigma \circ \psi^\gamma_\alpha(F)$ provided by $\mathbb{C}^\gamma_\alpha$, where σ is the resurrection map $\sigma_{\langle \eta^\gamma_\alpha, l^\gamma_\alpha\rangle}[M_{\langle \eta^\gamma_\alpha, l^\gamma_\alpha\rangle} \parallel \langle \mathrm{lh}\, \psi^\gamma_\alpha(F), 0\rangle]$ of $\mathbb{C}^\gamma_\alpha$.*

Proof. F is on the $\mathcal{M}_\alpha^{\mathcal{W}_\gamma}$-sequence, so there is a background extender H^* for $\sigma\circ\psi^\gamma_\alpha(F)$ provided by $\mathbb{C}^\gamma_\alpha$. By induction, the extender $E_\alpha^{\mathcal{W}_\gamma^*}$ used to exit $\mathcal{M}_\alpha^{\mathcal{W}_\gamma^*}$ comes from lifting and resurrecting $E_\alpha^{\mathcal{W}_\gamma}$. But F comes before $E_\alpha^{\mathcal{W}_\gamma}$, so H^* comes before $E_\alpha^{\mathcal{W}_\gamma^*}$ in $\mathbb{C}^\gamma_\alpha$. But letting $E_\alpha^{\mathcal{W}_\gamma^*} = (F_\theta)^{\mathbb{C}^\gamma_\alpha}$, we then have

$$\mathbb{C}^\gamma_\tau \restriction \theta = \mathbb{C}^\gamma_\alpha \restriction \theta$$

for all $\tau \geq \alpha$, and in particular, for $\tau = z(\gamma)$. Moreover, the part of the lifting and resurrecting maps acting on F does not change from α to $z(\gamma)$:

$$\sigma \circ \psi^\gamma_\alpha(F) = \sigma' \circ \psi^\gamma_{z(\gamma)}(F),$$

where σ' is appropriate for resurrecting $\psi^\gamma_{z(\gamma)}(F)$ in $\mathcal{M}^{\mathcal{W}_\gamma^*}_{z(\gamma)}$, and hence also by (4)(i) in $\mathcal{M}^{W_\gamma^*}_{z^*(\gamma)} = \mathcal{M}_\gamma^{\mathcal{U}^*}$. But our inductive hypothesis (4)(ii) yields

$$\begin{aligned} \psi^\gamma_{z(\gamma)}(F) &= \psi^\gamma_{z(\gamma)} \circ \sigma_\gamma(E_\gamma^{\mathcal{U}}) \\ &= \psi_\gamma^{\mathcal{U}}(E_\gamma^{\mathcal{U}}), \end{aligned}$$

so $\sigma\circ\psi^\gamma_\alpha(F) = \sigma'\circ\psi^\gamma_\tau(F) = \sigma'(\psi_\gamma^{\mathcal{U}}(E_\gamma^{\mathcal{U}})) = G$. Thus $H^* = G^*$. Hence $\alpha(W_\gamma^*, G^*) \leq \alpha$.

But suppose $G^* \in \mathbb{C}^\gamma_\xi$ for some $\xi < \alpha$. Since $\operatorname{lh} E^{\mathcal{W}_\gamma}_\xi < \operatorname{lh} F$, $\operatorname{lh}(E^{\mathcal{W}^*_\gamma}_\xi) < \operatorname{lh} G^*$, and so G^* occurs after $E^{W^*_\gamma}_\xi$ in $\mathbb{C}^\gamma_\xi$. So $\mathcal{M}^{\mathcal{W}^*_\gamma}_\beta$ does not compute $V_{\operatorname{lh} G^*}$ the same way that $\mathcal{M}^{\mathcal{W}^*_\gamma}_\xi$ does, for all $\beta > \xi$. This implies $G^* \notin \mathbb{C}^\gamma_\beta$, for all $\beta > \xi$, contrary to $G^* \in \mathbb{C}^\gamma_{z^*(\gamma)}$.

This shows $\alpha = \alpha(W^*_\gamma, G^*)$. In the course of the proof we also showed the rest of Claim 4.44. □

Claim 4.45 *1. The iteration tree in* $\operatorname{lift}(\mathcal{W}_\gamma \restriction (\alpha+1)^\frown \langle F\rangle, M_{\nu_0,k_0}, \mathbb{C})$ *is* $\mathcal{W}^*_\gamma \restriction (\alpha+1)^\frown \langle G^*\rangle$.

2. $\beta = \beta^{\mathcal{W}^*_\gamma, G^*}$.

Proof. Part 1 is just Claim 4.44 restated. Part 2 follows at once from the fact that the lifted tree is normal; cf. 2.45. □

Since $\alpha(W_\gamma, F) = \alpha(\mathcal{W}^*_\gamma, G)$ and $\beta^{\mathcal{W}_\gamma, F} = \beta^{\mathcal{W}^*_\gamma, G^*}$, we have $\phi_{\nu,\gamma+1} \subseteq \phi^*_{\nu,\gamma+1}$.

Remark 4.46 *If* $D^{\mathcal{U}} \cap [0,\gamma+1]_U = \varnothing$, *then* $\operatorname{lh} \mathcal{W}_{\gamma+1} = \operatorname{lh} W^*_{\gamma+1}$, *and* $\phi_{\nu,\gamma+1} = \phi^*_{\nu,\gamma+1}$.

We now show that (1) and (2) of Lemma 4.43 hold at $\gamma+1$. For this, we show by induction on ξ that for $\xi \le \operatorname{lh} \mathcal{W}_{\gamma+1}$, letting

$$\mathcal{S}^* = \mathcal{S}^*_{\gamma+1},$$

Induction Hypothesis $(\dagger)_\xi$**:**

(1) $\mathcal{S}^* \restriction \xi = \mathcal{W}^*_{\gamma+1} \restriction \xi$

(2) if $(\nu, \gamma+1]_U$ does not drop in model or degree, and $\phi_{0,\gamma+1}(\tau) < \xi$, then

  (a) $\langle \eta^{\gamma+1}_{\phi_{\nu,\gamma+1}(\tau)}, l^{\gamma+1}_{\phi_{\nu,\gamma+1}(\tau)} \rangle = \overset{*}{\pi}{}^{\nu,\gamma+1}_\tau(\langle \eta^\nu_\tau, l^\nu_\tau \rangle)$, and

  (b) $\psi^{\gamma+1}_{\phi_{\nu,\gamma+1}(\tau)} \circ \pi^{\nu,\gamma+1}_\gamma = \overset{*}{\pi}{}^{\nu,\gamma+1}_\tau \circ \psi^\nu_\tau$.

Note that the limit step in the inductive proof of $(\dagger)_\xi$ is trivial.

Base Case 1. $\xi = \alpha+1$.

We have $\mathcal{W}_{\gamma+1} \restriction (\alpha+1) = \mathcal{W}_\gamma \restriction (\alpha+1)$ and $\mathcal{W}^*_{\gamma+1} \restriction (\alpha+1) = \mathcal{W}^*_\gamma \restriction (\alpha+1)$. Since Lemma 4.43 holds at γ, we get $(\dagger)_\xi(1)$. For $(\dagger)_\xi(2)$, let $\phi_{\nu,\gamma+1}(\tau) < \alpha+1$. Then $\tau < \beta$ and $\phi_{\nu,\gamma+1}(\tau) = \tau$. Moreover $\pi^{\nu,\gamma+1}_\tau$ and $\overset{*}{\pi}{}^{\nu,\gamma+1}_\tau$ are the identity. So $(\dagger)_\xi(2)$ boils down to

$\langle \eta_\tau^{\gamma+1}, l_\tau^{\gamma+1}\rangle = \langle \eta_\tau^\nu, l_\tau^\nu\rangle$, and $\psi_\tau^{\gamma+1} = \psi_\tau^\nu$. This holds because $\mathcal{W}_\nu \restriction (\tau+1) = \mathcal{W}_{\gamma+1} \restriction (\tau+1)$, so their lifts are equal.

Base Case 2. $\xi = \alpha + 2$.

We have

$$\mathcal{W}_{\gamma+1} \restriction (\alpha+2) = \mathcal{W}_{\gamma+1} \restriction (\alpha+1)^\frown \langle F\rangle$$

and

$$\mathcal{W}^*_{\gamma+1} \restriction (\alpha+2) = \mathcal{W}^*_{\gamma+1} \restriction (\alpha+1)^\frown \langle G^*\rangle.$$

By Claim 4.44, G^* is the background extender for $\sigma \circ \psi_\alpha^{\gamma+1}(F)$ provided by $\mathbb{C}_\alpha^{\gamma+1}$. So

$$\begin{aligned} \mathcal{S}^* \restriction (\alpha+2) &= \mathcal{S}^* \restriction (\alpha+1)^\frown \langle G^*\rangle \\ &= \mathcal{W}^*_{\gamma+1} \restriction (\alpha+2), \end{aligned}$$

and we have $(\dagger)_\xi(1)$. (Note that G^* is applied to $\mathcal{M}_\beta^{\mathcal{S}^*}$ in $\mathcal{S}^*$, because lifting produces normal trees.)

For $(\dagger)_\xi(2)$, the new case to consider is $\tau = \beta$. Note that

$$\begin{aligned} \psi_\beta^\nu &= \psi_\beta^{\gamma+1}, \\ \pi_\beta^{\nu,\gamma+1} &= i_{\beta,\alpha+1}^{\mathcal{W}_{\gamma+1}} \end{aligned}$$

and

$$\overset{*}{\pi}{}_\beta^{\nu,\gamma+1} = i_{\beta,\alpha+1}^{\mathcal{W}^*_{\gamma+1}}.$$

The first because $\mathcal{W}_{\gamma+1} \restriction (\beta+1) = \mathcal{W}_\nu \restriction (\beta+1)$, and the second two by our definition of embedding normalization. (Note we are in the case that $(\beta, \alpha+1]_{\mathcal{W}_{\gamma+1}}$ is not a drop in model or degree.) But

$$\psi_{\alpha+1}^{\gamma+1} \circ i_{\beta,\alpha+1}^{\mathcal{W}_{\gamma+1}} = i_{\beta,\alpha+1}^{\mathcal{W}^*_{\gamma+1}} \circ \psi_\beta^{\gamma+1}$$

holds because lifting maps commute with the tree embedding in a conversion system. This gives

$$\psi_{\alpha+1}^{\gamma+1} \circ \pi_\beta^{\nu,\gamma+1} = \overset{*}{\pi}{}_\beta^{\nu,\gamma+1} \circ \psi_\beta^\nu$$

as desired.

If $\operatorname{lh} \mathcal{W}_\nu = \beta+1$ or $\gamma+1 \in D^{\mathcal{U}}$ or $\deg^{\mathcal{U}}(\gamma+1) < \deg^{\mathcal{U}}(\nu)$, then $\operatorname{lh} \mathcal{W}_{\gamma+1} = \alpha+2$, so we are done. So suppose $\operatorname{lh} \mathcal{W}_\nu > \beta+1$, and $(\nu, \gamma+1]_U$ is not a drop of any kind in $\mathcal{U}$.

Inductive Case 1. $(\dagger)_{\xi+1}$ holds, and $\xi \geq \alpha+1$.

We must prove $(\dagger)$ at $\xi+2$. We are assuming $\xi+1 < \operatorname{lh} \mathcal{W}_{\gamma+1}$. Let

$$E = E_\xi^{\mathcal{W}_{\gamma+1}}.$$

Let σ be the resurrection map for $\psi_\xi^{\gamma+1}(E)$ in $\mathbb{C}_\xi^{\gamma+1}$, the construction of $\mathcal{M}_\xi^{\mathcal{S}^*} = \mathcal{M}_\xi^{\mathcal{W}^*_{\gamma+1}}$. That is,

$$\sigma = \sigma_{\langle \eta_\xi^{\gamma+1}, l_\xi^{\gamma+1}\rangle}[M_{\langle \eta_\xi^{\gamma+1}, l_\xi^{\gamma+1}\rangle} | \langle \operatorname{lh} \psi_\xi^{\gamma+1}(E), 0\rangle].$$

Let

$$E^* = \text{background extender for } \sigma \circ \psi_\xi^{\gamma+1}(E) \text{ provided by } \mathbb{C}_\xi^{\gamma+1}.$$

So

$$\mathcal{S}^* {\restriction} (\xi+2) = \mathcal{S}^* {\restriction} (\xi+1)^\frown \langle E^* \rangle.$$

Claim 4.47 $E^* = E_\xi^{\mathcal{W}^*_{\gamma+1}}$.

Proof. Since $\xi \geq \alpha+1$, we can write

$$\xi = \phi_{\nu,\gamma+1}(\bar{\xi}), \quad \bar{\xi} \geq \beta$$

Let

$$\bar{E} = E_{\bar{\xi}}^{\mathcal{W}_\nu},$$

so that

$$E = \pi_{\bar{\xi}}^{\nu,\gamma+1}(\bar{E}).$$

Letting $H = \sigma \circ \psi_\xi^{\gamma+1}(E)$, we have

$$\begin{aligned} H &= \sigma \circ (\psi_\xi^{\gamma+1} \circ \pi_{\bar{\xi}}^{\nu,\gamma+1}(\bar{E})) \\ &= \sigma \circ (\overset{*}{\pi}{}_{\bar{\xi}}^{\nu,\gamma+1} \circ \psi_{\bar{\xi}}^{\nu}(\bar{E})) \end{aligned}$$

by induction. Let $\bar{\sigma}$ be the resurrection map for $\psi_{\bar{\xi}}^\nu(\bar{E})$ in $\mathbb{C}_{\bar{\xi}}^\nu$, that is,

$$\bar{\sigma} = \sigma_{\langle \eta_{\bar{\xi}}^\nu, l_{\bar{\xi}}^\nu\rangle}[M_{\langle \eta_{\bar{\xi}}^\nu, l_{\bar{\xi}}^\nu\rangle} | \langle \operatorname{lh} \psi_{\bar{\xi}}^\nu(\bar{E}), 0\rangle].$$

It is not hard to see that

$$\overset{*}{\pi}{}_{\bar{\xi}}^{\nu,\gamma+1}(\bar{\sigma}) = \sigma.$$

This is because $\overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\xi}}(\langle\eta^{\nu}_{\bar{\xi}}, l^{\nu}_{\bar{\xi}}\rangle) = \langle\eta^{\gamma+1}_{\xi}, l^{\gamma+1}_{\xi}\rangle$ by induction hypothesis (2)(a), and similarly $\overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\xi}}(\psi^{\nu}_{\bar{\xi}}(\bar{E})) = \psi^{\gamma+1}_{\xi}(\pi^{\nu,\gamma+1}_{\bar{\xi}}(\bar{E})) = \psi^{\gamma+1}_{\xi}(E)$. But then

$$\begin{aligned}
E^{\mathcal{W}^*_{\gamma+1}}_{\xi} &= \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\xi}}(E^{\mathcal{W}^*_{\nu}}_{\bar{\xi}}) \\
&= \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\xi}}(\text{background for } \bar{\sigma}(\psi^{\nu}_{\bar{\xi}}(\bar{E})) \text{ in } \mathbb{C}^{\nu}_{\bar{\xi}}) \\
&= \text{background for } \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\xi}}(\bar{\sigma}(\psi^{\nu}_{\bar{\xi}(\bar{E})})) \text{ in } \mathbb{C}^{\gamma+1}_{\xi} \\
&= \text{background for } \sigma(\overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\xi}}(\psi^{\nu}_{\bar{\xi}}(\bar{E}))) \text{ in } \mathbb{C}^{\gamma+1}_{\xi} \\
&= \text{background for } H \text{ in } \mathbb{C}^{\gamma+1}_{\xi} \\
&= E^*
\end{aligned}$$

as desired. $\square$

From Claim 4.47, we have that $\mathcal{S}^*\restriction(\xi+2)$ is the unique normal continuation of $\mathcal{S}^*\restriction(\xi+1) = \mathcal{W}^*_{\gamma+1}\restriction(\xi+1)$ via $E^{\mathcal{W}^*_{\gamma+1}}_{\xi}$. That is, $\mathcal{S}^*\restriction(\xi+2) = \mathcal{W}^*_{\gamma+1}\restriction(\xi+2)$.

It remains to show, keeping our previous notation:

Claim 4.48 $\psi^{\gamma+1}_{\xi+1} \circ \pi^{\nu,\gamma+1}_{\bar{\xi}+1} = \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\xi}+1} \circ \psi^{\nu}_{\bar{\xi}+1}$.

Proof. Both maps act on $\mathcal{M}^{\mathcal{W}_\nu}_{\bar{\xi}+1}$. The right hand side embeds it elementarily into $M_{\eta',l'}$ of $\mathbb{C}^{\gamma+1}_{\xi+1}$, where

$$\langle\eta', l'\rangle = \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\xi}+1}(\langle\eta^{\nu}_{\bar{\xi}+1}, l^{\nu}_{\bar{\xi}+1}\rangle)$$

The left hand side embeds $\mathcal{M}^{\mathcal{W}_\nu}_{\bar{\xi}+1}$ elementarily into $M_{\langle\eta^{\gamma+1}_{\xi+1}, l^{\gamma+1}_{\xi+1}\rangle}$ of $\mathbb{C}^{\gamma+1}_{\xi+1}$. So first we show $(\dagger)_{\xi+1}(2)(a)$:

Subclaim 4.48.1 $\langle\eta^{\gamma+1}_{\xi+1}, l^{\gamma+1}_{\xi+1}\rangle = \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\xi}+1}(\langle\eta^{\nu}_{\bar{\xi}+1}, l^{\nu}_{\bar{\xi}+1}\rangle)$.

Proof. Let

$$\begin{aligned}
\theta &= \mathcal{W}_{\gamma+1}\text{-pred}(\xi+1) \\
&= \mathcal{W}^*_{\gamma+1}\text{-pred}(\xi+1) \\
&= \mathcal{S}^*_{\gamma+1}\text{-pred}(\xi+1).
\end{aligned}$$

<u>Case 1</u>. $\text{crit}(\bar{E}) \geq \text{crit}(F_\gamma)$, or $\theta < \beta$.

This is the case in which $\phi_{\nu,\gamma+1}$ preserves tree predecessor, that is, $\theta = \phi_{\nu,\gamma+1}(\bar{\theta}) = \phi^*_{\nu,\gamma+1}(\bar{\theta})$ for $\bar{\theta} = \mathcal{W}_\nu\text{-pred}(\bar{\xi}+1)$. We have

$$\mathcal{M}^{\mathcal{W}_\nu}_{\bar{\xi}+1} = \text{Ult}(\bar{P}, \bar{E}),$$

where $\bar{P} \trianglelefteq \mathcal{M}_{\bar\theta}^{\mathcal{W}_\nu}$. Let

$$P = \pi_{\bar\theta}^{\nu,\gamma+1}(\bar{P}).$$

Embedding normalization leads to

$$\mathcal{M}_{\xi+1}^{\mathcal{W}_{\gamma+1}} = \mathrm{Ult}(P, E),$$

where recall $E = \pi_{\bar\xi}^{\nu,\gamma+1}(\bar{E})$. Letting ρ be the resurrection map for P in $\mathbb{C}_\theta^{\gamma+1}$, that is

$$\rho = \sigma_{\langle \eta_\theta^{\gamma+1}, l_\theta^{\gamma+1}\rangle}[\psi_\theta^{\gamma+1}(P)],$$

ρ maps $\psi_\theta^{\gamma+1}(P)$ into $(M_{\eta,l})^{\mathbb{C}_\theta^{\gamma+1}}$, where

$$\langle \eta, l\rangle = \mathrm{Res}_{\langle \eta_\theta^{\gamma+1}, l_\theta^{\gamma+1}\rangle}[\psi_\theta^{\gamma+1}(P)],$$

we have

$$\langle \eta_{\xi+1}^{\gamma+1}, l_{\xi+1}^{\gamma+1}\rangle = i_{\theta,\xi+1}^{\mathcal{W}^*_{\gamma+1}}(\langle \eta, l\rangle),$$

because $\mathcal{W}^*_{\gamma+1}|(\xi+2) = \mathcal{S}^*|(\xi+2)$ is a conversion system. Note that $\overset{*}{\pi}{}_{\bar\theta}^{\nu,\gamma+1}(\langle \eta_{\bar\theta}^{\gamma+1}, l_{\bar\theta}^{\gamma+1}\rangle) = \langle \eta_\theta^{\gamma+1}, l_\theta^{\gamma+1}\rangle$ by induction. (I.e. Subclaim 4.48.1 at $\bar\theta$ instead of $\bar\xi$.) Also, $\overset{*}{\pi}{}_{\bar\theta}^{\nu,\gamma+1}(\psi_{\bar\theta}^{\nu}(\bar{P})) = \psi_\theta^{\gamma+1}(\pi_{\bar\theta}^{\nu,\gamma+1}(\bar{P})) = \psi_\theta^{\gamma+1}(P)$. It follows that

$$\langle \eta, l\rangle = \overset{*}{\pi}{}_{\bar\theta}^{\nu,\gamma+1}(\mathrm{Res}_{\eta_{\bar\theta}^\nu, l_{\bar\theta}^\nu}[\psi_{\bar\theta}^\nu(\bar{P})]^{\mathbb{C}_{\bar\theta}^\nu}).$$

Thus

$$\begin{aligned}
\langle \eta_{\xi+1}^{\gamma+1}, l_{\xi+1}^{\gamma+1}\rangle &= i_{\theta,\xi+1}^{\mathcal{W}^*_{\gamma+1}}(\langle \eta, l\rangle)\\
&= i_{\theta,\xi+1}^{W^*_{\gamma+1}} \circ \overset{*}{\pi}{}_{\bar\theta}^{\nu,\gamma+1}(\mathrm{Res}_{\eta_{\bar\theta}^\nu, l_{\bar\theta}^\nu}[\psi_{\bar\theta}^\nu(\bar{P})])\\
&= \overset{*}{\pi}{}_{\bar\xi+1}^{\nu,\gamma+1} \circ i_{\bar\theta,\bar\xi+1}^{\mathcal{W}^*_\nu}(\mathrm{Res}_{\eta_{\bar\theta}^\nu, l_{\bar\theta}^\nu}[\psi_{\bar\theta}^\nu(\bar{P})])\\
&= \overset{*}{\pi}{}_{\bar\xi+1}^{\nu,\gamma+1}(\langle \eta_{\bar\xi+1}^\nu, l_{\bar\xi+1}^\nu\rangle),
\end{aligned}$$

as desired.

<u>Case 2</u>. Otherwise.

In this case, we must have $\beta \leq \theta$ and $\mathrm{crit}(\bar{E}) < \mathrm{crit}(F)$. It follows that $\theta = \beta$, and $\mathcal{W}_\nu$-pred$(\bar\xi+1) = \mathcal{W}_{\gamma+1}$-pred$(\bar\xi+1) = \beta$. The argument above works, with $\bar\theta = \theta = \beta$ and $\bar{P} = P$, and $\pi_{\bar\theta}^{\nu,\gamma+1}$ and $\overset{*}{\pi}{}_{\bar\theta}^{\nu,\gamma+1}$ being replaced by the identity map. (

If $\theta < \beta$ they are already the identity. This case is similar to the case $\theta < \beta$.) The relevant calculation is

$$\begin{aligned}
\langle \eta^{\gamma+1}_{\xi+1}, l^{\gamma+1}_{\xi+1} \rangle &= i^{\mathcal{W}^*_{\gamma+1}}_{\beta,\xi+1}(\mathrm{Res}_{\eta^{\gamma+1}_\beta, l^{\gamma+1}_\beta}[\psi^{\gamma+1}_\beta(P)]^{\mathbb{C}^{\gamma+1}_\beta}) \\
&= i^{\mathcal{W}^*_{\gamma+1}}_{\beta,\xi+1}(\mathrm{Res}_{\eta^{\nu}_\beta, l^{\nu}_\beta}[\psi^{\nu}_\beta(P)]^{\mathbb{C}^{\nu}_\beta}) \\
&= \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\xi}+1} \circ i^{\mathcal{W}^*_\nu}_{\beta,\bar{\xi}+1}(\mathrm{Res}_{\eta^{\nu}_\beta, l^{\nu}_\beta}[\psi^{\nu}_\beta(P)]^{\mathbb{C}^{\nu}_\beta}) \\
&= \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\xi}+1}(\langle \eta^{\nu}_{\bar{\xi}+1}, l^{\nu}_{\bar{\xi}+1} \rangle).
\end{aligned}$$

The first equation holds because $\mathcal{W}^*_{\gamma+1}\restriction(\xi+2) = \mathcal{S}^*\restriction(\xi+2)$ is a conversion system. The second comes from the fact that $\mathcal{W}^*_{\gamma+1}\restriction(\beta+1) = \mathcal{W}^*_\nu\restriction(\beta+1)$. The third comes from properties of embedding normalization. The last comes from $\mathcal{W}^*_\nu$ being a conversion system. $\square$

We now finish proving Claim 4.48. We keep the notation above. Let us assume that we are in Case 1. Let $x \in \mathcal{M}^{\mathcal{W}_\nu}_{\bar{\xi}+1}$ be arbitrary, and let

$$x = [a, f]^{\bar{P}}_{\bar{E}},$$

where $a \subseteq h_{\bar{E}}$ is finite and $f \in \bar{P}$. (We assume $k(\bar{P}) = 0$ for simplicity.) Then

$$\begin{aligned}
\psi^{\gamma+1}_{\xi+1} \circ \pi^{\nu,\gamma+1}_{\bar{\xi}+1}(x) &= \psi^{\gamma+1}_{\xi+1}(\pi^{\nu,\gamma+1}_{\bar{\xi}+1}([a,f]^{\bar{P}}_{\bar{E}})) \\
&= \psi^{\gamma+1}_{\xi+1}([\pi^{\nu,\gamma+1}_{\bar{\xi}}(a), \pi^{\nu,\gamma+1}_{\bar{\theta}}(f)]^{P}_{E})
\end{aligned}$$

(by the properties of embedding normalization, and the fact $\pi^{\nu,\gamma+1}_{\bar{\theta}}(\bar{P}) = P$ and $\pi^{\nu,\gamma+1}_{\bar{\xi}}(\bar{E}) = E$)

$$= [\sigma \circ \psi^{\gamma+1}_{\xi} \circ \pi^{\nu,\gamma+1}_{\bar{\xi}}(a), \rho \circ \psi^{\gamma+1}_{\theta} \circ \pi^{\nu,\gamma+1}_{\bar{\theta}}(f)]^{\mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{\theta}}_{E^*},$$

where σ resurrects $\psi^{\gamma+1}_\xi(E)$ and ρ resurrects $\psi^{\gamma+1}_\theta(P)$, as defined above. We have

$$\sigma = \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\xi}}(\bar{\sigma}), \quad \text{and} \quad \rho = \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\theta}}(\bar{\rho}).$$

Further

$$
\begin{aligned}
\overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar\xi+1} \circ \psi^{\nu}_{\bar\xi+1}(x) &= \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar\xi+1}(\psi^{\nu}_{\bar\xi+1}([a,f]^{\bar P}_{\bar E})) \\
&= \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar\xi+1}([\bar\sigma \circ \psi^{\nu}_{\bar\xi}(a), \bar\rho \circ \psi^{\nu}_{\bar\theta}(f)]^{\mathcal{M}^{\mathcal{W}^*_\nu}_{\bar\theta}}_{E^{\mathcal{W}^*_\nu}_{\bar\xi}}) \\
&= [\overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar\xi} \circ \bar\sigma \circ \psi^{\nu}_{\bar\xi}(a), \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar\theta} \circ \bar\rho \circ \psi^{\nu}_{\bar\theta}(f)]^{\mathcal{M}^{\mathcal{W}^*_{\nu+1}}_{\theta}}_{E^{\mathcal{W}^*_{\nu+1}}_{\xi}} \\
&= [\sigma \circ \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar\xi} \circ \psi^{\nu}_{\bar\xi}(a), \rho \circ \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar\theta} \circ \psi^{\nu}_{\bar\theta}(f)]^{\mathcal{M}^{\mathcal{W}^*_{\nu+1}}_{\theta}}_{E^*} \\
&= [\sigma \circ \psi^{\gamma+1}_{\xi} \circ \pi^{\nu,\gamma+1}_{\bar\xi}(a), \rho \circ \psi^{\gamma+1}_{\theta} \circ \pi^{\nu,\gamma+1}_{\bar\theta}(f)]^{\mathcal{M}^{\mathcal{W}^*_{\nu+1}}_{\theta}}_{E^*}.
\end{aligned}
$$

The first 4 lines come from the way embedding normalization and lifting work. The last line comes from our induction hypothesis.

We leave it to the reader to finish the proof in Case 2. This proves Claim 4.48.
□

Returning to the inductive proof of $(\dagger)_\xi$, we see that the limit case is trivial. We are left with

Inductive Case 2. ξ is a limit ordinal, and $(\dagger)_\xi$.

We must prove $(\dagger)_{\xi+1}$. We have $\mathcal{S}^*\restriction\xi = \mathcal{W}^*_{\gamma+1}\restriction\xi$. Since Σ^* normalizes well, the branch $[0,\xi]_{\mathcal{W}^*_{\gamma+1}}$ of $\mathcal{W}^*_{\gamma+1}$ produced by embedding normalization is equal to $\Sigma^*(\mathcal{S}^*\restriction\xi)$. Thus $\mathcal{S}^*\restriction(\xi+1) = \mathcal{W}^*_{\gamma+1}\restriction(\xi+1)$. One can then prove $(\dagger)_{\xi+1}$ by looking at how the objects it deals with come from the $\mathcal{M}^{\mathcal{W}_\nu}_\tau$ and $\mathcal{M}^{\mathcal{W}^*_\nu}_\tau$ for $\tau <_{W_\gamma} \phi^{-1}_{\nu,\gamma+1}(\xi)$, and using our induction hypothesis $(\dagger)_\xi$. We omit further detail.

This completes our inductive proof of (1) and (2) of Lemma 4.43. We have already proved (3) of Lemma 4.43. We now prove (4).

Recall that $z(\eta) = \mathrm{lh}\,\mathcal{W}_\eta - 1$. The following diagram summarizes the proof of (4).

$$
\begin{tikzcd}
\mathcal{M}^{\mathcal{U}}_{\gamma+1} \arrow[r, "\sigma_{\gamma+1}"] & \mathcal{M}^{\mathcal{W}_{\gamma+1}}_{z(\gamma+1)} \arrow[r, "\psi^{\gamma+1}_{z(\gamma+1)}"] & M_{\eta_{z(\gamma+1)}, l_{z(\gamma+1)}} \arrow[r, phantom, "\in"] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{z(\gamma+1)} \\
\mathcal{M}^{\mathcal{U}}_{\nu} \arrow[u, "E^{\mathcal{U}}_\gamma"] \arrow[r, "\sigma_\nu"'] & \mathcal{M}^{\mathcal{W}_\nu}_{z(\nu)} = R_\nu \arrow[u, "\pi^{\nu,\gamma+1}_{z(\nu)}"'] \arrow[r, "\psi^{\nu}_{z(\nu)}"'] & M_{\eta_{z(\nu)}, l_{z(\nu)}} \arrow[u, "\overset{*}{\pi}{}^{\nu,\gamma+1}_{z(\nu)}"'] \arrow[r, phantom, "\in"] & \mathcal{M}^{\mathcal{W}^*_\nu}_{z(\nu)} \arrow[u]
\end{tikzcd}
$$

That the square on the right commutes is $(\dagger)_{z(\gamma+1)}$. We have shown already that the square on the left commutes. We have that $\psi_\nu^{\mathcal{U}} = \psi_{z(\nu)}^{\mathcal{W}_\nu} \circ \sigma_\nu$ by induction. Further, the diagram

$$\begin{tikzcd}
\mathcal{M}_{\gamma+1}^{\mathcal{U}} \arrow[r, "\psi_{\gamma+1}^{\mathcal{U}}"] & M_{\eta_{z(\gamma+1)}, l_{z(\gamma+1)}} \in \mathcal{M}_{z(\gamma+1)}^{\mathcal{W}_{\gamma+1}^*} = \mathcal{M}^{\mathcal{U}_{\gamma+1}^*} \\
\mathcal{M}_\nu^{\mathcal{U}} \arrow[u, "i_{\nu,\gamma+1}^{\mathcal{U}}"] \arrow[r, "\psi_\nu^{\mathcal{U}}"'] & M_{\eta_{z(\nu)}, l_{z(\nu)}} \in \mathcal{M}_{z(\nu)}^{\mathcal{W}_\nu^*} = M^{\mathcal{U}_\nu^*} \arrow[u, "i_{\nu,\gamma+1}^{\mathcal{U}^*}"']
\end{tikzcd}$$

commutes, since it is part of the copy and conversion of $\mathcal{U}$ to $\mathcal{U}^*$. So $\psi_{\gamma+1}^{\mathcal{U}}$ agrees with $\psi_{z(\gamma+1)}^{\gamma+1} \circ \sigma_{\gamma+1}$ on ran $i_{\nu,\gamma+1}^{\mathcal{U}}$. But $\mathcal{M}_{\gamma+1}^{\mathcal{U}}$ is generated by ran $i_{0,\gamma+1}^{\mathcal{U}}$ union $\lambda_{E_\gamma^{\mathcal{U}}}$. For $a \in [\lambda_{E_\gamma^{\mathcal{U}}}]^{<\omega}$,

$$\psi_{z(\gamma+1)}^{\gamma+1} \circ \sigma_{\gamma+1}(a) = \psi_{z(\gamma)}^{\gamma} \circ \sigma_\gamma(a). \tag{$*$}$$

To see ($*$), note first $\sigma_\gamma \restriction \lambda_{E_\gamma^{\mathcal{U}}} = \sigma_{\gamma+1} \restriction \lambda_{E_\gamma^{\mathcal{U}}}$ by facts about embedding normalization. (See 3.49.) So it is enough to show that $\psi_{z(\gamma+1)}^{\gamma+1}$ agrees with $\psi_{z(\gamma)}^{\gamma}$ on λ_{F_γ}. But for $\alpha = \alpha_\gamma^{\mathcal{T},\mathcal{U}}$ as before, $W_\gamma \restriction (\alpha+1) = W_{\gamma+1} \restriction (\alpha+1)$. Also, $\lambda_{F_\gamma} < \lambda_{E_\alpha^{W_\gamma}}$. Thus for $\lambda = \lambda_{F_\gamma}$,

$$\begin{aligned}
\psi_{z(\gamma)}^{\gamma} \restriction \lambda &= \psi_\alpha^\gamma \restriction \lambda \\
&= \psi_\alpha^{\gamma+1} \restriction \lambda \\
&= \psi_{z(\gamma+1)}^{\gamma+1} \restriction \lambda.
\end{aligned}$$

This completes the proof of ($*$).

But $\psi_\gamma^{\mathcal{U}} = \psi_{z(\gamma)}^{\gamma} \circ \sigma_\gamma$ by induction, and $\psi_\gamma^{\mathcal{U}}$ agrees with $\psi_{\gamma+1}^{\mathcal{U}}$ on $\lambda_{E_\gamma^{\mathcal{U}}}$, by the properties of conversion systems. So $\psi_{\gamma+1}^{\mathcal{U}}$ agrees with $\psi_{z(\gamma+1)}^{\gamma+1} \circ \sigma_{\gamma+1}$ on $\lambda_{E_\gamma^{\mathcal{U}}}$, as desired.

This completes the proof of (4) in Lemma 4.43 in the case that $[0, \gamma+1]_U$ does not drop in model or degree, so that we have $z(\gamma) = \operatorname{lh} \mathcal{W}_{\gamma+1}^* - 1$ as well, and $\mathcal{M}_{\gamma+1}^{\mathcal{U}^*} = \mathcal{M}_{z(\gamma+1)}^{\mathcal{W}_{\gamma+1}^*}$. We leave the dropping case to the reader.

This completes the proof that if Lemma 4.43 holds at γ, then it holds at $\gamma+1$.

Now suppose γ is a limit ordinal. Let

$$\lambda = \sup\{\alpha^{\mathcal{T},\mathcal{U}} \mid \xi < \gamma\}.$$

So $W(\mathcal{T},\mathcal{U}\restriction\gamma) = \mathcal{W}_\gamma\restriction\lambda$, and $W(\mathcal{T}^*,\mathcal{U}^*\restriction\gamma) = \mathcal{W}^*_\gamma\restriction\lambda$. Also

$$\mathcal{S}^*_\gamma\restriction\lambda = \mathcal{W}^*_\gamma\restriction\lambda,$$

because $\mathcal{S}^*_\xi\restriction\alpha_\xi = \mathcal{W}^*_\xi\restriction\alpha_\xi = \mathcal{W}^*_\gamma\restriction\alpha_\xi$ for $\xi < \lambda$. Since Σ^* normalizes well, $[0,\lambda)_{W^*_\gamma} = \Sigma^*(\mathcal{W}^*_\gamma\restriction\lambda)$. Thus

$$\mathcal{S}^*_\gamma\restriction(\lambda+1) = \mathcal{W}^*_\gamma\restriction(\lambda+1).$$

We now go on to prove $(\dagger)_\xi$, for $\xi \geq \lambda$, by induction. The proof is similar to the one above. Having $(\dagger)_\xi$ for $\xi = \operatorname{lh}\mathcal{W}_\gamma$, we go on to prove (4) as above. We omit further detail.

This proves Lemma 4.43. $\square$

Now let $lh(\mathcal{U}) = \gamma+1$. So $W(\mathcal{T},\mathcal{U}) = \mathcal{W}_\gamma$ and $W(\mathcal{T}^*,\mathcal{U}^*) = \mathcal{W}^*_\gamma$. By Lemma 4.43, $\mathcal{W}_\gamma$ lifts to $\mathcal{W}^*_\gamma$, so $\mathcal{W}_\gamma$ is by Σ. Let $\tau = z(\gamma)$. Let $P = \mathcal{M}^{\mathcal{U}}_\gamma$, $R = \mathcal{M}^{\mathcal{W}_\gamma}_\tau$, and $S = \mathcal{M}^{\mathcal{W}^*_\gamma}_\tau$. We have $N = M_{\eta^{\mathcal{U}}_\gamma, l^{\mathcal{U}}_\gamma} = M_{\eta^\gamma_\tau, l^\gamma_\tau}$ in the construction of $\mathcal{M}^{\mathcal{U}^*}_\gamma = \mathcal{M}^{\mathcal{W}^*_\gamma}_\tau$, by Lemma 4.43. Moreover, the lemma tells us that $\psi^{\mathcal{U}}_\gamma = \psi^\gamma_\tau \circ \sigma_\gamma$. Let then Ω be the strategy for N induced by the construction of $\mathcal{M}^{\mathcal{U}^*}_\gamma$. Then

$$\begin{aligned}
\Sigma_{\langle\mathcal{T},\mathcal{U}\rangle,P} &= \Omega^{\psi^{\mathcal{U}}_\gamma}\\
&= \Omega^{\psi^\gamma_\tau\circ\sigma_\gamma}\\
&= (\Omega^{\psi^\gamma_\tau})^{\sigma_\gamma}\\
&= (\Sigma_{\mathcal{W}_\gamma,R})^{\sigma_\gamma}.
\end{aligned}$$

Thus Σ 2-normalizes well.

Finally, we must show that all tails of Σ 2-normalize well. It is enough to consider tails of the form $\Sigma_{\mathcal{T},Q}$, where $\mathcal{T}$ is normal on $M = M^{\mathbb{C}}_{\nu_0,k_0}$. Let

$$\operatorname{lift}(\mathcal{T},M,\mathbb{C}) = \langle\mathcal{T}^*, \langle\eta^{\mathcal{T}}_\xi, l^{\mathcal{T}}_\xi \mid \xi\leq\xi_0\rangle, \langle\psi^{\mathcal{T}}_\xi \mid \xi\leq\xi_0\rangle\rangle.$$

Let Ω be the iteration strategy for

$$Q^* = M^{i^{\mathcal{T}^*}_{0,\xi_0}(\mathbb{C})}_{\eta^{\mathcal{T}}_{\xi_0}, l^{\mathcal{T}}_{\xi_0}}$$

that is induced by $\Sigma^*_{\mathcal{T}^*,\mathcal{M}^{\mathcal{T}^*}_{\xi_0}}$. The argument we have just given shows that Ω 2-normalizes well. But $\Sigma_{\mathcal{T},Q}$ is by definition the pullback of $\Omega_{\mathcal{T}^*}$ via $\psi^{\mathcal{T}}_{\xi_0}$. So by 4.4, $\Sigma_{\mathcal{T},Q}$ 2-normalizes well.

This finishes our proof of Theorem 4.41. $\square$

Strong unique iterability yields strategies for coarse premice that normalize well for infinite stacks. In particular, assuming AD^+, if (M,Σ^*) is a coarse Γ-Woodin pair, then Σ^* normalizes well for countable stacks. We believe that by extending the proof of 4.41 one can show that normalizing infinite stacks commutes with lifting to a background universe. Thus if we assume in the hypothesis of Theorem 4.41 that Σ^* normalizes well for infinite stacks, we can conclude that the induced strategies $\Omega(\mathbb{C}, M, \Sigma^*)$ normalize well for infinite stacks.

4.5 Fine strategies that condense well

We show that if Σ^* is an iteration strategy for V that has strong hull condensation, then the strategies for premice induced by Σ^* via a full background extender construction also have strong hull condensation. The proof is routine, but we include it for the sake of completeness. The corresponding result for ordinary hull condensation was proved by Sargsyan in [30].

Theorem 4.49 *Let $N^* \models \mathsf{ZFC}$ + "$\mathbb{C}$ is a background construction". Let Σ^* be a (λ, θ)-iteration strategy for $(N^*, \vec{F}^{\mathbb{C}})$. Suppose that $\langle \nu, k\rangle < \mathrm{lh}(\mathbb{C})$, and Σ is the complete (λ, θ)- iteration strategy for $M^{\mathbb{C}}_{\nu,k}$ induced by Σ^*. Suppose finally that Σ^* has strong hull condensation; then Σ has strong hull condensation.*

Proof. We show first that Σ condenses properly on weakly normal trees. The proof that all its tails Σ_s do so as well is similar. We then deal with the pullback clause in the definition of strong hull condensaion.

Let $\mathcal{U}$ be a weakly normal iteration tree on $M = M^{\mathbb{C}}_{\nu_0,k_0}$ that is by Σ, and let $\Phi\colon \mathcal{T} \to \mathcal{U}$ be a tree embedding, with

$$\Phi = \langle u, \langle s_\beta \mid \beta < \mathrm{lh}(\mathcal{T})\rangle, \langle t_\beta \mid \beta + 1 < \mathrm{lh}(\mathcal{T})\rangle, p\rangle.$$

Let $N_\beta = N^\Phi_\beta$, so that $s_\beta\colon \mathcal{M}^{\mathcal{T}}_\beta \to N_\beta \trianglelefteq \mathcal{M}^{\mathcal{U}}_{v(\beta)}$. The reader will lose little by assuming that $\mathcal{T}$ and $\mathcal{U}$ are fully normal, so that $N_\beta = \mathcal{M}^{\mathcal{U}}_{v(\beta)}$. Nevertheless, we shall not make that simplification here.

We must see that $\mathcal{T}$ lifts to a tree by Σ^*. Let

$$\mathrm{lift}(\mathcal{U}, M, \mathbb{C}) = \langle \mathcal{U}^*, \langle \theta_\xi, m_\xi \mid \xi < \mathrm{lh}(\mathcal{U})\rangle, \langle \psi_\xi \mid \xi < \mathrm{lh}(\mathcal{U})\rangle\rangle.$$

It is enough to show that $\mathcal{T}$ lifts to a psuedo-hull of $\mathcal{U}^*$. For this, let

$$\mathrm{lift}(\mathcal{T}, M, \mathbb{C}) = \langle \mathcal{T}^*, \langle \eta_\xi, l_\xi \mid \xi < \mathrm{lh}(\mathcal{T})\rangle, \langle \varphi_\xi \mid \xi < \mathrm{lh}(\mathcal{T})\rangle\rangle.$$

Note that both $\mathcal{T}^*$ and $\mathcal{U}^*$ are fully normal. (See 2.45.) Let $\mathbb{C}_\alpha = i^{\mathcal{T}^*}_{0,\alpha}(\mathbb{C})$ and $\mathbb{D}_\alpha = i^{\mathcal{U}^*}_{0,\alpha}(\mathbb{C})$. Let $Q_\alpha = M^{\mathbb{C}_\alpha}_{\eta_\alpha, l_\alpha}$, and $X_\alpha = M^{\mathbb{D}_\alpha}_{\theta_\alpha, m_\alpha}$. Thus

$$\varphi_\alpha\colon \mathcal{M}^{\mathcal{T}}_\alpha \to Q_\alpha,$$

and

$$\psi_\alpha\colon \mathcal{M}^{\mathcal{U}}_\alpha \to X_\alpha$$

are the liftup maps of the two conversion systems. The map that resurrects $\varphi_\alpha(E^{\mathcal{T}}_\alpha)$ inside $\mathbb{C}_\alpha$ is

$$\sigma_\alpha = \sigma_{\eta_\alpha, l_\alpha}[M^{\mathbb{C}_\alpha}_{\eta_\alpha, l_\alpha} | \langle \mathrm{lh}(\varphi_\alpha(E^{\mathcal{T}}_\alpha)), 0\rangle].$$

Similarly, the resurrection map for $\psi_\alpha(E^{\mathcal{U}}_\alpha)$ is

$$\tau_\alpha = \sigma_{\theta_\alpha, m_\alpha}[M^{\mathbb{D}_\alpha}_{\theta_\alpha, m_\alpha} | \langle \mathrm{lh}(\varphi_\alpha(E^{\mathcal{U}}_\alpha)), 0\rangle].$$

For any background construction $\mathbb{D}$, if G is the last extender of $M^{\mathbb{D}}_{\nu,0}$, we write $B^{\mathbb{D}}(G) = F^{\mathbb{D}}_\nu$ for the background extender of G given by $\mathbb{D}$. (Note G is the last extender at most once.) Thus

$$E^{\mathcal{T}^*}_\alpha = B^{\mathbb{C}_\alpha} \circ \sigma_\alpha \circ \varphi_\alpha(E^{\mathcal{T}}_\alpha)$$

and

$$E^{\mathcal{U}^*}_\alpha = B^{\mathbb{D}_\alpha} \circ \tau_\alpha \circ \psi_\alpha(E^{\mathcal{U}}_\alpha).$$

Let us write R_α for the level of $\mathbb{C}_\alpha$ that has $\sigma_\alpha \circ \varphi_\alpha(E^{\mathcal{T}}_\alpha)$ as its last extender, so that σ_α maps an initial segment of Q_α to R_α. Similarly, let Y_α be such that $\tau_\alpha \circ \psi_\alpha(E^{\mathcal{U}}_\alpha)$ is the last extender of Y_α.

We shall construct a tree embedding $\Phi^*\colon \mathcal{T}^* \to \mathcal{U}^*$ by induction, with

$$\Phi^* = \langle u, \langle r_\beta \mid \beta < \mathrm{lh}(\mathcal{T})\rangle, \langle w_\beta \mid \beta + 1 < \mathrm{lh}(\mathcal{T})\rangle, q\rangle.$$

Notice here that $u^{\Phi^*} = u = u^{\Phi}$. Because Φ^* is to be a tree embedding, this completely determines the putative Φ^*, and what we have to show is just that Φ^* is indeed a tree embedding of $\mathcal{T}^*$ into $\mathcal{U}^*$.

For $\gamma \le \mathrm{lh}(\mathcal{T})$, let

$$\Phi^*_\gamma = \Phi^*\restriction\gamma = \langle u\restriction\{\xi \mid \xi + 1 < \gamma\}, \langle r_\beta \mid \beta < \gamma\rangle, \langle w_\beta \mid \beta + 1 < \gamma\rangle, q_\gamma\rangle.$$

Let v be the common "minimal realization" map of Φ and Φ^*, given by $v(0) = 0$, $v(\alpha + 1) = u(\alpha) + 1$, and $v(\lambda) = \sup_{\alpha<\lambda} v(\alpha)$ for λ a limit ordinal. We show by induction on γ that

(1) $\Phi^*\restriction\gamma$ is a tree embedding of $\mathcal{T}^*\restriction\gamma$ into $\mathcal{U}^*$,

(2) for $\alpha < \gamma$, $\psi_{v(\alpha)} \circ s_\alpha = r_\alpha \circ \varphi_\alpha$, and

(3) for $\alpha < \gamma$, $r_\alpha(Q_\alpha) = \psi_{v(\alpha)}(N_\alpha) \trianglelefteq X_{v(\alpha)}$.

Let $(*)_\gamma$ be the conjunction of (1)-(3). The following diagram illustrates the situation:

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{U}}_{u(\alpha)} \arrow[r, "\psi_{u(\alpha)}"] & X_{u(\alpha)} \in \mathcal{M}^{\mathcal{U}^*}_{u(\alpha)} \\
\mathcal{M}^{\mathcal{U}}_{v(\alpha)} \arrow[u, "i^{\mathcal{U}}_{v(\alpha),u(\alpha)}"] \arrow[r, "\psi_{v(\alpha)}"] & X_{v(\alpha)} \in \mathcal{M}^{\mathcal{U}^*}_{v(\alpha)} \arrow[u, "i^{\mathcal{U}^*}_{v(\alpha),u(\alpha)}"'] \\
\mathcal{M}^{\mathcal{T}}_{\alpha} \arrow[u, "s_\alpha"] \arrow[r, "\varphi_\alpha"'] \arrow[uu, bend left=60, "t_\alpha"] & Q_\alpha \in \mathcal{M}^{\mathcal{T}^*}_{\alpha} \arrow[u, "r_\alpha"'] \arrow[uu, bend right=60, "w_\alpha"']
\end{tikzcd}$$

Some care is needed in reading this diagram. The bottom rectangle is just (2) and (3) of our induction hypotheses, and is always valid, provided we understand that s_α may only be elementary as a map of $\mathcal{M}^{\mathcal{T}}_\alpha$ into a proper initial segment N_α of $\mathcal{M}^{\mathcal{U}}_{v(\alpha)}$. Similarly, $r_\alpha(Q_\alpha)$ may be a proper initial segment of $X_{v(\alpha)}$. (These would be the relics of gratuitous dropping along $[0,\alpha)_T$ or $[0,v(\alpha)]_U$.) The top rectangle involves only the conversion of $\mathcal{U}$ to $\mathcal{U}^*$, so our induction hypotheses are irrelevant. It is valid if and only if $(v(\alpha),u(\alpha)]_U$ does not drop (in model or degree), so that $i^{\mathcal{U}^*}_{v(\alpha),u(\alpha)}(X_{v(\alpha)}) = X_{u(\alpha)}$. In the case that $(v(\alpha),u(\alpha]_U$ drops, something like it is valid. We discuss that below.

To start with, Φ^*_1 is given by setting $v(0) = 0$ and $r_0 =$ identity map from $N^* = \mathcal{M}^{\mathcal{T}^*}_0$ to $N^* = \mathcal{M}^{\mathcal{U}^*}_0$.

If λ is a limit, and $(*)_\alpha$ for $\alpha < \lambda$, then

$$\Phi^*_\lambda = \bigcup_{\alpha<\lambda} \Phi^*_\alpha$$

in the obvious componentwise sense. It is clear that $(*)_\lambda$ holds.

If $\gamma = \lambda + 1$ for $\lambda < \mathrm{lh}(\mathcal{T})$ a limit such that $(*)_\lambda$, then $\Phi^*_{\lambda+1}$ is just Φ^*_λ together with the map r_λ, defined as follows. Recall that v preserves tree order, and

$$v(\lambda) = \sup_{\alpha<\lambda} v(\alpha).$$

For $\alpha <_T \lambda$ and $x \in \mathcal{M}^{\mathcal{T}^*}_\alpha$, we set

$$r_\lambda(i^{\mathcal{T}^*}_{\alpha,\lambda}(x)) = i^{\mathcal{U}^*}_{v(\alpha),v(\lambda)}(r_\alpha(x)).$$

Using (1) at $\gamma < \lambda$, we see that r_λ is well defined, elementary, and as required for $(*)_{\lambda+1}$.

Finally, suppose we have $\Phi^*_{\alpha+1}$ satisfying $(*)_{\alpha+1}$. The whole of $\Phi^*_{\alpha+2}$ is determined by $u(\alpha)$, which is already given to us, but we must see this choice works; that is, that $(*)_{\alpha+2}$ holds for the system it determines.

Let

$$\begin{aligned} G &= E^{\mathcal{T}}_\alpha, \\ G^* &= E^{\mathcal{T}^*}_\alpha = (B^{\mathbb{C}_\alpha} \circ \sigma_\alpha \circ \varphi_\alpha)(G), \\ H &= E^{\mathcal{U}}_{u(\alpha)}, \\ H^* &= E^{\mathcal{U}^*}_{u(\alpha)} = (B^{\mathbb{D}_{u(\alpha)}} \circ \tau_\alpha \circ \psi_{u(\alpha)})(H). \end{aligned}$$

Set also

$$w_\alpha = i^{\mathcal{U}^*}_{v(\alpha),u(\alpha)} \circ r_\alpha,$$

as we are forced to do. Note that $w_\alpha(\mathbb{C}_\alpha) = \mathbb{D}_{u(\alpha)}$. Lemma 5.3 below will tell us that the following claim is what we need.

Claim 4.50 *(a)* $\tau_{u(\alpha)} \circ \psi_{u(\alpha)} \circ i^{\mathcal{U}}_{v(\alpha),u(\alpha)} \circ s_\alpha \restriction (\mathrm{lh}(G)+1) = i^{\mathcal{U}^*}_{v(\alpha),u(\alpha)} \circ r_\alpha \circ \sigma_\alpha \circ \varphi_\alpha \restriction (\mathrm{lh}(G)+1)$.

(b) $w_\alpha(G^*) = H^*$.

Proof. We prove (a). Suppose first that $(v(\alpha), u(\alpha)]_U$ does not drop. In that case, $i^{\mathcal{U}^*}_{v(\alpha),u(\alpha)}(X_{v(\alpha)}) = X_{u(\alpha)}$, so the top rectangle in the diagram above is valid. Expanding the diagram, we have

$$\begin{tikzcd} \mathcal{M}^{\mathcal{U}}_{u(\alpha)} \arrow[r, "\psi_{u(\alpha)}"] & X_{u(\alpha)} \arrow[r, "\tau_{u(\alpha)}"] & Y_{u(\alpha)} \\ \mathcal{M}^{\mathcal{U}}_{v(\alpha)} \arrow[u, "i^{\mathcal{U}}_{v(\alpha),u(\alpha)}"] \arrow[r, "\psi_{v(\alpha)}"] & X_{v(\alpha)} \arrow[u, "i^{\mathcal{U}^*}_{v(\alpha),u(\alpha)}"'] \arrow[r, "\tau_{v(\alpha)}"] & Y_{v(\alpha)} \arrow[u, "i^{\mathcal{U}^*}_{v(\alpha),u(\alpha)}"'] \\ \mathcal{M}^{\mathcal{T}}_\alpha \arrow[u, "s_\alpha"] \arrow[r, "\varphi_\alpha"'] & Q_\alpha \arrow[u, "r_\alpha"'] \arrow[r, "\sigma_\alpha"'] & R_\alpha \arrow[u, "r_\alpha"'] \end{tikzcd}$$

Notice that $r_\alpha(\sigma_\alpha) = \tau_{v(\alpha)}$. So the diagram commutes, and in particular the two routes from $\mathcal{M}^{\mathcal{T}}_\alpha$ to $Y_{u(\alpha)}$ around the outer edges are the same. This gives us (a).

Suppose now that $(v(\alpha), u(\alpha)]_U$ drops. Let $I = s_\alpha(G)$. Since $H = \hat{\imath}^{\mathcal{U}}_{v(\alpha),u(\alpha)}(I)$, all extenders used along $(v(\alpha), u(\alpha]_U$ have critical points strictly below the current image of λ_I. For simplicity, let us assume there is just one such drop, at ξ, where $v(\alpha) <_U \xi \leq_U u(\alpha)$. (It doesn't matter whether or not the drop is gratuitous.) Let $\theta = U\text{-pred}(\xi)$. We have the following diagram:

$$
\begin{tikzcd}
\mathcal{M}^{\mathcal{U}}_{u(\alpha)} \arrow[r] \arrow[rr, bend left, "\psi_{u(\alpha)}"] & i^{\mathcal{U}^*}_{v(\alpha),u(\alpha)}(X_{v(\alpha)}) \arrow[r, "k"'] & X_{u(\alpha)} \arrow[r, "\tau_{u(\alpha)}"] & Y_{u(\alpha)} \\
\mathcal{M}^{\mathcal{U}}_{\xi} \arrow[u] \arrow[r] & {} \arrow[u, "i^{\mathcal{U}^*}_{\xi,u(\alpha)}"] \arrow[r, "h"] & X_{\xi} \arrow[u, "i^{\mathcal{U}^*}_{\xi,u(\alpha)}"] \arrow[r, "\tau_{\xi}"] & Y_{\xi} \arrow[u, "i^{\mathcal{U}^*}_{\xi,u(\alpha)}"] \\
\mathcal{M}^{\mathcal{U}}_{\theta} \arrow[u] \arrow[r, "\psi_\theta"] & X_\theta \arrow[u, "i^{\mathcal{U}^*}_{\theta,\xi}"] \arrow[r, "j"] & Z \arrow[u, "i^{\mathcal{U}^*}_{\theta,\xi}"] \arrow[r, "l"] & Y_\theta \arrow[u, "i^{\mathcal{U}^*}_{\theta,\xi}"] \\
\mathcal{M}^{\mathcal{U}}_{v(\alpha)} \arrow[u] \arrow[r, "\psi_{v(\alpha)}"] & X_{v(\alpha)} \arrow[u, "i^{\mathcal{U}^*}_{v(\alpha),\theta}"] \arrow[rr, "\tau_{v(\alpha)}"] & & Y_{v(\alpha)} \arrow[u, "i^{\mathcal{U}^*}_{v(\alpha),\theta}"] \\
\mathcal{M}^{\mathcal{T}}_{\alpha} \arrow[u, "s_\alpha"] \arrow[r, "\varphi_\alpha"] & Q_\alpha \arrow[u, "r_\alpha"] \arrow[rr, "\sigma_\alpha"] & & R_\alpha \arrow[u, "r_\alpha"]
\end{tikzcd}
$$

In the diagram, $j = \sigma_{\mu,n}[\psi_\theta(\mathcal{M}^{*,\mathcal{U}}_{\xi}]^{\mathbb{D}_\theta}$ resurrects the drop in $\mathcal{U}$, and $\tau_\theta = l \circ j$. We have $X_\xi = i^{\mathcal{U}^*}_{\theta,\xi}(Z)$, and $\tau_\xi = i^{\mathcal{U}^*}_{\theta,\xi}(l)$. Also, $h = i^{\mathcal{U}^*}_{\theta,\xi}(j)$ and $k = i^{\mathcal{U}^*}_{\xi,u(\alpha)}(h)$. The unlabelled vertical arrows on the far left are the maps of $\mathcal{U}$. Finally, $r_\alpha(\sigma_\alpha) = \tau_{v(\alpha)}$.

The facts we have just enumerated imply that all parts of the diagram commute on the image of $\text{lh}(G) + 1$. (For the square at the bottom left, this is our induction hypothesis.) The reason for restricting to the image of $\text{lh}(G) + 1$ is that the resurrection maps j, h, k and the τ's and σ_α are partial, defined on initial segments of the models displayed above. But all are defined on the image of $\text{lh}(G)+1$ in that model.

The fact that the two routes from $\mathcal{M}^{\mathcal{T}}_\alpha$ to $Y_{u(\alpha)}$ going along the outer edges are the same when restricted to $\text{lh}(G) + 1$ gives us part (a) of the claim.

Part (b) follows easily from the fact that the images of G in $Y_{u(\alpha)}$ along the two outer edges of the diagram are the same.

This proves Claim 4.50. □

By Lemma 5.3, there is a unique tree embedding Ψ from $\mathcal{T}^*\restriction(\alpha+2)$ to $\mathcal{U}^*$ that extends $\Phi^*_{\alpha+1}$ and satisfies $u^\Psi(\alpha) = u(\alpha)$. Let $\Phi^*_{\alpha+2}$ be this Ψ. We check now that $(*)_{\alpha+2}$ holds.

Let $\beta = T\text{-pred}(\alpha+1)$, and let $\tau = U\text{-pred}(u(\alpha)+1)$. Because Φ is a tree embedding, $\tau \in [v(\beta), u(\beta)]_U$. Let us assume for simplicity that there is no relevant dropping, that is,

(a) $(\alpha+1) \notin D^{\mathcal{T}}$, and

(b) $D^{\mathcal{U}} \cap [v(\beta), v(\alpha+1)] = \emptyset$.

So $\mathcal{M}^{\mathcal{T}}_{\alpha+1} = \mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_\beta, G)$ and $\mathcal{M}^{\mathcal{U}}_{v(\alpha+1)} = \mathrm{Ult}(\mathcal{M}^{\mathcal{U}}_\tau, H))$. Let $\rho = i^{\mathcal{U}}_{v(\beta),\tau)} \circ s_\beta$ and $\rho^* = i^{\mathcal{U}^*}_{v(\beta),\tau} \circ r_\beta$. The lifting construction yields $\mathcal{M}^{\mathcal{T}^*}_{\alpha+1} = \mathrm{Ult}(\mathcal{M}^{\mathcal{T}^*}_\beta, G^*))$ and $\mathcal{M}^{\mathcal{U}^*}_{v(\alpha+1)} = \mathrm{Ult}(\mathcal{M}^{\mathcal{U}^*}_\tau, H^*)$, moreover

$$X_{v(\alpha+1)} = i^{\mathcal{U}^*}_{v(\beta),v(\alpha+1)}(X_{v(\beta)}).$$

$r_{v(\alpha+1)}$ is given by the Shift Lemma:

$$r_{v(\alpha+1)}([a,f]^{\mathcal{M}^{\mathcal{T}^*}_\beta}_{G^*}) = [w_\alpha(a), \rho^*(f)]^{\mathcal{M}^{\mathcal{U}^*}_\tau}_{H^*}.$$

Here is a diagram of the situation.

$$\begin{tikzcd}
 & \mathcal{M}^{\mathcal{U}}_{v(\alpha+1)} \arrow[rr, "\psi_{v(\alpha+1)}"] & & X_{v(\alpha+1)} \in \mathcal{M}^{\mathcal{U}^*}_{v(\alpha+1)} \\
\mathcal{M}^{\mathcal{T}}_{\alpha+1} \arrow[ur, "s_{\alpha+1}"] \arrow[rr, "\varphi_{\alpha+1}"] & & Q_{\alpha+1} \in \mathcal{M}^{\mathcal{T}^*}_{\alpha+1} \arrow[ur, "r_{\alpha+1}"'] & \\
 & \mathcal{M}^{\mathcal{U}}_\tau \arrow[uu, "H"] \arrow[rr, "\psi_\tau"] & & X_\tau \in \mathcal{M}^{\mathcal{U}^*}_\tau \arrow[uu, "H^*"'] \\
 & \mathcal{M}^{\mathcal{U}}_{v(\beta)} \arrow[u] \arrow[rr, "\psi_{v(\beta)}"] & & X_{v(\beta)} \in \mathcal{M}^{\mathcal{U}^*}_{v(\beta)} \arrow[u] \\
\mathcal{M}^{\mathcal{T}}_\beta \arrow[uuu, "G"] \arrow[uur, "\rho"] \arrow[ur, "s_\beta"'] \arrow[rr, "\varphi_\beta"'] & & Q_\beta \in \mathcal{M}^{\mathcal{T}^*}_\beta \arrow[uuu, "G^*"'] \arrow[uur, "\rho^*"] \arrow[ur, "r_\beta"'] &
\end{tikzcd}$$

The diagram resembles the diagram associated to our proof the copying commutes with embedding normalization. That is not an accident, of course. Embedding normalization yields tree embeddings, and lifting to a background universe is similar to copying. We have simplified the diagram above by ignoring the fact that s_β is only elementary as a map into N_β, which may be a proper initial segment of $\mathcal{M}^{\mathcal{U}}_{v(\beta)}$. In that case, r_β maps Q_β to the corresponding initial segment $\psi_{v(\beta)}(N_\beta)$ of $X_{v(\beta)}$. Similarly, $s_{\alpha+1}$ and $r_{\alpha+1}$ will then map $\mathcal{M}^{\mathcal{T}}_{\alpha+1}$ and $Q_{\alpha+1}$ to proper initial segments of $\mathcal{M}^{\mathcal{U}}_{v(\alpha+1)}$ and $X_{v(\alpha+1)}$.

We are asked to show that $\psi_{v(\alpha+1)} \circ s_{\alpha+1} = \varphi_{\alpha+1} \circ r_{\alpha+1}$, in other words, that the rectangle on the top face of the cube commutes. We argue just as we did in the proof of 3.55. The rectangle on the bottom commutes by our induction hypothesis. The rectangle in front commutes because $\mathcal{T}^*$ comes from lifting $\mathcal{T}$ to the background universe. The diagram on the back face commutes because $\mathcal{U}^*$ comes from lifting $\mathcal{U}$. The maps on the left face commute because Φ is a tree embedding of $\mathcal{T}$ into $\mathcal{U}$. The maps on the right face commute because we obtained $r_{\alpha+1}$ from the Shift Lemma. (This of course is where we used that $H^* = w_\alpha(G^*)$.)

It is clear from these facts that the top rectangle commutes on $\mathrm{ran}(i^{\mathcal{T}}_{\beta,\alpha+1})$. Since $\mathcal{M}^{\mathcal{T}}_{\alpha+1}$ is generated by $\mathrm{ran}(i^{\mathcal{T}}_{\beta,\alpha+1}) \cup \lambda(G)$, it is enough to see that the top square commutes on $\lambda(G)$. But

$$\begin{aligned}
\psi_{v(\alpha+1)} \circ s_{\alpha+1} \restriction \lambda(G) &= \tau_{u(\alpha)} \circ \psi_{u(\alpha)} \circ i^{\mathcal{U}}_{v(\alpha),u(\alpha)} \circ s_\alpha \restriction \lambda(G) \\
&= i^{\mathcal{U}^*}_{v(\alpha),u(\alpha)} \circ r_\alpha \circ \sigma_\alpha \circ \varphi_\alpha \restriction \lambda(G) \\
&= r_{\alpha+1} \circ \varphi_{\alpha+1} \restriction \lambda(G).
\end{aligned}$$

Line 1 comes from the facts that $s_{\alpha+1}$ agrees with $i^{\mathcal{U}}_{v(\alpha),u(\alpha)} \circ s_\alpha$ on $\lambda(G)$ by the way it is defined using the Shift Lemma, and that $\psi_{v(\alpha+1)}$ agrees with $\tau_{u(\alpha)} \circ \psi_{u(\alpha)}$ on $\lambda(H)$ for a similar reason. Line 2 comes from Claim 4.50. Line 3 again comes from using the Shift Lemma, now at the level of $\mathcal{T}^*$ and $\mathcal{U}^*$.

This completes the proof that Σ condenses well on weakly normal trees. The proof that its tails do so as well is similar. Let us now consider the pullback condition, clause (b) of 4.6. For this, let us keep our previous notation, but assume that $\mathrm{lh}(\mathcal{T}) = \alpha+1$, $\mathrm{lh}(\mathcal{U}) = \beta+1$, and that $v(\alpha) \leq \beta$ and Φ has been extended by adding the t-map

$$\pi = i^{\mathcal{U}}_{v(\alpha),\beta} \circ s_\alpha.$$

Let us assume $J \trianglelefteq \mathrm{dom}(\pi)$, and let $K = \pi(J)$. We need to see that $(\Sigma_{\mathcal{U},K})^\pi = \Sigma_{\mathcal{T},J}$. For that, consider the diagram

$$
\begin{tikzcd}
\mathcal{M}^{\mathcal{U}}_{\beta} \arrow[r, "i"'] \arrow[rr, bend left, "\psi_\beta"] & i^{\mathcal{U}^*}_{\theta,\beta}(X_\theta) \arrow[r, "k"'] & X_\beta \arrow[r] & \mathcal{M}^{\mathcal{U}^*}_{\beta} \\
\mathcal{M}^{\mathcal{U}}_{\xi} \arrow[u] \arrow[r] & {} \arrow[u, "i^{\mathcal{U}^*}_{\xi,\beta}"] \arrow[r, "h"] & X_\xi \arrow[u, "i^{\mathcal{U}^*}_{\xi,\beta}"] \arrow[r] & \mathcal{M}^{\mathcal{U}^*}_{\xi} \arrow[u, "i^{\mathcal{U}^*}_{\xi,\beta}"] \\
\mathcal{M}^{\mathcal{U}}_{\theta} \arrow[u] \arrow[r, "\psi_\theta"] & X_\theta \arrow[u, "i^{\mathcal{U}^*}_{\theta,\xi}"] \arrow[r, "j"] & Z \arrow[u, "i^{\mathcal{U}^*}_{\theta,\xi}"] \arrow[r] & \mathcal{M}^{\mathcal{U}^*}_{\theta} \arrow[u, "i^{\mathcal{U}^*}_{\theta,\xi}"] \\
\mathcal{M}^{\mathcal{U}}_{v(\alpha)} \arrow[u] \arrow[r, "\psi_{v(\alpha)}"] & X_{v(\alpha)} \arrow[u, "i^{\mathcal{U}^*}_{v(\alpha),\theta}"] \arrow[rr] & & \mathcal{M}^{\mathcal{U}^*}_{v(\alpha)} \arrow[u, "i^{\mathcal{U}^*}_{v(\alpha),\theta}"] \\
\mathcal{M}^{\mathcal{T}}_{\alpha} \arrow[u, "s_\alpha"] \arrow[r, "\varphi_\alpha"] & Q_\alpha \arrow[u, "r_\alpha"] \arrow[rr] & & \mathcal{M}^{\mathcal{T}^*}_{\alpha} \arrow[u, "r_\alpha"]
\end{tikzcd}
$$

In the diagram, $j = \sigma_{\mu,n}[\psi_\theta(\mathcal{M}^{*,\mathcal{U}}_{\xi}]^{\mathbb{D}_\theta}$, and h and k are its images under the $\mathcal{U}^*$ embeddings. We are assuming for definiteness that $\mathcal{U}$ dropped once on $(v(\alpha), \beta]_U$, at its step from θ to ξ. The maps j, h, and k are defined only on initial segments of the models displayed, but all are defined on the image of J in that model.

Let $L = \varphi_\alpha(J)$ and $P = \psi_\beta(K)$. Let also $N = i(K) = k^{-1}(P)$. By the commutativity of the left column in the diagram, it is enough to see that the $\mathbb{D}_\beta$-induced strategy of P pulls back under $k \circ i_{v(\alpha),\beta} \circ r_\alpha$ to the $\mathbb{C}_\alpha$-induced strategy of L. The following claims show this. Put $Y = i^{\mathcal{U}^*}_{v(\alpha),\beta}(X_{v(\alpha)})$.

Claim 1. $\Omega(\mathbb{D}_\beta, Y, \Sigma^*_{\mathcal{U}^*\restriction\beta+1})_N = \Omega(\mathbb{D}_\beta, X_\beta, \Sigma^*_{\mathcal{U}^*\restriction\beta+1})^k_P$.

Proof. This follows at once from Lemma 2.47. □

Claim 2. $\Omega(\mathbb{D}_{v(\alpha)}, X_{v(\alpha)}, \Sigma^*_{\mathcal{U}^*\restriction v(\alpha)+1}) = \Omega(\mathbb{D}_\beta, Y, \Sigma^*_{\mathcal{U}^*\restriction\beta+1})^{i^{\mathcal{U}^*}_{v(\alpha),\beta}}$.

Proof. Let $\pi = i^{\mathcal{U}^*}_{v(\alpha),\beta}$. Because Σ^* has strong hull condensation, it is pullback consistent, so $\Sigma^*_{\mathcal{U}^*\restriction v(\alpha+1)} = (\Sigma^*_{\mathcal{U}^*\restriction\beta+1})^\pi$. But $\Omega(\mathbb{D}_\beta, Y, \Sigma^*_{\mathcal{U}^*\restriction\beta+1})^\pi = \Omega(\mathbb{D}_{v(\alpha)}, X_{v(\alpha)}, (\Sigma^*_{\mathcal{U}^*\restriction\beta+1})^\pi)$ by 2.50. □

Claim 3. $\Omega(\mathbb{C}_\alpha, Q_\alpha, \Sigma^*_{\mathcal{T}^*\restriction\alpha+1}) = \Omega(\mathbb{D}_{v(\alpha)}, X_{v(\alpha)}, \Sigma^*_{\mathcal{U}^*\restriction v(\alpha)+1})^{r_\alpha}$.

Proof. Since Σ^* has strong hull condensation, $\Sigma^*_{\mathcal{T}^*\restriction\alpha+1} = (\Sigma^*_{\mathcal{U}^*\restriction v(\alpha)+1})^{r_\alpha}$. We can therefore apply Corollary 2.50 again. $\square$

Let $\Lambda = \Omega(\mathbb{D}_\beta, X_\beta, \Sigma^*_{\mathcal{U}^*\restriction\beta+1})_P$. The claims imply that $\Omega(\mathbb{C}_\alpha, Q_\alpha, \Sigma^*_{\mathcal{T}^*\restriction\alpha+1})_L$ is the pullback of Λ under $k \circ i^{\mathcal{U}^*}_{v(\alpha),\beta} \circ r_\alpha$, and hence that $\Sigma_{\mathcal{T},J}$ is the pullback of Λ under $k \circ i^{\mathcal{U}^*}_{v(\alpha),\beta} \circ r_\alpha \circ \varphi_\alpha$. By commutativity, $\Sigma_{\mathcal{T},J}$ is the pullback of Λ under $\psi_\beta \circ i^{\mathcal{U}}_{v(\alpha,\beta} \circ s_\alpha$. But this means that it is the pullback of $\Sigma_{\mathcal{U}\restriction(\beta+1),K}$ under $i^{\mathcal{U}}_{v(\alpha),\beta} \circ s_\alpha$, as desired.

This completes the proof of Theorem 4.49.

$\square$

4.6 Pure extender pairs and strategy coherence

As we have just seen, background constructions in Γ-Woodin universes yield iteration strategies for premice that condense and normalize well. It seems that all the nice behavior of iteration strategies one could wish for follows from these two properties. We shall see this as we proceed. Because of that, the following is one of our central definitions.

Definition 4.51 (M, Ω) *is a* pure extender pair *with scope* H_δ *iff*

(1) M is a pure extender premouse, and $M \in H_\delta$,

(2) Ω is a complete iteration strategy for M, with scope H_δ, and

(3) Ω normalizes well, and has strong hull condensation.

We are only interested in the case that Ω is absolutely definable. In the most important context, M is countable, Ω has scope H_{ω_1}, and its absolute definability is witnessed by membership in a model of AD^+. At other times we are working under hypotheses that allow us to reach something close to this AD^+ context in a generic extension.

The proviso "scope H_δ" implies that Ω is an (ω, δ)-strategy. It would be more natural to require that Ω be a (δ, δ)-strategy, but then our comparison proof for pure extender pairs would need to go into normalizing infinite stacks.

It follows immediately from the definitions that any iterate of a pure extender pair is also a pure extender pair. That is, if (P, Σ) is a pure extender pair with scope H_δ, and s is a P-stack by Σ with last model Q, then (Q, Σ_s) is a pure extender pair with scope H_δ. We have already in effect proved another useful basic fact, namely, that elementary submodels of pure extender pairs are pure extender pairs. More precisely,

Lemma 4.52 *Let (M, Ω) be a pure extender pair with scope H_δ, and let $\pi\colon N \to M$ be weakly elementary, where N is a pure extender premouse; then (N, Ω^π) is a pure extender pair with scope H_δ.*

Proof. Clearly, Ω^π is a complete iteration strategy for N with scope H_δ. Ω^π normalizes well by 4.4, and has strong hull condensation by 4.10. $\square$

Another elementary fact is

Lemma 4.53 *Let (M, Ω) be a pure extender pair; then Ω is pullback consistent.*

Proof. We proved this in Lemma 4.9. $\square$

Concerning pairs with scope going beyond HC, the following lemmas will be useful. The first says that the strategy restricted to countable trees determines the strategy on all trees.

Lemma 4.54 *Let (P, Σ) and (P, Λ) be pure extender pairs with scope H_δ, and suppose that Σ and Λ agree on countable normal trees; then $\Sigma = \Lambda$.*

Proof. Otherwise we have a normal $\mathcal{T}$ of limit length by both Σ and Λ, with $\Sigma(\mathcal{T}) = b$ and $\Lambda(\mathcal{T}) = c$, and $b \neq c$. Let H be countable and transitive, and

$$\pi\colon H \to V_\gamma$$

be elementary, with γ large and everything relevant in $\mathrm{ran}(\pi)$. Let $\bar{P}, \bar{\mathcal{T}}, \bar{b}, \bar{c}$ in H be the collapses of $P, \mathcal{T}, b, c$. So $\bar{b} \neq \bar{c}$. Letting

$$\mathcal{U} = \pi\bar{\mathcal{T}},$$

it is easy to see that $\mathcal{U}^\frown\bar{b}$ is a pseudo-hull of $\mathcal{T}^\frown b$. (For example, the relevant u-map is just $\pi\restriction \mathrm{lh}(\mathcal{U})$.) Similarly, $\mathcal{U}^\frown\bar{c}$ is a pseudo-hull of $\mathcal{T}^\frown c$. But by strong hull condensation, $\mathcal{U}^\frown\bar{b}$ is by Σ and $\mathcal{U}^\frown\bar{c}$ is by Λ, so $\bar{b} = \bar{c}$ because the strategies agree on countable normal trees. This is a contradiction. $\square$

The reader should compare the following lemma to Proposition 2.32.

Lemma 4.55 *Let (P, Σ) be a pure extender pair with scope H_δ, and let $j\colon V \to M$ be elementary, where M is transitive and* $\mathrm{crit}(j) > |P|$; *then $j(\Sigma)$ and Σ agree on all trees in $j(H_\delta) \cap H_\delta$.*

Proof. Otherwise we have a normal tree $\mathcal{T}$ with distinct cofinal branches b and c such that $\mathcal{T}^\frown b$ is by Σ and $\mathcal{T}^\frown c$ is by $j(\Sigma)$. As in the proof of the last lemma, this gives us a countable normal tree $\mathcal{U}$ on P with distinct cofinal branches $\bar{b}$ and $\bar{c}$ such that $\mathcal{U}^\frown\bar{b}$ is a pseudo-hull of $\mathcal{T}^\frown b$ and $\mathcal{U}^\frown\bar{c}$ is a pseudo-hull of $\mathcal{T}^\frown c$. Thus $\Sigma(\mathcal{U}) = \bar{b}$. But since $\mathcal{U}$ is countable, and M is wellfounded,

$$M \models \mathcal{U}^\frown c \text{ is a pseudo-hull of } \mathcal{T}^\frown c.$$

Thus $j(\Sigma)(\mathcal{U}) = \bar{c}$. But $\mathcal{U}$ is countable, hence fixed by j, so $\Sigma(\mathcal{U}) = \bar{c}$, a contradiction. $\square$

For the remainder of this section, we look at one further elementary property, *strategy coherence*. To see what is at stake here, suppose (P, Σ) is a pure extender pair, and κ is a cardinal of P such that $\kappa \leq \rho(P)$. Let $\mathcal{T}$ be a tree on $P|\kappa$ that is according to $\Sigma_{P|\kappa}$. We can also think of $\mathcal{T}$ as a tree on P, or as a tree on $\mathrm{Ult}(P, E^P_\alpha)$ whenever $\mathrm{lh}(E^P_\alpha) > \kappa$. Does it follow from our definitions that, considered this way, $\mathcal{T}$ is by Σ? There is no reason to believe that an arbitrary complete strategy Σ would be coherent in this way, but we shall show that strong hull condensation and normalizing well guarantee it.

Given $\pi\colon P \to R$ weakly elementary, we can copy a P-stack s to an R-stack πs, until we reach an illfounded model on the πs side. Thus if Ω is a complete strategy for R, we have the complete pullback strategy Ω^π for P. We extend the construction slightly, so as to allow stronger ultrapowers on the R side than the copied ones. This will let us lift weakly normal trees to fully normal ones.

Let $\mathcal{T}$ be a weakly normal tree on the premouse P, and let $k = k(P)$. Let

$$\pi\colon P \to Q|\langle \nu, k\rangle$$

be weakly elementary; then we can copy $\mathcal{T}$ to a fully normal tree $\mathcal{U}$ on Q as follows. (We care most about the case $P = Q|\langle \nu, k\rangle$ and $\pi = \mathrm{id}$.) $\mathcal{U}$ has the same tree order as $\mathcal{T}$, so long as it is defined. Let P_α and Q_α be the α-th models, and E_α and F_α the α-th extenders, of $\mathcal{T}$ and $\mathcal{U}$. We shall have a weakly elementary

$$\pi_\alpha\colon P_\alpha \to Q_\alpha|\langle \nu_\alpha, k_\alpha\rangle.$$

Here $\pi_0 = \pi$, $\nu_0 = \nu$, and $k_0 = k$. We have the usual agreement and commutativity conditions:

(1) whenever $\beta \leq \alpha$, $\pi_\alpha \restriction \mathrm{lh}(E_\beta){+}1 = \pi_\beta \restriction \mathrm{lh}(E_\beta){+}1)$ and $Q_\alpha || \mathrm{lh}(F_\beta) = Q_\beta || \mathrm{lh}(F_\beta)$, and

(2) whenever $\beta \leq_T \alpha$, then $\pi_\alpha \circ \hat{\imath}^{\mathcal{T}}_{\beta,\alpha} = \hat{\imath}^{\mathcal{U}}_{\beta,\alpha} \circ \pi_\beta$.

(We do not demand any further coordination of the points at which the two trees drop. $\mathcal{T}$ may drop gratuitously where $\mathcal{U}$ does not, and $\mathcal{U}$ may drop where $\mathcal{T}$ does not because the dropping point is above some $\langle \nu_\alpha, k_\alpha \rangle$.) The successor step is the following. We are given E_α on M_α; set

$$F_\alpha = \pi_\alpha(E_\alpha),$$

or $F_\alpha = \dot{F}^{Q_\alpha|\langle \nu_\alpha, k_\alpha\rangle}$ if $E_\alpha = \dot{F}^{P_\alpha}$. Let $\beta = T\text{-pred}(\alpha+1) =$ least ξ such that $\kappa < \lambda(E_\xi)$, where $\kappa = \mathrm{crit}(E_\alpha)$. By (1) above, $\beta = U\text{-pred}(\alpha+1)$ according to the rules of weak normality for $\mathcal{U}$. Let

$$P_{\alpha+1} = \mathrm{Ult}(P_\beta|\langle \eta, l\rangle, E_\alpha),$$

and

$$Q_{\alpha+1} = \mathrm{Ult}(Q_\beta|\langle \gamma, n\rangle, F_\alpha),$$

where $\langle \eta, l\rangle$ is chosen by player I in $\mathcal{T}$, and $\langle \gamma, n\rangle$ is determined by normality. It is easy to see that

$$\langle \pi_\beta(\eta), l\rangle \leq_{\mathrm{lex}} \langle \gamma, n\rangle.$$

(If $\langle \eta, l\rangle = l(M_\beta)$, we understand $\pi_\beta(\eta) = \nu_\beta$ here, and we have $l = k_\beta$. Since π_β is weakly elementary, and no proper initial segment of M_β projects $\leq \kappa$, no proper initial segment of $Q_\beta|\langle \nu_\beta, l\rangle$ projects $\leq \pi_\beta(\kappa)$. But $\pi_\beta(\kappa) = \mathrm{crit}(F_\alpha)$, so $\langle \nu_\beta, l\rangle \leq_{\mathrm{lex}} \langle \gamma, n\rangle$. If $\langle \eta, l\rangle <_{\mathrm{lex}} l(P_\beta)$, a similar argument works.) We then set

$$\langle \nu_{\alpha+1}, k_{\alpha+1}\rangle = \hat{\imath}^{\mathcal{U}}_{\beta,\alpha+1}(\langle \pi_\beta(\eta), l\rangle)$$

and we have $\langle \nu_{\alpha+1}, k_{\alpha+1}\rangle \leq_{\mathrm{lex}} l(Q_{\alpha+1})$. $\pi_{\alpha+1}$ comes from the Shift Lemma definition:

$$\pi_{\alpha+1}([a, f]) = [\pi_\alpha(a), \pi_\beta(f)],$$

where the equivalence classes are in $\mathrm{Ult}(P_\beta|\langle \eta, l\rangle, E_\alpha)$ and $\mathrm{Ult}(Q_\beta|\langle \gamma, n\rangle, F_\alpha)$ respectively. The proof of the Shift Lemma tells us that $\pi_{\alpha+1}$ is weakly elementary. (Even if we had started with elementary maps, the case that $\langle \pi_\beta(\eta), l\rangle <_{\mathrm{lex}} \langle \gamma, n\rangle$ could lead to $\pi_{\alpha+1}$ not being fully elementary.)

