Normalizing Iteration Trees and Comparing Iteration Strategies

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Abstract

In this paper, we shall prove a general comparison lemma for iteration strategies. The comparison method involves iterating into a level of a background construction, one that has been done in a universe that is uniquely iterable in the appropriate sense. The proof that it succeeds relies heavily on an analysis the normalization of a stack of normal iteration trees.

We then use this comparison method to develop the basic theory of hod mice in the least branch hierarchy. Modulo the existence of iteration strategies, our results yield a fine structural analysis of $(\mathrm{HOD}|\theta)^M$, whenever M is a model of $\mathsf{AD}_{\mathbb{R}} + V = L(P(\mathbb{R}))$ that has no iteration strategies for mice with long extenders. In particular, $\mathrm{HOD}^M \models \mathsf{GCH}$, for such M.

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0 Introduction

In this paper, we shall prove a general comparison lemma for iteration strategies. We then use it to develop the basic theory of hod mice in the least branch hierarchy.[1] [2]

Our comparison lemma relies heavily on an analysis of the normalization of a finite stack of iteration trees. Recall that an iteration tree $\mathcal{W}$ on a premouse M is *normal* iff the extenders $E_\alpha^{\mathcal{W}}$ used in $\mathcal{W}$ have lengths increasing with α, and each $E_\alpha^{\mathcal{W}}$ is applied to the longest possible initial segment of the earliest possible model in $\mathcal{W}$. Suppose now $\vec{\mathcal{T}}$ is a finite stack of iteration trees, with $\mathcal{T}_0$ being a normal tree on M, and $\mathcal{T}_{i+1}$ being a normal tree on the last model of $\mathcal{T}_i$. Let N be the last model of the last tree. There is a natural attempt to construct a "minimal" normal iteration tree $\mathcal{W}$ on M having last model N. This attempt may break down by reaching an illfounded model. If it does not break down, it will in the end produce a model P and $\pi : N \to P$ such that $\pi \circ i^{\vec{\mathcal{T}}} = i^{\mathcal{W}}$. We call $\mathcal{W}$ the *embedding normalization of* $\vec{\mathcal{T}}$.

If $\vec{\mathcal{T}}$ is played according to a reasonable iteration strategy Σ, then $\mathcal{W}$ is also by Σ, so the $\mathcal{W}$-construction does not break down. Although it is embedding normalization that is important to us here, one can also ask whether there is a normal tree on M whose last model is equal to N. We shall show that this is true if M is an iterable premouse, and $\vec{\mathcal{T}}$ is a finite stack of finite trees. The proof gives that there is a full normalization of $\vec{\mathcal{T}}$ in other cases as well.

Some of our work on normalization was done earlier (but never written up) with Itay Neeman, and then later with Grigor Sargsyan. Fuchs, Neeman and Schindler ([5]) and Mitchell ([9]), and probably others, have considered the question. Much of what seems to be new in this part of the paper was done independently, and at roughly the same time, by Farmer Schlutzenberg. (See [26].) Schlutzenberg and the author have carried this work further, and in particular analyzed embedding normalization and full normalization for infinite stacks of normal trees. See [27].

The reasonableness of iteration strategies with respect to embedding normalization is isolated in

Definition 0.1 *Let Σ be an iteration strategy for a (hod) premouse M. We say that*

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Σ normalizes well *iff whenever* $\vec{\mathcal{T}}$ *is a finite stack of normal trees by* Σ*, and* $\mathcal{W}$ *is an embedding normalization of* $\vec{\mathcal{T}}$*, then* $\mathcal{W}$ *is by* Σ.

The concept is defined more fully in 3.1, and that should be considered the official definition of normalizing well.

Embedding normalization actually makes sense for coarse-structural stacks $\vec{\mathcal{T}}$ on coarse-structural M. Granted the appropriate form of UBH in V, the iteration strategy Σ^* for V normalizes well. In particular, if we assume AD^+, and then let V be a coarse Γ-Woodin model N_x^* as in Theorem 10.3 of [30] (due to Woodin), then the iteration strategy Σ^* we get for N_x^* normalizes well.

We shall show that the property of normalizing well passes from Σ^* (for V) to the iteration strategy of M, whenever M is a level of the (hod or pure-extender) mouse construction in V. The proof of this is like Sargsyan's proof that hull condensation passes to induced strategies (Lemma 2.9 of [16]). It is important here that we defined normalizing well in terms of embedding normalizations. We do show in [33] that if Σ is induced by Σ^* for N^* as above, and $\vec{\mathcal{T}}$ is by Σ, and $\mathcal{U}$ is its full normalization, then $\mathcal{U}$ is by Σ. However, the proof does not proceed by some direct, combinatorial route. It involves a comparison argument, and so cannot be used until a comparison theorem for iteration strategies has already been proved.

We shall define a slight strengthening of hull condensation, and show that it passes from Σ^* for V to Σ for $\mathcal{M}$, where Σ is the induced iteration strategy for a level $\mathcal{M}$ of a full background construction. We shall call this property *strong hull condensation*. The details are in §3.

With these properties in hand, we can state our strategy comparison theorem. We state first a version that has AD^+ as its hypothesis.

Let Σ be a strategy for M, $\vec{\mathcal{T}}$ a stack on M with last model P, and Q an initial segment of P; then $\Sigma_{\vec{\mathcal{T}},Q}$ is the $\vec{\mathcal{T}}$-tail of Σ restricted to stacks on Q. So for $\vec{\mathcal{U}}$ on Q:

$$\Sigma_{\vec{\mathcal{T}},Q}(\vec{\mathcal{U}}) = \Sigma(\vec{\mathcal{T}}^\frown\vec{\mathcal{U}}).$$

Strategy comparison involves lining up such tail strategies.