Of course, at limit steps $\lambda < \mathrm{lh}(\mathcal{T})$, we stop unless $[0, \lambda]_T$ is a wellfounded branch of $\mathcal{U}$. If it is, we get π_λ, ν_λ and k_λ from commutativity, and continue.

Definition 4.56 *Given $\pi\colon P \to Q|\langle \nu, k\rangle$ weakly elementary and $\mathcal{T}$ on P weakly normal,*

(a) $(\pi\mathcal{T})^+$ is the normal tree on Q defined above. We call it the (π, ν, k)-lift *of $\mathcal{T}$.*

(b) *When $P = Q|\langle \nu, k\rangle$ and π is the identity, we let $\mathcal{T}^+ = (\pi\mathcal{T})^+$.*

(c) *If Ω is a strategy for Q defined on normal trees, then $\Omega^{(\pi,\nu,k)}$ is the strategy on weakly normal trees given by pulling back: $\Omega^{(\pi,\nu,k)}(\mathcal{T}) = \Omega((\pi\mathcal{T})^+)$. When $P = Q|\langle \nu, k\rangle$ and $\pi = id$, we write $\Omega^{(\nu,k)}$ for $\Omega^{(\pi,\nu,k)}$.*

With $\pi =$ identity and $\langle \nu, k\rangle = l(P)$, we get a reduction of weakly normal trees on P to fully normal trees on P, and of P-stacks to finite maximal stacks of normal trees. So

Lemma 4.57 *Let P be a premouse that is θ-iterable for normal trees; then P is θ-iterable for weakly normal trees. If P is θ-iterable for maximal, finite stacks of normal trees, then P has a complete (ω, θ)-iteration strategy.*

We don't actually need this lemma, because background constructions give us directly strategies that apply to non-maximal stacks of merely weakly normal trees.

Definition 4.58 *Let Σ be a complete strategy for P; then we say (P, Σ) is* strategy coherent *iff whenever s is a P-stack by Σ with last model Q, then*

(a) *for any $\langle \nu, k\rangle \leq l(Q)$, $(\Sigma_{s,Q})^{(\nu,k)} = \Sigma_{s,Q|\langle \nu,k\rangle}$, and*

(b) *whenever $\mathcal{T}$ is a normal tree on Q by Σ_s, and $N \trianglelefteq \mathcal{M}_\alpha^\mathcal{T}$ and $N \trianglelefteq \mathcal{M}_\beta^\mathcal{T}$, then $\Sigma_{s,\mathcal{T}\restriction(\alpha+1),N} = \Sigma_{s,\mathcal{T}\restriction(\beta+1),N}$.*

We do need the following lemma.

Lemma 4.59 *Let (M, Σ) be a pure extender pair; then Σ is strategy coherent.*

Proof. We begin with part (a). Let s be an P-stack by Σ with last model Q, let $\Omega = \Sigma_{s,Q}$, and let $R = Q|\langle \nu, k\rangle$. Let $\mathcal{T}$ be a weakly normal tree on R such that for $\mathcal{U} = \mathcal{T}^+ = (\mathrm{id}, \nu, k)\mathcal{T}$, $\mathcal{U}$ is by Ω. We must see that $\mathcal{T}$ is by Ω_R. Let $R_\alpha = \mathcal{M}_\alpha^\mathcal{T}$, $Q_\alpha = \mathcal{M}_\alpha^\mathcal{U}$, and

$$\pi_\alpha \colon R_\alpha \to Q_\alpha|\langle \nu_\alpha, k_\alpha\rangle$$

be as in the construction above.

Let $\mathcal{T}_0$ be the Q-equivalent of $\mathcal{T}$; that is, $\mathcal{T}_0$ is the weakly normal tree on Q that uses the same extenders as $\mathcal{T}$, and always drops at least as far as R when it applies an extender to its base model Q. The construction above then gives us a tree embedding Φ from $\mathcal{T}_0$ into $\mathcal{U}$. Namely, $u^\Phi = v^\Phi =$ identity, $s_\alpha^\Phi = t_\alpha^\Phi = \pi_\alpha$, and $N_\alpha^\Phi = Q_\alpha|\langle \nu_\alpha, k_\alpha\rangle$.

Since (P, Σ) is a pure extender pair, Ω has strong hull condensation. This implies that $\mathcal{T}_0$ is by Ω. But Ω is mildly positional, so $\mathcal{T}$ is by Ω_R, as desired.

We now prove (b). Let $\mathcal{T}$ be normal on Q and by Σ_s, and let $N \lhd \mathcal{M}_\alpha^\mathcal{T}$ and $N \lhd \mathcal{M}_\beta^\mathcal{T}$. Let $\Psi_0 = \Sigma_{\mathcal{T}\restriction(\alpha+1),N}$ and $\Psi_1 = \Sigma_{\mathcal{T}\restriction(\beta+1),N}$, and let $\mathcal{U}$ be a normal tree of limit length on N that is by both Ψ_0 and Ψ_1. We may assume $\alpha < \beta$. Let $\gamma \leq \alpha$ be least such that $N \trianglelefteq \mathcal{M}_\gamma^\mathcal{T}$. Then $N \trianglelefteq \mathcal{M}_\gamma^\mathcal{T}|\xi$, where $\xi = \mathrm{lh}(E_\gamma^\mathcal{T})$, so by part (a) we may as well assume $N = \mathcal{M}_\gamma^\mathcal{T}||\xi$.

Looking at the normalization process, it is easy to see by induction on $\eta < \mathrm{lh}(\mathcal{U})$ that

$$W(\mathcal{T} \restriction \gamma + 1, \mathcal{U} \restriction \eta + 1) = W(\mathcal{T} \restriction \beta + 1), \mathcal{U} \restriction \eta + 1) \restriction \phi_{0,\eta}(\gamma),$$

where $\phi_{0,\eta}$ is the u-map of the tree embedding of $\mathcal{T} \restriction \beta + 1$ into $W(\mathcal{T} \restriction \beta + 1), \mathcal{U} \restriction \eta + 1)$. So

$$W(\mathcal{T} \restriction \gamma + 1, \mathcal{U}) = W(\mathcal{T} \restriction \beta + 1), \mathcal{U}),$$

and in parallel fashion,

$$W(\mathcal{T} \restriction \gamma + 1, \mathcal{U}) = W(\mathcal{T} \restriction \alpha + 1), \mathcal{U}).$$

But then let $b_0 = \Psi_0(\mathcal{U})$ and $b_1 = \Psi_1(\mathcal{U})$, and let

$$a_i = \mathrm{br}(b_i, \mathcal{T} \restriction (\gamma + 1), \mathcal{U})).$$

Since Σ_s normalizes well, $\Sigma_s(W(\mathcal{T} \restriction (\gamma+1), \mathcal{U})) = a_i$, for $i = 0, 1$. Thus $a_0 = a_1$. By 3.73, $b_0 = b_1$, as desired. $\square$

Recall that Σ is *positional* iff whenever s and t are stacks by Σ, and N is an initial segment of the last model of each, then $\Sigma_{s,N} = \Sigma_{t,N}$. Positionality clearly implies part (b) of strategy coherence. The techniques of [60] show that normalizing well and strong hull condensation together imply positionality, but the proof is not an elementary combinatorial one like that above.

It was in order to be able to prove part (a) of strategy coherence that defined very strong hull condensation using weakly normal trees, and dealt with the small extra awkwardness this brings to the proof that background-induced strategies have strong hull condensation. We shall use Lemma 4.59 in our comparison proof for iteration strategies. One can see on heuristic grounds that it must come up somewhere; one could not hope to compare an incoherent strategy with a coherent one.

Our work in the last few sections has shown how to reduce a complete strategy that normalizes well and has strong hull condensation to its action on normal trees. We summarize this now.

Theorem 4.60 *Let (P,Σ) and (P,Ψ) be pure extender pairs with scope H_δ, and such that Σ and Ψ agree on normal trees; then $\Sigma = \Psi$.*

Proof. Note first that Σ and Ψ agree on weakly normal trees. For if $\mathcal{T}$ is by Σ and weakly normal, then $\mathcal{T}^+$ is a normal tree by Σ because (P,Σ) is strategy coherent. So $\mathcal{T}^+$ is by Ψ, and hence $\mathcal{T}$ is by Ψ because Ψ is strategy coherent.

Now suppose $\langle \mathcal{T}, \mathcal{U}\rangle$ is a P-stack by Σ. Let Q be the last model of $\mathcal{T}$ and R the last model of $\mathcal{T}^+$, and let $\pi \colon Q \to N \trianglelefteq R$ come from the copying/lifting process. Then $\langle \mathcal{T}^+, (\pi\mathcal{U})^+\rangle$ is a maximal stack by Σ, because Σ is strategy coherent. But Σ and Ψ agree on maximal stacks by Proposition 4.2, so $\langle \mathcal{T}^+, (\pi\mathcal{U})^+\rangle$ is by Ψ. Also, $\mathcal{T}$ is by Ψ and $\Psi_{\mathcal{T},Q} = (\Psi^\pi_{\mathcal{T}^+,R}$ by strong hull condensation for Ψ. But $\pi\mathcal{U}$ is a psuedo-hull of $(\pi\mathcal{U})^+$, so $\langle \mathcal{T}^+, \pi\mathcal{U}\rangle$ is by Ψ, so $\mathcal{U}$ is by $\Psi_{\mathcal{T},Q}$, so $\langle \mathcal{T}, \mathcal{U}\rangle$ by by Ψ.

Clearly, this works for finite stacks of any length, so $\Sigma = \Psi$. □

In the next chapter we shall prove a basic comparison theorem for pure extender pairs. The following terminology helps smooth the statement of this theorem.

Definition 4.61 *Let (P,Σ) and (Q,Ψ) be pure extender pairs with common scope H_θ; then*

(a) $(P,\Sigma) \trianglelefteq (Q,\Psi)$ *iff $P \trianglelefteq Q$ and $\Sigma = \Psi_P$.*

(b) $(P,\Sigma) \triangleleft (Q,\Psi)$ *iff $P \triangleleft Q$ and $\Sigma = \Psi_P$.*

(c) (P,Σ) iterates past (Q,Ψ) *iff there is a normal tree $\mathcal{T}$ on P by Σ with last model R such that $(Q,\Psi) \trianglelefteq (R, \Sigma_{\mathcal{T},R})$. If P-to-R drops, or if $Q \triangleleft R$, then we say that* (P,Σ) iterates strictly past (Q,Ψ). *If $Q = R$ and P-to-R does not drop, then we say* (P,Σ) iterates to (Q,Ψ).

Note that if (P,Σ) iterates past (Q,Ψ), then the normal tree $\mathcal{T}$ on P witnessing this is determined completely by Q and Σ: it comes from iterating away least extender disagreements, with the Q side never moving. No strategy disagreements show up along the way, because there are no strategy disagreements at the end, and (P,Σ) is strategy coherent.

We shall show that assuming AD^+, for any two pairs (P,Σ) and (Q,Ψ) with scope HC, there is a pair (R,Ω) such that either

(i) (P,Σ) iterates to (R,Ω), and (Q,Ψ) iterates past (R,Ω), or

(ii) (Q,Ψ) iterates to (R,Ω), and (P,Σ) iterates past (R,Ω).

5 Comparing iteration strategies

The standard Comparison Theorem of inner model theory applies to mice. One statement of it is

Theorem 5.1 *Let P and Q be premice of size $\leq \theta$, and suppose Σ and Ψ are $\theta^+ + 1$-iteration strategies for P and Q respectively; then there are normal trees by Σ and $\mathcal{U}$ by Ψ of size θ, with last models R and S, such that either*

(a) $R \trianglelefteq S$, and P-to-R does not drop, or

(b) $S \trianglelefteq R$, and Q-to-S does not drop.

This theorem, and the comparison process behind it, are the main engines driving inner model theory, but they have a clear defect. We haven't really compared the data. We were given (P, Σ) and (Q, Ψ), and we only compared P with Q. Whether it is the P-side or the Q-side that comes out shorter could depend on which iteration strategies for P and Q we use. (See Proposition 6.26.)

The standard way to to avoid this problem when it might arise is to make assumptions that imply P and Q can have at most one iteration strategy. This is good enough for practical purposes in many situations, but it is unnatural, and leads to somewhat awkward devices like the Weak Dodd-Jensen Lemma. The better response would be to strengthen the Comparison Theorem by finding a process which will compare all the data.

In this chapter, we shall do that. The resulting comparison process is the key to developing the theory of a class of strategy mice sufficiently rich to analyze HOD in models of $\mathsf{AD}_{\mathbb{R}}+$ NLE. This theory is the practical payoff for the work we do here, but one can see without knowing anything about HOD in models of determinacy that we are filling a gap in basic inner model theory.

We shall prove the main comparison theorem for pairs (P, Σ) such that P is a pure extender premouse, in Jensen indexing, and Σ is a complete strategy for P that normalizes well and has strong hull condensation. The proof adapts easily to ms-indexing, and to hod mice. The good behavior of Σ is needed for the argument, and it is unlikely that one could drop it as a hypothesis. It does not seem to be a restrictive hypothesis; for example, every iterable P has an iteration strategy with these properties. (See Proposition 6.25.)

The first two sections contain some preliminary lemmas. The last contains the comparison argument.

5.1 Extending tree embeddings

We shall prove an elementary lemma on the extendibility of tree embeddings. Its proof uses

Proposition 5.2 *Let $\mathcal{S}$ be a normal tree, let $\delta \leq_S \eta$, and suppose that $P \trianglelefteq \mathcal{M}_\eta^{\mathcal{S}}$, but $P \ntrianglelefteq \mathcal{M}_\sigma^{\mathcal{S}}$ whenever $\sigma <_S \delta$. Suppose also that $P \in \operatorname{ran}(\hat{\imath}_{\delta,\eta}^{\mathcal{S}})$. Let*

$$\begin{aligned} \alpha = &\ \textit{least } \gamma \textit{ such that } P \trianglelefteq \mathcal{M}_\gamma^{\mathcal{S}} \\ = &\ \textit{least } \gamma \textit{ such that } o(P) < \operatorname{lh}(E_\gamma^{\mathcal{S}}) \textit{ or } \gamma = \eta, \end{aligned}$$

and

$$\beta = \textit{least } \gamma \in [0,\eta]_S \textit{ such that } o(P) < \operatorname{crit}(\hat{\imath}_{\gamma,\eta}^{\mathcal{S}}) \textit{ or } \gamma = \eta.$$

Then $\beta \in [\delta,\eta]_S$, and

(a) either $\beta = \alpha$, or $\beta = \alpha + 1$, and $\lambda(E_\alpha^{\mathcal{S}}) \leq o(P) < \operatorname{lh}(E_\alpha^{\mathcal{S}})$;

(b) if $P = \operatorname{dom}(E_\xi^{\mathcal{S}})$, then S-pred$(\xi+1) = \alpha = \beta$.

(We allow $\delta = \eta$, with the understanding $\hat{\imath}_{\delta,\delta}$ is the identity.)

Proof. By normality, for any $\gamma < \eta$, $P \trianglelefteq \mathcal{M}_\gamma^{\mathcal{S}}$ iff $\operatorname{lh}(E_\gamma^{\mathcal{S}}) > o(P)$. So the two characterizations of α are equivalent. Clearly, $P \trianglelefteq \mathcal{M}_\beta^{\mathcal{S}}$, and thus $\alpha \leq \beta$. We have that $o(P) \geq \operatorname{lh}(E_\sigma^{\mathcal{S}})$ for all $\sigma <_S \delta$, and hence by normality, for all $\sigma <_S \delta$ whatsoever. So $\delta \leq \alpha$, and $\beta \in [\delta,\eta]_S$.

Suppose $\alpha < \beta$; then $o(P) < \operatorname{lh}(E_\alpha^{\mathcal{S}})$, so $o(P) < \operatorname{lh}(E_\sigma^{\mathcal{S}})$ where σ is least such that $\alpha \leq \sigma$ and $\sigma + 1 \leq_S \beta$. If $o(P) < \lambda(E_\sigma^{\mathcal{S}})$, then because $\delta \leq \sigma$ and $P \in \operatorname{ran}(\hat{\imath}_{\delta,\eta}^{\mathcal{S}})$, we have $o(P) < \operatorname{crit}(E_\sigma^{\mathcal{S}})$, which contradicts our definition of β. So $\lambda(E_\sigma^{\mathcal{S}}) < o(P) < \operatorname{lh}(E_\sigma^{\mathcal{S}})$. If $\operatorname{crit}(\hat{\imath}_{\sigma+1,\eta}^{\mathcal{S}}) = \lambda(E_\sigma^{\mathcal{S}})$, then P is not in $\operatorname{ran}(\hat{\imath}_{\delta,\eta}^{\mathcal{S}})$, so $\operatorname{crit}(\hat{\imath}_{\sigma+1,\eta}^{\mathcal{S}}) > o(P)$, and thus $\beta = \sigma + 1$.

This yields (a). For (b), note that if $\lambda(E_\sigma^{\mathcal{S}}) \leq o(P) < \operatorname{lh}(E_\sigma^{\mathcal{S}})$, then P cannot be the domain of an extender used in $\mathcal{S}$. So we have $\alpha = \beta$. We have already observed that S-pred$(\xi+1) = \alpha$.

□

On extending tree embeddings, we have

Lemma 5.3 *Let $\Phi = \langle u, \langle s_\beta \mid \beta \leq \alpha\rangle, \langle t_\beta \mid \beta < \alpha\rangle, p\rangle$ be a tree embedding of $\mathcal{T}$ into $\mathcal{U}$, and let F be an extender on the $\mathcal{M}_\alpha^{\mathcal{T}}$-sequence such that $\operatorname{lh}(F) > \operatorname{lh}(E_\beta^{\mathcal{T}})$ for all $\beta < \alpha$. Let $\mathcal{T}^\frown\langle F\rangle$ be the unique putative normal tree $\mathcal{S}$ extending $\mathcal{T}$ such that $F = E_\alpha^{\mathcal{S}}$. Let $\xi < \operatorname{lh}(\mathcal{U})$; then the following are equivalent:*

(1) There is a tree embedding Ψ *of* $\mathcal{T}^\frown\langle F\rangle$ *into* $\mathcal{U}$ *such that* $\Phi \subseteq \Psi$ *and* $u^\Psi(\alpha) = \xi$*,*

(2) $v(\alpha) \leq_U \xi$*, and* $E^{\mathcal{U}}_\xi = \hat{\imath}^{\mathcal{U}}_{v(\alpha),\xi} \circ s_\alpha(F)$*.*

Moreover, there is at most one such Ψ*.*

Proof. It is easy to see from definition 3.27 that (1) implies (2).

Suppose that ξ witnesses that (2) holds. Set $u(\alpha) = \xi$ and $t_\alpha = \hat{\imath}^{\mathcal{U}}_{v(\alpha),\xi} \circ s_\alpha$. Clearly,

$$t_\alpha \restriction \lambda^{\mathcal{T}}_\alpha = s_\alpha \restriction \lambda^{\mathcal{T}}_\alpha,$$

and

$$\mathrm{crit}(\hat{\imath}^{\mathcal{U}}_{v(\alpha),\xi}) \geq \lambda^{\mathcal{U}}_{v(\alpha)}.$$

Let $p(F) = G = E^{\mathcal{U}}_\xi$. We shall find $s_{\alpha+1}$ such that $\Psi = \langle u, \langle s_\beta \mid \beta \leq \alpha+1\rangle, \langle t_\beta \mid \beta \leq \alpha\rangle, p\rangle$ is a tree embedding of $\mathcal{S} = \mathcal{T}^\frown\langle F\rangle$ into $\mathcal{U}$.

Let $\mu = \mathrm{crit}(F)$ and $\mu^* = \mathrm{crit}(G)$. Let

$$\beta = S\text{-pred}(\alpha+1) = \text{ least } \eta \text{ s.t. } \mu < \lambda^{\mathcal{T}}_{\eta+1},$$

and

$$\beta^* = U\text{-pred}(\xi+1) = \text{ least } \eta \text{ s.t. } \mu^* < \lambda^{\mathcal{U}}_{\eta+1}.$$

Let $\gamma = (\mu^+)^{\mathcal{M}^{\mathcal{T}}_\alpha | \mathrm{lh}(F)}$ and $P = \mathcal{M}^{\mathcal{T}}_\alpha | \gamma$. Similarly, let $\gamma^* = (\mu^{*,+})^{\mathcal{M}^{\mathcal{U}}_\xi | \mathrm{lh}(G)}$ and $P^* = \mathcal{M}^{\mathcal{U}}_\xi | \gamma^*$. So P is the domain of F (the sets measured by it), P^* is the domain of G, and $t_\alpha(P) = P^*$. The rules of normality tell us that

$$\beta = \text{ least } \eta \text{ s.t. } P = \mathcal{M}^{\mathcal{T}}_\eta | \gamma,$$

and

$$\beta^* = \text{ least } \eta \text{ s.t. } P^* = \mathcal{M}^{\mathcal{U}}_\eta | \gamma^*.$$

(P and P^* are passive, so these identities imply that γ and γ^* are passive stages in $\mathcal{M}^{\mathcal{T}}_\beta$ and $\mathcal{M}^{\mathcal{U}}_{\beta^*}$.) Suppose first that $\beta < \alpha$. We then have that $\mu < \lambda^{\mathcal{T}}_\alpha$, so

$$\begin{aligned} P^* &= t_\alpha(P) \\ &= s_\alpha(P) \\ &= t_\beta(P) \\ &= \hat{\imath}^{\mathcal{U}}_{v(\beta),u(\beta)} \circ s_\beta(P), \end{aligned}$$

where the last equalities hold because $\mu < \lambda_{E^{\mathcal{T}}_\beta}$. Thus P^* is in the range of $\hat{\imath}^{\mathcal{U}}_{v(\beta),u(\beta)}$. Proposition 5.2, with $\delta = v(\beta)$, $\eta = u(\beta)$, and P^* as its P then tells us that

$$\beta^* = \text{least } \eta \in [v(\beta), u(\beta)]_U \text{ such that crit } \hat{\imath}^{\mathcal{U}}_{\eta,u(\beta)} > \hat{\imath}^{\mathcal{U}}_{v(\beta),\eta} \circ s_\beta(\mu).$$

Let Q be the first level of $\mathcal{M}^{\mathcal{T}}_\beta$ beyond P that projects to or below μ, and let Q^* be the first level of $\mathcal{M}^{\mathcal{U}}_{\beta^*}$ beyond P^* that projects to or below μ^*. So $\mathcal{M}^{\mathcal{T}}_{\alpha+1} = \text{Ult}(Q, F)$ and $\mathcal{M}^{\mathcal{U}}_{\xi+1} = \text{Ult}(Q^*, G)$. Let

$$\rho = (\hat{\imath}^{\mathcal{U}}_{v(\beta),\beta^*} \circ s_\beta)\restriction Q.$$

We have that

$$\rho\restriction P = t_\beta\restriction P = s_\alpha\restriction P = t_\alpha\restriction P.$$

We can then set

$$s_{\alpha+1}([a, f]^Q_F) = [t_\alpha(a), \hat{\imath}^{\mathcal{U}}_{v(\beta),\beta^*} \circ s_\beta(f)]^{Q^*}_G,$$

as we are required to do by definition 3.27, and the Shift Lemma tells us that $s_{\alpha+1}$ as defined is indeed well-defined, elementary, and agrees with t_α as required in a tree embedding.

We must check clause (b) of definition 3.27. The new case involves F and G; we must see that $E \in \text{ran}(e^{\mathcal{S}}_\beta)$ iff $p(E) \in e^{\mathcal{U}}_{\beta^*}$. But for $E \in \text{Ext}(\mathcal{T})$,

$$\begin{aligned} E \in \text{ran}(e^{\mathcal{T}}_\beta) &\Leftrightarrow p(E) \in e^{\mathcal{U}}_{v(\beta)} \\ &\Leftrightarrow p(E) \in \text{ran}(e^{\mathcal{U}}_{\beta^*}). \end{aligned}$$

The right-to-left implication in line 2 holds because if $E \notin \text{ran}(e^{\mathcal{T}}_\beta)$ and $\text{lh}(E) < \text{lh}(E^{\mathcal{T}}_\beta)$, then E is incompatible with some $H \in \text{ran}(e^{\mathcal{T}}_\beta)$, so $p(E)$ is incompatible with $p(H) \in e^{\mathcal{U}}_{v(\beta)}$, so the right hand side of line 2 fails. On the other hand, if $\text{lh}(E) \geq \text{lh}(E^{\mathcal{T}}_\beta)$, then $\text{lh}(p(E)) \geq \text{lh}(p(E^{\mathcal{T}}_\beta)) = \text{lh}(E^{\mathcal{U}}_{u(\beta)})$, and since $\beta^* \leq u(\beta)$, again the right hand side of line 2 fails.

The case that $\alpha = \beta$ is similar. In this case, we apply the proposition to P^* with $\delta = v(\beta)$ and $\eta = \xi$. This gives us that

$$\beta^* = \text{least } \eta \in [v(\beta), \xi]_U \text{ such that crit } \hat{\imath}^{\mathcal{U}}_{\eta,\xi} > \hat{\imath}^{\mathcal{U}}_{v(\beta),\eta} \circ s_\beta(\mu).$$

We leave the remaining details to the reader. □

Remark 5.4 The proof gives a formula for the point of application of $E^{\mathcal{U}}_{u(\alpha)}$ under a tree embedding of $\mathcal{T}$ into $\mathcal{U}$, namely

$$U\text{-pred}(u(\alpha)+1) = \text{least } \eta \in [v(\beta), u(\beta)]_U \text{ such that crit } \hat{\imath}^{\mathcal{U}}_{\eta,u(\beta)} > \hat{\imath}^{\mathcal{U}}_{v(\beta),\eta} \circ s_\beta(\mu),$$

where

$$\beta = T\text{-pred}(\alpha+1) \text{ and } \mu = \text{crit}(E^{\mathcal{T}}_\alpha).$$

Remark 5.5 One can have the following situation, for $F = E^{\mathcal{T}}_\alpha$:

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{T}}_\alpha \arrow[d, "s_\alpha"'] \arrow[dr, "\rho"] \arrow[drr, "t_\alpha"] & & \\
\mathcal{M}^{\mathcal{U}}_{v(\alpha)} \arrow[r] & \mathcal{M}^{\mathcal{U}}_\gamma \arrow[r, "\hat{\imath}_{\gamma,u(\alpha)}"'] & \mathcal{M}^{\mathcal{U}}_{u(\alpha)}
\end{tikzcd}$$

It can happen that $\operatorname{dom}\rho = \mathcal{M}^{\mathcal{T}}_\alpha$, but $\operatorname{dom} t_\alpha = \mathcal{M}^{\mathcal{T}}_\alpha \restriction \operatorname{lh} F$, so $t_\alpha(F)$ is the last extender of $\mathcal{M}^{\mathcal{U}}_{u(\alpha)}$. In this case, $\hat{\imath}_{\gamma,u(\alpha)}$ is acting like a resurrection embedding, resurrecting $\rho(F)$, and $(\gamma, u(\alpha)]_U$ drops.

5.2 Resurrection embeddings as branch embeddings

We prove a technical lemma on normal iterations past levels of a background construction.

Let Σ be an iteration strategy for the premouse P_0, for finite stacks of normal trees, that normalizes well and has strong hull condensation. Suppose that Σ is universally Baire. Let $\mathbb{C}$ be a background construction above $|P_0|^+$, and $\langle \nu_0, k_0\rangle <$ length($\mathbb{C}$). Let us write $M_{\nu,k} = M^{\mathbb{C}}_{\nu,k}$. Suppose that whenever $\langle \nu, k\rangle <_{\text{lex}} \langle \nu_0, k_0\rangle$, $M^{\mathbb{C}}_{\nu,k}$ is not a Σ-iterate of P_0. It has been known since the mid-80s that whenever $\langle \nu, k\rangle \leq_{\text{lex}} \langle \nu_0, k_0\rangle$, only the P_0 side moves if we compare it with $M_{\nu,k}$ by least disagreement, using Σ to pick branches. See Lemma 2.52.

Thus for $\langle \nu, k\rangle \leq_{\text{lex}} \langle \nu_0, k_0\rangle$, we have

$$\begin{aligned}\mathcal{W}^*_{\nu,k} &= \text{unique shortest normal tree on } P_0 \text{ by } \Sigma \\ &\quad \text{with last model } Q \trianglerighteq M_{\nu,k}.\end{aligned}$$

Our technical lemma says that below $\langle \nu_0, k_0\rangle$, the resurrection embeddings of $\mathbb{C}$ are captured by branch embeddings of the $\mathcal{W}^*_{\nu,k}$.

Lemma 5.6 *Let* $\langle \theta, j\rangle \leq \langle \nu_0, k_0\rangle$*, and let* $P \trianglelefteq M^{\mathbb{C}}_{\theta,j}$*. Let* $\tau = \sigma_{\theta,j}[P]^{\mathbb{C}}$*, so that* $\tau : P \to M^{\mathbb{C}}_{\theta_0,j_0}$*, where* $\langle \theta_0, j_0\rangle = \mathrm{Res}_{\theta,j}[P]$*. Let*

$$\mathcal{T} = \mathcal{W}^*_{\theta,j}{\restriction}(\alpha+1), \text{ where } \alpha \text{ is least such that } \mathcal{M}^{\mathcal{W}^*_{\theta,j}}_\alpha \trianglerighteq P.$$

Then $\mathcal{T} = \mathcal{W}^*_{\theta_0,j_0}{\restriction}(\alpha+1)$*,* $\mathcal{W}^*_{\theta_0,j_0}$ *has last model* $\mathcal{M}^{\mathcal{W}^*_{\theta_0,j_0}}_\xi = M^{\mathbb{C}}_{\theta_0,j_0}$*, and* $\alpha \leq_{\mathcal{W}^*_{\theta_0,j_0}} \xi$*, and* $\tau = \hat{\imath}^{\mathcal{W}^*_{\theta_0,j_0}}_{\alpha,\xi}$*.*

We remark that our convention that $P \ntrianglelefteq Q$ when Q is active and $P = Q||o(Q)$ matters here. It could be that for α as in the lemma, $E = E^{\mathcal{W}^*_{\theta,j}}_{\alpha-1}$ is such that $\mathrm{lh}(E) = o(P)$. The resurrection embedding τ is given by a branch of $\mathcal{W}^*_{\theta_0,j_0}$ that has α in it, and may not have $\alpha-1$ in it, even though P is an initial segment of $\mathcal{M}^{\mathcal{W}^*_{\theta,j}}_{\alpha-1}$ in a weaker sense.

Definition 5.7 *If* M *is a premouse such that* $k(M) > 0$*, then* M^- *is the premouse that is equal to* M*, except that* $k(M^-) = k(M) - 1$*.*

Sublemma 5.7.1 *Suppose that* $M_{\nu,k}$ *is not* $k+1$*-sound. Let* $\pi : M^-_{\nu,k+1} \to M_{\nu,k}$ *be the anticore embedding. Let* $\xi_0 + 1 = \mathrm{lh}\,\mathcal{W}^*_{\nu,k+1}$*; then*

(*a*) $\mathcal{W}^*_{\nu,k}$ *has last model* $M_{\nu,k}$*,*

(*b*) $\mathcal{W}^*_{\nu,k+1} = \mathcal{W}^*_{\nu,k}{\restriction}(\xi_0+1)$*,*

(*c*) ξ_0 *is the least* γ *such that* $\mathrm{lh}\, E^{\mathcal{W}^*_{\nu,k}}_\gamma > \rho(M_{\nu,k})$*, and*

(*d*) *letting* $lh(W^*_{\nu,k}) = \xi_1 + 1$*, we have* $\xi_0 <_{\mathcal{W}^*_{\nu,k}} \xi_1$*, and* $\hat{\imath}^{\mathcal{W}^*_{\nu,k}}_{\xi_0,\xi_1} = \pi$*.*

Proof.

By definition, $\mathcal{M}^{\mathcal{W}^*_{\nu,k}}_{\xi_1} \trianglerighteq M_{\nu,k}$. But $M_{\nu,k}$ is not sound ($= k+1$-sound), so $\mathcal{M}^{\mathcal{W}^*_{\nu,k}}_{\xi_1} = M_{\nu,k}$. This gives (*a*).

The iteration $\mathcal{W}^*_{\nu,k}$ from P_0 to $M_{\nu,k}$ must have dropped. The last drop had to be to $M_{\nu,k+1}$, and it lies on the branch to $M_{\nu,k}$. So we can fix η such that

$$M_{\nu,k+1} = \mathrm{dom}\, \hat{\imath}^{W^*_{\nu,k}}_{\eta,\xi_1}, \text{ and } \hat{\imath}^{\mathcal{W}^*_{\nu,k}}_{\eta,\xi_1} = \pi.$$

We have that $M_{\nu,k+1} \trianglelefteq M^{W^*_{\nu,k}}_\eta$.

Letting $\rho = \rho(M_{\nu,k})$, we have that $M_{\nu,k+1}$ agrees with $M_{\nu,k}$ to $\rho^{+M_{\nu,k}} = \rho^{+M_{\nu,k+1}}$. Thus $\mathcal{W}^*_{\nu,k+1}$ and $\mathcal{W}^*_{\nu,k}$ use the same extenders E such that $\operatorname{lh} E \leq \rho$.

We claim that $\mathcal{W}^*_{\nu,k+1}$ uses no extenders E such that $\operatorname{lh}(E) > \rho$. For if $\mathcal{W}^*_{\nu,k+1}$ uses E such that $\operatorname{lh} E > \rho$, then the branch P_0-to-$\mathcal{M}_{\xi_0}^{\mathcal{W}^*_{\nu,k+1}}$ uses such an E, since $\xi_0 + 1 = \operatorname{lh} W^*_{\nu,k+1}$. $\operatorname{lh}(E) \leq o(M_{\nu,k+1})$ because $\mathcal{W}^*_{\nu,k+1}$ was of minimal length. But then $\rho \leq \operatorname{crit}(E)$ is impossible, because $\operatorname{dom}(E) \subseteq M_{\nu,k+1}$, and $M_{\nu,k+1}$ is sound. However, $\operatorname{crit}(E) < \rho$ is also impossible, since no model on the branch $[0, \xi_0]$ after E can project into $(\operatorname{crit}(E), \operatorname{lh} E)$.

So we have that $\mathcal{W}^*_{\nu,k+1} = \mathcal{W}^*_{\nu,k} \restriction \xi_0 + 1$. We have (a)-(c) of the sublemma already. For (d), we need to see $\xi_0 = \eta$. Since $M_{\nu,k+1} \trianglelefteq \mathcal{M}_\eta^{\mathcal{W}^*_{\nu,k}}$, $\xi_0 \leq \eta$. Suppose toward contradiction that $\xi_0 < \eta$. We then have that $o(M_{\nu,k+1}) \leq \operatorname{lh}(E_{\xi_0}^{\mathcal{W}^*_{\nu,k}})$ because $M_{\nu,k+1}$ is an initial segment of both $\mathcal{M}_{\xi_0}^{\mathcal{W}^*_{\nu,k}}$ and $\mathcal{M}_\eta^{\mathcal{W}^*_{\nu,k}}$. But let $\theta+1$ be the successor of η on the branch $[0, \xi_1]$ of $\mathcal{W}^*_{\nu,k}$, that is, $W^*_{\nu,k}$-pred$(\theta+1) = \eta$ and $\theta + 1 \leq_{W^*_{\nu,k}} \xi_1$. Then $M_{\nu,k+1} = (\mathcal{M}^*_{\theta+1})^{\mathcal{W}^*_{\nu,k}}$, and so $\operatorname{lh}(E_\eta^{\mathcal{W}^*_{\nu,k}}) \leq o(M_{\nu,k+1}) \leq \operatorname{lh}(E_{\xi_0}^{\mathcal{W}^*_{\nu,k}})$. Thus $\eta \leq \xi_0$, so $\eta \leq \xi_0$, a contradiction. $\square$

Proof. [Proof of Lemma 5.6] We go by induction on $\langle \theta, j \rangle$. Suppose Lemma 5.6 holds for $\langle \theta', j' \rangle <_{\text{lex}} \langle \theta, j \rangle$, as well as for all $Q \triangleleft P$, where $P \trianglelefteq M_{\theta,j}$. Let

$$\begin{aligned} \rho = \text{least } \kappa \text{ such that } \kappa = \rho_n(S) \text{ for some } S \trianglelefteq M_{\theta,j} \\ \text{such that } P \trianglelefteq S, \text{ and } n = k(S). \end{aligned}$$

(Here we do *not* mean $\kappa = \rho(S) = \rho_{n+1}(S)$, where $n = k(S)$.) Pick S to be the first such. We can assume that $\rho < o(P)$, as otherwise $\tau =$ identity, and all is trivial. Thus $k(S) > 0$.

The reader can check that $\sigma_{\theta,j}[S] \restriction P = \sigma_{\theta,j}[P] = \tau$. If $S \triangleleft M_{\theta,j}$, then we can find some $\langle \theta', j' \rangle <_{\text{lex}} \langle \theta, j \rangle$ such that $S = M_{\theta',j'}$. [Let $\langle \nu, k \rangle$ be least such that $S \trianglelefteq M_{\nu,k}$, and assume toward contradiction that $S \neq M_{\nu,k}$. We must have $k > 0$, as otherwise $S \trianglelefteq M_{\eta,j}$ for some $\eta < \nu$. Since $S \ntrianglelefteq M_{\nu,k-1}$, we have $M^-_{\nu,k} \neq M_{\nu,k-1}$, that is, the coring down is nontrivial. We must have $\rho_k(M_{\nu,k-1}) < \rho$, because $\rho_{k(S)}(S) = \rho$, and $S \triangleleft M_{\nu,k}$, so if $\rho \leq \rho_k(M_{\nu,k-1})$, then $S \trianglelefteq M_{\nu,k-1}$. so $\rho_k(M_{\nu,k-1}) < \rho$. $P \trianglelefteq M_{\nu,k}$ and $\rho_k(M_{\nu,k}) < \rho$, contrary to our definition of ρ.]

The argument above also shows that $\sigma_{\theta,j}[S] = \sigma_{\theta',j'}[S]$. So we can apply our induction hypothesis at θ', j'. Note that $\mathcal{W}^*_{\theta,j} \restriction (\alpha+1) = \mathcal{W}^*_{\theta',j'} \restriction (\alpha+1)$.

Thus we may assume $S = M_{\theta,j}$. So $j = k(S)$ and $j > 0$. If $\sigma_{\theta,j}[S] = \sigma_{\theta,j-1}[S]$, then as $\langle \theta, j-1 \rangle <_{\text{lex}} \langle \theta, j \rangle$, our induction hypothesis carries the day. Otherwise, we have that $M_{\theta,j-1}$ is not sound. Moreover

$$\sigma_{\theta,j}[S] = \pi \circ \sigma_{\theta,j-1}[S],$$

where $\pi : M^-_{\theta,j} \to M_{\theta,j-1}$ is the anticore embedding.

Let $\alpha + 1 = \operatorname{lh} \mathcal{W}^*_{\theta,j}$ and $\beta + 1 = \operatorname{lh} \mathcal{W}^*_{\theta,j-1}$. By the sublemma, $S \trianglelefteq \mathcal{M}_\alpha^{\mathcal{W}^*_{\theta,j}}$ and $M_{\theta,j-1} = \mathcal{M}_\beta^{W^*_{\theta,j-1}}$, $\alpha \leq_{W^*_{\theta,j-1}} \beta$, and

$$\pi = \hat{\imath}_{\alpha,\beta}^{\mathcal{W}^*_{\theta,j}}.$$

Also, $\mathcal{W}^*_{\theta,j}$ uses only extenders of $\operatorname{lh} \leq \rho$, so α is the least γ such that $P \trianglelefteq \mathcal{M}_\gamma^{W^*_{\theta,j}}$.

Remark 5.8 The reason that the statement of Lemma 5.6 does not have $\alpha + 1 = \operatorname{lh} \mathcal{W}^*_{\theta,j}$ is that that is clearly not always true. It becomes true when we reduce $\langle \theta, j \rangle$ to a $\langle \theta', j' \rangle$ with $S = M_{\theta',j'}$.

Let $P_1 = \pi(P)$. Let

$$\alpha_1 = \text{least } \gamma \text{ such that } P_1 \trianglelefteq \mathcal{M}_\gamma^{\mathcal{W}^*_{\theta,j-1}}.$$

We can assume $\operatorname{crit}(\pi) \leq o(P)$, as otherwise $P \trianglelefteq M_{\theta,j-1}$ and $\tau = \sigma_{\theta,j-1}[P]$, so we are done by induction.

Claim 5.8.1 *$\alpha <_{\mathcal{W}^*_{\theta,j-1}} \alpha_1 \leq_{W^*_{\theta,j-1}} \beta$.*

Proof. Let $\gamma \in (\alpha, \beta]_{\mathcal{W}^*_{\theta,j-1}}$ be least such that $o(P_1) < \operatorname{crit}(\hat{\imath}_{\gamma,\beta}^{\mathcal{W}^*_{\theta,j-1}})$. We claim that $\alpha_1 = \gamma$. Certainly, $P_1 \trianglelefteq \mathcal{M}_\gamma^{\mathcal{W}^*_{\theta,j-1}}$. Also, $P_1 \ntrianglelefteq \mathcal{M}_\alpha^{\mathcal{W}^*_{\theta,j-1}}$. Since P_1 is in the range of $\hat{\imath}_{\alpha,\beta}^{\mathcal{W}^*_{\theta,j-1}}$, we get $\alpha_1 = \gamma$. See the proof of Proposition 5.2. $\square$

The claim also showed that

$$\pi \restriction P = \hat{\imath}_{\theta,\alpha_1} \restriction P.$$

Now we apply our induction hypothesis to $P_1 \trianglelefteq M_{\theta,j-1}$. We get θ_0, j_0 such that

1. $\mathcal{W}^*_{\theta_0,j_0} \restriction (\alpha_1 + 1) = W^*_{\theta,j-1} \restriction (\alpha_1 + 1)$.
2. $\mathcal{W}^*_{\theta_0,j_0}$ has last model $M_{\theta_0,j_0} = \mathcal{M}_\xi^{\mathcal{W}^*_{\theta_0,j_0}}$, and
3. $\alpha_1 \leq_{W^*_{\theta_0,j_0}} \xi$, and $\sigma_{\theta,j-1}[P_1] = \hat{\imath}_{\alpha_1,\xi}^{\mathcal{W}^*_{\theta_0,j_0}}$.

But $\sigma_{\theta,j}[P] = \sigma_{\theta,j-1}[P_1] \circ \pi$. This yields $\sigma_{\theta,j}[P] = \hat{\imath}_{\alpha_1,\xi}^{\mathcal{W}^*_{\theta_0,j_0}} \circ \hat{\imath}_{\alpha,\alpha_1}^{\mathcal{W}^*_{\theta_0,j_0}} = \hat{\imath}_{\alpha,\xi}^{\mathcal{W}^*_{\theta_0,j_0}}$, as desired. $\square$

5.3 Iterating into a backgrounded strategy

In this section we prove the basic comparison theorem for pure extender pairs. In the next chapter we shall generalize it to least branch hod pairs, but all the main ideas occur in the pure extender proof.

The proof is based on proving $(*)(P,\Sigma)$, for pure extender pairs (P,Σ). This involves iterating (P,Σ) to a level $(M_{\nu,k},\Omega_{\nu,k})$ of some background construction $\mathbb{C}$. In the statement of $(*)(P,\Sigma)$, $\mathbb{C}$ is the construction of some coarse Γ-Woodin background universe N^* that captures Σ, but here we shall assume somewhat less about $\mathbb{C}$.

Definition 5.9 *Let $\vec{F}$ be coarsely coherent; then $\Omega^{\mathsf{UBH}}_{\vec{F}}$ is the partial iteration strategy for V: if $\vec{\mathcal{T}}^\frown\langle\mathcal{U}\rangle$ is a finite stack of normal $\vec{F}$-trees by $\Omega^{\mathsf{UBH}}_{\vec{F}}$ such that $\mathcal{U}$ has limit length, then*

$$\Omega^{\mathsf{UBH}}_{\vec{F}}(\vec{\mathcal{T}}^\frown\langle\mathcal{U}\rangle)=b \quad \text{iff} \quad b \textit{ is the unique cofinal, wellfounded branch of } \mathcal{U}.$$

So if V is strongly uniquely iterable for finite stacks of normal $\vec{F}$-trees, then $\Omega^{\mathsf{UBH}}_{\vec{F}}$ is total, and it is the unique iteration strategy witnessing this. Moreover, $\Omega^{\mathsf{UBH}}_{\vec{F}}$ normalizes well, and has strong hull condensation. The results of Chapter 3 show that this is the case if V is a coarse Γ-Woodin model, and $\vec{F}$ its distinguished coherent sequence, and under other hypotheses as well. But our notation allows the case that $\Omega^{\mathsf{UBH}}_{\vec{F}}$ is partial. $\Omega^{\mathsf{UBH}}_{\vec{F}}(\vec{\mathcal{T}}^\frown\langle\mathcal{U}\rangle)$ can fail to be defined because $\mathcal{U}$ has no cofinal wellfounded branch, or because it has more than one cofinal wellfounded branch.

Definition 5.10 *Let $\mathbb{C}$ be a background construction above κ. Suppose $M_{\nu,k}$ exists. Then $\Omega^{\mathbb{C}}_{\nu,k}$ is the partial strategy for $M_{\nu,k}$ induced by $\Omega^{\mathsf{UBH}}_{\vec{F}^{\mathbb{C}}}$, i.e.*

$$\vec{\mathcal{T}} \textit{ is by } \Omega^{\mathbb{C}}_{\nu,k} \quad \text{iff} \quad \text{lift}(\vec{\mathcal{T}},M_{\nu,k},\mathbb{C}) \textit{ is by } \Omega^{\mathsf{UBH}}_{\vec{F}^{\mathbb{C}}},$$

whenever $\vec{\mathcal{T}}$ is a finite stack of weakly normal trees on $M_{\nu,k}$.

So if V is strongly uniquely $(\omega,\theta,\vec{F}^{\mathbb{C}})$-iterable above κ, then $\Omega^{\mathbb{C}}_{\nu,k}$ is a complete strategy with scope H_θ that normalizes well and has strong hull condensation.

The following is essentially Theorem 1.15, but in the pure extender model case.

Theorem 5.11 *Let (P,Σ) be a pure extender pair with scope H_δ, where δ is inaccessible. Let $\mathbb{C}$ be a background construction above $|P|^+$ such that all $F^{\mathbb{C}}_\nu$ are in H_δ. Let $\langle\nu,k\rangle<\text{lh}(\mathbb{C})$, and suppose that (P,Σ) iterates past $(M^{\mathbb{C}}_{\eta,j},\Omega^{\mathbb{C}}_{\eta,j})$, for all $\langle\eta,j\rangle<_{\text{lex}}\langle\nu,k\rangle$; then (P,Σ) iterates past $(M^{\mathbb{C}}_{\nu,k},\Omega^{\mathbb{C}}_{\nu,k})$.*

Remark 5.12 Σ is total so if (P, Σ) iterates past $(M_{\nu,k}^{\mathbb{C}}, \Omega_{\nu,k}^{\mathbb{C}})$, then $\Omega_{\nu,k}^{\mathbb{C}}$ is total. So although did not assume unique iterability in the hypothesis of Theorem 5.11, we got the $\Omega_{\eta,l}^{\mathbb{C}}$ are total, until we reach an $M_{\nu,k}$ that is beyond Σ. Before that point, $\mathbb{C}$-lifted trees have unique cofinal wellfounded branches.

Theorem 5.11 yields at once a comparison theorem for pure extender pairs. The following is the pure extender case of our main strategy comparison theorem.

Theorem 5.13 *(Pure extender mouse pair comparison) Assume* AD^+*, and let* (P, Σ) *and* (Q, Ψ) *be pure extender pairs with scope* H_{ω_1}*; then there are countable normal trees* $\mathcal{T}$ *on* P *and* $\mathcal{U}$ *on* Q *by* Ψ*, with last models* R *and* S *respectively, such that either*

1. *P-to-R does not drop, $R \trianglelefteq S$, and $\Sigma_{\mathcal{T},R} = \Psi_{\mathcal{U},R}$, or*
2. *Q-to-S does not drop, $S \trianglelefteq R$, and $\Psi_{\mathcal{U},S} = \Sigma_{\mathcal{T},S}$.*

Proof. By the Basis Theorem of AD^+, we may assume that Code(Σ) and Code(Ψ) are Suslin and co-Suslin. (The paper [63] shows this directly, assuming only AD.) So we have a coarse Γ-Woodin tuple $(N^*, \lhd, S, T, \Sigma^*)$, where Γ is a pointclass big enough that Σ and Ψ are coded by sets of reals in Γ. We may assume P and Q are in N^*, and countable there. Let $\mathbb{C}$ be the maximal $\lhd$-construction of N^*. Since we are in the pure extender case, we have that $\mathbb{C}$ does not break down.

We now apply Theorem 4.20. This gives us a $\langle \nu, k\rangle$ such that $M_{\nu,k}^{\mathbb{C}}$ is a Σ-iterate of P, and P iterates by Σ past $M_{\eta,j}^{\mathbb{C}}$, for each $\langle \eta, j\rangle <_{\text{lex}} \langle \nu, k\rangle$. Similarly we have $\langle \mu, l\rangle$ such that $M_{\mu,l}^{\mathbb{C}}$ is a Ψ-iterate of Q, and Q iterates by Ψ past $M_{\eta,j}^{\mathbb{C}}$, for each $\langle \eta, j\rangle <_{\text{lex}} \langle \nu, k\rangle$. By Theorem 5.11, no strategy disagreements with the strategies in $\mathbb{C}$ show up in these iterations. So if $\langle \nu, k\rangle \leq_{\text{lex}} \langle \mu, l\rangle$, then by Theorem 5.11, we get conclusion (1), with $R = M_{\nu,k}^{\mathbb{C}}$ and $\Sigma_{\mathcal{T},R} = \Omega_{\nu,k}^{\mathbb{C}}$. If $\langle \mu, l\rangle \leq_{\text{lex}} \langle \nu, k\rangle$, then we get conclusion (2).

Let $\mathcal{T}, \mathcal{U}, R$, and S witness in N^* that either (1) or (2) holds. $\mathcal{T}$ and $\mathcal{U}$ are countable in V, and N^* is sufficiently correct that either (1) or (2) holds in V.

$\square$

Remark 5.14 When we generalize the comparison theorem for pure extender pairs to strategy mouse pairs in Chapter 5, we shall have to re-organize the proof a bit. Lemma 2.52 and Theorem 4.20 don't help in the strategy mouse context, so in effect we must prove the analogs of both Theorem 5.11 and Lemma 2.52 as part of one induction.

The rest of this chapter is devoted to the proof of Theorem 5.11.

Proof. [Proof of Theorem 5.11] The proof is by induction on $\langle \nu, k\rangle$. Suppose that (P_0, Σ) iterates past $M^{\mathbb{C}}_{\nu,k}$ for all $\langle \nu, k\rangle <_{\text{lex}} \langle \nu_0, k_0\rangle$. For $\langle \nu, k\rangle \leq_{\text{lex}} \langle \nu_0, k_0\rangle$, let

$$\begin{aligned}\mathcal{W}^*_{\nu,k} = &\text{ unique shortest normal tree on } P_0 \text{ by } \Sigma\\ &\text{ with last model } Q \trianglerighteq M_{\nu,k}.\end{aligned}$$

Let $M = M_{\nu_0,k_0}$, and let $\mathcal{U}$ be a normal tree on M that is of limit length, and is by both $\Sigma_{\mathcal{W}^*_{\nu_0,k_0},M}$ and $\Omega^{\mathbb{C}}_{\nu_0,k_0}$. Let

$$\text{lift}(\mathcal{U}, M, \mathbb{C}) = \langle \mathcal{U}^*, \langle \eta_\tau, l_\tau \mid \tau < \text{lh}\,\mathcal{U}\rangle, \langle \psi^{\mathcal{U}}_\tau \mid \tau < \text{lh}\,\mathcal{U}\rangle\rangle.$$

Lemma 5.15 *If b is a cofinal, wellfounded branch of $\mathcal{U}^*$, then $\Sigma_{\mathcal{W}^*_{\nu_0,k_0},M}(\mathcal{U}) = b$.*

Lemma 5.15 implies that $\mathcal{U}^*$ has at most one cofinal wellfounded branch. Moreover, that branch is identified by Σ, if it exists, and Σ is universally Baire. So a simple reflection argument will then give that $\mathcal{U}^*$ has a cofinal, wellfounded branch. From this we get that $\Sigma_{\mathcal{W}^*_{\nu_0,k_0},M}$ and $\Omega^{\mathbb{C}}_{\nu_0,k_0}$ agree on normal trees, and then by the proof of Theorem 4.60, they must agree on finite stacks of normal trees. (If we were assuming $\Omega^{\mathbb{C}}_{\nu_0,k_0}$ is total, we could simply quote 4.60 at this point.)

Proof. [Proof of Lemma 5.15] Let

$$\begin{aligned}S_\gamma &= \mathcal{M}^{\mathcal{U}^*}_\gamma\\ N^0_\gamma &= M^{S_\gamma}_{\eta_\gamma,l_\gamma} = M^{i^{\mathcal{U}^*}_{0,\gamma}(\mathbb{C})}_{\eta_\gamma,l_\gamma},\end{aligned}$$

so that

$$\psi^{\mathcal{U}}_\gamma : \mathcal{M}^{\mathcal{U}}_\gamma \to N^0_\gamma$$

is elementary. We have $M = \mathcal{M}^{\mathcal{U}}_0 = N^0_0$, and $\psi^{\mathcal{U}}_0 =$ identity. We write $(\mathcal{W}^*_{\nu,k})^{S_\gamma}$ for $\langle \nu, k\rangle \leq_{\text{lex}} i^{\mathcal{U}^*}_{0,\gamma}(\langle \nu_0, k_0\rangle)$ to stand for $i^{\mathcal{U}^*}_{0,\gamma}(\langle \eta, l\rangle \mapsto \mathcal{W}^*_{\eta,l})_{\nu,k}$. Note that

$$i^{\mathcal{U}^*}_{0,\gamma}(\Sigma) \cap S_\gamma = \Sigma \cap S_\gamma,$$

by Lemma 4.55. Also $i^{\mathcal{U}^*}_{0,\gamma}(P_0) = P_0$. Thus $(\mathcal{W}^*_{\nu,k})^{S_\gamma}$ is by Σ.

The statements above also make sense for b replacing γ. So $S_b = \mathcal{M}^{\mathcal{U}^*}_b$, $N^0_b = M^{S_b}_{\eta_b,l_b}$, $\psi^{\mathcal{U}}_b : \mathcal{M}^{\mathcal{U}}_b \to N^0_b$, etc. Set

$$\mathcal{W}^*_\gamma = (\mathcal{W}^*_{\eta_\gamma,l_\gamma})^{S_\gamma}$$

for $\gamma < \mathrm{lh}\,\mathcal{U}$ or $\gamma = b$. Let $z^*(\gamma)+1 = \mathrm{lh}(\mathcal{W}^*_\gamma)$ and put

$$N_\gamma = \mathcal{M}^{\mathcal{W}^*_\gamma}_{z^*(\gamma)}.$$

So $\mathcal{W}^*_0$ is our normal tree from P_0 to $N_0 \trianglerighteq M = N^0_0$ that is by Σ. We have $N^0_\gamma \trianglelefteq N_\gamma$ for all γ. If $\nu <_U \gamma$ and $(\nu,\gamma]_U$ does not drop, then $i^{\mathcal{U}^*}_{\nu,\gamma}(\mathcal{W}^*_\nu) = \mathcal{W}^*_\gamma$. (This is not the case if we have a drop.)

Now let's look at the embedding normalization of $\langle \mathcal{W}^*_0, \mathcal{U}^+\rangle$. This is a maximal normal stack, so our theory of embedding normalization applies to it. (If $N^0_0 = N_0$, then $\mathcal{U}^+ = \mathcal{U}$. In any case, $\mathcal{U}$ and $\mathcal{U}^+$ have the same tree order.) Set

$$\mathcal{W}_\gamma = W(\mathcal{W}^*_0, \mathcal{U}^+ \restriction (\gamma+1))$$

for $\gamma < \mathrm{lh}\,\mathcal{U}$, and

$$\mathcal{W}_b = W(\mathcal{W}^*_0, (\mathcal{U}^+)^\frown b).$$

So $\mathcal{W}_0 = \mathcal{W}^*_0$. The $\mathcal{W}_\gamma$'s are all by Σ, because Σ normalizes well and $\mathcal{U}^+ \restriction (\gamma+1)$ is by Σ. Suppose that $\mathcal{W}_b$ is by Σ, and let $\Sigma(\langle \mathcal{W}_0, \mathcal{U}^+\rangle) = c$; then $\mathcal{W}_c$ is by Σ because Σ normalizes well, so $\mathrm{br}(b, W_0, \mathcal{U}^+) = \mathrm{br}(c, W_0, \mathcal{U}^+)$, so $b = c$. Thus $\Sigma(\langle \mathcal{W}_0, \mathcal{U}^+\rangle) = b$, and hence $\Sigma(\langle \mathcal{W}_0, \mathcal{U}\rangle) = b$ by strategy coherence. This is what we want, so it is enough to show that $\mathcal{W}_b$ is by Σ.

We shall show

Sublemma 5.15.1 *$\mathcal{W}_b$ is pseudo-hull of $\mathcal{W}^*_b$.*

That is enough to yield Lemma 5.15, since $\mathcal{W}^*_b$ is by Σ, and Σ has strong hull condensation.

Proof. [Proof of Sublemma 5.15.1] We construct by induction on γ an extended tree embedding

$$\Phi_\gamma \colon \mathcal{W}_\gamma \to \mathcal{W}^*_\gamma.$$

We write $z(\gamma)+1 = \mathrm{lh}\,\mathcal{W}_\gamma$, and

$$\Phi_\gamma = \langle u^\gamma, \langle s^\gamma_\beta \mid \beta \le z(\gamma)\rangle, \langle t^\gamma_\beta \mid \beta \le z(\gamma)\rangle, p^\gamma\rangle.$$

The domain of u^γ is $z(\gamma)$. Let $v^\gamma = v^{\Phi_\gamma}$ be as in Definition 3.27. Then $\mathrm{dom}\,v^\gamma = z(\gamma)+1$. Because Φ_γ is an extended tree embedding, we have $v^\gamma(z(\gamma)) \le_{W^*_\gamma} z^*(\gamma)$, and the last t-map $t^\gamma_{z(\gamma)}$ from $\mathcal{M}^{\mathcal{W}_\gamma}_{z(\gamma)}$ to $\mathcal{M}^{\mathcal{W}^*_\gamma}_{z^*(\gamma)}$. We shall write simply t^γ for $t^\gamma_{z(\gamma)}$, and

$$R_\gamma = \mathcal{M}^{\mathcal{W}_\gamma}_{z(\gamma)},$$

so that

$$t^\gamma \colon R_\gamma \to N_\gamma$$

is the last t-map of Φ_γ. As we noted above, the last t-map of an extended tree embedding determines the whole of the tree embedding.

The embedding normalization process gives us extended tree embeddings

$$\Psi_{\nu,\gamma} \colon \mathcal{W}_\nu \to \mathcal{W}_\gamma,$$

defined when $\nu <_U \gamma$. We use $\phi_{\nu,\gamma}$ for the u-map of $\Psi_{\nu,\gamma}$, so that $\phi_{\nu,\gamma} : \mathrm{lh}\,\mathcal{W}_\nu \to \mathrm{lh}\,\mathcal{W}_\gamma$, the map being total if $(\nu,\gamma]_U$ does not drop in model or degree. We write $\pi_\tau^{\nu,\gamma}$ for the t-map $t_\tau^{\Psi_{\nu,\gamma}}$, so that

$$\pi_\tau^{\nu,\gamma} : \mathcal{M}_\tau^{\mathcal{W}_\nu} \to \mathcal{M}_{\phi_{\nu,\gamma}(\tau)}^{\mathcal{W}_\gamma}$$

elementarily, for $\nu <_U \gamma$ and $\tau \in \mathrm{dom}\,\phi_{\nu,\gamma}$. Let also $e_{\nu,\gamma} = p^{\Psi_{\nu,\gamma}}$, so that

$$e_{\nu,\gamma}(E_\alpha^{\mathcal{W}_\nu}) = E_{\phi_{\nu,\gamma}(\alpha)}^{\mathcal{W}_\gamma},$$

is the natural partial map from $\mathrm{Ext}(\mathcal{W}_\nu)$ to $\mathrm{Ext}(\mathcal{W}_\gamma)$. Let also

$$\sigma_\eta^1 : \mathcal{M}_\eta^{\mathcal{U}^+} \to R_\eta$$

be the natural map from $\mathcal{M}_\eta^{\mathcal{U}^+}$ to the last model of $\mathcal{W}_\eta$, and

$$F_\eta = \sigma_\eta^1(E_\eta^{\mathcal{U}^+}),$$

so that

$$\mathcal{W}_{\eta+1} = W(\mathcal{W}_\xi, \mathcal{W}_\eta, F_\eta)$$

where $\xi = U\text{-pred}(\eta+1)$. Finally,

$$\alpha_\eta = \text{least } \alpha \text{ such that } F_\eta \text{ is on the } \mathcal{M}_\alpha^{\mathcal{W}_\eta} \text{ sequence.}$$

We also have an extended tree embedding $\Psi_{\nu,\gamma}^* \colon \mathcal{W}_\nu^* \to \mathcal{W}_\gamma^*$ defined when $\nu <_U \gamma$ and $(\nu,\gamma]_U$ does not drop. The maps of $\Phi_{\nu,\gamma}^*$ are all restrictions of $i_{\nu,\gamma}^{\mathcal{U}^*}$, so we don't need to give them special names. Part of what we want to maintain as we define the Φ_γ is that in this case, the diagram

$$\begin{tikzcd}
\mathcal{W}_\gamma \arrow[r, "\Phi_\eta"] & \mathcal{W}_\gamma^* \\
\mathcal{W}_\nu \arrow[u, "\Psi_{\nu,\gamma}"] \arrow[r, "\Phi_\nu"] & \mathcal{W}_\nu^* \arrow[u, "\Psi_{\nu,\gamma}^*"']
\end{tikzcd}$$

commutes, in the appropriate sense. The other inductive requirements have to do with the agreement between Φ_η and Φ_ξ for $\eta \leq \xi$, and the fact that σ_η factors into $\psi^{\mathcal{U}}_\eta$. We spell the requirements out completely below.

Since $\mathcal{W}_0 = \mathcal{W}^*_0$, Φ_0 is trivial, consisting of identity embeddings.

Remark 5.16 Before going through the induction in technical detail, let us look at the definition of Φ_1 in a simple case. This case contains the main idea.

Let $F = E^{\mathcal{U}^+}_0 = E^{\mathcal{U}}_0 = \psi^{\mathcal{U}}_0(E^{\mathcal{U}}_0)$. Let G be the resurrection of F in $\mathbb{C}$, and suppose $G = F$ for simplicity. Let F^* be the background extender for F given by $\mathbb{C}$. Then $\mathcal{W}_1 = W(\mathcal{W}_0, F)$ and $\mathcal{W}^*_1 = i_{F^*}(\mathcal{W}_0)$. Let $\alpha = \alpha(\mathcal{W}_0, F)$. The last model of $\mathcal{W}^*_1$ is $i_{F^*}(M)$, and $i_{F^*}(M)$ agrees with $\mathrm{Ult}(M, F)$ up to $\mathrm{lh}(F) + 1$. (The "plus 1" part is important, and it is why we were careful about choosing our background extenders.) It follows that $\mathcal{W}^*_1$ uses F; in fact $\mathcal{W}_1 {\restriction} (\alpha+2) = \mathcal{W}^*_1 {\restriction} (\alpha+2)$, with $F = E^{\mathcal{W}_1}_{\alpha+1} = E^{\mathcal{W}^*_1}_{\alpha+1}$. This gives us the desired tree embedding from $\mathcal{W}_1$ to $\mathcal{W}^*_1$. For example, the map $p^1\colon \mathrm{Ext}(\mathcal{W}_1) \to \mathrm{Ext}(\mathcal{W}^*_1)$ is given by:

$$p^1(E) = E, \text{ if } E = E^{\mathcal{W}_1}_\xi \text{ for some } \xi \leq \alpha + 1,$$

and if there is no dropping at $\alpha + 1$,

$$p^1(e_{0,1}(E)) = i_{F^*}(E).$$

This is typical of the general successor step. Various maps that are the identity in this special case are no longer so in the general case. In particular, the resurrection maps may not be the identity. But the key is still that if $\mathcal{W}_{\gamma+1} = W(\mathcal{W}_\nu, \mathcal{W}_\gamma, F)$, and $H = \psi^{\mathcal{U}}_\gamma(E^{\mathcal{U}}_\gamma)$ is the blowup of F in the last model of $\mathcal{W}^*_\gamma$, and G is the resurrection of H inside S_γ, then $\mathcal{W}^*_{\gamma+1} = i_{G^*}(\mathcal{W}^*_\nu)$, and G is used in $\mathcal{W}^*_{\gamma+1}$. [There is a small revision to the first part of the conclusion in the dropping case.] In showing this, we shall need to know that the map resurrecting H to G appears as a branch embedding inside a certain normal tree $\mathcal{W}^{**}_\gamma$ extending $\mathcal{W}^*_\gamma$.

Setting $p^{\gamma+1}(F) = G$ determines everything. For we certainly want $p^{\gamma+1}$ to agree with p^γ on the extenders used before F in $\mathcal{W}_{\gamma+1}$. Moreover, we need to take a limit of the Φ_η's along branches of $\mathcal{U}$ in order to get past limit ordinals, and this requires that $p^{\gamma+1} \circ e_{\nu,\gamma+1} = i^{\mathcal{U}^*}_{\nu,\gamma+1} \circ p^\nu$. But this accounts for all the extenders in $\mathrm{dom}(p^{\gamma+1})$, so we have completely determined $p^{\gamma+1}$, and hence $\Phi_{\gamma+1}$, from Φ_ν.

The following little lemma says something about how $i^{\mathcal{U}^*}_{\nu,\gamma}(\mathcal{W}^*_\nu)$ sits inside $\mathcal{W}^*_\gamma$. In the language of tree embeddings, the map l it describes is just $s^{\Psi^*_{\nu,\gamma}}_\beta$.

Lemma 5.17 *Suppose $\nu <_U \gamma$, and $(\nu, \gamma]_U$ does not drop. Let $\beta \leq z(\nu)$; then*

$$\sup i^{\mathcal{U}^*}_{\nu,\gamma}``\beta \leq_{W^*_\gamma} i^{\mathcal{U}^*}_{\nu,\gamma}(\beta).$$

Moreover, setting $\theta = \sup i^{\mathcal{U}^*}_{\nu,\gamma}``\beta$, we have that $(\theta, i^{\mathcal{U}^*}_{\nu,\gamma}(\beta)]_{W^*_\gamma}$ does not drop, and there is a unique embedding $l \colon \mathcal{M}^{\mathcal{W}^*_\nu}_\beta \to \mathcal{M}^{\mathcal{W}^*_\gamma}_\theta$ such that

$$i^{\mathcal{W}^*_\gamma}_{\theta, i^{\mathcal{U}^*}_{\nu,\gamma}(\beta)} \circ l = i^{\mathcal{U}^*}_{\nu,\gamma} \restriction \mathcal{M}^{\mathcal{W}^*_\nu}_\beta.$$

Proof. We have

$$i^{\mathcal{U}^*}_{\nu,\gamma}(\mathcal{W}^*_\nu) = \mathcal{W}^*_\gamma$$

because $(\nu, \gamma]_U$ did not drop. If β is a successor ordinal, or $i^{\mathcal{U}^*}_{\nu,\gamma}$ is continuous at β, then $\theta = i^{\mathcal{U}^*}_{\nu,\gamma}(\beta)$ and all is trivial. Otherwise, let $\tau <_{W^*_\nu} \beta$ be the site of the last drop; then $i^{\mathcal{U}^*}_{\nu,\gamma}(\tau)$ is the site of the last drop in $[0, i^{\mathcal{U}^*}_{\nu,\gamma}(\beta)]_{W^*_\nu}$, and $i^{\mathcal{U}^*}_{\nu,\gamma}(\tau) <_{\mathcal{W}^*_\gamma} \theta$. Finally, we can define l by: if $\eta \in (\tau, \beta)_{W^*_\nu}$ and

$$\mu = i^{\mathcal{U}^*}_{\nu,\gamma}(\eta),$$

then

$$l(i^{\mathcal{W}^*_\nu}_{\eta,\beta}(x)) = i^{\mathcal{W}^*_\gamma}_{\mu,\theta}(i^{\mathcal{U}^*}_{\nu,\gamma}(x)).$$

It is easy to see that this works. □

The following diagram illustrates the lemma.

$$\begin{tikzcd}
 & \mathcal{M}^{\mathcal{W}^*_\gamma}_\theta \arrow[r, "j_1"] & \mathcal{M}^{\mathcal{W}^*_\gamma}_{i^{\mathcal{U}^*}_{\nu,\gamma}(\beta)} \\
P_0 \arrow[ur, "j_0"] \arrow[r, "j"'] & \mathcal{M}^{\mathcal{W}^*_\nu}_\beta \arrow[u, "l"'] \arrow[ur, "i^{\mathcal{U}^*}_{\nu,\gamma}"'] &
\end{tikzcd}$$

Here $j_1 \circ j_0 = i^{\mathcal{U}^*}_{\nu,\gamma}(j)$. (The diagram assumes j exists, which is of course not the general case.) j_0 is given by the downward closure of $\{i^{\mathcal{U}^*}_{\nu,\gamma}(E) \mid E \text{ is used in } [0, \beta)_{W^*_\nu}\}$. Again, l is just $s^{\Psi^*_{\nu,\gamma}}_\beta$.

We proceed to the general successor step. Suppose we are given Φ_η for $\eta \leq \gamma$, and let us define $\Phi_{\gamma+1}$. For any $\gamma + 1 < \mathrm{lh}\,\mathcal{U}$, let

- $H_\gamma = \psi^{\mathcal{U}}_\gamma(E^{\mathcal{U}}_\gamma)$,
- $X_\gamma = M^{S_\gamma}_{\eta_\gamma, l_\gamma} | \mathrm{lh}(H_\gamma) = N_\gamma | \mathrm{lh}(H_\gamma)$, and

- $\mathrm{res}_\gamma = \sigma_{\eta_\gamma, l_\gamma}[X_\gamma]$.

So res_γ is the map resurrecting $\psi^{\mathcal{U}}_\gamma(E^{\mathcal{U}}_\gamma)$ inside S_γ. Let also

- $Y_\gamma = M^{S_\gamma}_{\theta,j}$, where $\langle\theta, j\rangle = \mathrm{Res}_{\eta_\gamma,l_\gamma}[X_\gamma]$,
- $G_\gamma = \mathrm{res}_\gamma(H_\gamma)$, and
- $G^*_\gamma =$ background extender for G_γ in $i^{\mathcal{U}^*}_{0,\gamma}(\mathbb{C})$.

So $\mathrm{res}_\gamma \colon X_\gamma \to Y_\gamma$, G_γ is the last extender of Y_γ, and $G^*_\gamma = E^{\mathcal{U}^*}_\gamma$. Finally, let

$$\sigma^0_\gamma \colon \mathcal{M}^{\mathcal{U}}_\gamma \to K_\gamma \trianglelefteq \mathcal{M}^{\mathcal{U}^+}_\gamma$$

be the copy/lifting map, and set

$$\sigma_\gamma = \sigma^1_\gamma \circ \sigma^0_\gamma,$$

so that $\sigma_\gamma \colon \mathcal{M}^{\mathcal{U}}_\gamma \to K^1_\gamma \trianglelefteq R_\gamma$. To save notation below, we shall just write $\sigma_\gamma \colon \mathcal{M}^{\mathcal{U}}_\gamma \to R_\gamma$. Our induction hypothesis is

Induction Hypothesis †.

$(\dagger)_\gamma$ (*a*) For $\xi < \eta \leq \gamma$, $\Phi_\xi \restriction (\alpha_\xi + 1) = \Phi_\eta \restriction (\alpha_\xi + 1)$.

(*b*) For all $\eta \leq \gamma$, t^η is well defined; that is, $v^\eta(z(\eta)) \leq_{W^*_\eta} z^*(\eta)$.

(*c*) For $\nu < \eta \leq \gamma$, $s^\eta_{z(\eta)} \restriction (\mathrm{lh}\, F_\nu + 1) = \mathrm{res}_\nu \circ t^\nu \restriction (\mathrm{lh}\, F_\nu + 1)$.

(*d*) Let $\nu < \eta \leq \gamma$, and $\nu <_U \eta$, and suppose that $(\nu, \eta]_U$ does not drop. Let $i^* = i^{\mathcal{U}^*}_{\nu,\eta}$, and let $\tau = \phi_{\nu,\eta}(\xi)$; then

  (i) if $\xi < z(\nu)$, then $u^\eta(\tau) = i^*(u^\nu(\xi))$,

  (ii) if $\xi < z(\nu)$, setting $j = i^{\mathcal{W}^*_\nu}_{v^\nu(\xi), u^\nu(\xi)}$ and $k = i^{\mathcal{W}^*_\eta}_{v^\eta(\tau), u^\eta(\tau)}$, there is an embedding $l \colon \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(\xi)} \to \mathcal{M}^{\mathcal{W}^*_\eta}_{v^\eta(\tau)}$ such that $k \circ l = i^* \circ j$, and $s^\eta_\tau \circ \pi^{\nu,\eta}_\xi = l \circ s^\nu_\xi$, and

  (iii) if $\xi = z(\nu)$, then setting $j = i^{\mathcal{W}^*_\nu}_{v^\nu(\xi), z^*(\nu)}$ and $k = i^{\mathcal{W}^*_\eta}_{v^\eta(\tau), z^*(\eta)}$, there is an embedding $l \colon \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(\xi)} \to \mathcal{M}^{\mathcal{W}^*_\eta}_{v^\eta(\tau)}$ such that $k \circ l = i^* \circ j$, and $s^\eta_\tau \circ \pi^{\nu,\eta}_\xi = l \circ s^\nu_\xi$.

(*e*) For $\xi \leq \gamma$, $\psi^{\mathcal{U}}_\xi = t^\xi \circ \sigma_\xi$.

(*f*) . For all $\nu < \eta \leq \gamma$, Y_ν agrees with N_η strictly below $\mathrm{lh}\, G_\nu$. G_ν is on the Y_ν-sequence, but $\mathrm{lh}\, G_\nu$ is a cardinal of N_η.

Items (*a*), (*c*), and (*f*) are our agreement hypotheses on the Φ_ν.

Clauses (*c*) and (*f*) should be read with clause (*e*) in mind. By (*e*), for all $\eta \leq \gamma$,

$$G_\eta = t^\eta(F_\eta).$$

For $\nu < \eta \leq \gamma$, $\mathrm{res}_\nu \circ t^\nu$ maps $R_\nu | \mathrm{lh}\, F_\nu$ elementarily into Y_ν, and $s^\eta_{z(\eta)}$ maps $R_\eta \parallel \mathrm{lh}\, F_\nu$ elementarily into $N_\eta \parallel \mathrm{lh}(G_\nu)$. But dropping last extender predicates, the domain models are the same, and (*f*) says that the range models are the same. By (*c*), the maps agree on $\mathrm{lh}(F_\nu)$. (This also uses (*a*), and the agreement between s and t maps in a tree embedding.) The upshot is that $(\dagger)_\gamma$ implies

$$\mathrm{res}_\nu \circ t^\nu \restriction (R_\nu \parallel \mathrm{lh}\, F_\nu) = s^\eta_{z(\eta)} \restriction (R_\gamma \parallel \mathrm{lh}\, F_\nu),$$

for all $\nu < \eta \leq \gamma$.

Remark 5.18 Literally speaking, $(\dagger)_\gamma.(c)$ does not make sense, because $t^\nu(\mathrm{lh}\, F_\nu) \notin \mathrm{dom}(\mathrm{res}_\nu)$. Here and below, we are declaring that if $\sigma \colon P \to Q$ is a resurrection map, then $\sigma(o(P)) = o(Q)$.

Remark 5.19 In most cases, $(\dagger)_\gamma.(c)$ implies that if $\nu < \eta$, then t^η agrees with $\mathrm{res}_\nu \circ t^\nu$ on $\mathrm{lh}(F_\nu) + 1$. For letting $G_\nu = t^\eta_{\alpha_\nu}(F_\nu)$, we have that

$$\mathrm{crit}(\hat{\imath}^{\mathcal{W}^*_\eta}_{v^\eta(z(\eta)),z^*(\eta)}) \geq \lambda_{G_\nu}.$$

Thus in any case, $t^\eta = t^\eta_{z(\eta)}$ agrees with $s^\eta_{z(\eta)}$ on λ_{F_ν}, and thus with $\mathrm{res}_\nu \circ t^\nu$ on λ_{F_ν} by $(\dagger)_\gamma.(c)$. The stronger agreement will fail iff $\mathrm{crit}(\hat{\imath}^{\mathcal{W}^*_\eta}_{v^\eta(z(\eta)),z^*(\eta)}) = \lambda_{G_\nu}$. The reader can check that for this to happen, F_ν must be the last extender used in $\mathcal{W}_\eta$, so that $\eta = \nu + 1$, and $z(\eta) = \alpha_\nu + 1$.

Item (*d*) captures the commutativity hypothesis $\Phi_\eta \circ \Psi_{\nu,\eta} = \Psi^*_{\nu,\eta} \circ \Phi_\nu$. It is written out in terms of the component maps of these tree embeddings; the map l in part (*d*) is $(s_{v^\nu(\xi)})^{\Psi^*_{\nu,\eta}}$. $(\dagger)_\gamma.(d)$(i) says that $p^\eta(e_{\nu,\eta}(E)) = i^{\mathcal{U}^*}_{\nu,\eta}(p^\nu(E))$. Here is a diagram to go with the rest of this clause. In the diagram, $\tau = \phi_{\nu,\eta}(\xi)$. The far right assumes $u^\nu(\xi)$ exists, that is, $\xi < z(\nu)$.

$$\begin{array}{ccccc}
\mathcal{M}_\tau^{\mathcal{W}_\eta} & \xrightarrow{s^\eta_\tau} & \mathcal{M}^{\mathcal{W}^*_\eta}_{v^\eta(\tau)} & \xrightarrow{k} & \mathcal{M}^{\mathcal{W}^*_\eta}_{u^\eta(\tau)} \\
\uparrow \pi^{\nu,\eta}_\xi & & \uparrow l & & \uparrow i^{\mathcal{U}^*}_{\nu,\eta} \\
\mathcal{M}_\xi^{\mathcal{W}_\nu} & \xrightarrow{s^\nu_\xi} & \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(\xi)} & \xrightarrow{j} & \mathcal{M}^{\mathcal{W}^*_\nu}_{u^\nu(\xi)}
\end{array}$$

Here j and k are the branch embeddings of $\mathcal{W}^*_\nu$ and $\mathcal{W}^*_\eta$. There is a similar diagram when $\xi = z(\nu)$, with $z^*(\nu)$ and $z^*(\eta)$ replacing $u^\nu(\xi)$ and $u^\eta(\tau)$.

Remark 5.20 The embedding along the bottom row of the diagram above is either t^ν_ξ or t^ν, depending on whether $\xi < z(\nu)$. The embedding along the top is either t^η_τ or t^η. So $(\dagger)_\gamma.(d)$ implies that

$$t^\eta_{\phi_{\nu,\eta}(\xi)} \circ \pi^{\nu,\eta}_\xi = i^{\mathcal{U}^*}_{\nu,\eta} \circ t^\nu_\xi$$

if $\xi < z(\nu)$, and

$$t^\eta \circ \pi^{\nu,\eta}_{z(\nu)} = i^{\mathcal{U}^*}_{\nu,\eta} \circ t^\nu.$$

Remark 5.21 $(\dagger)_\gamma$ implies that for $\nu < \eta \leq \gamma$,

$$t^\eta_{\alpha_\nu} \restriction (\mathrm{lh}\, F_\nu + 1) = \mathrm{res}_\nu \circ t^\nu \restriction (\mathrm{lh}\, F_\nu + 1).$$

This is because $\alpha_\nu < z(\eta)$, and $F_\nu = E^{\mathcal{W}_\eta}_{\alpha_\nu}$. So on $\mathrm{lh}(F_\nu) + 1$, $t^\eta_{\alpha_\nu}$ agrees with $s^\eta_{z(\eta)}$ by the agreement properties of tree embeddings (3.35), and hence with $\mathrm{res}_\nu \circ t^\nu$ by $(\dagger)_\gamma.(c)$.

If $\alpha_\nu < z(\nu)$, then since Φ_ν is a tree embedding, $t^\nu \restriction (\mathrm{lh}\, E^{\mathcal{W}_\nu}_{\alpha_\nu} + 1) = t^\nu_{\alpha_\nu}(\restriction \mathrm{lh}\, E^{W_\nu}_{\alpha_\nu} + 1)$. But $\mathrm{lh}\, F_\nu < \mathrm{lh}\, E^{\mathcal{W}_\nu}_{\alpha_\nu}$, so t^ν and $t^\nu_{\alpha_\nu}$ agree on $\mathrm{lh}(F_\nu) + 1$.

Thus $t^\nu_{\alpha_\nu} \neq t^{\nu+1}_{\alpha_\nu}$ in general. (In fact, always.) The two maps agree up to $\mathrm{lh}(F_\nu)$ if res_ν is the identity on $t^\nu_{\alpha_\nu}(\mathrm{lh}(F_\nu))$, but they need not agree past that, and they do not agree below that if res_ν is not the identity that far. They may map into different models.

This is all consistent with $(\dagger)_\gamma.(a)$, because $t^\nu_{\alpha_\nu}$ is not part of $\Phi_\nu \restriction (\alpha_\nu + 1)$. The map t^ξ_η is recording how the extender $E^{\mathcal{W}_\xi}_\eta$ is blown up into $\mathcal{W}^*_\xi$. As we go from ν to $\nu + 1$, $E^{\mathcal{W}_\nu}_{\alpha_\nu}$ is replaced by $F_\nu = E^{\mathcal{W}_{\nu+1}}_{\alpha_\nu}$. So the map blowing it up must be changed somewhat — even below $\mathrm{lh}\, F_\nu$, if there is resurrection going on in S_ν. But $E^{\mathcal{W}_\nu}_{\alpha_\nu}$ is not part of $\mathcal{W}_\nu \restriction (\alpha_\nu + 1)$, so this does not affect (a).