Theorem 0.2 *Assume* AD^+*, and for* M *and* N *be countable (hod or pure-extender) premice, with Suslin-co-Suslin* $(\omega, \omega, \omega_1)$*-iteration strategies* Σ *and* Ω *respectively. Suppose* Σ *and* Ω *normalize well and have strong hull condensation. Then there are countable normal trees* $\mathcal{T}$ *on* M *and* $\mathcal{U}$ *on* N *by* Σ *and* Ω*, with last models* P *and* Q *respectively, such that either*

1. $P \trianglelefteq Q$*, and* $\Sigma_{\mathcal{T},P}$ *agrees with* $\Omega_{\mathcal{U},P}$ *on finite stacks of normal trees, or*

2. $Q \trianglelefteq P$*, and* $\Sigma_{\mathcal{U},\mathcal{Q}}$ *agrees with* $\Omega_{\mathcal{T},Q}$ *on finite stacks of normal trees.*

This seems to be new even in the case of pure extender premice. Of course, if we drop the strategy-agreement condition, it becomes in that case the usual Comparison Lemma.

By *hod premouse* we mean here what we call in section 5 below a *least branch hod premouse*. In earlier work, Woodin, Sargsyan, and the author have developed the theory of hod mice in a different hierarchy, the “rigidly layered” or “extender biased” hierarchy. See [38], [30], [16], [17], [18], and [32]. This hierarchy becomes quite complicated once one reaches the level of strong cardinals that are limits of Woodin cardinals, and it is not known how to properly define it much past that. Moreover, the extent of extender bias is controlled by the background determinacy model M whose HOD is being analyzed, so that there are different notions of hod mouse corresponding to different M, and we do not have such elementary condensation results as "the first level of P satisfying the sentence φ is countable”. The least branch hierarchy is much simpler and more uniform. It has condensation properties like those of the pure extender hierarchy. There is no extender bias; one simply tells the model P being built, at essentially every stage, a branch for the first iteration tree $\mathcal{T}$ it has constructed that is according to the strategy it is being told, and such that it has not been told a branch for $\mathcal{T}$ yet. We give the detailed definition in section 5.

The “least branch” idea originates in unpublished work of Woodin. The new comparison process is what makes it possible to use this hierarchy to analyze HOD^M, for $M \models \mathsf{AD}^+$, in the short extender realm. We believe it will some day be possible to use it in the long extender realm as well.

By a *least branch hod pair* we mean a pair (M, Σ) such that M is a least branch premouse, and Σ is an iteration strategy for M (generally defined on countable stacks of countable normal trees) that normalizes well and has strong hull condensation. The full definition is given in 5.16. If M is a pure extender premouse, and Σ normalizes well and has strong hull condensation, then we call (M, Σ) a *pure extender pair*. The full definition is 5.19. A pair of one of the two types we call a *mouse pair*. Theorem 0.2 says that assuming AD^+, any two mouse pairs of the same type can be compared.

We prove 0.2 by putting M and N into a common Γ-Woodin universe N^*, where Σ and Ω are in $\Gamma \cap \check{\Gamma}$. We then iterate (M, Σ) and (N, Ω) into levels of the full background construction (of the appropriate type) of N^*. Here are some definitions encapsulating the method.

Definition 0.3 *Let (M, Σ) and (N, Ω) be mouse pairs of the same type; then*

(a) (M, Σ) iterates past (N, Ω) *iff there is a normal iteration tree $\mathcal{T}$ by Σ on M*

with last model Q such that $N \trianglelefteq Q$, and $\Sigma_{\mathcal{T},N} = \Omega$.

(b) (M, Σ) iterates to (N, Ω) *iff there are $\mathcal{T}$ and Q as in (a), and moreover, $N = Q$, and the branch M-to-Q of $\mathcal{T}$ does not drop.*

(c) (M, Σ) iterates strictly past (N, Ω) *iff it iterates past (N, Ω), but not to (N, Ω).*

Definition 0.4 (AD^+) *Let (P, Σ) be a mouse pair; then $(*)(P, \Sigma)$ is the assertion:*

Let N^* be any coarse Γ-Woodin model with iteration strategy Ψ as in 10.1 of [30] (so Γ is inductive-like and has the scale property), such that $P \in HC^{N^*}$, and $\Sigma \in \Gamma \cap \check{\Gamma}$ is Suslin captured by (N^*, Ψ). Let $\mathbb{C}$ be a background construction done in N^* of the appropriate type, and let (R, Φ) be a level of $\mathbb{C}$. Suppose that (P, Σ) iterates strictly past all levels of $\mathbb{C}$ that are strictly earlier than (R, Φ); then (P, Σ) iterates past (R, Φ).

The conclusion of 0.4 asserts: suppose the comparison of P with R has produced a normal tree $\mathcal{T}$ on P with last model Q, with $\mathcal{T}$ by Σ, and $Q|\eta = R|\eta$; then $\Sigma_{\mathcal{T},Q|\eta}$ and $\Phi_{R|\eta}$ agree on finite stacks of normal trees. Thus the least disagreement between Q and R is an extender disagreement. Moreover, if E on Q and F on R are the extenders involved in it, then $F = \varnothing$.

We shall show (cf. Theorem 4.10 below)

Theorem 0.5 *Assume AD^+; then $(*)(P, \Sigma)$ holds, for all mouse pairs (P, Σ).*

We note

Proposition 0.6 *Theorem 0.5 implies Theorem 0.2.*

Proof. Let (M, Σ) and (N, Ω) be as in the hypotheses of 0.2. Let (N^*, Ψ) witness $(*)(M, \Sigma)$ and $(*)(N, \Omega)$ simultaneously. Let $\mathbb{C}$ be the full background extender construction of N^* of the appropriate type.

Claim 0.6.1 *There is a level R of $\mathbb{C}$ such that R is a Σ-iterate of M.*

Proof. Suppose first $\mathbb{C}$ breaks down, in that it has a least level Q such that Q is not ω-solid. Since M is ω-solid, and this is preserved by iteration, Q is not an initial segment of an iterate of M. By $(*)(M, \Sigma)$, M iterates to a proper initial segment of Q, with no strategy disagreement. This implies that some R properly before Q in $\mathbb{C}$ is a Σ iterate of M.

If $\mathbb{C}$ never breaks down, then let $Q = (N_\delta)^{\mathbb{C}}$, where δ is the Woodin of N^*. Then M cannot Σ-iterate past Q by the usual universality argument. (Note here $\Sigma \in \Gamma \cap \check{\Gamma}$.) So M iterates to a proper initial segment of Q, and thus some R properly before Q in $\mathbb{C}$ is a Σ-iterate of M. $\square$

Claim 0.6.2 *There is a level S of $\mathbb{C}$ such that S is an Ω-iterate of N.*

Proof. Symmetric. $\square$

Notice that the iterations provided by our two claims do not drop. Letting R and S be the last models, we may assume without lost of generality that R is before S in $\mathbb{C}$, or $R = S$. Let $\mathcal{T}$ be the normal tree on M by Σ with last model R. Let $\mathcal{U}$ be the normal tree on N by Ω that comes from comparing N with R. It is clear that $\mathcal{T}$ and $\mathcal{U}$ witness the conclusion of 0.2. $\square$

Least branch hod pairs can be used to analyze HOD in models of AD^+, provided that there are enough such pairs.

Definition 0.7 *(AD^+)*

(a) Hod Pair Capturing *(HPC) is the assertion: for every Suslin-co-Suslin set A, there is a least branch hod pair (P, Σ) such that A is definable from parameters over $(HC, \in, \Sigma)$.*

(b) $L[E]$ capturing *(LEC) is the assertion: for every Suslin-co-Suslin set A, there is a pure extender pair (P, Σ) such that A is definable from parameters over $(HC, \in, \Sigma)$.*

An equivalent (under AD^+) formulation would be that the sets of reals coding strategies of the type in question, under some natural map of the reals onto HC, are Wadge cofinal in the Suslin-co-Suslin sets of reals. The restriction to Suslin-co-Suslin sets A is necessary, for AD^+ implies that if (P, Σ) is a pair of one of the two types, then the codeset of Σ is Suslin and co-Suslin. This is proved in [33].

Remark 0.8 HPC is a cousin of Sargsyan's *Generation of Full Pointclasses*. See [16] and [17], §6.1.

Assuming AD^+, LEC is equivalent to the well known Mouse Capturing: for reals x and y, x is ordinal definable from y iff x is in a pure extender mouse over y. This equivalence is shown in [30]. (See especially Theorem 16.6.) Using the results of this paper, one can show that under AD^+, LEC implies HPC. See 5.70 below. We do not know whether HPC implies LEC. This may be a hint that whether LEC holds is the more fundamental question.

Theorem 0.9 *Assume $\mathsf{AD}_{\mathbb{R}}$ and HPC; then $V_\theta \cap HOD$ is the universe of a least branch premouse.*

We believe Theorem 0.9 remains true if $\mathsf{AD}_{\mathbb{R}}$ is weakened to AD^+ in its hypothesis, but we do not have a proof. We shall prove an approximation to Theorem 0.9 in §7. The full theorem is proved in [33].

The natural conjecture is that LEC and HPC hold in all models of AD^+ that have not reached an iteration strategy for a premouse with a long extender. They cannot hold past that, of course.

Definition 0.10 NLE *("No long extenders") is the assertion: there is no countable, $\omega_1 + 1$-iterable pure extender premouse M such that there is a long extender on the M-sequence.*

Conjecture 0.11 *Assume* AD^+ *and* NLE*; then* LEC.

Conjecture 0.12 *Assume* AD^+ *and* NLE*; then* HPC.

As we remarked above, 0.11 implies 0.12. Conjecture 0.11 is equivalent to a slight strengthening of the usual Mouse Set Conjecture MSC. (The hypothesis of MSC is that there is no iteration strategy for a pure extender premouse with a superstrong, which is slightly stronger than NLE.) Hence by [16], both LEC and HPC hold in models of AD^+ that are below the minimal model of $\mathsf{AD}_{\mathbb{R}}$ + "θ is regular". By [18], they hold in all models of AD^+ below the minimal model of AD^+ + "the largest Suslin cardinal belongs to the Solovay sequence".

The mouse pairs witnessing LEC and HPC are produced in background extender constructions. One important context in which such constructions can be done is described in the following theorem.

Theorem 0.13 *Assume* AD^+*, let Γ be an inductive-like pointclass with the scale property, and such that all sets in $\check{\Gamma}$ are Suslin. Let (N^*, Ψ) be a coarse Γ-Woodin together with its unique Γ-fullness preserving strategy. (cf. 10.1 of [16]) Let M be a level of the (hod or pure extender) full background construction of N^*, then letting Σ be the strategy for M induced by Ψ,*

- (a) *Σ normalizes well and has strong hull condensation,*
- (b) *$(^*)(M, \Sigma)$, and*
- (c) *$\mathcal{M}$ is ω-solid.*

We stated part (a) above. We shall prove it in §3. That (a) $\Rightarrow$ (b) is Theorem 0.5. We shall prove 0.5 in §4; see also §5.4. We prove part (c) of Theorem 0.13 in §5.7.

For pure extender mice, it is a standard theorem. The proof for hod mice resembles the proof for pure extender mice, but there are some extra difficulties in adapting the comparison process implicit in the proof of 0.6 to the comparison of phalanxes. Finally, the various pieces of the proof of Theorem 0.13 are gathered together in §5.8.