In defining $\Phi_{\gamma+1}$, we shall make use of 5.6, which implies that res_γ is present in a branch embedding of some $(\mathcal{W}^*_{\nu,k})^{S_\gamma}$. Let

$$\tau_\gamma = \text{least } \xi \text{ such that } X_\gamma \trianglelefteq \mathcal{M}_\xi^{\mathcal{W}^*_\gamma}.$$

Let's also drop some subscripts for now, by setting

$$\langle F, H, G, G^*, X, \tau\rangle = \langle F_\gamma, H_\gamma, G_\gamma, G^*_\gamma, X_\gamma, \tau_\gamma\rangle.$$

Claim 5.22 *1. If $\alpha_\gamma = z(\gamma)$, then $\tau \in [v^\gamma(\alpha_\gamma), z^*(\gamma)]_{W^*_\gamma}$,*

2. If $\alpha_\gamma < z(\gamma)$, then $\tau \in [v^\gamma(\alpha_\gamma), u^\gamma(\alpha_\gamma)]_{W^*_\gamma}$.

Proof.

1. If $\alpha_\gamma = z(\gamma)$, then $v^\gamma(\alpha_\gamma) \leq_{W^*_\gamma} z^*(\gamma)$. $t^\gamma(F) = \hat{\imath}^{\mathcal{W}^*_\gamma}_{v(\alpha_\gamma),z^*(\gamma)} \circ s^\gamma_{z(\gamma)}(F)$ is on the sequence of $\mathcal{M}^{\mathcal{W}^*_\gamma}_{z^*(\gamma)}$. Since $\mathrm{lh}\, E^{\mathcal{W}_\gamma}_\xi < \mathrm{lh}\, F$ for all $\xi < \alpha_\gamma$, $\mathrm{lh}(p^\gamma(E^{\mathcal{W}_\gamma}_\xi)) < \mathrm{lh}\, t^\gamma(F)$ for all $\xi < \alpha_\gamma$. Cofinally many extenders used in $[0, v(\alpha_\gamma))_{\mathcal{W}^*_\gamma}$ are in $\mathrm{ran}\, p^\gamma$, which gives $\mathrm{lh}\, s^\gamma_{z(\gamma)}(F) > \mathrm{lh}\, E^{\mathcal{W}^*_\gamma}_\xi$ for all $\xi < v^\gamma(\alpha_\gamma)$. So $v^\gamma(\alpha_\gamma)$ is less than or equal to the least τ such that $t^\gamma(F)$ is on the $M^{\mathcal{W}^*_\gamma}_\tau$ sequence. That τ is the least η such that $t^\gamma(F) = \hat{\imath}^{\mathcal{W}^*_\gamma}_{v(\alpha_\gamma),\eta} \circ s^\gamma_{z(\gamma)}(F)$, so that $\tau \in [v^\gamma(\alpha_\gamma), z^*(\gamma)]_{W^*_\gamma}$. (See proposition 5.2.)

2. If $\alpha_\gamma < z(\gamma)$, then $t^\gamma(F) = t^\gamma_{\alpha_\gamma}(F) = \hat{\imath}^{W^*_\gamma}_{v^\gamma(\alpha_\gamma),u^\gamma(\alpha_\gamma)} \circ s^\gamma_{\alpha_\gamma}(F)$. In this case

$$\tau = \text{least } \beta \in [v(\alpha_\gamma), u(\alpha_\gamma)]_{W^*_\gamma} \text{ such that } \mathrm{crit}(\hat{\imath}_{\tau,u(\alpha_\gamma)}) > \hat{\imath}_{v(\alpha_\gamma),\tau}(\mathrm{lh}\, F).$$

   This can be shown as in 1. We omit the details.

□

By Lemma 5.6, there is a normal tree $\mathcal{W}^{**}_\gamma$ such that

(*i*) $\mathcal{W}^{**}_\gamma$ is by Σ, and extends $\mathcal{W}^*_\gamma \restriction (\tau + 1)$,

(*ii*) letting $\xi_\gamma = \mathrm{lh}\, \mathcal{W}^{**}_\gamma - 1$, G is on the $\mathcal{M}^{\mathcal{W}^{**}_\gamma}_{\xi_\gamma}$ sequence, and not on the $\mathcal{M}^{\mathcal{W}^{**}_\gamma}_\alpha$ sequence for any $\alpha < \xi_\gamma$,

(*iii*) $\tau \leq_{\mathcal{W}^{**}_\gamma} \xi_\gamma$, and $\hat{\imath}^{\mathcal{W}^{**}_\gamma}_{\tau,\xi_\gamma} \restriction (\mathrm{lh}\, H + 1) = \mathrm{res}_\gamma \restriction (\mathrm{lh}\, H + 1)$.

Let

$$N^*_\gamma = \mathcal{M}^{\mathcal{W}^{**}_\gamma}_{\xi_\gamma}.$$

We shall show that $\mathcal{W}^{**}_\gamma$ is an initial segment of $\mathcal{W}^*_{\gamma+1}$, and that G is used in $\mathcal{W}^*_{\gamma+1}$. (So $G_\gamma = E^{\mathcal{W}^*_{\gamma+1}}_{\xi_\gamma}$.) By induction, the same has been true at all $\nu < \gamma$. That is, we have

Induction Hypothesis $(\dagger)_\gamma$**.**

$(\dagger)_\gamma$ $\quad (g)$. For all $\nu < \gamma$, $\mathcal{W}^{**}_\nu$ is an initial segment of $\mathcal{W}^*_\gamma \restriction (v^\gamma(\alpha_\gamma)+1)$. The last model of $\mathcal{W}^{**}_\nu$ is $N^*_\nu = \mathcal{M}^{\mathcal{W}^{**}_\nu}_{\xi_\nu}$, and $Y_\nu \trianglelefteq N^*_\nu$.

Here is a diagram showing where G came from, in the case that $\alpha_\gamma = z(\gamma)$.

$$\begin{tikzcd}
& & (N_\gamma, H) & \\
& & & (\mathcal{M}^{\mathcal{W}^{**}_\gamma}_{\xi_\gamma}, G) \\
& & (\mathcal{M}^{\mathcal{W}^*_\gamma}_{\tau_\gamma}, H) \arrow[uu, "k"] \arrow[ur, "l"] & \\
\mathcal{M}^{\mathcal{U}}_\gamma \arrow[r, "\sigma_\gamma"] & (R_\gamma, F) \arrow[uuur, "t^\gamma"] \arrow[r, "s^\gamma_{\alpha_\gamma}"] & \mathcal{M}^{\mathcal{W}^*_\gamma}_{v^\gamma(\alpha_\gamma)} \arrow[u] & \\
& & P_0 \arrow[ull, "\mathcal{W}_\gamma"] \arrow[ul] \arrow[u, "\mathcal{W}^*_\gamma"] \arrow[uuur, "\mathcal{W}^{**}_\gamma"'] &
\end{tikzcd}$$

Here k is the branch embedding of $\mathcal{W}^*_\gamma$, and it is the identity on $\mathrm{lh}(H)+1$. l is the branch embedding of $\mathcal{W}^{**}_\gamma$, and it agrees with res_γ on $\mathrm{lh}(H)+1$.

If $\alpha_\gamma < z(\gamma)$, then the corresponding diagram is:

$$
\begin{tikzcd}
 & & (N_\gamma, H) & & (\mathcal{M}^{\mathcal{W}^*_\gamma}_{u^\gamma(\alpha_\gamma)}, H) & \\
 & & & & & (\mathcal{M}^{\mathcal{W}^{**}_\gamma}_{\xi_\gamma}, G) \\
 & & & & (\mathcal{M}^{\mathcal{W}^*_\gamma}_{\tau_\gamma}, H) \arrow[uu, "k"] \arrow[ur, "l"] & \\
\mathcal{M}^{\mathcal{U}}_\gamma \arrow[r, "\sigma_\gamma"] & (R_\gamma, F) \arrow[uuur, "t^\gamma"] & & (\mathcal{M}^{\mathcal{W}_\gamma}_{\alpha_\gamma}, F) \arrow[r, "s^\gamma_{\alpha_\gamma}"] \arrow[uuur, "t^\gamma_{\alpha_\gamma}"] & \mathcal{M}^{\mathcal{W}^*_\gamma}_{v^\gamma(\alpha_\gamma)} \arrow[u] & \\
 & & & & P_0 \arrow[u, "\mathcal{W}^*_\gamma"] \arrow[uuur, "\mathcal{W}^{**}_\gamma"'] \arrow[ul, "\mathcal{W}_\gamma"] \arrow[ulll, "\mathcal{W}_\gamma"] \arrow[ullll, "\mathcal{W}_\gamma"] &
\end{tikzcd}
$$

Here again, k is the branch embedding of $\mathcal{W}^*_\gamma$, and it is the identity on $\mathrm{lh}(H)+1$. l is the branch embedding of $\mathcal{W}^{**}_\gamma$, and it agrees with res_γ on $\mathrm{lh}(H)+1$. R_γ and $\mathcal{M}^{\mathcal{W}_\gamma}_{\alpha_\gamma}$ agree up to $\mathrm{lh}(F)+1$, and t^γ agrees with $t^\gamma_{\alpha_\gamma}$ on $\mathrm{lh}(F)+1$. (In fact, on $\lambda_{E^{\mathcal{W}_\gamma}_{\alpha_\gamma}}$.)

In either case, we get

Claim 5.23 $\mathrm{res}_\gamma \circ t^\gamma$ *agrees with* $\hat{\imath}^{\mathcal{W}^{**}_\gamma}_{v^\gamma(\alpha_\gamma),\xi_\gamma} \circ s^\gamma_{\alpha_\gamma}$ *on* $\mathrm{lh}(F)+1$.

Proof. Suppose $\alpha_\gamma < z(\gamma)$. Let k and l be as in the diagram above. Then for $\eta \le \mathrm{lh}(F)$,

$$
\begin{aligned}
\mathrm{res}_\gamma \circ t^\gamma(\eta) &= \mathrm{res}_\gamma \circ t^\gamma_{\alpha_\gamma}(\eta) \\
&= \mathrm{res}_\gamma \circ (k \circ \hat{\imath}^{\mathcal{W}^*_\gamma}_{v^\gamma(\alpha_\gamma),\tau_\gamma} \circ s^\gamma_{\alpha_\gamma})(\eta) \\
&= \mathrm{res}_\gamma \circ (\hat{\imath}^{\mathcal{W}^*_\gamma}_{v^\gamma(\alpha_\gamma),\tau_\gamma} \circ s^\gamma_{\alpha_\gamma})(\eta) \\
&= l \circ (\hat{\imath}^{\mathcal{W}^*_\gamma}_{v^\gamma(\alpha_\gamma),\tau_\gamma} \circ s^\gamma_{\alpha_\gamma})(\eta) \\
&= \hat{\imath}^{\mathcal{W}^{**}_\gamma}_{v^\gamma(\alpha_\gamma),\xi_\gamma} \circ s^\gamma_{\alpha_\gamma}(\eta),
\end{aligned}
$$

as desired. The calculation when $\alpha_\gamma = z(\gamma)$ is similar. $\square$

Now let

$$\nu = U\text{-pred}(\gamma + 1).$$

Thus we have

$$S_{\gamma+1} = \text{Ult}(S_\nu, G^*),$$

where G^* is the background extender for $G = G_\gamma$ provided by $i^{\mathcal{U}^*}_{0\gamma}(\mathbb{C})$. We write

$$i_{G^*} = i^{\mathcal{U}^*}_{\nu,\gamma+1}$$

for the canonical embedding.

Case 1. $(\nu, \gamma + 1]_U$ does not drop in model or degree.

In this case, we have

$$\langle \eta_{\gamma+1}, l_{\gamma+1} \rangle = i_{G^*}(\langle \eta_\nu, l_\nu \rangle)$$
$$N_{\gamma+1} = i_{G^*}(N_\nu)$$

and

$$\mathcal{W}^*_{\gamma+1} = i_{G^*}(\mathcal{W}^*_\nu).$$

Our goal is to define $\Phi_{\gamma+1}$, and with it $t^{\gamma+1}$, so that the following diagram is realized (among other things).

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{U}}_{\gamma+1} \arrow[rr, "\sigma_{\gamma+1}"] \arrow[rrrr, bend left, "\psi^{\mathcal{U}}_{\gamma+1}"] & & R_{\gamma+1} \arrow[rr, "t^{\gamma+1}"] & & N_{\gamma+1} & S_{\gamma+1} \\
\mathcal{M}^{\mathcal{U}}_{\nu} \arrow[u] \arrow[r, "\sigma_\nu"] & R_\nu \arrow[ur, "\pi^{\nu,\gamma+1}_{z(\nu)}"] \arrow[rr, "t^\nu"] & & N_\nu \arrow[ur, "i_{G^*}"] & & S_\nu \arrow[u, "i_{G^*}"'] \\
& & P_0 \arrow[ull] \arrow[ul, "\mathcal{W}_\nu"] \arrow[uu, "\mathcal{W}_{\gamma+1}"'] & & P_0 \arrow[ul, "\mathcal{W}^*_\nu"] \arrow[uu, "\mathcal{W}^*_{\gamma+1}"'] &
\end{tikzcd}$$

As we remarked in the case $\gamma+1=1$, it is important to see that the resurrection of the blowup of F, which is in our case G, is used in $\mathcal{W}^*_{\gamma+1}$.

Claim 5.24 *(a)* $\mathcal{W}^*_{\gamma+1}\restriction\xi_\gamma = \mathcal{W}^{**}_\gamma\restriction\xi_\gamma$.

(b) $G = E^{\mathcal{W}^*_{\gamma+1}}_{\xi_\gamma}$.

Proof. Let $\mu = \mathrm{crit}(F)$, where $F = F_\gamma$. Let $\sigma_\gamma(\bar\mu) = \mu$, where $\bar\mu = \mathrm{crit}(E^{\mathcal{U}}_\gamma)$. Since $\mathcal{U}$ does not drop at $\gamma+1$, no level of $M^{\mathcal{U}}_\nu$ beyond $\mathrm{lh}\, E^{\mathcal{U}}_\nu$ projects to or below $\bar\mu$. So no level of R_ν beyond $\mathrm{lh}\, F_\nu$ projects to or below μ. So no level of N_ν beyond $\mathrm{lh}\, H_\nu$ projects to or below $t^\nu(\mu)$. Thus res_ν is the identity on $t^\nu(\mu)^{+N_\nu}$, and $N^*_\nu\restriction(t^\nu(\mu)^+)^{N^*_\nu} = N_\nu\restriction(t^\nu(\mu)^+)^{N_\nu}$. Also, $(t^\nu(\mu)^+)^{N^*_\nu} < \lambda_{G_\nu}$. Thus

$$N_\nu|t^\nu(\mu)^{+N_\nu} = N^*_\nu|t^\nu(\mu)^{+N^*_\nu} = N_\gamma|t^\nu(\mu)^{+N_\gamma}.$$

But also, if $\nu < \gamma$, then no proper initial segment of $\mathcal{M}^{\mathcal{U}}_\gamma$ projects to or below $\mathrm{lh}\, E^{\mathcal{U}}_\nu$, so no proper initial segment of N_γ projects to or below $\mathrm{lh}\, G_\nu$, so $\mathrm{res}_\gamma = \mathrm{id}$ on $\mathrm{lh}\, G_\nu$, and $N_\gamma|t^\gamma(\mu^+)^{N_\gamma} = N^*_\gamma|t^\gamma(\mu^+)^{N^*_\gamma}$.. Thus in both cases ($\nu < \gamma$ and $\nu = \gamma$),

$$N_\gamma|t^\gamma(\mu^+)^{N_\gamma} = N^*_\gamma|t^\gamma(\mu^+)^{N^*_\gamma}.$$

Letting $\lambda = t^\gamma(\mu^+)^{N^*_\gamma}$, we have then that $i_{G^*}(N_\gamma|\lambda) = i_{G^*}(N^*_\gamma|\lambda)$. But $\mathrm{Ult}(N^*_\gamma, G)$ agrees with $i_{G^*}(N^*_\gamma|\lambda)$ up to $\mathrm{lh}\, G + 1$. (We chose G^* so that they would agree *at* $\mathrm{lh}\, G$.) Thus

$$N_{\gamma+1} \parallel \mathrm{lh}\, G = N^*_\gamma \parallel \mathrm{lh}\, G$$

and $\mathrm{lh}\, G$ is a cardinal in $N_{\gamma+1}$. Since $W^*_{\gamma+1}$ and W^{**}_γ are normal trees by the same strategy Σ, we get Claim 5.24. $\square$

By Lemma 5.3, there is a unique tree embedding Ψ of $\mathcal{W}_{\gamma+1}\restriction(\alpha_\gamma + 2)$ into $\mathcal{W}^*_{\gamma+1}$ such that Ψ extends $\Phi_\gamma\restriction(\alpha_\gamma + 1)$, and $u^\Psi(\alpha_\gamma) = \xi_\gamma$, or equivalently, $p^\Psi(F) = G$. We let $\Phi_{\gamma+1}\restriction(\alpha_\gamma + 2)$ be the unique such Ψ.

In order to establish the proper notation related to $\Phi_{\gamma+1}\restriction(\alpha_\gamma + 2)$, as well as its relationship to Φ_ν, we shall now just run through the proof of Lemma 5.3 again.

Let's keep our notation $\mu = \mathrm{crit}(F)$, and write

$$\mu^* = t^\nu(\mu) = t^\gamma(\mu) = \mathrm{crit}(G).$$

Let

$$\beta = \beta^{\mathcal{W}_\nu, F},$$

so that F is applied to $\mathcal{M}^{\mathcal{W}_\nu}_\beta = \mathcal{M}^{\mathcal{W}_{\gamma+1}}_\beta$ in $\mathcal{W}_{\gamma+1}$. Let

$$\beta^* = W^*_{\gamma+1}\text{-pred}(\xi_\gamma + 1),$$

so that G is applied to $\mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{\beta^*} = \mathcal{M}^{\mathcal{W}^{**}_\gamma}_{\beta^*}$ in $\mathcal{W}^*_{\gamma+1}$.

Claim 5.25 *(a) $\beta^* \le \tau_\nu$, and $\mathcal{M}^{\mathcal{W}^*_\nu}_{\beta^*} = \mathcal{M}^{\mathcal{W}^{**}_\nu}_{\beta^*} = \mathcal{M}^{\mathcal{W}^*_\gamma}_{\beta^*} = \mathcal{M}^{\mathcal{W}^{**}_\gamma}_{\beta^*} = \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{\beta^*}$.*

(b) $\beta^* = \mu^*$.

(c) If $\beta < z(\nu)$, then $\beta^* \in [v^\nu(\beta), u^\nu(\beta)]_{W^*_\nu}$.

(d) If $\beta = z(\nu)$, then $\beta^* \in [v^\nu(\beta), z^*(\nu)]_{W^*_\nu}$.

Proof. Let P be the domain of F and P^* the domain of G; that is,

$$P = R_\gamma|(\mu^+)^{R_\gamma}$$

and

$$P^* = N_\gamma|(t^\gamma(\mu)^+)^{N_\gamma} = N^*_\gamma|(t^\gamma(\mu)^+)^{N^*_\gamma}.$$

(N_γ agrees with N^*_γ this far because we are not dropping when we apply F.) By the rules of normality,

$$\beta^* = \text{least } \alpha \text{ such that } P^* = \mathcal{M}^{\mathcal{W}^{**}_\gamma}_\alpha|o(P^*).$$

Put another way, $\mathcal{W}_\gamma^{**} \restriction \beta^* + 1$ is unique shortest normal tree on P_0 by Σ such that P^* is an initial segment of its last model, and $o(P^*)$ is passive in its last model. But we showed in the proof of Claim 5.24 that $P^* = N_\nu^* | o(P^*)$, and $o(P^*) < \lambda_{G_\nu}$. We also showed that $(\mathrm{res}_\nu) \restriction P^* =$ identity. Thus $P^* = N_\nu | o(P^*)$, and $o(P^*) < \lambda_{H_\nu}$. So P^* is a passive initial segment of the last models of $\mathcal{W}_\nu^*, \mathcal{W}_\nu^{**}, \mathcal{W}_\gamma^*, \mathcal{W}_\gamma^{**}$, and $\mathcal{W}_{\gamma+1}^*$. Thus all these trees agree up to $\beta^* + 1$. As $o(P^*) < \mathrm{lh}(H_\nu)$, $\beta^* \leq \tau_\nu$. This yields (a).

For (b), note that μ^* is a cardinal of S_γ, so $|\mathcal{M}_\alpha^{\mathcal{W}_\gamma^*}| < \mu^*$ in S_γ, for all $\alpha < \mu^*$. It follows that $\mu^* \leq \beta^*$, and if $s = e_{\mu^*}^{\mathcal{W}_\gamma^*}$ is the branch extender, then $s \colon \mu^* \to V_{\mu^*}$. If $\mu^* + 1 = \mathrm{lh}(W_\gamma^*)$ or $\lambda(E_{\mu^*})^{\mathcal{W}_\gamma^*} > \mu^*$, then $\beta^* \leq \mu^*$. So we may assume that $E = E_{\mu^*}^{\mathcal{W}_\gamma^*}$ exists, and $\lambda_E = \mu^*$. This implies $P^* = \mathcal{M}_{\mu^*}^{\mathcal{W}_\gamma^*} || \mathrm{lh}(E)$.

Working in S_γ, let

$$\mathcal{V} = i_{G^*}(\mathcal{W}_\gamma^*).$$

and

$$i_{G^*}(\mathcal{M}_{\mu^*}^{\mathcal{W}_\gamma^*}) = \mathcal{M}_\theta^{\mathcal{V}},$$

where $\theta = i_{G^*}(\mu^*)$. Since $s = i_{G^*}(s) \restriction \mu^*$, we have that $\mathcal{M}_{\mu^*}^{\mathcal{W}_\gamma^*} = \mathcal{M}_{\mu^*}^{\mathcal{V}}$, $\mu^* \in [0, \theta)_V$, and $[\mu^*, \theta)_V$ has no drops. Thus $\mathcal{M}_{\mu^*}^{\mathcal{W}_\gamma^*}$ agrees with $\mathcal{M}_\theta^{\mathcal{V}}$ up to their common value of $\mu^{*,+}$, and in particular, E is on the $\mathcal{M}_\theta^{\mathcal{V}}$-sequence. It follows that E is on the sequence of $i_{G^*}(P^*)$. But now let

$$k \colon \mathrm{Ult}(P^*, G) \to i_{G^*}(P^*)$$

be the canonical factor map. We have that $\mathrm{crit}(k) = \lambda_G$, and in particular, $\mathrm{crit}(k) > o(P^*)$. Since $o(P^*)$ is passive in $\mathrm{Ult}(P^*, G)$, it must be passive in $i_{G^*}(P^*)$, contrary to our assumption that E is indexed there. This proves (b).

For (c): if $\beta < z(\nu)$, then $\mu < \lambda_{E_\beta^{\mathcal{W}_\nu}}$, so

$$\begin{aligned} \mu^* &= t^\nu(\mu) = t_\beta^\nu(\mu) \\ &= \hat{i}^{\mathcal{W}_\nu^*}_{v^\nu(\beta), u^\nu(\beta)} \circ s_\beta^\nu(\mu). \end{aligned}$$

Also, $\mu^* < \lambda(E_{u^\nu(\beta)}^{\mathcal{W}_\nu^*})$, so $\beta^* \leq u^\nu(\beta)$ and $P^* \lhd M_{u^\nu(\beta)}^{\mathcal{W}_\nu^*} \parallel \lambda(E_{u^\nu(\beta)}^{\mathcal{W}_\nu^*})$. But since

$$P^*, \mu^* \in \mathrm{ran}\, i^{\mathcal{W}_\nu^*}_{v^\nu(\beta), u^\nu(\beta)}$$

(we don't actually need $\hat{i}$ because in this case $[v^\nu(\beta), u^\nu(\beta)]_{W_\nu^*}$ does not drop), we get

$$\beta^* = \text{least } \alpha \in [v^\nu(\beta), u^\nu(\beta)]_{W_\nu^*} \text{ such that } \mathrm{crit}(i^{\mathcal{W}_\nu^*}_{\alpha, u^\nu(\beta)}) > i^{\mathcal{W}_\nu^*}_{v^\nu(\beta), \alpha}(s_\beta^\nu(\mu)) \text{ or } \alpha = u^\nu(\beta)..$$

Proposition 5.2 essentially proves this, but the situation is not quite the same, so we repeat the argument.

First, note that $v^\nu(\beta) \le \beta^*$. For if $E = E_\eta^{\mathcal{W}_\nu}$ is used in $[0,\beta)_{W_\nu}$, then $\lambda_E \le \mu$, and thus $\lambda_{p^\nu(E)} = t^\nu_\eta(\lambda_E) \le t^\nu(\lambda_E) \le t^\nu(\mu) = \mu^*$. This implies $v^\nu(\beta) \le \beta^*$.

We have by the agreement of $\mathcal{W}_\gamma^{**}$ with $\mathcal{W}_\nu^*$ up to $\beta^* + 1$ that

$$\beta^* = \text{least } \alpha \text{ such that } P^* = \mathcal{M}_\alpha^{\mathcal{W}_\nu^*} | o(P^*).$$

Let α be least such that $\alpha \in [v^\nu(\beta), u^\nu(\beta)]_{\mathcal{W}_\nu^*}$ and $\operatorname{crit}(i^{\mathcal{W}_\nu^*}_{\alpha, u^\nu(\beta)}) > i^{\mathcal{W}_\nu^*}_{v^\nu(\beta),\alpha}(s^\nu_\beta(\mu))$ or $\alpha = u^\nu(\beta)$. We want to see $\beta^* = \alpha$. Since $P^* = \mathcal{M}^{\mathcal{W}_\nu^*}_{u^\nu(\beta)} | o(P^*)$, we have $\beta^* \le \alpha$. We must see $\alpha \le \beta^*$. If $\alpha = v^\nu(\beta)$, this holds, so assume $\alpha > v^\nu(\beta)$.

If $\sigma < \alpha$ and $E_\sigma^{\mathcal{W}_\nu^*}$ is used in $[0,\alpha)_{W_\nu^*}$, then $\lambda(E_\sigma^{\mathcal{W}_\nu^*}) \le o(P^*)$. This is true if $\sigma + 1 \le v^\nu(\beta)$ because $v^\nu(\beta) \le \beta^*$. If $v^\nu(\beta) < \sigma + 1$, then E_σ is used in $(v^\nu(\beta), \alpha]_{W_\nu^*}$, and since $P^* \in \operatorname{ran} i^{\mathcal{W}_\nu^*}_{v^\nu(\beta), u^\nu(\beta)}$, $o(P^*) < \operatorname{crit}(E_\sigma)$, and α was not least.

It follows that $\operatorname{lh}(E_\sigma^{\mathcal{W}_\nu^*}) \le o(P^*)$ for all $\sigma < \alpha$ such that $E_\sigma^{\mathcal{W}_\nu^*}$ is used in $[0,\alpha)_{W_\nu^*}$, and hence for all $\sigma < \alpha$ whatsoever. So if $\sigma < \alpha$, $P^* \ne \mathcal{M}_\sigma^{\mathcal{W}_\nu^*} | o(P^*)$, as E_σ is on the sequence of the latter model, but not of the former. Thus $\alpha \le \beta^*$, as desired.

This gives *(c)*. The proof of *(d)* is similar. □

With regard to part (b) of the claim: it is perfectly possible that β is a successor ordinal. We can even have $\beta = \alpha + 1$, where $\lambda_{E_\alpha} = \mu$. In this case $v^\nu(\beta) < \beta^* = \mu^*$, and $s^\nu_\beta(\mu) < \mu^*$ as well. So $\beta^* = \mu^*$ is strictly between $v^\nu(\beta)$ and either $u^\nu(\beta)$ or $z^*(\nu)$, as the case may be. This is a manifestation of the fact that the tree embeddings Φ_ν are very far from being onto, when $\nu > 0$.

Claim 5.26 *1. If $\beta < z(v)$, then $\beta^* =$ least $\alpha \in [v^\nu(\beta), u^\nu(\beta)]_{W_\nu^*}$ such that $\operatorname{crit}(i^{\mathcal{W}_\nu^*}_{\alpha, u^\nu(\beta)}) > i^{\mathcal{W}_\nu^*}_{v^\nu(\beta),\alpha}(s^\nu_\beta(\mu))$.*

2. If $\beta = z(v)$, then $\beta^* =$ least $\alpha \in [v^\nu(\beta), z^*(\nu)]_{W_\nu^*}$ such that $\operatorname{crit}(i^{\mathcal{W}_\nu^*}_{\alpha, z^*(\nu)}) > i^{\mathcal{W}_\nu^*}_{v^\nu(\beta),\alpha}(s^\nu_\beta(\mu))$.

3. In either case, the embeddings t^ν, $\operatorname{res}_\nu \circ t^\nu$, and $i^{\mathcal{W}_\nu^*}_{v^\nu(\beta),\beta^*} \circ s^\nu_\beta$ all agree on the domain of F.

Proof. This is what we actually showed in Claim 5.25. The following diagram illustrates the situation when $\beta < z(\nu)$.

$$\begin{tikzcd}
 & & N_\nu & & \mathcal{M}^{\mathcal{W}^*_\nu}_{u^\nu(\beta)} \\
 & & & & \mathcal{M}^{\mathcal{W}^*_\nu}_{\beta^*} \arrow[u, "k"'] \\
\mathcal{M}^{\mathcal{U}}_\nu \arrow[r, "\sigma_\nu"] & R_\nu \arrow[ur, "t^\nu"] & & \mathcal{M}^{\mathcal{W}_\nu}_\beta \arrow[r, "s^\nu_\beta"] \arrow[uur, "t^\nu_\beta"] & \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(\beta)} \arrow[u]
\end{tikzcd}$$

We have shown that both k and res_ν are the identity on the domain of G, that is, on $t^\nu(\mu)^+$ of $\mathcal{M}^{\mathcal{W}^*_\nu}_{\beta^*}$. The agreement of t^ν with t^ν_β on $\mathrm{lh}(E^{\mathcal{W}_\nu}_\beta)$, which is strictly greater than $(\mu^+)^{R_\gamma}$, completes the proof. The case that $\beta = z(\nu)$ is similar. $\square$

Now let

$$\rho = i^{\mathcal{W}^*_\nu}_{v^\nu(\beta),\beta^*} \circ s^\nu_\beta,$$

so that $\rho : \mathcal{M}^{\mathcal{W}_\nu}_\beta \to \mathcal{M}^{\mathcal{W}^*_\nu}_{\beta^*}$. On the domain of F, ρ agrees with t^ν and with $\mathrm{res}_\nu \circ t^\nu$. We can then define $\Phi_{\gamma+1}$ at $\alpha_\gamma + 1$. That is, we set

$$\begin{aligned}
u^{\gamma+1}\restriction\alpha_\gamma &= u^\gamma\restriction\alpha_\gamma, \\
p^{\gamma+1}\restriction\mathrm{Ext}(W_\gamma\restriction\alpha_\gamma) &= p^\gamma\restriction\mathrm{Ext}(W_\gamma\restriction\alpha_\gamma), \\
s^{\gamma+1}_\eta &= s^\gamma_\eta \qquad \text{for } \eta \le \alpha_\gamma,
\end{aligned}$$

and

$$t^{\gamma+1}_\eta = t^\gamma_\eta \qquad \text{for } \eta < \alpha_\gamma.$$

Then we set

$$\begin{aligned}
u^{\gamma+1}(\alpha_\gamma) &= \xi_\gamma, \\
p^{\gamma+1}(F) &= G,
\end{aligned}$$

and let $s^{\gamma+1}_{\alpha_\gamma+1}$ be given by the Shift Lemma,

$$s^{\gamma+1}_{\alpha_\gamma+1}([a,f]^{\mathcal{M}^{\mathcal{W}_\nu}_\beta}_F) = [\mathrm{res}_\gamma \circ t^\gamma(a), \rho(f)]^{\mathcal{M}^{\mathcal{W}^*_\nu}_{\beta^*}}_G.$$

We have shown that ρ agrees with $\mathrm{res}_\nu \circ t^\nu$ on the domain of F. By $(\dagger)_\gamma$, ρ agrees with t^γ on the domain of F. Since res_γ is the identity on the domain of H (cf. 5.24), ρ agrees with $\mathrm{res}_\gamma \circ t^\gamma$ on the domain of F, and we can apply the Shift Lemma here. Let us also set

$$t^{\gamma+1}_{\alpha_\gamma} = \hat{\imath}^{\mathcal{W}^{**}_\gamma}_{v^\gamma(\alpha_\gamma),\xi_\gamma} \circ s^\gamma_{\alpha_\gamma}.$$

Then $t^{\gamma+1}\colon \mathcal{M}^{\mathcal{W}_{\gamma+1}}_{\alpha_\gamma} \to \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{u^{\gamma+1}(\alpha_\gamma)} = \mathcal{M}^{\mathcal{W}^{**}_\gamma}_{\xi_\gamma}$, and $t^{\gamma+1}$ agrees with $\mathrm{res}_\gamma \circ t^\gamma$ on $\mathrm{lh}(F)+1$, by claim 5.23.

This gives us $\Phi_{\gamma+1}\restriction(\alpha_\gamma+2)$.

Claim 5.27 $\Phi_{\gamma+1}\restriction(\alpha_\gamma+2)$ *is a tree embedding of* $\mathcal{W}_{\gamma+1}\restriction(\alpha_\gamma+2)$ *into* $\mathcal{W}^*_{\gamma+1}\restriction(\xi_\gamma+2)$, *and extends* $\Phi_\gamma\restriction(\alpha_\gamma+1)$.

Proof. We checked some of the tree embedding properties as we defined $\Phi_{\gamma+1}$. We must still check that $t^{\gamma+1}_{\alpha_\gamma}$ satisfies properties (d) and (e) of definition 3.27. Noting that $E^{\mathcal{W}_\gamma}_{\alpha_\gamma} = F$ and that $t^{\gamma+1}_{\alpha_\gamma}$ agrees with $\mathrm{res}_\gamma \circ t^\gamma$ on $\mathrm{lh}(F)+1$, this is easy to do. See the proof of lemma 5.3. $\square$

We can define the remainder of the maps $u^{\gamma+1}$ and $p^{\gamma+1}$ of $\Phi_{\gamma+1}$ right now. If $\beta \le \xi < z(\nu)$, then we set

$$u^{\gamma+1}(\phi_{\nu,\gamma+1}(\xi)) = i_{G^*}(u^\nu(\xi)),$$

and

$$p^{\gamma+1}(e_{\nu,\gamma+1}(E)) = i_{G^*}(p^\nu(E)),$$

for $E = E^{\mathcal{W}_\nu}_\xi$. Note that this then holds true for any E, since if $E = E^{\mathcal{W}_\nu}_\xi$ for some $\xi < \beta$, then $p^{\gamma+1}(e_{\nu,\gamma+1}(E)) = p^{\gamma+1}(E) = p^\nu(E) = i^{G^*}(p^\nu(E))$.

The definition of the s and t-maps of $\Phi_{\gamma+1}$, and the proof that everything fits together properly, must be done by induction.

As we define $\Phi_{\gamma+1}$, we shall also check the applicable parts of $(\dagger)_{\gamma+1}$. We begin with

Claim 5.28 $\Phi_{\gamma+1}\restriction(\alpha_\gamma+2)$ *satisfies the applicable clauses of* $(\dagger)_{\gamma+1}$.

Proof. We have $\Phi_{\gamma+1}\restriction(\alpha_\gamma+1) = \Phi_\gamma\restriction(\alpha_\gamma+1)$ by construction, which yields $(\dagger)_{\gamma+1}$(a).

Suppose that $(\dagger)_{\gamma+1}$(b) is applicable, that is, that $z(\gamma+1) = \alpha_\gamma+1$. So $z(\nu) = \beta$. We have $v^{\gamma+1}(\alpha_\gamma+1) = \xi_\gamma+1$. So what we must see is that $\xi_\gamma+1 \le_{W^*_{\gamma+1}} z^*(\gamma+1)$. That is, we must see that G is used on the branch to $z^*(\gamma+1)$. We are in the non-dropping case, so $z^*(\gamma+1) = i_{G^*}(z^*(\nu))$. The relevant diagram here is

$$
\begin{tikzcd}
& \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{i_{G^*}(\beta^*)} \arrow[r] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{z^*(\gamma+1)} \\
\mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{\xi_\gamma+1} \arrow[ur, "\sigma"] & & \\
& \mathcal{M}^{\mathcal{W}^*_\nu}_{\beta^*} \arrow[ul, "i_G"] \arrow[uu, "i_{G^*}"'] \arrow[r] & \mathcal{M}^{\mathcal{W}^*_\nu}_{z^*(\nu)} \\
& \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(\beta)} \arrow[u] &
\end{tikzcd}
$$

If s is the branch extender $s = e^{\mathcal{W}^*_\nu}_{\beta^*}$, then $i_{G^*}(s(i)) = s(i)$ for all $i \in \mathrm{dom}(s)$, and thus $s \subseteq e^{\mathcal{W}^*_{\gamma+1}}_{i_{G^*}(\beta^*)}$. It follows that

$$\mathcal{M}^{\mathcal{W}^*_\nu}_{\beta^*} = \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{\beta^*},$$

and that

$$i_{G^*} \restriction \mathcal{M}^{\mathcal{W}^*_\nu}_{\beta^*} = i^{\mathcal{W}^*_{\gamma+1}}_{\beta^*, i_{G^*}(\beta^*)}.$$

The factor map σ in our diagram is the identity on the generators of G. It follows that G is compatible with the first extender used in $i^{\mathcal{W}^*_{\gamma+1}}_{\beta^*, i_{G^*}(\beta^*)}$, and thus G is that extender, as desired.

Turning to $(\dagger)_{\gamma+1}$(c), the new applicable cases are (ii) and (iii), when $\xi = \beta$ and $\tau = \alpha_\gamma + 1$. Let us suppose that it is (ii) that applies, that is, that $\beta < z(\nu)$. The last paragraph showed that G is used on the branch to $i_{G^*}(\beta^*)$ in this case as well. We have the diagram

$$\begin{tikzcd}
 & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{i_{G^*}(\beta^*)} \arrow[r, "\varphi"] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{u^{\gamma+1}(\alpha_\gamma+1)} \\
\mathcal{M}^{\mathcal{W}_{\gamma+1}}_{\alpha_\gamma+1} \arrow[r, "s^{\gamma+1}_{\alpha_\gamma+1}"] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{\xi_\gamma+1} \arrow[u, "\sigma"'] & \\
 & \mathcal{M}^{\mathcal{W}^*_\nu}_{\beta^*} \arrow[u, "i_G"'] \arrow[r, "h"] & \mathcal{M}^{\mathcal{W}^*_\nu}_{u^\nu(\beta)} \arrow[uu, "i^{\mathcal{U}^*}_{\nu,\gamma+1}"'] \\
\mathcal{M}^{\mathcal{W}_\nu}_\beta \arrow[uu, "i_F"'] \arrow[ur, "\rho"] \arrow[r, "t^\nu_\beta"] & \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(\beta)} \arrow[u, "f"'] &
\end{tikzcd}$$

Here $\pi^{\nu,\gamma+1}_\beta = i_F^{\mathcal{M}^{\mathcal{W}_\nu}_\beta}$. The branch embeddings $\varphi \circ \sigma$ of $\mathcal{W}^*_{\gamma+1}$ and $h \circ f$ of $\mathcal{W}^*_\nu$ play the roles of k and j in $(\dagger)_\gamma.(d)$. The role of l is played by $i_G \circ f$. The diagram commutes, so we are done. The case $\beta = z(\nu)$ is similar.

Turning to $(\dagger)_\gamma.(c)$, it is enough to show that $s^{\gamma+1}_{\alpha_\gamma+1}$ agrees with $\mathrm{res}_\gamma \circ t^\gamma$ on $\mathrm{lh}(F)+1$. But this follows from the Shift Lemma.

We turn to $(\dagger)_\gamma.(e)$, that $\psi^{\mathcal{U}}_{\gamma+1} = t^{\gamma+1} \circ \sigma_{\gamma+1}$. This is applicable when $z(\gamma+1) = \alpha_\gamma + 1$, and hence since we didn't drop, $z(\nu) = \beta$. So $\mathcal{M}^{\mathcal{W}_\nu}_\beta = R_\nu$, $\mathcal{M}^{\mathcal{W}_{\gamma+1}}_{\alpha_\gamma+1} = R_{\gamma+1}$, $\mathcal{M}^{\mathcal{W}^*_\nu}_{z^*(\nu)} = N_\nu$, and $\mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{z^*(\gamma+1)} = N_{\gamma+1}$. Expanding the diagram immediately above a little, while making these substitutions, we get

$$
\begin{tikzcd}
 & & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{i_{G^*}(\beta^*)} \arrow[r, "\varphi"] & N_{\gamma+1} \\
\mathcal{M}^{\mathcal{U}}_{\gamma+1} \arrow[r, "\sigma_{\gamma+1}"] \arrow[urrr, bend left, "\psi^{\mathcal{U}}_{\gamma+1}"] & R_{\gamma+1} \arrow[r, "s^{\gamma+1}_{\alpha_\gamma+1}"] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{\xi_\gamma+1} \arrow[u, "\sigma"'] & \\
 & & \mathcal{M}^{\mathcal{W}^*_\nu}_{\beta^*} \arrow[u, "i_G"'] \arrow[r, "h"] & N_\nu \arrow[uu, "i^{\mathcal{U}^*}_{\nu,\gamma+1}"'] \\
\mathcal{M}^{\mathcal{U}}_\nu \arrow[uu, "i^{\mathcal{U}}_{\nu,\gamma+1}"] \arrow[r, "\sigma_\nu"] \arrow[urrr, bend right=60, "\psi^{\mathcal{U}}_\nu"'] & R_\nu \arrow[uu, "i_F"'] \arrow[ur, "\rho"] \arrow[r, "s^\nu_\beta"] & \mathcal{M}^{\mathcal{W}^*_\nu}_{\upsilon^\nu(\beta)} \arrow[u] &
\end{tikzcd}
$$

We have $t^{\gamma+1} = \varphi \circ \sigma \circ s^{\gamma+1}_{\alpha_\gamma+1}$ and $t^\nu = h \circ \rho$.

Note first that $\psi^{\mathcal{U}}_{\gamma+1}$ agrees with $t^{\gamma+1} \circ \sigma_{\gamma+1}$ on $\operatorname{ran}(i^{\mathcal{U}}_{\nu,\gamma+1})$. This is because

$$
\begin{aligned}
\psi^{\mathcal{U}}_{\gamma+1} \circ i^{\mathcal{U}}_{\nu,\gamma+1} &= i^{\mathcal{U}^*}_{\nu,\gamma+1} \circ \psi^{\mathcal{U}}_\nu \\
&= i^{\mathcal{U}^*}_{\nu,\gamma+1} \circ (h \circ \rho \circ \sigma_\nu)
\end{aligned}
$$

(by $(\dagger)_\nu$)

$$
= t^{\gamma+1} \circ \sigma_{\gamma+1} \circ i^{\mathcal{U}}_{\nu,\gamma+1}.
$$

The last equality holds because of the commutativity of the non-ψ part of the diagram.

$\mathcal{M}^{\mathcal{U}}_{\gamma+1}$ is generated by $\operatorname{ran}(i^{\mathcal{U}}_{\nu,\gamma+1}) \cup \lambda$, where $\lambda = \lambda_{E^{\mathcal{U}}_\gamma}$. So it is now enough to show that $\psi^{\mathcal{U}}_{\gamma+1}$ agrees with $t^{\gamma+1} \circ \sigma_{\gamma+1}$ on λ. But note

$$
\begin{aligned}
\psi^{\mathcal{U}}_{\gamma+1} \restriction \lambda &= \operatorname{res}_\gamma \circ \psi^{\mathcal{U}}_\gamma \restriction \lambda \\
&= \operatorname{res}_\gamma \circ t^\gamma \circ \sigma_\gamma \restriction \lambda
\end{aligned}
$$

(by $(\dagger)_\gamma$)

$$= t^{\gamma+1} \circ \sigma_\gamma \restriction \lambda$$

(because $t^{\gamma+1}$ agrees with $\mathrm{res}_\gamma \circ t^\gamma$ on λ_F)

$$= t^{\gamma+1} \circ \sigma_{\gamma+1} \restriction \lambda.$$

The last equality holds because σ_γ agrees with $\sigma_{\gamma+1}$ on $\mathrm{lh}(F)+1$, by our earlier work on normalization. This proves $(\dagger)_{\gamma+1}$(e).

For $(\dagger)_{\gamma+1}$(f), note that $N_{\gamma+1}$ agrees with $N^*_\gamma = \mathcal{M}^{\mathcal{W}^{**}_\gamma}_{\xi_\gamma}$ below $\mathrm{lh}(G)$, and the latter is a cardinal in $N_{\gamma+1}$. This and $(\dagger)_\gamma$(f) give us what we want.

This proves Claim 5.28.

$\square$

For the rest, we define $\Phi_{\gamma+1} \restriction \eta + 1$, for $\alpha_\gamma + 1 < \eta \leq z(\gamma+1)$, by induction on η, and verify that it is a tree embedding. At the same time, we prove those clauses in $(\dagger)_{\gamma+1}$ that make sense by stage η. The agreement clauses *(a)*, *(c)*, and (f) already make sense once we have $\Phi_{\gamma+1} \restriction (\alpha_\gamma + 2)$, and we have already verified them. So we must consider clauses *(b)*, *(d)*, and *(e)*.

First, suppose we are given $\Phi_{\gamma+1} \restriction (\eta+1)$, where $\alpha_\gamma + 2 \leq \eta + 1 < z(\gamma+1)$. We must define $\Phi_{\gamma+1} \restriction (\eta+2)$. Let

$$\phi_{\nu,\gamma+1}(\tau) = \eta,$$
$$E = E^{\mathcal{W}_{\gamma+1}}_\eta,$$

and

$$K = E^{\mathcal{W}_\nu}_\tau.$$

Let

$$E^* = p^{\gamma+1}(E) \text{ and } K^* = p^\nu(K).$$

We have already defined $p^{\gamma+1}$ so that $i_{G^*}(K^*) = E^*$, and $u^{\gamma+1}(\eta) = i_{G^*}(u^\nu(\tau))$. We can simply apply lemma 5.3 to obtain $\Phi_{\gamma+1} \restriction (\eta+2)$ from $\Phi_{\gamma+1} \restriction (\eta+1)$. For we have the diagram from $(\dagger)_{\gamma+1}$(c).

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{W}_{\gamma+1}}_\eta \arrow[r, "s^{\gamma+1}_\eta"] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\eta)} \arrow[r] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{u^{\gamma+1}(\eta)} \\
\mathcal{M}^{\mathcal{W}_\nu}_\tau \arrow[u, "\pi^{\nu,\gamma+1}_\tau"] \arrow[r, "s^\nu_\tau"] & \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(\tau)} \arrow[u, "l"] \arrow[r] & \mathcal{M}^{\mathcal{W}^*_\nu}_{u^\nu(\tau)} \arrow[u, "i^{\mathcal{U}^*}_{\nu,\gamma+1}"]
\end{tikzcd}$$

Taking $\xi = u^{\gamma+1}(\eta)$, we see from the commutativity of this diagram that $E_\xi^{\mathcal{W}^*_{\gamma+1}} = i^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\eta),\xi} \circ s^{\gamma+1}_\eta(E_\eta^{\mathcal{W}_{\gamma+1}})$. Thus the condition (2) in 5.3 is fulfilled, and we can let $\Phi_{\gamma+1}\restriction(\eta+2)$ be the unique tree embedding of $\mathcal{W}_{\gamma+1}\restriction(\eta+2)$ into $\mathcal{W}^*_{\gamma+1}$ that extends $\Phi_{\gamma+1}\restriction(\eta+1)$, and maps E to $i_{G^*}(p^\nu(K))$.

We now verify the applicable parts of $(\dagger)_{\gamma+1}$. The proofs are like the successor case $\eta = \alpha_\gamma$ that we have already done. We consider first clause *(d)*. The new case to consider is $\xi = \tau+1$. We have $\phi_{\nu,\gamma+1}(\tau+1) = \eta+1$. Let $\sigma = W_\nu\text{-pred}(\tau+1)$ and $\theta = W_{\gamma+1}\text{-pred}(\eta+1)$ index the places K and E are applied. Let σ^* and θ^* index the models in $\mathcal{W}^*_\nu$ and $\mathcal{W}^*_{\gamma+1}$ to which K^* and E^* are applied. Let us write $i^* = i_{G^*}$. We have $i^*(K^*) = E^*$ and $i^*(\sigma^*) = \theta^*$.

For purposes of drawing the following diagram, we assume $\tau+1 < z(\nu)$. The situation is

$$\begin{tikzcd}
& \mathcal{M}^{\mathcal{W}_{\gamma+1}}_{\eta+1} \arrow[r, "s^{\gamma+1}_{\eta+1}"] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\eta+1)} \arrow[r, dashed] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{u^{\gamma+1}(\eta+1)} \\
& \mathcal{M}^{\mathcal{W}_\nu}_{\tau+1} \arrow[u, "\pi^{\nu,\gamma+1}_{\tau+1}"'] \arrow[r, "s^\nu_{\tau+1}"] & \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(\tau+1)} \arrow[u, "h"] \arrow[r, dashed] & \mathcal{M}^{\mathcal{W}^*_\nu}_{u^\nu(\tau+1)} \arrow[u, "i^*"'] \\
\mathcal{M}^{\mathcal{W}_{\gamma+1}}_\theta \arrow[uur, "E"] \arrow[rr, "s^{\gamma+1}_\theta"'] & & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\theta)} \arrow[r, dashed] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{\theta^*} \arrow[uul, "E^*"] \\
\mathcal{M}^{\mathcal{W}_\nu}_\sigma \arrow[u, "\pi"] \arrow[uur, "K"'] \arrow[rr, "s^\nu_\sigma"'] & & \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(\sigma)} \arrow[u, "l"'] \arrow[r, dashed] & \mathcal{M}^{\mathcal{W}^*_\nu}_{\sigma^*} \arrow[u, "i^*"'] \arrow[uul, "K^*"] \\
P_0 \arrow[u] \arrow[uu, bend left] & & & P_0 \arrow[u] \arrow[uu, bend right]
\end{tikzcd}$$

There are two cases being covered in this diagram:

(Case A.) $\text{crit}(F) \le \text{crit}(K)$. In this case, $\theta = \phi_{\nu,\gamma+1}(\sigma)$, and $\pi = \pi^{\nu,\gamma+1}_\sigma$. The map l in our diagram is given by the part of $(\dagger)_\gamma.(d)$ we have already verified.

(Case B.) $\text{crit}(K) < \text{crit}(F)$. In this case, $\theta = \sigma \leq \beta$, where $\beta = \beta^{\mathcal{W}_\nu, F}$. Moreover, $\mathcal{W}_\nu \restriction (\sigma + 1) = \mathcal{W}_{\gamma+1} \restriction (\theta + 1)$, and π is the identity. Moreover, $\beta \leq \alpha_\nu$ by the way normalization works, so the part of $(\dagger)_\gamma.(a)$ we have already verified tells us that $s^\nu_\sigma = s^{\gamma+1}_\theta$, and $\mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(\sigma} = \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\theta)}$. We take l to be the identity as well. In other words, the bottom left rectangle in the diagram above consists of identity embeddings.

We also have $\text{dom}(E) = \text{dom}(K) < \text{crit}(i^*)$ in this case (though $E \neq K$ is perfectly possible). So then $\text{dom}(E^*) = \text{dom}(K^*)$, which implies that $\mathcal{M}^{\mathcal{W}^*_\nu}_{\sigma^*} = \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{\theta^*}$, and $i^* \restriction \mathcal{M}^{\mathcal{W}^*_\nu}_{\sigma^*}$ is the identity. Thus the bottom right rectangle also consists of identity embeddings. (It is however possible that $u^\nu(\sigma) \neq u^{\gamma+1}(\sigma)$ in this case.)

In both cases, our job is to define h so that it fits into the diagram as shown. Using the notation just established, we can handle the cases in parallel.

We define h using the Shift Lemma:

$$h([a, f]^{\mathcal{M}^{\mathcal{W}^*_\nu}_{\sigma^*}}_{K^*}) = [i^*(a), i^*(f)]^{\mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{\theta^*}}_{E^*}.$$

Note here that $i^*(u^\nu(\tau)) = u^{\gamma+1}(\eta)$ by our induction hypotheses, so i^* maps $\mathcal{M}^{\mathcal{W}^*_\nu}_{u^\nu(\tau)}$, the model where we found K^*, elementarily into $\mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{u^{\gamma+1}(\eta)}$, the model that had E^*. So the Shift Lemma gives us h, and that $h \circ i_{K^*} = i_{E^*} \circ i^*$.

We shall leave it to the reader to show that the rectangle on the upper right of our diagram commutes. If s is the branch extender of $[0, u^\nu(\tau + 1)]_{W^*_\nu}$ and t is the branch extender of $[0, u^{\gamma+1}(\eta + 1)]_{W^*_{\gamma+1}}$, then $i^*(s) = t$. Moreover, if $s(a) = K^*$ and $t(b) = E^*$, then $i^*(s \restriction (a + 1)) = t \restriction (b + 1)$. This implies that the upper right rectangle commutes.

So we are left to show that $h \circ s^\nu_{\tau+1} = s^{\gamma+1}_{\eta+1} \circ \pi^{\nu,\gamma+1}_{\tau+1}$. Let $x = [b, f]^{\mathcal{M}^{\mathcal{W}_\nu}_\sigma}_K$ be in $\mathcal{M}^{\mathcal{W}_\nu}_{\tau+1}$. Then

$$\begin{aligned}
h \circ s^\nu_{\tau+1}(x)) &= h(s^\nu_{\tau+1}([b, f]^{\mathcal{M}^{\mathcal{W}_\nu}_\sigma}_K)) \\
&= h([t^\nu_\tau(b), i^{\mathcal{W}^*_\nu}_{v^\nu(\sigma),\sigma^*} \circ s^\nu_\sigma(f)]^{\mathcal{M}^{\mathcal{W}^*_\nu}_{\sigma^*}}_{K^*} \\
&= [i^* \circ t^\nu_\tau(b)), i^* \circ i^{\mathcal{W}^*_\nu}_{v^\nu(\sigma),\sigma^*} \circ s^\nu_\sigma(f)]^{\mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{\theta^*}}_{E^*}.
\end{aligned}$$

The second step uses our definition of $s^\nu_{\tau+1}$. On the other hand,

$$
\begin{aligned}
s^{\gamma+1}_{\eta+1} \circ \pi^{\nu,\gamma+1}_{\tau+1}(x) &= s^{\gamma+1}_{\eta+1}(\pi^{\nu,\gamma+1}_{\tau+1}([b,f]^{\mathcal{M}^{\mathcal{W}_\nu}_\sigma}_K)) \\
&= s^{\gamma+1}_{\eta+1}([\pi^{\nu,\gamma+1}_{\tau}(b), \pi(f)]^{\mathcal{M}^{\mathcal{W}_{\gamma+1}}_\theta}_E) \\
&= [t^{\gamma+1}_\eta \circ \pi^{\nu,\gamma+1}_\tau(b), i^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\theta),\theta^*} \circ s^{\gamma+1}_\theta \circ \pi(f)]^{\mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{\theta^*}}_{E^*}.
\end{aligned}
$$

Now let's compare the two expressions above. The function f is moved the same way in both cases because the bottom rectangles in the diagram above commute. That is,

$$i^* \circ i^{\mathcal{W}^*_\nu}_{v^\nu(\sigma),\sigma^*} \circ s^\nu_\sigma = i^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\theta),\theta^*} \circ s^{\gamma+1}_\theta \circ \pi.$$

So we just need to see that

$$t^{\gamma+1}_\eta \circ \pi^{\nu,\gamma+1}_\tau = i^* \circ t^\nu_\tau.$$

But this follows from the part of $(\dagger)_{\gamma+1}(d)$ that we have already verified. The relevant diagram is

$$
\begin{tikzcd}
\mathcal{M}^{\mathcal{W}_{\gamma+1}}_\eta \arrow[r, "s^{\gamma+1}_\eta"] \arrow[rr, bend left, "t^{\gamma+1}_\eta"] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\eta)} \arrow[r, dashed] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{u^{\gamma+1}(\eta)} \\
\mathcal{M}^{\mathcal{W}_\nu}_\tau \arrow[u, "\pi^{\nu,\gamma+1}_\tau"] \arrow[r, "s^\nu_\tau"] \arrow[rr, bend right, "t^\nu_\tau"'] & \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(\tau)} \arrow[u] \arrow[r, dashed] & \mathcal{M}^{\mathcal{W}^*_\nu}_{u^\nu(\tau)} \arrow[u, "i^*"']
\end{tikzcd}
$$

Thus we have verified the new case of $(\dagger)_{\gamma+1}(d)$ that is applicable to $\Phi_{\gamma+1}\restriction(\eta+2)$.

We turn to $(\dagger)_{\gamma+1}(e)$. If it is applicable, then $z(\gamma+1) = \eta+1$, and because we did not drop, $z(\nu) = \tau+1$. We must show that $\psi^{\mathcal{U}}_{\gamma+1} = t^{\gamma+1} \circ \sigma_{\gamma+1}$. We have $R_{\gamma+1} = \mathcal{M}^{\mathcal{W}_{\gamma+1}}_{\eta+1}$, and $R_\nu = \mathcal{M}^{\mathcal{W}_\nu}_{\tau+1}$. Making these substitutions and expanding the upper part of the diagram above, we get

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{U}}_{\gamma+1} \arrow[r, "\sigma_{\gamma+1}"] & R_{\gamma+1} \arrow[r, "s^{\gamma+1}_{z(\gamma+1)}"] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(z(\gamma+1))} \arrow[r] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{z^*(\gamma+1)} = N_{\gamma+1} \\
\mathcal{M}^{\mathcal{W}_\nu}_{\nu} \arrow[u] \arrow[r, "\sigma_\nu"] & R_\nu \arrow[u, "\pi^{\nu,\gamma+1}_{z(\nu)}"] \arrow[r, "s^{\nu}_{z(\nu)}"] & \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(z(\nu))} \arrow[u, "h"] \arrow[r] & \mathcal{M}^{\mathcal{W}^*_\nu}_{z^*(\nu)} = N_\nu \arrow[u, "i^*"]
\end{tikzcd}$$

The embedding across the bottom row is $t^\nu \circ \sigma_\nu$, and hence by induction, it is $\psi^{\mathcal{U}}_\nu$. The embedding across the top row is $t^{\gamma+1} \circ \sigma_{\gamma+1}$. The diagram commutes, so

$$\begin{aligned}
\psi^{\mathcal{U}}_{\gamma+1} \circ i^{\mathcal{U}}_{\nu,\gamma+1} &= i^{\mathcal{U}^*}_{\nu,\gamma} \circ \psi^{\mathcal{U}}_\nu \\
&= i^* \circ t^\nu \circ \sigma_\nu. \\
&= t^{\gamma+1} \circ \sigma_{\gamma+1} \circ i^{\mathcal{U}}_{\nu,\gamma+1}.
\end{aligned}$$

Thus $t^{\gamma+1} \circ \sigma_{\gamma+1}$ agrees with $\psi^{\mathcal{U}}_{\gamma+1}$ on $\mathrm{ran}(i^{\mathcal{U}}_{\nu,\gamma+1})$. So it will be enough to show the two embeddings agree on $\lambda = \lambda_{E^{\mathcal{U}}_\gamma}$. For that, we calculate exactly as we did in the case $\eta = \alpha_\gamma + 1$:

$$\begin{aligned}
\psi^{\mathcal{U}}_{\gamma+1} \restriction \lambda &= \mathrm{res}_\gamma \circ \psi^{\mathcal{U}}_\gamma \restriction \lambda \\
&= \mathrm{res}_\gamma \circ t^\gamma \circ \sigma_\gamma \restriction \lambda \\
&= t^{\gamma+1} \circ \sigma_\gamma \restriction \lambda \\
&= t^{\gamma+1} \circ \sigma_{\gamma+1} \restriction \lambda.
\end{aligned}$$

The last equality holds because σ_γ agrees with $\sigma_{\gamma+1}$ on $\mathrm{lh}(F)+1$, by our earlier work on normalization. This proves $(\dagger)_\gamma.(e)$.

Finally, suppose that λ is a limit ordinal, and we have defined $\Phi_{\gamma+1} \restriction \eta$ for all $\eta < \lambda$. Then we set

$$\Phi_{\gamma+1} \restriction \lambda = \bigcup_{\eta < \lambda} \Phi_{\gamma+1} \restriction \eta.$$

We are of course assuming $\Phi_{\gamma+1} \restriction \eta$ is a subsystem of $\Phi_{\gamma+1} \restriction \beta$ whenever $\eta < \beta$, and the tree embedding properties clearly pass through limits, so this gives us a tree embedding of $\mathcal{W}_{\gamma+1} \restriction \lambda$ into $\mathcal{W}^*_{\gamma+1} \restriction \lambda$.

In order to define $\Phi_{\gamma+1} \restriction (\lambda + 1)$, for $\lambda \leq z(\gamma+1)$ a limit ordinal, let τ be such that

$$\lambda = \phi_{\nu,\gamma+1}(\tau).$$

Consider $r = \hat{p}^{\gamma+1}(e^{\mathcal{W}_{\gamma+1}}_\lambda)$. Since $\Phi_{\gamma+1} \restriction \lambda$ is a tree embedding, $\hat{p}^{\gamma+1}$ is $\subseteq$-preserving on $\mathcal{W}^{\mathrm{ext}}_{\gamma+1}$. Thus r is the extender of some branch b of $\mathcal{W}^*_{\gamma+1}$. In fact, b is the downward closure of $\{i_{G^*}(v^\nu(\xi)) \mid \xi <_{W_\nu} \tau\}$. Recall that the v-maps preserve tree order, so that $\{i_{G^*}(v^\nu(\xi)) \mid \xi <_{W_\nu} \tau\}$ is contained in the branch $[0, i_{G^*}(v^\nu(\tau)]_{W^*_{\gamma+1}}$ of $\mathcal{W}^*_{\gamma+1}$. So

$$v^{\gamma+1}(\lambda) = \sup\{i_{G^*}(v^\nu(\xi)) \mid \xi <_{W_\nu} \tau\}.$$

Moreover, we can define $s^{\gamma+1}_\lambda : \mathcal{M}^{\mathcal{W}_{\gamma+1}}_\lambda \to \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\lambda)}$ using the commutativity given by (c) of definition 3.27:

$$s^{\gamma+1}_\lambda(i^{\mathcal{W}_{\gamma+1}}_{\theta,\lambda}(x)) = i^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\theta),v^{\gamma+1}(\lambda)}(s^{\gamma+1}_\theta(x).$$

It is easy to verify the agreement of $s^{\gamma+1}_\lambda$ with earlier embeddings specified in clause (d) of 3.27. Thus $\Phi_{\gamma+1}\restriction(\lambda+1)$ is a tree embedding.

We must check that the applicable parts of $(\dagger)_{\gamma+1}$ hold. Let us keep the notation of the last paragraph. For part *(b)*, we must consider the case $z(\gamma+1) = \lambda$. We have not dropped in $(\nu, \gamma+1]_U$, so $z(\nu) = \tau$, and $v^\nu(\tau) \leq_{W^*_\nu} z^*(\nu)$ by $(\dagger)_\nu$. We showed that $v^{\gamma+1}(\lambda) \leq_{W^*_{\gamma+1}} i_{G^*}(v^\nu(\tau))$ in the last paragraph. So $v^{\gamma+1}(\lambda) \leq_{W^*_{\gamma+1}} i_{G^*}(z^*(\nu)) = z^*(\gamma+1)$, as desired.

For $(\dagger)_{\gamma+1}$*(d)*, the new case is $\xi = \tau$, and $\lambda = \phi_{\nu,\gamma+1}(\tau)$. Everything in sight commutes, so things work out. Let's work them out. Setting $i^* = i^{\mathcal{U}^*}_{\nu,\gamma+1}$, and letting k be the branch embedding from $\mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\lambda)}$ to $\mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{i^*(v^\nu(\tau))}$, the relevant diagram is

$$\begin{tikzcd}
 & & \mathcal{M}^{\mathcal{W}_{\gamma+1}}_\lambda \arrow[r, "s^{\gamma+1}_\lambda"] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\lambda)} \arrow[r, "k"] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{i^*(v^\nu(\tau))} & \\
 & & \mathcal{M}^{\mathcal{W}_\nu}_\tau \arrow[u, "\pi^{\nu,\gamma+1}_\tau"'] \arrow[r, "s^\nu_\tau"] & \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(\tau)} \arrow[u, "l"] & & \\
 & \mathcal{M}^{\mathcal{W}_{\gamma+1}}_\theta \arrow[uur] \arrow[rrr, "s^{\gamma+1}_\theta"'] & & & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\theta)} \arrow[uul] & \\
 & \mathcal{M}^{\mathcal{W}_\nu}_\sigma \arrow[u, "\pi^{\nu,\gamma+1}_\sigma"'] \arrow[uur] \arrow[rrr, "s^\nu_\sigma"'] & & & \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(\sigma)} \arrow[u, "i^*"'] \arrow[uul] & \\
P_0 \arrow[uur] \arrow[ur] & & & & & P_0 \arrow[uul] \arrow[ul]
\end{tikzcd}$$

Here we are taking $\theta = \phi_{\nu,\gamma+1}(\sigma)$, where $\sigma <_{W_\nu} \tau$, and σ is sufficiently large that $\phi_{\nu,\gamma+1}$ preserves tree order above σ. We also take σ to be a successor ordinal, so that $i^*(v^\nu(\sigma)) = v^{\gamma+1}(\tau)$. The map l is defined by

$$l(i^{\mathcal{W}_\nu^*}_{v^\nu(\sigma),v^\nu(\tau)}(x)) = i^{\mathcal{W}_{\gamma+1}^*}_{v^{\gamma+1}(\theta),v^{\gamma+1}(\lambda)}(i^*(x)).$$

(Where of course we are taking the union over all such successor ordinals σ.) If we draw the same diagram with τ replaced by some sufficiently large $\tau_0 <_{W_\nu} \tau$ and λ replaced by $\lambda_0 = \phi_{\nu,\gamma+1}(\tau_0)$, then all parts of our diagram commute, because we have verified $(\dagger)_{\gamma+1}$ that far already. Since all these approximating diagrams commute, l is well-defined, and the diagram displayed commutes. Moreover, it is easy to check that $k \circ l = i^* \restriction \mathcal{M}^{\mathcal{W}_\nu^*}_{v^\nu(\tau)}$. Thus we have $(\dagger)_{\gamma+1}(d)$.

The proof of $(\dagger)_{\gamma+1}(e)$ is exactly the same as it was in the successor case, so we omit it.

Remark 5.29 Actually, that proof seems to show that $(\dagger)_\gamma.(e)$ is redundant, in that it follows from the other clauses.

Thus $t^{\gamma+1} \circ \sigma_{\gamma+1}$ agrees with $\psi^{\mathcal{U}}_{\gamma+1}$ on $\mathrm{ran}(i^{\mathcal{U}}_{\nu,\gamma})$. So it will be enough to show the two embeddings agree on $\lambda_{E^{\mathcal{U}}_\gamma}$. For that, it is enough to see $t^{\gamma+1}$ agrees with t^ν on λ_F. But in fact, $t^{\gamma+1}$ agrees with t^ν on $\mathrm{lh}(F_\xi)$, for all $\xi < \nu$, so we are done.

This completes our work associated to the definition of $\Phi_{\gamma+1} \restriction \lambda + 1$, for $\lambda > \alpha_\gamma$ a limit. Thus we have completed the definition of $\Phi_{\gamma+1}$, and the verification of $(\dagger)_{\gamma+1}$, in Case 1.

Case 2. $(\nu, \gamma+1]_U$ drops, in either model or degree.

Let

$$\begin{aligned}\bar\mu &= \mathrm{crit}(E^{\mathcal{U}}_\gamma),\\ \bar P &= \mathrm{dom}(E^{\mathcal{U}}_\gamma),\\ \bar Q &= \text{ first level of } \mathcal{M}^{\mathcal{U}}_\nu \text{ beyond } \bar P\\ &\quad \text{that projects to or below } \bar\mu.\end{aligned}$$

We have that

$$\bar P = \mathcal{M}^{\mathcal{U}}_\nu|(\bar\mu^+)^{\mathcal{M}^{\mathcal{U}}_\nu|\mathrm{lh}(E^{\mathcal{U}}_\nu)} = \mathcal{M}^{\mathcal{U}}_\gamma|(\bar\mu^+)^{\mathcal{M}^{\mathcal{U}}_\gamma|\mathrm{lh}(E^{\mathcal{U}}_\gamma)}.$$

Let

$$\begin{aligned}
\mu &= \sigma_\nu(\bar{\mu}) &&= \mathrm{crit}(F),\\
P &= \sigma_\nu(\bar{P}) &&= \mathrm{dom}(F),\\
Q &= \sigma_\nu(\bar{Q}) &&= \text{ first level of } R_\nu \text{ beyond } P\\
&&& \quad \text{that projects to or below } \mu.
\end{aligned}$$

Since σ_ν agrees with σ_γ on $\mathrm{lh}(F_\nu)$, we can replace σ_ν by σ_γ in the first two equations. (But if $\nu < \gamma$, then $\bar{Q} \notin \mathrm{dom}(\sigma_\gamma)$.) We have that

$$P = R_\nu|(\mu^+)^{R_\nu|\,\mathrm{lh}(F_\nu)} = R_\gamma|(\mu^+)^{R_\gamma|\,\mathrm{lh}(F)}.$$

In this case, $z(\gamma+1) = \alpha_\gamma + 1$, and

$$\mathcal{W}_{\gamma+1} = \mathcal{W}_\gamma \restriction (\alpha_\gamma + 1)^\frown \langle \mathrm{Ult}(Q, F)\rangle.$$

Claim A. $\mathrm{res}_\gamma \circ t^\gamma$ agrees with $\mathrm{res}_\nu \circ t^\nu$ on λ_{F_ν}.

Proof. This is clear if $\nu = \gamma$. But if $\nu < \gamma$, then t^γ agrees with $\mathrm{res}_\nu \circ t^\nu$ on λ_{F_ν} by $(\dagger)_\gamma(c)$. (See the remarks after the statement of $(\dagger)_\gamma$.) But also, res_γ is the identity on $\mathrm{res}_\nu \circ t^\nu(\lambda_{F_\nu})$, because $\nu < \gamma$. This yields the claim. $\square$

We have $H = t^\gamma(F)$ and $G = \mathrm{res}_\gamma(G)$. We have that $\mathrm{res}_\gamma \colon N_\gamma|\,\mathrm{lh}(H) \to N^*_\gamma|\,\mathrm{lh}(G)$, and that res_γ agrees with $\hat{\imath}^{\mathcal{W}^{**}_\gamma}_{\tau_\gamma,\xi_\gamma}$ on $\mathrm{lh}(H)$. Let

$$\begin{aligned}
Q^* &= M^{S_\nu}_{\eta,l}, \text{ where } \langle \eta, l\rangle = \mathrm{Res}_{\eta_\nu,l_\nu}[t^\nu(Q)]^{S_\nu},\\
\sigma^* &= \sigma_{\eta_\nu,l_\nu}[t^\nu(Q)]^{S_\nu},\\
\mu^* &= \sigma^*(t^\nu(\mu)), \text{ and}\\
P^* &= \sigma^*(t^\nu(P)).
\end{aligned}$$

σ^* is a partial resurrection map at stage ν. We had $\mathrm{res}_\nu \colon N_\nu|\,\mathrm{lh}(H_\nu) \to N^*_\nu|\,\mathrm{lh}(G_\nu)$. σ^* resurrects more, namely $t^\nu(Q)$, but doesn't trace it as far back in $i^{\mathcal{U}^*}_{0,\nu}(\mathbb{C})$. Because no proper level of $t^\nu(Q)$ projects to $t^\nu(\mu)$, σ^* agrees with res_ν on $t^\nu(P)$. So

$$\sigma^* \circ t^\nu \restriction P = \mathrm{res}_\nu \circ t^\nu \restriction P = \mathrm{res}_\gamma \circ t^\gamma \restriction P,$$

the last equality being Claim A. The embeddings displayed also agree at P, where they have value P^*. Note that $P = \mathrm{dom}(F)$ and $P^* = \mathrm{dom}(G)$.

We have that Q^* is the last model of $(W^*_{\eta,l})^{S_\nu}$. Set

$$\mathcal{T}^* = (\mathcal{W}^*_{\eta,l})^{S_\nu}.$$

Lemma 5.6 tells us that $\mathcal{T}^*$ has the following form. Let ξ be least such that $t^\nu(Q) \trianglelefteq \mathcal{M}^{\mathcal{W}^*_\nu}_\xi$. Then $\mathcal{T}^* \restriction \xi+1 = \mathcal{W}^*_\nu \restriction \xi+1$, and letting $\mathrm{lh}(\mathcal{T}^*) = \eta+1$, $\xi \leq_{T^*} \eta$ and $\sigma^* = \hat{i}^{\mathcal{T}^*}_{\xi,\eta}$.

We have that

$$\begin{aligned} \mathcal{W}^*_{\gamma+1} &= i_{G^*}(\mathcal{T}^*), \text{ and} \\ N_{\gamma+1} &= i_{G^*}(Q^*), \end{aligned}$$

by the way that lifting to the background universe works in the dropping case. As in the non-dropping case, the key is

Claim B.

(i) $\mathcal{W}^*_{\gamma+1} \restriction \xi_\gamma + 1 = \mathcal{W}^{**}_\gamma \restriction \xi_\gamma + 1$, and

(ii) $G = E^{\mathcal{W}^*_{\gamma+1}}_{\xi_\gamma}$.

Proof. We have that $\mathrm{dom}(G) = \mathrm{res}_\gamma \circ t^\gamma(P) = \mathrm{res}_\nu \circ t^\nu(P)$ by claim A, so $\mathrm{dom}(G) = \sigma^* \circ t^\nu(P) = P^* = Q^* | (\mu^{*+})^{Q^*}$. P is $\mathcal{M}^{\mathcal{W}_\gamma}_{\alpha_\gamma} | \mathrm{lh}(F)$ cut off at its μ^+. So P^* is $\mathrm{res}_\gamma \circ t^\gamma(\mathcal{M}^{\mathcal{W}_\gamma}_{\alpha_\gamma} | \mathrm{lh}(F))$, cut off at its $(\mu^*)^+$, that is, P^* is $\mathcal{M}^{\mathcal{W}^{**}_\gamma}_{\xi_\gamma} | \mathrm{lh}(G)$, cut off at $(\mu^*)^+$.

Thus Q^* agrees with $\mathcal{M}^{\mathcal{W}^{**}_\gamma}_{\xi_\gamma} | \mathrm{lh}(G)$ up to their common value for $(\mu^*)^+$. It follows that $i_{G^*}(Q^*)$ agrees with $\mathrm{Ult}(\mathcal{M}^{\mathcal{W}^{**}_\gamma}_{\xi_\gamma} | \mathrm{lh}(G), G)$ up to $\mathrm{lh}(G) + 1$, with the agreement at $\mathrm{lh}(G)$ holding by our having chosen a minimal G^* for G. Claim B now follows from the fact that $\mathcal{W}^{**}_\gamma$ and $\mathcal{W}^*_{\gamma+1}$ are normal trees by the same strategy. $\square$

We now get $\Phi_{\gamma+1}$ by setting $p^{\gamma+1}(F) = G$, and applying Lemma 5.3. We must see that $(\dagger)_{\gamma+1}$ holds. Part (a) is clear.

Let $\beta^* = W_{\gamma+1}\text{-pred}(\xi_\gamma)$.

Claim C.

(1) $\mathrm{lh}(\mathcal{T}^*) = \beta^* + 1$, and $Q^* = \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{\beta^*}$.

(2) $\beta^* = \mu^*$, and if $s = s^{\mathcal{T}^*}_{\mu^*}$, then $s \colon \mu^* \to V_{\mu^*}$.

Proof. By definition, β^* is the least α such that $\mathcal{M}_\alpha^{\mathcal{W}^*_{\gamma+1}}|o(P^*) = P^*$. But Q^* is the last model of $\mathcal{T}^*$, and $P^* = Q^*|o(P^*)$, so since $\mathcal{T}^*$ and $\mathcal{W}^*_{\gamma+1}$ are normal trees by the same strategy, $\beta^* < \text{lh}(\mathcal{T}^*)$ and $\mathcal{M}^{\mathcal{T}^*}_{\beta^*} = \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{\beta^*}$. This gives (1).

Part (2) is proved exactly as in case 1. □

Now consider $(\dagger)_{\gamma+1}$(b). We have $v^{\gamma+1}(\alpha_\gamma+1) = \xi_\gamma+1$, and $z^*(\gamma+1) = i_{G^*}(\mu^*)$. So we must see that $\xi_\gamma + 1 \leq_{W^*_{\gamma+1}} i_{G^*}(\mu^*)$, that is, that G is used on the branch of $\mathcal{W}^*_{\gamma+1}$ to $i_{G^*}(\mu^*)$. But if $s = e^{\mathcal{T}^*}_{\mu^*}$, then $s = i_{G^*}(s)\restriction\mu^*$, so μ^* is on the branch of $W^*_{\gamma+1}$ to $i_{G^*}(\mu^*)$. Moreover, $i_{G^*}(s)(\mu^*)$ is compatible with G, so it is equal to G, as desired.

$(\dagger)_{\gamma+1}$(d) is vacuous, because we have dropped. We shall leave the agreement conditions (c) and (f) to the reader, and consider (e). That is, we show $\psi^{\mathcal{U}}_{\gamma+1} = t^{\gamma+1}\circ\sigma_{\gamma+1}$. The relevant diagram is

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{U}}_{\gamma+1} \arrow[r, "\sigma_{\gamma+1}"] & R_{\gamma+1} \arrow[rr, "s^{\gamma+1}_{\alpha_\gamma+1}"] & & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{\xi_\gamma+1} \arrow[r, "k"] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{i_{G^*}(\mu^*)} \\
\bar{Q} \arrow[u, "\hat{\imath}^{\mathcal{U}}_{\nu,\gamma+1}"] \arrow[r, "\sigma_\nu"] & Q \arrow[u, "i_F"] \arrow[r, "t^\nu"] & t^\nu(Q) \arrow[r, "\sigma^*"] & Q^* \arrow[u, "i_G"] \arrow[ur, "i_{G^*}"] & \\
 & & P_0 \arrow[u, "\mathcal{W}^*_\nu"] \arrow[ur, "\mathcal{T}^*"] & &
\end{tikzcd}$$

Here $k = i^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\alpha_\gamma+1),z^*(\gamma+1)}$. Thus the embedding along the top row is $t^{\gamma+1}\circ\sigma_{\gamma+1}$. The lifting process defines $\psi^{\mathcal{U}}_{\gamma+1}$ by

$$\psi_{\gamma+1}([a,f]^{\bar{Q}}_{E^{\mathcal{U}}_\gamma}) = [res_\gamma\circ\psi_\gamma(a), \sigma^*\circ\psi_\nu(f)]^{Q^*}_{G^*},$$

where we have dropped a few superscripts for readability. Let us write $\hat{\imath}$ for $\hat{\imath}^{\mathcal{U}}_{\nu,\gamma+1}$. Then $\psi_{\gamma+1}$ agrees with $t^{\gamma+1}\circ\sigma_{\gamma+1}$ on $\text{ran}(\hat{\imath})$, because

$$\begin{aligned}
t^{\gamma+1}\circ\sigma_{\gamma+1}\circ\hat{\imath} &= i_{G^*}\circ\sigma^*\circ t^\nu\circ\sigma_\nu \\
&= i_{G^*}\circ\sigma^*\circ\psi_\nu \\
&= \psi_{\gamma+1}\circ\hat{\imath}.
\end{aligned}$$

The first line comes from the commutativity of the diagram, the second from $(\dagger)_\nu$(e), and the last from the definition of $\psi_{\gamma+1}$.

So it is enough to see that $\psi_{\gamma+1}$ agrees with $t^{\gamma+1} \circ \sigma_{\gamma+1}$ on λ, where $\lambda = \lambda(E^{\mathcal{U}}_{\gamma})$. But note that $t^{\gamma+1} = k \circ s^{\gamma+1}_{\alpha_\gamma+1}$, and $\text{crit}(k) \geq \lambda_G$. So $t^{\gamma+1}$ agrees with the Shift Lemma map $s^{\gamma+1}_{\alpha_\gamma+1}$ on λ_F. Thus $t^{\gamma+1}$ agrees with $\text{res}_\gamma \circ t^\gamma$ on λ_F. So we can calculate

$$\begin{aligned}\psi_{\gamma+1}\restriction\lambda &= \text{res}_\gamma \circ \psi_\gamma\restriction\lambda \\ &= res_\gamma \circ t^\gamma \circ \sigma_\gamma\restriction\lambda \\ &= t^{\gamma+1} \circ \sigma_{\gamma+1}\restriction\lambda.\end{aligned}$$

The second line comes from $(\dagger)_\gamma)$(*e*), and the third from our argument above, together with the fact $\sigma_\gamma\restriction\lambda = \sigma_{\gamma+1}\restriction\lambda$.

This finishes case 2, and hence the definition of $\Phi_{\gamma+1}$ and verification of $(\dagger)_{\gamma+1}$.

We leave the detailed definition of Φ_λ and verification of $(\dagger)_\lambda$, for λ a limit ordinal or $\lambda = b$, to the reader. The normalization $\mathcal{W}_\lambda$ is a direct limit of the $\mathcal{W}_\nu$ for $\nu \in [0, \lambda)_U$. The tree $\mathcal{W}^*_\lambda$ is $i^{\mathcal{U}^*}_{\nu,\lambda}(\mathcal{W}^*_\nu)$, for ν past the last drop. So it is a direct limit too. We define Φ_λ to be the direct limit of the Φ_ν for $\nu \in [0, \lambda)_U$ past the last drop. Part (*d*) of $(\dagger)$ tells us we can do that. We omit further detail.

This finishes our proof of Sublemma 5.15.1, that $\mathcal{W}_b$ is a psuedo-hull of $\mathcal{W}^*_b$. $\square$

That in turn proves Lemma 5.15 $\square$

Lemma 5.30 *Let $M = M_{\nu_0,k_0}$, and let $\mathcal{U}$ be a normal tree on M that is of limit length, and is by both $\Sigma_{\mathcal{W}^*_{\nu_0,k_0},M}$ and $\Omega^{\mathbb{C}}_{\nu_0,k_0}$. Let*

$$\text{lift}(\mathcal{U}, M, \mathbb{C}) = \langle \mathcal{U}^*, \langle \eta_\tau, l_\tau \mid \tau < \text{lh}\mathcal{U}\rangle, \langle \psi^{\mathcal{U}}_\tau \mid \tau < \text{lh}\mathcal{U}\rangle\rangle;$$

then $\mathcal{U}^*$ has a cofinal, wellfounded branch.

Proof. Let $\pi\colon H \to V_\theta$ be elementary, where H is countable and transitive, and θ is sufficiently large, and everything relevant is in $\text{ran}(\pi)$. Let $\mathcal{S} = \pi^{-1}(\mathcal{U})$, $\mathcal{S}^* = \pi^{-1}(\mathcal{U}^*)$, and $\mathcal{T} = \pi^{-1}(\mathcal{W}^*_{\nu_0,k_0})$.

Because Σ is universally Baire, $\pi^{-1}(\Sigma) = \Sigma \cap H$, so $\langle \mathcal{T}, \mathcal{S}\rangle$ is by Σ. Moreover, letting

$$b = \Sigma(\langle \mathcal{T}, \mathcal{S}\rangle),$$

we have that $b \in H$. (Because $b \in H[g]$ for all g on $\text{Col}(\omega, \tau)$, for $\tau \in H$ sufficiently large.) It will be enough to see that $\mathcal{M}^{\mathcal{S}^*}_b$ is wellfounded, as then the elementarity of π yields a cofinal wellfounded branch of $\mathcal{U}^*$.