One can also prove a version of Theorem 0.13 for the least-branch hod mouse construction of V, provided that V is iterable by the strategy of choosing unique cofinal wellfounded branches for nice, normal iteration trees.

One must be careful in formulating unique iterability and UBH to restrict to nice, normal trees on V. Let us say that a tree is *nice* (or *strongly closed*) if all its extenders have length = strength an inaccessible-but-not-measurable cardinal in the model from which they are taken. The restriction to nice trees is needed to avoid some counterexamples to UBH due to Woodin. (See [39], [14], and section 3.) It is quite plausible to the author that UBH for stacks of nice, normal trees on V is true.

Woodin has shown that if κ is supercompact, and this form of UBH holds, then V is iterable by the strategy of choosing unique cofinal wellfounded branches for nice, normal iteration trees with all critical points above κ. We shall prove this in section 3. We also show that if V is iterable for nice, normal trees by the strategy of choosing unique cofinal wellfounded branches, and $\vec{F}$ is any coarsely coherent sequence of extenders, then V is iterable for stacks of normal $\vec{F}$-trees by the strategy of choosing unique cofinal wellfounded branches. (An $\vec{F}$-tree is an iteration tree all of whose extenders come from $\vec{F}$.) The resulting $\vec{F}$-iteration strategy normalizes well.

With these preliminaries, we can state our theorem about hod-mouse constructions done in V, assuming the existence of very large cardinals.

Theorem 0.14 *Suppose V is normally iterable above μ by the strategy of choosing unique cofinal wellfounded branches. Suppose there is a $j\colon V \to N$ such that for $\kappa = \mathrm{crit}(j)$, $\kappa > \mu$, $V_{j(\kappa)} \subseteq N$, and $j(\kappa)$ is inaccessible; then there is a canonical inner model M such that $M \models$ "There is a superstrong cardinal", and $M \models$ "I am iterable".*

Corollary 0.15 *Let μ be supercompact, and that* UBH *holds for nice, normal iteration trees on V with all critical points $> \mu$. Suppose also there is a $j\colon V \to N$ such that for $\kappa = \mathrm{crit}(j)$, $\kappa > \mu$, $V_{j(\kappa)} \subseteq N$, and $j(\kappa)$ is inaccessible; then there is a canonical inner model M such that $M \models$ "There is a superstrong cardinal", and $M \models$ "I am iterable".*

At bottom, the proof of 0.14 is the same as that for 0.13. We give it in §5.8. The inner model M of 0.14 is a hod premouse in the least-branch hierarchy. The hypothesis of the theorem requires a little more than a superstrong cardinal in V, but it seems quite likely one could make do with just a superstrong above μ.

One can arrange that the hod mouse M of theorem 0.14 has a limit of Woodin cardinals λ above its superstrong. Its derived model $D(M, \lambda)$ is then a model of $\mathsf{AD}_{\mathbb{R}}$ in which there is an iteration strategy for a hod mouse with a superstrong cardinal. The usual methods for computing HOD show that in fact

$$\mathrm{HOD}^{D(M,\lambda)} \models \mathsf{GCH} + \text{ there is a superstrong cardinal.}$$

One can realize $D(M, \lambda)$ as a Wadge cut in Hom_∞ by using an $\mathbb{R}$-genericity iteration. This leads to

Theorem 0.16 *Suppose V is normally iterable above κ by the strategy of choosing unique cofinal wellfounded branches. Suppose there is a superstrong cardinal $\lambda > \kappa$, and suppose there are arbitrarily large Woodin cardinals; then there is a Wadge cut Γ in Hom_∞ such that $L(\Gamma, \mathbb{R}) \models \mathsf{AD}_{\mathbb{R}}$, and*

$$HOD^{L(\Gamma,\mathbb{R})} \models \mathsf{GCH} + \textit{ there is a superstrong cardinal.}$$

We shall prove this theorem in the last section of the paper. Of course, there are much stronger statements of this kind that are well known, and likely to be true. The theorem does make a point: granted unique iterability for V, the HOD's of AD^+ models can be hod mice with superstrongs.

Remark 0.17 It is well known that if there is a supercompact cardinal, and there are arbitrarily large Woodin cardinals, and the appropriate form of UBH holds, then there is a Hom_∞ iteration strategy for a pure extender premouse M such that M has a long extender on its sequence. So we have a Wadge minimal cut Γ_0 in Hom_∞ such that $L(\Gamma_0, \mathbb{R}) \not\models \mathsf{NLE}$. We show in [33] that if $L(\Gamma, \mathbb{R})$ is a proper Wadge cut in $L(\Gamma_0, \mathbb{R})$, then both LEC and HPC hold in $L(\Gamma, \mathbb{R})$. Using 0.9, this yields a significant strengthening of Theorem 0.16.

In what follows, we shall give fairly complete proofs of the theorems above. The paper is long, partly because we wanted to check things carefully, and partly because we are looking more closely at the construction of iteration strategies in [10] (FSIT), and there are many details there. However, the main new idea behind our strategy-comparison theorem is quite simple. We describe it now.

The first step is to focus on proving (*)(P, Σ). That is, rather than directly comparing two strategies, we iterate them both into a common background construction and its strategy. In the comparison-of-mice context, this method goes back to Kunen

([7]), and was further developed by Mitchell and Baldwin ([2]). The first proof of comparison for pure extender mice with Woodin cardinals had this form, and Woodin and Sargsyan had used the method for strategy comparison in the hod mouse context. All these comparisons could be replaced by direct comparisons of the two mice or strategies involved, but in the general case of comparison of strategies, there are serious advantages to the indirect approach. There is no need to decide what to do if one encounters a strategy disagreement, because one is proving that that never happens. The comparison process is just the usual one of comparing least extender disagreements. Instead of the dual problems of designing a process and proving it terminates, one has a given process, and knows why it should terminate: no strategy disagreements show up. The problem is just to show this. These advantages led the author to focus, since 2009, on trying to prove $(*)(P, \Sigma)$.

The main new idea that makes this possible is motivated by Sargsyan's proof in [16] that if Σ has branch condensation, then $(*)(P, \Sigma)$ holds. Branch condensation is too strong to hold once P has extenders overlapping Woodin cardinals; we cannot conclude that $\Sigma(\mathcal{T}) = b$ from having merely realized $\mathcal{M}_b^{\mathcal{T}}$ into a Σ-iterate of P. We need some kind of realization of the entire phalanx $\Phi(\mathcal{T}^\frown b)$ in order to conclude that $\Sigma(\mathcal{T}) = b$. This leads to a weakening of branch condensation that one might call "phalanx condensation", in which one asks for a family of branch-condensation-like realizations having some natural agreement with one another. Phalanx condensation is still strong enough to imply $(*)(P, \Sigma)$, and might well be true in general for background-induced strategies. Unfortunately, Sargsyan's construction of strategies with branch condensation does not seem to yield phalanx condensation in the more general case. For one thing, it involves comparison arguments, and in the general case, this looks like a vicious circle. It was during one of the author's many attempts to break into this circle that he realized that certain properties related to phalanx condensation, namely normalizing well and strong hull condensation, could be obtained directly for background-induced strategies, and that these properties suffice for $(*)(P, \Sigma)$.

Let us explain this last part briefly. Suppose that we are in the context of Theorem 0.5. We have a premouse P with iteration strategy Σ that normalizes well and has strong hull condensation. We have N a premouse occuring in the fully backgrounded construction of N^*, where $P \in \mathrm{HC}^{N^*}$ and N^* captures Σ. We compare P with N by iterating away the least extender disagreement. It has been known since 1985 that only P will move. We must prove that no strategy disagreement shows up.

Suppose we have produced an iteration tree $\mathcal{T}$ on P with last model Q, and that $Q|\alpha = N|\alpha$, and that $\mathcal{U}$ is a tree on $R = Q|\alpha = N|\alpha$ played by both $\Sigma_{\mathcal{T},Q|\alpha}$

(the tail of Σ) and Ω, the N^*-induced strategy for N. Let $\mathcal{U}$ have limit length, and let $b = \Omega(\mathcal{U})$. We must see $b = \Sigma(\langle \mathcal{T}, \mathcal{U} \rangle)$. For this, we look at the embedding normalization $W(\mathcal{T}, \mathcal{U})$ of $\langle \mathcal{T}, \mathcal{U} \rangle$, which also has limit length. We shall see:

(1) b generates (modulo $\mathcal{T}$) a unique cofinal branch a of $W(\mathcal{T}, \mathcal{U})$ (see §2.7).

(2) Letting $i_b^* : N^* \to N_b^*$ come from lifting $i_b^{\mathcal{U}}$ to N^* via the iteration-strategy construction of [10], we have that $W(\mathcal{T}, \mathcal{U})^\frown \langle a \rangle$ is a pseudo-hull of $i_b^*(\mathcal{T})$. This is the key step in the proof. It is carried out in section 4.3.

(3) $i_b^*(\Sigma) \subseteq \Sigma$ because Σ was Suslin-co-Suslin captured by N^*, so $i_b^*(\mathcal{T})$ is by Σ.

(4) Thus $W(\mathcal{T}, \mathcal{U})^\frown \langle a \rangle$ is by Σ, because Σ has strong hull condensation.

(5) Since a determines b (see §2.7), and Σ normalizes well, we must then have $\Sigma(\langle \mathcal{T}, \mathcal{U} \rangle) = b$, as desired.

Here is a diagram of the situation:

$$\begin{tikzcd}
M_b^{\mathcal{U}} \arrow[r] & M_\infty^{\mathcal{W}_b} \arrow[r] & i_b^*(R) & N_b^* \\
R \arrow[u, "\mathcal{U}"] & & R \arrow[u, "i_b^*"'] & N^* \arrow[u, "i_b^*"'] \\
P \arrow[u, "\mathcal{T}"] \arrow[uur, "\mathcal{W}_b"] \arrow[uurr, "i_b^*(\mathcal{T})"] & & &
\end{tikzcd}$$

Figure 0.1: Proof of $(*)(P, \Sigma)$. $\mathcal{W}_b$ is a psuedo-hull of $i_b^*(\mathcal{T})$.

Remark 0.18 We suspect that the existing iterability proofs will adapt to the hod-mouse hierarchy. So the following seem accessible:

1. Suppose κ is supercompact, and there is a Woodin limit of Woodins above κ; then there is a canonical inner model satisfying "There is a Woodin limit of Woodins, and I am iterable".

2. Assume PFA; then there is a canonical inner satisfying "There is a λ that is a limit of Woodins and $<\lambda$-strongs, and I am iterable".

Project 1 would use [11], 2 would use [1].

Historical note. The author proved the main theorems of this paper in Spring 2015. They have been circulated as a handwritten manuscript since July 2015. Something close to the present typewritten version has been circulated since April 2016.

1 Preliminaries

Inner model theory deals with canonical objects, but inner model theorists have presented them in various ways. The conventions we use here are all fairly common. For basic fine structural notions such as projecta, cores, standard parameters, fine ultrapowers, and degrees of elementarity, we shall follow the paper [23] by Schindler and Zeman. We shall use Jensen indexing for the sequences of extenders from which premice are constructed; see for example Zeman's book [40]. The construction of premice using background extenders comes ultimately from Mitchell-Steel [10], but the precise definitions and notation we use come from Neeman-Steel [15]. Here is some further detail.

1.1 Extenders and ultrapowers

Our notation for extenders is standard.