By [19], $\mathcal{S}^*$ has a cofinal, wellfounded branch c. The proof of Sublemma 5.15.1 shows that $\mathcal{W}_c$ is a psuedo-hull of $\mathcal{W}^*_c$, where $\mathcal{W}_c = W(\mathcal{T}, \mathcal{S}^\frown c)$ and $\mathcal{W}^*_c = i^{\mathcal{S}^*}_c(\mathcal{T})$. That is because we can run the construction of Φ_c in H; we don't need $c \in H$ to do that. But then $\mathcal{W}_c^*$ is by Σ, so $\mathcal{W}_c$ is by Σ by strong hull condensation, and $c = \Sigma(\langle \mathcal{T}, \mathcal{S} \rangle)$ since Σ normalizes well. Thus $c = b$, and $\mathcal{M}_b^{\mathcal{S}^*}$ is wellfounded, as desired. $\square$

We can now finish the proof of Theorem 5.11. We have just shown that $\Sigma_{\mathcal{W}^*_{\nu_0,k_0},M}$ agrees with $\Omega^{\mathbb{C}}_{\nu_0,k_0}$ on normal trees. We must see that they agree on finite stacks $\vec{\mathcal{T}}$ of normal trees. But for such $\vec{\mathcal{T}}$,

$$\begin{aligned} \vec{\mathcal{T}} \text{ is by } \Omega^{\mathbb{C}}_{\nu_0,k_0} &\Leftrightarrow \mathrm{lift}(\vec{\mathcal{T}}) \text{ is by } \Omega^{\mathsf{UBH}}_{F^{\mathbb{C}}} \\ &\Leftrightarrow W(\mathrm{lift}(\vec{\mathcal{T}})) \text{ is by } \Omega^{\mathsf{UBH}}_{F^{\mathbb{C}}} \\ &\Leftrightarrow \mathrm{lift}(W(\vec{\mathcal{T}})) \text{ is by } \Omega^{\mathsf{UBH}}_{F^{\mathbb{C}}} \\ &\Leftrightarrow W(\vec{\mathcal{T}}) \text{ is by } \Sigma. \end{aligned}$$

The first equivalence is our definition of $\Omega^{\mathbb{C}}_{\nu_0,k_0}$. The second comes from the fact that $\Omega^{\mathsf{UBH}}_{F^{\mathbb{C}}}$ normalizes well on its domain. (This is implicit in the results of Chapter 3, section 2.) The third comes from the fact that embedding normalization commutes with lifting to the background universe, which we proved in the proof of Theorem 4.41. The last comes from the agreement of Σ with $\Omega^{\mathbb{C}}_{\nu_0,k_0}$ on normal trees.

This finishes the proof of Theorem 5.11. $\square$

6 Fine structure for the least-branch hierarchy

We now adapt the definitions and results of the previous sections to mice that are being told their own background-induced iteration strategy.

The particular kind of strategy mice dealt with in this book we call *least branch hod mice.* Paired with their iteration strategies, they become *least branch hod pairs.* Least branch hod pairs and pure extender pairs share many basic properties, and so we define a *mouse pair* to be a pair of one of the two varieties. Section 3 discusses some of the basic properties of mouse pairs.

The deeper results about least branch hod pairs require a comparison theorem. The proof of our comparison theorem for pure extender pairs generalizes in a straightforward way to least branch hod pairs, provided that we have background constructions for them that do not break down. The main problem is to show that. That is, one must show that the standard parameter of any pair reached in such a construction is solid and universal.

One might worry that the usual solidity and universality proofs require a comparison, so we are being led into a vicious circle. But this is not a problem, because if (M, Σ) has been reached in some construction, and $\mathbb{C}$ is the maximal hod pair construction of some coarse Γ-Woodin mouse that captures Σ, then $\mathbb{C}$ cannot break down until it has reached an iterate of (M, Σ). This means we do have enough backgrounded hod pairs to show the comparisons involved in the solidity and universality proofs do terminate.

But we do in fact confront a new problem in adapting the usual solidity/universality proof. Namely, when we compare (M, H, ρ) with M, we must do so by iterating them into some background construction $\mathbb{C}$, and so disagreements will very often happen when the two sides agree with each other, but not with $\mathbb{C}$. If we proceed naively, this renders invalid the usual argument that we can't end up above M on both sides. Our solution is to modify the way the phalanx is iterated, so that sometimes we move the whole phalanx up, including its exchange ordinal. Schlutzenberg has, independently and earlier, developed and used this idea in another context.

Sections 4 through 7 are devoted to background constructions of least branch hod pairs, and the proof that all their levels have well behaved standard parameters.

6.1 Least branch premice

A *least branch premouse* (lpm) is a variety of acceptable J-structure. Acceptable J-structures are structures of the form $(J_\alpha^A, \in, A \cap J_\alpha^A)$ that are amenable, and satisfy a local form of GCH. The basic fine structural notions, like projecta, standard parameters, and solidity witnesses, can be defined at this level of generality, and various elementary facts involving them proved. This is done in [41], and we assume familiarity with that material here. See the preliminaries section for more.

The language $\mathcal{L}_0$ of least branch premice should therefore have symbols $\in$ and $\dot{A}$. It is more convenient in our situation to have $\in$, predicate symbols $\dot{E}, \dot{F}, \dot{\Sigma}, \dot{B}$, and constant symbol $\dot{\gamma}$. If M is an lpm, then $M = (N, k)$, where N is an amenable structure for $\mathcal{L}_0$, and $k = k(M)$. We often identify M with N. The predicates and constant of N can be amalgamated in some fixed way into a single amenable $\dot{A}^{\mathcal{M}}$. So we are within the framework of [41]. $o(M)$ is of course the ordinal height of M. We let $\hat{o}(M)$ be the α such that $o(M) = \omega\alpha$. The index of M is

$$l(M) = \langle \hat{o}(M), k(M) \rangle.$$

If $\langle \nu, l \rangle \leq_{\text{lex}} l(M)$, then $M|\langle \nu, l \rangle$ is the initial segment N of M with index $l(N) = \langle \nu, l \rangle$. (So $\dot{E}^N = \dot{E}^M \cap N$, $\dot{F}^N = \dot{E}^M_\nu$, $\dot{\Sigma}^N = \dot{\Sigma}^M \cap N$, and $\dot{B}^N$ is determined by $\dot{\Sigma}^M$ is a way that will become clear shortly.) In order that M be an lpm, all its initial segments N must be $k(N)$-sound. If $\nu \leq \hat{o}(M)$, then we write $M|\nu$ for $M|\langle \nu, 0 \rangle$.

As with ordinary premice, if M is an lpm, then $\dot{E}^M$ is the sequence of extenders that go into constructing M, and $\dot{F}^M$ is either empty, or codes a new extender being added to our model by M. $\dot{F}^M$ must satisfy the Jensen conditions; that is, if $F = \dot{F}^M$ is nonempty (i.e., M is *extender-active*), then $M \models \text{crit}(F)^+$ exists, and for $\mu = \text{crit}(F)^{+M}$, $o(M) = i^M_F(\mu)$. $\dot{F}^M$ is just the graph of $i^M_F \restriction (M|\mu)$. M must satisfy the Jensen initial segment condition (ISC). That is, the *whole* initial segments of $\dot{F}^M$ must appear in $\dot{E}^M$. If there is a largest whole proper initial segment, then $\dot{\gamma}^M$ is its index in $\dot{E}^M$. Otherwise, $\dot{\gamma}^M = 0$. Finally, an lpm M must be *coherent*, in that $i^M_F(\dot{E}^M) \restriction o(M) + 1 = \dot{E}^M {}^\frown \langle \emptyset \rangle$.

In other words, the conditions for adding extenders to M are just as in Jensen.

The predicates $\dot{\Sigma}^M$ and $\dot{B}^M$ are used to record information about an iteration strategy Ω for M. The strategy Ω will be determined by its action on normal trees, in an absolute way, so that we need only tell the model we are building how Ω acts on normal trees, and then the model itself can recover the action of Ω on the various non-normal trees it sees. Since this simplifies the notation, it is what we shall do.

Let us write $M|\langle \nu, -1 \rangle$ for $(M|\langle \nu, 0 \rangle)^-$; that is, for $M|\langle \nu, 0 \rangle$ with its last extender predicate set to $\emptyset$.

Definition 6.1 *An M-*tree *is a triple $s = \langle \nu, k, \mathcal{T} \rangle$ such that*

(1) $\langle \nu, k \rangle \leq_{\text{lex}} l(M)$, and

(2) $\mathcal{T}$ is a normal iteration tree on $M|\langle \nu, k \rangle$.

We allow here $\mathcal{T}$ to be empty. The case $k = -1$ allows us to drop by throwing away a last extender predicate. Given an M-tree s we write $s = \langle \nu(s), k(s), \mathcal{T}(s) \rangle$. We write $M_\infty(s)$ for the last model of $\mathcal{T}(s)$, if it has one. We say $\mathrm{lh}(\mathcal{T}(s))$ is the length of s.

What we shall feed into an lpm M is information about how its iteration strategy acts on M-trees.

$\dot{\Sigma}^M$ is a predicate that codes the strategy information added at earlier stages, with $\dot{\Sigma}^M(s, b)$ meaning that $\mathcal{T}(s)$ is a normal tree on $M|\langle \nu(s), k(s) \rangle$ of limit length, and $\mathcal{T}(s)^\frown b$ is according to the strategy. We write $\Sigma^M_{\nu,k}$ for the partial iteration strategy for $M|\langle \nu, k \rangle$ determined by $\dot{\Sigma}^M$. We write

$$\begin{aligned} \Sigma^M(s) = b &\text{ iff } \dot{\Sigma}^M(s, b) \\ &\text{ iff } \Sigma^M_{\nu(s),k(s)}(\mathcal{T}(s)) = b. \end{aligned}$$

We say that s is according to Σ^M iff $\mathcal{T}(s)$ is according to $\Sigma^M_{\nu(s),k(s)}$.

We now describe how strategy information is coded into the $\dot{B}^M$ predicate. Here we use the $\mathfrak{B}$-operator discovered by Schlutzenberg and Trang in [46]. In the original version of this paper, we made use of a different coding, one that has fine-structural problems. The authors of [65] discovered those problems. The discussion to follow is taken from [65].

Definition 6.2 *M is branch-active (or just B-active) iff*

(a) there is a largest $\eta < o(M)$ such that $M|\eta \models \mathsf{KP}$, and letting $N = M|\eta$,

(b) there is a $<_N$-least N-tree s such that s is by Σ^N, $\mathcal{T}(s)$ has limit length, and $\Sigma^N(s)$ is undefined.

(c) for N and s as above, $o(M) \leq o(N) + lh(\mathcal{T}(s))$.

Note that being branch-active can be expressed by a Σ_2 sentence in $\mathcal{L}_0 - \{\dot{B}\}$. This contrasts with being extender-active, which is not a property of the premouse with its top extender removed. In contrast with extenders, we know when branches must be added before we do so.

Definition 6.3 *Suppose that M is branch-active. We set*

$$\begin{aligned} &\eta^M = \text{the largest } \eta \text{ such that } M|\eta \models \mathsf{KP}, \\ &b^M = \{\alpha \mid \eta^M + \alpha \in \dot{B}^M\}, \\ &s^M = \text{least } M|\eta^M\text{-tree such that } \dot{\Sigma}^{M|\eta^M} \text{ is undefined, and} \\ &\nu^M = \text{unique } \nu \text{ such that } \eta^M + \nu = o(M). \end{aligned}$$

Moreover, for $s = s^M$,

(1) M is a potential lpm *iff b^M is a cofinal branch of $\mathcal{T}(s)\restriction\nu^M$.*

(2) M is honest *iff $\nu^M = \mathrm{lh}(\mathcal{T}(s))$, or $\nu^M < lh(\mathcal{T}(s))$ and $b^M = [0, \nu^M)_{T(s)}$.*

(3) M is an lpm iff M is an honest potential lpm.

(4) M is strategy-active *iff $\nu^M = \mathrm{lh}(\mathcal{T}(s))$.*

We demand of an lpm M that if M is not $\dot{B}$-active, then $\dot{B}^M = \emptyset$.

The $\dot{\Sigma}$ predicate of an lpm grows at strategy-active stages. More precisely, suppose that $\hat{o}(Q)$ is a successor ordinal, and $M = Q|(\hat{o}(Q)-1)$. If M is strategy-active, then in order for Q to be an lpm, we must have

$$\dot{\Sigma}^Q = \dot{\Sigma}^M \cup \{\langle s, b^M\rangle\},$$

while if M is not strategy-active, we must have $\dot{\Sigma}^Q = \dot{\Sigma}^M$. If $\hat{o}(Q)$ is a limit ordinal, then we require that $\dot{\Sigma}^Q = \bigcup_{\eta<\hat{o}(Q)} \dot{\Sigma}^{Q|\eta}$. We see then that if M is an lpm and $\nu < \hat{o}(M)$, then $\dot{\Sigma}^{M|\nu} \subseteq \dot{\Sigma}^M$, and $M|\nu$ is strategy-active iff $\dot{\Sigma}^{M|\nu} \neq \dot{\Sigma}^M$.

This completes our definition of what it is for M to be a least-branch premouse, the definition being by induction on the hierarchy of M.

Definition 6.4 *M is a least branch premouse (lpm) iff M is an acceptable J structure meeting the requirements stated above.*

Notice that if M is an lpm, then no level of M is both $\dot{B}$-active and extender-active, because $\dot{B}$-active stages are additively decomposable.

Returning to the case that M is branch-active, note that η^M is a Σ_0^M singleton, because it is the least ordinal in $\dot{B}^M$ (because 0 is in every branch of every iteration tree), and thus s^M is also a Σ_0^M singleton. We have separated honesty from the other conditions because it is not expressible by a Q-sentence, whereas the rest is. Honesty is expressible by a Boolean combination of Σ_2 sentences. See 6.9 below.

The original version of this book required that when $o(M) < \eta^M + lh(\mathcal{T}(s))$, $\dot{B}^M$ is empty, whereas here we require that it code $[0, o(M))_{T(s)}$, in the same way that $\dot{B}^M$ will have to code a new branch when $o(M) = \eta^M + lh(\mathcal{T}(s))$. Of course, $[0, \nu^M)_{T(s)} \in M$ when $o(M) < \eta^M + lh(\mathcal{T}(s))$ and M is honest, so the current $\dot{B}^M$ seems equivalent to the original $\dot{B}^M = \emptyset$. However, $\dot{B}^M = \emptyset$ leads to Σ_1^M being too weak, with the consequence that a Σ_1 hull of M might collapse to something that is not an lpm. (The hull could satisfy $o(H) = \eta^H + lh(\mathcal{T}(s^H))$, even though $o(M) < \eta^M + lh(\mathcal{T}(s^M))$. But then being an lpm requires $\dot{B}^H \neq \emptyset$.) Our current choice for $\dot{B}^M$ solves that problem.

Remark 6.5 Suppose N is an lpm, and $N \models \mathsf{KP}$. It is very easy to see that $\dot{\Sigma}^N$ is defined on all N-trees s that are by $\dot{\Sigma}^N$ iff there are arbitrarily large $\xi < o(N)$ such that $N|\xi \models \mathsf{KP}$. If M is branch-active, then η^M is a successor admissible; moreover, we do add branch information, related to exactly one tree, at each successor admissible. Waiting until the next admissible to add branch information is just a convenient way to make sure we are done coding in the branch information for a given tree before we move on to the next one. One could go faster.

We say that an lpm M is (fully) passive if $\dot{F}^M = \emptyset$ and $\dot{B}^M = \emptyset$.

We would like to see that being an lpm is preserved by the appropriate embeddings. Q-formulae are useful for that.

Definition 6.6 *A* rQ-formula *of* $\mathcal{L}_0$ *is a conjunction of formulae of the form*

(a) $\forall u \exists v(u \subseteq v \wedge \varphi)$, *where* φ *is a* Σ_1 *formula of* $\mathcal{L}_0$ *such that* u *does not occur free in* φ,

or of the form

(b) "$\dot{F} \neq \emptyset$, *and for* $\mu = \mathrm{crit}(\dot{F})^+$, *there are cofinally many* $\xi < \mu$ *such that* ψ*", where* ψ *is* Σ_1.

Formulae of type (a) are usually called Q-formulae. Being a passive lpm can be expressed by a Q-sentence, but in order to express being an extender-active lpm, we need type (b) clauses, in order to say that the last extender is total. rQ formulae are π_2, and hence preserved downward under Σ_1-elementary maps. They are preserved upward under Σ_0 maps that are *strongly cofinal*.

Definition 6.7 *Let* M *and* N *be* $\mathcal{L}_0$*-structures and* $\pi \colon M \to N$ *be* Σ_0 *and cofinal. We say that* π *is* strongly cofinal *iff* M *and* N *are not extender active, or* M *and* N *are extender active, and* $\pi``(\mathrm{crit}(\dot{F})^+)^M$ *is cofinal in* $(\mathrm{crit}(\dot{F})^+)^N$.

It is easy to see that

Lemma 6.8 *rQ formulae are preserved downward under Σ_1-elementary maps, and upward under strongly cofinal Σ_0-elementary maps.*

Lemma 6.9 *(a) There is a Q-sentence φ of $\mathcal{L}_0$ such that for all transitive $\mathcal{L}_0$ structures M, $M \models \varphi$ iff M is a passive lpm.*

(b) There is a rQ-sentence φ of $\mathcal{L}_0$ such that for all transitive $\mathcal{L}_0$ structures M, $M \models \varphi$ iff M is an extender-active lpm.

(c) *There is a Q-sentence φ of $\mathcal{L}_0$ such that for all transitive $\mathcal{L}_0$ structures M, $M \models \varphi$ iff M is a potential branch-active lpm.*

Proof. (Sketch.) We omit the proofs of (a) and (b). For (c), note that "$\dot{B} \neq \emptyset$" is Σ_1. One can go on then to say with a Σ_1 sentence that if η is least in $\dot{B}$, then $M|\eta$ is admissible, and s^M exists. One can say with a Π_1 sentence that $\{\alpha \mid \dot{B}(\eta + \alpha)\}$ is a branch of $\mathcal{T}(s)$, perhaps of successor order type. One can say that $\dot{B}$ is cofinal in the ordinals with a Q-sentence. Collectively, these sentences express the conditions on potential lpm-hood related to $\dot{B}$. That the rest of M constitutes an extender-passive lpm can be expressed by a Π_1 sentence. □

Corollary 6.10 *(a) If M is a passive (resp. extender-active, potential branch-active) lpm, and $\mathrm{Ult}_0(M, E)$ is wellfounded, then $\mathrm{Ult}_0(M, E)$ is a passive (resp.extender-active, potential branch-active) lpm.*

(b) *Suppose that M is a passive (resp. extender-active, potential branch-active) lpm, and $\pi \colon H \to M$ is Σ_1-elementary; then H is a passive (resp. potential branch-active) lpm.*

(c) *Let $k(M) = k(H) = 0$, and $\pi \colon H \to M$ be Σ_2 elementary; then H is a branch-active lpm iff M is a branch-active lpm.*

Proof. rQ-sentences are preserved upward by strongly cofinal Σ_0 embeddings, so we have (a). They are Π_2, hence preserved downward by Σ_1- elementary embeddings, so we have (b).

It is easy to see that honesty is expressible by a Boolean combination of Σ_2 sentences, so we get (c).

□

Part (c) of Corollary 6.10 is not particularly useful. In general, our embeddings will preserve honesty of a potential branch active lpm M because $\dot{\Sigma}^M$ and $\dot{B}^M$ are determined by a complete iteration strategy for M that has strong hull condensation. So the more useful preservation theorem in the branch-active case applies to *hod pairs*, rather than to hod premice. See 6.13 below.

Remark 6.11 The following examples show that the preservation reults of 6.10 are optimal in certain respects.

(1) Let M be an extender-active lpm, and $N = \mathrm{Ult}_0(M, E)$, where E is a long extender over M whose space is $(\mathrm{crit}(\dot{F})^+)^M$, so that the canonical embedding $\pi \colon M \to N$ is discontinuous at $(\mathrm{crit}(\dot{F})^+)^M$. Then π is cofinal and Σ_0, so that

M and N satisfy the same Q-sentences, but N is not an lpm, because its last extender is not total. π is not strongly cofinal, of course.

(2) The interpolation arguments in [37] yield examples of $\pi\colon M \to N$ being a weakly elementary (with $k(M) = k(N) = 0$), and N being an extender-active lpm, but M not being an lpm. Again, M falls short in that its last extender is not total.

The copying construction, and the lifting argument in the iterability proof, do give rise to maps that are only weakly elementary. However, in those cases we know the structures on both sides are lpms for other reasons. On the other hand, core maps and ultrapower maps are fully elementary, so we can apply (a) and (b) of Corollary 6.10 to them. We do need to do this.

6.2 Least branch hod pairs

If M is an lpm, then iteration trees on M can be understood in the same fine structural sense as iteration trees on ordinary premice. We are interested in least branch premice M that have well-behaved iteration strategies Ω, strategies that normalize well and have strong hull condensation. Another aspect of the good behavior of Ω is that all Ω-iterates of M are least branch premice whose strategy predicate is consistent with the appropriate tail of Ω.

It is really the pair (M, Ω) to which our definitions and results apply.

Definition 6.12 *(M, Ω) is a* least branch hod pair *(lbr hod pair) with scope H_δ iff*

(1) M is a least branch premouse,

(2) Ω is a complete iteration strategy for M, with scope H_δ,

(3) Ω normalizes well, and has strong hull condensation, and

(4) If s is by Ω and has last model N, then N is an lpm, and $\dot{\Sigma}^N \subseteq \Omega_s$.

Of course, δ as in (2) is determined by Ω.

We say that (M, Ω) is *self-consistent* just in case it has property (4).

Definition 6.12 assumes we have made sense of embedding normalization and tree embeddings as they apply to iteration trees on least branch premice. The definitions and basic results that apply to pure extender premice go over word-for-word, so we shall simply assume it has been done.

There is one small difference in the two situations, in that the class of lpms is not closed under Σ_0 ultrapowers or Σ_1 elementary embeddings, because of the branch-honesty requirement. But we will always be dealing with hulls or iterates of *pairs*, and lpm-hood is preserved in that context. For iterates, that is just part of clause (4) of 6.12. In the case of hulls, it is part of the following lemma.

Lemma 6.13 *Let (M, Ω) be a least branch hod pair with scope H_δ, let $\pi\colon N \to M$ be weakly elementary, and suppose that if $\dot{F}^N \neq \emptyset$, then $\dot{F}^N$ is total over N; then (N, Ω^π) is a lbr hod pair with scope H_δ.*

Proof. N is an lpm by 6.10, except perhaps when M is branch-active. In this case, N is a potential branch-active lpm, and we must see that N is honest.

So let $\nu = \nu^N$, $b = b^N$, and $\mathcal{T} = \mathcal{T}(s^N)$. If $\nu = \mathrm{lh}(\mathcal{T})$, there is nothing to show, so assume $\nu < \mathrm{lh}(\mathcal{T})$. We must show that $b = [0, \nu)_T$. We have by induction that for $Q = N|\eta^N$, (Q, Ω^π_Q) is an lbr hod pair, and in particular, that it is self-consistent. Thus $\mathcal{T}$ is by Ω^π, and so we just need to see that for $\mathcal{U} = \mathcal{T}\restriction\nu$, $\mathcal{U}^\frown b$ is by Ω^π, or equivalently, that $\pi\mathcal{U}^\frown b$ is by Ω. But it is easy to see that $\pi\mathcal{U}^\frown b$ is a psuedo-hull of $\pi(\mathcal{U})^\frown b^M$, and Ω has strong hull condensation, so we are done.

Thus N is an lpm. Ω^π is a complete iteration strategy defined on all N-stacks in H_δ, where H_δ is the scope of (M, Ω). Ω^π normalizes well by the the proof of 4.4, and has strong hull condensation by the proof of 4.10.

Finally, we must show that (N, Ω^π) is self-consistent. Let P be a Ω^π iterate of N, via the stack s. Let Q be the corresponding iterate of M via πs, and let $\tau\colon P \to Q$ be the copy map. Then

$$\begin{aligned}\mathcal{U} \text{ is by } \dot{\Sigma}^P &\Rightarrow \tau(\mathcal{U}) \text{ is by } \dot{\Sigma}^Q\\ &\Rightarrow \tau(\mathcal{U}) \text{ is by } \Omega_{\pi s, Q}\\ &\Rightarrow \tau\mathcal{U} \text{ is by } \Omega_{\pi s, Q}\\ &\Rightarrow \mathcal{U} \text{ is by } (\Omega^\pi)_{s,P},\end{aligned}$$

as desired. □

Definition 6.12 records the properties of the hod pairs we construct needed to prove the comparison theorem and the existence of cores. The other properties one might hope for seem to follow from these, as they did in the case of pure extender pairs, and by the same proofs. For example, from the proofs of 4.9, 4.59, and 4.60, we get

Lemma 6.14 *Let (M, Ω) be an lbr hod pair with scope H_θ; then* *(a)* (M, Ω) *is pullback consistent and strategy coherent, and*

(b) *if* (M, Ψ) *is an lbr hod pair with scope* H_θ *such that* Ψ *and* Ω *agree on normal trees, then* $\Psi = \Omega$.

Inspired by these and many other similarities, we define

Definition 6.15 (M, Ω) *is a* mouse pair *iff* (M, Ω) *either a pure extender pair, or an lbr hod pair.*

The reader will naturally ask whether there are other classes of strategy pairs (M, Σ) which behave like the two classes we have isolated here. The answer is positive. The remarks to follow were stimulated by a suggestion by Hugh Woodin.

One can vary how much of Σ gets encoded into $\dot{\Sigma}^M$, and when that is done. One can think each of these variations as associated to some Σ_1 formula $\varphi(v)$. Roughly, a φ-premouse M starts to encode a branch for $\mathcal{T}$ when it reaches some α such that $M|\alpha \models \varphi[\mathcal{T}]$. Pure extender premice are φ-premice, for $\varphi =$ "$v \neq v$". Least branch premice are φ-premice, for φ a Σ_1 formula that can be abstracted from §5.1. Other Σ_1 formulae would lead to classes that might be called "φ-mouse pairs". The requirements of normalizing well, strong hull condensation, and self-consistency are the same for all classes of φ-mouse pairs. What varies is how much of the strategy Σ is encoded into M, and when that is done.

We should note that the rigidly layered hod pairs of [30] are not φ-mouse pairs, because the condition governing branch insertion is not first order. φ-mouse pairs have the condensation properties of pure extender pairs, while rigidly layered hod pairs do not.

The analysis of HOD in models of AD^+ that do not satisfy $\mathsf{AD}_{\mathbb{R}}$ may need φ-mouse pairs, for φ not one of the two formulae we have given privileged status in Definition 6.15. But this is speculation right now, and we have no real applications for classes of mouse pairs beyond those identified in 6.15, so we have avoided the extra generality.

6.3 Mouse pairs and the Dodd-Jensen Lemma

Mouse is generally taken to mean *iterable premouse*, and the Comparison Lemma is taken to say that any two mice M and N can be compared as to how much information they contain. But in fact, how M and N are compared depends on which iteration strategies witnessing their iterability are chosen. There is no mouse order on iterable premice, even of the pure extender variety, unless we make restrictive assumptions which imply that the iteration strategy is unique. The canonical information levels of the mouse order are occupied not by mice, but by mouse pairs. These pairs are the objects to which the Comparison Lemma, the Dodd-Jensen Lemma, and the other basic results of inner model theory apply. In the special case that M can have at most one strategy, we don't need to make the pair explicit, but in general, we do.

Let us introduce some terminology that reflects this point of view. We have already used some of it as it applies to pure extender pairs. (See 4.61.)

Definition 6.16 *Let (P, Σ) and (Q, Ω) be mouse pairs.*

(a) $(P, \Sigma) \trianglelefteq (Q, \Omega)$ *iff* $P \trianglelefteq Q$ *and* $\Sigma = \Omega_Q$.

(b) $\pi \colon (P, \Sigma) \to (Q, \Omega)$ *is* elementary *(resp.* weakly elementary*) iff π is elementary (resp. weakly elementary) as a map from P to Q, and $\Sigma = \Omega^\pi$,*

(c) A (normal, weakly normal) iteration tree on (P, Σ) *is a (normal, weakly normal) iteration tree $\mathcal{T}$ on P such that $\mathcal{T}$ is by Σ. The α^{th}* pair of $\mathcal{T}$ *is* $(\mathcal{M}_\alpha^\mathcal{T}, \Sigma_{\mathcal{T}\restriction\alpha+1})$.

(d) A (P, Σ)-stack *is a P-stack by Σ. If s is a (P, Σ)-stack with last model Q, then the* last pair *of s is* $(Q, \Sigma_{s,Q})$.

(e) (Q, Ψ) *is an* iterate of (P, Σ) *iff there is a (P, Σ)-stack with last pair (Q, Ψ). If s can be taken to be a single normal tree, then (Q, Ψ) is a* normal iterate *of (P, Σ). If s can be taken so that P-to-Q in s does not drop, then (Q, Ψ) is a* non-dropping iterate *of (P, Σ).*

(f) $(P, \Sigma) \leq^* (Q, \Omega)$ *iff there is an iterate (R, Ψ) of $Q, \Omega)$ and an elementary* $\pi \colon (P, \Sigma) \to (R, \Psi)$*. We call $\leq^*$ the* mouse pair order*.*

Notice that the natural agreement of pairs in a normal tree on (P, Σ) follows at once from strategy coherence. Here are some further elementary facts stated in this language.

Lemma 6.17 *Let (P, Σ) be a mouse pair with scope H_δ, and let (Q, Ω) be an iterate of (P, Σ); then (Q, Ω) is a mouse pair with scope H_δ.*

Proof. Iterates of pure extender premice are pure extender premice, and normalizing well and strong hull condensation are defined so that they pass to tail strategies. If M is an lpm, then N is an lpm by clause (4) of 6.12. The properties in (3) and (4) of 6.12 clearly pass to tail strategies. $\square$

In the mouse pair language, the elementarity of iteration maps amounts to pullback consistency. So we have

Lemma 6.18 *Let (P,Σ) be a mouse pair, and let s be a (P,Σ)-stack; then the iteration maps of s are elementary in the category of mouse pairs. That is, if $Q = \mathcal{M}_\alpha^{\mathcal{T}_m(s)}|\langle \nu, k\rangle$ and $\pi\colon Q \to \mathcal{M}_\infty(s)$ is the iteration map of s, then for $t = s{\restriction}(m-1)^\frown\langle(\nu_m(s), k_m(s), \mathcal{T}_m(s){\restriction}(\alpha+1))\rangle$, π is elementary as a map from $(Q, \Sigma_{t,Q})$ to $(\mathcal{M}_\infty(s), \Sigma_s)$.*

The appropriate statement of the Dodd-Jensen Lemma on the minimality of iteration maps is:

Theorem 6.19 *(Dodd-Jensen Lemma) Let (P,Σ) be an mouse pair, let (Q,Ω) be an iterate of (P,Σ) via the stack s, and let $\pi\colon (P,\Sigma) \to (Q,\Omega)$ be weakly elementary; then*

(a) the branch P-to-Q of s does not drop, and

(b) letting $i_s\colon P \to Q$ be the iteration map, for all $\eta < o(P)$, $i_s(\eta) \le \pi(\eta)$.

We omit the well known proof. Notice that it requires the assumption that $\Sigma^\pi_{s,Q} = \Sigma$. This was at one time a nontrivial restriction on the applicability of the Dodd-Jensen Lemma, and led to the Weak Dodd-Jensen Lemma of [27]. Now that we can compare iteration strategies, the restriction is less important.

We get the Dodd-Jensen corollary on the uniqueness of iteration maps.

Corollary 6.20 *Let (P,Σ) be a mouse pair, (Q,Ω) a non-dropping iterate of (P,Σ) via the stack s, and suppose $(Q,\Omega) \trianglelefteq (R,\Psi)$, where (R,Ψ) is an iterate of (P,Σ) via the stack t; then*

(a) $(Q,\Omega) = (R,\Psi)$, and the branch P-to-R of t does not drop, and

(b) letting i_s and i_t be the two iteration maps, $i_s = i_t$.

In the language of mouse pairs, the Comparison Lemma reads

Theorem 6.21 *(Comparison Lemma) Assume AD^+, and let (P,Σ) and (Q,Ψ) be mouse pairs with scope HC of the same type; then there are iterates (R,Λ) of (P,Σ) and (S,Ω) of (Q,Ψ), obtained via normal trees $\mathcal{T}$ on P and $\mathcal{U}$ on Q, such that either*

(1) $(R,\Lambda) \trianglelefteq (S,\Omega)$ and P-to-R does not drop, or

(2) $(S,\Omega) \trianglelefteq (R,\Lambda)$ and Q-to-S does not drop.

We proved this for pure extender pairs in 5.13, and we shall give the proof for least branch hod pairs in 6.54. For now let us assume it. We get

Corollary 6.22 *Assume* AD^+*; then*

(a) For (P, Σ) *and* (Q, Ψ) *mouse pairs with scope HC of the same type,*

$$(P, \Sigma) <^* (Q, \Psi) \Leftrightarrow \exists (R, \Omega) \exists \pi [(R, \Omega) \text{ is a dropping iterate of } (Q, \Psi) \\ \text{and } \pi \colon (P, \Sigma) \to (R, \Omega) \text{ is weakly elementary}].$$

(b) When restricted to a fixed type, $\leq^*$ *is a prewellorder of mouse pairs with scope HC.*

Proof. The left-to-right direction of (a) follows from the Comparison Lemma. The right-to-left direction follows from Dodd-Jensen. For (b), the Comparison Lemma implies that $\leq^*$ is linear. That it is wellfounded follows from (a), using the proof of the Dodd-Jensen Lemma. □

For the record

Definition 6.23 *Let* (P, Σ) *be a mouse pair; then* Σ *is* positional *iff whenever* (Q, Ψ) *and* (R, Ω) *are iterates of* (P, Σ)*, and* $Q = R$*, then* $\Psi = \Omega$*.*

The property is clearly related to what is called being positional in [30]. In the present context, with gratuitous dropping allowed, it implies clause (b) of strategy coherence.

[48] proves

Lemma 6.24 *Assume* AD^+*, and let* (P, Σ) *be a mouse pair with scope HC; then* Σ *is positional.*

Fortunately, this lemma is not needed in the proof of the Comparison Lemma 6.21. Its proof instead relies on a comparison argument.

Here are two propositions that explain the relationship between pure extender mice and pure extender pairs.

Proposition 6.25 *Assume* AD^+*, and let* P *be a countable,* ω_1*-iterable pure extender premouse; then there is a* Σ *such that* (P, Σ) *is a pure extender pair.*

Proof. Let Ψ be an arbitrary ω_1 iteration strategy for P. We may assume Ψ is Suslin and co-Suslin by Woodin's Basis Theorem. (See [52], Theorem 7.1.) Thus there is a coarse Γ-Woodin mouse $(N^*, \lhd, S, T, \Sigma^*)$ that captures Ψ. Working in N^*, we get that P iterates by Ψ to a level (Q, Ψ) of the pure extender pair construction of N^*. Let $\pi \colon P \to Q$ be the iteration map; then (P, Ψ^π) is a pure extender pair. $\square$

Proposition 6.26 *Assume* AD^+, LEC, *and* $\theta_0 < \theta$; *then there are pure extender pairs* (P, Σ) *and* (P, Ω) *such that* $(P, \Sigma) <^* (P, \Omega)$.

Proof.(Sketch.) By LEC, there is a pure extender pair (P, Ω) such that Ω is not ordinal definable from a real. Fix such a pair. By the Basis Theorem, there is a Σ such that (P, Σ) is a pure extender pair, and Σ is ordinal definable from a real. Suppose toward contradiction that $(P, \Omega) \leq^* (P, \Sigma)$; then

$$\Omega = (\Sigma_s)^\pi$$

for some stack s and iteration map π. Thus Ω is ordinal definable from a real, contradiction. $\square$

It follows that under the hypotheses of 6.26, there are pure extender pairs (P, Σ) and (P, Ω) such that for some R, P iterates normally by Σ to a proper initial segment of R, and normally by Ω to a proper extension of R.

The Dodd-Jensen Lemma hypothesis that $\Sigma^\pi_{s,P} = \Sigma$ is too restrictive for use in the proof of solidity and universality of standard parameters. For that proof, we need the Weak Dodd-Jensen Lemma of [27].

Note that the proofs we have given that background induced strategies normalize well and have strong hull condensation actually yield (ω_1, ω_1) strategies Ω such that each Ω^*_s, for $\mathrm{lh}(s) < \omega_1$, normalizes well and has strong hull condensation. Here Ω^*_s is the complete strategy, defined on finite stacks t, given by $\Omega^*_s(t) = \Omega(s^\frown t)$. We need this in the weak Dodd-Jensen argument to come.

Let N be a countable pure extender premouse or lpm, and $\langle e_i \mid i < \omega \rangle$ enumerate the universe of N. A map $\pi \colon N \to M$ is *$\vec{e}$-minimal* just in case π is elementary, and whenever $\sigma \colon N \to M | \langle \eta, k \rangle$ is elementary, then $\langle \eta, k \rangle = l(M)$, and if $\sigma \neq \pi$, then for i least such that $\sigma(e_i) \neq \pi(e_i)$, we have $\pi(e_i) < \sigma(e_i)$ (in the order of construction). A complete strategy Ω for N has the *weak Dodd-Jensen property relative to $\vec{e}$* iff whenever $M = M_\infty(s)$ for some stack s by Ω, and there is some elementary embedding fom N to an initial segment of M, then the branch N-to-M of s does not drop, and the iteration map i^s is $\vec{e}$-minimal.

Lemma 6.27 *(Weak Dodd-Jensen) Let* (M, Ω) *be a mouse pair with scope* H_δ, *and let* $\vec{e}$ *be an enumeration of the universe of* M *in order type* ω. *Suppose that* Ω *is* *defined on all countable M-stacks s from H_δ, and that for any such s having a last model, $(M_\infty(s), \Omega_s)$ is an lbr hod pair. Then there is a countable M-stack s by Ω having last model $N = M_\infty(s)$, and an elementary $\pi \colon M \to N$, such that*

(1) $(N, (\Omega_s)^\pi)$ is a mouse pair, and

(2) $(\Omega_s)^\pi$ has the weak Dodd-Jensen property relative to $\vec{e}$.

Proof. The proof from [27] goes over verbatim. Notice here that any such $(N, (\Omega_s)^\pi)$ is an lbr hod pair, by 6.17 and 6.13. □

The proofs of Lemmas 4.54 and 4.55 go over from pure extender pairs to least branch hod pairs with no change. We get

Lemma 6.28 *Let (P, Σ) and (P, Λ) be mouse pairs with scope H_δ, and suppose that Σ and Λ agree on countable normal trees; then $\Sigma = \Lambda$.*

Lemma 6.29 *Let (P, Σ) be a mouse pair pair with scope H_δ, and let $j \colon V \to M$ be elementary, where M is transitive and $\mathrm{crit}(j) > |P|$; then $j(\Sigma)$ and Σ agree on all trees in $j(H_\delta) \cap H_\delta$.*

We have stated the elementary results about lbr hod pairs in this section as results about mouse pairs, because that is their natural context. We are mainly interested in lbr hod pairs for the rest of this book, so we shall return to that level of generality.

6.4 Background constructions

It is easy to modify the background constructions of pure extender premice described in the preliminaries chapter so that they produce least branch hod pairs. The background conditions for adding an extender are unchanged. If we have reached the stage at which $M_{\nu,0}$ is to be defined, then our construction, together with an iteration strategy for the background universe, will have provided us with complete iteration strategies $\Omega_{\eta,l}$ for $M_{\eta,l}$, for all $\eta < \nu$. We must assume here that the background universe knows how to iterate itself for trees that are of the form $\mathrm{lift}(\mathcal{T}, M_{\eta,l}, \mathbb{C})_0$. Each $(M_{\eta,l}, \Omega_{\eta,l})$ will be a least branch hod pair. If $M_{\nu,0}$ is to be branch-active according to the lpm requirements, then we use the appropriate $\Omega_{\eta,l}$ to determine $\dot{B}^{M_{\nu,0}}$.

The additional strategy predicates in our structures affect what we mean by cores and resurrection, but otherwise nothing much changes.

As before, $M_{\nu,k+1}$ is the core of $M_{\nu,k}$. We shall need to show that the standard parameter of $M_{\nu,k}$ behaves well, so that this core is sound, and agrees with $M_{\nu,k}$ up to ξ, where $\xi = (\rho(M_{\nu,k})^+)^{M_{\nu,k}}$. Letting $\pi\colon M_{\nu,k+1} \to M_{\nu,k}$ be the uncoring map, and $\gamma < \xi$, this requires that $(\Omega_{\nu,k})_{\langle \gamma, l\rangle}$ agree with the π-pullback of $(\Omega_{\nu,k})_{\langle \pi(\gamma), l\rangle}$ on all stacks belonging to $M_{\nu,k}|\xi$. We shall show this, but we shall not show that these two strategies agree on all $M_{\nu,k}|\langle \gamma, l\rangle$-stacks in V. We doubt that is true in general, but we do not have a counterexample.

The simplest sort of iterability hypothesis under which we can carry out such a construction is the following.

Definition 6.30 $\mathsf{IH}_{\kappa,\delta}$ *is the assertion: for any coarsely coherent* $\vec{F}$ *such that all* F_ν *have critical point* $> \kappa$*, and belong to* V_δ*,* $(V, \vec{F})$ *is strongly uniquely* (δ, δ)*- iterable.*

Assuming AD^+, we have by Corollary 4.19 that whenever $(N^*, \delta, S, T, \lhd, \Sigma^*)$ is a coarse Γ-Woodin tuple, then $L(N^*, \lhd, S, T) \models \mathsf{IH}_{\omega,\delta}$, where δ is the Γ-Woodin of N^*. So we could be doing our background construction inside this model.

Now let δ be inaccessible, w be a wellorder of V_δ, and $\kappa < \delta$. Let us assume $\mathsf{IH}_{\kappa,\delta}$ for a while; we shall relax this assumption later. A *least branch w-construction above* κ is a full background construction in which, as before, the background extenders are nice, have critical points $> \kappa$, cohere with w, have strictly increasing strengths, and are minimal (first in Mitchell order, then in w). The index of the last pair in our construction is some $\langle \nu, k\rangle \leq_{\text{lex}} \langle \delta, 0\rangle$.

More precisely, such a construction $\mathbb{C}$ consists of least branch premice $M^{\mathbb{C}}_{\nu,k}$ and extenders $F^{\mathbb{C}}_\nu$. The length $\text{lh}(\mathbb{C})$ of $\mathbb{C}$ is the least $\langle \nu, k\rangle$ such that $M^{\mathbb{C}}_{\nu,k}$ is not defined. $M_{0,0}$ is the passive premouse with universe V_ω, and $\Omega_{0,0}$ is its unique iteration strategy. The indices are pairs $\langle \nu, k\rangle \leq_{\text{lex}} \langle \delta, 0\rangle$ such that $-1 \leq k \leq \omega$.

$\mathbb{C}$ determines resurrection maps $\text{Res}_{\nu,k}$ and $\sigma_{\nu,k}$ for $\langle \nu, k\rangle <_{\text{lex}} \text{lh}(\mathbb{C})$, in the same way as before: we define $\text{Res}_{\nu,k+1}$, $\sigma_{\nu,k+1}$ by

1. If $N = M_{\nu,k+1}$, then $\text{Res}_{\nu,k+1}[N] = \langle \nu, k+1\rangle$ and $\sigma_{\nu,k+1}[N] = \text{identity}$.

2. If $N \lhd M_{\nu,k+1}|(\rho^+)^{M_{\nu,k+1}}$, where $\rho = \rho(M_{\nu,k})$, then $\text{Res}_{\nu,k+1}[N] = \text{Res}_{\nu,k}[N]$ and $\sigma_{\nu,k+1}[N] = \sigma_{\nu,k}[N]$.

3. Otherwise, letting $\pi : M_{\nu,k+1} \to M_{\nu,k}$ be the anti-core map, $\text{Res}_{\nu,k+1}[N] = \text{Res}_{\nu,k}[\pi(N)]$ and $\sigma_{\nu,k+1} = \sigma_{\nu,k}[\pi(N)] \circ \pi$.

For the definition of $\text{Res}_{\nu,0}$ and $\sigma_{\nu,0}$ see [1]. The resurrection maps are fully elementary, and their agreement properties are the same as before.

The definitions of *conversion system*, and of the particular conversion system $\text{lift}(\mathcal{T}, M, \mathbb{C}) = \langle \mathcal{T}^*, \langle (\eta_\xi, l_\xi) \mid \xi < \text{lh}\,\mathcal{T}\rangle, \langle \pi_\xi \mid \xi < \text{lh}\,\mathcal{T}\rangle\rangle$, for $\mathcal{T}$ weakly normal, do not change. The lift of an M-stack is essentially the stack of the lifts, as before.

$\operatorname{lift}(s, M, \mathbb{C})_0$ is the stack on V that is one component of $\operatorname{lift}(s, M, \mathbb{C})$, and it is a maximal stack of fully normal trees. Conversion systems treat gratuitous dropping like ordinary dropping.

The sequence $\langle F_\nu^{\mathbb{C}} \mid \dot{F}^{M_{\nu,0}} \neq \emptyset\rangle$ of background extenders is coarsely coherent, so by $\mathsf{IH}_{\kappa,\delta}$, V is strongly uniquely $(\delta, \delta, \vec{F})$-iterable. Let Σ^* be the iteration strategy witnessing this. Σ^* then induces a complete strategy $\Omega(\mathbb{C}, M, \Sigma^*)$ with scope H_δ for $M = M_{\nu,k}^{\mathbb{C}}$, for each $\langle \nu, k\rangle < \operatorname{lh}(\mathbb{C})$. That is,

$$s \text{ is by } \Omega(\mathbb{C}, M, \Sigma^*) \text{ iff } \operatorname{lift}(s, M, \mathbb{C})_0 \text{ is by } \Sigma^*.$$

We write

$$\Omega_{\nu,k}^{\mathbb{C}} = \Omega(\mathbb{C}, M_{\nu,k}^{\mathbb{C}}, \Sigma^*)\restriction\text{finite stacks},$$

for the (ω, δ)-iteration strategy determined by $\Omega(\mathbb{C}, M_{\nu,k}^{\mathbb{C}}, \Sigma^*)$.

Remark 6.31 For example, let $s = \langle \beta, l, \mathcal{T}\rangle$ be an $M_{\nu,k}$-stack of length one, and let $N = M_{\nu,k}|\langle \beta, l\rangle$. Let

$$\langle \eta, l\rangle = \operatorname{Res}_{\nu,k}[N], \text{ and } \sigma = \sigma_{\nu,k}[N].$$

So σ is elementary from N to $M_{\eta,l}$. Then letting

$$\operatorname{lift}(\sigma\mathcal{T}, M_{\eta,l}, \mathbb{C}, \Sigma^*) = \langle \mathcal{T}^*, \langle (\eta_\xi, l_\xi) \mid \xi < \operatorname{lh}\mathcal{T}\rangle, \langle \pi_\xi \mid \xi < \operatorname{lh}\mathcal{T}\rangle\rangle,$$

we have that

$$\langle \beta, l, \mathcal{T}\rangle \text{ is by } \Omega_{\nu,k}^{\mathbb{C}} \text{ iff } \mathcal{T}^* \text{ is by } \Sigma^*.$$

If $Q = \mathcal{M}_\xi^{\mathcal{T}}$ is the last model of $\mathcal{T}$, and $\tau : Q \to M_\xi^{\sigma\mathcal{T}}$ is the copy map, then $\pi_\xi \circ \tau$ maps Q into a model of the construction $i_{0,\xi}^{\mathcal{T}^*}(\mathbb{C})$. This enables us to define $\Omega_{\nu,k}$ on stacks extending s; for example, if $t = s^\frown\langle \gamma, n, \mathcal{U}\rangle$, then we handle the possibly gratuitous drop in Q by resurrecting $\pi_\xi(Q|\langle \gamma, n\rangle)$ from the stage $\pi_\xi(Q)$ inside $i_{0,\xi}^{\mathcal{T}^*}(\mathbb{C})$, just as above. Etc.

Our construction determines in this way complete iteration strategies $\Omega_{\nu,k}^{\mathbb{C}}$ for $M_{\nu,k}^{\mathbb{C}}$, defined on stacks in H_δ, for each $\langle \nu, k\rangle < \operatorname{lh}(\mathbb{C})$. We demand that $(M_{\nu,k}, \Omega_{\nu,k})$ be a least branch hod pair; otherwise we stop the construction and leave $M_{\nu,k}$ undefined.

Suppose now we have $M_{\nu,k}$ and $\Omega_{\nu,k}$, with $k \geq 0$. Let $\rho = \rho(M_{\nu,k})$ and $p = p(M_{\nu,k})$ be the $k+1$-st projectum and parameter. Let u be either the sequence of solidity witnesses for $p_k(M_{\nu,k})$, or that sequence together with $\rho_{k-1}(M_{\nu,k})$ if the latter is $< o(M_{\nu,k})$. Let

$$\pi \colon N \to M_{\nu,k}$$

where N is transitive and

$$\operatorname{ran}(\pi) = \operatorname{Hull}_{k+1}^{M_{\nu,k}}(\rho \cup \{p, u\}).$$

We shall prove, for $k \geq 0$,

$(\dagger)_{\nu,k}$.

(a) $M_{\nu,k}|(\rho^+)^{M_{\nu,k}} = N|(\rho^+)^N$, and

(b) $\pi^{-1}(p)$ is solid over N.

Items (a) and (b) of $(\dagger)$ are the universality and solidity of the standard parameter. They are needed to see that the iteration maps of $\Omega_{\nu,k+1}$ are elementary, which goes into the proof that the lifting maps in the construction of $\Omega_{\nu,k+1}$ are weakly elementary. So we need (a) and (b) before we can define $\Omega_{\nu,k+1}$.

Corollary 6.66 below proves $(\dagger)_{\nu,k}$ (for $k \geq 0$) under the assumption that for every countable M and $\pi\colon M \to M_{\nu,k}$ elementary, letting $\Psi = (\Omega_{\nu,k})^\pi \restriction \mathrm{HC}$, $L(\Psi, \mathbb{R}) \models \mathsf{AD}^+$. Note here that Ψ is κ-Universally Baire, where κ is our lower bound on the critical points of background extenders, by the uniqueness implicit in $\mathsf{IH}_{\kappa,\delta}$. So $L(\Psi, \mathbb{R}) \models \mathsf{AD}^+$ follows from there being infinitely many Woodin cardinals below κ. (If we are already assuming AD^+, and the construction $\mathbb{C}$ takes place inside a coarse Γ-Woodin mouse, then the argument is slightly different.)

If $(M_{\nu,k}, \Omega_{\nu,k})$ satisfies $(\dagger)_{\nu,k}$, then we let

$$M_{\nu,k+1} = \text{ transitive collapse of } \operatorname{Hull}_{k+1}^{M_{\nu,k}}(\rho \cup \{p, u\}),$$

with $k(M_{\nu,k+1}) = k + 1$. The lifting procedure and our iterability hypothesis $\mathsf{IH}_{\kappa,\delta}$ yield a complete iteration strategy

$$\Omega_{\nu,k+1} = \Omega_{\nu,k+1}^{\mathbb{C}}$$

for $M_{\nu,k+1}$ on stacks in H_δ.

Lemma 6.32 *[$\mathsf{IH}_{\kappa,\delta}$] Assume $\mathbb{C}$ satisfies $(\dagger)_{\nu,k}$; then*

(1) $(M_{\nu,k+1}, \Omega_{\nu,k+1})$ is a least branch hod pair, and

(2) setting $\gamma = (\rho^+)^{M_{\nu,k}}$, $(\Omega_{\nu,k})_{\langle\gamma,0\rangle} = (\Omega_{\nu,k+1})_{\langle\gamma,0\rangle}$.

Proof. Part (2) is an immediate consequence of the fact that for $\xi < (\rho^+)^{M_{\nu,k}}$ and $Q = M_{\nu,k}|\langle \xi, l\rangle$, $\mathrm{Res}_{\nu,k}[Q] = \mathrm{Res}_{\nu,k+1}[Q]$ and $\sigma_{\nu,k}[Q] = \sigma_{\nu,k+1}[Q]$.

For part (1), we repeat the proofs that background induced strategies normalize well and have strong hull condensation (4.41 and 4.49) that we gave in the pure extender model case. What is left is to show that $(M_{\nu,k+1}, \Omega_{\nu,k+1})$ is self-consistent.

For this, let $(N, \Omega) = (M_{\nu,k+1}, \Omega_{\nu,k+1})$, and let s be a stack on N by Ω, with last model P. Let $\mathcal{T} \in \dot{\Sigma}^P$. We must see that $\mathcal{T}$ is by Ω_s. Let

$$s^* = \mathrm{lift}(s, N, \mathbb{C}),$$

and let R be the last model of s^*. Let Σ^* be the unique $\vec{F}^{\mathbb{C}}$-iteration strategy for V, so that $\Sigma^*_{s^*,R}$ is the unique $\vec{F}^{\mathbb{D}}$ strategy for R, where $\mathbb{D}$ is the image of $\mathbb{C}$ in R. We have

$$\pi \colon N \to Q$$

where Q is a model of the construction of R. Let Ψ be the strategy for Q induced by the construction of R. We have that

$$\Omega_s = \Psi^\pi,$$

because this is how Ω is induced by Σ^*. So we are done if we show that $\pi\mathcal{T}$ is by Ψ.

But $\pi(\mathcal{T}) \in \dot{\Sigma}^Q$, so $\pi(\mathcal{T})$ is by Ψ because (Q, Ψ) is an lbr hod pair in R. Moreover, Ψ has strong hull condensation, not just in R, but in V. (That is because a psuedo-hull $\mathcal{W}$ of some $\mathcal{U}$ by Ψ lifts to a psuedo-hull $\mathcal{W}^*$, of some $\mathcal{U}^*$ by $\Sigma^*_{s^*,R}$, and even if $\mathcal{W}$ and $\mathcal{W}^*$ are not in R, $\Sigma^*_{s^*,R}$ chooses unique-in-V cofinal wellfounded branches, so $\mathcal{W}^*$ is by $\Sigma^*_{s^*,R}$, and hence $\mathcal{W}$ is by Ψ.) Since $\pi\mathcal{T}$ is a hull of $\pi(\mathcal{T})$, $\pi\mathcal{T}$ is by Ψ, as desired.

□

If $(\dagger)_{\nu,k}$ is not the case, then we stop the construction, leaving $M_{\nu,k+1}$ undefined.

Suppose now that $(\dagger)_{\nu,k}$ holds for all $k < \omega$. For $k < \omega$ sufficiently large, $M_{\nu,k} = M_{\nu,k+1}$[21], and we set

$$M_{\nu,\omega} = \text{eventual value of } M_{\nu,k} \text{ as } k \to \omega,$$

and

$$M_{\nu+1,0} = \text{rud closure of } M_{\nu,\omega} \cup \{M_{\nu,\omega}\},$$
$$\text{arranged as a fully passive premouse.}$$

$\Omega_{\nu+1,0} = \Omega^{\mathbb{C}}_{\nu+1,0}$ is the $\mathbb{C}$-induced strategy. The proof of Lemma 6.32 gives

[21]Except of course that the distinguished degree of soundness differs.

Lemma 6.33 *[$\mathsf{IH}_{\kappa,\delta}$] Suppose $(\dagger)_{\nu,k}$ holds for all $k < \omega$; then $(M_{\nu+1,0}, \Omega_{\nu+1,0})$ is an lbr hod pair with scope H_δ.*

Finally, if ν is a limit, put

$\mathcal{M}^{<\nu}$ = unique fully passive structure P such that for all premice N,
$N \lhd P$ iff $N \lhd M_{\alpha,l}$ for all sufficiently large $\langle \alpha, l\rangle < \langle \nu, 0\rangle$.

Case 1. $M^{<\nu}$ is branch active.

Let $M = M^{<\nu}$, and $b = \Omega^{<\nu}(s^M)$; then

$$M_{\nu,0} = (M^{<\nu}, \emptyset, B),$$

where $B = \{\eta^M + \gamma \mid \gamma \in b \wedge \eta^M + \gamma < o(M)\}$.

Case 2. There is an F such that $(M^{<\nu}, F, \emptyset)$ is an lpm, $\mathrm{crit}(F) \geq \kappa$, and there is a certificate for F, in the sense of Definition 2.1 of [29].

As we remarked, cases 1 and 2 are mutually exclusive. We shall prove

$(\dagger)_{\nu,\mathbf{-1}}$. There is at most one F such that $(M^{<\nu}, F, \emptyset)$ is an lpm, $\mathrm{crit}(F) \geq \kappa$, and F admits a certificate in the sense of Definition 2.1 of [29].

This is the Bicephalus Lemma; see Corollary 7.5. We are now allowed either to set

$$M_{\nu,0} = (M^{<\nu}, \emptyset, \emptyset),$$

that is, to pass on the opportunity to add F, or to set

$$M_{\nu,0} = (M^{<\nu}, \emptyset, F).$$

In the latter case, we add the same demands of our certificate as we had in Definition 2.42, and again choose $F^{\mathbb{C}}_\nu$ to be the unique certificate for F such that

(∗) $F^{\mathbb{C}}_\nu$ is a certificate for F, minimal in the Mitchell order among all certificates for F, and w-least among all Mitchell order minimal certificates for F.

Thus the sequence of all $\vec{F}^{\mathbb{C}}$ of all $F^{\mathbb{C}}_\nu$ is coarsely coherent. By a $\mathbb{C}$-iteration, we mean a $\vec{F}^{\mathbb{C}}$-iteration in the sense explained above.

Case 3. Otherwise.

Then we set

$$M_{\nu,0} = (M^{<\nu}, \emptyset, \emptyset).$$

In any case, $\Omega_{\nu,0}$ is the $\mathbb{C}$-induced strategy for $M_{\nu,0}$. We get

Lemma 6.34 *[$\mathsf{IH}_{\kappa,\delta}$] Let ν be a limit ordinal, and suppose that $(\dagger)_{\alpha,j}$ holds for all $\alpha < \nu$ and $j < \omega$; then $(M_{\nu,0}, \Omega_{\nu,0})$ is an lbr hod pair.*

This finishes the definition of what it is for $\mathbb{C}$ to be a least branch w- construction above κ.

We may wish to restrict our choice of background extenders to members of some coarsely coherent sequence $\vec{F}$ given in advance. Such $\mathbb{C}$ we call *least branch $\vec{F}$-constructions.* Any w-construction $\mathbb{C}$ is a least branch $\vec{F}^{\mathbb{C}}$-construction. Also, it is not necessary that the iteration strategy for the background universe used in the construction pick unique wellfounded branches. What we need is that it normalizes well, has strong hull condensation, and is moved to its tails by its own iteration maps. So let us drop our hypothesis $\mathsf{IH}_{\kappa,\delta}$, and make the following definitions.

Definition 6.35 *A* coarse strategy premouse *is a structure* $(M, \vec{F}, \Sigma)$ *such that* $(M, \vec{F})$ *is a coarse extender premouse, and* $\Sigma \in M$*, and for some* θ*, the following hold in* M*:*

(a) θ *is inaccessible and* $\vec{F} \in V_\theta$*,*

(b) Σ *is a* $(\theta, \theta, \vec{F})$*-iteration strategy for* V *that normalizes well and has strong hull condensation, and*

(c) if $i \colon V \to N$ *is an iteration map associated to the stack* s *by* Σ*, then* $i(\Sigma) \subseteq \Sigma_s$*.*

Inside a coarse strategy premouse $(M, \vec{F}, \Sigma)$ we can do least branch $(\vec{F}, \Sigma)$-constructions. These are sequences $\mathbb{C} = \langle M_{\nu,k}, \Omega_{\nu,k}, F_\nu \rangle$, where the $M_{\nu,k}$ are formed as above using background extenders $F_\nu \in \vec{F}$, and the $\Omega_{\nu,k}$ are given by $\Omega_{\nu,k} = \Omega(\mathbb{C}, M_{\nu,k}, \Sigma)$. In order to show that $\mathbb{C}$ does not break down, we need a further assumption about the countable elementary submodels of $(M, \vec{F}, \Sigma)$.

The next definition is meant to be considered in the AD^+ context.

Definition 6.36 *A* coarse strategy pair *is a pair* $\langle (M, \vec{F}, \Sigma), \Sigma^* \rangle$ *such that*

(a) $(M, \vec{F}, \Sigma)$ *is a countable coarse strategy premouse,*

(b) Σ^* *is a complete* (ω_1, ω_1) *iteration strategy for* $(M, \vec{F})$ *that normalizes well and has strong hull condensation, and*

(c) if $i \colon M \to N$ *is the iteration map associated to a stack* s *by* Σ^**, then* $i(\Sigma) \subseteq \Sigma^*_s$*.*

The proofs of 6.32, 6.33, and 6.34 show

Lemma 6.37 *Let* $\langle (M, \vec{F}, \Sigma), \Sigma^* \rangle$ *be a coarse strategy pair, and let* $\mathbb{C}$ *be an* $(\vec{F}, \Sigma)$*-construction done in* M*; then*

(1) *if* $(\dagger)_{\nu,k}$ *then* $(M_{\nu,k+1}, \Omega_{\nu,k+1})$ *is a least branch hod pair with scope HC, and setting* $\gamma = (\rho^+)^{M_{\nu,k}}$*,* $(\Omega_{\nu,k})_{\langle\gamma,0\rangle} = (\Omega_{\nu,k+1})_{\langle\gamma,0\rangle}$*,*

(2) *if* $(\dagger)_{\nu,k}$ *holds for all* $k < \omega$*, then* $(M_{\nu+1,0}, \Omega_{\nu+1,0})$ *is a least branch hod pair with scope HC, and*

(3) *if* ν *is a limit ordinal and* $(\dagger)_{\alpha,k}$ *holds for all* $\alpha < \nu$ *and* $k < \omega$*, then* $(M_{\nu,0}, \Omega_{\nu,0})$ *is a least branch hod pair with scope HC.*

Notice here that the pairs referred to in (1)-(3) have scope all of HC, even though they come from a construction done in the countable model M. This is because Σ extends to Σ^*, and Σ^* has scope HC.

Definition 6.38 $\mathbb{C}$ *is a* least branch background construction *iff* $\mathbb{C}$ *is a least branch* $(\vec{F}, \Sigma)$*-construction, for some* $\vec{F}$ *and* Σ*. We say* $\mathbb{C}$ *is* maximal *iff it never passes on an opportunity to add an extender.*

Definition 6.39 *A least branch background construction* $\mathbb{C}$ *is* pathological *iff for some* $\langle\nu, k\rangle$*,* $(\dagger)_{\nu,k}$ *is false.*

A pathological construction is one that reaches a pair $(M_{\nu,k}, \Omega_{\nu,k})$ whose standard parameter does not behave well, or that reaches a stage $\langle\nu, -1\rangle$ at which the Bicephalus Lemma fails.[22]

We shall show that assuming AD^+, if $\langle(M, \vec{F}, \Sigma), \Sigma^*\rangle$ is a coarse strategy pair, and $\mathbb{C}$ is a least branch $(\vec{F}, \Sigma)$-construction done in M, then $\mathbb{C}$ is not pathological. Lemma 6.37 is a preliminary step in that direction. The remaining steps are taken in Theorem 6.57 on the existence of cores, and in 7.3, the Bicephalus Lemma.

The existence of coarse strategy pairs under AD^+ comes from

Theorem 6.40 *Assume* AD^+*, and let* (M, Σ^*) *be a coarse* Γ*-Woodin pair. Let* $\delta = \delta^M$*, and* $\vec{F} \subseteq V_\delta^M$ *be such that* $M \models$ *"*$\vec{F}$ *is coarsely coherent". Suppose* $\delta < \theta < \alpha$ *with* θ *and* α *inaccessible in* M*. Let* $P = V_\alpha^M$ *and* $\Sigma = \Sigma^* \cap V_\theta^M$*; then* $\langle(P, \vec{F}, \Sigma), \Sigma^*\rangle$ *is a coarse strategy pair.*

[22]It is not clear that we need to stop our construction because of a bicephalus pathology. We might continue by not adding any extenders to $M^{<\nu}$, or by picking one of the certified extenders and adding it. However, the existence of a bicephalus pathology would cause problems later, in the argument that a certified extender that coheres with $M^{<\mu}$ must satisfy the Jensen initial segment condition. Without this, we can't show the model we construct reaches even a Woodin cardinal, or, in the Γ-Woodin background model case, is universal.

Proof. Since Σ^* is guided by C_Γ Q-structures, $\Sigma^* \cap V_\theta^M \in M$, that is, $\Sigma \in P$. Moreover, M believes it is strongly uniquely $(\theta, \theta, \vec{F})$-iterable by Σ, so P believes Σ has strong hull condensation, normalizes well, and moves itself to its tails under iteration. Thus $(P, \vec{F}, \Sigma)$ is a coarse strategy premouse. The remaining clauses of definition 6.36 follow from the fact that Σ^* witnesses that M is strongly uniquely (ω_1, ω_1)-iterable in V. $\square$

If we are starting with ZFC and very large cardinals, together with $\mathsf{IH}_{\kappa,\delta}$, we can use

Theorem 6.41 *Assume* ZFC *plus* $\mathsf{IH}_{\kappa,\delta}$*, and that there are $\lambda < \mu < \kappa$ such that λ is a limit of Woodin cardinals, and μ is measurable. Let w be a wellorder of V_δ, $\mathbb{C}$ be a w-construction above κ, and let Ω be the unique $\vec{F}^{\mathbb{C}}$-iteration strategy for V; then there is a coarse strategy premouse of the form $(N, \vec{F}^{\mathbb{C}}, \Omega \restriction N)$ such that $V_\delta^N = V_\delta$. Moreover, whenever*

$$\pi\colon (M, \vec{F}^{\mathbb{D}}, \Sigma) \to (N, \vec{F}^{\mathbb{C}}, \Omega \restriction N)$$

is elementary, with M countable transitive, then letting $\Sigma^* = \Omega^\pi \restriction HC$,

(a) $\langle (M, \vec{F}^{\mathbb{D}}, \Sigma), \Sigma^* \rangle$ *is a coarse strategy pair, and*

(b) $L(\mathbb{R}, \Sigma^*) \models \mathsf{AD}^+$.

Proof. We leave it to the reader to find N such that $(N, \vec{F}^{\mathbb{C}}, \Omega \restriction N)$ is a coarse strategy premouse. Since π is elementary, $(M, \vec{F}^{\mathbb{D}}, \Sigma)$ is a coarse strategy premouse. $\langle (M, \vec{F}^{\mathbb{D}}, \Sigma), \Sigma^* \rangle$ is a coarse strategy pair because strong hull condensation, normalizing well, and self-consistency pull back under π. Finally, Σ^* is κ-universally Baire by $\mathsf{IH}_{\kappa,\delta}$. Since we have λ and μ, we get that $L(\mathbb{R}, \Sigma^*) \models \mathsf{AD}^+$. $\square$

Theorem 6.41 makes theorems about the constructions of coarse strategy pairs proved assuming AD^+ applicable in the ZFC context. Whatever was true of $\mathbb{C}$ in N is true of $\pi^{-1}(\mathbb{C})$ in M.

Remark 6.42 If $\mathbb{C}$ is a maximal, non-pathological, pure extender w-construction above κ, then the map $\langle \nu, k \rangle \mapsto M_{\nu,k}^{\mathbb{C}}$ is ordinal definable. w enters into picking the $F_\nu^{\mathbb{C}}$, but not into defining the $M_{\nu,k}^{\mathbb{C}}$, by the Bicephalus Lemma. But it is not at all clear that if $\mathbb{C}$ is a maximal, non-pathological, least branch w-construction, then each $M_{\nu,k}^{\mathbb{C}}$ is ordinal definable. The problem lies in the use of w to pick background extenders. Although our strategy for V is unique, different choices for the $F_\nu^{\mathbb{C}}$ lead to different ways of lifting trees on $M_{\nu,k}^{\mathbb{C}}$ to V, and hence possibly different candidates for $\Omega_{\nu,k}^{\mathbb{C}}$. Information about $\Omega_{\nu,k}$ is being recorded in later $M_{\mu,l}$, making it possible that they are not ordinal definable.

Remark 6.43 Let $M = M^{\mathbb{C}}_{\nu,k}$ and $\Omega = \Omega^{\mathbb{C}}_{\nu,k}$, and suppose $M \models \mathsf{ZFC}$. Then $\Omega{\restriction}M$ is definable over M, by a definition that is uniform in $\langle \nu, k\rangle$. That is because the restriction of Ω to normal trees in M is given by $\dot{\Sigma}^M$, and that determines its restriction to stacks of normal trees because Ω normalizes well, and that determines its restriction to stacks of weakly normal trees in M because Ω is strategy coherent.

6.5 Comparison and the hod pair order

We can adapt Theorem 5.11 to hod pairs.

Definition 6.44 *Let (M, Σ) and (N, Ω) be mouse pairs; then*

(a) (M, Σ) iterates past (N, Ω) *iff there is a normal iteration tree $\mathcal{T}$ by Σ on M with last model Q such that $N \trianglelefteq Q$, and $\Sigma_{\mathcal{T},N} = \Omega$.*

(b) (M, Σ) iterates to (N, Ω) *iff there are $\mathcal{T}$ and Q as in (a), and moreover, $N = Q$, and the branch M-to-Q of $\mathcal{T}$ does not drop.*

(c) (M, Σ) iterates strictly past (N, Ω) *iff it iterates past (N, Ω), but not to (N, Ω).*

The normal tree $\mathcal{T}$ above is completely determined by N and Σ; it must come by iterating away the least extender disagreement. (M, Σ) and (N, Ω) are strategy coherent and self-consistent, so (M, Σ) iterates past (N, Ω) if and only if (i) no strategy disagreements show up as we iterate, (ii) no non-empty extenders from N participate in least disagreements, so that N does not move, and (iii) N is an initial segment of the final model on the M-side.

The following notation is convenient: let $\mathbb{C}$ be a construction such that $M^{\mathbb{C}}_{\nu,0}$ is extender-active; then

$$(M^{\mathbb{C}}_{\nu,-1}, \Omega^{\mathbb{C}}_{\nu,-1}) = (M^{<\nu}, \Omega^{<\nu}).$$

Setting $\gamma = \hat{o}(M^{\mathbb{C}}_{\nu,0})$, we can write this $(M^{\mathbb{C}}_{\nu,-1}, \Omega^{\mathbb{C}}_{\nu,-1}) = (M^{\mathbb{C}}_{\nu,0}|\langle\gamma, -1\rangle, (\Omega^{\mathbb{C}}_{\nu,0})_{\langle\gamma,-1\rangle})$.

Adapting the proof of Theorem 5.11, we get

Theorem 6.45 *Suppose that $(V, \vec{F}, \Lambda)$ is a coarse strategy premouse, with $\vec{F} \subseteq V_\delta$, where δ is inaccessible. Let (P, Σ) be a least branch hod pair with scope H_δ such that $|P| < \mathrm{crit}(E)$ for all E on $\vec{F}$. Let $\mathbb{C}$ be a $(\vec{F}, \Lambda)$- construction, and let $\langle \nu, k\rangle < \mathrm{lh}(\mathbb{C})$ be such that (P, Σ) iterates strictly past $(M^{\mathbb{C}}_{\eta,j}, \Omega^{\mathbb{C}}_{\eta,j})$, for all $\langle \eta, j\rangle <_{\mathrm{lex}} \langle \nu, k\rangle$; then (P, Σ) iterates past $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$.*

Remark 6.46 It is not possible that (P, Σ) iterates to $(M^{\mathbb{C}}_{\nu,-1}, \Omega^{\mathbb{C}}_{\nu,-1})$, for some ν such that $F^{\mathbb{C}}_\nu \neq \emptyset$. For if so, then in $\mathrm{Ult}(V, F^{\mathbb{C}}_\nu)$, (P, Σ) would iterate strictly past $(M^{\mathbb{C}}_{\nu,-1}, \Omega^{\mathbb{C}}_{\nu,-1})$, contradiction.

Remark 6.47 It follows by our work realizing resurrection embeddings as branch embeddings that if M iterates to $M^{\mathbb{C}}_{\nu,l+1}$, then it iterates strictly past $M^{\mathbb{C}}_{\nu,l}$. This terminology might be a bit confusing at first, because the iteration tree $\mathcal{T}$ from M to $M^{\mathbb{C}}_{\nu,l+1}$ is an initial segment of the tree $\mathcal{U}$ from M to $M^{\mathbb{C}}_{\nu,l}$. Along the branch of $\mathcal{U}$ from M to $M^{\mathbb{C}}_{\nu,l}$ we dropped once, at $M^{\mathbb{C}}_{\nu,l+1}$, from degree $l+1$ to degree l. That drop meant that M iterates past, but not to, $M_{\nu,l}$. This is the case even if $M^{\mathbb{C}}_{\nu,l} = M^{\mathbb{C}}_{\nu,l+1}$ as an lpm, with only the attached soundness level changing. Then $\mathcal{U}$ would be $\mathcal{T}$, together with one gratuitous drop in degree at the end.

Remark 6.48 We do not know whether there can be more than one $\langle \nu, k\rangle$ such that (P, Σ) iterates to $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$.

Theorem 6.45 easily implies theorem 1.15 of the introduction:

Theorem 6.49 *Assume* AD^+, *and let* (P, Σ) *be a least branch hod pair; then* $(*)(P, \Sigma)$ *holds.*

Proof. Let N^* be a coarse Γ-Woodin model that Suslin-co-Suslin captures Σ, as in the hypothesis of $(*)(P, \Sigma)$. We can then simply apply 6.45 inside N^*. $\square$

In order to apply $(*)(P, \Sigma)$, we need to know that there are coarse Γ-Woodin models whose maximal hod-pair construction does not break down before they absorb (P, Σ). The following lemma will help with that.

Lemma 6.50 *Assume* $\mathsf{IH}_{\kappa,\delta}$, *and let let* $\mathbb{C}$ *be a least branch construction above* κ. *Suppose that* $M^{\mathbb{C}}_{\nu,k}$ *exists. Let* (P, Σ) *be a least branch hod pair with scope* H_δ *such that* $o(P) < \kappa$*; then for any* ν, k*:*

(a) if (P, Σ) *iterates strictly past all* $(M^{\mathbb{C}}_{\mu,l}, \Omega^{\mathbb{C}}_{\mu,l})$ *such that* $\mu < \nu$, *then* $\mathbb{C}$ *satisfies* $(\dagger)_{\nu,-1}$, *and*

(b) if (P, Σ) *iterates strictly past* $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$, *then* $\mathbb{C}$ *satisfies* $(\dagger)_{\nu,k}$.

Proof. For (a), suppose toward contradiction that $F_0 \neq F_1$, and for $i \in \{0, 1\}$, $(M^{<\nu}, F_i, \emptyset)$ is an lpm, $\mathrm{crit}(F_i) \geq \kappa$, and F_i is certifiable, in the sense of Definition 2.1 of [29]. It follows that for $i \in \{0, 1\}$ there is a construction $\mathbb{C}_i$ such that $M^{\mathbb{C}_i}_{\nu,0} = (M^{<\nu}, F_i, \emptyset)$, and for all $\mu < \nu$ and k, $(M^{\mathbb{C}_i}_{\mu,k}, \Omega^{\mathbb{C}_i}_{\mu,k}) = (M^{\mathbb{C}}_{\mu,k}, \Omega^{\mathbb{C}}_{\mu,k})$. It follows from

Theorem 6.45 that (P, Σ) iterates past both $(M^{\mathbb{C}_0}_{\nu,0}, \Omega^{\mathbb{C}_0}_{\nu,0})$ and $(M^{\mathbb{C}_1}_{\nu,0}, \Omega^{\mathbb{C}_1}_{\nu,0})$. This is impossible, for it has to be the same iteration, but $F_0 \neq F_1$.

For (b), we have a normal tree $\mathcal{T}$ on P by Σ, with last model $N = \mathcal{M}^{\mathcal{T}}_{\gamma}$, such that either

(i) $M^{\mathbb{C}}_{\nu,k}$ is a proper initial segment of N, or

(ii) $M^{\mathbb{C}}_{\nu,k} = N$, and $[0, \gamma]_T$ drops (in model or degree).

We claim that in either case, $\mathbb{C}$ satisfies $(\dagger)_{\nu,k}$, a contradiction.

Let μ and s be the projectum and standard parameter of $M_{\nu,k}$. (That is, the $k+1$-st.) In case (i), $M_{\nu,k}$ is sound, so (a) and (b) of $(\dagger)_{\nu,k}$ hold trivially.

Suppose we are in case (ii), and let $Q = \mathcal{M}^{\mathcal{T}}_{\theta} | \langle \hat{o}(Q), k \rangle$ be the last structure we drop to in $[0, \gamma]_T$. So $k(Q) = k$, and Q is sound (i.e. $k+1$ sound), and setting

$$i = \hat{\imath}^{\mathcal{T}}_{\theta,\gamma},$$

we have that $i \colon Q \to N$ is elementary, and

$$\rho(Q) = \rho(N) = \mu \leq \mathrm{crit}(i).$$

Since there was no further dropping, Q and N agree to their common value for μ^+. Also, i maps $p(Q)$ to s, so s is solid. This gives us (a) and (b) of $(\dagger)_{\nu,k}$. □

From this we get

Theorem 6.51 *Assume* AD^+*, and let* (P, Σ) *be an lbr hod pair with scope HC. Let* $\mathrm{Code}(\Sigma) \in \Gamma$*, and let* $(N^*, \delta, S, T, \lhd, \Psi)$ *be a coarse* Γ*-Woodin tuple, and let* $\mathbb{C}$ *be the maximal least branch construction of* N^**; then there is an* $\langle \nu, k \rangle$ *such that*

(i) $\nu < \delta$*,*

(ii) $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$ *exists (that is, the construction has not broken down yet), and*

(iii) there is a normal $\mathcal{T}$ *such that* (P, Σ) *iterates via* $\mathcal{T}$ *to* $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$*.*

Remark 6.52 Clause (iii) of the conclusion can be understood as a truth in N^* about $\Sigma \cap N^*$. But letting $(\Omega^{\mathbb{C}}_{\nu,k})^*$ be the strategy on all stacks in V that is induced by $\mathbb{C}$ and Ψ, (iii) implies that in V, $\Sigma_{\mathcal{T}, M_{\nu,k}} = (\Omega^{\mathbb{C}}_{\nu,k})^*$.

Proof. If not, then by applying 6.45 and 6.50 in N^*, we have that $\mathbb{C}$ does not break down at all, and P iterates past $M^{\mathbb{C}}_{\delta^*,0}$ in N^*. The proof of universality at a Woodin cardinal in the pure extender premouse case (see 2.53 and 4.20) then leads to a contradiction. □

We can now show that under AD^+, any two least branch hod pairs are comparable. First, some notation for cutpoint initial segments:

Definition 6.53 *For M and N lpms, we write $M \trianglelefteq^{\mathrm{ct}} N$ iff $M \trianglelefteq N$, and whenever E is on the N-sequence and $\mathrm{lh}(E) \geq o(M)$, then $\mathrm{crit}(E) > o(M)$.*

Theorem 6.54 *Assume AD^+, and let (P, Σ) and (Q, Ψ) be lbr hod pairs with scope HC; then there are normal trees $\mathcal{T}$ and $\mathcal{U}$ by Σ and Ψ respectively, with last models R and S respectively, such that either*

(a) $R \trianglelefteq^{\mathrm{ct}} S$, $\Sigma_{\mathcal{T},R} = \Psi_{\mathcal{U},R}$, and the branch P-to-R of $\mathcal{T}$ does not drop, or

(b) $S \trianglelefteq^{\mathrm{ct}} R$, $\Psi_{\mathcal{U},S} = \Sigma_{\mathcal{T},S}$, and the branch Q-to-S of $\mathcal{U}$ does not drop.

Proof. We find Γ-Woodin background universe N^* having universally Baire representations for both strategies. Letting $\mathbb{C}$ be the maximal least branch construction of N^*, we have that there are $\langle \nu, k \rangle$ and $\langle \mu, l \rangle$ such that (P, Σ) normally iterates to $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$ and strictly past all earlier pairs, while (Q, Ψ) normally iterates to $(M^{\mathbb{C}}_{\mu,l}, \Omega^{\mathbb{C}}_{\mu,l})$ and strictly past all earlier pairs. If say $\langle \nu, k \rangle \leq_{\mathrm{lex}} \langle \mu, l \rangle$, then $(Q, \Psi))$ normally iterates past $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$, and the latter is a normal, nondropping iterate of (P, Σ). By perhaps using one more extender on the Q-side, we can arrange that $M^{\mathbb{C}}_{\nu,k}$ is a cutpoint of the last model. This yields a successful comparison of type (a). If $\langle \mu, l \rangle \leq_{\mathrm{lex}} \langle \nu, k \rangle$, then we have a successful comparison of type (b).

□

Theorem 6.54 was phrased in the language of mouse pairs in 6.21. We get at once

Corollary 6.55 *Assume AD^+, and let (M, Ω) be an lbr hod pair with scope HC; then every real in M is ordinal definable.*

It is natural to ask whether M satisfies "every real is ordinal definable". Borrowing Lemma 8.1 from the future, we have

Theorem 6.56 *Assume AD^+, and let $M, \Omega)$ be an lbr hod pair with scope HC. Suppose $M \models \mathsf{ZFC}$ + "δ is Woodin". Working in M, let UB be the collection of δ-universally Baire sets; then*

$$M \models \text{ there is a } (\Sigma^2_1)^{\mathsf{UB}} \text{ wellorder of } \mathbb{R}.$$

Proof.. Working in M, let $N \in C$ iff $N \trianglelefteq M$ and $\rho(N) = \omega$. We claim that N is in C if and only if there is a Ψ such that (N, Ψ) is an lbr hod pair, and Ψ is δ-universally Baire.

For let $N \in C$. By Lemma 8.1, Ω_N is δ-universally Baire in M. Clearly, (N, Ω_N) is an lbr hod pair in M.

Conversely, let (N, Ψ) be an lbr hod pair in M such that $\rho(N) = \omega$, and Ψ is δ universally Baire in M. Let S be the first initial segment of M that projects to ω and is such that $S \notin N$. We apply Theorem 6.45 in M. Letting $\mathbb{C}$ be the maximal construction below δ in M, neither side can iterate past $M_{\langle \delta, 0 \rangle}$ because δ is Woodin. It is easy to see then that there must be a $\langle \nu, k \rangle$ such that both (N, Ψ) and $S, \Omega_S)$ iterate to $M^{\mathbb{C}}_{\langle \nu, k \rangle}$; otherwise we would get $N \in S$ or $S \in N$. This then implies $S = N$, as desired. (It also implies $\Psi = \Omega_S$, by pullback consistency.)

This easily yields the theorem. $\square$

Theorem 6.56 stands in contrast to the situation with pure extender mice, which can satisfy "not all reals are ordinal definable". (See for example [39].) We shall show in Chapter 7 that $V = \mathrm{HOD}$ holds in any hod mouse with arbitrarily large Woodin cardinals, and in fact, a version of $V = K$ holds true.

One feature of our comparison process is that we may often use the same extender on both sides. That does not happen in an ordinary comparison of premice by iterating least disagreements. This feature can be awkward. What we gain is that we never encounter strategy disagreements in our comparison process. A comparison process that involves iterating away strategy disagreements as we encounter them (such as the process of [30]) will also often use the same extender on both sides. But such a process (if we knew one in general) might have some advantages. For example, it might be possible to get by without assuming the existence of a Γ-Woodin background universe, where Σ_0 and Σ_1 are in Γ. It might also give better bounds on the lengths of comparisons between uncountable pairs.