Definition 1.1 *Let M be transitive and rudimentarily closed; then $E = \langle E_a \mid a \in [\theta]^{<\omega}\rangle$ is a (κ, θ)-extender over M with spaces $\langle \mu_a \mid a \in [\theta]^{<\omega}\rangle$ if and only if*

(1) Each E_a is an (M, κ)-complete ultrafilter over $P([\mu_a]^{|a|}) \cap M$, with μ_a being the least μ such that $[\mu]^{|a|} \in E_a$.

(2) (Compatibility) For $a \subseteq b$ and $X \in M$, $X \in E_a \Leftrightarrow X^{ab} \in E_b$.

(3) (Uniformity)$\mu_{\{\kappa\}} = \kappa$.

(4) (Normality) If $f \in M$ and $f(u) < \max(u)$ for E_a a.e. u, then there is a $\beta < \max(u)$ such that for $E_{a\cup\{\beta\}}$ a.e. u, $f^{a,a\cup\{\beta\}}(u) = u^{a,a\cup\{\beta\}}$.

The unexplained notation here can be found in [23, §8]. We shall often identify E with the binary relation $(a, X) \in E$ iff $X \in E_a$. One can also identify it with the other section-function of this binary relation, which is essentially the function $X \mapsto i^M_E(X) \cap \theta$. We call θ the *length* of E, and write $\theta = \mathrm{lh}(E)$. The *space of E* is

$$\mathrm{sp}(E) = \sup\{\mu_a \mid a \in [\mathrm{lh}(E)]^{<\omega}\}.$$

The *domain of E* is the family of sets it measures, that is, $\mathrm{dom}(E) = \{X \mid \exists a(a, X) \in E\}$. If M is a premouse of some kind, we also write $M|\eta = \mathrm{dom}(E)$, where η is least such that $\forall(a, X) \in E)X \in M|\eta$. By acceptability, $\eta = \sup(\{\mu_a^{+,M} \mid a \in [\theta]^{<\omega}\})$. The critical point of a (κ, θ) extender is κ, and we use either $\mathrm{crit}(E)$ or κ_E to denote it.

Given an extender E over M, we form the Σ_0 ultrapower

$$\mathrm{Ult}_0(M,E) = \{[a,f]^M_E \mid a \in [\mathrm{lh}(E)]^{<\omega} \text{ and } f \in M\},$$

as in [23, 8.4]. Our M will always be rudimentarily closed and satisfy the Axiom of Choice, so we have Los' theorem for Σ_0 formulae, and the canonical embedding

$$i^M_E \colon M \to \mathrm{Ult}_0(M,E)$$

is cofinal and Σ_0-elementary, and hence Σ_1-elementary. By normality, $a = [a, \mathrm{id}]^M_E$, so $\mathrm{lh}(E)$ is included in the (always transitivized) wellfounded part of $\mathrm{Ult}_0(M,E)$. More generally,

$$[a,f]^M_E = i^M_E(f)(a).$$

If $X \subseteq \mathrm{lh}(E)$, then $E \restriction X = \{(a,X) \in E \mid a \subseteq X\}$. $E \restriction X$ has the properties of an extender, except possibly normality, so we can form $\mathrm{Ult}_0(M, E\restriction X)$, and there is a natural factor embedding $\tau\colon \mathrm{Ult}_0(M, E \restriction X) \to \mathrm{Ult}_0(M,E)$ given by

$$\tau([a,f]^M_{E\restriction X}) = [a,f]^M_E.$$

In the case that $X = \nu > \kappa_E$ is an ordinal, $E \restriction \nu$ is an extender, and $\tau \restriction \nu$ is the identity. We say ν is a *generator of* E iff ν is the critical point of τ, that is, $\nu \neq [a,f]^M_E$ whenever $f \in M$ and $a \subseteq \nu$. Let

$$\nu(E) = \sup(\{\nu + 1 \mid \nu \text{ is a generator of } E\ \}).$$

So $\nu(E) \leq \mathrm{lh}(E)$, and E is equivalent to $E\restriction\nu(E)$, in that the two produce the same ultrapower.

We write $\lambda(E)$ or λ_E for $i^M_E(\kappa_E)$. Note that although E may be an extender over more than one M, $\mathrm{sp}(E), \kappa_E, \mathrm{lh}(E), \mathrm{dom}(E), \nu(E)$, and $\lambda(E)$ depend only on E itself. If N is another transitive, rudimentarily closed set, and $P(\mu_a) \cap N = P(\mu_a) \cap M$ for all $a \in [\mathrm{lh}(E)]^{<\omega}$, then E is also an extender over N; moreover i^M_E agrees with i^N_E on $\mathrm{dom}(E)$. However, i^M_E and i^N_E may disagree beyond that. We say E is *short* iff $\nu(E) \leq \lambda(E)$. It is easy to see that E is short iff $\mathrm{lh}(E) \leq \sup(i^M_E``((\kappa^+_E)^M))$. If E is short, then all its interesting measures concentrate on the critical point. When E is short, i^M_E is continuous at $\kappa^{+,M}$, and if M is a premouse, then $\mathrm{dom}(E) = M|\kappa^{+,M}_E$. In this paper, we shall deal almost exclusively with short extenders. If we start with $j\colon M \to N$ with critical point κ, and an ordinal ν such that $\kappa < \nu \leq o(N)$, then for $a \in [\nu]^{<\omega}$ we let μ_a be the least μ such that $a \subseteq j(\mu)$, and for $X \subseteq [\mu_a]^{|a|}$ in M, we put

$$(a,X) \in E_j \Leftrightarrow a \in j(X).$$

E_j is an extender over M, called the (κ,ν) *extender derived from* j. We have the diagram

$$\begin{tikzcd}
M \arrow[rr, "j"] \arrow[drr, "i_E^M"'] & & N \\
& & \mathrm{Ult}(M,E) \arrow[u, "k"']
\end{tikzcd}$$

where $i = i^M_{E_j}$, and

$$k(i(f)(a)) = j(f)(a).$$

$k{\restriction}\nu$ is the identity. If E is an extender over M, then E is derived from i^M_E.