For example, Grigor Sargsyan has pointed out that our results leave the following question open. Suppose that (P, Σ) and (Q, Ψ) are pure extender pairs with scope H_δ, where δ is Woodin, and that $o(P) = o(Q) = \omega_1$. Suppose that whenever $i \colon V \to N$ with N transitive, then $i(\Sigma) \subseteq \Sigma$ and $i(\Psi) \subseteq \Psi$. Our results show that (P, Σ) and (Q, Ψ) have a common iterate (R, Λ) such that one of P-to-R and Q-to-R does not drop. Can we find such an (R, Λ) with $o(R) = \omega_1$? The standard "weasel comparison" proof shows that one can find iterates (R, Λ_0) and (R, Λ_1) such that $o(R) = \omega_1$, but if one demands that $\Lambda_0 = \Lambda_1$, the question is open, and our strategy-comparison theorem does not answer it.

6.6 The existence of cores

As in the case of ordinary premice, we can formulate our solidity and universality results abstractly, in a theorem about least branch premice having sufficiently good iteration strategies.

Theorem 6.57 *(The existence of cores.) Let M be a countable lpm, and let Ψ be an iteration strategy for M defined on all countable M-stacks by Σ. Suppose that whenever s is a countable M-stack by Ψ having last model N, then (N, Ψ_s) is a least branch hod pair. Suppose that Ψ is coded by a set of reals that is Suslin and co-Suslin in some $L(\Gamma, \mathbb{R})$, where $L(\Gamma, \mathbb{R}) \models \mathsf{AD}^+$. Let $\rho = \rho(M)$ and $r = p(M)$ be the projectum and standard parameter of M, and let*

$$H = \text{ transitive collapse of } Hull^M(\rho \cup r);$$

then

(1) r is solid, and

(2) $H|(\rho^+)^H = M|(\rho^+)^M$.

Remark 6.58 We don't need the full strength of a model of AD^+ with Ψ in it.

Proof. Let q be the longest solid initial segment of r. Let $r = q \cup s$, where either $s = \emptyset$ or $\min(q) > \max(s)$. Let

$$\alpha_0 = \text{ least } \beta \text{ such that } \mathrm{Th}^M_{k+1}(\beta \cup q) \notin M.$$

Here $k = k(M)$. We may assume $\alpha_0 \in M$, as otherwise $r = \emptyset$ and $\alpha_0 = \rho(M) = o(M)$, in which case the theorem is trivially true. Let

$$K = \text{ transitive collapse of } \mathrm{Hull}^M(\alpha_0 \cup q),$$

and let $\pi \colon K \to M$ be the collapse map. We may assume that $\alpha_0 \in K$, as otherwise $K \lhd M$, so $\mathrm{Th}^M_{k+1}(\alpha_0 \cup q) \in M$.

Claim 0.

(a) If $q = r$, then $\rho = \alpha_0$.

(b) If $q \neq r$, then $\rho < \alpha_0 \leq \max(s)$.

(c) $K \models \alpha_0$ is a cardinal.

Proof. (a) is clear. For (b), let W be the solidity witness for $q \cup \{\max(s)\}$, that is, the transitive collapse of $\mathrm{Hull}^M(\max(s) \cup q)$. We are assuming $W \notin M$. This implies that $\mathrm{Th}^M_{k+1}(\max(s) \cup q) \notin M$. [Proof: Suppose $T = \mathrm{Th}^M_{k+1}(\max(s) \cup q)$ is in M. Note $\max(s)$ is a cardinal of W, and $\max(s) = crit(\pi)$, where $\pi \colon W \to M$ is the uncollapse. So $T \in M|\pi(\alpha)$, and $M|\pi(\alpha) \models \mathsf{KP}$. So $W \in M|\pi(\alpha)$.] Thus $\alpha_0 \leq \max(s)$.

We have $\rho < \alpha_0$ because otherwise $p(M) = q$.

(c) is clear if $\alpha_0 = \rho$. So we may assume $\pi \neq \mathrm{id}$. (c) is clear if $\alpha_0 = \mathrm{crit}(\pi)$, so we may assume $\alpha_0 < \mathrm{crit}(\pi)$. Suppose $f \colon \beta \to \alpha_0$ is a surjection, with $\beta < \alpha_0$ and

$f \in K$. Let $\pi(f)$ be $r\Sigma_{k+1}^M$ definable from parameters in $\gamma \cup q$, where $\beta < \gamma < \alpha_0$. Then from $\mathrm{Th}_{k+1}^M(\gamma \cup q)$ one can easily compute $\mathrm{Th}_{k+1}^M(\alpha_0 \cup q)$, so $\mathrm{Th}_{k+1}^M(\gamma \cup q) \notin M$, contrary to the minmality of α_0.

□

We shall show that if $q \neq r$, then $\mathrm{Th}_{k+1}^M(\alpha_0 \cup q) \in M$. This implies $q = r$, so r is solid. We then show that K satisfies conclusion (2). The argument is based on comparing the phalanx (M, K, α_0) with M, as usual.

Let $M = \{e_i \mid i < \omega\}$ be an enumeration of M in which for some n, $r = \langle e_0, ..., e_n\rangle$ (in descending order, so $e_0 = \max(r)$). By Lemma 6.27, we may assume that Ψ has the weak Dodd-Jensen property relative to $\vec{e}$. This involves replacing Ψ by a pullback of one of its tails, but we stay with the same M, and it is the first order theory of M that matters in (1) and (2).

Remark 6.59 Under the additional hypothesis that Ψ has the weak Dodd-Jensen property relative to some $\vec{e}$, we can strengthen the strategy agreement part of (2) to: for $\gamma = (\rho^+)^M$, $\Psi_{\langle\gamma,0\rangle} = (\Psi^\pi)_{\langle\gamma,0\rangle}$.

In the comparison argument, we iterate both M and (M, K, α_0) into the models of a common background construction. Additional phalanxes (N, L, β) may appear above (M, K, α_0) in its tree.

The background construction is the following. Working in our model of AD^+ having Ψ in it, let $(N^*, \delta^*, S, T, \lhd, \Sigma^*)$ be a coarse Γ-Woodin tuple, with M countable in N^* and $\mathrm{Code}(\Psi)$ in Γ. Let $\mathbb{C}$ be the maximal $\lhd$-construction done in N^*. (N^*, Σ^*). $\mathbb{C}$ may break down before stage δ^*, but by Theorem 6.51 it absorbs (M, Ψ) before that. In other words, letting

$$M_{\eta,l} = M_{\eta,l}^{\mathbb{C}} \text{ and } \Omega_{\eta,l} = \Omega_{\eta,l}^{\mathbb{C}},$$

we have

Claim 1. Let $k = k(M)$. There is an $\eta < \delta^*$ such that $\langle \eta, k\rangle \leq l(\mathbb{C})$, and (M, Ψ) iterates to $(M_{\eta,k}, \Omega_{\eta,k})$, and strictly past all earlier pairs in $\mathbb{C}$.

Let us fix $k_0 = k(M)$, and $\eta_0 < \delta^*$ and $\mathcal{U}$ a normal tree on M with last model M_{η_0,k_0} witnessing Claim 1. For each $\langle \nu, l\rangle \leq_{\mathrm{lex}} \langle \eta_0, k_0\rangle$, let $\mathcal{U}_{\nu,l}$ be the unique normal tree on M witnessing that (M, Ψ) iterates strictly past $(M_{\nu,l}, \Omega_{\nu,l})$.

We now want to compare (M, K, α_0) with the $M_{\nu,l}$ for $\langle \nu, l\rangle \leq_{\mathrm{lex}} \langle \eta_0, k_0\rangle$. For each such $\langle \nu, l\rangle$ we shall define a "psuedo iteration tree" $\mathcal{S}_{\nu,l}$ on (M, K, α_0). We shall have complete strategies attached to the models of $\mathcal{S}_{\nu,l}$, and as before, the key will be that no strategy disagreements with $\Omega_{\nu,l}$ show up, and that $M_{\nu,l}$ does not move.

The rules for forming $\mathcal{S}_{\nu,l}$ will be the usual ones for iterating a phalanx, with the exception that at certain steps we are allowed to move the whole phalanx up. (We don't throw away the phalanxes we had before, we just create a new one.) Whenever we introduce a new phalanx, we continue the construction of $\mathcal{S}$ by looking at the least disagreement between its second model and $M_{\nu,l}$.

Fix ν and l. Let us write $\mathcal{U} = \mathcal{U}_{\nu,l}$. At the same time that we define $\mathcal{S} = \mathcal{S}_{\nu,l}$, we shall copy it to a normal tree $\mathcal{T} = \mathcal{T}_{\nu,l}$ on M that is by Ψ. We allow a bit of padding in $\mathcal{T}$; that is, occasionally $\mathcal{M}_\theta^\mathcal{T} = \mathcal{M}_{\theta+1}^\mathcal{T}$. We shall have copy maps

$$\pi_\theta \colon \mathcal{M}_\theta^\mathcal{S} \to \mathcal{M}_\theta^\mathcal{T}$$

with the usual commutativity and agreement properties. We should write $\pi_\theta^{\nu,l}$ here, but will omit the superscripts when we can. The strategy we attach to $\mathcal{M}_\theta^\mathcal{S}$ is

$$\Sigma_\theta = (\Psi_{\mathcal{T}\restriction\theta+1})^{\pi_\theta}.$$

We shall have that $(\mathcal{M}_\theta^\mathcal{S}, \Sigma_\theta)$ is an lbr hod-pair. Finally, we have ordinals $\lambda_\theta^\mathcal{S}$ for each $\theta < \mathrm{lh}(\mathcal{S})$ that measure agreement between the models of $\mathcal{S}$, and tell us which one we should apply the next extender to.[23]

We start with

$$\mathcal{M}_0^\mathcal{S} = M, \mathcal{M}_1^\mathcal{S} = K, \text{ and } \lambda_0^\mathcal{S} = \alpha_0,$$

and

$$\mathcal{M}_0^\mathcal{T} = \mathcal{M}_1^\mathcal{T} = M.$$

We let $\pi_0 =$ identity, and let $\pi_1 \colon K \to M$ be the uncollapse map. Since $\mathrm{crit}(\pi_1) \geq \alpha_0 = \lambda_0^\mathcal{S}$, π_0 and π_1 agree up to the relevant exchange ordinal. We think of 0 and 1 as distinct roots of $\mathcal{S}$. One additional root will be created each time we move a phalanx up, and only then.

As we proceed, we define what it is for a node θ of $\mathcal{S}$ to be *unstable*. We shall have that if θ is unstable, then $0 \leq_S \theta$ and $[0, \theta]_S$ does not drop. We then set

$$\alpha_\theta = \sup i_{0,\theta}^\mathcal{S} \text{``} \alpha_0.$$

The idea is that θ is unstable iff $(\mathcal{M}_\theta^\mathcal{S}, \mathcal{M}_{\theta+1}^\mathcal{S}, \alpha_\theta)$ is a phalanx that we are allowed to move up. If θ is unstable, then $\theta + 1$ is stable, and a new root in $\mathcal{S}$, that is, there are no $\xi <_S \theta + 1$. These are the only roots, except for 0. Our first unstable node is 0, and 1 is stable.

The padding in $\mathcal{T}$ corresponds exactly to the unstable nodes of $\mathcal{S}$, in that θ is unstable iff $\mathcal{M}_\theta^\mathcal{T} = \mathcal{M}_{\theta+1}^\mathcal{T}$.

We maintain by induction on the construction of $\mathcal{S}$ that the current last model is stable, and conversely, every stable model is the last model at some stage. So really,

[23]Earlier we defined $\lambda_\alpha^\mathcal{T}$, for $\mathcal{T}$ a normal iteration tree, to be the sup of the Jensen generators on the branch $[0, \alpha)_T$. (See 2.13.) Our use of the notation now is a different one. Psuedo-trees are not normal trees, so there is not a literal conflict. But if $\mathcal{S}$ is a psuedo-tree, then $\lambda_\alpha^\mathcal{S}$ corresponds to $\lambda_{\alpha+1}^\mathcal{T}$ in the normal case, and not to $\lambda_\alpha^\mathcal{T}$.

we are defining $\mathcal{S}^\eta$, which has a stable last model, by induction on η, sometimes adding two models at once, and taking $\mathcal{S} = \bigcup_\eta \mathcal{S}^\eta$. We shall suppress the superscript η, however. All extenders used in $\mathcal{S}$ will be taken from stable nodes. We also maintain that if $\mathcal{M}_\theta^{\mathcal{S}}$ has been defined, then

Induction hypotheses. If θ is unstable, then

(i) $0 \leq_S \theta$, the branch $[0, \theta]_S$ does not drop in model or degree,

(ii) $\lambda_\theta^{\mathcal{S}} \leq \alpha_\theta \leq \rho_k(\mathcal{M}_\theta^{\mathcal{S}})$, where $k = k(M)$,

(iii) every $\tau \leq_S \theta$ is unstable,

(iv) there is a ξ such that $\mathcal{M}_\theta^{\mathcal{S}} = \mathcal{M}_\xi^{\mathcal{U}}$,

(v) $\rho(\mathcal{M}_\theta^{\mathcal{S}}) = \sup i_{0,\theta}^{\mathcal{S}}``\rho$,

(vi) $\alpha_\theta =$ least β such that $\mathrm{Th}_{k+1}^{\mathcal{M}_\theta^{\mathcal{S}}}(\beta \cup i_{0,\theta}^{\mathcal{S}}(q)) \notin \mathcal{M}_\theta^{\mathcal{S}}$.

Item (ii) explains why $[0, \theta]_S$ does not drop in model or degree, for an extender applied to $\mathcal{M}_\theta^{\mathcal{S}}$ must have critical point $< \lambda_\theta^{\mathcal{S}}$. Concerning item (iv), notice

Claim 2. If $0 \leq_S \theta$, and $[0, \theta]_S$ does not drop in model or degree, and $\mathcal{M}_\theta^{\mathcal{S}} = \mathcal{M}_\xi^{\mathcal{U}}$, then then $[0, \xi]_U$ does not drop in model or degree; moreover $i_{0,\theta}^{\mathcal{S}} = i_{0,\xi}^{\mathcal{U}}$.

Proof. This follows as usual the weak Dodd-Jensen property of Ψ. If for example that $[0, \xi]_U$ drops, then $i_{0,\theta}^{\mathcal{S}}$ maps M elementarily into a dropping Ψ-iterate of M, contradiction. Similarly, $i_{0,xi}^{\mathcal{U}}$ must be "to the left of" $i_{0,\theta}^{\mathcal{S}}$ with respect to $\vec{e}$. But also, $\pi_\theta \circ i_{0,\xi}^{\mathcal{U}}$ is an elementary map from M to $\mathcal{M}_\theta^{\mathcal{T}}$, so $i_{0,\theta}^{\mathcal{T}} = \pi_\theta \circ i_{0,\theta}^{\mathcal{S}}$ is to its left. So $i_{0,\theta}^{\mathcal{S}}$ is to the left of $i_{0,\xi}^{\mathcal{U}}$, so $i_{o,\theta}^{\mathcal{S}} = i_{0,\xi}^{\mathcal{U}}$. □

The following notation will be useful. For any node γ of $\mathcal{S}$, let

$$\mathrm{st}(\gamma) = \text{ least stable } \theta \text{ such that } \theta \leq_S \gamma,$$

and

$$\mathrm{rt}(\gamma) = \begin{cases} S\text{-pred}(\mathrm{st}(\gamma)) & \text{if } S\text{-pred}(\mathrm{st}(\gamma)) \text{ exists} \\ \mathrm{st}(\gamma) & \text{otherwise.} \end{cases}$$

Note that if θ is unstable and $\theta + 1 \leq_S \gamma$, then $\mathrm{rt}(\gamma) = \theta + 1$. If θ is the largest unstable ordinal $\leq_S \gamma$, then $\mathrm{rt}(\gamma) = \theta$. Finally, if there are unstable ordinals $\leq_S \gamma$, but no largest one, then $\mathrm{rt}(\gamma) = \sup\{\theta \mid \theta \leq_S \gamma \text{ and } \theta \text{ is unstable }\}$.

The construction of $\mathcal{S}$ can end in one of two ways:

(1) We reach a stable θ such that either

(a) $M_{\nu,l} \lhd \mathcal{M}_\theta^\mathcal{S}$ or

(b) $\mathcal{M}_\theta^\mathcal{S} \unlhd M_{\nu,l}$, and $[\mathrm{rt}(\theta), \theta]_S$ does not drop in model or degree.

In both cases, the full external strategies will be lined up, by Lemma 6.64 below. Case 1(b) constitutes a successful comparison of (M, K, α_0) with M, which iterated past $M_{\nu,l}$ via $\mathcal{U}$. So in case 1(b), we leave $\mathcal{S}_{\eta,m}$ undefined for all $\langle \eta, m\rangle >_{\mathrm{lex}} \langle \nu, l\rangle$. In case 1(a) our phalanx has iterated strictly past $M_{\nu,l}$, and so we go one to define $\mathcal{S}_{\nu,l+1}$.

There is a second way the construction of $\mathcal{S}$ can end.

(2) We reach a stable θ such that for some ξ, $\mathcal{M}_\theta^\mathcal{S} = \mathcal{M}_\xi^\mathcal{U}$, and neither $[rt(\theta), \theta]_S$ nor $[0, \xi]_U$ has dropped in model or degree. Moreover, letting $Q = \mathcal{M}_\theta^\mathcal{S} | \langle \hat{o}(\mathcal{M}_\theta^\mathcal{S}), -1\rangle$ be the result of removing the last extender predicate, we have that $Q \unlhd M_{\nu,l}$.

If $\mathcal{M}_\theta^\mathcal{S}$ is not extender-active, then this is the same as case 1(b) above (and we must have $\langle \nu, l\rangle = \langle \eta_0, k_0\rangle$). But if $\mathcal{M}_\theta^\mathcal{S}$ is extender-active, it is a new way to end. We think of it as a successful comparison, and leave $\mathcal{S}_{\eta,m}$ undefined for all $\langle \eta, m\rangle >_{\mathrm{lex}} \langle \nu, l\rangle$. Note that in the extender-active case, we have not actually lined up the strategies of $\mathcal{M}_\theta^\mathcal{S}$ and $\mathcal{M}_\xi^\mathcal{U}$. We've lined up the part of them that acts on Q, and we've lined up the last extender predicates themselves, but not how the strategies act on trees involving the last extender.

In both case (1) and case (2), the last model of $\mathcal{S}$ is $\mathcal{M}_\theta^\mathcal{S}$.

Claim 3. Induction hypotheses (i)-(vi) hold for $\theta = 0$ and $\theta = 1$.

Proof.. (i)-(vi) are trivial for $\theta = 0$, and vacuous for $\theta = 1$. □

The rules for extending $\mathcal{S}$ at successor steps are the following. Suppose $\mathcal{M}_\gamma^\mathcal{S}$ is the current last model, so that γ is stable, and suppose the construction is not required to stop by (1) or (2) above. So we have a least disagreement between $\mathcal{M}_\gamma^\mathcal{S}$ and $M_{\nu,l}$. Suppose the least disagreement involves only an extender E from the $\mathcal{M}_\gamma^\mathcal{S}$ sequence. By this we mean: letting $\tau = \mathrm{lh}(E)$,

- $M_{\nu,l} | \langle \tau, 0\rangle = \mathcal{M}_\gamma^\mathcal{S} | \langle \tau, -1\rangle$, and
- $(\Omega_{\nu,l})_{\langle \tau,0\rangle} = (\Sigma_\gamma)_{\langle \tau,-1\rangle}$.

Lemma 6.64 below proves that this is the case. Set

$$\lambda_\gamma^\mathcal{S} = \lambda_E.$$

Let ξ be least such that $\mathrm{crit}(E) < \lambda_\xi^\mathcal{S}$. We declare that S-pred$(\gamma+1) = \xi$. Let $\langle \beta, n\rangle$ be lex least such that either $\rho(\mathcal{M}_\xi^\mathcal{S} | \langle \beta, n\rangle) \leq \mathrm{crit}(E)$, or $\langle \beta, n\rangle = \langle \hat{o}(\mathcal{M}_\xi^\mathcal{S}), k(\mathcal{M}_\xi^\mathcal{S})\rangle$. We set

$$\mathcal{M}_{\gamma+1}^\mathcal{S} = \mathrm{Ult}(\mathcal{M}_\xi^\mathcal{S} | \langle \beta, n\rangle, E),$$

and let $\hat{\imath}^{\mathcal{S}}_{\xi,\gamma+1}$ be the canonical embedding. We let

$$\mathcal{M}^{\mathcal{T}}_{\gamma+1} = \mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_{\xi}|\langle \pi_\xi(\beta), n\rangle, \pi_\gamma(E)),$$

and let $\pi_{\gamma+1}$ be given by the Shift Lemma, as usual. If ξ is stable, or if $\langle \beta, n\rangle <_{\mathrm{lex}} \langle \hat{o}(\mathcal{M}^{\mathcal{S}}_\xi), k(\mathcal{M}^{\mathcal{S}}_\xi)\rangle$, then we declare $\gamma+1$ to be stable, and we just go on now to look at least disagreement between $\mathcal{M}^{\mathcal{S}}_{\gamma+1}$ and $M_{\nu,l}$. Nothing unusual has happened.

Induction hypotheses (i)-(vi) concern only unstable nodes, so they are vacuously true at $\theta = \gamma+1$.

Remark 6.60 There is an anomalous case to consider here. It occurs also in the solidity proof for ordinary premice, where Schindler and Zeman found the arguments that take care of it. (See [42].) This case only occurs when $\alpha_0 = \mathrm{lh}(F)$, for some extender F from the M-sequence. Equivalently, for some (all) unstable ξ, $\alpha_\xi = \mathrm{lh}(F)$ for some F from the M-sequence. Then we could have an unstable ξ and a γ such that S-pred$(\gamma+1) = \xi$, and $\mathrm{crit}(E^{\mathcal{S}}_\gamma) = \lambda(F)$, where F is the last extender of $\mathcal{M}^{\mathcal{S}}_\xi|\alpha_\xi$. Thus $\langle \beta, n\rangle = \langle \alpha_\xi, 0\rangle$, and $\mathcal{M}^{\mathcal{S}}_{\gamma+1} = \mathrm{Ult}(\mathcal{M}^{\mathcal{S}}_\xi|\langle \alpha_\xi, 0\rangle)$ is not an lpm, because F is a missing whole initial segment of $i^{\mathcal{S}}_{\xi,\gamma+1}(F)$. But this is ok. The next disagreement will force us to apply $i^{\mathcal{S}}_{\xi,\gamma+1}(F)$ to $\mathcal{M}^{\mathcal{S}}_\xi$, and that will produce an lpm; moreover, $\lambda(E^{\mathcal{S}}_\gamma) = \lambda(i^{\mathcal{S}}_{\xi,\gamma+1}(F))$, so $\gamma+1$ is now a dead node. One can cope with the fact that $i^{\mathcal{S}}_{\xi,\gamma+1}(F)$ has a missing whole initial segment in the termination arguments; the argument is the same as that of Schindler-Zeman. We shall not give any further details of this anomalous case here.

Now suppose ξ is unstable, and $\langle \beta, n\rangle = \langle \hat{o}(\mathcal{M}^{\mathcal{S}}_\xi), k(\mathcal{M}^{\mathcal{S}}_\xi)\rangle$. (Since $\alpha_0 \in M$, this means the anomalous case does not occur.) We look to see whether $\mathcal{M}^{\mathcal{S}}_{\gamma+1}$ is also a model of $\mathcal{U}$. If not, then again we declare $\gamma+1$ to be stable, and go on. Our new last node $\gamma+1$ is stable, so (i)-(vi) are vacuous for $\theta = \gamma+1$.

Finally, if $\mathcal{M}^{\mathcal{S}}_{\gamma+1}$ is also a model of $\mathcal{U}$, then we declare $\gamma+1$ to be unstable, and $\gamma+2$ to be stable. Set

$$\mathcal{M}^{\mathcal{S}}_{\gamma+2} = \text{ transitive collapse of } \mathrm{Hull}^{\mathcal{M}^{\mathcal{S}}_{\gamma+1}}(\alpha_{\gamma+1} \cup i^{\mathcal{S}}_{0,\gamma+1}(q)).$$

Let also $\sigma_{\gamma+1} \colon \mathcal{M}^{\mathcal{S}}_{\gamma+2} \to \mathcal{M}^{\mathcal{S}}_{\gamma+1}$ be the collapse map, and

$$\mathcal{M}^{\mathcal{T}}_{\gamma+2} = \mathcal{M}^{\mathcal{T}}_{\gamma+1}, \text{ and}$$

$$\pi_{\gamma+2} = \pi_{\gamma+1} \circ \sigma_{\gamma+1}.$$

Our new last node is stable. Our induction hypothesis (i) holds for $\theta = \gamma+1$ because it held for $\theta = \xi$, and because $\lambda_\xi \le \alpha_\xi$. (iii) is clear. For (ii), we must define $\lambda_{\gamma+1}$. Suppose that there is a least disagreement between $\mathcal{M}^{\mathcal{S}}_{\gamma+2}$ and $M_{\nu,l}$, and lemma 6.64 applies to it, so it involves only some F from the sequence of $\mathcal{M}^{\mathcal{S}}_{\gamma+2}$. If there is no such F, $\mathcal{M}^{\mathcal{S}}_{\gamma+2}$ is the last model of $\mathcal{S}$, and we leave $\lambda^{\mathcal{S}}_{\gamma+1}$ as undefined as $\lambda^{\mathcal{S}}_{\gamma+2}$ is. If F exists, we set

$$\lambda^{\mathcal{S}}_{\gamma+2} = \lambda(F),$$

and

$$\lambda^{\mathcal{S}}_{\gamma+1} = \inf(\lambda^{\mathcal{S}}_{\gamma+2}, \alpha_{\gamma+1}).$$

This insures that (ii) holds at $\theta = \gamma + 1$. It also insures that $\lambda_\gamma < \lambda_{\gamma+1} \leq \lambda_{\gamma+2}$, so that the λ's remain nondecreasing, which is something we want. $\pi_{\gamma+2}$ agrees with $\pi_{\gamma+1}$ on $\lambda^{\mathcal{S}}_{\gamma+1}$, as required. $(\mathcal{M}^{\mathcal{S}}_{\gamma+1}, \mathcal{M}^{\mathcal{S}}_{\gamma+2}, \alpha^{\mathcal{S}}_{\gamma+1})$ is the result of moving up the phalanx.

Remark 6.61 It is possible that $\lambda_{\gamma+1} = \lambda_{\gamma+2}$, and $\mathrm{lh}(F) < \alpha_{\gamma+1}$. Indeed, this will happen a lot. In this case, F will immediately move the phalanx $(\mathcal{M}^{\mathcal{S}}_{\gamma+1}, \mathcal{M}^{\mathcal{S}}_{\gamma+2}, \alpha_{\gamma+1})$ up again. Moreover, since $\lambda_{\gamma+1} = \lambda_{\gamma+2}$, no extender ever gets applied to $\mathcal{M}^{\mathcal{S}}_{\gamma+2}$. It is a "dead node". The phalanx $(\mathcal{M}^{\mathcal{S}}_{\gamma+1}, \mathcal{M}^{\mathcal{S}}_{\gamma+2}, \alpha_{\gamma+1})$ may get moved up repeatedly, along various branches, but that doesn't really involve $\mathcal{M}^{\mathcal{S}}_{\gamma+2}$. After contributing F, it became irrelevant.

Induction hypothesis (iv) is clear. Next we verify (v) and (vi). For this we need

Claim 4. For $a \subset \lambda_E$ finite, $E_a \in \mathcal{M}^{\mathcal{S}}_\xi$.

Proof. Let $\mathcal{M}^{\mathcal{S}}_{\gamma+1} = \mathcal{M}^{\mathcal{U}}_\mu$. By claim 2, $[0,\mu]_U$ does not drop, and $s^{\mathcal{S}}_{\gamma+1} = s^{\mathcal{U}}_\mu$. It follows that E is also used in $\mathcal{U}$. Say $E = E^{\mathcal{U}}_\beta$. Let $\kappa = \mathrm{crit}(E)$. We have

$$\sup_{\tau<\xi} \lambda_\tau \leq \kappa < \lambda_\xi,$$

because we are applying E to $\mathcal{M}^{\mathcal{S}}_\xi$.

Suppose first that E is not the last extender of $\mathcal{M}^{\mathcal{S}}_\gamma$. Then $E_a \in \mathcal{M}^{\mathcal{S}}_\gamma$, and since $\kappa < \lambda^{\mathcal{S}}_\xi \leq \lambda^{\mathcal{S}}_{\xi+1}$, $E_a \subseteq \mathcal{M}^{\mathcal{S}}_{\xi+1}|\lambda^{\mathcal{S}}_{\xi+1}$. Thus by the agreement of models in $\mathcal{S}$, $E_a \in \mathcal{M}^{\mathcal{S}}_{\xi+1}$. If $\alpha_\xi = \mathrm{crit}(\sigma_\xi)$, then α_ξ is a cardinal of $\mathcal{M}^{\mathcal{S}}_{\xi+1}$. If $\mathcal{M}^{\mathcal{S}}_\xi = \mathcal{M}^{\mathcal{S}}_{\xi+1}$, we get $E_a \in \mathcal{M}^{\mathcal{S}}_\xi$, as desired. If not, then $\kappa < \alpha_\xi \leq \mathrm{crit}(\sigma_\xi)$, and $\mathrm{crit}(\sigma_\xi)$ is a cardinal of $\mathcal{M}^{\mathcal{S}}_{\xi+1}$, so $E_a \subseteq \mathcal{M}_{\xi+1}|\,\mathrm{crit}(\sigma_\xi)$, which yields $E_a \in \mathcal{M}^{\mathcal{S}}_\xi$, as desired.

Suppose next that E is the last extender of $\mathcal{M}^{\mathcal{S}}_\gamma$, and the branch to γ of $\mathcal{S}$ has dropped. Let η be the site of the last drop, i.e. η is least such that $\hat{\imath}^{\mathcal{S}}_{\eta,\gamma}$ maps the full $\mathcal{M}^{\mathcal{S}}_\eta$ elementarily to $\mathcal{M}^{\mathcal{S}}_\gamma$. Then $\kappa \in \mathrm{ran}(\hat{\imath}^{\mathcal{S}}_{\eta,\gamma})$, and $\gamma \geq (\xi + 1)$. This implies $\eta > \xi$. (Proof: $\eta \leq_S \xi$ is impossible since $[0,\xi]_S$ does not drop. So if $\eta < \xi$, and F is the first extender used in $(\eta,\gamma]_S$ such that $\lambda_F > \kappa$, then F is applied to $\mathcal{M}^{\mathcal{S}}_\tau$ where $\tau < \xi$. So $\mathrm{crit}(F) < \lambda_\tau \leq \kappa$, and $\kappa \notin \mathrm{ran}(\hat{\imath}^{\mathcal{S}}_{\eta,\gamma}.)$ Thus $\mathrm{crit}(\hat{\imath}^{\mathcal{S}}_{\eta,\gamma}) > \kappa$. Letting $\tau = S\text{-pred}(\eta)$, this easily yields $E_a \in \mathcal{M}^{\mathcal{S}}_\tau$. Then we can argue as we did in the preceding paragraph under the hypothesis that $E_a \in \mathcal{M}^{\mathcal{S}}_\gamma$, and we get $E_a \in \mathcal{M}^{\mathcal{S}}_\xi$ as desired.

Thus we may assume that E is the last extender of $\mathcal{M}^{\mathcal{S}}_\gamma$, and the branch of $\mathcal{S}$ to γ (i.e. either $[0,\gamma]_S$ or $[\mathrm{rt}(\gamma),\gamma]_S$) does not drop in model or degree. By a parallel argument, we may assume that E is the last extender of $\mathcal{M}^{\mathcal{U}}_\beta$, and the branch $[0,\beta]_U$

does not drop in model or degree. But that means we stop our construction for reason (2), with $\mathcal{M}_\gamma^{\mathcal{S}}$ being the last model of $\mathcal{S}$, contrary to our assumption. This proves Claim 4.

□

It is precisely in order to insure Claim 4 that we stop the construction for reason (2).

Claim 5. Items (v) and (vi) of our induction hypotheses hold.

Proof. Let $i = i^{\mathcal{S}}_{\xi,\gamma+1}$, and $k = k_0 = k(M)$. Consider first (vi). For $\beta \leq \alpha_\xi$, let

$$T_\beta = \mathrm{Th}_{k+1}^{\mathcal{M}_\xi^{\mathcal{S}}}(\beta \cup i^{\mathcal{S}}_{0,\xi}(q)),$$

and for $\beta \leq \alpha_{\gamma+1}$, let

$$R_\beta = \mathrm{Th}_{k+1}^{\mathcal{M}_{\gamma+1}^{\mathcal{S}}}(\beta \cup i^{\mathcal{S}}_{0,\gamma+1}(q)).$$

If $\beta < \alpha_\xi$, then $T_\beta \in \mathcal{M}_\xi^{\mathcal{S}}$, and we can use $i(T_\beta)$ to compute $R_{i(\beta)}$, as usual with solidity witnesses. Since $\alpha_{\gamma+1} = \sup i``\alpha_\xi$, this gives half of (vi). For the other half, assume $R = R_{\alpha_{\gamma+1}}$ is in $\mathcal{M}^{\mathcal{S}}_{\alpha_{\gamma+1}}$, say

$$R = [a, f]_E^{\mathcal{M}_\xi^{\mathcal{S}}}.$$

Letting $T = T_{\alpha_\xi}$, we then have $\langle \varphi, \mu \rangle \in T$ iff $\langle \varphi, i(\mu) \rangle \in R$ iff for E_a almost every u, $\langle \varphi, \mu \rangle \in f(u)$. Since $E_a \in \mathcal{M}_\xi^{\mathcal{S}}$, $T \in \mathcal{M}_\xi^{\mathcal{S}}$, a contradiction.

Consider now (v). Let $t = p(\mathcal{M}_\xi^{\mathcal{S}})$ and $\sigma = \rho(\mathcal{M}_\xi^{\mathcal{S}})$ be the standard parameter and projectum. Let $\tau = \sup i``\sigma$.

Remark 6.62 Our proof shows that $i^{\mathcal{S}}_{0,\xi}(q)$ is an initial segment of t, but it does not show $t = i^{\mathcal{S}}_{0,\xi}(r)$. The standard parameter could move down in its non-solid region.

Let for any $\beta, x \in \mathcal{M}_\xi^{\mathcal{S}}$

$$T_\beta(x) = \mathrm{Th}_{k+1}^{\mathcal{M}_\xi^{\mathcal{S}}}(\beta \cup \{x\}),$$

and for $\beta, x \in \mathcal{M}_{\gamma+1}^{\mathcal{S}}$, let

$$R_\beta(x) = \mathrm{Th}_{k+1}^{\mathcal{M}_{\gamma+1}^{\mathcal{S}}}(\beta \cup \{x\}).$$

If $R_\tau(i(t)) \in \mathcal{M}_{\gamma+1}^{\mathcal{S}}$, say $R_\tau(i(t)) = [a, f]$, then using E_a we can compute $T_\sigma(t)$ inside $\mathcal{M}_\xi^{\mathcal{S}}$, contradiction. Thus $\rho(\mathcal{M}_{\gamma+1}^{\mathcal{S}}) \leq \tau$. On the other hand, let $\kappa \leq \beta < \sigma$ and $x = [a, f]$ in $\mathrm{Ult}(\mathcal{M}_\xi^{\mathcal{S}}, E)$. Then $T_\beta(f) \in \mathcal{M}_\xi^{\mathcal{S}}$, and we can compute $R_{i(\beta)}(x)$ from $i(T_\beta(f))$ in $\mathcal{M}_{\gamma+1}^{\mathcal{S}}$. (First, compute $R_{i(\beta)}(i(f))$. Then note $x = i(f)(a)$, and $a \subset i(\beta)$.) Since $\mathrm{ran}(i)$ is cofinal in τ, we get $\tau \leq \rho(\mathcal{M}_{\gamma+1}^{\mathcal{S}})$.

This proves Claim 5. □

Now let θ be a limit ordinal, and let $b = \Psi(\mathcal{T}\restriction\theta)$ be the branch of $\mathcal{T}$ chosen by Ψ. b may have pairs of the form $\gamma, \gamma+1$ in it where $\mathcal{M}_\gamma^{\mathcal{T}} = \mathcal{M}_{\gamma+1}^{\mathcal{T}}$; this occurs precisely when $\gamma \in b$ is unstable. By construction, the set of such pairs is an initial segment of b that is closed as a set of ordinals.

Suppose first

Case 1. There is a largest $\eta \in b$ such that η is unstable.

Fix this η. There are two subcases.

1(b) for all $\gamma \in b - (\eta + 1)$, $\mathrm{rt}(\gamma) = \eta + 1$. In this case, $b - (\eta + 1)$ is a branch of $\mathcal{S}$. We let $\mathcal{S}$ choose this branch, that is,
$$[\eta + 1, \theta)_S = b - (\eta + 1),$$
and let $\mathcal{M}^{\mathcal{S}}_\theta$ be the direct limit of the $\mathcal{M}^{\mathcal{S}}_\gamma$ for $\gamma \in b - (\eta + 1)$ sufficiently large. The branch embeddings $\hat{\imath}^{\mathcal{S}}_{\gamma,\theta}$, for $\gamma \geq \eta$ in b, are as usual. $\pi_\theta \colon \mathcal{M}^{\mathcal{S}}_\theta \to \mathcal{M}^{\mathcal{T}}_\theta$ is given by the fact that the copy maps commute with the branch embeddings. We declare θ to be stable.

1(b) for all $\gamma \in b - (\eta + 1)$, $\mathrm{rt}(\gamma) = \eta$. We let $\mathcal{S}$ choose
$$[0, \theta)_S = (b - \eta) \cup [0, \eta]_S,$$
and let $\mathcal{M}^{\mathcal{S}}_\theta$ be the direct limit of the $\mathcal{M}^{\mathcal{S}}_\gamma$ for $\gamma \in b$ sufficiently large. The branch embeddings $\hat{\imath}^{\mathcal{S}}_{\gamma,\theta}$, for $\gamma \geq \eta$ in b, are as usual. $\pi_\theta \colon \mathcal{M}^{\mathcal{S}}_\theta \to \mathcal{M}^{\mathcal{T}}_\theta$ is given by the fact that the copy maps commute with the branch embeddings. Again, we declare θ to be stable.

In this case, θ is stable, so (i)-(vi) still hold.

Case 2. There are boundedly many unstable ordinals in b, but no largest one.

Let η be the sup of the unstable ordinals in b. We let $\mathcal{S}$ choose
$$[0, \theta_S] = (b - \eta) \cup [0, \eta]_S,$$
etc. Again, we declare θ to be stable, and (i)-(vi) still hold.

Case 3. There are arbitrarily large unstable ordinals in b. In this case b is a disjoint union of pairs $\{\gamma, \gamma + 1\}$ such that γ is unstable and $\gamma + 1$ is stable. That is, in $\mathcal{S}$ we have been moving our phalanx up all along b. We set
$$[0, \theta)_S = \{\xi \in b \mid \xi \text{ is unstable }\},$$
and let $\mathcal{M}^{\mathcal{S}}_\theta$ be the direct limit of the $\mathcal{M}^{\mathcal{S}}_\xi$ for $\xi \in b$ unstable. There is no dropping of any kind in $[0, \theta)_S$. The branch embeddings $i^{\mathcal{S}}_{\gamma,\theta}$ and the copy map π_θ are as usual. If $\mathcal{M}^{\mathcal{S}}_\theta$ is not a model of $\mathcal{U}$, then we declare θ to be stable. Otherwise, we declare θ to be unstable, and set
$$\mathcal{M}^{\mathcal{S}}_{\theta+1} = \text{ transitive collapse of } \mathrm{Hull}^{\mathcal{M}^{\mathcal{S}}_\theta}(\lambda^{\mathcal{S}}_\theta \cup i^{\mathcal{S}}_{0,\theta}(q)).$$
$\lambda^{\mathcal{S}}_\theta$ is defined as it was in the unstable successor case: first we define $\lambda_{\theta+1}$, then set
$$\lambda^{\mathcal{S}}_\theta = \inf(\lambda^{\mathcal{S}}_{\theta+1}, \alpha_\theta).$$

Let also

$$\sigma_\theta \colon \mathcal{M}^{\mathcal{S}}_{\theta+1} \to \mathcal{M}^{\mathcal{S}}_{\theta}$$

be the collapse map, and

$$\mathcal{M}^{\mathcal{T}}_{\theta+1} = \mathcal{M}^{\mathcal{T}}_{\theta}, \text{ and}$$

$$\pi_{\theta+1} = \pi_\theta \circ \sigma_\theta.$$

$\pi_{\theta+1}$ agrees with π_θ on $\lambda^{\mathcal{S}}_\theta$, as desired.

(i)-(iv) are clear. Items (v) and (vi) are routine.

We shall use the following proposition in the next section.

Proposition 6.63 *Let θ be a limit ordinal such that θ is stable in $\mathcal{S}_{\nu,l}$, but every $\xi <_{S_{\nu,l}} \theta$ is unstable in $\mathcal{S}_{\nu,l}$; then* $\mathrm{cof}(\theta) = \omega$.

Proof. Let $t = e^{\mathcal{S}_{\nu,l}}_\theta$ be the branch extender of $[0,\theta)_S$, and $\lambda = \mathrm{dom}(t)$. By hypothesis, $t{\restriction}\eta \in \mathcal{U}^{\mathrm{ext}}_{\nu,l}$ for all $\eta < \lambda$, but $t \notin \mathcal{U}^{\mathrm{ext}}_{\nu,l}$. For $\eta < \lambda$, let ξ_η be such that

$$t{\restriction}\eta = s^{\mathcal{U}_{\nu,l}}_{\xi_\eta}.$$

Then $\eta < \gamma$ implies $s^{\mathcal{U}}_{\xi_\eta} \subseteq s^{\mathcal{U}}_{\xi_\gamma}$, and hence $\xi_\eta <_U \xi_\gamma$. Letting $\mu = \sup(\{\xi_\eta \mid \eta < \lambda\})$, and b be the branch of $\mathcal{U}{\restriction}\mu$ determined by the ξ_η's, we have that t is the branch extender of b in $\mathcal{U}$, so $b \neq s^{\mathcal{U}}_\mu$, so $b \neq [0,\mu)_U$. This implies $\mathrm{cof}(\mu) = \omega$, so $\mathrm{cof}(\lambda) = \omega$, so $\mathrm{cof}(\theta) = \omega$, as desired. □

This finishes our construction of the psuedo-tree $\mathcal{S}_{\nu,l}$, and its lift $\mathcal{T}_{\nu,l}$. Notice that every extender used in $\mathcal{S}$ was taken from the sequence of a stable node. Every stable node, except the last model of $\mathcal{S}$, contributes exactly one extender to be used. The last model of $\mathcal{S}$ is stable.

Recall that we assumed that the construction never reached a strategy disagreement between the current model of $\mathcal{S}_{\nu,l}$ and $(M_{\nu,l}, \Omega_{\nu,l})$, and that the extender disagreements involved only empty extenders on the $M_{\nu,l}$ side. Let us record this in a lemma.

Lemma 6.64 *Let $\gamma < \mathrm{lh}(\mathcal{S})$, where $\mathcal{S} = \mathcal{S}_{\nu,l}$ is defined as above; then either*

(1) $(\mathcal{M}^{\mathcal{S}}_\gamma, \Sigma_\gamma) \trianglelefteq (M_{\nu,l}, \Omega_{\nu,l})$, *or*

(2) $(M_{\nu,l}, \Omega_{\nu,l}) \triangleleft (\mathcal{M}^{\mathcal{S}}_\gamma, \Sigma_\gamma)$, *or*

(3) there is a nonempty extender E on the $\mathcal{M}^{\mathcal{S}}_\gamma$ sequence such that, setting $\tau =$ $\mathrm{lh}(E)$,

  (i) $\dot{E}^{M_{\nu,l}}_\tau = \emptyset$, *and*

(ii) $(\Sigma_\gamma)_{\langle\tau,-1\rangle} = (\Omega_{\nu,l})_{\langle\tau,0\rangle}$.

So far as we can see, the lemma can only be proved by going back through the proof of Theorem 5.11, and extending the arguments so that they apply to $\mathcal{S}_{\nu,l}$. That involves generalizing strong hull condensation to psuedo-trees like $\mathcal{S}$, and normalizing well to stacks $\langle\mathcal{S},\mathcal{U}\rangle$, where $\mathcal{U}$ is a normal tree on the last model of $\mathcal{S}$. Then we need to run the construction of 5.11, showing that $W(\mathcal{S},\mathcal{U}^\frown b)$ is a psuedo-hull of $i^*_b(\mathcal{S})$, where b is the branch of $\mathcal{U}$ chosen by $\Omega_{\nu,l}$. There is nothing new in these arguments, but it does not seem possible to get by with quoting our earlier results. We therefore defer the proof of Lemma 6.64 to the next section.

Claim 6. For some $\langle\nu,l\rangle \leq_{\text{lex}} \langle\eta_0,k_0\rangle$, the construction of $\mathcal{S}_{\nu,l}$ stops for either reason 1(b) (that is, $\mathcal{M}^{\mathcal{S}}_\infty \trianglelefteq M_{\nu,l}$), or reason (2).

Proof. If not, then the construction of $\mathcal{S} = \mathcal{S}_{\eta_0,k_0}$ must reach some $\mathcal{M}^{\mathcal{S}}_\theta$ such that M_{η_0,k_0} is a proper initial segment of $\mathcal{M}^{\mathcal{S}}_\theta$. But M_{η_0,k_0} is a Ψ-iterate of M via a branch of $\mathcal{U}_{\eta_0,k_0}$ that does not drop; let j be the iteration map. We have π_θ from $\mathcal{M}^{\mathcal{S}}_\theta$ to the last model of $\mathcal{T}_{\eta_0,k_0}$. Then $\pi_\theta\circ j$ maps M elementarily into a proper initial segment of the last model of $\mathcal{T}_{\eta_0,k_0}$, contrary to the weak Dodd-Jensen property of Ψ. $\square$

The following weaker version of induction hypotheses (v) and (vi) holds more generally.

Claim 7. Let $\mathcal{U} = \mathcal{U}_{\nu,l}$ for some ν, l. Suppose $[0,\eta]_U$ does not drop in model or degree, and let $i = i^{\mathcal{U}}_{0,\eta}$; then

(a) for any $\beta < \alpha_0$, $\text{Th}^{\mathcal{M}^{\mathcal{U}}_\eta}_{k_0+1}(i(\beta)\cup i(q)) \in \mathcal{M}^{\mathcal{U}}_\eta$,

(b) $\sup i``\rho(M) \leq \rho(\mathcal{M}^{\mathcal{U}}_\eta) \leq i(\rho(M))$, and

(c) if $q \neq r$, then $\text{Th}^{\mathcal{M}^{\mathcal{U}}_\eta}_{k_0+1}(\rho(\mathcal{M}^{\mathcal{U}}_\eta)\cup i(q)) \in \mathcal{M}^{\mathcal{U}}_\eta$.

Proof. Part (a) holds because $i(\text{Th}^M_{k_0+1}(\beta\cup q))$ can be used to compute $\text{Th}^{\mathcal{M}^{\mathcal{U}}_\eta}(i(\beta)\cup i(q))$. Part (b) is proved in Claim 5 of the proof of Theorem 6.2 of [23]. If $q \neq r$, then $\rho < \alpha_0$, and $\rho(\mathcal{M}_\eta) \leq i^{\mathcal{U}}_{0,\eta}(\rho)$, so we get (c) by using (a) with $\beta = \rho$. ‘ $\square$

Let us now fix ν, l as in Claim 6, and let $\mathcal{S} = \mathcal{S}_{\nu,l}$, $\mathcal{U} = \mathcal{U}_{\nu,l}$, and $\mathcal{T} = \mathcal{T}_{\nu,l}$. Let $\text{lh}(\mathcal{S}) = \theta + 1$. We have that $[\text{rt}(\theta),\theta]_S$ does not drop in model or degree. If $0 \leq_S \theta$, this implies that $[0,\theta]_S$ does not drop in model or degree.

Claim 8. For some unstable ξ, $\text{rt}(\theta) = \xi + 1$.

Proof. If not, then $0 \leq_S \theta$, and $[0,\theta]_S$ does not drop. If $\mathcal{S}$ ended for reason 1(b), then $\mathcal{M}^{\mathcal{S}}_\theta \trianglelefteq \mathcal{M}^{\mathcal{U}}_\delta$ for some δ. But then $\mathcal{M}^{\mathcal{U}}_\delta = \mathcal{M}^{\mathcal{S}}_\theta$ and $[0,\delta]_U$ does not drop, by weak Dodd-Jensen. If $\mathcal{S}$ ended for reason (2), then again $\mathcal{M}^{\mathcal{U}}_\delta = \mathcal{M}^{\mathcal{S}}_\theta$ and $[0,\delta]_U$ does not drop.

Standard weak Dodd-Jensen arguments give

$$i^{\mathcal{S}}_{0,\theta} = i^{\mathcal{U}}_{0,\delta}.$$

(This involves copying over to $\mathcal{T}$ in one direction.) But the extenders used in each of these branches can be recovered from the embeddings, using the hull and definability properties. So

$$s^{\mathcal{S}}_\theta = s^{\mathcal{U}}_\delta.$$

Now let η be least such that η is stable and $\eta \leq_S \theta$. Then $s^{\mathcal{S}}_\eta = s^{\mathcal{S}}_\theta \restriction \gamma = s^{\mathcal{U}}_\delta \restriction \gamma$, for some γ. But there is τ such that $s^{\mathcal{U}}_\tau = s^{\mathcal{U}}_\delta \restriction \gamma$. Thus $\mathcal{M}^{\mathcal{S}}_\eta = \mathcal{M}^{\mathcal{U}}_\tau$. If η is a limit ordinal, then by the rules in limit case 3, η was declared unstable, contradiction. If S-pred$(\eta) = \mu$, then μ is unstable, and our rules in the successor case declare η to be unstable. So in any case, we have a contradiction. $\square$

Fix ξ as in Claim 8. Since ξ is unstable, we can fix τ such that $\mathcal{M}^{\mathcal{U}}_\tau = \mathcal{M}^{\mathcal{S}}_\xi$. Fix also $\gamma \geq \tau$ such that $M_{\nu,l} \trianglelefteq \mathcal{M}^{\mathcal{U}}_\gamma$, and hence $\mathcal{M}^{\mathcal{S}}_\theta \trianglelefteq \mathcal{M}^{\mathcal{U}}_\gamma$. Set

$$\mu = \rho(\mathcal{M}^{\mathcal{S}}_{\xi+1}),$$

and

$$t = \sigma_\xi^{-1}(i^{\mathcal{S}}_{0,\xi}(q)).$$

Claim 9. Either

(i) $\mu = \alpha_\xi$, or

(ii) $\mu < \alpha_\xi \leq \mathrm{crit}(\sigma_\xi)$, and $\mathrm{crit}(\sigma_\xi) = (\mu^+)^{\mathcal{M}^{\mathcal{S}}_{\xi+1}}$.

Proof. By induction hypothesis (vi), $\mathrm{Th}^{\mathcal{M}^{\mathcal{S}}_{\xi+1}}_{k_0+1}(\alpha_\xi \cup t) \notin \mathcal{M}^{\mathcal{S}}_{\xi+1}$, and therefore $\mu \leq \alpha_\xi$.

Suppose $\mu < \alpha_\xi$. We can then find some finite $p \subset \alpha_\xi$ such that $\mathrm{Th}^{\mathcal{M}^{\mathcal{S}}_{\xi+1}}_{k_0+1}(\mu \cup p \cup t) \notin \mathcal{M}^{\mathcal{S}}_{\xi+1}$. Since $\max(p) < \alpha_\xi$, we get from (vi) that $R = \mathrm{Th}^{\mathcal{M}^{\mathcal{S}}_\xi}_{k_0+1}(\mu \cup p \cup i^{\mathcal{S}}_{0,\xi}(q)) \in \mathcal{M}^{\mathcal{S}}_\xi$. If $R \in \mathcal{M}^{\mathcal{S}}_{\xi+1}$, then we have a contradiction, so assume $R \notin \mathcal{M}^{\mathcal{S}}_{\xi+1}$. Since R is essentially a subset of μ, we get (ii) of Claim 9. $\square$

Claim 10. $\mu = \rho(\mathcal{M}^{\mathcal{S}}_\theta)$.

Proof. This follows easily from the fact that all extenders used in $[\xi+1,\theta]_S$ are close to the model to which they are applied, and $\text{crit}(i^{\mathcal{S}}_{\xi+1,\theta}) \geq \alpha_\xi$. $\square$

Claim 11.

(i) $\mathcal{M}^{\mathcal{S}}_\theta = \mathcal{M}^{\mathcal{U}}_\gamma$, and $[0,\gamma]_U$ does not drop in model or degree.

(ii) If $\tau \leq \eta < \gamma$, then $\text{lh}(E^{\mathcal{U}}_\eta) \geq \alpha_\xi$.

Proof. We have by (vi) that

$$\text{Th}^{\mathcal{M}^{\mathcal{S}}_\xi}_{k_0+1}(\alpha_\xi \cup i^{\mathcal{S}}_{0,\xi}(q)) \notin \mathcal{M}^{\mathcal{S}}_\xi.$$

Suppose $\mathcal{M}^{\mathcal{S}}_\theta \lhd \mathcal{M}^{\mathcal{U}}_\gamma$. We have that $[\xi+1,\theta]_S$ does drop in model or degree, and $\text{crit}(i^{\mathcal{S}}_{\xi+1,\theta}) \geq \alpha_\xi$, so we get

$$\text{Th}^{\mathcal{M}^{\mathcal{S}}_\xi}_{k_0+1}(\alpha_\xi \cup i^{\mathcal{S}}_{0,\xi}(q)) = \text{Th}^{\mathcal{M}_{\xi+1}}_{k_0+1}(\alpha_\xi \cup t) \in \mathcal{M}^{\mathcal{U}}_\gamma.$$

Set

$$R = \text{Th}^{\mathcal{M}_{\xi+1}}_{k_0+1}(\alpha_\xi \cup t).$$

Note that if $E^{\mathcal{S}}_{\xi+1}$ exists (i.e. $\theta \neq \xi+1$), then $\text{lh}(E^{\mathcal{S}}_{\xi+1}) \geq \alpha_\xi$. This is because otherwise $\lambda^{\mathcal{S}}_\xi = \lambda^{\mathcal{S}}_{\xi+1}$, so $\xi+1$ is a dead node of $\mathcal{S}$, and $\xi+1 <_S \theta$ is impossible. So in any case, $\mathcal{M}^{\mathcal{S}}_\theta$ agrees with $\mathcal{M}^{\mathcal{S}}_\xi$ below α_ξ. It follows that $\mathcal{M}^{\mathcal{U}}_\gamma$ agrees with $\mathcal{M}^{\mathcal{S}}_\xi$ below α_ξ, and hence with $\mathcal{M}^{\mathcal{U}}_\tau$ below α_ξ. Thus all $E^{\mathcal{U}}_\mu$ for $\tau \leq \mu < \gamma$ have length $\geq \alpha_\xi$. But R is essentially a subset of α_ξ, and $R \in \mathcal{M}^{\mathcal{U}}_\gamma$, so $R \in \mathcal{M}^{\mathcal{U}}_\tau$, contradiction.

Thus $\mathcal{M}^{\mathcal{U}}_\gamma = \mathcal{M}^{\mathcal{S}}_\theta$. The argument also proved (ii).

To see that $[0,\gamma]_U$ does not drop, suppose not, and let the last drop in $[0,\gamma]_U$ occur at $\eta+1$. We must have $\eta+1 \leq \tau$, as otherwise $R \in \mathcal{M}^{\mathcal{U}}_\tau$. But then $\rho(\mathcal{M}^{\mathcal{U}}_\gamma) \leq \text{crit}(E^{\mathcal{U}}_\eta) < \lambda(E^{\mathcal{U}}_\eta) < \alpha_\xi$, which yields $\rho(\mathcal{M}^{\mathcal{S}}_\theta) = \rho(\mathcal{M}^{\mathcal{U}}_\gamma) < \mu$, by Claim 9. This contradicts Claim 10. $\square$

Claim 12. $i^{\mathcal{S}}_{\xi+1,\theta}(t) = i^{\mathcal{U}}_{0,\gamma}(q)$.

Proof. Let β be the first (i.e. largest) element of q such that $i^{\mathcal{U}}_{0,\gamma}(\beta) \neq i^{\mathcal{S}}_{\xi+1,\theta} \circ \sigma_\xi^{-1} \circ i^{\mathcal{S}}_{0,\xi}(\beta)$. If

$$i^{\mathcal{U}}_{0,\gamma}(\beta) < i^{\mathcal{S}}_{\xi+1,\theta} \circ \sigma_\xi^{-1} \circ i^{\mathcal{S}}_{0,\xi}(\beta),$$

then

$$\pi_\theta \circ i^{\mathcal{U}}_{0,\gamma}(\beta) < \pi_\theta \circ i^{\mathcal{S}}_{\xi+1,\theta} \circ \sigma_\xi^{-1} \circ i^{\mathcal{S}}_{0,\xi}(\beta) = i^{\mathcal{T}}_{0,\theta}(\beta).$$

The maps on the two sides above agree at all earlier elements of q, and $\vec{e}$ started out with r, so this contradicts the weak Dodd-Jensen property of Ψ relative to $\vec{e}$. On the other hand, suppose

$$i^{\mathcal{U}}_{0,\gamma}(\beta) > i^{\mathcal{S}}_{\xi+1,\theta} \circ \sigma_\xi^{-1} \circ i^{\mathcal{S}}_{0,\xi}(\beta).$$

Let $\bar{\beta} = \sigma_\xi^{-1} \circ i^{\mathcal{S}}_{0,\xi}(\beta)$, and $u = t - (\bar{\beta}+1)$. Since q is solid at β, and $i^{\mathcal{S}}_{\xi+1,\theta}(u) = i^{\mathcal{U}}_{0,\gamma}(q - (\beta+1))$, we get that

$$\mathrm{Th}^{\mathcal{M}^{\mathcal{S}}_\theta}_{k_0+1}(i^{\mathcal{S}}_{\xi+1,\theta}((\bar{\beta}+1) \cup i^{\mathcal{S}}_{0,\xi}(u)) \in \mathcal{M}^{\mathcal{S}}_\theta.$$

It follows that $\mathrm{Th}^{\mathcal{M}^{\mathcal{S}}_\theta}_{k_0+1}(\alpha_\xi \cup i^{\mathcal{S}}_{\xi+1,\theta}(t)) \in \mathcal{M}^{\mathcal{S}}_\theta$. But the theory is a subset of $\alpha_x i$, and it is equal to $\mathrm{Th}^{\mathcal{M}^{\mathcal{S}}_{\xi+1}}_{k_0+1}(\alpha_\xi \cup t)$. So $\mathrm{Th}^{\mathcal{M}^{\mathcal{S}}_\xi}_{k_0+1}(\alpha_\xi \cup i^{\mathcal{S}}_{0,\xi}(q)) \in \mathcal{M}_\xi$, contradiction. □

Claim 13. Let η be such that $\eta + 1 \leq_U \gamma$ and $\eta \geq \tau$; then $\alpha_\xi \leq \mathrm{crit}(E^{\mathcal{U}}_\eta)$.

Proof. Let $E = E^{\mathcal{U}}_\eta$ and $\beta = U\text{-pred}(\eta+1)$. Let $\kappa = \mathrm{crit}(E)$, and suppose $\kappa < \alpha_\xi$. We have $\mathrm{lh}(E) \geq \alpha_\xi$ by Claim 11.

If $\rho(\mathcal{M}^{\mathcal{U}}_\beta) \leq \kappa$, then $\rho(\mathcal{M}^{\mathcal{U}}_\beta) = \rho(\mathcal{M}^{\mathcal{U}}_\gamma) = \mu$, and so we have $\mu < \alpha_\xi$, and thus (ii) of Claim 9 holds, and $(\mu^+)^{\mathcal{M}^{\mathcal{S}}_\xi} > \alpha_\xi$. Now if F is used in $[0,\xi)_S$, then $\lambda(F) < \alpha_\xi$, and so $\lambda(F) \leq \mu \leq \kappa$. Thus if $\beta < \tau$, then $\lambda(E^{\mathcal{U}}_\beta) \leq \mu \leq \kappa$, contradiction. So $\beta = \tau$. But then $P(\mu)^{\mathcal{M}^{\mathcal{S}}_\xi} = P(\mu)^{\mathcal{M}^{\mathcal{U}}_\tau} = P(\mu)^{\mathcal{M}^{\mathcal{U}}_\gamma} = P(\mu)^{\mathcal{M}^{\mathcal{S}}_\theta} = P(\mu)^{\mathcal{M}^{\mathcal{S}}_{\xi+1}}$, which contradicts (ii) of Claim 9.

Thus $\kappa < \rho(\mathcal{M}^{\mathcal{U}}_\beta)$. But then

$$\alpha_\xi \leq \sup i_E \text{“}(\kappa^+)^{\mathcal{M}^{\mathcal{U}}_\beta} \leq \rho(\mathcal{M}^{\mathcal{U}}_\gamma) = \mu \leq \alpha_\xi,$$

so $\alpha_\xi = \mu = \mathrm{lh}(E)$. If $q \neq r$, then (c) of Claim 7, applied with $\eta = \gamma$, implies that $\mathrm{Th}^{\mathcal{M}^{\mathcal{U}}_\gamma}_{k_0+1}(\alpha_\xi \cup i^{\mathcal{U}}_{0,\gamma}(q)) \in \mathcal{M}^{\mathcal{U}}_\gamma$. Hence $\mathrm{Th}^{\mathcal{M}^{\mathcal{S}}_{\xi+1}}_{k_0+1}(\alpha_\xi \cup t) \in \mathcal{M}^{\mathcal{S}}_{\xi+1}$, a contradiction. On the other hand, if $q = r$, then $\alpha_\xi = \rho(\mathcal{M}^{\mathcal{S}}_\xi)$ is a cardinal of $\mathcal{M}^{\mathcal{S}}_\xi$, so $\sup i_E \text{“}(\kappa^+)^{\mathcal{M}^{\mathcal{U}}_\beta} = \mathrm{lh}(E) > \alpha_\xi$, contrary to the inequality displayed above. □

It follows from Claim 13 that $\tau \leq_U \gamma$, and either $\tau = \gamma$ or $\mathrm{crit}(i^{\mathcal{U}}_{\tau,\gamma}) \geq \alpha_\xi$. In either case

$$(\mu^+)^{\mathcal{M}^{\mathcal{S}}_\xi} = (\mu^+)^{\mathcal{M}^{\mathcal{U}}_\tau} = (\mu^+)^{\mathcal{M}^{\mathcal{U}}_\gamma} = (\mu^+)^{\mathcal{M}^{\mathcal{S}}_\theta} = (\mu^+)^{\mathcal{M}^{\mathcal{S}}_{\xi+1}},$$

and all models displayed agree to their common value for μ^+. In particular,

$$\mathcal{M}^{\mathcal{S}}_\xi|(\mu^+)^{\mathcal{M}^{\mathcal{S}}_\xi} = \mathcal{M}^{\mathcal{S}}_{\xi+1}|(\mu^+)^{\mathcal{M}^{\mathcal{S}}_{\xi+1}}.$$

It follows then from Claim 9 that

$$\mu = \alpha_\xi.$$

Claim 14. r is solid; that is, $q = r$.

Proof. If not, then $\rho(M) < \alpha_0$. It follows by Claim 7 that

$$\rho(\mathcal{M}^{\mathcal{U}}_\tau) < \sup i^{\mathcal{U}}_{0,\tau} \text{“}\alpha_0 = \sup i^{\mathcal{S}}_{0,\xi} \text{“}\alpha_0 = \alpha_\xi = \mu = \rho(\mathcal{M}^{\mathcal{S}}_\theta) = \rho(\mathcal{M}^{\mathcal{U}}_\gamma).$$

However, $\mathrm{crit}(i^{\mathcal{U}}_{\tau,\gamma}) \geq \alpha_\xi$ or $\gamma = \tau$, so $\rho(\mathcal{M}^{\mathcal{U}}_\tau) = \rho(\mathcal{M}^{\mathcal{U}}_\gamma)$. This is a contradiction. □

By Claim 14, $\alpha_0 = \rho$. It follows from (v) and (vi) that for all unstable η, $\alpha_\eta = \rho(\mathcal{M}^{\mathcal{S}}_\eta)$. Moreover, by the usual preservation of solid parameters, $i^{\mathcal{S}}_{0,\eta}(r)$ is the standard parameter of $\mathcal{M}^{\mathcal{S}}_\eta$. In particular, this is true when $\eta = \xi$. That tells us that the parameter of $\mathcal{M}^{\mathcal{S}}_\xi$ is universal:

Claim 15. $i^{\mathcal{S}}_{0,\xi}(r)$ is universal over $\mathcal{M}^{\mathcal{S}}_\xi$; that is, $\mathcal{M}^{\mathcal{S}}_\xi|\eta = \mathcal{M}^{\mathcal{S}}_{\xi+1}|\eta$, where $\eta = (\alpha^+_\xi)^{\mathcal{M}^{\mathcal{S}}_\xi}$.

Proof. This follows from the fact that $\mathcal{M}^{\mathcal{S}}_\theta = \mathcal{M}^{\mathcal{U}}_\gamma$, and $\mathrm{crit}(i^{\mathcal{S}}_{0,\theta}) \geq \alpha_\xi$ and $\mathrm{crit}(i^{\mathcal{U}}_{0,\gamma}) \geq \alpha_\xi$ (and neither branch drops). $\square$

If $\xi = 0$, we are done.

Claim 16. r is universal; that is, $K|(\rho^+)^M = M|(\rho^+)^M$.

Proof. Let us assume $k_0 = 0$ and $\hat{o}(M)$ is a limit ordinal to simplify the fine structure a bit. We may also assume $\xi > 0$.

Suppose first that ρ is regular in M. Let $N \lhd M|(\rho^+)^M$, $\rho(N) = \rho$, and $B \subseteq \rho$ code $\mathrm{Th}^N_n(\rho(N) \cup p(N))$ for $n = k(N)$. We must show $N \lhd K$, and that is equivalent to

(*) For some Σ_1 formula φ, some $b < \rho$, and some $\sigma < \hat{o}(M)$, there is a unique $\langle P, C\rangle$ such that:

 (a) $P \lhd M|\sigma$ and $C \subseteq \rho(P)$ codes $\mathrm{Th}^P_n(\rho(P) \cup p(P))$ for $n = k(P)$, and

 (b) $\mathcal{M}|\sigma \models \varphi[P, C, b, r]$.

 Moreover, for the unique such $\langle P, C\rangle$, we have $C \cap \rho = B$.

We can express (*) as
$$M \models \psi[B, \rho, r],$$
where ψ is Σ_1. Let $i = i^{\mathcal{S}}_{0,\xi}$, and note that $i\colon M \to \mathcal{M}^{\mathcal{S}}_\xi$ is elementary, that is, cofinal and Σ_1-elementary. Moreover, $i(\rho) = \sup i``\rho = \alpha_\xi$, because ρ is regular in M. By Claim 15
$$\mathcal{M}^{\mathcal{S}}_\xi \models \psi[i(B), i(\rho), i(r)].$$
Thus $M \models \psi[B, \rho, r]$, as desired.

Now assume that ρ is singular in M. It will then be enough to show that $P(\rho)^M \subseteq K$. This is because if $\pi\colon K \to M$ is the collapse map, then $\mathrm{crit}(\pi) > \rho$, as otherwise $\mathrm{crit}(\pi) = \rho$ is regular in K, and hence regular in M because $P(\rho)^M \subseteq K$. It follows that $\mathrm{crit}(\pi) \geq (\rho^+)^K = (\rho^+)^M$, which yields Claim 16.

So let $B \subseteq \rho$, $B \in M$, and $B \notin K$. We show by induction on $\eta \leq_S \xi$ that $i^{\mathcal{S}}_{0,\eta}(B) \notin \mathcal{M}^{\mathcal{S}}_{\eta+1}$. The case η is a limit ordinal is easy, so assume $S\text{-pred}(\eta) = \beta$, let $E = E^{\mathcal{S}}_{\eta-1}$, and let $A = i^{\mathcal{S}}_{0,\beta}(B) \cap \alpha_\beta$. So $A \notin \mathcal{M}^{\mathcal{S}}_{\beta+1}$. Let us write i_E for $i^{\mathcal{S}}_{\beta,\eta}$, and let

$s = i^{\mathcal{S}}_{0,\beta}(r)$. Suppose toward contradiction that $i_E(A) \cap \alpha_\eta \in \mathcal{M}^{\mathcal{S}}_{\eta+1}$; then we have some $b < \alpha_\eta$, some C, and some Σ_1 formula φ such that

$$\mathcal{M}^{\mathcal{S}}_\eta \models \text{ } C \text{ is the unique } D \text{ such that } \varphi(D, b, i_E(s)),$$

and $C \cap \alpha_\eta = i_E(A) \cap \alpha_\eta$. Fix b, C, and φ. There are cofinally many ordinals in $\mathcal{M}^{\mathcal{S}}_\beta$ that are Σ_1 definable from parameters in $\alpha_\beta \cup s$, so we can find such an ordinal σ such that

$$\mathcal{M}^{\mathcal{S}}_\eta | i_E(\sigma) \models \text{ } C \text{ is the unique } D \text{ such that } \varphi(D, b, i_E(s)),$$

But now let

$$b = [a, f]^{\mathcal{M}^{\mathcal{S}}_\beta}_E.$$

For E_a almost every u,

$$\mathcal{M}^{\mathcal{S}}_\beta | \sigma \models \text{ there is a unique } D \text{ such that } \varphi(D, f(u), s).$$

Let C_u be the unique such D, when it exists. The function $u \mapsto C_u$ is definable over $\mathcal{M}^{\mathcal{S}}_\beta | \sigma$ from f and s. Since $\alpha_\eta = \sup i_E``\alpha_\beta$, we may assume that $f \in \mathcal{M}^{\mathcal{S}}_\beta | \alpha_\beta$. ($\alpha_\beta$ is a singular cardinal of $\mathcal{M}^{\mathcal{S}}_\beta$ in the present case.) Moreover, $E_a \in \mathcal{M}^{\mathcal{S}}_\beta | \alpha_\beta$ by Claim 4. Then for $\delta < \alpha_\beta$,

$$\delta \in A \Leftrightarrow \text{ for } E_a \text{ a.e. } u, \, \delta \in C_u.$$

This defines A over $\mathcal{M}^{\mathcal{S}}_\beta | \sigma$ from f, s, and E_a. That implies $A \in \mathcal{M}^{\mathcal{S}}_{\beta+1}$, a contradiction. □

This completes the proof of Theorem 6.57, modulo Lemma 6.64. □

Corollary 6.65 *Assume* AD^+*, and let* $\langle (M, \vec{F}, \Sigma), \Sigma^* \rangle$ *be a coarse strategy pair. Let* $\mathbb{C}$ *be an* $(\vec{F}, \Sigma)$*-construction done in* M*; then for any* $\langle \nu, k \rangle < \mathrm{lh}(\mathbb{C})$ *such that* $k \geq 0$*,* $(\dagger)_{\nu,k}$ *holds, that is, the standard parameter of* $M^{\mathbb{C}}_{\nu,k}$ *is solid and universal.*

Corollary 6.66 *Assume* $\mathsf{IH}_{\kappa,\delta}$*, and there are infinitely many Woodin cardinals below* κ*. Let* w *be a wellorder of* V_δ*, and let* $\mathbb{C}$ *be a* w*-construction above* κ*; then for any* $\langle \nu, k \rangle < \mathrm{lh}(\mathbb{C})$ *such that* $k \geq 0$*,* $(\dagger)_{\nu,k}$ *holds, that is, the standard parameter of* $M^{\mathbb{C}}_{\nu,k}$ *is solid and universal.*

Proof. We can use Corollary 6.65, inside a model of AD^+ we get by the method of Remark 6.41. That is, if 6.66 is false, then we have a countable M and $\pi \colon M \to M^{\mathbb{C}}_{\nu,k}$ elementary such that the standard parameter of M is either non-solid or non-universal. We have that (M, Ω^π) is a least branch hod pair by 6.13. Standard arguments using unique iterability show that Ω^π is $< \kappa$-homogeneously Suslin. Because we have assumed that there are infinitely many Woodin cardinals below κ, $L(\Omega^\pi, \mathbb{R}) \models \mathsf{AD}^+$. Thus the hypotheses of 6.57 are satisfied, and the standard parameter of M is solid and universal, a contradiction. □

We can prove a condensation lemma for lbr hod pairs by the same method. Rather than attempt a general statement, we shall content ourselves with the following simple one, since it is what we need in the next section. The author and Nam Trang have proved a stronger condensation theorem in [65], and used it to generalize the Schimmerling-Zeman characterization of $\{\kappa \mid M \models \square_\kappa\}$ to the case that M is a least branch hod mouse. The proof in [65] is given in much greater detail than we give here; moreover, it yields condensation for mouse pairs, not just condensation for mice.

Theorem 6.67 *(***Condensation lemma***) Let M be a countable lpm, and let Ψ be a complete iteration strategy for M defined on all countable M-stacks by Σ. Suppose that whenever s is a countable M-stack by Ψ having last model N, then (N, Ψ_s) is a least branch hod pair. Suppose that Ψ is coded by a set of reals that is Suslin and co-Suslin in some $L(\Gamma, \mathbb{R})$, where $L(\Gamma, \mathbb{R}) \models \mathsf{AD}^+$. Let*

$$\pi \colon H \to M$$

be elementary, with $\mathrm{crit}(\pi) = \rho(H) < \rho(M)$, and H being $k(H)+1$-sound. Suppose also that $\rho(H)$ is a limit cardinal of H; then $H \trianglelefteq M$.

Proof.(Sketch.) We proceed as in the proof of 6.57. Let $\mathbb{C}$ be the construction of some Ψ-Woodin model N^*. We have $\langle \eta_0, k_0 \rangle$ such that (M, Ψ) iterates to $(M^{\mathbb{C}}_{\eta_0,k_0}, \Omega^{\mathbb{C}}_{\eta_0,k_0})$. We may assume that Ψ has the weak Dodd-Jensen property relative to some $\vec{e}$.

For $\langle \nu, l \rangle \leq_{\text{lex}} \langle \eta_0, k_0 \rangle$ we define a psuedo iteration tree $\mathcal{S}_{\nu,l}$ which iterates the phalanx $(M, H, \rho(H))$. $\mathcal{S}_{\nu,l}$ is defined exactly as it was in the proof of 6.57, with one exception with regard to how we move phalanxes up. Note that because $\rho(H) < \rho(M)$, we have $H \in M$.(The theory coding H is a bounded $r\Sigma^M_{k(M)+1}$ subset of $\rho(M)$, hence in M. Since $M|\rho(M) \models \mathsf{KP}$, $H \in M|\rho(M)$.) Now suppose $\gamma+1$ is unstable, and $\xi = S\text{-pred}(\gamma+1)$. We have $\mathcal{M}^{\mathcal{S}}_{\gamma+1} = \mathrm{Ult}(\mathcal{M}^{\mathcal{S}}_\xi, E_\gamma)$ as before. We then set

$$\mathcal{M}^{\mathcal{S}}_{\gamma+2} = i^{\mathcal{S}}_{0,\gamma+1}(H),$$

and

$$\alpha^{\mathcal{S}}_{\gamma+1} = i^{\mathcal{S}}_{0,\gamma+1}(\rho(H)).$$

We have

$$\sigma_{\gamma+1} \colon M^{\mathcal{S}}_{\gamma+2} \to \mathcal{M}^{\mathcal{S}}_{\gamma+1}$$

determined by: $\sigma_{\gamma+1} \restriction \alpha^{\mathcal{S}}_{\gamma+1}$ is the identity, and $\sigma_{\gamma+1}(i^{\mathcal{S}}_{0,\gamma+1}(p(H)) = i^{\mathcal{S}}_{0,\gamma+1}(\pi(p(H))$. If H is not an initial segment of M, then $\mathcal{M}^{\mathcal{S}}_{\gamma+2}$ is not an initial segment of $\mathcal{M}^{\mathcal{S}}_{\gamma+1}$, so we have successfully moved the bad situation up.

There is a similar change at unstable limit ordinals θ. We set $\mathcal{M}^{\mathcal{S}}_{\theta+1} = i^{\mathcal{S}}_{0,\theta}(H)$ and $\alpha^{\mathcal{S}}_\theta = i^{\mathcal{S}}_{0,\theta}(\rho(H))$, etc.

The rest of the construction of $\mathcal{S}_{\nu,l}$, and its conditions for termination, are the same as in the proof of 6.57. Again, the key lemma is the counterpart of Lemma

6.64, according to which no strategy disagreements show up, and least extender disagreements involve only empty extenders on the $M_{\nu,l}^{\mathbb{C}}$ side. We shall prove this lemma in the next section.

We argue as before that for some ν, l, the construction of $\mathcal{S}_{\nu,l}$ terminates at a stable θ such that $\mathcal{M}_\theta^{\mathcal{S}} \trianglelefteq \mathcal{M}_\gamma^{\mathcal{U}}$, where $\mathcal{U} = \mathcal{U}_{\nu,l}$. (We no longer have $\mathcal{M}_\gamma^{\mathcal{U}} \trianglelefteq \mathcal{M}_\theta^{\mathcal{S}}$, as the proof of that used that $K \notin M$, whereas $H \in M$.) Using weak Dodd-Jensen, We get that for some unstable ξ, $\mathrm{rt}(\theta) = \xi + 1$.

Let $\mathcal{M}_\tau^{\mathcal{U}} = \mathcal{M}_\xi^{\mathcal{S}}$. We have that $\mathrm{lh}(E_\tau^{\mathcal{U}}) \geq \lambda_{\xi+1}^{\mathcal{S}}$, as otherwise $\xi + 1$ would have been dead. But in the present case, $\lambda_{\xi+1}^{\mathcal{S}}$ is a limit cardinal of $\mathcal{M}_\xi^{\mathcal{S}} = \mathcal{M}_\tau^{\mathcal{U}}$, so $\mathrm{lh}(E_\tau^{\mathcal{U}}) > \lambda_{\xi+1}^{\mathcal{S}}$.

Now we simply follow the proofs of Claims 1-4 in the proof of Theorem 8.2 of [23]. We get from that that $\mathcal{M}_{\xi+1}^{\mathcal{S}}$ is a proper initial segment of $\mathcal{M}_\gamma^{\mathcal{U}}$. This implies there are no cardinals of $\mathcal{M}_\gamma^{\mathcal{U}}$ strictly between $\lambda_{\xi+1}^{\mathcal{S}}$ and $o(\mathcal{M}_{\xi+1}^{\mathcal{S}})$. It follows that $\mathrm{lh}(E_\tau^{\mathcal{U}}) \geq o(\mathcal{M}_{\xi+1}^{\mathcal{S}})$, so that $\mathcal{M}_{\xi+1}^{\mathcal{S}} \trianglelefteq \mathcal{M}_\tau^{\mathcal{U}} = \mathcal{M}_\xi^{\mathcal{S}}$. But then, as we observed above, $H \trianglelefteq M$, as desired.

□

We get at once

Corollary 6.68 *Assume* AD^+*, and let* $\langle (M, \vec{F}, \Sigma), \Sigma^* \rangle$ *be a coarse strategy pair. Let* $\mathbb{C}$ *be an* $(\vec{F}, \Sigma)$*-construction done in* M*, and let* $M = M_{\nu,k}^{\mathbb{C}}$*. Let*

$$\pi \colon H \to M$$

be elementary, with $\mathrm{crit}(\pi) = \rho(H) < \rho(M)$*, and* H *being* $k(H) + 1$*-sound. Suppose also that* $\rho(H)$ *is a limit cardinal of* H*; then* $H \trianglelefteq M$*.*

Corollary 6.69 *Assume* $\mathsf{IH}_{\kappa,\delta}$*, and there are infinitely many Woodin cardinals below* κ*. Let* w *be a wellorder of* V_δ*, let* $\mathbb{C}$ *be a* w*-construction above* κ*, and let* $M = M_{\nu,k}^{\mathbb{C}}$*. Let*

$$\pi \colon H \to M$$

be elementary, with $\mathrm{crit}(\pi) = \rho(H) < \rho(M)$*, and* H *being* $k(H) + 1$*-sound. Suppose also that* $\rho(H)$ *is a limit cardinal of* H*; then* $H \trianglelefteq M$*.*

6.7 Some successful background constructions

Let us assume Theorem 7.3, the Bicephalus Lemma, throughout this section.

In the AD^+ context, we get that Γ-Woodin constructions do not break down.

Theorem 6.70 *Assume* AD^+*, let* $(N^*, \delta, S, T, \lhd, \Sigma^*)$ *be a coarse* Γ*-Woodin tuple, and let* $\mathbb{C}$ *be a least branch* $\lhd$*-construction in* $L[N^*, S, T, \lhd]$ *with all* $F_\nu^{\mathbb{C}} \in N^*$*; then*

$\mathbb{C}$ is not pathological in $L[N^*, S, T, \lhd]$. In fact, letting $M = M^{\mathbb{C}}_{\nu,k}$, and letting Ω be the canonical extension of $\Omega^{\mathbb{C}}_{\nu,k}$ to all M-stacks in HC; then

1. *(M, Ω) is a least branch hod pair, with scope HC,*

2. *$(*)(M, \Omega)$, and*

3. *M has a core; that is, $p(M)$ is solid and universal.*

Proof. We have a coarse strategy pair $\langle (N, \vec{F}^C, \Sigma), \Sigma^* \rangle$ such that $V^N_\delta = V^{N^*}_\delta$, so by 6.37, 6.65, and the Bicephalus Lemma, $\mathbb{C}$ is not pathological in $L[N^*, S, T, \lhd]$.

The canonical extension Ω of $\Omega^{\mathbb{C}}_{\nu,k}$ is just the strategy for M induced by lifting to N^* and using Σ there. Σ acts on all stacks of trees in HC, not just those in N^*, and we don't need that the stack is in N^* to define its lift to N^*.

Since Σ^* witnesses that $L[N^*, S, T, \lhd]$ is strongly uniquely (ω_1, ω_1) iterable in V, it has strong hull condensation, normalizes well, and moves to its tails under its own iteration maps. (See 4.21 and 4.32.) By our work in Chapter 3 (see 4.41 and 4.49), Ω has strong hull condensation and normalizes well. By the proof of 6.32, whenever s is a stack by Ω with last model Q, then $\dot{\Sigma}^Q \subseteq \Omega_s$. Thus (M, Ω) is an lbr hod pair.

That (1) implies (2) is Theorem 6.49. That (1) implies (3) is Theorem 6.57.

□

We have shown that least branch constructions done in a coarse Γ Woodin model do not break down, but we are missing a proof that such constructions go far enough; that is, a proof of HPC. Borrowing 7.3 from the next chapter, we do get

Theorem 6.71 *Assume* AD^+*; then* LEC *implies* HPC*.*

Proof. It is enough to show that whenever (P, Σ) is a pure extender mouse pair with scope HC, then there is an lbr hod pair (Q, Ψ) with scope HC such that Σ is definable from parameters over $(\mathrm{HC}, \in, \Psi)$.

So fix (P, Σ), and let $(N^*, \delta, S, T, \lhd, \Phi)$ be a coarse Γ-Woodin tuple, with P countable in N^* and $\mathrm{Code}(\Sigma)$ in Γ. Let $\mathbb{C}$ be the maximal $\lhd$- construction of $L[N^*, S, T, \lhd]$, with last pair

$$(Q, \Psi) = (M^{\mathbb{C}}_{\delta,0}, \Omega^{\mathbb{C}}_{\delta,0}).$$

Note here that $\mathbb{C}$ does not break down, by 6.57 and 7.3. Since Φ has scope all of HC, it induces an extension of Ψ with scope HC. We call this extension Ψ as well.

Now let $\mathbb{D}$ be the pure extender $L[E]$ construction of Q, where nice extenders from the Q-sequence are used as backgrounds. By 6.57 and 7.3, $\mathbb{D}$ never breaks down, and each $(M^{\mathbb{D}}_{\nu,k}, \Omega^{\mathbb{D}}_{\nu,k})$ is a pure extender pair in Q, and hence can be canonically extended to such a pair in N^*. Working in N^*, we can compare (P, Σ) with each $(M^{\mathbb{D}}_{\nu,k}, \Omega^{\mathbb{D}}_{\nu,k})$. Because the background extenders of $\mathbb{D}$ are assigned background extenders over N^* by $\mathbb{C}$, we can repeat the proof of $(^*)(P, \Sigma)$, so (P, Σ) iterates past $(M^{\mathbb{D}}_{\nu,k}, \Omega^{\mathbb{D}}_{\nu,k})$, provided it iterates strictly past all earlier levels of $\mathbb{D}$.

By the Q-filtered backgrounding again, (P, Σ) cannot iterate past $(M^{\mathbb{D}}_{\delta,0}, \Omega^{\mathbb{D}}_{\delta,0})$. It follows that (P, Σ) iterates to some $(M^{\mathbb{D}}_{\nu,k}, \Omega^{\mathbb{D}}_{\nu,k})$. This is true in N^*, but it is also true in V of (P, Σ) and the canonical extension (M, Ω) of $(M^{\mathbb{D}}_{\nu,k}, \Omega^{\mathbb{D}}_{\nu,k})$, because N^* is sufficiently correct. But then Σ is projective in Ω, and Ω is projective in Ψ, so we are done. $\square$

Remark 6.72 We do not see how to show that under AD^+, HPC implies LEC. That, together with 6.71, suggests that one should try to prove HPC by proving the ostensibly stronger LEC.

We now look at constructions done in a model of the Axiom of Choice, under strong large cardinal hypotheses. Here we must assume unique iterability. We shall show that under such assumptions, least branch constructions can produce hod pairs (M, Ω) such that $M \models$ "there is a subcompact cardinal".

Definition 6.73 *A cardinal κ is* subcompact *iff for all $A \subseteq H_{\kappa^+}$, there are μ, B, and j such that*

(a) $\mu < \kappa$ and $B \subseteq H_{\mu^+}$,

(b) $j\colon (H_{\mu^+}, \in, B) \to (H_{\kappa^+}, \in, A)$ is elementary, and

(c) $\mu = \mathrm{crit}(j)$.

Subcompactness was introduced by Jensen. It is interesting in part because it can be represented by short extenders[24], but it is strong enough that if κ is subcompact, then $\neg\Box_\kappa$. The main theorem of [37] is that in iterable pure extender models, $\neg\Box\kappa$ if and only if κ is subcompact. If κ is subcompact, then the set

$$S = \{i_E(\mu^+) \mid E \text{ is a superstrong } (\mu, \kappa)\text{-extender}\}$$

is stationary in κ^+.[25] Jensen showed that in iterable pure extender models, the stationarity of S is equivalent to subcompactness. (See [37].)

[24]Let E be the (μ, κ)-extender of j; then i_E also satisfies (b) and (c) of 6.73.

[25]To see this, let A be a given club, apply the definition to get j and μ, and then let $E = E_j \restriction \kappa$.

Subcompactness is close to the limit of the large cardinal properties that can be represented by short extenders, and it is thus close to the limit of the large cardinal properties exhibited in the strategy mice whose theory is developed in this book.

The large cardinal hypothesis of the following theorem is just beyond those that can be captured by short extenders.

Theorem 6.74 *Suppose*

(i) $j\colon V \to N$ *is elementary,* $\kappa = \mathrm{crit}(j)$*, and* $\delta = j(\kappa)$*,*

(ii) $V_\delta \cup \{E_j \restriction \delta\} \subseteq N$*,*

(iii) $\mathsf{IH}_{\mu,\delta}$ *holds, where* $\mu < \kappa$*, and*

(iv) w_0 *is a wellorder of* V_κ*,* $w = j(w_0)$*, and* $\mathbb{C}$ *is a maximal least branch* w*-construction above* μ*.*

Then $\mathbb{C}$ *is not pathological, and*

(a) $M^{\mathbb{C}}_{\delta,0} \models \kappa$ *is subcompact, and*

(b) $M^{\mathbb{C}}_{\delta,0} \models$ *there are arbitrarily large superstrong cardinals.*

Proof. That $\mathbb{C}$ is not pathological follows from 6.66 and the Bicephalus Lemma. Thus $M = M^{\mathbb{C}}_{\delta,0}$ exists. We show first that κ is subcompact in M

Let $A \subseteq (\kappa^+)^M$ and $A \in M$. It will be enough to show that δ is $j(A)$-subcompact in $j(M)$.

Our choice of w guarantees that $j(w) \cap V_\delta = w$. It follows then that $j(\mathbb{C})\restriction\langle\delta, 0\rangle = \mathbb{C}$. Thus

$$M = M^{j(\mathbb{C})}_{\delta,0}.$$

But this implies that

$$M = j(M)|\langle\delta, 0\rangle.$$

To see that, let us call η a β-closure point of $\mathbb{C}$ iff $\eta = o(M^{\mathbb{C}}_{\eta,0})$, $\eta < \beta$, and η is a cardinal of $M^{\mathbb{C}}_{\beta,0}$. Note that this implies $M^{\mathbb{C}}_{\eta,0} \trianglelefteq M^{\mathbb{C}}_{\beta,0}$. The set B_β of β-closure points of $\mathbb{C}$ is closed in β. If β is a cardinal of V, it is club in β. But then

$$\begin{aligned} B^{\mathbb{C}}_\kappa &= j(B^{\mathbb{C}}_\kappa) \cap \kappa \\ &= B^{j(\mathbb{C})}_\delta \cap \kappa \\ &= B^{\mathbb{C}}_\delta \cap \kappa, \end{aligned}$$

so $\kappa \in B^{\mathbb{C}}_{\delta}$, so $\delta \in B^{j(\mathbb{C})}_{j(\delta)}$, or in other words, δ is a closure point of $j(\mathbb{C})$. That implies $M = j(M)|\langle\delta, 0\rangle$.

Let
$$E = \{(a, X) \mid a \in [\delta]^{<\omega} \wedge X \in P([\kappa]^{|a|})^M \wedge a \in j(X)\}$$
be the length δ extender of j, restricted to M.

Claim. If $\eta \leq \delta$ and $E\restriction\eta$ is whole, then the trivial completion of $E\restriction\eta$ is on the $j(M)$-sequence.

Proof. We prove this by induction on η. Suppose we know it for $\beta < \eta$, and let F be the trivial completion of $E\restriction\eta$, and $\gamma = i^M_F(\kappa^{+,M})$. Assume first that $\eta < \delta$. We have that $\mathrm{Ult}(M, F) = \mathrm{Ult}(M, E\restriction\eta)$, and there is a natural factor embedding
$$\sigma\colon \mathrm{Ult}(M, F) \to \mathrm{Ult}(M, E)$$
such that $\sigma\restriction\eta = \mathrm{id}$, and $\sigma(\eta) = \delta$. Since η is a limit cardinal of $\mathrm{Ult}(M, F)$, we have that η is a limit cardinal of M. Using the Condensation lemma 6.69 applied to σ, we get that
$$\mathrm{Ult}(M, F)|\langle\gamma, -1\rangle = \mathrm{Ult}(M, E)\restriction\langle\gamma, -1\rangle = M\restriction\langle\gamma, -1\rangle.$$
Since η is a cardinal of M, there must be a stage of $\mathbb{C}$ at which we have $M|\langle\eta, 0\rangle = M^{\mathbb{C}}_{\nu,0}$. After this stage, no projectum drops strictly below η, and stages which project to η are initial segments of M. Thus there is a ν such that
$$(M^{<\nu})^{\mathbb{C}} = M|\langle\gamma, -1\rangle.$$
But then $(M^{<\nu}, F, \emptyset)$ is an lpm. (Coherence we verified above, and the Jensen initial segment condition holds by our induction hypothesis.) Moreover, F has a background certificate that shifts w to itself, namely $E_j\restriction\mu$, for μ the least inaccessible cardinal strictly greater than η. By the Bicephalus Lemma,
$$M^{\mathbb{C}}_{\nu,0} = (M^{<\nu}, F, \emptyset).$$
Since η is a cardinal of M and $M^{\mathbb{C}}_{\nu,0}$ projects to η, $M^{\mathbb{C}}_{\nu,0} \lhd M$. Thus F is on the M-sequence. Since $\eta < \delta$, it is on the $j(M)$-sequence.

Now we take the case $\eta = \delta$, that is, $F = E$. Again, let $\gamma = i^M_E(\kappa^{+,M}) = i^{M|\delta}_E(\kappa^{+,M}$ be the length of the Jensen completion of E. The factor embedding from $\mathrm{Ult}(M, E)$ to $j(M)$ has critical point $\geq \gamma$, and thus $\mathrm{Ult}(j(M)|\gamma, E)$ agrees with $j(M)$ strictly below γ. E satisfies the Jensen initial segment condition by the claim applied to $\eta < \delta$. To get a background certificate E^* for E in N, simply take
$$E^* = j_1(E_j\restriction\delta)\restriction\lambda,$$
where $j_1 = j(j)$ and λ is the least inaccessible of N above δ. This clearly works, so by the Bicephalus Lemma, E is on the sequence of $j(M)$. $\square$

Let $i_E\colon (M|\kappa^{+,M}, A) \to \mathrm{Ult}((M|\kappa^{+,M}, A), E) = (j(M)||\,\mathrm{lh}(E), B)$ be the canonical fully elementary embedding. Let $\sigma\colon\ \mathrm{Ult}((M, A), E) \to (j(M), j(A))$ be the factor embedding. Since $\mathrm{crit}(\sigma) = \mathrm{lh}(E)$ and σ is elementary, we see that $(j(M)||\,\mathrm{lh}(E), B) \prec (j(M)||\delta^{+,j(M)}, j(A))$. Thus E witnesses that δ is $j(A)$-subcompact in N.

To see that δ is a limit of superstrong cardinals in M, it is enough to see that $M|\kappa \models$ "there are arbitrarily large superstrong cardinals", for then we can apply j to this fact. But κ is subcompact in M, and it is quite easy to see that if κ is subcompact, then $V_\kappa \models$ "there are arbitrarily large superstrong cardinals".

$\square$

7 Phalanx iteration into a backgrounded construction

In this chapter we prove that there are no nontrivial iterable bicephali, and we prove Lemma 6.64, thereby completing the proofs of theorems 6.70 and 6.74. Both results involve showing that certain bicephali and phalanxes iterate into background constructions in the same way that ordinary lbr hod pairs do.

We shall also use such a phalanx-comparison argument to show that if (M, Ω) is an lbr hod-pair such that $M \models \mathsf{ZFC}$ + "there are arbitrarily large Woodin cardinals", then whenever g is $\mathbb{P}$-generic over M, $M[g] \models$ " UBH holds for all nice, normal iteration trees that use extenders from $\dot{E}^M$ with critical points strictly above $|\mathbb{P}|^M$". That implies that Ω determines itself on generic extensions of M. We shall use this in the next section to show that if λ is a limit of cutpoint Woodin cardinals in M, and N is a derived model of M below λ, then HOD^N is an Ω-iterate of M.

7.1 The Bicephalus Lemma

Definition 7.1 *An* lpm-bicephalus *is a structure* $\mathcal{B} = (B, \in, \dot{E}^{\mathcal{B}}, \dot{\Sigma}^{\mathcal{B}}, F, G)$ *such that both* $(B, \in, \dot{E}^{\mathcal{B}}, \dot{\Sigma}^{\mathcal{B}}, F, \emptyset)$ *and* $(B, \in, \dot{E}^{\mathcal{B}}, \dot{\Sigma}^{\mathcal{B}}, G, \emptyset)$ *are extender-active least branch premice. We say that* $\mathcal{B}$ *is* nontrivial *iff* $F \neq G$*.*

We shall usually drop "lpm" from "lpm-bicephalus".

We think of $\mathcal{B}$ as a structure in the language with $\in$ and predicate symbols $\dot{\Sigma}, \dot{E}, \dot{F}$, and $\dot{G}$. We let

$$\mathcal{B}^- = (B, \in, \dot{E}^{\mathcal{B}}, \dot{\Sigma}^{\mathcal{B}}, \emptyset, \emptyset)$$

be the lpm obtained by removing both top extenders. (To be pedantic, $\mathcal{B}$ and $\mathcal{B}^-$ have different languages.) The degree of $\mathcal{B}$ is zero, i.e. $k(\mathcal{B}) = 0$. For $\nu < o(\mathcal{B}) = \hat{o}(\mathcal{B})$, we set $\mathcal{B}|\langle \nu, l\rangle = \mathcal{B}^-|\langle \nu, l\rangle$. The extender sequence of $\mathcal{B}$ is $\dot{E}^{\mathcal{B}}$ together with $\dot{F}^{\mathcal{B}}$ and $\dot{G}^{\mathcal{B}}$; it's not actually a sequence.

A $\mathcal{B}$*-tree* is a tuple $\langle \nu, k, \mathcal{T}\rangle$ such that $\langle \nu, k\rangle \leq_{\text{lex}} \langle \hat{o}(\mathcal{B}), 0\rangle$, and $\mathcal{T}$ is a weakly normal tree on $\mathcal{B}|\langle \nu, k\rangle$. That is, $\mathcal{M}_0^{\mathcal{T}} = \mathcal{B}|\langle \nu, k\rangle$, the extenders used in $\mathcal{T}$ are length-increasing and nonoverlapping along branches, and $E_\alpha^{\mathcal{T}}$ must come from the sequence of $\mathcal{M}_\alpha^{\mathcal{T}}$. If $\mathcal{M}_\alpha^{\mathcal{T}}$ is a bicephalus, this means that the extenders from $\dot{E}^{\mathcal{M}_\alpha}$ together with $\dot{F}^{\mathcal{M}_\alpha}$ and $\dot{G}^{\mathcal{M}_\alpha}$ are eligible. A $\mathcal{B}$*-stack* is a sequence $\langle (\nu_i, k_{i,i}) \mid i \leq n\rangle$ such that $\langle \nu_0, k_{0,0}\rangle$ is a $\mathcal{B}$-tree, and $\langle \nu_{i+1}, k_{i+1}, \mathcal{T}_{i+1}\rangle$ is a $\mathcal{M}_\infty(\mathcal{T}_i)$-tree. A *complete strategy* for $\mathcal{B}$ is a strategy Ω defined on all $\mathcal{B}$-stacks s by Ω such that $s \in N$, for some set N. N is called the scope of Ω.

Definition 7.2 *A* bicephalus pair *is a pair* $(\mathcal{B}, \Omega)$ *such that* $\mathcal{B}$ *is an lpm-bicephalus, and* Ω *is a complete strategy for* $\mathcal{B}$.

Tail strategies are given by $\Omega_s(t) = \Omega(s^\frown t)$. We use $\Omega_{s,N}$ and Ω_N as before. We write Ω^- for $\Omega_{\mathcal{B}^-}$, the complete strategy for $\mathcal{B}^-$ induced by Ω.

We can define the notions of normalizing well, having strong hull condensation, and being self-consistent for bicephalus pairs just as we did before.

The main theorem about bicephali is that there aren't any interesting ones.

Theorem 7.3 *Let* $(\mathcal{B}, \Psi)$ *be a bicephalus pair, where* Ψ *has scope HC. Suppose that* $L(\Psi, \mathbb{R}) \models \mathsf{AD}^+$. *Suppose also that* Ψ *normalizes well and has strong hull condensation, and that* $(\mathcal{B}, \Psi)$ *is self-consistent; then* $\dot{F}^{\mathcal{B}} = \dot{G}^{\mathcal{B}}$.

Proof. Let us assume toward contradiction that $\dot{F}^{\mathcal{B}} \neq \dot{G}^{\mathcal{B}}$.

We work in $L(\Psi, \mathbb{R})$. Fix an inductive-like pointclass Γ_0 with the scale property such that Ψ is coded by a set of reals in $\Gamma_0 \cap \check{\Gamma}_0$. We then fix a "coarse Γ_0-Woodin" tuple $(N^*, \Sigma^*, \delta^*, \tau)$, as in theorem 10.1 of [54]. So $N^* \models \delta^*$ is Woodin, and Σ^* is an (ω_1, ω_1) iteration strategy for $N^*|\delta^*$, and fixing a universal Γ_0 set U, $i(\tau)^g = U \cap i(N^*)[g]$ for all g on $\mathrm{Col}(\omega, i(\delta^*))$, whenever i is an iteration map by Σ^*. We also have that the restriction of Σ^* to trees that are definable over $N^*|\delta^*$ is in N^*. We can assume that there is an $\vec{F}$ such that

(a) $N^* \models \vec{F}$ is coarsely coherent,

(b) δ^* is Woodin in N^* via extenders from $\vec{F}$, and

(c) $N^* \models$ "I am strongly uniquely $\vec{F}$-iterable for stacks of trees in V_{δ^*}."

Working now in N^*, let $\mathbb{C}$ be the $\vec{F}$-maximal least branch hod pair construction done in N^*. The construction lasts until we reach some $\langle \nu, k \rangle < \langle \delta^*, 0 \rangle$ such that $(\dagger)_{\nu,k}$ fails, or until we reach $\langle \nu, k \rangle = \langle \delta^*, 0 \rangle$. Let $\langle \eta_0, l_0 \rangle$ be this $\langle \nu, k \rangle$. We write

$$M_{\nu,l} = M^{\mathbb{C}}_{\nu,l} \text{ and } \Omega_{\nu,l} = \Omega^{\mathbb{C}}_{\nu,l},$$

for $\langle \nu, l \rangle \leq \langle \eta_0, l_0 \rangle$.

We now compare $(\mathcal{B}, \Psi)$ with itself, by comparing two versions of it with $(M_{\nu,l}, \Omega_{\nu,l})$. The result will be two trees $\mathcal{S}_{\nu,l}$ and $\mathcal{T}_{\nu,l}$, each on $\mathcal{B}$ and by Ψ. We show that only the two $\mathcal{B}$ sides move in our coiteration, and that no strategy disagreements show up. This is done by induction on $\langle \nu, l \rangle$. It is not possible for our coiterations to terminate because $\mathcal{B}$ is nontrivial, so we end up with $\mathcal{B}$ iterating past $M^{\mathbb{C}}_{\eta_0,l_0}$. This leads to a contradiction.

Let $\mathcal{C}$ be a premouse. For $\eta < \hat{o}(\mathcal{C})$, we let $E^{\mathcal{C}}_\eta = \dot{E}^{\mathcal{C}}_\eta$, and for $\eta = \hat{o}(\mathcal{C})$, we let $E^{\mathcal{C}}_\eta = \dot{F}^{\mathcal{C}}$. If $\mathcal{C}$ is a bicephalus, and $\eta < \hat{o}(\mathcal{C})$, then we set $E^{\mathcal{C}}_\eta = \dot{E}^{\mathcal{C}}_\eta$. If $\eta = \hat{o}(\mathcal{C})$, we leave $E^{\mathcal{C}}_\eta$ undefined.

Fix $\langle \nu, l\rangle$, and suppose we have defined $\mathcal{S}_{\mu,k}$ and $\mathcal{T}_{\mu,k}$ for all $\langle \mu, k\rangle <_{\text{lex}} \langle \nu, l\rangle$. (The trees are empty until $\mathbb{C}$ has gone well past $0^\sharp$.) We define normal trees $\mathcal{S} = \mathcal{S}_{\nu,l}$ and $\mathcal{T} = \mathcal{U}_{\nu,l}$ on $\mathcal{B}$ by induction. At stage α, we have $\mathcal{S}^\alpha$ and $\mathcal{T}^\alpha$ with last models

$$\mathcal{C} = \mathcal{M}^{\mathcal{S}^\alpha}_\infty \text{ and } \mathcal{D} = \mathcal{M}^{\mathcal{T}^\alpha}_\infty.$$

We do not assume $\text{lh}(\mathcal{S}^\alpha) = \text{lh}(\mathcal{T}^\alpha)$.

Case 1. $(M_{\nu,l}, \Omega_{\nu,l}) \trianglelefteq \mathcal{C}$ and $(M_{\nu,l}, \Omega_{\nu,l}) \trianglelefteq \mathcal{D}$.

In this case, we must have that either $(M_{\nu,l}, \Omega_{\nu,l}) \triangleleft \mathcal{C}$, or the branch of $\mathcal{S}_{\nu,l}$ to $\mathcal{C}$ has dropped, because $\mathcal{C}$ is a bicephalus and $M_{\nu,l}$ is not. Similarly on the $\mathcal{D}$ side. (Our claim 0 below implies we never get "half" of a bicephalus lining up with an $M_{\nu,l}$.) We stop the construction of $\mathcal{S}_{\nu,l}$ and $\mathcal{T}_{\nu,l}$, and go on to $\mathcal{S}_{\nu,l+1}$ and $\mathcal{T}_{\nu,l+1}$.

Case 2. Otherwise.

Here the main claim is

Claim 0. There is a γ such that

(a) $M_{\nu,l}|\langle \gamma, 0\rangle$ is extender-passive,

(b) $M_{\nu,l}|\langle \gamma, 0\rangle = \mathcal{C}|\langle \gamma, -1\rangle = \mathcal{D}|\langle \gamma, -1\rangle$, and $(\Omega_{\nu,l})_{\langle \gamma, 0\rangle} = \Psi_{\mathcal{S}^\alpha, \langle \gamma, -1\rangle} = \Psi_{\mathcal{T}^\alpha, \langle \gamma, -1\rangle}$, and

(c) at least one of $\mathcal{C}|\langle \gamma, 0\rangle$ and $\mathcal{D}|\langle \gamma, 0\rangle$ is extender-active.

We defer proof of Claim 0 for now.

Let $\gamma = \gamma(\alpha)$ be the unique γ as in Claim 0. We get $\mathcal{S}^{\alpha+1}$ and $\mathcal{T}^{\alpha+1}$ as follows. Let $\eta = o(M_{\nu,l}|\langle \gamma, 0\rangle)$. Let

$$\mathcal{C} = \mathcal{M}^{\mathcal{S}^\alpha}_\xi \text{ and } \mathcal{D} = \mathcal{M}^{\mathcal{T}^\alpha}_\tau.$$

Suppose $\eta < o(\mathcal{C})$, or $\eta = o(\mathcal{C})$ but $\mathcal{C}$ is not a bicephalus, because $[0, \xi]_S$ dropped. We set

$$E^{\mathcal{S}^{\alpha+1}}_\xi = E^{\mathcal{C}}_\eta,$$

if $E^{\mathcal{C}}_\eta \neq \emptyset$, with $\mathcal{S}^{\alpha+1}$ then determined by normality. If $E^{\mathcal{C}}_\eta = \emptyset$, then $\mathcal{S}^{\alpha+1} = \mathcal{S}^\alpha$. Similarly, if $\eta < o(\mathcal{D})$ or $\mathcal{D}$ is not a bicephalus, then we set

$$E^{\mathcal{T}^{\alpha+1}}_\tau = E^{\mathcal{D}}_\eta,$$

if $E^{\mathcal{D}}_\eta \neq \emptyset$, with $\mathcal{T}^{\alpha+1}$ then determined by normality. If $E^{\mathcal{D}}_\eta = \emptyset$, then $\mathcal{T}^{\alpha+1} = \mathcal{T}^\alpha$.

If $\eta = o(\mathcal{C})$ and $\mathcal{C}$ is a bicephalus, then if $E_\tau^{\mathcal{T}^{\alpha+1}}$ has already been determined, we let $E_\eta^{\mathcal{S}^{\alpha+1}}$ be the first of $\dot{F}^{\mathcal{C}}$ and $\dot{G}^{\mathcal{C}}$ that is different from $E_\tau^{\mathcal{T}^{\alpha+1}}$. If also $o(\mathcal{D}) = \eta$ and $\mathcal{D}$ is a bicephalus, then we set $E_\xi^{\mathcal{S}^{\alpha+1}} = \dot{F}^{\mathcal{C}}$, and

$$E_\tau^{\mathcal{T}^{\alpha+1}} = \begin{cases} \dot{F}^{\mathcal{D}} & \text{if } \dot{F}^{\mathcal{D}} \neq \dot{F}\mathcal{C} \\ \dot{G}^{\mathcal{D}} & \text{otherwise.} \end{cases}$$

Our definitions guarantee that if one of $E_\xi^{\mathcal{S}}$ and $E_\tau^{\mathcal{T}}$ is a top extender of a bicephalus, then $E_\xi^{\mathcal{S}} \neq E_\tau^{\mathcal{T}}$.

This finishes the definition of $\mathcal{S}^{\alpha+1}$ and $\mathcal{T}^{\alpha+1}$. The limit steps in the construction of $\mathcal{S}_{\nu,l}$ and $\mathcal{T}_{\nu,l}$ are determined by Ψ. Note that $\alpha < \beta \Rightarrow \gamma(\alpha) < \gamma(\beta)$; that is, the common lined up part keeps lengthening.

Eventually, we reach Case 1 above, and the construction of $\mathcal{S}_{\nu,l}$ and $\mathcal{T}_{\nu,l}$ stops. $(\mathcal{B}, \Psi)$ has iterated strictly past $(M_{\nu,l}, \Omega_{\nu,l})$, in two ways. As in the proof of 6.50, this implies $(\dagger)_{\nu,l}$. (When $l = -1$ as well.) It follows then that

$$\eta_0 = \delta^* \text{ and } l_0 = 0.$$

However, $(\mathcal{B}, \Psi)$ cannot iterate past $M_{\delta^*,0}$, by the usual universality argument. Note here that we have $(\dagger)_{\nu,-1}$ for all $\nu < \delta^*$, so the extenders added to the $M_{\nu,-1}$ are unique, and the universality argument applies. This contradiction completes the proof, modulo Claim 0.

Proof of Claim 0. (Sketch) We repeat the proof of Theorem 5.11. Virtually nothing changes, so we shall just mention the main points here.

The main change is the following. We used many times in the proof of 5.11 that for premice Q and R, and Σ an iteration strategy for Q, there is at most one iteration tree $\mathcal{T}$ by Σ such that $R \trianglelefteq M_\alpha(\mathcal{T})$ for $\alpha+1 = \mathrm{lh}(\mathcal{T})$, and $R \ntrianglelefteq \mathcal{M}_\alpha^{\mathcal{T}}$ whenever $\alpha + 1 < \mathrm{lh}(\mathcal{T})$. This uniqueness for normal iterations past a given R clearly fails for bicephali; let $Q = \mathcal{B}$ and $R = \mathrm{Ult}(\mathcal{B}, \dot{F}^{\mathcal{B}})$. What saves us is that in our siuation, with $Q = \mathcal{B}$ and R some initial segment of $M_{\nu,l}$, the trees $\mathcal{S}_{\nu,l}$ and $\mathcal{T}_{\nu,l}$ are being defined together in a way that completely specifies which extender to use at each step on both sides, whether that extender is from the top pair of a bicephalus or not. Moreover, this specification is absolute.

Definition 7.4 *Let R be a premouse, and suppose $\mathcal{S}$ and $\mathcal{T}$ are normal iteration trees on $\mathcal{M}$ of lengths $\alpha + 1$ and $\beta + 1$ respectively such that*

(a) *α is the least ξ such that $R \trianglelefteq \mathcal{M}_\xi^{\mathcal{S}}$,*

(b) *β is the least ξ such that $R \trianglelefteq \mathcal{M}_\xi^{\mathcal{T}}$,*

(c) *$\mathcal{S}$ and $\mathcal{T}$ are by Ψ, and*

(d) *the extenders used in $\mathcal{S}$ and $\mathcal{T}$ are chosen according to the rules above, with R playing the role of $M_{\nu,l}$.*

Then we call $(\mathcal{S}, \mathcal{T})$ *the* (R, Ψ)- coiteration.

Subclaim A.

(1) If $R_0 \trianglelefteq R_1$, and $(\mathcal{S}_i, \mathcal{T}_i)$ is the (R_i, Ψ)-coiteration, then $\mathcal{S}_0$ is an initial segment of $\mathcal{S}_1$ and $\mathcal{T}_0$ is an initial segment of $\mathcal{T}_1$.

(2) If S_0 and S_1 are transitive models of ZFC such that $\mathcal{B}, R \in S_i$ and $\Psi \cap S_i \in S_i$ for $i = 0, 1$, and $S_0 \models (\mathcal{S}, \mathcal{T})$ is the $(R, \Psi \cap S_0)$-coiteration, then $S_1 \models (\mathcal{S}, \mathcal{T})$ is the $(R, \Psi \cap S_1)$-coiteration.

Proof. This is obvious. □

Let us assume that Claim 0 is true for $\langle \eta, k \rangle <_{\text{lex}} \langle \nu, l \rangle$. Let $\langle \gamma^*, k^* \rangle$ be least $\langle \gamma, k \rangle$ such that either $(M_{\nu,l}|\langle \gamma, k \rangle, (\Omega_{\nu,l})_{\langle \gamma,k \rangle}) \neq (\mathcal{C}|\langle \gamma, k \rangle, \Psi_{\mathcal{S}^\alpha, \langle \gamma,k \rangle})$, or $(M_{\nu,l}|\langle \gamma, k \rangle, (\Omega_{\nu,l})_{\langle \gamma,k \rangle}) \neq (\mathcal{D}|\langle \gamma, k \rangle, \Psi_{\mathcal{T}^\alpha, \langle \gamma,k \rangle})$. We show first that we are not in the bad case for extender disagreement.

Subclaim B. It is not the case that $k^* = 0$ and $\dot{F}^{M_{\nu,l}|\langle \gamma_0, 0 \rangle} \neq \emptyset$.

Proof. Suppose otherwise, and let $F = \dot{F}^{M_{\nu,l}}|\langle \gamma^*, 0 \rangle$.

We claim first that $l = 0$. For suppose $l = k + 1$. F cannot be on the sequence of $M_{\nu,k}$, since otherwise $\mathcal{S}_{\nu,k}$ would agree with $\mathcal{S}_{\nu,l}$ on all extenders used with length $< \text{lh}(F)$, and similarly for $\mathcal{T}_{\nu,k}$ and $\mathcal{U}_{\nu,l}$. But this would mean Claim 0 failed at $\langle \nu, k \rangle$, contrary to our induction hypothesis. It follows that $M_{\nu,k}$ is not sound. That implies that $M_{\nu,k}$ is the last model of $\mathcal{S}_{\nu,k}$, along a branch that dropped to $M_{\nu,l}$. Similarly, $M_{\nu,k}$ is the last model of $\mathcal{T}_{\nu,k}$, along a branch that dropped to $M_{\nu,l}$. Let α be least such that $M_{\nu,l} \trianglelefteq \mathcal{M}_\alpha^{\mathcal{S}_{\nu,k}}$ and β be least such that $M_{\nu,l} \trianglelefteq \mathcal{M}_\beta^{\mathcal{T}_{\nu,k}}$. From Subclaim A(1), we see that $\mathcal{S}_{\nu,l} = \mathcal{S}_{\nu,k} \restriction (\alpha + 1)$ and $\mathcal{T}_{\nu,l} = \mathcal{T}_{\nu,k} \restriction (\beta + 1)$. Thus $M_{\nu,l}$ is the last model of $\mathcal{S}_{\nu,l}$ and $\mathcal{T}_{\nu,l}$, contradiction.

But then F must be the last extender of $M_{\nu,0}$, for otherwise F is on the sequence of some $M_{\eta,k}$ with $\eta < \nu$, and Claim 0 would fail at $\langle \eta, k \rangle$, contrary to induction hypothesis.

So suppose that $M_{\nu,0}$ is extender-active, with last extender F. Suppose $\mathcal{S} = \mathcal{S}_{\nu,0}^\alpha$ and $\mathcal{T} = \mathcal{T}_{\nu,0}^\alpha$ have last models $\mathcal{C}$ and $\mathcal{D}$ respectively, and

$$(M_{\nu,-1}, \Omega_{\nu,-1}) = (\mathcal{C}|\langle \nu, -1 \rangle, \Psi_{\mathcal{S}, \langle \nu,-1 \rangle}) = (\mathcal{D}|\langle \nu, -1 \rangle, \Psi_{\mathcal{T}, \langle \nu,-1 \rangle}).$$

So $(\mathcal{S}, \mathcal{T})$ is the $(M_{\nu,-1}, \Psi)$-coiteration. We want to show that F is on the sequences of $\mathcal{C}$ and $\mathcal{D}$, and not as a top extender of a bicephalus in either case. For this, let

$$j\colon V \to \mathrm{Ult}(V, F_\nu^{\mathbb{C}})$$

be the canonical embedding, and $\kappa = \mathrm{crit}(j)$. ($V = N^*$ at this moment.) We have that $M_{\nu,-1} \trianglelefteq j(M_{\nu,-1})$ by coherence. (Note $j(M_{\nu,-1})|\nu$ is extender passive.) $j(\mathcal{S}, \mathcal{T})$ is the $(j(M_{\nu,-1}), \Psi)$ coiteration, because $j(\Psi) \subseteq \Psi$. So by Subclaim A, $\mathcal{S}$ is an initial segment of $j(\mathcal{S})$ and $\mathcal{T}$ is an initial segment of $j(\mathcal{T})$.

We have that $\mathcal{M}_\kappa^{\mathcal{S}} = \mathcal{M}_\kappa^{j(\mathcal{S})}$ and $j{\restriction}\mathcal{M}_\kappa^{\mathcal{S}} = i_{\kappa,j(\kappa)}^{j(\mathcal{S})}$, so F is compatible with the first extender G used in $[\kappa, j(\kappa)]_{j(\mathcal{S})}$. $M_{\nu,-1} \triangleleft \mathcal{M}_{j(\kappa)}^{j(\mathcal{S})}$, so G cannot be a proper initial segment of F. But F is not on the sequence of $\mathcal{M}_{j(\kappa)}^{j(\mathcal{S})}$, so F cannot be a proper initial segment of G. Hence $F = G$. Since $\mathcal{S} = j(\mathcal{S}){\restriction}(\xi + 1)$, where $\mathcal{C} = \mathcal{M}_\xi^{\mathcal{S}}$, we have that F is on the sequence of $\mathcal{C}$.

Similarly, F is on the sequence of $\mathcal{D}$, and used in $j(\mathcal{T})$. But then applying our observation above in $j(V)$, we see that it is not the case that $\mathcal{C}$ is a bicephalus and F is one of its top extenders, or that $\mathcal{D}$ is a bicephalus and F is one of its top extenders. $\square$

By Subclaim B, we may assume that

$$M_{\nu,l}|\langle\gamma^*, k^*\rangle = \mathcal{C}|\langle\gamma^*, k^*\rangle = \mathcal{D}|\langle\gamma^*, k^*\rangle,$$

but there is a strategy disagreement. The situation is symmetric, so we may assume

$$(\Omega_{\nu,l})_{\langle\gamma^*,k^*\rangle} \neq \Psi_{\mathcal{T}^\alpha,\langle\gamma^*,k^*\rangle}.$$

Let

$$M = M_{\nu,l}|\langle\gamma^*, k^*\rangle.$$

We consider first the case that $M = M_{\nu,l}$, then we reduce to this case using the pullback consistency of Ψ. We derive a contradiction in the case $M = M_{\nu,l}$ by repeating the proof of Theorem 5.11. We shall try to keep the notation close to that in the proof of 5.11.

Let $(\mathcal{S}, \mathcal{T})$ be the (M, Ψ)-coiteration of $\mathcal{B}$. So M is an initial segment of both last models, but $\Omega_{\nu,l} \neq \Psi_{\mathcal{T},M}$. Note that M is an lpm, not a bicephalus. We suppose for simplicity that our strategies diverge on a single weakly normal tree $\mathcal{U}$ on M. That is, letting

$$\Omega = (\Omega_{\nu,l})_{\langle\gamma^*,k^*\rangle},$$

$\mathcal{U}$ is by both Ω and $\Psi_{\mathcal{T},M}$, but

$$\Omega(\mathcal{U}) \neq \Psi(\langle\mathcal{T}, \mathcal{U}\rangle).$$

Let $b = \Omega(\mathcal{U})$. For $\gamma < \mathrm{lh}(\mathcal{U})$ we have the embedding normalizations

$$\mathcal{W}_\gamma = W(\mathcal{T}, \mathcal{U}{\restriction}(\gamma + 1)) \text{ and } \mathcal{W}_b = W(\mathcal{T}, \mathcal{U}^\frown b).$$

These are defined just as they were for trees on premice of the ordinary or least branch variety. The fact that $\mathcal{U}$ is only weakly normal affects nothing. We adopt all the previous notation; for example, R_γ is the last model of $\mathcal{W}_\gamma$, and $\sigma_\gamma\colon \mathcal{M}^{\mathcal{U}}_\gamma \to R_\gamma$ is the natural map.

Ω is defined by lifting to V. Let

$$\mathrm{lift}(\mathcal{U}, M_{\nu,l}|\langle\gamma^*, k^*\rangle, \mathbb{C}) = \langle \mathcal{U}^*, \langle \eta_\tau, l_\tau \mid \tau < \mathrm{lh}\,\mathcal{U}\rangle, \langle \psi^{\mathcal{U}}_\tau \mid \tau < \mathrm{lh}\,\mathcal{U}\rangle\rangle.$$

Here $\langle \eta_0, l_0\rangle = \langle \nu, l\rangle$ and $\psi^{\mathcal{U}}_0 = \text{ id }$. Let

$$S_\gamma = \mathcal{M}^{\mathcal{U}^*}_\gamma,$$

and for $\langle \mu, k\rangle \leq_{\mathrm{lex}} \langle \nu, l\rangle$ let

$$(\mathcal{V}^*_{\mu,k}, \mathcal{W}^*_{\mu,k}) = \text{ the } (M_{\mu,k}, \Psi)\text{-coiteration of } \mathcal{B},$$

For $\gamma < \mathrm{lh}(\mathcal{U})$ or $\gamma = b$, let

$$(\mathcal{V}^*_\gamma, \mathcal{W}^*_\gamma) = (\mathcal{V}^*_{\eta_\gamma,l_\gamma}, \mathcal{W}^*_{\eta_\gamma,l_\gamma})^{S_\gamma}.$$

So if $[0,\gamma]_U$ does not drop in model or degree, $(\mathcal{V}^*_\gamma, \mathcal{W}^*_\gamma) = i^{\mathcal{U}^*}_{0,\gamma}((\mathcal{S}, \mathcal{T}))$.

We define by induction tree embeddings Φ_γ from $\mathcal{W}_\gamma$ into $\mathcal{W}^*_\gamma$, for $\gamma < \mathrm{lh}(\mathcal{U})$ or $\gamma = b$, just as before. Let

$$\Phi_\gamma = \langle u^\gamma, \langle t^{0,\gamma}_\beta \mid \beta \leq z(\gamma)\rangle, \langle t^{1,\gamma}_\beta \mid \beta < z(\gamma)\rangle, p^\gamma\rangle.$$

Let us just say a few words about how to obtain $\Phi_{\gamma+1}$, because this is where the main point lies.

We have $t^\gamma\colon R_\gamma \to N_\gamma$, where N_γ is the last model of $\mathcal{W}^*_\gamma$. Let $F = \sigma_\gamma(E^{\mathcal{U}}_\gamma)$, and let $\mu = U\text{-pred}(\gamma+1)$. (Sadly, we can't use "$\nu$" for this ordinal.) So $\mathcal{W}_{\gamma+1} = W(\mathcal{W}_\mu, F)$. Let us assume for simplicity that $(\mu, \gamma+1]_U$ is not a drop in model or degree. Let

$$\mathrm{res}_\gamma = (\sigma_{\eta_\gamma,l_\gamma}[M_{\eta_\gamma,l_\gamma}|\langle \mathrm{lh}\,\psi^{\mathcal{U}}_\gamma(E^{\mathcal{U}}_\gamma), 0\rangle])^{S_\gamma},$$

and let

$$G = \mathrm{res}_\gamma(t^\gamma(F)).$$

We have $t^\gamma \circ \sigma_\gamma = \psi^{\mathcal{U}}_\gamma$, so $G = \mathrm{res}_\gamma(\psi^{\mathcal{U}}_\gamma(E^{\mathcal{U}}_\gamma)$. Let G^* be the background extender for G provided by $i^{\mathcal{U}^*}_{0,\gamma}(\mathbb{C})$, so that

$$S_{\gamma+1} = \mathrm{Ult}(S_\mu, G^*).$$

Since we are not dropping,

$$\mathcal{W}^*_{\gamma+1} = i_{G^*}(\mathcal{W}^*_\mu),$$

where $i_{G^*} = i^{\mathcal{U}^*}_{\mu,\gamma+1}$. The main thing we need to see is that G is used in $\mathcal{W}^*_{\gamma+1}$.

Let $P = N_\gamma|\langle \mathrm{lh}(t^\gamma(F), 0\rangle$, θ be least such that $P \trianglelefteq \mathcal{M}^{\mathcal{V}^*_\gamma}_\theta$, and τ least such that $P \trianglelefteq \mathcal{M}^{\mathcal{W}^*_\gamma}_\tau$. Let $(\mathcal{V}^{**}_\gamma, \mathcal{W}^{**}_\gamma)$ be the $(\mathrm{res}_\gamma(P), \Psi)$-coiteration of $\mathcal{B}$. By the counterpart of Lemma 5.6,

(i) $\mathcal{W}^{**}_\gamma$ extends $\mathcal{W}^*_\gamma\restriction(\tau+1)$,

(ii) letting $\xi = \mathrm{lh}\,\mathcal{W}^{**}_\gamma - 1$, G is on the $\mathcal{M}^{\mathcal{W}^{**}_\gamma}_\xi$ sequence, and not on the $\mathcal{M}^{\mathcal{W}^{**}_\gamma}_\alpha$ sequence for any $\alpha < \xi$,

(iii) $\tau \leq_{W^{**}_\gamma} \xi$, and $\hat{\imath}^{\mathcal{W}^{**}_\gamma}_{\tau,\xi}\restriction(\mathrm{lh}\,t^\gamma(F)+1) = \mathrm{res}_\gamma\restriction(\mathrm{lh}\,t^\gamma(F)+1)$, and

(iv) similarly for $\mathcal{V}_\gamma^{**}$ vis-a-vis $\mathcal{V}_\gamma^*$.

$P, \mathrm{res}_\gamma(P)$, and N_μ all agree up to $\mathrm{dom}(G)$, so

$$\mathrm{res}_\gamma(P)|\langle \mathrm{lh}(G), -1\rangle \trianglelefteq i_{G^*}(N_\mu),$$

and $i_{G^*}(N_\mu)|\langle \mathrm{lh}(G), 0\rangle$ is extender-passive, by coherence. We then get that $\mathcal{V}_\gamma^{**}$ is an initial segment of $\mathcal{V}_{\gamma+1}^*$, $\mathcal{W}_\gamma^{**}$ is an initial segment of $\mathcal{W}_{\gamma+1}^*$ and G is used in both $\mathcal{V}_{\gamma+1}^*$ and $\mathcal{W}_{\gamma+1}^*$. It matters here that $\mathrm{res}_\gamma(P)$ is a premouse, not a bicephalus, so both trees are forced to use G by our rules.

Now let $M = M_{\nu,l}|\langle \gamma^*, l^*\rangle$, where $\langle \gamma^*, l^*\rangle <_{\mathrm{lex}} \langle \hat{o}(M_{\nu,l}), l\rangle$. Let

$$\langle \nu_0, l_0\rangle = \mathrm{Res}_{\nu,l}[M] \text{ and } \pi = \sigma_{\nu,l}[M].$$

$(\Omega_{\nu,l})_M$ is defined by $(\Omega_{\nu,l})_M = \Omega^\pi_{\nu_0,l_0}$. By induction, the (M_{ν_0,l_0}, Ψ) coiteration is a pair $(\mathcal{V}^*, \mathcal{W}^*)$ such that M_{ν_0,l_0} is the last model of $\mathcal{W}^*$, and $\Omega_{\nu_0,l_0} = \Psi_{\mathcal{W}^*, M_{\nu_0,l_0}}$. By the counterpart of Lemma 5.6, the last drop along the main branch of $\mathcal{W}^*$ was to M, and the branch embedding is the resurrection map π, that is,

$$\pi = \hat{i}^{\mathcal{W}^*}_{\xi,\theta} : M \to M_{\nu_0,l_0}.$$

Here ξ is least such that $M \trianglelefteq \mathcal{M}_\xi^{\mathcal{W}^*}$, so the (M, Ψ) coiteration $(\mathcal{S}, \mathcal{T})$ of $\mathcal{B}$ is such that

$$\mathcal{W}^* \restriction (\xi + 1) = \mathcal{T}.$$

But then

$$\begin{aligned}\Psi_{\mathcal{T},M} &= (\Psi_{\mathcal{W}^*, M_{\nu_0,l_0}})^{\hat{i}^{\mathcal{W}^*}_{\xi,\theta}}\\ &= (\Omega_{\nu_0,l_0})^\pi\\ &= (\Omega_{\nu,l})_M.\end{aligned}$$

The first equality holds because Ψ normalizes well and has strong hull condensation, and is therefore pullback consistent.

This finishes our proof of 7.3.

□

Corollary 7.5 *Assume $\mathsf{IH}_{\kappa,\delta}$, and there are infinitely many Woodin cardinals below κ. Let w be a wellorder of V_δ, let $\mathbb{C}$ be a w-construction above κ; then $\mathbb{C}$ gives rise to no nontrivial bicephali. That is, if $\langle \nu, -1\rangle < \mathrm{lh}(\mathbb{C})$, then $\mathbb{C}$ satisfies $(\dagger)_{\nu,-1}$.*

7.2 Proof of Lemma 6.64

Let us assume AD^+ throughout this section. Our proof of 6.64 follows closely the proof of Theorem 5.11. We begin by discussing tree embeddings and normalization for psuedo-trees.

Let (M, Λ) be an lbr hod pair, and let

$$K = \text{ transitive collapse of } h_M\text{“}(\alpha_0 \cup q),$$

where q is a finite set of ordinals, and let

$$\pi \colon K \to M$$

be the collapse map. So π is elementary, and $\alpha_0 \leq \mathrm{crit}(\pi)$. The assumptions of 6.64 imply that α_0 is a cardinal of K, so we assume this. We have a pullback iteration strategy

$$\Sigma = \Lambda^{(\mathrm{id},\pi)}$$

for $(M, K, \alpha_))$, obtained by using $\mathrm{id}\colon M \to M$ and $\pi \colon K \to M$ to lift $\mathcal{S}$ on (M, K, α_0) to a tree $\mathcal{T} = (\mathrm{id}, \pi)\mathcal{S}$ on M, then choosing the branch chosen by Λ. That is

$$\Sigma(\mathcal{S}) = \Lambda((\mathrm{id}, \pi)\mathcal{S}).$$

Σ is actually a strategy for a stronger iteration game than the usual game producing a normal tree on a phalanx. Namely, Σ wins $\mathcal{G}_0$, where in $\mathcal{G}_0$ the opponent, player I, plays not just the extenders $E^{\mathcal{S}}_\gamma$, but also decides whether nodes are unstable. We demand that if I declares θ unstable, then he must have declared all $\tau <_S \theta$ unstable, and $0 \leq_S \theta$, and $[0, \theta]_S$ does not drop in model or degree. We then set $\alpha_\theta = \sup i^{\mathcal{S}}_{0,\theta}``\alpha_0$ and let $\mathcal{M}^{\mathcal{S}}_{\theta+1}$ be the transitive collapse of $h_{\mathcal{M}^{\mathcal{S}}_\theta}``(\alpha_\theta \cup i^{\mathcal{S}}_{0,\theta}(q))$. I must then declare $\theta+1$ to be stable, and take his next extender from $\mathcal{M}^{\mathcal{S}}_{\theta+1}$. If I declares θ to be stable, he must take his next extender from $\mathcal{M}^{\mathcal{S}}_\theta$. The rest of $\mathcal{G}_0$ is as in the normal iteration game. Let us call a play $\mathcal{V}$ of $\mathcal{G}_0$ in which no one has yet lost a *psuedo iteration tree* on (M, K, α_0).

The psuedo-tree occurring in the proof of 6.57 was a play of $\mathcal{G}_0$ in which I followed certain rules for picking his extenders and declaring nodes unstable. But for now, we do not assume I is playing in any such special way.

Remark 7.6 We can generalize $\mathcal{G}_0$ much further, to a game in which I is allowed to gratuitously drop to Skolem hulls whenever he pleases. With some minimal conditions, Ψ will pull back to a strategy for this game. We don't need that generality, so we won't go into it.

Let us define strong hull condensation. The changes we need to make in order to accomodate psuedo-trees are straightforward, but we may as well spell them out.

If $\mathcal{T}$ is a psuedo-tree on (M, K, α_0), then we set

$$\mathrm{stab}(\mathcal{T}) = \{\beta < \mathrm{lh}(\mathcal{T}) \mid \beta \text{ is } \mathcal{T}\text{-stable }\}.$$

We let $\mathrm{Ext}(\mathcal{T})$ be the set of extenders used, and $\mathcal{T}^{\mathrm{ext}}$ the extender tree of $\mathcal{T}$. $\mathcal{T}$ is determined by $\mathrm{stab}(\mathcal{T})$ and $\mathrm{Ext}(\mathcal{T})$. (Psuedo-trees are normal, and their last nodes are stable, by definition.). If β is an unstable node of $\mathcal{T}$, we write

$$\alpha^{\mathcal{T}}_\beta = \sup(i^{\mathcal{T}}_{0,\beta}``\alpha_0).$$

Definition 7.7 *For $\mathcal{T}$ a psuedo-tree, we put $\xi \leq^*_T \eta$ iff* *(a) $\xi \leq_T \eta$, or*

(b) there is a $\gamma \leq_T \eta$ such that ξ and γ are stable roots of $\mathcal{T}$, and $\xi - 1 \leq_T \gamma - 1$.

In case (b), we let $i^{\mathcal{T}}_{\xi,\eta} \colon \mathcal{M}^{\mathcal{T}}_{\xi} \to \mathcal{M}^{\mathcal{T}}_{\eta}$ be given by

$$\hat{\imath}^{\mathcal{T}}_{\xi,\eta} = \hat{\imath}^{\mathcal{T}}_{\gamma,\eta} \circ (\tau^{-1} \circ i^{\mathcal{T}}_{\xi-1,\gamma-1} \circ \sigma),$$

where $\sigma \colon \mathcal{M}_{\xi} \to \mathcal{M}_{\xi-1}$ and $\tau \colon \mathcal{M}_{\gamma} \to \mathcal{M}_{\gamma-1}$ are the maps from the Skolem hulls.

Notice that $i^{\mathcal{T}}_{\xi,\gamma} = (\tau^{-1} \circ i^{\mathcal{T}}_{\xi-1,\gamma-1} \circ \sigma)$ is total in case (b), because $i^{\mathcal{T}}_{0,\gamma-1}(q)$ is in $\operatorname{ran}(\tau)$. Recall here that for θ unstable,

$$\alpha^{\mathcal{T}}_{\theta} = \alpha_{\theta} = \sup i^{\mathcal{T}}_{0,\theta}\text{“}\alpha,$$

and

$$\mathcal{M}_{\theta+1} = \text{ collapse of } h_{\mathcal{M}_{\theta}}\text{“}(\alpha_{\theta} \cup i_{0,\theta}(q)).$$

So in case (b), we also get that $i_{\xi-1,\gamma-1} \restriction \alpha_{\xi-1} = i_{\xi,\gamma} \restriction \alpha_{\xi-1}$. Here is a diagram:

$$\begin{tikzcd}
 & \mathcal{M}^{\mathcal{T}}_{\eta} \\
\mathcal{M}^{\mathcal{T}}_{\gamma-1} & \mathcal{M}^{\mathcal{T}}_{\gamma} \arrow[l, "\tau"'] \arrow[u, "\hat{\imath}_{\gamma,\eta}"'] \\
\mathcal{M}^{\mathcal{T}}_{\xi-1} \arrow[u, "\hat{\imath}_{\xi-1,\gamma-1}"] & \mathcal{M}^{\mathcal{T}}_{\xi} \arrow[l, "\sigma"'] \arrow[u, "\hat{\imath}_{\xi,\gamma}"'] \\
\mathcal{M}^{\mathcal{T}}_{0} \arrow[u, "\hat{\imath}_{0,\xi-1}"] & \mathcal{M}^{\mathcal{T}}_{1} \arrow[l] \arrow[u, "\hat{\imath}_{1,\xi}"']
\end{tikzcd}$$

Thus the stable roots of $\mathcal{T}$ have a branch structure themselves, with 1 at its root.

As before, a tree embedding will have u, v, t, and s maps. The u maps connect exit extenders, but we shall also define them at unstable α such that $E^{\mathcal{T}}_{\alpha+1}$ exists. $v(\alpha)$ is the least ξ on the branch in $\leq^*_T$ to $u(\alpha)$ such that $\mathcal{M}^{\mathcal{T}}_{\alpha}$ is naturally embedded into $\mathcal{M}^{\mathcal{U}}_{\xi}$. The t and s maps are the corresponding maps on models.

Definition 7.8 *Let $\mathcal{T}$ and $\mathcal{U}$ be (normal) psuedo-iteration trees on (M, K, α_0). A* tree embedding *of $\mathcal{T}$ into $\mathcal{U}$ is a system*

$$\langle u, \langle s_{\beta} \mid \beta < \operatorname{lh}\mathcal{T}\rangle, \langle t_{\beta} \mid \beta + 1 < \operatorname{lh}\mathcal{T} \wedge \beta \in \operatorname{stab}(\mathcal{T})\rangle, p\rangle$$

such that

1. $\alpha \in \mathrm{dom}(u)$ *iff* $\alpha \in \mathrm{stab}(\mathcal{T})$ *and* $\alpha + 1 < \mathrm{lh}(\mathcal{T})$ *or* $\alpha \notin \mathrm{stab}(\mathcal{T})$ *and* $\alpha + 2 < \mathrm{lh}(\mathcal{T})$. *For any* α, β, $\alpha < \beta \Rightarrow u(\alpha) < u(\beta)$, *and for* $\alpha \in \mathrm{dom}(u)$,
   (a) $\alpha \in \mathrm{stab}(\mathcal{T}) \Leftrightarrow u(\alpha) \in \mathrm{stab}(\mathcal{U})$, *and*
   (b) *if* $\alpha \notin \mathrm{stab}(\mathcal{T})$, *then* $u(\alpha) = \mathrm{rt}(u(\alpha+1)) - 1$.

2. $p : \mathrm{Ext}(\mathcal{T}) \to \mathrm{Ext}(\mathcal{U})$ *is such that* E *is used before* F *on the same branch of* $\mathcal{T}$ *iff* $p(E)$ *is used before* $p(F)$ *on the same branch of* $\mathcal{U}$. *Thus* p *induces* $\hat{p} : \mathcal{T}^{\mathrm{ext}} \to \mathcal{U}^{\mathrm{ext}}$.

3. *Let* $v : \mathrm{lh}\,\mathcal{T} \to \mathrm{lh}\,\mathcal{U}$ *be given by* $v(0) = 0$, $v(\lambda) = \sup_{\alpha<\lambda} v(\alpha)$ *for* λ *a limit, and*
$$v(\alpha+1) = \begin{cases} u(\alpha)+1 & \text{if } \alpha \in \mathrm{stab}(\mathcal{T}) \\ v(\alpha)+1 & \text{otherwise.} \end{cases}$$
Then v *preserves* $\leq^*_T$, *and*
   (i) $\alpha \in \mathrm{stab}(\mathcal{T}) \Leftrightarrow v(\alpha) \in \mathrm{stab}(\mathcal{U})$, *and*
   (ii) $\alpha \in \mathrm{dom}(u) \Rightarrow v(\alpha) \leq^* u(\alpha)$.

4. *For any* β,
$$s_\beta : M^{\mathcal{T}}_\beta \to M^{\mathcal{U}}_{v(\beta)}$$
is total and elementary. Moreover, for $\alpha <^*_T \beta$,
$$s_\beta \circ \hat{\imath}^{\mathcal{T}}_{\alpha,\beta} = \hat{\imath}^{\mathcal{U}}_{v(\alpha),v(\beta)} \circ s_\alpha.$$
In particular, the two sides have the same domain. Further, if β *is unstable, then*
$$s_\beta(\alpha^{\mathcal{T}}_\beta) = \alpha^{\mathcal{U}}_{v(\beta)}$$

5. *For* $\alpha \in \mathrm{dom}(u)$,
$$t_\alpha = \hat{\imath}^{\mathcal{U}}_{v(\alpha),u(\alpha)} \circ s_\alpha,$$
and if $\alpha \in \mathrm{stab}(\mathcal{T})$, *then*
$$\begin{aligned} p(E^{\mathcal{T}}_\alpha) &= t_\alpha(E^{\mathcal{T}}_\alpha) \\ &= E^{\mathcal{U}}_{u(\alpha)}. \end{aligned}$$

6. *If* $\beta \notin \mathrm{stab}(\mathcal{T})$,
$$s_{\beta+1} = \sigma^{-1} \circ s_\beta \circ \tau,$$
where $\tau\colon \mathcal{M}^{\mathcal{T}}_{\beta+1} \to \mathcal{M}^{\mathcal{T}}_\beta$ *and* $\sigma\colon \mathcal{M}^{\mathcal{U}}_{v(\beta)+1} \to \mathcal{M}^{\mathcal{U}}_{v(\beta)}$ *are the Skolem hull maps. (Note* $v(\beta+1) = v(\beta)+1$ *when* β *is unstable, by 3 above.) In other words,* $s_{\beta+1}$ *agrees with* s_β *on* $\alpha^{\mathcal{T}}_\beta$, *and maps the collapse of* $i^{\mathcal{T}}_{0,\beta}(q)$ *to the collapse of* $i^{\mathcal{U}}_{0,v(\beta)}(q)$.

7. *If* $\beta = T\text{-pred}(\alpha+1)$ *(and hence* $\alpha \in \mathrm{stab}(\mathcal{T}) \cap \mathrm{dom}(u)$*)* , *then letting* $\beta^* = U\text{-pred}(u(\alpha)+1)$,
$$v(\beta) \leq^*_U \beta^* \leq^*_U u(\beta),$$
and
$$s_{\alpha+1}([a,f]^P_{E^{\mathcal{T}}_\alpha}) = [t_\alpha(a), \hat{\imath}^{\mathcal{U}}_{v(\beta),\beta^*} \circ s_\beta(f)]^{P^*}_{E^{\mathcal{U}}_{u(\alpha)}},$$
where $P \trianglelefteq M^{\mathcal{T}}_\beta$ *is what* $E^{\mathcal{T}}_\alpha$ *is applied to, and* $P^* \trianglelefteq M^{\mathcal{U}}_{\beta^*}$ *is what* $E^{\mathcal{U}}_{u(\alpha)}$ *is applied to.*

Here is a diagram that goes with the last clause of the definition, in the case that $\alpha+1$ and β are both $\mathcal{T}$-unstable.

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{T}}_{\alpha+2} \arrow[rrr, "s_{\alpha+2}"] \arrow[dr] & & & \mathcal{M}^{\mathcal{U}}_{v(\alpha+2)} \arrow[dl] & \\
& \mathcal{M}^{\mathcal{T}}_{\alpha+1} \arrow[r, "s_{\alpha+1}"] & \mathcal{M}^{\mathcal{U}}_{v(\alpha+1)} & & \\
& & & & \mathcal{M}^{\mathcal{U}}_{u(\beta+1)} \\
& & \mathcal{M}^{\mathcal{U}}_{\beta^*} \arrow[uu, "E^{\mathcal{U}}_{u(\alpha)}"'] & \mathcal{M}^{\mathcal{U}}_{\beta^*+1} \arrow[l] \arrow[uuu, "\hat{\imath}^{\mathcal{U}}_{\beta^*+1,v(\alpha+2)}"'] \arrow[ur] & \\
& \mathcal{M}^{\mathcal{T}}_{\beta} \arrow[uuu, "E^{\mathcal{T}}_\alpha"] \arrow[r, "s_\beta"] \arrow[ur] & \mathcal{M}^{\mathcal{U}}_{v(\beta)} \arrow[u, "i^{\mathcal{U}}_{v(\beta),\beta^*}"'] & & \\
\mathcal{M}^{\mathcal{T}}_{\beta+1} \arrow[uuuuu, "\hat{\imath}^{\mathcal{T}}_{\beta+1,\alpha+2}"] \arrow[ur] \arrow[rrr, "s_{\beta+1}"] & & & \mathcal{M}^{\mathcal{U}}_{v(\beta+1)} \arrow[ul] \arrow[uu, "i^{\mathcal{U}}_{v(\beta+1),\beta^*+1}"'] &
\end{tikzcd}$$

Remark 7.9 The slightly new feature is the following. If $\Psi\colon \mathcal{T} \to \mathcal{U}$ is a tree embedding of psuedo-trees, and N is a stable root of $\mathcal{T}$, and P is a stable root immediately above it in $\mathcal{T}$, then we want Ψ to lift the process whereby we got the embedding from N to P of $\mathcal{T}$. This embedding came from an ultrapower of the backup model M for N. To make it possible to copy such ultrapowers, Ψ must have associated (M, N) to (M^*, N^*), where N^* is a stable root of $\mathcal{U}$, and M^* is its backup model. This leads to clause 7 in the definition.

The agreement of maps in a tree embedding is given by

Lemma 7.10 *Let $\langle u, \langle s_\beta \mid \beta < \mathrm{lh}\,\mathcal{T}\rangle, \langle t_\beta \mid \beta+1 < \mathrm{lh}\,\mathcal{T} \wedge \beta \in \mathrm{stab}(\mathcal{T})\rangle, p\rangle$ be a tree embedding of $\mathcal{T}$ into $\mathcal{U}$; then for $\xi < \beta < \mathrm{lh}(\mathcal{T})$*

(a) if $\xi \in \mathrm{stab}(\mathcal{T})$, then $s_\beta \restriction \mathrm{lh}(E^{\mathcal{T}}_\xi) + 1 = t_\xi \restriction \mathrm{lh}(E^{\mathcal{T}}_\xi) + 1$.

(b) if $\xi \notin \mathrm{stab}(\mathcal{T})$ and $\xi+1 < \beta$, then $s_\beta \restriction \inf(\alpha^{\mathcal{T}}_\xi, \mathrm{lh}(E^{\mathcal{T}}_{\xi+1})+1) = t_\xi \restriction \inf(\alpha^{\mathcal{T}}_\xi, \mathrm{lh}(E^{\mathcal{T}}_{\xi+1})+1)$.

The proof is an easy induction that we omit here. Part (a) comes from the fact that s_β agrees with the map from $\mathrm{lh}(E^{\mathcal{T}}_\xi)+1$ to $\mathrm{lh}(E^{\mathcal{U}}_{u(\xi)})+1$ that is an input for the Shift Lemma. In case (a), that map is t_ξ. In case (b) the same proof shows that s_β agrees with $t_{\xi+1}$ on $\mathrm{lh}(E^{\mathcal{T}}_{\xi+1})+1$. But it is easy to see that t_ξ agrees with $t_{\xi+1}$ on $\alpha^{\mathcal{T}}_\xi$ when ξ is unstable. (They may disagree at $\alpha^{\mathcal{T}}_\xi$.) This gives us (b).

Definition 7.11 *Let Σ be a winning strategy for* II *in $\mathcal{G}_0$; then Σ has* strong hull condensation *iff whenever $\mathcal{U}$ is a psuedo-tree according to Σ, and there is a tree embedding from $\mathcal{T}$ into $\mathcal{U}$, then $\mathcal{T}$ is according to Σ.*

Lemma 7.12 *Let (M, Λ) be an lbr hod pair, let $\pi\colon K \to M$ with $\mathrm{crit}(\pi) \geq \alpha$ and $K = h_K``(\alpha \cup q)$ for some finite set q of ordinals. Let $\Sigma = \Lambda^{(id,\pi)}$ be the pullback strategy for* II *in the game $\mathcal{G}_0$ on (M, K, α); then Σ has strong hull condensation.*

Proof.(Sketch.) This is like the proof of 4.10. If $\mathcal{U}$ is a play by Σ, and $\mathcal{T}$ is a psuedo-hull of $\mathcal{U}$, then $(\mathrm{id}, \pi)\mathcal{T}$ is a psuedo-hull of $(\mathrm{id}, \pi)\mathcal{U}$. □

Definition 7.11 does not have the clause on pullback strategies that is part of the definition of strong hull condensation for ordinary strategies. This is just because we don't have a use for it. We believe that Lemma 7.12 holds for the stronger property.

We turn now to normalization.

Let $\mathcal{G}$ be the game in which I and II play $\mathcal{G}_0$ until someone loses, or I decides that they should play the game $G^+(N, \omega, \omega_1)$ for producing finite stacks of weakly normal trees on the last model N of their play of $\mathcal{G}_0$. Clearly, we can pull back Λ via (id, π) to a winning strategy for II in this game. We again call this strategy Σ, and write

$$\Sigma = \Lambda^{(\mathrm{id},\pi)}$$

for it.

Let M, K, α_0, q, and π be as above. Let $\mathcal{V}$ be a psuedo-tree on (M, K, α_0) with last model N, and $s = \langle(\nu_i, k_i, \mathcal{U}_i) | i \leq n\rangle$ an N-stack. We can define the embedding normalization $\mathcal{W} = W(\mathcal{V}, s)$ in essentially the same way that we did when no psuedo-trees were involved. For example, suppose that s consists of just one normal tree $\mathcal{U}$ on N. Being the last model, N has been declared stable in $\mathcal{V}$. We define

$$\mathcal{W}_\gamma = W(\mathcal{V}, \mathcal{U}\restriction(\gamma+1))$$

by induction on γ. Each $\mathcal{W}_\gamma$ is a psuedo-tree with last model R_γ, and we have $\sigma_\gamma\colon \mathcal{M}^{\mathcal{U}}_\gamma \to R_\gamma$. We also have tree embeddings

$$\Psi_{\nu,\gamma}\colon \mathcal{W}_\nu \to \mathcal{W}_\gamma,$$

defined when $\nu <_U \gamma$. $\Psi_{\nu,\gamma}$ is partial iff $(\nu, \gamma]_U$ drops somewhere. We call its u-map $\phi_{\nu,\gamma}$, and its t-maps are $\pi^{\nu,\gamma}_\xi$.

We set $\mathcal{W}_0 = \mathcal{W}$. The successor step is given by

$$\mathcal{W}_{\gamma+1} = \mathcal{W}_\gamma\restriction(\theta+1)^\frown\langle F\rangle^\frown i_F``(\mathcal{W}^{\geq\beta}_\nu),$$

where $F = \sigma_\gamma(E^{\mathcal{U}}_\gamma)$, $\theta = \alpha_F$ is the least *stable* node of $\mathcal{W}_\gamma$ such that F is on the $\mathcal{M}^{\mathcal{W}_\gamma}_\theta$-sequence,and $\nu = U\text{-pred}(\gamma+1)$. Let

$$\beta(\mathcal{W}_\nu, \mathcal{W}_\gamma, F) = \begin{cases} \text{least } \eta \text{ such that } \operatorname{crit}(F) < \lambda^{\mathcal{W}_\nu}_\eta & \text{if there is such an } \eta \\ \operatorname{lh}(\mathcal{W}_\nu) - 1 & \text{otherwise.} \end{cases}$$

Set $\beta = \beta(\mathcal{W}_\nu, \mathcal{W}_\gamma, F)$. It is easy to see that $\beta \leq \theta$, and

$$\mathcal{W}_\nu\restriction\beta+1 = \mathcal{W}_\gamma\restriction\beta+1 = \mathcal{W}_{\gamma+1}\restriction\beta+1.$$

This is because between ν and γ, all the $\mathcal{W}_\eta$ used the same extenders E such that $\lambda(E) < \operatorname{lh}(F_\nu)$.

Let us assume for simplicity that $(\nu, \gamma+1]_U$ does not drop. We have $\phi\colon \operatorname{lh}(\mathcal{W}_\nu) \to \operatorname{lh}(\mathcal{W}_{\gamma+1})$ given by, for $\xi \in \operatorname{stab}(\mathcal{W}_\nu)$,

$$\phi(\xi) = \begin{cases} \xi & \text{if } \xi < \beta \\ (\theta+1) + (\xi - \beta) & \text{otherwise.} \end{cases}$$

For $\eta \leq_U \nu$, we let $\phi_{\eta,\gamma+1} = \phi \circ \phi_{\eta,\nu}$. A node η of $\mathcal{W}_{\gamma+1}$ is stable just in case $\eta \leq \theta$ and η is stable as a node of $\mathcal{W}_\gamma$, or $\eta = \phi(\xi)$, where ξ is stable as a node of $\mathcal{W}_\nu$. (The stable nodes are just those having exit extenders, so there is no other reasonable choice here.) For $\xi < \beta$, $\pi^{\nu,\gamma+1}_\xi$ is the identity. We define by induction on $\xi \geq \beta$ the models $\mathcal{M}^{\mathcal{W}_{\gamma+1}}_{\varphi(\xi)}$ and maps $\pi_\xi\colon \mathcal{M}^{\mathcal{W}_\nu}_\xi \to \mathcal{M}^{\mathcal{W}_{\gamma+1}}_{\varphi(\xi)}$ as before.

For example, suppose $\xi = \beta$. We let

$$\mathcal{M}^{\mathcal{W}_{\gamma+1}}_{\theta+1} = \operatorname{Ult}(\mathcal{M}^{\mathcal{W}_\nu}_\beta, F),$$

and let π_β be the canonical embedding, so that

$$\pi_\beta = i^{\mathcal{W}_{\gamma+1}}_{\beta,\theta+1}.$$

If β is stable in $\mathcal{W}_\nu$, then $E^{\mathcal{W}_{\gamma+1}}_{\theta+1} = \pi_\beta(E^{\mathcal{W}_\nu}_\beta)$, and

$$\mathcal{M}^{\mathcal{W}_{\gamma+1}}_{\theta+2} = \operatorname{Ult}(P, E^{\mathcal{W}_{\gamma+1}}_{\theta+1}),$$

where P is the appropriate initial segment of some $\mathcal{M}^{\mathcal{W}_\nu}_\tau$. We determine $\pi_{\beta+1}$ using the Shift Lemma as before. (I.e., $\pi_{\beta+1}([a, f]) = [\pi_{\theta+1}(a), \pi_\tau(f)]$ if $\tau \neq \beta$, or if

$\tau = \beta$ and $\mathrm{crit}(F) \le \mathrm{crit}(E_{\theta+1}^{\mathcal{W}_{\gamma+1}})$. Otherwise, $\pi_{\beta+1}([a, f]) = [\pi_{\theta+1}(a), f]$.) So nothing changes.

On the other hand, if β is unstable in $\mathcal{W}_\nu$, then $\theta + 1$ is unstable in $\mathcal{W}_{\gamma+1}$. We set

$$(\alpha_{\theta+1})^{\mathcal{W}_{\gamma+1}} = \sup i_{0,\theta+1}^{\mathcal{W}_{\gamma+1}} “(\alpha_0),$$

and as we must,

$$\mathcal{M}_{\theta+2}^{\mathcal{W}_{\gamma+1}} = \text{ collapse of Hull}^{\mathcal{M}_{\theta+1}^{\mathcal{W}_{\gamma+1}}}(\alpha_{\theta+1} \cup i_{0,\theta+1}^{\mathcal{W}_{\gamma+1}}(q)).$$

Let σ be the uncollapse map. Let $\tau \colon \mathcal{M}_{\beta+1}^{\mathcal{W}_\nu} \to \mathcal{M}_{\beta}^{\mathcal{W}_\nu}$ be the uncollapse map. Note that $\mathcal{W}_\nu \restriction (\beta+2) = \mathcal{W}_\gamma \restriction (\beta+2) = \mathcal{W}_{\gamma+1} \restriction (\beta+2)$ in the present case. We set

$$\pi_{\beta+1} = \sigma^{-1} \circ \pi_\beta \circ \tau.$$

If $\beta + 2 = \mathrm{lh}(\mathcal{W}_\nu)$ then we are done defining $\mathcal{W}_{\gamma+1}$ and $\Psi_{\nu,\gamma+1}$. If not, we set $E_{\theta+2}^{\mathcal{W}_{\gamma+1}} = \pi_{\beta+1}(E_{\beta+1}^{\mathcal{W}_\nu})$. We have $\lambda_\beta^{\mathcal{W}_\nu} = \inf(\alpha_\beta^{\mathcal{W}_\nu}, \lambda(E_{\beta+1}^{\mathcal{W}_\nu}))$, and we set

$$\lambda_{\theta+1}^{\mathcal{W}_{\gamma+1}} = \inf(\alpha_{\theta+1}^{\mathcal{W}_{\gamma+1}}, \lambda(E_{\theta+2}^{\mathcal{W}_{\gamma+1}})).$$

It is easy to see that $\mathcal{M}_{\theta+2}^{\mathcal{W}_{\gamma+1}} | \lambda_{\theta+1} = \mathcal{M}_{\theta+1}^{\mathcal{W}_{\gamma+1}} | \lambda_{\theta+1}$. (We are ignoring the anomalous case here.)[26] We also have

$$\pi \restriction \lambda_\beta^{\mathcal{W}_\nu} = \pi_{\beta+1} \restriction \lambda_\beta^{\mathcal{W}_\nu},$$

which is the agreement we need to continue defining $\mathcal{W}_{\gamma+1}$ and $\Psi_{\nu,\gamma+1}$.

Let us check that $\Psi_{\nu,\gamma+1}$ satisfies the clauses in Definition 7.8 that are relevant so far. These involve the behavior of its maps at $\alpha \le \beta$ if β is stable in $\mathcal{W}_\nu$, and at $\alpha \le \beta + 1$ otherwise.

Clause (1) is clear. The v-map of $\Psi_{\nu,\gamma+1}$ is given by

$$v(\xi) = \begin{cases} \xi & \text{if } \xi \le \beta \\ (\theta+1) + (\xi - \beta) & \text{otherwise.} \end{cases}$$

We can then see that (3) of 7.8 holds. The case to check here is (3) at β and possibly $\beta + 1$. But $v(\beta) = \beta$, and $\beta \in \mathrm{stab}(\mathcal{W}_\nu)$ iff $\beta \in \mathrm{stab}(\mathcal{W}_{\gamma+1})$, so (i) holds. (ii) holds at β because $v(\beta) = \beta <_{W\gamma+1} \theta + 1 = \phi(\beta)$. If $\beta \notin \mathrm{stab}(\mathcal{W}_\nu)$, then $v(\beta+1) = \beta+1$, and $\beta + 1 \le^*_{W_{\gamma+1}} \theta + 2 = \phi(\beta+1)$, so (ii) holds at $\beta + 1$.

Clause (4) is trivial at this stage, because $s_\alpha = \mathrm{id}$ for $\alpha \le \beta$, and for $\alpha = \beta + 1$ if β is unstable. Clause (5) is also trivial at $\alpha < \beta$, because all maps are then the identity. At β, it only applies if β is stable, and then it amounts to $\pi_\beta = i_{\beta,\theta+1}^{\mathcal{W}_{\gamma+1}}$, which is indeed how we defined π_β. If β is unstable in $\mathcal{W}_\nu$, then clause (5) requires that $\pi_{\beta+1} = \hat{\imath}_{\beta+1,\theta+2}^{\mathcal{W}_{\gamma+1}} \circ s_{\beta+1}$. But $s_{\beta+1} = \mathrm{id}$, and $\pi_{\beta+1} = \sigma^{-1} \circ \hat{\imath}_{\beta,\theta+1}^{\mathcal{W}_{\gamma+1}} \circ \tau = \hat{\imath}_{\beta+1,\theta+2}^{\mathcal{W}_{\gamma+1}}$, so (5) is satisfied.

[26] $\lambda_\alpha^{\mathcal{W}_\xi}$ is an agreement ordinal. It corresponds to $\lambda(E_\alpha^{\mathcal{T}}) = \lambda_{\alpha+1}^{\mathcal{T}}$ in normal trees $\mathcal{T}$.

The rest of the definition proceeds as above, defining $\pi_\xi \colon \mathcal{M}_\xi^{\mathcal{W}_\nu} \to \mathcal{M}_{\phi(\xi)}^{\mathcal{W}_{\gamma+1}}$ using the Shift Lemma and the appropriate earlier π_τ. If ξ is unstable in $\mathcal{W}_\nu$, we then go on to define $\pi_{\xi+1} \colon \mathcal{M}_{\xi+1}^{\mathcal{W}_\nu} \to \mathcal{M}_{\phi(\xi+1)}^{\mathcal{W}_{\gamma+1}}$ as we did above. At limit steps, we take direct limits.

This gives us $\mathcal{W}_{\gamma+1}$ and $\Psi_{\nu,\gamma+1} \colon \mathcal{W}_\nu \to \mathcal{W}_{\gamma+1}$. At limit ordinals λ, we let $\mathcal{W}_\lambda$ be the direct limit of the $\mathcal{W}_\nu$ for $\nu <_U \lambda$, under the $\Psi_{\nu,\mu}$. Finally, $W(\mathcal{V},\mathcal{U}) = \mathcal{W}_\gamma$, where $\gamma + 1 = \mathrm{lh}(\mathcal{U})$.

If $\mathcal{V}$ is a psuedo-tree and $s^\frown\langle\mathcal{U}\rangle$ is a maximal normal stack on the last model of $\mathcal{V}$, then

$$W(\mathcal{V}, s^\frown\langle\mathcal{U}\rangle) = W(W(\mathcal{V}, s), \sigma\mathcal{U}),$$

where σ is the natural embedding from the last model of $\mathcal{V}$ to the last model of $W(\mathcal{V}, s)$ that we get from the normalization process. That is, we normalize stacks "bottom up".

Remark 7.13 One might look at normalizing stacks of psuedo-trees, but we are not doing that. $\mathcal{W}(\mathcal{V}, s)$ is defined only when s is a stack of ordinary trees.

This finishes our discussion of the normalization $W(\mathcal{V}, s)$, for $\mathcal{V}$ a psuedo-tree on (M, K, α), and s a maximal stack on the last model of $\mathcal{V}$. We say that strategy Σ for the game $\mathcal{G}$ *normalizes well* iff whenever $\langle\mathcal{V}, s\rangle$ is according to Σ, then $W(\mathcal{V}, s)$ is according to Σ.

Lemma 7.14 *Let (M, Λ) be an lbr hod pair, and K, π, q, α_0 be as above. Let $\Sigma = \Lambda^{(id,\pi)}$; then Σ normalizes well.*

Proof.(Sketch.) If $\mathcal{V}$ is a psuedo-tree, and $\mathcal{U}$ is a normal tree on the last model of $\mathcal{V}$, let us write

$$(\mathrm{id}, \pi)\langle\mathcal{V},\mathcal{U}\rangle = \langle(\mathrm{id}, \pi)\mathcal{V}, \sigma\mathcal{U}\rangle,$$

where σ is the copy map acting on the last model of $\mathcal{V}$. Just for the space of this proof, to keep things straight, let's write $\hat{W}$ for the embedding normalization operation on psuedo-trees defined above.

Λ itself normalizes well. But normalizing commutes with copying in this context, as it did in the case of ordinary iteration trees. That is

$$(\mathrm{id}, \pi)\hat{W}(\mathcal{V},\mathcal{U}) = W((\mathrm{id}, \pi)\langle\mathcal{V},\mathcal{U}\rangle).$$

So

$$\begin{aligned} \hat{W}(\mathcal{V},\mathcal{U}) \text{ is by } \Sigma \ &\Leftrightarrow (\mathrm{id}, \pi)\hat{W}(\mathcal{V},\mathcal{U}) \text{ is by } \Lambda \\ &\Leftrightarrow W((\mathrm{id}, \pi)\mathcal{V}, \sigma\mathcal{U}) \text{ is by } \Lambda \\ &\Leftrightarrow \langle\mathrm{id}, \pi)\mathcal{V}, \sigma\mathcal{U}\rangle \text{ is by } \Lambda \\ &\Leftrightarrow \langle\mathcal{V},\mathcal{U}\rangle \text{ is by } \Sigma\ , \end{aligned}$$

as desired. See the proof of Theorem 4.4. $\square$

Let us turn now to the proof of Lemma 6.64. We were given an lbr hod pair (M, Σ), but it works better with the current notation to call that pair (M, Λ), so let's make that switch. We are also given K, α_0, π, and q as above. We have the pullback strategy

$$\Sigma = \Lambda^{(\mathrm{id},\pi)}$$

for $\mathcal{G}$ on (M, K, α_0), and Σ normalizes well and has strong hull condensation. We have a coarse Γ-Woodin tuple $(N^*, \delta^*, ..., \lhd, \Sigma^*)$ such that $\mathrm{Code}((M, \Lambda)) \in \Gamma$, and $\mathbb{C}$ is a $\lhd$-construction in N^* such that (M, Λ) iterates to $(M^{\mathbb{C}}_{\eta_0,j_0}, \Omega^{\mathbb{C}}_{\eta_0,j_0})$. For $\langle \eta, j \rangle \le \langle \eta_0, j_0 \rangle$, we set [27]

$\mathcal{V}_{\eta,j} =$ tree of minimal length whereby (M, Λ) iterates past $(M_{\eta,j}, \Omega_{\eta,j})$.

We also had psuedo-trees $\mathcal{S}_{\eta,j}$ on (M, K, α_0) formed by certain rules.

Definition 7.15 *For an lpm R, we say that $(\mathcal{T}, \mathcal{V})$ is the* (Σ, Λ, R)-coiteration *(of $(M, K, \alpha_{)})$ with M) iff*

(a) $\mathcal{T}$ is a psuedo-tree by Σ on (M, K, α) with last model P,

(b) $\mathcal{V}$ is a normal tree by Λ on M with last model Q,

(c) $R \trianglelefteq P$ and $R \trianglelefteq Q$, and $\mathcal{T}$ and $\mathcal{V}$ are of minimal length such that this is true, and

(d) stability (and hence the next model) in $\mathcal{T}$ is determined by the rules we have given: θ is unstable iff $[0, \theta]_T$ does not drop, and $e^{\mathcal{T}}_\theta = e^{\mathcal{V}}_\tau$ for some τ.

We remark that the internal strategy $\dot{\Sigma}^R$ is relevant in (c), but no external strategy agreement is relevant. (c) tells us that $\mathcal{V}$ and $\mathcal{W}$ proceed by hitting the least extender disagreement with R, and that the corresponding R-extenders are all empty.

We had fixed $\langle \nu_0, k_0 \rangle \le_{\mathrm{lex}} \langle \eta_0, j_0 \rangle$ such that for each $\langle \eta, j \rangle <_{\mathrm{lex}} \langle \nu_0, k_0 \rangle$, the $(\Sigma, \Lambda, M_{\eta,j})$-coiteration (M, K, α_0) with M exists, and moreover, the last model on both sides is strictly longer than $M_{\eta,j}$, and no external strategy disagreements show up on either side.[28] We are trying to show that the $(\Sigma, \Lambda, M_{\nu_0,k_0})$-coiteration exists, and that no external strategy disagreements show up on the (M, K, α_0) side. That is,

Lemma 7.16 *Let $\mathcal{T}$ be an initial segment of $\mathcal{S}_{\nu_0,k_0}$ with stable last node, and let N_0 be the last model of $\mathcal{T}$; then either*

[27] We called it $\mathcal{U}_{\eta,j}$ before, but $\mathcal{V}_{\eta,j}$ works better now. j_0 was formerly k_0.

[28] We called this pair $\langle \nu, l \rangle$ before, but we want to free up those letters for other use below.

(1) $(N_0, \Sigma_{\mathcal{T}}) \trianglelefteq (M_{\nu_0,k_0}, \Omega_{\nu_0,k_0})$, *or*

(2) $(M_{\nu_0,k_0}, \Omega_{\nu_0,k_0}) \triangleleft (N_0, \Sigma_{\mathcal{T}})$, *or*

(3) there is a nonempty extender E on the N_0 sequence such that, setting $\tau = \mathrm{lh}(E)$,

   (i) $\dot{E}_\tau^{M_{\nu_0,k_0}} = \emptyset$, *and*

   (ii) $(\Sigma_{\mathcal{T}})_{\langle \tau,-1\rangle} = (\Omega_{\nu_0,k_0})_{\langle \tau,0\rangle}$.