The *Jensen completion* of a short extender E over some M is the $(\kappa_E, i^M_E((\kappa_E^+)^M))$ extender derived from i^M_E. E and its Jensen completion E^* are equivalent, in that $\nu(E) = \nu(E^*)$, and $E = E^*{\restriction}\mathrm{lh}(E)$.

1.2 Pure extender premice

Our main results apply to premice of various kinds, both hod premice and pure extender premice, with λ-indexing or ms-indexing for their extender sequences. The comparison theorem for iteration strategies that is our first main goal holds in all these contexts. Although the proof of this theorem requires a detailed fine-structural analysis, the particulars of the fine structure don't affect anything important. We shall prove it first in the case of iteration strategies for pure extender premice with λ-indexing. The essential equivalence of λ-indexing with ms-indexing has been carefully demonstrated by Fuchs in [3] and [4].

The reader should see [1, Def. 2.4] for further details on the following definition. A *Jensen premouse* is a pair

$$M = \langle \hat{M}, k \rangle,$$

where

$$\hat{M} = \langle J^{\vec{E}}_\alpha, \in, \vec{E}, \gamma, F \rangle$$

is an acceptable structure with various properties, and $k \leq \omega$. The language $\mathcal{L}_0$ of $\hat{M}$ has $\in$, predicate symbols $\dot{E}$ and $\dot{F}$, and a constant symbol $\dot{\gamma}$. We call $\mathcal{L}_0$ the *language of (pure extender) premice*. We write $k = k(M)$; it marks the level of the Levy hierarchy over $\hat{M}$ at which we are considering this structure, and we demand that $\hat{M}$ be $k(M)$-sound. So what we are calling a premouse is just a premouse in the usual sense, paired with a degree of soundness that it has. We usually abuse notation by identifying M with $\hat{M}$.

Abusing notation this way, we set $o(M) = \mathrm{ORD} \cap M$, so that $o(M) = \omega\alpha$ for M as displayed. (The [23] convention differs slightly here.) We write $\hat{o}(M)$ for α itself. The index of M is

$$l(M) = \langle \hat{o}(M), k(M) \rangle.$$

If $\langle \nu, l \rangle \leq_{\text{lex}} l(M)$, then $M|\langle \nu, l\rangle$ is the initial segment N of M with index $l(N) = \langle \nu, l \rangle$. (So $\dot{E}^N = \dot{E}^M \cap N$, and $\dot{F}^N = \dot{E}^M_\nu$.) If $\nu \leq \hat{o}(M)$, then we write $M|\nu$ for $M|\langle \nu, 0\rangle$. We write $M||\nu$, or sometimes $M|\langle \nu, -1\rangle$, for the structure that agrees with $M|\nu$ except possibly on the interpretation of $\dot{F}$, and satisfies $\dot{F}^{M||\nu} = \emptyset$. By convention, $k(M||\nu) = 0$.

Definition 1.2 *If P and Q are Jensen premice, then $P \trianglelefteq Q$ iff there are μ and l such that $P = Q|\langle \mu, l\rangle$. Also, $P \triangleleft Q$ iff $P \trianglelefteq Q$ and $P \neq Q$.*

Thus if P and Q have the same universe, but $k(P) < k(Q)$, then $P \triangleleft Q$. Also, if P is passive and Q is active at $o(P)$, then it is not the case that $P \trianglelefteq Q$. So for example, if Q is active, then $Q||o(Q) \not\trianglelefteq Q$, where $Q||o(Q)$ is Q with its last extender predicate removed. Other conventions would be possible, but this one works best here.

If M is a Jensen premouse, then $\dot{E}^M$ is a sequence of extenders, and $\dot{F}^M$ is either empty, or codes a new extender being added to our model by M. The main requirements are

(1) (λ-indexing) If $F = \dot{F}^M$ is nonempty (i.e., M is *active*), then $M \models \mathrm{crit}(F)^+$ exists, and for $\mu = \mathrm{crit}(F)^{+M}$, $o(M) = i^M_F(\mu)$. $\dot{F}^M$ is just the graph of $i^M_F \restriction (M|\mu)$.

(2) (Coherence) $i^M_F(\dot{E}^M)\restriction o(M) + 1 = \dot{E}^M{}^\frown\langle \emptyset \rangle$.

(3) (Initial segment condition, J-ISC) If G is a whole proper initial segment of F, then the Jensen completion of G must appear in $\dot{E}^M$. If there is a largest whole proper initial segment, then $\dot{\gamma}^M$ is the index of its Jensen completion in $\dot{E}^M$. Otherwise, $\dot{\gamma}^M = 0$.

(4) If N is an initial segment of M, then N is $k(N)$ sound.

Here an initial segment $G = F\restriction\eta$ of F is *whole* iff $\eta = \lambda_G$. Since Jensen premice are acceptable J-structures, the basic fine structural notions apply to them, so clause (4) above makes sense.

Figure 1.1 illustrates a common situation, one that occurs at successor steps in an iteration tree, for example.