Proof. Suppose $\mathcal{T}$ and N_0 are a counterexample. Since (1) and (2) fail, there is a least disagreement between $(N_0, \Sigma_{\mathcal{T}})$ and $(M_{\nu_0,k_0}, \Omega_{\nu_0,k_0})$, and since (3) fails, the least disagreement either involves a nonempty extender from M_{ν_0,k_0}, or is a strategy disagreement.

Suppose first that (3) fails because there is a nonempty extender on the M_{ν_0,k_0} side at the least disagreement between $(N_0, \Sigma_{\mathcal{T}})$ with $(M_{\nu_0,k_0}, \Omega_{\nu_0,k_0})$. As in the proof of the Bicephalus Lemma, we can reduce to the case that $k_0 = 0$, and the least disagreement involves $F = \dot{F}^{M_{\nu_0,0}}$, with $F \neq \emptyset$. Letting $\mathcal{V} = \mathcal{V}_{\nu_0,0}$, we then have that $(\mathcal{T}, \mathcal{V})$ is the $(\Sigma, \Lambda, M_{\nu_0,-1})$- coiteration. Let P and Q be the last models of $\mathcal{T}$ and $\mathcal{V}$. So

$$(M_{\nu_0,-1}, \Omega_{\nu_0,-1}) = (P|\langle \nu_0, -1\rangle, \Sigma_{\mathcal{T},\langle \nu_0,-1\rangle}) = (Q, \Lambda_{\mathcal{V},\langle \nu_0,-1\rangle}).$$

Let

$$j\colon N^* \to \mathrm{Ult}(N^*, F^{\mathbb{C}}_{\nu_0})$$

be the canonical embedding, and $\kappa = \mathrm{crit}(j)$. We have that $M_{\nu_0,-1} \trianglelefteq j(M_{\nu_0,-1})$ by coherence. (Note $j(M_{\nu_0,-1})|\nu_0$ is extender passive.) $j(\mathcal{T}, \mathcal{V})$ is the $(\Sigma, \Lambda, j(M_{\nu_0,-1}))$ coiteration, because $j(\Lambda) \subseteq \Lambda$, and hence $j(\Sigma) \subseteq \Sigma$. So $\mathcal{V}$ is an initial segment of $j(\mathcal{V})$. But then $\mathcal{T}$ is an initial segment of $j(\mathcal{T})$, because the relevant conditions for declaring stability are the same in N^* and $j(N^*)$.

We have that $\mathcal{M}^{\mathcal{T}}_\kappa = \mathcal{M}^{j(\mathcal{T})}_\kappa$ and $j \restriction \mathcal{M}^{\mathcal{T}}_\kappa = i^{j(\mathcal{T})}_{\kappa,j(\kappa)}$, so F is compatible with the first extender G used in $[\kappa, j(\kappa)]_{j(T)}$. $M_{\nu_0,-1} \triangleleft \mathcal{M}^{j(\mathcal{T})}_{j(\kappa)}$, so G cannot be a proper initial segment of F. But F is not on the sequence of $\mathcal{M}^{j(\mathcal{T})}_{j(\kappa)}$, so F cannot be a proper initial segment of G. Hence $F = G$, and F is used in $j(\mathcal{T})$. Since $\mathcal{T} = j(\mathcal{T})\restriction(\xi + 1)$, where $P = \mathcal{M}^{\mathcal{T}}_\xi$, we have that F is on the sequence of P, contradiction.

So we may assume that we have $J \trianglelefteq M_{\nu_0,k_0}$ such that

$$J \trianglelefteq N_0,$$

but there is a strategy disagreement, that is

$$(\Omega_{\nu_0,k_0})_J \neq \Sigma_{\mathcal{T},J}.$$

Note that $(J, (\Omega_{\nu_0,k_0})_J)$ and $(J, \Sigma_{\mathcal{T},J})$ are lbr hod pairs. (In the case of $(J, \Sigma_{\mathcal{T},J})$, this is because the pair is elementarily embedded, as a mouse pair, into some iterate of

(M, Λ).) Thus the two strategies are determined by their actions on normal trees, and we can fix a single normal tree $\mathcal{U}$ on J of limit length such that $(\Omega_{\nu_0,k_0})_J(\mathcal{U}) \neq \Sigma_{\mathcal{T},J}(\mathcal{U})$. Again we consider first the case that $J = M_{\nu_0,k_0}$, then we reduce to this case using the pullback consistency of Σ. Let $\Omega = \Omega_{\nu_0,k_0}$ and

$$b = \Omega(\mathcal{U}).$$

We derive a contradiction in the case $J = M_{\nu_0,k_0}$ by repeating the proof of Theorem 5.11. Large stretches of that proof can be simply copied, and that is basically what we are going to do. We shall try to condense things enough that the new points stand out. We have set up the notation to mimic that in the proof of 5.11. To make the correspondence better, let us now set

$$\mathcal{W}^*_{\eta,j} = \mathcal{S}_{\eta,j},$$

and forget about our prior $\mathcal{S}$ notation.

Let

$$\mathrm{lift}(\mathcal{U}, M_{\nu_0,k_0}, \mathbb{C}) = \langle \mathcal{U}^*, \langle \eta_\tau, l_\tau \mid \tau < \mathrm{lh}\,\mathcal{U}\rangle, \langle \psi^{\mathcal{U}}_\tau \mid \tau < \mathrm{lh}\,\mathcal{U}\rangle\rangle.$$

Remembering to forget our previous use of "$\mathcal{S}$", for $\gamma < \mathrm{lh}(\mathcal{U})$ or $\gamma = b$, let

$$S_\gamma = \mathcal{M}^{\mathcal{U}^*}_\gamma,$$

$$N^0_\gamma = M^{S_\gamma}_{\eta_\gamma,l_\gamma} = M^{i^{\mathcal{U}^*}_{0,\gamma}(\mathbb{C})}_{\eta_\gamma,l_\gamma},$$

so that

$$\psi^{\mathcal{U}}_\gamma : \mathcal{M}^{\mathcal{U}}_\gamma \to N^0_\gamma$$

is elementary. For $\gamma + 1 < \mathrm{lh}\,\mathcal{U}$, let res_γ be the map resurrecting $\psi^{\mathcal{U}}_\gamma(E^{\mathcal{U}}_\gamma)$ inside S_γ, namely

$$\mathrm{res}_\gamma = (\sigma_{\eta_\gamma,l_\gamma}[M_{\eta_\gamma,l_\gamma}|\langle \mathrm{lh}\,\psi^{\mathcal{U}}_\gamma(E^{\mathcal{U}}_\gamma), 0\rangle])^{S_\gamma}.$$

We have $M_{\nu_0,k_0} = \mathcal{M}^{\mathcal{U}}_0 = N^0_0$, and $\psi^{\mathcal{U}}_0 = $ identity. For $\langle \eta, j\rangle \leq_{\mathrm{lex}} i^{\mathcal{U}^*}_{0,\gamma}(\langle \nu_0, k_0\rangle)$, we let $(\mathcal{W}^*_{\eta,j})^{S_\gamma} = i^{\mathcal{U}^*}_{0,\gamma}(\langle \mu, l\rangle \mapsto \mathcal{W}^*_{\mu,l})_{\eta,j}$ and $(\mathcal{V}_{\eta,j})^{S_\gamma} = i^{\mathcal{U}^*}_{0,\gamma}(\langle \mu, l\rangle \mapsto \mathcal{V}_{\mu,l})_{\eta,j}$. Note that $i^{\mathcal{U}^*}_{0,\gamma}(\Lambda) \cap S_\gamma = \Lambda \cap S_\gamma$, so that the $\mathcal{V}^{S_\gamma}_{\eta,j}$ and $(\mathcal{W}^*_{\eta,j})^{S_\gamma}$ are by Λ and Σ, respectively.

Set

$$(\mathcal{W}^*_\gamma, \mathcal{V}_\gamma) = (\mathcal{W}^*_{\eta_\gamma,l_\gamma}, \mathcal{V}_{\eta_\gamma,l_\gamma})^{S_\gamma},$$

for $\gamma < \mathrm{lh}\,\mathcal{U}$ or $\gamma = b$. Let $z^*(\gamma) + 1 = \mathrm{lh}(\mathcal{W}^*_\gamma)$, and put

$$N_\gamma = \mathcal{M}^{\mathcal{W}^*_\gamma}_{z^*(\gamma)}.$$

So $\mathcal{W}^*_0 = \mathcal{T}$ is our psuedo-tree on (M, K, α_0) by Σ. Its last model is N_0, and $M_{\nu_0,k_0} = J = N^0_0 \trianglelefteq N_0$. We want to normalize $\langle \mathcal{T}, \mathcal{U}\rangle$, but it may not be a maximal stack, so we replace $\mathcal{U}$ with $\mathcal{U}^+$. This yields a maximal normal stack, so our theory of embedding normalization applies to it. Set

$$\mathcal{W}_\gamma = W(\mathcal{W}^*_0, \mathcal{U}^+ \upharpoonright (\gamma + 1))$$

for $\gamma < \mathrm{lh}\mathcal{U}$, and

$$\mathcal{W}_b = W(\mathcal{W}_0^*, (\mathcal{U}^+)^\frown b).$$

So $\mathcal{W}_0 = \mathcal{W}_0^* = \mathcal{T}$. The $\mathcal{W}_\gamma$'s are all by Σ, because Σ normalizes well and $\mathcal{U}^+ \restriction (\gamma+1)$ is by $\Sigma_{\mathcal{T},N_0}$. Since Σ normalizes well, it is enough to show that $\mathcal{W}_b$ is by Σ, for then $\Sigma_{\mathcal{T},N_0}(\mathcal{U}^+) = b$, so $\Sigma_{\mathcal{T},J}(\mathcal{U}) = b$, as desired. Since Σ has strong hull condensation, it is enough to show

Sublemma 7.16.1 *$\mathcal{W}_b$ is pseudo-hull of $\mathcal{W}_b^*$.*

Proof. As before, we define by induction on γ, for $\gamma < \mathrm{lh}(\mathcal{U})$ or $\gamma = b$, tree embeddings

$$\Phi_\gamma \colon \mathcal{W}_\gamma \to \mathcal{W}_\gamma^*.$$

Let

$$\Phi_\gamma = \langle u^\gamma, \langle s_\beta^\gamma \mid \beta \leq z(\gamma)\rangle, \langle t_\beta^\gamma \mid \beta < z(\gamma)\rangle, p^\gamma\rangle.$$

Φ_γ can be extended, in that $v^\gamma(z(\gamma)) \leq^*_{W_\gamma^*} z^*(\gamma)$, and we let

$$t^\gamma = i^{\mathcal{W}_\gamma^*}_{v^\gamma(z(\gamma)),z^*(\gamma)} \circ s^\gamma_{z(\gamma)}$$

be the final t-map of the extended tree embedding. Letting $R_\gamma = \mathcal{M}^{\mathcal{W}_\gamma}_{z(\gamma)}$ we have that

$$t^\gamma \colon R_\gamma \to N_\gamma.$$

Again, the rest of Φ_γ is actually determined by t^γ. It is also determined by u^γ, and by p^γ.

The embedding normalization process gives us extended tree embeddings

$$\Psi_{\nu,\gamma} \colon \mathcal{W}_\nu \to \mathcal{W}_\gamma,$$

defined when $\nu <_U \gamma$. We use $\phi_{\nu,\gamma}$ for the u-map of $\Psi_{\nu,\gamma}$, so that $\phi_{\nu,\gamma} \colon \mathrm{lh}(\mathcal{W}_\nu) \to \mathrm{lh}(\mathcal{W}_\gamma)$, the map being total if $(\nu,\gamma]_U$ does not drop in model or degree. We let $\pi_\tau^{\nu,\gamma}$ be the t-map $t_\tau^{\Psi_{\nu,\gamma}}$, so that

$$\pi_\tau^{\nu,\gamma} : \mathcal{M}_\tau^{\mathcal{W}_\nu} \to \mathcal{M}^{\mathcal{W}_\gamma}_{\phi_{\nu,\gamma}(\tau)}$$

elementarily, for $\nu <_U \gamma$ and $\tau \in \mathrm{dom}\,\phi_{\nu,\gamma}$. Let also:

- $\sigma_\eta^1 : \mathcal{M}_\eta^{\mathcal{U}^+} \to R_\eta$ be the embedding normalization map,
- $\sigma_\eta^0 \colon \mathcal{M}_\eta^{\mathcal{U}} \to \mathcal{M}_\eta^{\mathcal{U}^+}$ be the copy map,[29]
- $\sigma_\eta = \sigma_\eta^1 \circ \sigma_\eta^0$,

[29] σ_η^0 may only be elementary as a map into some proper initial segment of R_η.

- $F_\eta = \sigma_\eta(E^{\mathcal{U}}_\eta)$, and
- $\bar{\xi}_\eta =$ least α such that F_η is on the $\mathcal{M}^{\mathcal{W}_\eta}_\alpha$ sequence.[30]

Thus $\mathcal{W}_{\gamma+1} = W(\mathcal{W}_\nu, \mathcal{W}_\gamma, F_\gamma)$, where $\nu = U$-pred$(\gamma+1)$.

We also have an extended tree embedding $\Psi^*_{\nu,\gamma} \colon \mathcal{W}^*_\nu \to \mathcal{W}^*_\gamma$ defined when $\nu <_U \gamma$ and $(\nu,\gamma]_U$ does not drop. The maps of $\Psi^*_{\nu,\gamma}$ are all restrictions of $i^{\mathcal{U}^*}_{\nu,\gamma}$, so we don't give them special names. As before, we maintain by induction that the diagram

$$\begin{tikzcd}
\mathcal{W}_\gamma \arrow[r, "\Phi_\eta"] & \mathcal{W}^*_\gamma \\
\mathcal{W}_\nu \arrow[u, "\Psi_{\nu,\gamma}"'] \arrow[r, "\Phi_\nu"] & \mathcal{W}^*_\nu \arrow[u, "\Psi^*_{\nu,\gamma}"']
\end{tikzcd}$$

commutes, in the appropriate sense.

Our induction hypothesis is

Induction Hypothesis $(\dagger)_\gamma$**.**

(1) (a) For $\nu < \eta \leq \gamma$, $\Phi_\nu \restriction (\bar{\xi}_\nu + 1) = \Phi_\eta \restriction (\bar{\xi}_\nu + 1)$.

(b) For all $\eta \leq \gamma$, t^η is well defined; that is, $v^\eta(z(\eta)) \leq^*_{W^*_\eta} z^*(\eta)$.

(c) For $\nu < \eta \leq \gamma$, $s^\eta_{z(\eta)} \restriction (\mathrm{lh}\, F_\nu + 1) = \mathrm{res}_\nu \circ t^\nu \restriction (\mathrm{lh}\, F_\nu + 1)$.

(2) Let $\nu < \eta \leq \gamma$, and $\nu <_U \eta$, and suppose that $(\nu, \eta]_U$ does not drop; then $\Phi_\eta \circ \Psi_{\nu,\eta} = \Psi^*_{\nu,\eta} \circ \Phi_\nu$.

(3) For $\xi \leq \gamma$, $\psi^{\mathcal{U}}_\xi = t^\xi \circ \sigma_\xi$.

(4) For all $\nu < \gamma$, N^*_ν agrees with N_γ strictly below $\mathrm{lh}\, G_\nu$. G_ν is on the N^*_ν-sequence, but $\mathrm{lh}\, G_\nu$ is a cardinal of N_γ. W^{**}_ν is an initial segment of $W^*_\gamma \restriction (v^\gamma(\bar{\xi}_\gamma) + 1)$.

Let us check that the embeddings in clause (3) fit together plausibly. $t_\xi \colon R_\xi \to N_\xi$, and $\psi^{\mathcal{U}}_\xi \colon \mathcal{M}^{\mathcal{U}}_\xi \to N^0_\xi \trianglelefteq N_\xi$. But $\sigma_\xi \colon \mathcal{M}^{\mathcal{U}}_\xi \to K_\xi \trianglelefteq R_\xi$, so indeed the embeddings fit together plausibly.

[30] We called this ordinal α_η before, but that would clash with our notation for exchange ordinals in psuedo-trees.

We shall explain the terms in clause (4) shortly. The precise meaning of clause (2) can be given by writing it out in terms of the component maps, as we did in (d) in the proof of 5.11. We leave it to the reader to do that.

We now describe how to obtain $\Phi_{\gamma+1}$ from the Φ_α for $\alpha \leq \gamma$.

We have $t^\gamma \colon R_\gamma \to N_\gamma$, where N_γ is the last model of W^*_γ. Let $F = F_\gamma$, and let $\nu = U\text{-pred}(\gamma+1)$. So $\mathcal{W}_{\gamma+1} = W(\mathcal{W}_\nu, W_\gamma, F)$. Let us assume for simplicity that $(\nu, \gamma+1]_U$ is not a drop in model or degree. Let

- $H = H_\gamma = t^\gamma(F)$,
- $Q = N_\gamma | \langle \mathrm{lh}(H), 0\rangle$,
- $G = G_\gamma = \mathrm{res}_\gamma(H)$, and
- $G^* =$ background extender for G in $i^{\mathcal{U}^*}_{0,\gamma}(\mathbb{C})$.

We have $t^\gamma \circ \sigma_\gamma = \psi^{\mathcal{U}}_\gamma$, so $G = \mathrm{res}_\gamma(\psi^{\mathcal{U}}_\gamma(E^{\mathcal{U}}_\gamma))$, so

$$S_{\gamma+1} = \mathrm{Ult}(S_\nu, G^*).$$

Since we are not dropping, $W^*_{\gamma+1} = i_{G^*}(W^*_\nu)$, where $i_{G^*} = i^{\mathcal{U}^*}_{\nu,\gamma+1}$. The first thing we need to see is that G is used in $W^*_{\gamma+1}$.

Lemma 5.6 on capturing resurrection embeddings works also for our system of psuedo-trees:

Claim 0. Let τ be least in $\mathrm{stab}(\mathcal{W}^*_\gamma)$ such that $Q \trianglelefteq \mathcal{M}^{\mathcal{W}^*_\gamma}_\tau$, and θ least such that $Q \trianglelefteq \mathcal{M}^{\mathcal{V}_\gamma}_\theta$. Let $(\mathcal{W}^{**}_\gamma, \mathcal{V}^{**}_\gamma)$ be the $(\Sigma, \Lambda, \mathrm{res}_\gamma(Q))$-coiteration of (M, K, α_0) with M; then

(i) $\mathcal{W}^{**}_\gamma$ extends $\mathcal{W}^*_\gamma \restriction (\tau+1)$,

(ii) letting $\xi = \mathrm{lh}(\mathcal{W}^{**}_\gamma) - 1$, G is on the $\mathcal{M}^{\mathcal{W}^{**}_\gamma}_\xi$ sequence, and not on the $\mathcal{M}^{\mathcal{W}^{**}_\gamma}_\alpha$ sequence for any $\alpha < \xi$,

(iii) $\tau \leq_{W^{**}_\gamma} \xi$, and $\hat{\imath}^{W^{**}_\gamma}_{\tau,\xi} \restriction (\mathrm{lh}(t^\gamma(F)) + 1) = \mathrm{res}_\gamma \restriction (\mathrm{lh}(t^\gamma(F)) + 1)$, and

(iv) similarly for $\mathcal{V}^{**}_\gamma$ vis-a-vis $\mathcal{V}_\gamma$.

Proof. (Sketch.) Part (iv) literally follows from Lemma 5.6 [31], because the $\mathcal{V}_{\eta,j}$ do not depend on the $\mathcal{W}^*_{\eta,j}$. For parts (i)-(iii), one simply repeats the proof of 5.6.

Item (i) includes the agreement on stability declarations and next models. The point is that the $(\Sigma, \Lambda, \mathrm{res}_\gamma(Q))$-coiteration reaches models extending Q on both sides

[31] Apart from the fact that we are now dealing with a least branch construction.

by the proof of Lemma 5.6. Let η be least such that $\eta \leq_{W^{**}_\gamma} \xi$ and $Q \trianglelefteq \mathcal{M}^{\mathcal{W}^{**}_\gamma}_\eta$. We have that from the proof of 5.6 that

$$\hat{i}^{W^{**}_\gamma}_{\eta,\xi} \restriction (\operatorname{lh} t^\gamma(F) + 1) = \operatorname{res}_\gamma \restriction (\operatorname{lh} t^\gamma(F) + 1).$$

The proof also shows that either $\eta = \xi$, or the first ultrapower taken in $(\eta, \xi]_{W^{**}_\gamma}$ involves a drop in model or degree. In either case, η is stable in $\mathcal{W}^{**}_\gamma$. Let also δ be least such that $Q \trianglelefteq \mathcal{M}^{\mathcal{V}^{**}_\gamma}_\delta$. We then have that $(\mathcal{W}^{**}_\gamma \restriction (\eta + 1), \mathcal{V}^{**}_\gamma \restriction (\delta + 1))$ is the (Σ, Λ, Q) coiteration. But $Q \trianglelefteq N_\gamma$, so this is an initial segment of the $(\Sigma, \Lambda, N_\gamma)$ coiteration, that is, of $(\mathcal{W}^*_\gamma, \mathcal{V}_\gamma)$. This implies $\eta = \tau$ and $\delta = \theta$. $\square$

Let

- $\xi_\gamma = \operatorname{lh}(\mathcal{W}^{**}_\gamma) - 1$
- $\tau_\gamma =$ least τ such that $Q \trianglelefteq \mathcal{M}^{\mathcal{W}^*_\gamma}_\tau$ and τ is stable in $\mathcal{W}^*_\gamma$, and
- $N^*_\gamma = \mathcal{M}^{\mathcal{W}^{**}_\gamma}_{\xi_\gamma}$.

With these definitions, clause (4) of $(\dagger)_\gamma$ now makes sense. Note $\mathcal{W}^*_\gamma \restriction \tau_\gamma + 1 = \mathcal{W}^{**}_\gamma \restriction \tau_\gamma + 1$, and ξ_γ is the least stable α of $\mathcal{W}^{**}_\gamma$ such that G is on the sequence of $\mathcal{M}^{\mathcal{W}^{**}_\gamma}_\alpha$.

$Q, \operatorname{res}_\gamma(Q)$, and N_ν all agree up to $\operatorname{dom}(G)$, so

$$\operatorname{res}_\gamma(Q) | \langle \operatorname{lh}(G), -1 \rangle \trianglelefteq i_{G^*}(N_\nu) = N_{\gamma+1},$$

and $i_{G^*}(N_\nu) | \langle \operatorname{lh}(G), 0 \rangle$ is extender-passive, by coherence. $(\mathcal{W}^{**}_\gamma, \mathcal{V}^{**}_\gamma)$ is the $\operatorname{res}_\gamma(Q)$ coiteration, so G is on the sequence of the last model on both sides. We then get that $\mathcal{V}^{**}_\gamma$ is an initial segment of $\mathcal{V}_{\gamma+1}$, $\mathcal{W}^{**}_\gamma$ is an initial segment of $\mathcal{W}^*_{\gamma+1}$ and G is used in both $\mathcal{V}_{\gamma+1}$ and $\mathcal{W}^*_{\gamma+1}$.

We define

$$\Phi_{\gamma+1} \restriction \bar{\xi}_\gamma + 1 = \Phi_\gamma \restriction \bar{\xi}_\gamma + 1,$$

and this is ok because $\mathcal{W}_\gamma \restriction (\bar{\xi}_\gamma + 1) = \mathcal{W}_{\gamma+1} \restriction (\bar{\xi}_\gamma + 1)$ and $\mathcal{W}^*_\gamma \restriction v^\gamma(\bar{\xi}_\gamma + 1) = \mathcal{W}^*_{\gamma+1} \restriction v^{\gamma+1}(\bar{\xi}_\gamma + 1)$. We set

$$u^{\gamma+1}(\bar{\xi}_\gamma) = \xi_\gamma,$$

so that

$$p^{\gamma+1}(F) = G.$$

Let us set $\bar{\xi} = \bar{\xi}_\gamma$ and $\xi = \xi_\gamma$.

Let

$$\beta = \beta(\mathcal{W}_\nu, \mathcal{W}_\gamma, F) = W_{\gamma+1}\text{-pred}(\bar{\xi} + 1),$$

and

$$\beta^* = W^*_{\gamma+1}\text{-pred}(\xi + 1).$$

Let us verify that β^* is located where it should be in $\mathcal{W}^*_{\gamma+1}$ according to Definition 7.8. Basically, we just run through the proof of Sublemma 5.15.1, taking into account the stability structure now present. So let

- $\bar{\kappa} = \text{crit}(E^{\mathcal{U}}_\gamma)$, and $\bar{P} = \mathcal{M}^{\mathcal{U}}_\gamma | \bar{\kappa}^+ = \mathcal{M}^{\mathcal{U}}_\nu | \bar{\kappa}^+ = \text{dom}(E^{\mathcal{U}}_\gamma)$,
- $\kappa = \text{crit}(F)$, and $P = \mathcal{M}^{\mathcal{W}_\nu}_\beta | \kappa^+ = \mathcal{M}^{\mathcal{W}_\gamma}_\beta | \kappa^+ = \mathcal{M}^{\mathcal{W}_{\gamma+1}}_\beta | \kappa^+ = \text{dom}(F)$, and
- $\kappa^* = \text{crit}(G)$, and $P^* = \text{dom}(G)$.

In these formulae, the successor cardinals are evaluated in the corresponding models, of course. Recall here that $\mathcal{W}_\nu \restriction \beta + 1 = \mathcal{W}_\gamma \restriction \beta + 1 = \mathcal{W}_{\gamma+1} \restriction \beta + 1$.

Claim 1. σ_ν agrees with σ_γ on $\text{lh}(E^{\mathcal{U}}_\nu)$, and $\sigma_\nu(\bar{P}) = \sigma_\gamma(\bar{P}) = P$.

Proof. We have that σ^0_ν agrees with σ^0_γ on $\text{lh}(E^{\mathcal{U}}_\nu)$ by the agreement of copy maps, and σ^1_ν agrees with σ^1_γ on $\text{lh}(\sigma^0(E^{\mathcal{U}}_\nu))$ by the agreement of the embedding normalization maps in $W(\mathcal{T}, \mathcal{U}^+)$. (Cf. 3.49.) This proves the first part. But $\bar{P} \lhd \mathcal{M}^{\mathcal{U}}_\nu | \lambda(E^{\mathcal{U}}_\nu)$, so $\sigma_\nu(\bar{P}) = \sigma_\gamma(\bar{P})$, and $\sigma_\gamma(E^{\mathcal{U}}_\gamma) = F$, so $\sigma_\gamma(\bar{P}) = P$. □

Claim 2. $t^\nu(P) = t^\gamma(P) = P^*$, and $t^\nu \restriction P = t^\gamma \restriction P$.

Proof. Because $[\nu, \gamma+1)_U$ does not drop, whenever $\mathcal{M}^{\mathcal{U}}_\nu | \text{lh}(E^{\mathcal{U}}_\nu) \trianglelefteq X \lhd \mathcal{M}^{\mathcal{U}}_\nu$, then $\rho(X) > \bar{\kappa}$. This implies that whenever $R_\nu | \text{lh}(F_\nu) \trianglelefteq X \lhd R_\nu$, then $\rho(X) > \kappa$. It follows that

$$\text{res}_\nu \restriction (t^\nu(P) \cup \{t^\nu(P)\}) = \text{ id}.$$

If $\nu < \gamma$, we also get for the same reason

$$\text{res}_\gamma \restriction (t^\gamma(P) \cup \{t^\gamma(P)\}) = \text{ id}.$$

This implies

$$t^\gamma(P) = \text{res}_\gamma \circ t^\gamma(P) = \text{dom}(\text{res}_\gamma \circ t^\gamma(F)) = \text{dom}(G) = P^*.$$

But also $\psi^{\mathcal{U}}_\gamma \restriction \lambda(E^{\mathcal{U}}_\nu) = \text{res}_\nu \circ \psi^{\mathcal{U}}_\nu \restriction \lambda(E^{\mathcal{U}}_\nu)$ by the properties of conversion systems. So we get

$$\begin{aligned} t^\gamma(P) &= t^\gamma \circ \sigma_\gamma(\bar{P}) \\ &= \psi^{\mathcal{U}}_\gamma(\bar{P}) \\ &= \text{res}_\nu \circ \psi^{\mathcal{U}}_\nu(\bar{P}) \\ &= \text{res}_\nu \circ t^\nu \circ \sigma_\nu(\bar{P}) \\ &= t^\nu(P). \end{aligned}$$

The same calculation shows that $t^\gamma \restriction P = t^\nu \restriction P$. □

Claim 3. If β is stable in $\mathcal{W}_\nu$, and $\beta < z(\nu)$, then $v^\nu(\beta) \leq^*_{W^*_\nu} \beta^* \leq^*_{W^*_\nu} u^\nu(\beta)$.

Consider the diagram

$$\begin{tikzcd}
 & \mathcal{M}^{\mathcal{W}^*_\nu}_{u^\nu(\beta)} \\
 & \mathcal{M}^{\mathcal{W}^*_\nu}_{\eta} \arrow[u, "\hat{\imath}_{\eta,u^\nu(\beta)}"'] \\
\mathcal{M}^{\mathcal{W}_\nu}_{\beta} \arrow[ruu, "t^\nu_\beta"] \arrow[r, "s^\nu_\beta"'] & \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(\beta)} \arrow[u, "\hat{\imath}_{v^\nu(\beta),\eta}"']
\end{tikzcd}$$

By 7.10, t^ν agrees with t^ν_β on $\mathcal{M}^{\mathcal{W}_\nu}_\beta | \lambda(E^{\mathcal{W}_\nu}_\beta)$, and $P \lhd \mathcal{M}^{\mathcal{W}_\nu}_\beta | \lambda(E^{\mathcal{W}_\nu}_\beta)$, so

$$t^\nu_\beta(P) = P^*.$$

Let $\eta \in [v^\nu(\beta), u^\nu(\beta)]_{W_\nu}$ be least such that either $\eta = u^\nu(\beta)$ or $\mathrm{crit}(\hat{\imath}_{\eta,u^\nu(\beta)}) > \kappa^*$. Thus

$$\hat{\imath}_{v^\nu(\beta),\eta} \circ s^\nu_\beta(P) = P^*,$$

and all extenders used in $\mathcal{W}^*_\nu \restriction \eta + 1$ have length $< \kappa^*$.

We claim that $\lambda(E^{\mathcal{W}^*_\nu}_\eta) > \kappa^*$. If $\eta = u^\nu(\beta)$, this holds because $\kappa < \lambda(E^{\mathcal{W}_\nu}_\beta)$, and t^ν_β preserves that fact. If $\eta < u^\nu(\beta)$, then $\kappa^* < \mathrm{crit}(\hat{\imath}_{\eta,u^\nu(\beta)}) < \lambda(E^{\mathcal{W}^*_\nu}_\eta)$, so again our claim is correct. The claim tells us that $\beta^* \leq \eta$.

On the other hand, if $\alpha < \eta$ and α is stable in $\mathcal{W}^*_\nu$, then $\mathrm{lh}(E^{\mathcal{W}^*_\nu}_\alpha) < \kappa^*$. This is true by definition for those α such that $\alpha + 1 \leq^*_{W^*_\nu} \eta$, but the lengths of these special E_α are cofinal in $\{\mathrm{lh}(E^{\mathcal{W}^*_\nu}_\alpha) \mid \alpha < \eta \wedge \alpha \in \mathrm{stab}(\mathcal{W}^*_\nu)\}$. This tells us that if $\alpha < \eta$ and α is stable, then $\alpha < \beta^*$.

We claim $\eta = \beta^*$. What is left to rule out is that β^* is unstable, and $\beta^* + 1 = \eta$. Supposing this holds, we get that $\beta = \theta + 1$, where θ is unstable in $\mathcal{W}_\nu$. We have $\alpha^{\mathcal{W}_\nu}_\theta \leq \kappa$ because F is applied to $\mathcal{M}^{\mathcal{W}_\nu}_\beta$. Thus $s^\nu_\beta(\kappa) \leq s^\nu_\beta(\alpha^{\mathcal{W}_\nu}_\theta) = \alpha^{\mathcal{W}^*_\nu}_{v^\nu(\theta)}$. But $\alpha^{\mathcal{W}^*_\nu}_{\beta^*} = \sup(i^{\mathcal{W}^*_\nu}_{v^\nu(\theta),\beta^*} \text{“} \alpha^{\mathcal{W}^*_\nu}_{v^\nu(\theta)})$. It follows that

$$\alpha^{\mathcal{W}^*_\nu}_{\beta^*} \leq i^{\mathcal{W}^*_\nu}_{v^\nu(\theta),\beta^*} \circ s^\nu_\theta(\kappa) = \kappa^*.$$

But then G is not applied to $\mathcal{M}^{\mathcal{W}^*_\nu}_{\beta^*}$ in $\mathcal{W}^*_{\gamma+1}$, contradiction. □

Claim 4. If $\beta = z(\nu)$, then $v^\nu(\beta) \leq^*_{W^*_\nu} \beta^* \leq^*_{W^*_\nu} z^*(\nu)$.

Proof. If $\beta = z(\nu)$, then β must be stable. The proof of Claim 3 then works with small changes. □

Note that Claims (3) and (4) imply that if β is stable in $\mathcal{W}_\nu$, then β^* is stable in $\mathcal{W}^*_\nu$.

Claim 5. If β is unstable in $\mathcal{W}_\nu$ and $\beta+1 < z(\nu)$, then β^* is unstable in $\mathcal{W}^*_\nu$, and $v^\nu(\beta) \leq^* \beta^* \leq^* u^\nu(\beta)$ in $\mathcal{W}^*_\nu$.

Proof. Let $\lambda = \lambda^{\mathcal{W}_\nu}_\beta = \inf(\alpha^{\mathcal{W}_\nu}_\beta, \lambda(E^{\mathcal{W}_\nu}_{\beta+1}))$. By 7.10, t^ν agrees with t^ν_β on $\mathcal{M}^{\mathcal{W}_\nu}_\beta|\lambda$. Since $P \lhd \mathcal{M}^{\mathcal{W}_\nu}_\beta|\lambda$, we have again

$$t^\nu_\beta(P) = P^*.$$

Let $\eta \in [v^\nu(\beta{+}1), u^\nu(\beta{+}1)]_{W_\nu}$ be least such that either $\eta = u^\nu(\beta{+}1)$ or $\mathrm{crit}(\hat{\imath}_{\eta,u^\nu(\beta+1)}) > \kappa^*$. Thus

$$\hat{\imath}_{v^\nu(\beta+1),\eta} \circ s^\nu_{\beta+1}(P) = P^*,$$

and all extenders used in $\mathcal{W}^*_\nu \restriction \eta+1$ have length $< \kappa^*$.

Note that $s^\nu_\beta \restriction \alpha^{\mathcal{W}_\nu}_\beta = s^{\mathcal{W}_\nu}_{\beta+1}$, and $\kappa < \alpha^{\mathcal{W}_\nu}_\beta$. All extenders used in $[v^\nu(\beta+1), \eta]_{W^*_\nu}$ have critical point below the current image of $s^\nu_{\beta+1}(\kappa)$, hence below the current image of $s^\nu_\beta(\alpha^{\mathcal{W}_\nu}_\beta)$. Thus all these extenders are moving up the current image of the phalanx indexed at $(v^\nu(\beta), v^\nu(\beta+1))$. It follows that $\eta = \gamma+1$, where γ is unstable in $\mathcal{W}^*_\nu$, and $v^\nu(\beta) \leq^* \gamma \leq^* u^\nu(\beta)$.

It is now easy to see that $\gamma = \beta^*$, so that Claim 5 holds. $\square$

Claim 6. If β is unstable in $\mathcal{W}_\nu$ and $\beta+1 = z(\nu)$, then β^* is unstable in $\mathcal{W}^*_\nu$, and $v^\nu(\beta) \leq^*_{W^*_\nu} \beta^* \leq^*_{W^*_\nu} z^*(\nu) - 1$.

Proof. The proof of Claim 5 works here. $\square$

We let $v^{\gamma+1}(\bar{\xi}+1) = \xi+1$. We need to see

Claim 7. $\bar{\xi}+1 \in \mathrm{stab}(\mathcal{W}_{\gamma+1})$ if and only if $\xi+1 \in \mathrm{stab}(\mathcal{W}^*_{\gamma+1}$.

Proof. We have that

$$\begin{aligned}\bar{\xi}+1 \in \mathrm{stab}(\mathcal{W}_{\gamma+1}) &\Leftrightarrow \beta \in \mathrm{stab}(\mathcal{W}_\nu)\\ &\Leftrightarrow \beta^* \in \mathrm{stab}(\mathcal{W}^*_\nu)\\ &\Leftrightarrow \xi+1 \in \mathrm{stab}(\mathcal{W}^*_{\gamma+1}).\end{aligned}$$

The first line holds because $\Phi_{\nu,\gamma+1}$ is a tree embedding. The second line was proved in Claims (3)-(6). Toward the last line, suppose first that $\beta^* \in \mathrm{stab}(\mathcal{W}^*_\nu)$. Since $\mathcal{W}^*_\nu \restriction \beta^*+1 = \mathcal{W}^*_{\gamma+1} \restriction \beta^*+1$, and $\mathcal{V}_\nu$ uses the same extenders of length $< o(P^*)$ as $\mathcal{V}_{\gamma+1}$ does, we get that $\beta^* \in \mathrm{stab}(\mathcal{W}^*_{\gamma+1})$. But $\beta^* \leq^* \xi+1$ in $\mathcal{W}^*_{\gamma+1}$, so $\xi+1 \in \mathrm{stab}(\mathcal{W}^*_{\gamma+1})$.

Conversely, suppose β^* is unstable in $\mathcal{W}^*_\nu$. The agreement noted in the last paragraph shows that β^* is unstable in $\mathcal{W}^*_{\gamma+1}$. Now recall that $(\mathcal{W}^{**}_\gamma, \mathcal{V}^{**}_\gamma)$ is the $(\Sigma, \Lambda, \mathrm{res}_\gamma(Q))$ coiteration. Letting $\rho+1 = \mathrm{lh}(\mathcal{V}^* *_\gamma)$, we have that G is on the sequence of $\mathcal{M}^{\mathcal{V}^{**}_\gamma}_\rho$, but not on the sequence of any earlier model. It follows that

$$\mathcal{V}_{\gamma+1} \restriction (\rho+1) = \mathcal{V}^{**}_\gamma,$$

and

$$E^{\mathcal{V}_{\gamma+1}}_\rho = G.$$

Since β^* is unstable in $\mathcal{W}^*_{\gamma+1}$, we have τ such that
$$\mathcal{M}_\tau^{\mathcal{V}_{\gamma+1}} = \mathcal{M}_{\beta^*}^{\mathcal{W}^*_{\gamma+1}}.$$
But then G must be applied to $\mathcal{M}_\tau^{\mathcal{V}_{\gamma+1}}$ in $\mathcal{V}_{\gamma+1}$, leading to
$$\mathcal{M}_{\tau+1}^{\mathcal{V}_{\gamma+1}} = \mathcal{M}_{\xi+1}^{\mathcal{W}^*_{\gamma+1}},$$
so that $\xi+1$ is unstable in $\mathcal{W}^*_{\gamma+1}$, as desired. □

The map $s^{\gamma+1}_{\bar{\xi}+1}$ of $\Phi_{\gamma+1}$ is given by the Shift Lemma, as clause (7) in the definition of tree embeddings requires. If $\bar{\xi}+1$ is unstable in $\mathcal{W}_{\gamma+1}$, this also determines $s^{\gamma+1}_{\bar{\xi}+2}$.

The rest of $\Phi_{\gamma+1}$ is determined by
$$u^{\gamma+1}(\phi_{\nu,\gamma+1}(\eta)) = i_{G^*}(u^\nu(\eta)).$$
$u^{\gamma+1}$ preserves stability, because u^ν and $\phi_{\nu,\gamma+1}$ do, and i_{G^*} is elementary. One must check that the associated $v^{\gamma+1}$ also preserves stability. Here we use proposition 6.63. Let $\phi = \phi_{\nu,\gamma+1}$. In general, $v^{\gamma+1}(\phi(\eta)) = \sup i_{G^*}``v^\nu(\eta)$. However, if $\phi(\eta)$ is a stable limit ordinal in $\mathcal{W}_{\gamma+1}$, then η is stable in $\mathcal{W}_\nu$, so $\mathrm{cof}(\eta) = \mathrm{cof}(\phi(\eta)) = \omega$. But then $\mathrm{cof}(v^\nu(\eta)) = \omega$, so i_{G^*} is continuous at $v^\nu(\eta)$. Thus $v^{\gamma+1}(\phi(\eta)) = i_{G^*}(v^\nu(\eta))$, hence $v^{\gamma+1}(\phi(\eta))$ is stable in $\mathcal{W}^*_{\gamma+1}$ by the elementarity of i_{G^*}.

This proves Sublemma 7.16.1. □

That in turn proves Lemma 7.16, or what is the same, Lemma 6.64 of Chapter 5. □

7.3 UBH holds in hod mice

In this section, we adapt the proof in [51] that a form of UBH is true in pure extender models. We show thereby that whenever (M, Ω) is an lbr hod pair with scope HC, and Ω is Suslin-co-Suslin in some model of AD^+, then UBH for nice, normal iteration trees holds in M. As in the pure extender case, the proof involves a comparison of phalanxes of the form $\Phi(\mathcal{T}^\frown b)$ and $\Phi(\mathcal{T}^\frown c)$.

We shall use this theorem to show that if (M, Ω) is as above, and λ is a limit of Woodin cardinals in M, then for each $\xi < \lambda$ there is a term $\tau \in M$ such that for all g generic over M for a poset belonging to $M|\lambda$,
$$\tau^g = \Omega_{M|\xi} \cap (M|\lambda)[g].$$
This generic interpretability result is important in showing that the HOD of the derived model of M below λ is an iterate of $M|\lambda$. It has other uses as well.

Definition 7.17 *Let M be a premouse such that $M \models \mathsf{ZFC}^-$; then an M-*nice tree *is a normal iteration tree $\mathcal{T}$ on M such that for all $\alpha < \mathrm{lh}(\mathcal{T})$,*

(1) $\mathcal{M}^{\mathcal{T}}_\alpha \models$ "$E^{\mathcal{T}}_\alpha$ is a nice extender", and

(2) $E_\alpha^{\mathcal{T}} = F \restriction \mathrm{lh}(E_\alpha^{\mathcal{T}})$, *for some* F *on the sequence of* $\mathcal{M}_\alpha^{\mathcal{T}}$.

Notice here that if $\mathcal{T}$ is M-nice, then $E_\alpha^{\mathcal{T}}$ cannot be on the sequence of $\mathcal{M}_\alpha^{\mathcal{T}}$, because in Jensen-indexed premice, the extenders of the sequence are never nice. Nevertheless, if (M, Σ) is a mouse pair and $\mathcal{T}$ is an M-nice tree on $M, \Sigma)$, then the pairs in $\mathcal{T}$ have the extender and strategy agreement properties that a tree using extenders from the sequences would have. That is, is $\alpha < \beta$, then $(\mathcal{M}_\alpha, \Sigma_{\mathcal{T}\restriction\alpha+1})|\,\mathrm{lh}(E_\alpha^{\mathcal{T}}) = (\mathcal{M}_\beta, \Sigma_{\mathcal{T}\restriction\beta+1})|\,\mathrm{lh}(E_\alpha^{\mathcal{T}})$. [32]

Theorem 7.18 *Assume* AD^+, *and let* (M, Ω) *be a least branch hod pair with scope HC. Suppose* $M \models \mathsf{ZFC}^-$, *and* Ω *is coded by a Suslin-co-Suslin set of reals. Let* δ *be a cutpoint of* M, $\mu > \delta$ *a regular cardinal of* M, *and let* $\mathcal{T}$ *be an* M*-nice tree such that*

(a) $\mathcal{T}$ *has all critical points* $> \delta$, *and*

(b) $\mathcal{T} \in (M|\mu)[g]$, *for some* g *that is* M*-generic over* $\mathrm{Col}(\omega, \delta)$*;*

then

$$M[g] \models \mathcal{T} \textit{ has at most one cofinal, wellfounded branch.}$$

Remark 7.19 Our proof of this theorem can be extended without much more work to cover plus two trees $\mathcal{T}$, as does the theorem of [51] it generalizes. We don't see how to make it work for arbitrary non-dropping trees.

Proof. Suppose not. Let $\dot{\mathcal{T}} \in M|\mu$ be the M-least name such that 1 forces $\dot{\mathcal{T}}$ to be a counterexample. Let g be M-generic over $\mathrm{Col}(\omega, \delta)$, and $\mathcal{T} = \dot{\mathcal{T}}^g$. $\mathcal{T}$ is countable in $M|\mu[g]$. Let

$$\pi\colon N \to M|\mu$$

be elementary, and such that $\mathrm{crit}(\pi) > \delta$, and N is pointwise definable from ordinals $\leq \delta$. Thus $\dot{\mathcal{T}} \in \mathrm{ran}(\pi)$. Let

$$\hat{\pi}\colon N[g] \to (M|\mu)[g]$$

be the canonical extension of π, and let

$$\hat{\pi}(\mathcal{S}) = \mathcal{T}.$$

By assumption, $\mathcal{T}$ has distinct, cofinal, wellfounded branches in $(M|\mu)[g]$, so we have b, c such that

$$N[g] \models \text{ } b \text{ and } c \text{ are distinct cofinal, wellfounded branches of } \mathcal{S}.$$

[32] $\mathrm{lh}(E_\alpha^{\mathcal{T}})$ is inaccessible in $\mathcal{M}_\alpha^{\mathcal{T}}$, so it is not an index.

Let $\Phi(\mathcal{S}^\frown b)$ be the phalanx $(\langle \mathcal{M}_\alpha^{\mathcal{S}} \mid \alpha < \mathrm{lh}(\mathcal{S})\rangle^\frown \langle \mathcal{M}_b^{\mathcal{S}}\rangle, \langle \mathrm{lh}(E_\alpha^{\mathcal{S}}) \mid \alpha < \mathrm{lh}(\mathcal{S})\rangle)$. We get an iteration strategy for $\Phi(\mathcal{S}^\frown b)$ by finding maps with sufficient agreement that embed its models into M.

Claim 0. There are $\pi_\alpha, \gamma_\alpha$ for $\alpha < \mathrm{lh}(\mathcal{S})$, and π_b, such that π_b and the π_α are the identity on $\delta + 1$, and for all α,

(1) $\pi_b \colon \mathcal{M}_b^{\mathcal{S}} \to M|\mu$,

(2) $\pi_\alpha \colon \mathcal{M}_\alpha^{\mathcal{S}} \to M|\gamma_\alpha$, and

(3) $\pi_\alpha \restriction \mathrm{lh}(E_\alpha^{\mathcal{S}}) = \pi_b \restriction \mathrm{lh}(E_\alpha^{\mathcal{S}})$.

Proof. The proof is given, under slightly different strength hypotheses on the $E_\alpha^{\mathcal{T}}$, in [51, §3]. See especially the proof of Theorem 3.3.[33] □

Our iteration strategy for $\Phi(\mathcal{S}^\frown b)$ is then just the pullback of Ω under the π_α, for $\alpha < \mathrm{lh}(\mathcal{S})$ or $\alpha = b$. Call this strategy Ψ.

Similarly, we have

Claim 1. There are $\sigma_\alpha, \xi_\alpha$ for $\alpha < \mathrm{lh}(\mathcal{S})$, and σ_c, such that σ_c and the σ_α are the identity on $\delta + 1$, and for all α,

(1) $\sigma_c \colon \mathcal{M}_c^{\mathcal{S}} \to M|\mu$,

(2) $\sigma_\alpha \colon \mathcal{M}_\alpha^{\mathcal{S}} \to M|\xi_\alpha$, and

(3) $\sigma_\alpha \restriction \mathrm{lh}(E_\alpha^{\mathcal{S}}) = \sigma_c \restriction \mathrm{lh}(E_\alpha^{\mathcal{S}})$.

We then get an iteration strategy for the phalanx $\Phi(\mathcal{S}^\frown c)$ by pulling back Ω under the maps σ_α, for $\alpha < \mathrm{lh}(\mathcal{S})$ or $\alpha = c$. Call this iteration strategy Σ.

Let $(N^*, \Sigma^*, \delta^*)$ be a coarse Γ Woodin model, where Ω is coded by a $\Gamma \cap \check{\Gamma}$ set of reals. We assume that the various countable objects we have encountered so far are countable in N^*. In particular, $M[g]$, $\Phi(\mathcal{S}^\frown b)$, $\Phi(\mathcal{S}^\frown c)$, and the maps from claims (1) and (2) are countable in N^*. Let $\mathbb{C}$ be a maximal w-construction below δ^* in N^*. We

[33] Here is a sketch. The copy maps $\psi_\alpha \colon \mathcal{M}_\alpha^{\mathcal{S}} \to \mathcal{M}_\alpha^{\mathcal{T}}|\mu$ are all restrictions of π, as is the copy map $\psi_b \colon \mathcal{M}_b^{\mathcal{S}} \to \mathcal{M}_b^{\mathcal{T}}|\mu$. ($\mu$ is fixed by the maps of $\mathcal{T}$.) Letting $\nu_\alpha = \sup \psi_\alpha``\mathrm{lh}(E_\alpha^{\mathcal{S}})$, we have $\nu_\alpha < \mathrm{lh}(E_\alpha)^{\mathcal{T}}$. Using Condensation inside $\mathcal{M}_\alpha^{\mathcal{T}}$, we then get $\xi_\alpha < \mathrm{lh}(E_\alpha^{\mathcal{T}})$ and $\phi_\alpha \colon \mathcal{M}_\alpha^{\mathcal{S}} \to \mathcal{M}_b^{\mathcal{T}}|\xi_\alpha$ such that ϕ_α agrees with ψ_α, and hence π, on $\mathrm{lh}(E_\alpha^{\mathcal{S}})$. The ϕ_α are in $\mathcal{M}_b^{\mathcal{T}}$. An absoluteness argument done in the wellfounded model $\mathcal{M}_b^{\mathcal{T}}$ then gives us the Claim, but with $\mathcal{M}_b^{\mathcal{T}}$ replacing M. Pulling back under $i_b^{\mathcal{T}}$, we get the Claim itself.

compare $\Phi(\mathcal{S}^\frown b)$ with $\Phi(\mathcal{S}^\frown c)$ by defining, for each ν, l, the $(\Psi, \Sigma, M^{\mathbb{C}}_{\nu,l})$-*coiteration* (of $\Phi(\mathcal{S}^\frown b)$ with $\Phi(\mathcal{S}^\frown c)$). This is a pair of psuedo trees $(\mathcal{U}_{\nu,l}{}^\frown\mathcal{W}_{\nu,l}, \mathcal{U}_{\nu,l}{}^\frown\mathcal{V}_{\nu,l})$ according to Ψ and Σ respectively, obtained by iterating away least disagreements with $M^{\mathbb{C}}_{\nu,l}$, as in the proof of Theorem 6.57. The process of moving a phalanx up is a little different, so let us look at it briefly.

The first phase in the coiteration consists in moving $\Phi(\mathcal{S}^\frown b)$ and $\Phi(\mathcal{S}^\frown b)$ up by an ordinary normal iteration tree on $M|\delta = N|\delta$. Note δ is a cutpoint of of $N = \mathcal{M}^{\mathcal{S}}_0$, and Ψ and Σ both agree with $\Omega_{\delta,0}$ for trees on $N|\delta$. We let $\mathcal{U} = \mathcal{U}_{\nu,l}$ be the unique normal tree on $N|\delta$ that is by $\Omega_{\langle\delta,0\rangle}$ and has last model $P = M_{\nu,l}|\langle\delta_0, 0\rangle$, with the strategy agreement $\Omega_{\mathcal{U},P} = (\Omega^{\mathbb{C}}_{\nu,l})_{\langle\delta_0,0\rangle}$. There is such a $\mathcal{U}$ by Theorem 6.45. We assume here that $\langle\nu, l\rangle$ is large enough that $(N|\delta, \Omega_{\langle\delta,0\rangle})$ does not iterate past $(M_{\nu,l}, \Omega_{\nu,l})$. We wish now to define $\mathcal{W} = \mathcal{W}_{\nu,l}$ and $\mathcal{V} = \mathcal{V}_{\nu,l}$.

Thinking of $\mathcal{U}$ as a tree on N, its last model is

$$Q = \mathcal{M}^{\mathcal{U}}_{\tau_0} = \mathcal{M}^{\mathcal{W}}_0 = \mathcal{M}^{\mathcal{V}}_0.$$

$P = Q|\delta_0$ is a cutpoint initial segment of Q, and Q is pointwise definable from the ordinals $< \delta_0$. (In most cases, $\tau_0 = \delta_0$.) Letting E be the branch extender of $i^{\mathcal{U}}_{0,\tau_0}$, we move up our two phalanxes by setting, for $\alpha < \theta$,

- $\mathcal{M}^{\mathcal{W}}_\alpha = \mathcal{M}^{\mathcal{V}}_\alpha = \mathrm{Ult}(\mathcal{M}^{\mathcal{S}}_\alpha, E)$,
- $\rho_\alpha = i^{\mathcal{M}^{\mathcal{S}}_\alpha}_E(\mathrm{lh}(E^{\mathcal{S}}_\alpha))$,
- $\mathcal{M}^{\mathcal{W}}_\theta = \mathrm{Ult}(\mathcal{M}^{\mathcal{S}}_b, E)$, and
- $\mathcal{M}^{\mathcal{V}}_\theta = \mathrm{Ult}(\mathcal{M}^{\mathcal{S}}_c, E)$.

The rest of $\mathcal{W}$ and $\mathcal{V}$ will be psuedo-trees on the phalanxes $(\langle\mathcal{M}^{\mathcal{W}}_\xi \mid \xi \leq \theta\rangle, \langle\rho_\xi \mid \xi < \theta\rangle)$ and $(\langle\mathcal{M}^{\mathcal{V}}_\xi \mid \xi \leq \theta\rangle, \langle\rho_\xi \mid \xi < \theta\rangle)$. A *root* of $\mathcal{W}$ or $\mathcal{V}$ is an ordinal $\xi \leq \theta$. If $\xi = \theta$, the root is stable, and if $\xi < \theta$ the root is unstable. At any stage, the current last models of $\mathcal{W}$ and $\mathcal{V}$ are stable. If $\mathcal{M}^{\mathcal{W}}_\gamma$ is the current last model of $\mathcal{W}$ at some stage, then we let $E^{\mathcal{W}}_\gamma$ be the first extender on its sequence that is part of a disagreement with $M^{\mathbb{C}}_{\nu,l}$. Similarly on the $\mathcal{V}$ side. We show that the corresponding extender on $M^{\mathbb{C}}_{\nu,l}$ is empty, and no strategy disagreements ever show up. If there is no disagreement, the construction of $\mathcal{W}_{\nu,l}$ is complete, and similarly on the $\mathcal{V}$ side.

We shall also have ordinals $\lambda^{\mathcal{W}}_\alpha$ and $\lambda^{\mathcal{V}}_\alpha$ that tell us what model in $\mathcal{W}$ or $\mathcal{V}$ we should apply a given extender to.[34] If α is stable in $\mathcal{W}$ and $E^{\mathcal{W}}_\alpha$ exists (that is, the construction of $\mathcal{W}$ is not finished), then

$$\lambda^{\mathcal{W}}_\alpha = \lambda(E^{\mathcal{W}}_\alpha).$$

[34] Again, these correspond to $\lambda^{\mathcal{T}}_{\alpha+1}$ when $\mathcal{T}$ is normal.

If α is unstable in $\mathcal{W}$, then there is a least stable $\gamma \geq \alpha$. Suppose again $E_\gamma^{\mathcal{W}}$ exists, as otherwise the construction of $\mathcal{W}$ is done. Since α is unstable, we will have a unique unstable root $\eta < \theta$ such that $\eta \leq_W \alpha$, and $[\eta, \alpha]_W$ will not drop. We then set then we set

$$\lambda_\alpha^{\mathcal{W}} = \inf(i_{\eta,\alpha}^{\mathcal{W}}(\rho_\eta), \lambda(E_\gamma^{\mathcal{W}})).$$

Similarly for $\lambda_\alpha^{\mathcal{V}}$. The extenders used in $\mathcal{W}$ have increasing λ's, so if $\alpha < \beta$, then $\mathcal{M}_\alpha^{\mathcal{W}} | \lambda_\alpha^{\mathcal{W}} = \mathcal{M}_\beta^{\mathcal{W}} | \lambda_\alpha^{\mathcal{W}}$. [35]

Now let us look at the general successor step. Suppose $\mathcal{M}_\gamma^{\mathcal{W}}$ is the current last model of $\mathcal{W}$, and hence is stable. Let

$$E = E_\gamma^{\mathcal{W}}$$

be the least disagreement between $\mathcal{M}_\gamma^{\mathcal{W}}$ and $M_{\nu,l}$. Again, we are assuming such an agreement exists, it is not a strategy disagreement, and it does not involve an extender on the $M_{\nu,l}$ sequence. Set

$$\lambda_\gamma^{\mathcal{W}} = \lambda(E),$$

and for unstable α such that γ is the least stable above α, let $\lambda_\alpha^{\mathcal{W}}$ be defined as above. Let $\kappa = \text{crit}(E)$, and α be least such that $\kappa < \lambda_\alpha^{\mathcal{W}}$. We set $\alpha = W\text{-pred}(\theta + 1)$. If α is stable we just proceed as usual, creating one new model $\mathcal{M}_{\gamma+1}^{\mathcal{W}}$, which is stable. Similarly, if α is unstable but $\text{Ult}(\mathcal{M}_\alpha, E)$ does not occur in $\mathcal{V}$, we create only one new model, and it is stable.[36] So suppose α is unstable, and $\text{Ult}(\mathcal{M}_\alpha^{\mathcal{W}}, E)$ does occur in $\mathcal{V}$.

Let β be least such that $\alpha < \beta$ and β is stable. (E.g. if $\alpha < \theta$, then $\beta = \theta$.) For $0 \leq \xi \leq (\beta - \alpha)$, we set

$$\mathcal{M}_{(\gamma+1)+\xi}^{\mathcal{W}} = \text{Ult}(\mathcal{M}_{\alpha+\xi}^{\mathcal{W}}, E).$$

If $\xi < (\beta - \alpha)$, we declare that $\gamma + 1 + \xi$ is unstable, and we declare that $\gamma + 1 + (\beta - \alpha)$ is stable. $\gamma + 1 + (\beta - \alpha)$ is the new last node of $\mathcal{W}$, from which we shall take the next extender.

By induction, we have that for every node ξ of $\mathcal{W}$, there is a unique root $\tau \leq \theta$ such that $\tau \leq_W \xi$. If ξ is unstable, then so is τ; that is, $\tau < \theta$. Moreover, if ξ is unstable, then $[\tau, \xi)_W$ does not drop in model or degree, and $\lambda_\xi^{\mathcal{W}} \leq i_{0,\xi}^{\mathcal{W}}(\rho_\tau)$.

At limit steps in the construction of $\mathcal{W}$, we use Ψ to pick a branch a of the form $[\tau, \gamma)_W$, where $\tau \leq \theta$ is a root. We take γ to be stable unless every $\xi \in a$ is unstable (so a does not drop), and $\mathcal{M}_\gamma^{\mathcal{W}}$ is a model of $\mathcal{V}$. (Equivalently, $e_\gamma^{\mathcal{W}} = e_\eta^{\mathcal{V}}$, for some η.) In this case, we declare γ to be unstable. For ξ such that $\tau + \xi \leq \theta$, we set

$$\mathcal{M}_{\gamma+\xi}^{\mathcal{W}} = \text{Ult}(\mathcal{M}_{\tau+\xi}^{\mathcal{W}}, E),$$

[35] For example, suppose there is a ξ such that $\lambda(E_\theta^{\mathcal{W}}) \leq \rho_\xi$, and let ξ be the least such. Then $\lambda(E_\theta^{\mathcal{W}}) < \rho_\xi$, and $E_\theta^{\mathcal{W}}$ is actually on the $\mathcal{M}_\xi^{\mathcal{W}}$ sequence. If $\xi < \alpha \leq \theta$, then $\lambda_\alpha^{\mathcal{W}} = \lambda_\xi^{\mathcal{W}}$, and the net effect of our definition of the λ's is that no extender will ever be applied later to $\mathcal{M}_\alpha^{\mathcal{W}}$.

[36] At this point, we already know what extenders with length $\leq \text{lh}(E)$ are used in $\mathcal{V}$.

where E is the branch extender of a. If $\tau + \xi < \theta$), then $\gamma + \xi$ is unstable, and $\gamma + (\theta - \tau)$ is stable. We take the next extender from $\mathcal{M}^{\mathcal{W}}_{\gamma+(\theta-\tau)}$.

The construction of $\mathcal{V}$ proceeds in completely parallel fashion; indeed, nothing in our situation has distinguished b from c. Although the constructions of $\mathcal{W}$ and $\mathcal{V}$ determine stability by looking at each other, the reader can check that there is no circularity: when it comes time to determine whether γ is stable in $\mathcal{W}$, the relevant part of $\mathcal{V}$ is already determined.

As in §6.2, the maps π_α, for $\alpha < \mathrm{lh}(\mathcal{S})$ or $\alpha = b$, yield a pullback strategy for a more general iteration game on $\Phi(\mathcal{S}^\frown b)$. We also call this strategy Ψ. In the more general game, I makes stability declarations and creates new models according to the rules above. Of course, there are no $M_{\nu,l}$ and $\mathcal{V}$ in the setting of the general game. I picks the next extender E freely (subject to normality), and if E is to be applied to an unstable $\mathcal{M}_\alpha$, I may decide whether $\mathrm{Ult}(\mathcal{M}_\alpha, E)$ is stable as he pleases. If he decides against stability, he must create new models as above. At limit γ such that the branch to γ II has chosen consists of unstable nodes, I is again free to decide whether γ is stable. If he decides for unstability, he must create new models in the way we have described.

Similarly, the σ_α for $\alpha < \mathrm{lh}(\mathcal{S})$ or $\alpha = c$ yield a pullback strategy Σ for the more general game on $\Phi(\mathcal{S}^\frown c)$.

Remark 7.20 Our process of moving phalanxes up amounts to a step of full normalization. We could have used a step of embedding normalization instead, and thereby arranged that our $\mathcal{W}$ and $\mathcal{V}$ are actually normal iteration trees on N. $\mathcal{W}$ and $\mathcal{V}$ would then be *meta-iterates* of $\mathcal{S}^\frown b$ and $\mathcal{S}^\frown c$, in the sense of [48]. That paper contains a proof of Theorem 7.18 that rests on the theory of meta-iteration trees.

Let us consider how the coiteration can terminate. Let
$$Z = \mathrm{Th}^N(\delta),$$
and
$$Z_0 = \mathrm{Th}^Q(\delta_0) = i^{\mathcal{U}}_{0,\tau_0}(Z).$$
Q is pointwise definable from ordinals $< \delta_0$, so it is completely determined by Z_0. All critical points in $\mathcal{S}$ are above δ, so $Z = \mathrm{Th}^{\mathcal{M}^{\mathcal{S}}_\alpha}(\delta)$ for all $\alpha < \mathrm{lh}(\mathcal{S})$, and also for $\alpha = b$ or $\alpha = c$. Thus for all $\xi \leq \theta$,
$$Z_0 = \mathrm{Th}^{\mathcal{M}^{\mathcal{W}}_\xi}(\delta_0) = \mathrm{Th}^{\mathcal{M}^{\mathcal{V}}_\xi}(\delta_0).$$
Moreover, for all η, the critical points of $E^{\mathcal{W}}_\eta$ or $E^{\mathcal{V}}_\eta$ (if they exist) are $> \delta_0$.

Motivated by this, let us call $\langle \nu, l\rangle$ *relevant* iff

(a) $(Q|\delta_0, \Omega_{\mathcal{U},\langle\delta_0,0\rangle}) = (M^{\mathbb{C}}_{\nu,l}|\langle\delta_0, 0\rangle, (\Omega^{\mathbb{C}}_{\nu,l})_{\langle\delta_0,0\rangle})$,

(b) δ_0 is a cardinal cutpoint of $M^{\mathbb{C}}_{\nu,l}$, and

(c) for no proper initial segment R of $M^{\mathbb{C}}_{\nu,l}$ do we have $Z_0 = \text{Th}^R(\delta_0)$.

Let us call $\langle \nu, l\rangle$ *exact* iff it is relevant, and $Z_0 = \text{Th}^{M^{\mathbb{C}}_{\nu,l}}(\delta_0)$.

If $\langle \nu, l\rangle$ is relevant, then neither $\mathcal{W}_{\nu,l}$ nor $\mathcal{V}_{\nu,l}$ can reach a last model that is a proper initial segment of $M_{\nu,l}$. Let us state explicitly the lemma on stationarity of background constructions we have been using

Lemma 7.21 *If $\langle \nu, l\rangle$ is relevant, then in the $(\Psi, \Sigma, M^{\mathbb{C}}_{\nu,l})$ coiteration, no strategy disagreements show up, and no nonempty extender on the $M^{\mathbb{C}}_{\nu,l}$ side is part of a least disagreement.*

Proof. (*Sketch.*) This proof is like the proofs of 5.11 and 7.16 we gave earlier. We show that the strategies Ψ and Σ normalize well and have strong hull condensation, in the appropriate senses. We then show there are no strategy disagreements by taking a candidate disagreement at some $\mathcal{U}$ on some stable model $\mathcal{M}^{\mathcal{W}}_{\gamma}$, letting $b = (\Omega_{\nu,l})_J(\mathcal{U})$, and showing that the normalization of $\langle \mathcal{W}\restriction\gamma + 1, \mathcal{U}^\frown b\rangle$ tree-embeds into a psuedo-tree by Ψ. This involves an inductive construction like that in the proofs of 5.15.1 and 7.16.1. $\square$

Claim 2. There is an exact $\langle \nu, l\rangle <_{\text{lex}} \langle \delta^*, 0\rangle$.

Proof. Otherwise $\langle \delta^*, 0\rangle$ is relevant, so the $(\Psi, \Sigma, \mathcal{M}^{\mathbb{C}}_{\delta^*,0})$ coiteration produces $(\mathcal{W}, \mathcal{V})$ with last models extending $\mathcal{M}^{\mathbb{C}}_{\delta^*,0}$. This contradicts the universality of $\mathcal{M}^{\mathbb{C}}_{\delta^*,0}$. $\square$

Now let $\langle \nu, l\rangle$ be the unique exact pair. Z_0 contains statements which collectively assert that $\rho_\omega = \text{OR}$, and $\text{Th}^{M_{\nu,l}}(\delta_0) = Z_0$, so $l = 0$. We have also that $M_{\nu,0} \models \mathsf{ZFC}^-$. Z_0 is Σ_1 over $M_{\nu+1,0}$, so $\rho(M_{\nu+1,0}) = \delta_0$.

Let $\mathcal{W} = \mathcal{W}_{\nu,0}$ and $\mathcal{V} = \mathcal{V}_{\nu,0}$ have lengths γ_0 and γ_1.

Claim 3. $\mathcal{M}^{\mathcal{W}}_{\gamma_0} = \mathcal{M}^{\mathcal{V}}_{\gamma_1} = M_{\nu,0}$; moreover, the branches of $\mathcal{W}$ and $\mathcal{V}$ to γ_0 and γ_1 do not drop.

Proof. Neither side can iterate to a proper initial segment of $M_{\nu,0}$ because $\langle \nu, 0\rangle$ is relevant. Neither side can iterate strictly past $M_{\nu,0}$ because $\langle \nu, 0\rangle$ is exact. $\square$

Let $\eta_0 \leq_W \gamma_0$ and $\eta_1 \leq_V \gamma_1$ be the roots of the two trees below γ_0 and γ_1. Let
$$i_0 \colon Q \to \mathcal{M}^{\mathcal{W}}_{\eta_0} \text{ and } i_1 \colon Q \to \mathcal{M}^{\mathcal{V}}_{\eta_1}$$

be the embeddings given by the fact that $Z_0 = \mathrm{Th}^{\mathcal{M}^{\mathcal{W}}_{\eta_0}}(\delta_0) = \mathrm{Th}^{\mathcal{M}^{\mathcal{V}}_{\eta_1}}(\delta_0)$. These are just the lifts under $i^{\mathcal{U}}_{0,\tau_0}$ of the branch embeddings $i^{\mathcal{S}}_{0,\eta_0}$ and $i^{\mathcal{S}}_{0,\eta_1}$. We have that

$$i^{\mathcal{W}}_{\eta_0,\gamma_0} \circ i_0 = i^{\mathcal{V}}_{\eta_1,\gamma_1} \circ i_1,$$

since both embeddings are the embedding given by Q being the transitive collapse of $\mathrm{Hull}^{M_{\nu,0}}(\delta_0)$.

We now get a contradiction using the hull and definability properties in $M_{\nu,0}$ as usual.

Definition 7.22 *For M an lpm, we say that M has the* definability property at α *iff α is first order definable over M from some ordinals $b \in [\alpha]^{<\omega}$, and write* $\mathrm{Def}(M,\alpha)$ *in this case. We say that M has the* hull property at α *iff whenever $A \subset \alpha$ and $A \in M$, there is a $B \in M$ such that B is definable over M from some $b \in [\alpha]^{<\omega}$, and $B \cap \alpha = A$. We write* $\mathrm{Hp}(M,\alpha)$ *in this case.*

Claim 4. $\eta_0 = \eta_1$.

Proof. Suppose otherwise. Let

$$j_0 \colon \mathcal{M}^{\mathcal{S}}_{\eta_0} \to \mathrm{Ult}(\mathcal{M}^{\mathcal{S}}_{\eta_0}, E) = \mathcal{M}^{\mathcal{W}}_{\eta_0},$$

and

$$j_1 \colon \mathcal{M}^{\mathcal{S}}_{\eta_1} \to \mathrm{Ult}(\mathcal{M}^{\mathcal{S}}_{\eta_1}, E) = \mathcal{M}^{\mathcal{V}}_{\eta_1}$$

be the canonical embeddings. Suppose first that η_0 and η_1 are incomparable in $\mathcal{S}$, and let $F = E^{\mathcal{S}}_{\alpha}$ and $G = E^{\mathcal{S}}_{\beta}$, where $\alpha+1 \le_S \eta_0$, $\beta+1 \le_S \eta_1$, $\alpha \neq \beta$, and $S\text{-pred}(\alpha+1) = S\text{-pred}(\beta+1) = \xi$. We may assume $\mathrm{lh}(F) < \mathrm{lh}(G)$, or equivalently, $\alpha < \beta$. Let $\lambda = \sup\{\lambda(E^{\mathcal{S}}_{\nu}) \mid \nu+1 \le_S \xi\}$. Letting $\kappa_0 = \mathrm{crit}(F)$, we have

$$\kappa_0 = \text{ least } \mu \geq \lambda \text{ such that } \neg\mathrm{Def}(\mathcal{M}^{\mathcal{S}}_{\eta_0}, \mu).$$

Because the generators of j_0 (i.e. the generators of E) are contained in δ_0, we get

$$\begin{aligned} j_0(\kappa_0) &= \text{ least } \mu \geq j_0(\lambda) \text{ such that } \neg\mathrm{Def}(\mathcal{M}^{\mathcal{W}}_{\eta_0}, \mu) \\ &= \text{ least } \mu \geq j_0(\lambda) \text{ such that } \neg\mathrm{Def}(M_{\nu,0}, \mu). \end{aligned}$$

To see the first line, note that $\neg\mathrm{Def}(\mathcal{M}^{\mathcal{S}}_{\eta_0}, \kappa_0)$ because F was used on the branch to η_0, and j_0 is fully elementary so it preserves this. On the other hand, and $\mu < j_0(\kappa_0)$ is of the form $j_0(f)(a)$, where f is definable over $\mathcal{M}^{\mathcal{S}}_{\xi}$ from ordinals $< \lambda$, and $a \in [\delta_0]^{<\omega}$. The second line comes from using $i^{\mathcal{W}}_{\eta_0,\gamma_0}$ to move up to $\mathcal{M}^{\mathcal{W}}_{\gamma_0} = M_{\nu,0}$. Note for this that $j_0(\kappa_0) < j_0(\mathrm{lh}(F)) = \rho_\alpha$, and $\rho_\alpha \leq \mathrm{crit}(i^{\mathcal{W}}_{\eta_0,\gamma_0})$ because $\alpha < \eta_0$. Similarly, letting $\kappa_1 = \mathrm{crit}(G)$, we get

$$\begin{aligned} j_1(\kappa_1) &= \text{ least } \mu \geq j_1(\lambda) \text{ such that } \neg\mathrm{Def}(\mathcal{M}^{\mathcal{V}}_{\eta_1}, \mu) \\ &= \text{ least } \mu \geq j_1(\lambda) \text{ such that } \neg\mathrm{Def}(M_{\nu,0}, \mu). \end{aligned}$$

So $j_0(\kappa_0) = j_1(\kappa_1)$. But $\kappa_0, \kappa_1 < \mathrm{lh}(F)$, and $j_0 \restriction (\mathrm{lh}(F)+1) = j_1 \restriction (\mathrm{lh}(F)+1)$, so $\kappa_0 = \kappa_1$.

It is not hard to see that

$$\mathrm{lh}(F) = \text{ least cardinal } \mu > \kappa_0 \text{ such that } \mathrm{Hp}(\mathcal{M}^{\mathcal{S}}_{\eta_0}, \mu),$$

and

$$\mathrm{lh}(G) = \text{ least cardinal } \mu > \kappa_0 \text{ such that } \mathrm{Hp}(\mathcal{M}^{\mathcal{S}}_{\eta_1}, \mu).$$

Here cardinals are in the sense of $\mathcal{M}^{\mathcal{S}}_{\eta_0}$ and $\mathcal{M}^{\mathcal{S}}_{\eta_0}$, of course.[37] Using $i^{\mathcal{W}}_{\eta_0,\gamma_0} \circ j_0$ and $i^{\mathcal{V}}_{\eta_1,\gamma_1} \circ j_1$ to move up to $M_{\nu,0}$, and considering the hull property there, we get as above that $j_0(\mathrm{lh}(F)) = j_1(\mathrm{lh}(G))$. But $j_0(\mathrm{lh}(F) = j_1(\mathrm{lh}(F))$, so $\mathrm{lh}(F) = \mathrm{lh}(G)$. However, G was used strictly after F in $\mathcal{S}$, so $\mathrm{lh}(F) < \mathrm{lh}(G)$, contradiction.[38]

We are left to consider the case $\eta_0 <_S \eta_1$. Let G be the extender used in $[0, \eta_1)_S$ and applied to $\mathcal{M}^{\mathcal{S}}_{\eta_0}$. Let $\kappa_1 = \mathrm{crit}(G)$, and let $\lambda = \sup\{\lambda(E^{\mathcal{S}}_\alpha) \mid \alpha + 1 \leq_S \eta_0\}$ be the set of generators of $\mathcal{M}^{\mathcal{S}}_{\eta_0}$. Then again,

$$\begin{aligned} j_1(\kappa_1) &= \text{ least } \mu \geq j_1(\lambda) \text{ such that } \neg\mathrm{Def}(\mathcal{M}^{\mathcal{V}}_{\eta_1}, \mu) \\ &= \text{ least } \mu \geq j_1(\lambda) \text{ such that } \neg\mathrm{Def}(\mathcal{M}^{\mathcal{V}}_{\gamma_1}, \mu). \end{aligned}$$

Note that γ_0 is stable, and η_0 is unstable, so $\eta_0 <_W \gamma_0$. Let F be the extender used in $[\eta_0, \gamma_0)_W$ and applied to $\mathcal{M}^{\mathcal{W}}_{\eta_0}$. Let

$$\kappa_0 = \mathrm{crit}(F).$$

If $\kappa_0 < j_1(\lambda)$, then $\kappa_0 < \rho_\alpha$ for some $\alpha < \eta_0$, so F should have been applied to an earlier model of $\mathcal{W}$. Thus $j_1(\lambda) \leq \kappa_0$, and since $\mathcal{M}^{\mathcal{W}}_{\eta_0}$ has the definability property everywhere above $j_1(\lambda)$, using $i^{\mathcal{W}}_{\eta_0,\gamma_0}$ we see that κ_0 is the least $\mu \geq j_1(\lambda)$ such that $\neg\mathrm{Def}(M_{\nu,0}, \mu)$. Thus

$$\kappa_0 = j_1(\kappa_1).$$

But $F = E^{\mathcal{W}}_\eta$ for some $\eta \geq \theta$, so

$$j_1(\mathrm{lh}(G)) < \sup_{\alpha<\theta} \rho_\alpha < \lambda(F).$$

An easy induction shows that $\mathcal{M}^{\mathcal{W}}_\eta$ does not project strictly below $\sup_\alpha < \theta\rho_\alpha$, so we get that $F \restriction j_1(\mathrm{lh}(G)) \in \mathrm{Ult}(\mathcal{M}^{\mathcal{W}}_{\eta_0}, F)$, so the hull property fails in $\mathrm{Ult}(\mathcal{M}^{\mathcal{W}}_{\eta_0}, F)$ at $j_1(\mathrm{lh}(G))$. Moving up by $i^{\mathcal{W}}_{\eta_0,\gamma_0}$, the hull property fails in $M_{\nu,0}$ at $j_1(\mathrm{lh}(G))$.

However, $\mathcal{M}^{\mathcal{S}}_{\xi_1}$ does have the hull property at $\mathrm{lh}(G)$. This gives $\mathrm{Hp}(M^{\mathcal{V}}_{\eta_1}, j_1(\mathrm{lh}(G)))$, and thus $\mathrm{Hp}(M_{\nu,0}, j_1(\mathrm{lh}(G))$, noting here that $\mathrm{crit}(i^{\mathcal{V}}_{\eta_1,\gamma_1}) \geq j_1(\mathrm{lh}(G))$. This is a con-

[37] Proof: for μ a cardinal such that $\kappa_0 < \mu < \mathrm{lh}(F)$, $F \restriction \mu$ yields a subset of μ that is not definable in $\mathcal{M}^{\mathcal{S}}_{\alpha+1} = \mathrm{Ult}(\mathcal{M}^{\mathcal{S}}_\xi, F)$ from ordinals $< \mu$, as otherwise the factor embedding would show $F \restriction \mu$ is in its own ultrapower. On the other hand, every point in $\mathcal{M}^{\mathcal{S}}_{\alpha+1}$ is definable from ordinals $< \mathrm{lh}(F)$. Since $\mathrm{crit}(i^{\mathcal{S}}_{\alpha+1,\eta_0}) > \mathrm{lh}(F)$, we get the first line displayed. The second is proved in parallel fashion.

[38] We could also identify $\mathrm{lh}(F)$ as the least ordinal $> \kappa_0$ definable in $\mathcal{M}^{\mathcal{S}}_{\eta_0}$ from ordinals $< \kappa_0$. This uses that $\mathrm{lh}(F)$ is not a critical point in $\mathcal{T}$, which follows from niceness. That would let us avoid the hull property in proving 7.18. The hull property seems to be needed in proving the plus-two version of 7.18.

tradiction. □

Claim 5. $\eta_0 < \theta$.

Proof. Otherwise $\eta_0 = \eta_1 = \theta$. Let F be the first extender used in $b - c$ and G the first extender used in $c - b$. We get a contradiction just as we did in the proof of Claim 4, in the case η_0 and η_1 were S-incomparable. □

Now let s be the increasing enumeration of the extenders used in $(\eta_0, \gamma_0)_W$ and t the increasing enumeration of the extenders used in $(\eta_0, \gamma_1)_V$. We show by induction on ξ that $s(\xi) = t(\xi)$. For given that $s \restriction \xi = t \restriction \xi$, we have that $\mathrm{Ult}(\mathcal{M}_{\eta_0}^{\mathcal{W}}, s \restriction \xi)$ is pointwise definable from $\sup_{\alpha < \xi} \lambda(s(\alpha))$, so $s(\xi)$ is the least whole initial segment of the extender of the natural embedding from $\mathrm{Ult}(\mathcal{M}_{\eta_0}^{\mathcal{W}}, s \restriction \xi)$ to $M_{\nu,0}$. $t(\xi)$ is the least whole initial segment of the same extender, so $s(\xi) = t(\xi)$.

Thus $s = t$. But this implies that γ_0 and γ_1 are unstable, a contradiction. That completes the proof of Theorem 7.18. □

8 HOD in the derived model of a hod mouse

In this chapter, we show that if D is the derived determinacy model associated to a hod pair (M, Σ), then HOD^D is a least branch premouse. This is Theorem 8.8 below. The proof also shows that HOD^D is an initial segment of an iterate of M. This implies that, under an iterability hypothesis, there are determinacy models whose HOD has a fine structure, and yet is rich enough to satisfy "there is a subcompact cardinal". This is Theorem 8.11 below.

We must assume here some of the basic facts about universally Baire sets, homogeneously Suslin sets, and derived determinacy models. The material covered in [52] is more than sufficient. See also [17].

We show in section 5 that reasonably closed hod mice satisfy $\mathsf{V} = \mathsf{K}$, in a certain natural sense. We then close the chapter with a short survey of further results on the structure of HOD in determinacy models that have been proved by the methods of this book.

8.1 Generic interpretability

We shall need the following *generic interpretability* theorem. Its proof follows the same basic outline as Sargsyan's proof of the corresponding fact for rigidly layered hod pairs below LSA.(See [30] and [32].) [39]

Theorem 8.1 *(Generic interpretability) Assume AD^+, and let (P, Σ) be an lbr hod pair with scope HC, and such that Σ is coded by a Suslin-co-Suslin set of reals. Let*

$$P \models \mathsf{ZFC}^- + \delta \text{ is Woodin;}$$

then there is a term $\tau \in P$ such that whenever $i\colon P \to Q$ is the iteration map associated to a non-dropping P-stack s by Σ, and g is $\mathrm{Col}(\omega, < i(\delta))$-generic over Q, then

$$i(\tau)^g = \Sigma_{s,<i(\delta)} \restriction HC^{Q[g]}.$$

Proof. For $\xi < \eta < \delta$ and $k < \omega$, we shall define a term $\tau_{\xi,k,\eta}$ such that whenever g is P-generic over $\mathrm{Col}(\omega, \eta)$, then $\tau^g_{\xi,k,\eta} = \Sigma_{\langle \xi,k \rangle} \restriction \mathrm{HC}^{P[g]}$. We then take τ to be the join of the $\tau_{\xi,k,\eta}$. Clearly then $\tau^g = \Sigma_{<\delta} \restriction \mathrm{HC}^{P[g]}$ whenever g is $\mathrm{Col}(\omega, < \delta)$ generic over P. It will be clear that this property of τ is preserved by Σ-iteration.

[39] The main difference is that our mice have extenders overlapping Woodin cardinals, which means we can't use Q-structures to determine Σ on small generic extensions of (M, Σ) in the way Sargsyan did. It is at this point that we use Theorem 7.18 on UBH in $M[g]$. The proof of that theorem used a phalanx comparison, as any proof of generic interpretability at the level of extenders overlapping Woodin cardinals would probably need to do.

So fix $\xi < \eta < \delta$ and $k < \omega$. Let g be P-generic over $\mathrm{Col}(\omega, \eta)$. We shall define $\Sigma_{\langle \xi,k\rangle} \restriction \mathrm{HC}^{P[g]}$ from $\xi, k, P|\delta$ and g. The definition will be uniform in g, giving us the desired term.

Let $\mu = (\eta^+)^P$. We may assume that μ is a cutpoint of P. For if not, let E be the first extender on the P-sequence such that $\mathrm{crit}(E) < \mu < \mathrm{lh}(E)$, and set $Q = \mathrm{Ult}(P, E)$. Then μ is a cutpoint of Q, $\mathrm{HC}^{P[g]} = \mathrm{HC}^{Q[g]}$, and by strategy coherence, $\Sigma_{\langle E\rangle, \langle \xi,k\rangle} = \Sigma_{\langle \xi,k\rangle}$. A definition of $\Sigma_{\langle E\rangle,\langle \xi,k\rangle} \restriction \mathrm{HC}^{Q[g]}$ from $Q|i_E(\delta), \xi, k$, and g will then give the desired definition of $\Sigma_{\langle \xi,k\rangle} \restriction \mathrm{HC}^{P[g]}$. So we assume μ is a cutpoint of P.

Let w be the canonical wellorder of $P|\delta$, and working in P, let $\mathbb{C}$ be a w-construction of length δ that is above μ, and such that

(i) Each $F_\nu^{\mathbb{C}}$ is a P-nice extender, and

(ii) $\mathbb{C}$ adds extenders whenever possible, subject to (i).

Our background condition has the consequence that for any $\mathcal{T}$ on $M_{\nu,k}^{\mathbb{C}}$, the iteration tree $\mathcal{T}^*$ on P that is part of $\mathrm{lift}(\mathcal{T}, M_{\nu,k}, \mathbb{C})$ is a P-nice tree. So by 7.18, if $\mathcal{T} \in P[g]$, then UBH holds for $\mathcal{T}^*$.

We also have CBH for P-nice trees $\mathcal{S}$ on P such that $\mathcal{S} \in P$. This is because $\mathcal{S}$ induces naturally a tree $\mathcal{S}^+$ with the same tree order that uses extenders from the P-sequence. We have that $b = \dot{\Sigma}^P(\mathcal{S}^+)$ is defined, in P, and wellfounded as a branch of $\mathcal{S}^+$. But then b is wellfounded as a branch of $\mathcal{S}$.[40] Thus in P, the $\Omega_{\nu,l}^{\mathbb{C}}$ are total. In P, they are induced by $\dot{\Sigma}^P$, but $\dot{\Sigma}^P \subseteq \Sigma$, and Σ is total on V. So Σ induces a total-on-V strategy $\Omega_{\nu,l}^*$ for $M_{\nu,l}$ such that $\Omega_{\nu,l}^{\mathbb{C}} \subseteq \Omega_{\nu,l}^*$. The $\Omega_{\nu,l}^*$ are Suslin-co-Suslin in V because Σ is. Since they are induced by Σ, they have strong hull condensation and normalize well. In fact, each $(M_{\nu,l}^{\mathbb{C}}, \Omega_{\nu,l}^*)$ is an lbr hod pair in V. Moreover, $V \models \mathsf{AD}^+$, so in V we can carry out the comparisons needed to see each $(M_{\nu,l}, \Omega_{\nu,l}^*)$ has a core. Thus $(M_{\nu,l}, \Omega_{\nu,l})$ has a core in P, and $\mathbb{C}$ does not break down in P.

Claim 1. In P, there is a $\nu < \delta$ such that $(P|\langle \xi, k\rangle, \dot{\Sigma}_{\langle \xi,k\rangle}^P)$ iterates to $(M_{\nu,k}^{\mathbb{C}}, \Omega_{\nu,k}^{\mathbb{C}})$.

Proof. Suppose not. Working in P, we claim that for all $\langle \nu, l\rangle$ such that $\nu < \delta$, $(P|\langle \xi, k\rangle, \dot{\Sigma}_{\langle \xi,k\rangle}^P)$ iterates strictly past $(M_{\nu,l}, \Omega_{\nu,l})$. This almost follows from the comparison theorem 6.45. However, to simply quote 6.45, we would need to know that $\dot{\Sigma}_{\langle \xi,k\rangle}^P$ is $< \delta$ universally Baire in P. That is part of the theorem we are proving now. Nevertheless, the proof of 6.45 works here. The consequence of universal

[40] With very little work, one can show that the trivial completion of $E_\alpha^{\mathcal{S}}$ is on the sequence of $\mathcal{M}_\alpha^{\mathcal{S}}$, so that we can take $\mathcal{S}^+ = \mathcal{S}$.

Baireness we need is just that if $\mathcal{T}$ is a normal tree by $\dot{\Sigma}^P_{\langle\xi,k\rangle}$, and $i\colon P \to Q$ is an iteration map by Σ with $\text{crit}(i) > \xi$, then $i(\mathcal{T})$ is by $\dot{\Sigma}^P_{\langle\xi,k\rangle}$. This much is true by the strategy coherence of Σ.

But then $(P|\langle\xi,k\rangle, \dot{\Sigma}^P_{\langle\xi,k\rangle})$ iterates past $M_{\delta,0}$ in P. This contradicts universality at Woodin cardinals, Theorem 2.53. □

Let $\mathcal{T}$ be the normal tree by $\dot{\Sigma}^P_{\langle\xi,k\rangle}$ whose last model is $M^{\mathbb{C}}_{\nu,k}$ given by claim 1, and let $i\colon P|\langle\xi,k\rangle \to M_{\nu,k}$ be its canonical embedding.

Claim 2. $\Sigma_{\mathcal{T},M_{\nu,k}} = \Omega^*_{\nu,k}$.

Proof. The proof that the two strategies agree on all trees in P actually shows that they agree on all trees in V. [Let $\mathcal{U}$ be by both strategies, and $b = \Omega^*_{\nu,k}(\mathcal{T})$. Let $\mathcal{U}^*$ be the tree according to Σ that is part of $\text{lift}(\mathcal{U}, M_{\nu,k}, \mathbb{C})$; again, we do not need $\mathcal{U} \in P$ to make sense of lifting. Then $W(\mathcal{T}, \mathcal{U}^\frown b)$ is a psuedo-hull of $i^{\mathcal{U}^*}_b(\mathcal{T})$ by our previous calculations. However, $i^{\mathcal{U}^*}_b(\mathcal{T})$ is by $\Sigma_{\langle\xi,k\rangle}$ by strategy coherence, so $W(\mathcal{T}, \mathcal{U}^\frown b)$ is by $\Sigma_{\langle\xi,k\rangle}$ because $\Sigma_{\langle\xi,k\rangle}$ normalizes well, so $b = \Sigma_{\mathcal{T},M_{\nu,k}}(\mathcal{U})$.] □

Now let $\mathcal{U}$ be a normal tree on $P|\langle\xi,k\rangle$ of limit length that is according to $\Sigma_{\langle\xi,k\rangle}$, and such that $\mathcal{U}$ is countable in $P[g]$. We wish to find $\Sigma_{\langle\xi,k\rangle}(\mathcal{U})$ in $P[g]$, and define it from the relevant parameters. But $\Sigma_{\langle\xi,k\rangle}$ is pullback consistent, so

$$\begin{aligned}\Sigma_{\langle\xi,k\rangle}(\mathcal{U}) = b &\text{ iff } \Sigma_{\mathcal{T},M_{\nu,k}}(i\mathcal{U}) = b\\ &\text{ iff } \Omega^*_{\nu,k}(i\mathcal{U}) = b.\end{aligned}$$

So it will be enough to show

Claim 3. If $\mathcal{S}$ is countable in $P[g]$, of limit length, and by $\Omega^*_{\nu,k}$, and $b = \Omega^*_{\nu,k}(\mathcal{S})$, then $b \in P[g]$. Moreover, b is uniformly definable over $P[g]$ from $\mathcal{S}$ and $\mathbb{C}$.

Proof. Let $\mathcal{S}^*$ be the P-nice tree on P that it part of $\text{lift}(\mathcal{S}, M_{\nu,k}, \mathbb{C})$. It is enough to show $b \in P[g]$, and to define there from the relevant parameters, uniformly.

We know from 7.18 that in $P[g]$, $\mathcal{S}^*$ has at most one cofinal, wellfounded branch. Since all critical points in $\mathcal{S}^*$ are strictly above μ, we can think of $\mathcal{S}^*$ as a P-nice tree on $P[g]$. Then by [19], since $\mathcal{S}^*$ is countable in $P[g]$, it has exactly one cofinal wellfounded branch b in $P[g]$. Moreover, again by [19], $\mathcal{S}^*$ is continuously illfounded off b. It follows that $b = \Sigma(\mathcal{S}^*)$, and therefore $b = \Omega^*_{\nu,k}(\mathcal{S})$, as desired. □

This completes the proof of Lemma 8.1. □

8.2 Mouse limits

Assume AD^+, and let (M, Ω) be a mouse pair with scope HC. Suppose s and t are stacks by Ω on M with last models P and Q such that M-to-P and M-to-Q do not drop. By 6.54 and Dodd-Jensen, we can then find stacks u and v by Ω_s and Ω_t with a common last model such that neither stack drops getting to N, and such that $\Omega_{s^\frown u} = \Omega_{t^\frown v}$. By Dodd-Jensen, for any such s, t, u, and v, $i_u \circ i_s = i_v \circ i_t$, where these are the the iteration maps in question. Thus we have a well-defined direct limit system.

Definition 8.2 *Let (P, Σ) be a mouse pair; then*

(1) $\mathcal{F}(P, \Sigma)$ is the collection of all (Q, Ψ) such that there is an P-stack s by Σ with last model Q, such that P-to-Q does not drop, and $\Psi = \Sigma_s$.

(2) For $(Q, \Psi) \in \mathcal{F}(P, \Sigma)$, $\pi_{(P,\Sigma),(Q,\Psi)} \colon P \to Q$ is the unique iteration map given by any and all stacks by Σ.

(3) $M_\infty(P, \Sigma)$ is the direct limit of $\mathcal{F}(P, \Sigma)$ under the $\pi_{(Q,\Psi),(R,\Phi)}$.

(4) $\pi_{(P,\Sigma),\infty} \colon P \to M_\infty(P, \Sigma)$ is the direct limit map.

Of course, $M_\infty(P, \Sigma) = M_\infty(Q, \Psi)$ for all $(Q, \Psi) \in \mathcal{F}(P, \Sigma)$. Clearly, if $(P, \Sigma) \equiv^* (Q, \Psi)$, then $M_\infty(P, \Sigma) = M_\infty(Q, \Psi)$.[41] Thus $M_\infty(P, \Sigma) \in$ HOD, being definable from the rank of (P, Σ) in the mouse order. In fact, this is true uniformly, in the sense that letting

(1) $m_e(\alpha) = X$ iff there is a pure extender pair (P, Σ) of mouse rank α such that $X = M_\infty(P, \Sigma)$, and

(2) $m_h(\alpha) = X$ iff here is a least branch hod pair (P, Σ) of mouse rank α such that $X = M_\infty(P, \Sigma)$,

we have

$$m_e, m_h \in \text{HOD}.$$

Assuming $\mathsf{AD}_\mathbb{R} + \mathsf{HPC}$, one can show that HOD $= L[m_h]$. This is not a very useful representation however, as it does not seem to lead to a fine structure for HOD. We do not know whether $L[m_e]$ has any natural identity, assuming say $\mathsf{AD}_\mathbb{R} + \mathsf{LEC}$.

Another simple fact worth noting is

Proposition 8.3 *(AD^+) Let (P, Σ) and (P, Ψ) be mouse pairs with scope HC such that (P, Σ) is mouse-equivalent to (P, Ψ) and $\pi_{(P,\Sigma),\infty} = \pi_{(P,\Psi),\infty}$; then $\Sigma = \Psi$.*

[41] The converse is also true; see [63][Proposition 2.2].

Proof. By our comparison theorems, the two pairs have a common iterate (Q, Ω). Let $i\colon (P, \Sigma) \to (Q, \Omega)$ and $j\colon (P, \Psi) \to (Q, \Omega)$ be the two iteration maps. Then

$$\begin{aligned}\pi_{(Q,\Omega),\infty} \circ i &= \pi_{(P,\Sigma),\infty} \\ &= \pi_{(P,\Psi),\infty} \\ &= \pi_{(Q,\Omega),\infty} \circ j.\end{aligned}$$

This implies that $i = j$. But then by pullback consistency, $\Sigma = \Omega^i = \Omega^j = \Psi$, as desired. □

Thus assuming AD^+, every mouse pair with scope HC is ordinal definable from a countable sequence of ordinals. On the other hand, a mouse pair (P, Σ) such that $\theta_0 \leq o(M_\infty(P, \Sigma))$ cannot be ordinal definable from a real.

In order to compute HOD, we must relate different mouse limits. The concept of fullness helps do that.

Definition 8.4 *Assume* AD^+*, and let* (P, Σ) *be a mouse pair with scope HC. We say that* (P, Σ) *is* full *iff* Σ *is Suslin-co-Suslin, and*

(a) $P \models \mathsf{ZFC}^-$*,* P *has a largest cardinal* δ*, and* $k(P) = 0$*, and*

(b) whenever s *is a* P*-stack by* Σ *with last model* Q*, and the branch* P*-to-*Q *of* s *does not drop, and* $i_s\colon P \to Q$ *is the iteration map, then there is no mouse pair* (R, Φ) *such that* Φ *is Suslin-co-Suslin,* $Q \trianglelefteq^{\mathrm{ct}} R$*,* $\rho(R) \leq i_s(\delta)$*, and* $\Phi_Q = \Sigma_s$*.*

This notion is sometimes called *mouse-fullness.*[42] [43] The following lemma explains its importance in relating mouse limits to one another.

Lemma 8.5 *Let* (P, Σ) *and* (N, Ψ) *be mouse pairs of the same type such that* $(P, \Sigma) \leq^* (N, \Psi)$*, and suppose that* (P, Σ) *is full; then letting* $\gamma = o(M_\infty(P, \Sigma))$

$$M_\infty(P, \Sigma) = M_\infty(N, \Psi)|\gamma,$$

and γ *is a successor cardinal cutpoint of* $M_\infty(N, \Psi)$*.*

Proof. Let (P, Σ) be full, and suppose that $(P, \Sigma) \leq^* (N, \Psi)$. Comparing the two leads to (Q, Λ) a nondropping, normal iterate of (P, Σ) and (R, Φ) a normal iterate of (N, Ψ) such that $(Q, \Lambda) \trianglelefteq (R, \Phi)$. By perhaps taking one additional ultrapower on the N side, we can arrange that Q is a cutpoint of R. But then $o(Q) \leq \rho(R)$,

[42] It is customary to define fullness for P itself, and then say that Σ is *fullness-preserving* iff (P, Σ) is full in the sense of our definition.

[43] *OD-fullness* is the intensionally stronger requirement that whenever $(Q, \Psi_{s,Q})$ is as in (b) of 8.4, and A is a bounded subset of $o(Q)$ that is ordinal definable from $(Q, \Psi_{s,Q})$, then $A \in Q$. Under an appropriate mouse capturing hypothesis, the two are equivalent.

and consequently N-to-R does not drop in $\mathcal{U}$. Because the iterations did not drop, we have $M_\infty(P,\Sigma) = M_\infty(Q,\Lambda)$ and $M_\infty(N,\Psi) = M_\infty(R,\Phi)$.

But $o(Q)$ is a successor cardinal cutpoint of R, and $o(Q) \leq \rho(R)$. Also, $\Lambda = \Phi_Q$. It follows then that $M_\infty(Q,\Lambda)$ is a successor cardinal cutpoint of $M_\infty(R,\Phi)$, and that $o(M_\infty(Q,\Lambda) \leq \rho(M_\infty(R,\Phi)$. $\square$

Corollary 8.6 *(*AD^+*) Let* (P,Σ) *and* (N,Ψ) *be full mouse pairs; then*

$$\begin{aligned}(P,\Sigma) \leq^* (N,\Psi) &\text{ iff } o(M_\infty(P,\Sigma)) \leq o(M_\infty(N,\Psi))\\ &\text{ iff } M_\infty(P,\Sigma) \trianglelefteq^{ct} M_\infty(N,\Psi).\end{aligned}$$

8.3 HOD as a mouse limit

We shall show that in the derived model of a hod mouse, HOD can be represented as a mouse limit.

We shall need the following notions associated to derived models. Working in ZFC, suppose that λ is a limit of Woodin cardinals. Let g be $\mathrm{Col}(\omega,<\lambda)$-generic over V. We set

$$\mathbb{R}^*_g = \bigcup\{\mathbb{R}\cap V[g(\restriction\omega\times\alpha)] \mid \alpha<\lambda\},$$

and

$$\mathrm{Hom}^*_g = \{p[T]\cap\mathbb{R}^*_g \mid \exists\alpha<\lambda(V[g\restriction(\omega\times\alpha)] \models T \text{ is } <\lambda\text{-absolutely complemented } \}.$$

The symmetry of the forcing tells us that $\mathbb{R}^*_g = \mathbb{R}\cap L(\mathbb{R}^*_g, \mathrm{Hom}^*_g)$. The sets in Hom^*_g are those that have $<\lambda$-homogeneously Suslin representations in some intermediate collapse, which is is equivalent to having a $<\lambda$-universally Baire representation in some intermediate collapse because λ is a limit of Woodin cardinals. Homogeneous Suslinity implies determinacy for sets in Hom^*_g, and with more work, that every set in Hom^*_g has a scale in Hom^*_g. Stationary tower forcing helps us pass from absolute definitions to absolutely complementing trees. In the end, we get

Theorem 8.7 (Woodin) *(*ZFC*) Suppose* λ *is a limit of Woodin cardinals, and let* g *be* $\mathrm{Col}(\omega,<\lambda)$*-generic over* V*; then*

$$L(\mathbb{R}^*_g, Hom^*_g) \models \mathsf{AD}^+,$$

and

$$A \in Hom^*_g \Leftrightarrow (L(\mathbb{R}^*_g, Hom^*_g) \models A \textit{ is Suslin and co-Suslin}),$$

for all $A \subseteq \mathbb{R}^*_g$.

The theorem was proved by Woodin in the late 1980s, as part of a more general theorem known as the Derived Model Theorem. See for example [52].

We want to look at the derived model construction in the case that our ground model is a least branch hod mouse. What we get is

Theorem 8.8 *Assume AD^+, and let (M, Ψ) be an lbr hod pair with scope HC, and such that Ψ is coded by a Suslin-co-Suslin set of reals. Suppose*

$$M \models \mathsf{ZFC} + \lambda \textit{ is a limit of Woodin cardinals.}$$

Let g be $\mathrm{Col}(\omega, <\lambda)$-generic over M; then

$$L(\mathbb{R}^*_g, Hom^*_g) \models \mathsf{AD}_{\mathbb{R}}.$$

and

(a) if λ is a limit of cutpoints in M, then then there is an iteration map $i \colon M \to M_\infty(s)$ coming from a stack s on $M|\lambda$ by Ψ such that

$$HOD^{L(\mathbb{R}^*_g, Hom^*_g)} = L[M_\infty(s)|i(\lambda)],$$

and

(b) if $\kappa < \lambda$ is least so that $o(\kappa) \geq \lambda$ in M, then there is an iteration map $i \colon M \to M_\infty(s)$ coming from a stack s on $M|\lambda$ by Ψ such that

$$HOD^{L(\mathbb{R}^*_g, Hom^*_g)} = L[M_\infty(s)|i(\kappa)].$$

Proof. The techniques here are pretty well known. Let (M, Ψ) and g be as in the hypotheses. For $\nu < \lambda$, let

$$\Psi^g_{\langle \nu, k\rangle} = \Psi_{\langle \nu, k\rangle} \restriction \mathrm{HC}^{M(\mathbb{R}^*_g)}.$$

Fixing a coding of elements of HC by reals, we can identify $\Psi^g_{\langle \nu, k\rangle}$ with a subset of $\mathbb{R}^*_g$. Our first two claims say that the $\Psi^g_{\langle \nu, k\rangle}$ witness that HPC holds in $L(\mathbb{R}^*_g, \mathrm{Hom}^*_g)$.

Claim 1. If $\nu < \lambda$, then $\Psi^g_{\langle \nu, k\rangle} \in \mathrm{Hom}^*_g$.

Proof. Let $h = g \cap \mathrm{Col}(\omega, <\nu^+)$. In $M[h]$ we have, for each $\mu < \lambda$, a term τ such that for all l that are $\mathrm{Col}(\omega, \mu)$-generic over $M[h]$,

$$\tau^l = \Psi_{\langle \nu, k\rangle} \restriction \mathrm{HC}^{M[h][l]}.$$

For the specific such term τ given to us by Theorem 8.1, it is not hard to see that for all sufficiently large γ,

$$M[h] \models \text{ there are club many generically } \tau\text{-correct hulls of } V_\gamma.$$

That is, in $M[h]$, whenever N is countable and transitive, and

$$\pi \colon N[h] \to (M|\gamma)[h]$$

is elementary, and everything relevant is in $\mathrm{ran}(\pi)$, and

$$\pi(\langle \bar{\tau}, \bar{\mu}\rangle) = \langle \tau, \mu\rangle,$$

then for any l that is $\mathrm{Col}(\omega, \bar{\mu})$-generic over N,

$$\bar{\tau}^l = \Psi_{\langle \nu,k \rangle} \cap \mathrm{HC}^{N[l]}.$$

The proof of this is similar to the proof of Theorem 5.1 of [54]. Working in M, let $\mathbb{C}$ be the background construction and

$$i \colon M|\langle \nu, k\rangle \to M^{\mathbb{C}}_{\eta,k}$$

be the iteration map by $\Psi_{\langle \nu,k \rangle}$ that is described in τ. Let $\bar{\mathbb{C}} = \pi^{-1}(\mathbb{C})$ and $\bar{i} = \pi^{-1}(i)$, etc. So these are described in $\bar{\tau}$. Suppose $\mathcal{U}$ is according to $\bar{\tau}^l$. Let

$$\mathcal{W} = \mathrm{lift}^N(\bar{i}\mathcal{U})$$

be the nice tree on N that is given to us by $\bar{\tau}^l$. $\mathcal{W}$ is countable and nice in $N[h,l]$, so by 7.18, it picks unique cofinal wellfounded branches there. This implies that $\mathcal{W}$ is continuously illfounded off the branches it chooses. But then $\pi\mathcal{W}$ is continuously illfounded off the branches it chooses, so $\pi\mathcal{W}$ is by Ψ. But lifting commutes with copying, so

$$\begin{aligned}\pi\mathcal{W} &= \pi\, \mathrm{lift}^N(\bar{i}\mathcal{U})\\ &= \mathrm{lift}^M((\pi \circ \bar{i})\mathcal{U})\\ &= \mathrm{lift}^M(i\mathcal{U}).\end{aligned}$$

Note here that π is the identity on the base model of $\mathcal{U}$, so $\pi \circ \bar{i}$ agrees with $\pi(\bar{i}) = i$ on the base model of $\mathcal{U}$. This gives the last equality.

So $\mathrm{lift}^M(i\mathcal{U})$ is by Ψ, and hence $i\mathcal{U}$ is by $(\Omega^{\mathbb{C}}_{\eta,k})^{h,l}$. But we saw in the proof of 8.1 that this means $i\mathcal{U}$ is by the tail strategy$(\Psi_{\langle \nu,k \rangle})_{\mathcal{T}, M^{\mathbb{C}}_{\eta,k}}$, where $\mathcal{T}$ is the tree giving rise to i. Since $\Psi_{\langle \nu,k \rangle}$ is pullback consistent, $\mathcal{U}$ is by $\Psi_{\langle \nu,k \rangle}$, as desired.

It is easy to go from a club of $< \lambda$-generically τ-correct hulls to a $< \lambda$-absoutely complemented tree projecting to τ^h whenever h is $< \lambda$-generic. (See [52].) This proves Claim 1. $\square$

Claim 2. The $\Psi^g_{\langle \nu,k \rangle}$, for $\nu < \lambda$ are Wadge-cofinal in Hom^*_g.

Proof. Let $\eta < \lambda$ and

$$M[g\restriction(\omega \times \eta)] \models \text{ } T \text{ and } T^* \text{ are } < \lambda\text{-absolute complements.}$$

Let $\eta < \delta < \lambda$, and $M \models \delta$ is Woodin. Let $\mu = (\delta^{++})^M$. Put $\pi \in \mathcal{I}$ iff there is a non-dropping, normal iteration tree $\mathcal{U}$ on $M|\mu$ such that

(i) $\mathcal{U}$ is by $\Psi^g_{\langle \mu,0 \rangle}$, with last model N,

(ii) all critical points in $\mathcal{U}$ are strictly above η, and

(iii) $\pi \colon M[g\restriction(\omega \times \eta)] \to N[g\restriction(\omega \times \eta)]$ is the lift of the iteration map.

Standard arguments show that for $x \in \mathbb{R}^*_g$,

$$x \in p[T] \Leftrightarrow \exists \pi \in \mathcal{I}(x \in p[\pi(T \cap (\omega \times \delta^{+,M}))]).$$

This shows that $p[T]$ is projective in $\Psi^g_{\langle\mu,0\rangle}$. This easily implies the claim. $\square$

Claim 3. Let η be a successor cardinal of M, and $\eta < \lambda$; then $(M|\eta, \Psi^g_{\langle\eta,0\rangle})$ is a full lbr hod pair in $L(\mathbb{R}^*_g, \mathrm{Hom}^*_g)$.

Proof. $(M|\eta, \Psi_{\langle\eta,0\rangle})$ is an lbr hod pair in V, so $(M|\eta, \Psi^g_{\langle\eta,0\rangle})$ is an lbr hod pair in $L(\mathrm{Hom}^*_g, \mathbb{R}^*_g)$. We must see that $(M|\eta, \Psi_{\langle\eta,0\rangle})$ is full. In short, this is true because non-dropping iterations of $M|\eta$ carry the rest of M along on top, and the resulting iterates of M can compute truth in the derived model of M by consulting their own derived models.[44]

Let us fill in our sketch. Suppose toward contradiction that in $L(\mathbb{R}^*_g, \mathrm{Hom}^*_g)$ we have

(i) an $M|\eta$-stack s by $\Psi_{\langle\eta,0\rangle}$ with last model Q, such that the branch $M|\eta$-to-Q of s does not drop, and

(ii) an lbr hod pair (R, Φ) such that Φ is Suslin-co-Suslin, $Q \trianglelefteq^{\mathrm{ct}} R$, $\rho(R) < o(Q)$, and $\Phi_Q = \Psi_{s,Q}$.

Let (R, Φ) be the minimal such pair in the mouse order, and let
$$T_R = \mathrm{Th}^R_{k+1}(\rho(R) \cup p(R)),$$
where $k = k(R)$, be the theory coding the core of R.

Since η is a cardinal of M, s is in fact an M-stack, and regarding it this way, it has a last model S such that $Q \trianglelefteq S$, and the branch M-to-S of s does not drop. Since $o(Q)$ is a cardinal of S and $\rho(R) < o(Q)$, if $T_R \in S$ then $T_R \in Q$. But then $\rho(R)^{+,R} < o(Q)$ because T_R collapses it, and $\rho^{+,R}(R)$ is not a cardinal of Q for the same reason. But $Q \trianglelefteq R$, contradiction. We conclude that $T_R \notin S$.

However, working in V now, we can find an $\mathbb{R}^*_g$-genericity iteration of $S|\lambda$ by Ψ_s so that all its critical points are strictly above $o(Q)$. Let W be the final model of this genericity iteration; then we have h being $\mathrm{Col}(\omega, <\lambda)$ generic over W so that
$$\mathbb{R}^*_h = \mathbb{R}^*_g.$$
Moreover, as in Claim 2, the strategies $(\Psi_s)^h_{\langle\nu,k\rangle}$ for $\nu < \lambda$ are Wadge cofinal in Hom^*_h, and clearly $(\Psi_s)^h_{\langle\nu,k\rangle} = (\Psi_s)^h_{\langle\nu,k\rangle}$. It follows that
$$\mathrm{Hom}^*_h = \mathrm{Hom}^*_g.$$
Thus we realized our derived model of M as a derived model of its iterate W.

We show that T_R is ordinal definable in $L(\mathbb{R}^*_g, \mathrm{Hom}^*_g)$ from Q and $(\Psi_Q)^h$. But by generic interpretability, $(\Psi_Q)^h$ is definable in $W(\mathbb{R}^*_h)$ from parameters in W. By

[44] We are showing that $(M|\eta, \Psi_{\langle\eta,0\rangle})$ is not just mouse-full, but OD-full. But we are in the derived model of a mouse, where the two are equivalent, so that is not surprising.

the homogeniety of the forcing, we then get that $T_R \in W$, and hence $T_R \in S$, contradiction.

So working in $L(\mathbb{R}^*_g, \mathrm{Hom}^*_g)$, let (P, Σ) be an lbr hod pair of minimal mouse rank such that $Q \trianglelefteq^{\mathrm{ct}} P$, $\Sigma_Q = \Psi_{s,Q}$, and $\rho(P) < o(Q)$. Let $T_P = \mathrm{Th}^P_{k+1}(\rho(P) \cup p(P))$. The following claim finishes our proof.

Subclaim 3.1. $T_P = T_R$.

Proof. We work in $L(\mathbb{R}^*_g, \mathrm{Hom}^*_g)$. Since (R, Φ) and (P, Σ) are mouse minimal with respect to the same property, they have a common iterate (N, Λ), via normal trees $\mathcal{T}$ and $\mathcal{U}$ that do not drop along their main branches. Because neither side drops, we have

$$k(R) = k(N) = k(P).$$

Let k be the common value. Let $i = i^{\mathcal{T}}$ and $j = i^{\mathcal{U}}$ be the two main branch embeddings. Because Q is a cutpoint on both sides, and $o(Q)$ is Σ_k-regular on both sides[45], we get that

$$i \restriction o(Q) = j \restriction o(Q).$$

But then $\rho(R) =$ least α such that $i(\alpha) \geq \rho(N) =$ least α such that $j(\alpha) \geq \rho(N)$. So $\rho(R) = \rho(P)$. Also

$$i(p(R)) = p(N) = j(p(P)).$$

Since i and j are elementary (hence Σ_{k+1} elementary), we get that $i``T_R = T_N = j``T_P$, so $T_R = T_P$. □

This proves Claim 3. □

We define in $L(\mathbb{R}^*_g, \mathrm{Hom}^*_g)$:

$$\mathcal{F} = \{(P, \Sigma) \mid (P, \Sigma) \text{ is a full lbr hod pair.}\}$$

For $(P, \Sigma), (Q, \Psi) \in \mathcal{F}$,

$$(P, \Sigma) \prec^* (Q, \Psi) \text{ iff } \exists (R, \Phi)[(R, \Phi) \trianglelefteq^{\mathrm{ct}} (Q, \Psi) \wedge (P, \Sigma) \text{ iterates to } (R, \Phi)].$$

If $(P, \Sigma) \prec^* (Q, \Psi)$, then

$$\pi_{(P,\Sigma),(Q,\Psi)} \colon P \to R \trianglelefteq^{\mathrm{ct}} Q$$

is the iteration map. By Dodd-Jensen, it is well-defined, that is, independent of the choice of stack witnessing that (P, Σ) iterates to some $(R, \Phi) \trianglelefteq^{\mathrm{ct}} (Q, \Psi)$. The π's commute, and $\prec^*$ is directed by Lemma 8.5, so we have a direct limit system. Set

$$M_\infty = \text{ direct limit of } (\mathcal{F}, \prec^*) \text{ under the } \pi_{(P,\Sigma),(Q,\Psi)},$$

and let

$$\pi_{(P,\Sigma),\infty} \colon P \to M_\infty$$

[45]Otherwise $\rho_k(R) < o(Q)$ or $\rho_k(P) < o(Q)$, contrary to minimality.

be the direct limit map. Another way to characterize M_∞ is that it is the lpm N of minimal height such that for all $(P,\Sigma)\in\mathcal{F}$, $M_\infty(P,\Sigma)\trianglelefteq^{\mathrm{ct}} M_\infty$. Our two definitions of $\pi_{(P,\Sigma),\infty}$ are consistent with one another..

Let us write

$$\begin{aligned}\Theta &= o(\mathrm{Hom}^*_g)\\ &= \sup\{|W| \mid W \text{ is a prewellorder of } \mathbb{R}^*_g \text{ in } \mathrm{Hom}^*_g\ \}.\end{aligned}$$

Θ is also the Wadge ordinal of Hom^*_g.

Claim 4. $o(M_\infty)\le\Theta$.

Proof. This follows immediately from Claim 1. □

Clearly $\Theta\le\theta^{L(\mathbb{R}^*_g,\mathrm{Hom}^*_g)}$. In fact

Claim 5. $o(M_\infty)=\Theta=\theta^{L(\mathbb{R}^*_g,\mathrm{Hom}^*_g)}$.

Proof. We need only show that $\theta^{L(\mathbb{R}^*_g,\mathrm{Hom}^*_g)}\le o(M_\infty)$. The proof is essentially due to G. Hjorth. (See [10].)

Let $\tau<\theta^{L(\mathbb{R}^*_g,\mathrm{Hom}^*_g)}$, and let $f\colon\mathbb{R}^*_g\to\tau$ be a surjection. f is ordinal definable in $L(\mathbb{R}^*_g,\mathrm{Hom}^*_g)$ from some $B\in\mathrm{Hom}^*_g$, and by Claim 2, we can take our B to be of the form $(\Psi^g_{\langle\eta,0\rangle},z)$ for some cardinal η of M and some real $z\in\mathbb{R}^*_g$. By amalagamating the f_z associated to all possible z, we can eliminate z from the definition.

So we can fix

$$B=\Psi^g_{\langle\eta,0\rangle},$$

where η is a cardinal of M, and

$$f\colon\mathbb{R}^*_g\to\tau$$

a surjection, and a formula $\varphi(u,v,w)$ and ordinal α such that

$$f(x)=\xi \text{ iff } L_\alpha(\mathbb{R}^*_g,\mathrm{Hom}^*_g)\models\varphi[x,\xi,B].$$

Let $M_0=\mathrm{Ult}(M,E)$, for E the first extender on M overlapping η, if there is one. Let $M_0=M$ otherwise. Let

$$\delta_0=\text{ least }\delta>\eta\text{ such that }M_0\models\delta\text{ is Woodin.}$$

So η and δ_0 are cutpoints of M_0. Letting $N=M_0|(\delta_0^+)^{M_0}$ and $\Phi=\Psi_{\langle E\rangle,N}$ or $\Phi=\Psi_N$ as appropriate, we have that $(N,\Phi)\in\mathcal{F}$. We shall show that

$$\pi_{(N,\Phi),\infty}(\delta_0)\ge\tau.$$

Remark 8.9 Let $\theta(B)$ be the sup of the lengths of $\mathrm{OD}(B)$ prewellorders of $\mathbb{R}$, in $L(\mathbb{R}^*_g,\mathrm{Hom}^*_g)$ of course. Since α and φ are arbitrary so far, we are showing that $\pi_{(N,\Phi),\infty}(\delta_0)\ge\theta(B)$. We believe that a little more work shows that $\pi_{(N,\Phi),\infty}(\delta_0)=\theta(B)$. See [63] for more along these lines.

To see this, it is more convenient to consider the relativised direct limit system $\mathcal{F}^\eta(N,\Phi)$, in which all iterations must be strictly above η. It is not hard to see that $\mathcal{F}^\eta(N,\Phi)$ is directed. Let $M^\eta_\infty(N,\Phi)$ be its direct limit, and $\pi^\eta_{(N,\Phi),\infty}$ be the direct limit map. We shall show

$$\tau \leq \pi^\eta_{(N,\Phi),\infty}(\delta_0).$$

Since $\mathcal{F}^\eta(N,\Phi)$ is a subsystem of the full $\mathcal{F}(N,\Phi)$, this is enough.

Working in V, let

$$\mathbb{R}^*_g = \{x_i \mid i < \omega\},$$

and let s be a run of $G^+(N,\omega,\omega_1)$ by Φ that is cofinal in $\mathcal{F}^\eta(N,\Phi)$, so that

$$N_\omega = M^\eta_\infty(N,\Phi),$$

where N_ω is the direct limit along s, and $i^s_{0,\omega} = \pi^\eta_{(N,\Phi),\infty}$. Let $N_0 = N$, and N_k be the last model of $s{\restriction}k$, for $k > 0$. Let $\delta_k = i^s_{0,k}(\delta_0)$. We can arrange that whenever $i < k$, then $x_i \in N_k[H]$, for some H that is generic over N_k for the extender algebra at δ_k.

We have $N_0 \trianglelefteq^{\mathrm{ct}} M_0$. The stack s is according to Ψ_{M_0}, so thinking of s as a stack on M_0, and letting M_k be the last model of $s{\restriction}k$ in this context, we have

$$N_k \trianglelefteq^{\mathrm{ct}} M_k,$$

and

$$i_{k,l}\colon M_k \to M_l$$

the iteration map given by s, for $k,l \leq \omega$.

Now we do the usual dovetailed $\mathbb{R}^*_g$- genericity iterations, iterating each $(M_k, \Psi_{s{\restriction}k,M_k})$, strictly above δ_k to (Q_k,Ω_k), and arranging that $L(\mathrm{Hom}^*_g,\mathbb{R}^*_g)$ is also a derived model of Q_k. Let

$$j_k\colon M_k \to Q_k$$

be the map of the $\mathbb{R}^*_g$ genericity iteration, and let

$$\sigma_{k,l}\colon Q_k \to Q_l$$

be the copy map, which exists because we dovetailed the genericity iterations together. (See for example the proof of Theorem 6.29 of [66] for the details of this well-known construction.) Here is a diagram.

$$\begin{tikzcd}
Q_0 \arrow[r, "\sigma_{0,k}"] & Q_k \arrow[r, "\sigma_{k,\omega}"] & Q_\omega \\
M_0 \arrow[u, "j_0"] \arrow[r, "i_{0,k}"] & M_k \arrow[u, "j_k"] \arrow[r, "i_{k,\omega}"] & M_\omega \arrow[u, "j_\omega"] \\
N_0 \arrow[u, "\mathrm{id}"] \arrow[r, "i_{0,k}"] & N_k \arrow[u, "\mathrm{id}"] \arrow[r, "i_{k,\omega}"] & N_\omega \arrow[u, "\mathrm{id}"]
\end{tikzcd}$$

We have for each $k < \omega$ a Q_k-generic h_k such that $\mathbb{R}^*_{h_k} = \mathbb{R}^*_g$ and $\mathrm{Hom}^*_{h_k} = \mathrm{Hom}^*_g$. The latter holds because for each $\xi < \lambda$, the critical points in j_k are eventually above $j_k(\xi)$, and the initial segment of the iteration that gets us to this point acts only on some $M|\gamma$ for $\gamma < \lambda$. This tells us that $(\Omega_k)^{h_k}_{\langle j_k(\xi),0\rangle}$ is projective in $\Psi^g_{\langle\gamma,0\rangle}$. That implies $\mathrm{Hom}^*_{h_k} \subseteq \mathrm{Hom}^*_g$. The reverse inclusion comes from the fact that each $\Psi_{\langle\gamma,0\rangle}$ is a pullback of some $\Omega_{\langle\xi,0\rangle}$.

Note that we have for each $k < \omega$ a term $\dot{B}_k \in Q_k$ such that

$$\dot{B}_k^{Q_k[l]} = B$$

for all l that are $\mathrm{Col}(\omega, <\lambda)$ generic over Q_k and such that $\mathbb{R}^*_l = \mathbb{R}^*_g$. Moreover,

$$\sigma_{k,n}(\dot{B}_k) = \dot{B}_n$$

for $k < n < \omega$. This follows from 8.1, the fact that all embeddings in the diagram above have critical point $> \eta$, and strategy coherence. Let $\mathbb{W}_k$ be the extender algebra of Q_k at δ_k, and put

$$\begin{aligned} \xi \in Y_k \text{ iff } Q_k \models & \exists b \in \mathbb{W}_k[b \Vdash (\mathrm{Col}(\omega,<\lambda) \Vdash \check{\xi} \text{ is the} \\ & \text{least } \gamma \text{ such that } L_{\check{\alpha}}(\mathrm{Hom}^*_{\dot{G}}, \mathbb{R}^*_{\dot{G}}) \models \varphi[\dot{x}, \gamma, \dot{B}_k])] \end{aligned}$$

Because $\mathbb{W}_k$ has the δ_k-chain condition in Q_k,

$$Q_k \models |Y_k| < \delta_k.$$

Now we define an order preserving map

$$p\colon \tau \to \pi^{\eta}_{(N,\Phi),\infty}(\delta_0) = i_{0,\omega}(\delta_0).$$

Let $\xi < \tau$, and pick any x such that $f(x) = \xi$. Let $k < \omega$ be sufficiently large that

(i) $x = x_i$ for some $i < k$, and

(ii) for $k \leq m \leq n < \omega$, $\sigma_{m,n}(\alpha) = \alpha$ and $\sigma_{m,n}(\xi) = \xi$.

Since Q_ω is wellfounded, we can find such a k. By (i), x is $\mathbb{W}_k$-generic over Q_k. It follows that $\xi \in Y_k$; say that

$$\xi = \text{ the } \gamma\text{-th element of } Y_k$$

in its increasing enumeration. We then set

$$p(\xi) = i_{k,\omega}(\gamma) = \sigma_{k,\omega}(\gamma).$$

We must check that $p(\xi)$ is independent of the choice of x, and that p is order preserving. For this, let $f(y) = \tau$. Let $k_{x,\xi}$ and $k_{y,\tau}$ be as in (i) and (ii) above, for (x, ξ) and (y, τ) respectively. Let $\gamma_{x,\xi}$ and $\gamma_{y,\tau}$ be the corresponding γ's. Taking $n \geq \max(k_{x,\xi}, k_{y,\tau})$, we have $\xi, \tau \in Y_n$, and

$$\xi = \text{ the } \sigma_{k_{x,\xi},n}(\gamma_{x,\xi})\text{-th element of } Y_n.$$

This is because $\sigma_{k_{x,\xi},n}(\xi) = \xi$. Similarly,

$$\tau = \text{ the } \sigma_{k_{y,\tau},n}(\gamma_{y,\tau})\text{-th element of } Y_n.$$

So

$$\begin{aligned} \xi \leq \tau \text{ iff } & i_{k_{x,\xi},n}(\gamma_{x,\xi}) \leq i_{k_{y,\tau},n}(\gamma_{y,\tau}) \\ \text{iff } & i_{k_{x,\xi},\omega}(\gamma_{x,\xi}) \leq i_{k_{y,\tau},\omega}(\gamma_{y,\tau}), \end{aligned}$$

as desired. This proves Claim 5. □

From the fact that $\Theta = \theta^{L(\mathbb{R}^*_g, \mathrm{Hom}^*_g)}$ we get at once that $L(\mathbb{R}^*_g, \mathrm{Hom}^*_g) \cap P(\mathbb{R}^*_g) = \mathrm{Hom}^*_g$. Thus in $L(\mathbb{R}^*_g, \mathrm{Hom}^*_g)$, all sets are Suslin, and therefore we get

Claim 6. $L(\mathbb{R}^*_g, \mathrm{Hom}^*_g) \models \mathsf{AD}_\mathbb{R}$.

Suppose that $(P, \Sigma) \in \mathcal{F}$, and let $\tau = o(M_\infty(P, \Sigma))$. The proof of Claim 5 showed that for some $\gamma < \lambda$, $(M|\gamma, \Psi^g_{\langle\gamma,0\rangle}) \in \mathcal{F}$ and $\tau < o(M_\infty(M|\gamma, \Psi^g_{\langle\gamma,0\rangle}))$. But this implies that $(P, \Sigma) \leq^* (M|\gamma, \Psi^g_{\langle\gamma,0\rangle})$. It follows then that the iterates of proper initial segments of $(M|\lambda, \Psi_{M|\lambda})$ are $\prec^*$-cofinal in $\mathcal{F}$.

This gives

Claim 7. There is a stack s on $M|\lambda$ of length ω that does not drop along its main branch, with canonical embedding $i_s \colon M \to M_\infty(s)$, such that

(a) for $n < \omega$, $s \restriction n \in (\mathrm{HC})^{L(\mathbb{R}^*_g, \mathrm{Hom}^*_g)}$,

(b) $M_\infty \trianglelefteq M_\infty(s)$, and

(c) if λ is a limit of cutpoints in M, then $i_s(\lambda) = o(M_\infty)$, and

(d) if κ is the least $< \lambda$-strong of M, then $i_s(\kappa) = o(M_\infty)$.

Proof. Let $\langle (P_i, \Lambda_i) \mid i < \omega \rangle$ be $\prec^*$-increasing and cofinal in $\mathcal{F}$. Let $(M, \Psi) = (Q_0, \Phi_0)$. Given $s{\restriction}i$ with last model (Q_i, Φ_i), let $s(i)$ be a normal tree on Q_i that comes from comparing (P_i, Λ_i) with some cardinal initial segment below λ of (Q_i, Φ_i) that is strictly greater that than (P_i, Λ_i) in the mouse order. There is such an initial segment by the remarks above. Let (Q_{i+1}, Φ_{i+1}) be the last pair of $s(i)$.

We do the comparison in such a way that (P_i, Λ_i) iterates to a cutpoint (N, Ω) of (Q_{i+1}, Φ_{i+1}). It follows that $i_{s{\restriction}i+1,\infty}$ agrees with the iteration map $\pi_{(N,\Omega),\infty}$ on N. This tells us that

$$\pi_{(P_i,\Lambda_i),\infty}``o(P_i) \subseteq i_{s{\restriction}i+1,\infty}(o(N)).$$

This implies that $M_\infty \trianglelefteq M_\infty(s)$. Also, N is a cutpoint, so $o(N)$ is below the least $< \lambda$-strong of Q_{i+1}, if there is one. Thus $o(M_\infty) \le i_{s{\restriction}0,\infty}(\kappa)$, where κ is the least $< \lambda$-strong, if there is one.

The cutpoint successor cardinal initial segments (N, Ω) of (Q_i, Φ_i) below λ are all in $\mathcal{F}$, and so $o(M_\infty)(N, \Omega) = i_{s{\restriction}i,\infty}(o(N)) < o(M_\infty)$ for such (N, Ω). It follows that

$$o(M_\infty) = \sup\{i_{s{\restriction}i,\infty}(o(N)) \mid i < \omega \wedge N \trianglelefteq^{\mathrm{ct}} Q_i|\lambda\}.$$

So if λ is a limit of cutpoints in M, and hence in each Q_i, then we get $i_s(\lambda) = o(M_\infty)$. If κ is the least strong to λ in M, we get $i_s(\kappa) = o(M_\infty)$. □

Claim 8. $\mathrm{HOD}^{L(\mathbb{R}^*_g, \mathrm{Hom}^*_g)} = L[M_\infty]$.

Proof. Let us write HOD for $\mathrm{HOD}^{L(\mathbb{R}^*_g, \mathrm{Hom}^*_g)}$, and θ for $\theta^{L(\mathbb{R}^*_g, \mathrm{Hom}^*_g)}$. It is clear that $M_\infty \in \mathrm{HOD}$, so what we must show is that $\mathrm{HOD} \subseteq L[M_\infty]$.

We use here

Lemma 8.10 (Woodin) *Assume $\mathsf{AD}_\mathbb{R} + V = L(P(\mathbb{R})$; then there is a definable (from no parameters) set $A \subseteq \theta$ such that $HOD = L[A]$.*

Fix A as in the lemma, and let $\varphi(v)$ be such that

$$\xi \in A \text{ iff } L(\mathbb{R}^*_g, \mathrm{Hom}^*_g) \models \varphi[\xi].$$

It is enough to show that $A \in L[M_\infty]$. For that, let s be a stack as in Claim 7, and let (Q_i, Φ_i) be the last model of $s{\restriction}i$. Let κ_i be the least $< \lambda$-strong of Q_i if there is one, and otherwise let $\kappa_i = \lambda$. We define $A_i \subseteq \kappa_i$ by

$$\xi \in A_i \text{ iff } i_{s{\restriction}i,\infty}(\xi) \in A.$$

We claim that A_i is definable over $L[Q_i|\kappa_i]$, uniformly in i. The definition is displayed in the following equivalence: for any ξ,

$$\xi \in A_i$$

if and only if

$L[Q_i|\kappa_i] \models \forall\alpha, h[(h$ is $\mathrm{Col}(\omega, <\kappa_i)$-generic and α is a cardinal cutpoint of $Q_i|kappa_i)$

$$\Rightarrow L(\mathbb{R}^*_h, \mathrm{Hom}^*_h) \models \varphi[\pi(\xi)], \text{where}$$

$$\pi = \pi^{L(\mathbb{R}^*_h,\mathrm{Hom}^*_h)}_{(Q_i|\alpha,\Lambda),\infty}, \text{ for } \Lambda = (\dot{\Sigma}^{Q_i})^h_{Q_i|\alpha}.]$$

Let us write the display above as $L[Q_i|\kappa_i] \models \varphi_0[\xi]$.

We give the well-known proof of the equivalence. Let $\alpha > \xi$ be a cutpoint of Q_i. Via an $\mathbb{R}^*_g$-genericity iteration of Q_i above α, we can find

$$\sigma\colon Q_i \to S$$

and h generic over S for $\mathrm{Col}(\omega, <\sigma(\kappa_i))$ such that

$$L(\mathbb{R}^*_g, \mathrm{Hom}^*_g) = L(\mathbb{R}^*_h, \mathrm{Hom}^*_h).$$

The only slight wrinkle here is that if $\kappa_i < \lambda$, out genericity iteration must weave in infinitely many steps at which we move the image of κ_i up by an extender with that critical point. Note $S|\alpha = Q_i|\alpha$, and the two are assigned the same strategy in their respective pairs. Call that strategy Λ.

We then get that

$$\begin{aligned} L[Q_i|\kappa_i] \models \varphi_0[\xi] &\text{ iff } L[S|\sigma(\kappa_i)] \models \varphi_0[\xi] \\ &\text{ iff } \pi_{(Q_i|\alpha,\Lambda),\infty}(\xi) \in A \\ &\text{ iff } \xi \in A_i, \end{aligned}$$

as desired.

Since $i_{s\restriction k,s\restriction l}$ is elementary, we get that $i_{s\restriction k,s\restriction l}(A_k) = A_l$ whenever $k < l$. This implies that $i_{s\restriction k,\infty}(A_k) = A$ for all k. But then A is definable over $L[M_\infty(s)|i_{s\restriction 0,\infty}(\kappa_0)]$ by the same formula that defined A_0 over $L[Q_0|\kappa_0]$. So $A \in L[M_\infty]$, as desired. □

Claim 8 completes the proof of Theorem 8.8.

□

By combining Theorem 8.8 with our earlier results on the existence of hod pairs with large cardinals, we get

Theorem 8.11 *Suppose there is $j\colon V \to M$ with* $\mathrm{crit}(j) = \kappa$ *and* $V_{j(\kappa)+1} \subseteq M$. *Suppose* $\mathsf{IH}_{\mu,j(\kappa)}$ *hold for some $\mu < \kappa$, and that there are $\lambda < \nu < \kappa$ such that λ is a limit of Woodin cardinals, and ν is measurable. Then there is a Wadge cut Γ in $Hom_{<\lambda}$ such that $L(\Gamma, \mathbb{R}) \models \mathsf{AD}_{\mathbb{R}}$, and*

$$HOD^{L(\Gamma,\mathbb{R})} \models \mathsf{GCH} + \textit{ there is a subcompact cardinal.}$$

Proof. Under the hypotheses of 8.11, we have shown in Theorem 6.74 that there is an lbr hod pair (M, Ψ) with scope HC such that for some λ, $M \models$ "λ is a limit of cutpoint Woodins, and there is a subcompact $< \lambda$." Moreover, we have that Code(Ψ) is $\mathrm{Hom}_{<\lambda}$. So we can apply 8.8, and we get that the HOD of the derived model $D(M, < \lambda)$ is an iterate of M, and satisfies "there is a subcompact cardinal". But then via an $\mathbb{R}$-genericity iteration M-to-M^*, we can realize $D(M^*, < \lambda)$ as $L(\Gamma, \mathbb{R})$, for some $\Gamma \subsetneq \mathrm{Hom}_{<\lambda}$. This proves the theorem.

$\square$

8.4 HOD mice satisfy $V = K$

We shall show that if (H, Ω) is an lbr hod pair such that $H \models \mathsf{ZFC}+$ "there are arbitrarily large Woodin cardinals", then in a certain natural sense, $H \models V = K$. This sense derives from the definition of K below one Woodin cardinal that uses thick sets at a regular cardinal, as in [13]. The definition has a generically absolute version, so that in a certain sense, $H = K^{H[g]}$, whenever g is set-generic over H.

Pure extender mice do not in general satisfy even $V = \mathrm{HOD}$, much less $V = K$. The basic problem is that they may not know how to iterate themselves.[46] In this respect, strategy mice are more natural; they know who they are, so to speak.

Definition 8.12 *Let α be a regular cardinal, and P be a premouse; then we say P is α^+-*universal *iff*

(1) $P \models$ " α is the largest cardinal",

(2) $o(P) = \alpha^+$, and

(3) $\{\eta \mid E^P_\eta \neq \emptyset\}$ is not stationary in α^+.

Of course, P determines α, so we write $\alpha = \alpha^P$. We say that P is universal iff P is α^+-universal, where $\alpha = \alpha^P$. One could make these definitions in the case α is singular, or α is subcompact, but then some complicating cases arise.

Lemma 8.13 *Let (P, Σ) be a mouse pair with scope H_λ, where λ is a limit of Woodin cardinals, and suppose that whenever $\pi \colon Q \to P$ is elementary, and Q is countable, then Σ^π is $< \lambda$-universally Baire. Suppose also that P is α^+-universal, where $\alpha^+ < \lambda$; then (P, Σ) is full.*

[46]See [38], [39], and [45] for results on the extent to which $V = K$ and $V = \mathrm{HOD}$ hold in pure extender mice.

Proof. Suppose not, and let (Q, Λ) be a nondropping iterate of (P, Σ), and let $(Q, \Lambda) \trianglelefteq^{\text{ct}} (R, \Psi)$, where (R, Ψ) is a mouse pair with scope H_λ such that $\rho(R) < o(Q)$. Let

$$i\colon P \to Q$$

be the iteration map, so that $i(\alpha)$ is the largest cardinal of Q, and $\operatorname{ran}(i)$ is α-club in $o(Q)$. Our plan is to use i to pull back R to a mouse that collapses $o(P)$, as one does in the arguments that show $\square$ fails when one has sufficiently strong embeddings.

For this, we need the condensation results of [65]. Those results were proved under AD^+ for mouse pairs with scope HC. They apply here because they are first order requirements on P, so it is enough to see that they hold for countable Q such that there is an elementary $\pi\colon Q \to P$. But then our hypotheses imply that (Q, Σ^π) is extends to a mouse pair with scope HC in the derived model of V below λ, so we can apply [65] in that derived model.

Assume first that $k(R) = 0$ and R is passive, and let $p = p_1(R)$. For $\xi < o(Q)$ let γ_ξ be least such that $h_{R|\gamma_\xi}``(i(\alpha) \cup p) \cap [\xi, o(Q)) \neq \emptyset$. Let

$$N_\xi = \text{ transitive collapse of } h_{R|\gamma_\xi}``(i(\alpha) \cup p),$$

and let

$$\pi_\xi\colon N_\xi \to R|\gamma_\xi$$

be the anticollapse. Letting $\tau_\xi = \operatorname{crit}(\pi_\xi)$, it is easy to see that $\pi_\xi(\tau_\xi) = o(P)$, and $N_\xi \in Q$. In fact, if τ_ξ is not an index on the P-sequence, then by [65],

$$N_\xi \trianglelefteq Q.$$

By the non-subcompactness clause of universality, we have an α-club $C \subseteq \operatorname{ran}(i)$ such that for all $\xi \in C$, $\xi = \tau_\xi$ and $\xi \in \operatorname{ran}(i)$. For $\xi \in C$, $\rho(N_\xi) = \rho_1(N_\xi) = i(\alpha)$, and

$$p(N_\xi) = \pi_\xi^{-1}(p).$$

For $\xi < \eta$ with both in C, we have a natural

$$\sigma_{\xi,\eta}\colon N_\xi \to N_\eta,$$

determined by $\sigma_{\xi,\eta}\restriction\tau_\xi = \text{ id }$, and $\sigma_{\xi,\eta}(p(N_\xi)) = p(N_\eta)$. The full R is just the direct limit of the N_ξ, for $\xi \in C$, under the $\sigma_{\xi,\eta}$.

Now we pull back to P. Let D be an α-club in $o(P)$ such that $i``D \subseteq C$. For $\xi \in D$, let $M_\xi \trianglelefteq P$ be such that

$$i(M_\xi) = N_{i(\xi)},$$

and let

$$\varphi_{\xi,\eta}\colon M_\xi \to M_\eta$$

be given by $\varphi_{\xi,\eta} = i^{-1} \circ \sigma_{\xi,\eta} \circ i$. Note here that $N_{i(\xi)}$ is definable from $i(\xi)$ as the first level of Q collapsing $i(\xi)$ to $i(\alpha)$, so $N_{i(\xi)} \in \operatorname{ran}(i)$, and M_ξ is the first level of P

collapsing ξ to α. Note also that

$$\varphi_{\xi,\eta}(p(M_\xi)) = p(M_\eta).$$

Letting M be the direct limit of the M_ξ and r be the common value of $\varphi_{\xi,\infty}(p(M_\xi))$, for ξ in D, we see that h_M"$(\alpha \cup r)$ contains α^+, which is a contradiction.

If $k(R) = 0$ but R is active, the proof is similar. The important point is that $\mathrm{crit}(\dot{F}^R) > o(Q)$, because Q is a cutpoint of R. Thus we do not get involved with protomice, and the condensation result of [65] applies. If $k(R) > 0$, then again the proof is similar, again using the condensation result of [65]. We use the fact that $r\Sigma_{k+1}$ over R is the same as Σ_1 over the mastercode structure R^k to find our approximations N_ξ to R.

□

Definition 8.14 *Let P be universal, and $\alpha = \alpha^P$. Then*

(1) Γ *is* thick *iff there is an α-club set $C \subseteq \alpha^+$ such that $C \subseteq \Gamma$.*

(2) P has the definability property *at β iff for all thick sets Γ, $\beta \in Hull^P(\Gamma \cup \beta)$.*

(3) P has the hull property *at β iff for all thick sets Γ, letting N be the transitive collapse of $Hull^P(\Gamma \cup \beta)$, $P(\beta)^P = P(\beta)^N$.*

(4) $\mathrm{Def}^P = \bigcap\{Hull^P(\Gamma) \mid \Gamma$ *is thick* $\}$.

(5) P is very sound *iff* $P = \mathrm{Def}^P$.

It is easy to see that P has the definability property at all $\beta < \gamma$ iff $\gamma \subseteq \mathrm{Def}^P$. Thus P is very sound iff P has the definability property at all $\beta < \alpha^+$. Every α^+-universal P has the definability property at all $\beta \in [\alpha, \alpha^+)$, because the critical point of any $\pi \colon N \to P$ is a cardinal of P. Thus P is very sound iff P has the definability property at all $\beta < \alpha$.

The following is a uniqueness lemma for very sound hod pairs. We shall formulate it as a first order fact about least branch hod mice. One could abstract the first order properties of such mice that we shall use in its proof, but we are not going to do that.

Lemma 8.15 *(AD^+) For any lbr hod pair (W, Ψ) with scope HC, the following is true in W: whenever (P, Σ) and (Q, Λ) be lbr hod pairs with scope H_λ, where λ is a limit of Woodin cardinals, and P and Q are α^+-universal and very sound, where $\alpha < \lambda$, then $(P, \Sigma) = (Q, \Lambda)$.*

Proof. We work inside W. Let Σ_0 and Λ_0 be the restrictions of Σ and Λ to V_δ, where $\alpha < \delta < \lambda$ and δ is Woodin. We show that $(P, \Sigma_0) = (Q, \Lambda_0)$, and since δ is arbitrary, this is enough. Let w be a wellorder of V_δ, and let $\mathbb{C}$ be the maximal w-construction. $\mathbb{C}$ does not break down before it has reached non-dropping iterates of (P, Σ_0) and (Q, Λ_0). Let (M, Ω) be the first pair in $\mathbb{C}$ that is a non-dropping iterate of one of these two, and assume without loss of generality that it is (P, Σ_0) that iterates to (M, Ω), while (Q, Λ_0) iterates past (M, Ω), perhaps not strictly.

Let $\mathcal{T}$ be the normal tree on P with last model M, and

$$i : P \to M$$

the canonical embedding. i is given by an extender all of whose measures concentrate on bounded subsets of α, so i is continuous at points of cofinality α. It follows that $\mathrm{ran}(i)$ is α-club in $o(M)$. Let $\mathcal{U}$ be the tree whereby Q iterates past M, with last model R such that

$$M \trianglelefteq^{\mathrm{ct}} R.$$

By 8.13, (P, Σ) is full, so branch Q-to-R of $\mathcal{U}$ does not drop, and we have an iteration map

$$j : Q \to R.$$

Note that the generators of j are contained in $i(\alpha)$, because $i(\alpha)$ is the largest cardinal of M. (In the worst case, the branch Q-to-R uses a last extender F such that $\mathrm{lh}(F) = o(M)$, but even then, $\lambda(F) = o(M)$.) Note also that j is continuous at α, because α is regular but not measurable in Q.

We claim that $M = R$. For if not, $j(\alpha) \geq o(M)$. But $j(\alpha)$ has cofinality (in V) α, while $o(M)$ has cofinality α^+. Thus we have some $\beta < \alpha$ such that $o(M) \leq j(\beta)$. But

$$j(\beta) \subseteq \{j(f)(a) \mid f \in Q|\alpha \wedge a \in [i(\alpha)]^{<\omega}\}.$$

For each $f \in Q|\alpha$, let

$$\gamma_f = \sup\{j(f)(a) \mid a \in [i(\alpha)]^{<\omega} \wedge j(f)(a) < o(M)\}.$$

Since $o(M)$ is regular in R, $\gamma_f < o(M)$. Since $o(M)$ is the sup over f of the γ_f, the V cofinality of $o(M)$ is $\leq \alpha$. This is a contradiction.

So $M = R$, and thus $i(\alpha) = j(\alpha)$. By the continuity of i and j at points of cofinality α, we have an α-club set $C \subseteq \alpha^+$ such that $i(\xi) = j(\xi)$ for all $\xi \in C$. This implies that $\mathrm{Hull}^P(C)$ is isomorphic to $\mathrm{Hull}^Q(C)$. By very soundness, $P = \mathrm{Hull}^P(C)$ and $Q = \mathrm{Hull}^Q(C)$. So $P = Q$, and then $i = j$ because the two agree on the generating set C. But then $\Sigma_0 = \Omega^i = \Omega^j = \Lambda_0$.

□

Lemma 8.16 *(AD$^+$) Let (W, Ψ) be an lbr hod pair with scope HC, and suppose $W \models$ "there are no 1-extendible cardinals". Working inside W, let α be regular but not subcompact, and $\alpha < \lambda$, where λ is a limit of Woodin cardinals. Let $P = W|\alpha^+$ and $\Sigma = \Psi_P$; then (P, Σ) is very sound.*

Proof. We work in W. It is enough to see that P has the definability property at all $\beta < \alpha$. So suppose β is least such that P does not have the definability property at β.

Claim 1. P has the hull property at all $\gamma < \beta$.

Proof. Let $\gamma < \beta$, and let $C \subseteq o(P)$ such that letting H be the transitive collapse of $\mathrm{Hull}^P(C \cup \gamma)$, $P(\gamma)^H \neq P(\gamma)^P$. Equivalently,

$$\gamma^{+,H} < \gamma^{+,P} = \gamma^+.$$

Let $\pi \colon H \to P$ be the anticollapse, and note that $\mathrm{crit}(\pi) > \gamma$ by the definability property at γ. Both (H, Σ^π) and (P, Σ) are α^+-universal. Letting $\delta > \alpha$ be Woodin, and $\mathbb{C}$ be a a construction of length δ that uses nice background extenders from the W-sequence having critical points above α, the proof of Lemma 8.15 shows that there is a pair (M, Ω) of $\mathbb{C}$ such that both (P, Σ) and $H, \Sigma^\pi)$ iterate to (M, Ω).

Let i and j be the two iteration maps. There is an α-club D in α^+ such that $i{\restriction}D = j{\restriction}D$. We can find $c \in D^{<\omega}$ such that $\pi^{-1}(c) = c$, and a Skolem term τ such that $\tau^P[c] = \gamma$. But then $\tau^H[c] = \gamma$, so $i(\gamma) = i(\tau^P[c]) = \tau^M[i(c)] = \tau^M[j(c)] = j(\tau^H[c]) = j(\gamma)$.

It follows that $i(\gamma^+) = j(\gamma^{+,H})$. But i is continuous at γ^+ and j is continuous at $\gamma^{+,H}$, so $\gamma^{+,H}$ must have cofinality γ^+, contradiction. $\square$

Claim 2. P has the hull property at β.

Proof. By Claim 1, β is a cardinal of P. Again, suppose toward contradiction that we have

$$\pi \colon H \to P$$

with $\mathrm{ran}(\pi)$ thick and $\mathrm{crit}(\pi) \geq \beta$ and $\beta^{+,H} < \beta^{+,P} = \beta^+$. We want to use the comparison argument in the first claim, but there is a problem if β has measurable cofinality in H or P. To deal with this, we simply take an ultrapower. Namely, set

$$G = \begin{cases} H & \text{if } H \models \mathrm{cof}(\beta) \text{ is not measurable,} \\ \mathrm{Ult}(H, U) & \text{if } H \models U \text{ is the order 0 measure on } \mathrm{cof}(\beta), \end{cases}$$

and

$$Q = \begin{cases} P & \text{if } P \models \mathrm{cof}(\beta) \text{ is not measurable,} \\ \mathrm{Ult}(P, U) & \text{if } P \models U \text{ is the order 0 measure on } \mathrm{cof}(\beta). \end{cases}$$

(We mean to take ultrapowers here if β is itself measurable.) Let $i_0 \colon P \to Q$ and $j_0 \colon G \to H$ be the canonical embeddings (possibly the identity, of course). Let $\Phi = \Sigma_Q$ and let $\Lambda = (\Sigma^\pi)_G$ be the strategies for Q and G.

Again, we have a construction $\mathbb{C}$, and a level (M, Ω) of $\mathbb{C}$ such that both (Q, Φ) and (G, Λ) iterate into (M, Ω). Let $i_1 \colon Q \to M$ and $j_1 \colon G \to M$ be the iteration maps. Let $i = i_1 \circ i_0$ and $j = j_1 \circ j_0$. So i is an iteration map from P to M, and j is an iteration map from H to M.

There is an α-club D in α^+ such that $i \restriction D = j \restriction D$, and $\pi \restriction D$ is the identity. Since P has the definability property at all $\xi < \beta$, we have $i \restriction \beta = j \restriction \beta$.

If β has non-measurable cofinality in P and H, then i and j are continuous at β, so $i(\beta) = j(\beta)$. This implies $i(\beta^+) = j(\beta^{+,H})$, which gives the same cofinality mismatch as before.

But more generally, let us just note that i_1 is continuous at $\beta = \sup(i_0``\beta)$ and $j_1$ is continuous at $\beta = \sup(j_0``\beta)$. So

$$\begin{aligned} i_1(\beta) &= i_1(\sup(i_0``\beta)) \\ &= \sup(i_1 \circ i_0``\beta) \\ &= \sup(j_1 \circ j_0``\beta) \\ &= j_1(\sup(j_0``\beta)) \\ &= j_1(\beta)). \end{aligned}$$

So $i_1(\beta^{+,Q}) = j_1(\beta^{+,G})$, and i_1 and j_1 are continuous at successor cardinals, so $\beta^{+,Q}$ has the same V-cofinality as $\beta^{+,G}$. But $\beta^{+,G} \le \beta^{+,H}$, so it has V-cofinality $\le \beta$. On the other hand, $\beta^{,Q} = \beta^+$, since the map $A \mapsto i_0(A) \cap \beta$ is one-to-one on $P(\beta) \cap P$, and $\beta^{+,P} = \beta^+$. So we still have our cofinality mismatch, contradiction. □

Now since P does not have the definability property at β, we have $\pi \colon H \to P$ such that $\mathrm{ran}(\pi)$ is thick, and $\beta = \mathrm{crit}(\pi)$. But $H|\beta^+ = P|\beta^+$, and $\pi \in P$ because we are working in W, and $P = W|\alpha^+$. Thus $W \models$ "π is a 1-extendibility embedding". This is contrary to our hypotheses on W. □

Remark 8.17 It seems likely that no lbr hod mouse can satisfy "there is a 1-extendible cardinal", but we have not proved this.

The following definition is meant to be employed inside hod mice satisfying ZFC and having arbitrarily large Woodin cardinals.

Definition 8.18 *Let P be a least branch premouse and α be a cardinal; then we say P is K-like at α iff P is α^+-universal and very sound, and for δ the least Woodin cardinal $> \alpha$, there is a Σ such that (P, Σ) is an lbr hod pair with scope H_{δ^+}.*

Theorem 8.19 *(AD^+) Let (H, Ω) be an lbr hod pair with scope HC, and suppose $H \models \mathsf{ZFC}+$ "there are arbitrarily large Woodin cardinals, and there are no 1-extendible cardinals". Let g be generic over H for a poset of size $< \nu$ in H, and let α be a successor cardinal of H above ν; then in $H[g]$, the following are equivalent:*

(1) P is K-like at α,

(2) $P = H|\alpha^+$.

Proof. Lemmas 8.15 and 8.16 show that the equivalence is true in H itself. In $H[g]$ we must work above the size of the forcing.[47] We leave it to the reader to think through that case. □

Definition 8.20 *We say that K^ν* exists *iff*

(1) for every successor cardinal $\alpha > \nu$, there is a unique lpm $K^\nu(\alpha)$ such that $K^\nu(\alpha)$ is K-like at α, and

(2) if $\nu < \alpha < \beta$ and α, β are successor cardinals, then $K^\nu(\alpha) \trianglelefteq K^\nu(\beta)$.

If K^ν exists, then we set $K^\nu = \bigcup_\alpha K^\nu(\alpha)$.

So a hod mouse H as in the theorem satisfies $V = K^0 = K^\nu$ for all ν. It follows that H satisfies $V = \text{HOD}$. In set generic extensions of H, $H = K^\nu$ for all sufficiently large ν, so $H \subseteq \text{HOD}^{H[g]}$. Thus H is the generic HOD, or gHOD, of its generic multiverse.[48] It follows that $P \models V = \text{HOD}$.

This should be compared with

Theorem 8.21 (Woodin [68]) *Assume $\mathsf{AD}_\mathbb{R} + V = L(P(\mathbb{R}))$; then HOD $\models V =$ HOD, and HOD$|\theta$ is the generic HOD of its own generic multiverse.*

This result is significantly more general than what we have proved, in that it applies to $\mathsf{AD}_\mathbb{R}$ models that have iteration strategies for mice with long extenders, and are therefore beyond the HOD analysis we have developed here. Our proof that HOD $\models \mathsf{V} =$ HOD does have extra information in it, in the short-extender region to which it applies.

[47] This is provably necessary, because of the local nature of "K-like at α". If $\delta_0 < \delta_1$ are Woodin cardinals, and if $j: H \to M \subseteq H[g]$ comes from a $\mathbb{P}_{\delta_1}$-stationary tower ultrapower, it will be initial segments of M that are K-like at $\alpha < j(\delta_0)$ in the sense of $H[g]$.

[48] See [9].

8.5 Further results

Our analysis of HOD in the derived model D of a HOD mouse was based on the fact that $D \models \mathsf{HPC}$. (This was the content of the first two claims in the proof of Theorem 8.8.) We used further facts about the way we had derived D, but with more work, one can avoid an appeal to them. Thus we get

Theorem 8.22 ([63]) *Assume* $\mathsf{AD}_{\mathbb{R}}$ *and* HPC*; then* $V_\theta \cap HOD$ *is the universe of a least branch premouse.*

Concerning the mouse capturing hypothesis of this theorem, we have

Theorem 8.23 ([63]) *Assume* AD^+*; then*

(a) if HPC *holds, then for any* $\Gamma \subseteq P(\mathbb{R})$*,* $L(\Gamma, \mathbb{R}) \models \mathsf{HPC}$*, and*

(b) if LEC *holds, then for any* $\Gamma \subseteq P(\mathbb{R})$*,* $L(\Gamma, \mathbb{R}) \models \mathsf{LEC}$*, and*

(c) if there is an ω_1 *iteration strategy for a countable pure extender premouse with a long extender on its sequence, then for any* $\Gamma \subseteq P(\mathbb{R})$ *such that* $L(\Gamma, \mathbb{R}) \models \mathsf{NLE}$*, we have* $L(\Gamma, \mathbb{R}) \models \mathsf{LEC}$*, and hence* $L(\Gamma, \mathbb{R}) \models \mathsf{HPC}$*.*

Part (c) is pretty strong evidence that $\mathsf{AD}^+ + \mathsf{NLE}$ implies LEC, and hence HPC. Whether this is in fact true is perhaps the main open problem in the theory to which this book contributes. Parts (a) and (b) suggest that one ought to try to prove this via an induction on the Wadge hierarchy, and that is a natural thing to try on other counts, too. There are partial results in this direction, but the situation is in sufficient flux that it seems wisest not to attempt a discussion of them.

The proof of 8.22 gives a characterization of the Solovay sequence in terms of the Woodin cardinals in HOD.

Definition 8.24 *For any set* X*,* $\theta(X)$ *is the least ordinal* α *such that there is no ordinal definable surjection of* X *onto* α*.*

If there is an ordinal definable map from X onto $X \times X$, then $\theta(X)$ is the supremum of the surjective images of X under maps that are ordinal definable from some parameter in X. This is our case of interest.

Definition 8.25 *(*AD^+*.) The* Solovay sequence $\langle \theta_\alpha \mid \alpha \leq \Omega \rangle$ *is given by*

$\theta_0 = \theta(\mathbb{R})$,

and if $\theta_\alpha < \theta$, *then*

$$\theta_{\alpha+1} = \theta(\mathbb{R} \cup \{A\}), \text{ for any (all) } A \text{ of Wadge rank } \theta_\alpha,$$
$$\theta_\lambda = \bigcup_{\alpha<\lambda} \theta_\alpha.$$

Ω *is the least* β *such that* $\theta_\beta = \theta$.

Assuming AD^+, if $\theta_\alpha < \theta$, then

$$\theta_{\alpha+1} = \theta(P(\theta_\alpha)).$$

This is easy to see, using the Coding Lemma and the fact that every set of reals of Wadge rank θ_α is θ_α-Suslin. The Solovay sequence is an important feature of any model AD^+, one that is tied to the pattern of scales in the model. It is definable, so it is in HOD. In fact, it has a natural identity within HOD.

Assume $\mathsf{AD}_\mathbb{R} + \mathsf{HPC}$. The proof of 8.22 then gives a canonical least branch premouse $\mathcal{H}$ whose universe is V_θ^{HOD}. We have shown in the last section that in fact $\mathcal{H}$ is definable over $(V_\theta^{\text{HOD}}, \in)$, as the union of all universal, very sound premice. Let us say that δ is a *cutpoint of HOD* iff δ is a cutpoint of $\mathcal{H}$, in the sense that there is no extender E on the $\mathcal{H}$-sequence such that $\text{crit}(E) < \delta \leq \text{lh}(E)$.[49] It is east to see that if δ is Woodin and a cutpoint of HOD, then there are no extenders on the $\mathcal{H}$-sequence with critical point δ.

Theorem 8.26 **([63])** *Assume* $\mathsf{AD}_\mathbb{R} + V = L(P(\mathbb{R})) + \mathsf{HPC}$*; then the following are equivalent:*

(1) δ *is a cutpoint Woodin cardinal of HOD,*

(2) $\delta = \theta_0$*, or* $\delta = \theta_{\alpha+1}$ *for some* α*.*

In particular, θ_0 *is the least Woodin cardinal in HOD.*

That θ_0 and the $\theta_{\alpha+1}$ are Woodin in HOD is due to Woodin, cf. [15]. Woodin also proved an approximation to the statement that they are cutpoints of HOD (unpublished). The rest of (2) $\to$ (1), and all of (1) $\to$ (2), comes from [63].

One can characterize the next Woodin cardinal of HOD in terms of a modified Solovay sequence. The following definition is due to Grigor Sargsyan.[50]

[49] Presumably, every extender in HOD that coheres with the $\mathcal{H}$-sequence is actually on that sequence, but no one has actually proved this, so far as we know.

[50] One might call this the *Sargsyan sequence*.

Definition 8.27 *Assume* AD^+*. We set*

$$\begin{aligned}\eta_0 &= \theta(^\omega\omega) = \theta_0,\\ \eta_{\alpha+1} &= \theta(^\omega\kappa),\ \textit{where } \kappa = (\eta_\alpha)^{+,HOD},\\ \eta_\lambda &= \bigcup_{\alpha<\lambda}\eta_\alpha.\end{aligned}$$

One can show

Theorem 8.28 ([63]) *Assume* $\mathsf{AD}_{\mathbb{R}} + \mathsf{HPC}$*; then for any* $\delta < \theta$*,* δ *is a successor Woodin cardinal of HOD iff* $\delta = \eta_{\alpha+1}$ *for some* α*.*

Of course, "successor Woodin" means "least Woodin above some ordinal". The Sargsyan sequence may grow more slowly than the Solovay sequence. Assuming $\mathsf{AD}_{\mathbb{R}}+\mathsf{HPC}$, Theorem 8.28 implies that this happens if and only if HOD has extenders overlapping Woodin cardinals.

It is also interesting to see what strong determinacy theories are true in the derived models of lbr hod pairs (P, Σ) such that P reaches reasonably large cardinals. There are some results in this direction in [63].

The key to the theorems above is an analysis of optimal Suslin representations for mouse pairs. That in turn rests on a strengthening of strong hull condensation that [48] calls *very strong hull condensation*. Roughly speaking, this property amounts to condensation under *weak tree embeddings*, a more general kind of tree embedding than the kind we have defined in 3.27.[51] [48] shows

Theorem 8.29 ([48]) *Assume* AD^+*, and let* (P, Σ) *be a mouse pair with scope HC; then* Σ *has very strong hull condensation.*

Given a stack $\langle \mathcal{T}, \mathcal{U}\rangle$ on P with last model Q, there is a natural attempt $X(\mathcal{T}, \mathcal{U})$ at a normal tree on P with last model Q. We say that Σ *fully normalizes well* iff whenever $\langle \mathcal{T}, \mathcal{U}\rangle$ is by Σ, then $X(\mathcal{T}, \mathcal{U})$ exists and is by Σ, and $\Sigma_{\langle \mathcal{T},\mathcal{U}\rangle} = \Sigma_{X(\mathcal{T},\mathcal{U})}$. (See [48].) The construction of $X(\mathcal{T}, \mathcal{U}$ produces a weak tree embedding from $X(\mathcal{T}, \mathcal{U})$ into $W(\mathcal{T}, \mathcal{U})$. Thus Theorem 8.29 yields

Corollary 8.30 ([48]) *Assume* AD^+*, and let* (P, Σ) *be a mouse pair with scope HC; then*

[51] In a weak tree embedding, the connection between exit extenders required by 3.27(d) is loosened. Rather than require that $t_\alpha(E^{\mathcal{T}}_\alpha) = E^{\mathcal{U}}_{u(\alpha)}$, we require that $t_\alpha(E^{\mathcal{T}}_\alpha)$ be connected to $E^{\mathcal{U}}_{u(\alpha)}$ inside $\mathcal{M}^{\mathcal{U}}_{u(\alpha)}$ via a sequence of fine structural hulls. This sequence of hulls is an abstract version of the sequence that occurred in Claim 3.3 of our proof of full normalizability of trees of length two.

(a) Σ fully normalizes well, and

(b) Σ is positional.

From the proof of Corollary 8.30 we obtain a normal tree $\mathcal{U}(P, \Sigma)$ on P that has last model $M_\infty(P, \Sigma)$, and is such that all its countable weak hulls are by Σ. This then gives us a Suslin representation for the fragment of Σ that is actually used in forming $M_\infty(P, \Sigma)$: to justify a countable tree $\mathcal{T}$ on P, we search for a weak tree embedding of $\mathcal{T}$ into $\mathcal{U}(P, \Sigma)$.

Not all of Σ is actually used in forming $M_\infty(P, \Sigma)$. Let us call a normal tree $\mathcal{T}$ *relevant* iff $\mathcal{T}$ is by Σ, and there is a normal $\mathcal{S}$ by Σ such that $\mathcal{T} \subseteq \mathcal{S}$, and $\mathcal{S}$ has a last model Q, and the branch P-to-Q does not drop. Call a P-stack s relevant if for $i+1 < \mathrm{dom}(s)$, the branch of $\mathcal{T}_i(s)$ to $M_\infty(\mathcal{T}_i(s))$ does not drop, and for $i+1 = \mathrm{dom}(s)$, $\mathcal{T}_i(s)$ is relevant. Let Σ^{rel} be the restriction of Σ to relevant trees. The Σ-iterations that go into forming $M_\infty(P, \Sigma)$ are all relevant, so Σ^{rel} is what we need to construct $M_\infty(P, \Sigma)$ and $\mathcal{U}(P, \Sigma)$. Moreover, $\mathcal{U}(P, \Sigma)$ acts as a kind of universal tree by Σ^{rel}, in that all countable trees by Σ^{rel} can be weakly embedded into it. This leads to

Theorem 8.31 **([63])** *Assume AD^+, and let (P, Σ) be a mouse pair with scope HC. Let κ be the cardinality of $o(M_\infty(P, \Sigma))$, and let Code(Σ^{rel}) be the set of reals coding stacks by Σ^{rel}; then*

(a) Code(Σ^{rel}) and its complement are κ-Suslin, and

(b) Code(Σ) is not α-Suslin, for any $\alpha < \kappa$.

In particular, κ is a Suslin cardinal.

Part (b) of the Theorem 8.31 follows at once from the Kunen-Martin theorem, and the fact that there is a wellfounded relation W on $\mathbb{R}$ of rank at least $o(M_\infty(P, \Sigma))$ such that W is arithmetic in Code(Σ). [Let $(t, b) W (s, a)$ iff s and t are stacks by Σ with last models M and N, $s \subseteq t$, P-to-N does not drop, and $i^t_{M,N}(a) > b$.]

The one can show the irrelevant part of Σ is also Suslin, but perhaps not $o(M_\infty(P, \Sigma))$-Suslin. (It is possible that $M_\infty(P, \Sigma) = P$, because there are no non-dropping iterations of P!) So one gets

Theorem 8.32 **([63])** *Assume AD^+, and let (P, Σ) be a mouse pair with scope HC. and let Code(Σ) be the set of reals coding stacks by Σ; then Code(Σ) and its complement are Suslin.*

Note here that since Σ is total on stacks by Σ, if Code(Σ) is β-Suslin, then so is its complement.

Theorem 8.31 implies that $|o(M_\infty(P,\Sigma))|$ is a Suslin cardinal.[52] With more work along the same lines, one can show that for any cutpoint τ of $M_\infty(P,\Sigma)$, $|\tau|$ is a Suslin cardinal. In recent unpublished work, S. Jackson and G. Sargsyan have shown that all Suslin cardinals arise below $o(M_\infty(P,\Sigma))$ arise this way. So we have

Theorem 8.33 (Jackson, Sargsyan, S.) *Assume* AD^+, *let* (P,Σ) *be a mouse pair, and let* $\kappa \le o(M_\infty(P,\Sigma))$. *The following are equivalent:*

(a) κ *is a Suslin cardinal,*

(b) $\kappa = |\tau|$, *where* τ *is a cutpoint of* $M_\infty(P,\Sigma)$ *or* $\tau = o(M_\infty(P,\Sigma))$.

The proof that (a) implies (b) by Jackson and Sargsyan shows that if κ is a regular Suslin cardinal, then κ itself is a cutpoint of $M_\infty(P,\Sigma)$. It is open whether that is also true for the other Suslin cardinals, the problematic case being when κ is the next Suslin cardinal after some regular Suslin cardinal.

The correspondence between iteration strategies and definable scales is central to descriptive inner model theory. Theorem **??** captures one aspect of it.

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