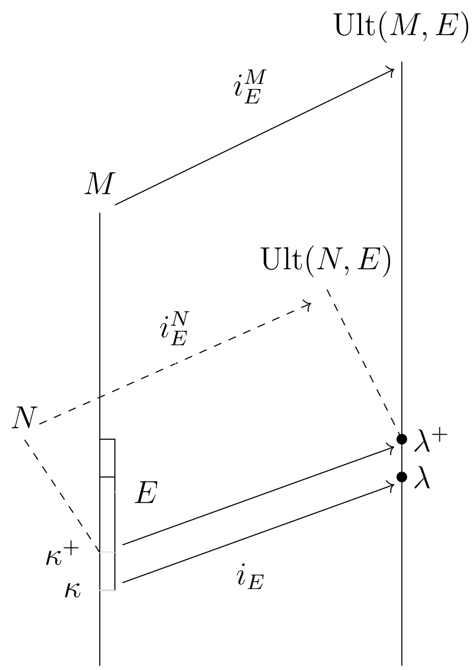


Figure 1.1: E is on the coherent sequence of M, $\kappa = \text{crit}(E)$, and $\lambda = \lambda(E)$. $P(\kappa)^M = P(\kappa)^N = \text{dom}(E)$, so Ult$(M, E)$ and Ult(N, E) make sense. The ultrapowers agree with M below lh(E), and with each other below lh$(E) + 1$.

There is a significant strengthening of the Jensen initial segment condition (3) above. If M is an active premouse, then we set

$$\nu(M) = \max(\nu(\dot{F}^M), \text{crit}(\dot{F}^M)^{+,M}).$$

$\dot{F}^M \restriction \nu(M)$ is equivalent to $\dot{F}^M$, and so it is not in M. But

Definition 1.3 *Let M be an active premouse with last extender F; then M satisfies the* ms-ISC *(or is* ms-solid*) iff for any $\eta < \nu(M)$, $F \restriction \eta \in M$.*

Theorem 1.4 *(*ms-ISC*) Let M be an active premouse with last extender F, and suppose M is 1-sound and $(1, \omega, \omega_1 + 1)$-iterable; then M is ms-solid.*

This is essentially the initial segment condition of [10], but stated for Jensen premice. [10] goes on to say that the trivial completion of $F \restriction \eta$ is either on the M-sequence, or an ultrapower away. This is correct unless $F \restriction \eta$ is type Z. If $F \restriction \eta$ is type Z, then it is the extender of $F \restriction \xi$-then-U, where ξ is its largest generator, and U is an ultrafilter on ξ, and we still get $F \restriction \eta \in M$. (See [24]. Theorem 2.7 of [24] is essentially 1.4 above.)

If M is active, we let its *initial segment ordinal* be

$$\iota(M) = \sup(\{\eta + 1 \mid \dot{F}^M \restriction \eta \in M\}).$$

So M is ms-solid iff $\iota(M) = \nu(M)$. Theorem 1.4 becomes false when its soundness hypothesis is removed, since if $N = Ult_0(M, E)$ where $\nu(M) \leq \text{crit}(E) < \lambda_F$, then $\iota(N) = \iota(M) = \nu(M)$, but $\text{crit}(E) < \nu(N)$.

We shall not use ms-premice, so henceforth we shall refer to Jensen premice as premice, or later, when we need to distinguish them from hod premice, as *pure extender premice.*

1.3 Projecta and cores

If $M = (N, k)$ is a premouse, then N is a k-sound acceptable J-structure. Thus the projecta $\rho_i(N)$ and standard parameters $p_i(N)$ exist for all $i \leq k+1$, as do the reducts ("Σ_i mastercodes") $N^i = N^{i,p_i(N)}$. As in [23], if $i \leq k$, then

$$\rho_{i+1}(N) = \rho_1(N^i),$$

and

$$p_{i+1}(N) = p_i(N)^\frown\langle r\rangle,$$

where r is the lexicographically least descending sequence of ordinals from which a new subset of $\rho_1(N^i)$ can be Σ_1 defined over N^i. Clearly, $ORD \cap N^i = \rho_i(N)$, and $r \subseteq [\rho_{i+1}(N), \rho_i(N))$. If $i < k$, then r is solid, so each $\alpha \in r$ has a standard solidity witness $W^{\alpha,p_i(N)}_{N^i}$ that belongs to N^i.

Definition 1.5 *(a) If Q is an amenable J-structure, then h^1_Q is its canonical Σ_1 Skolem function.*

(b) If M is a premouse and $n \leq k(M)+1$, then h^{k+1}_M is the $r\Sigma_{k+1}$ Skolem function obtained by iteratively composing Σ_1 Skolem functions of reducts. (Cf. [23], 5.4.)

(c) Let $M = (N, k)$ be a premouse and $\alpha < \rho_k(N)$ and $r \in [\rho_k(M)]^{<\omega}$; then

$$W^{\alpha,r}_M = \text{ transitive collapse of } h^{k+1}_N``(\alpha \cup r \cup p_k(M)).$$

When $\alpha \in p_{k+1}(M)$ and $r = p_{k+1}(M) - (\alpha+1)$, we call $W^{\alpha,r}_M$ the standard solidity witness *for α.*

Abusing notation, we speak of $\rho_i(M), M^i$, etc., instead of $\rho_i(N), N^i$, etc. Finally, if $k < \omega$, we set

$$\rho(M) = \rho_{k+1}(M), p(M) = p_{k+1}(M), \text{ and } h_M = h_M^{k+1},$$

where $k = k(M)$, and call them *the* projectum, parameter, and Skolem function of M. Let

$$\mathcal{C}(M) = \mathcal{C}_{k(M)+1}(M) = \text{ transitive collapse of } h_M``(\rho(M) \cup p(M)),$$

considered as an $\mathcal{L}_0$-structure. Let $\pi\colon \mathcal{C}(M) \to M$ be the anticollapse, and $t = \pi^{-1}(p(M))$. We say that M is $k+1$ *solid*, or M *has a core*, iff $p_{k+1}(M)$ is $k+1$ universal over M, and t is $k+1$ solid over $\mathcal{C}(M)$. This implies that t is $k+1$ universal over $\mathcal{C}(M)$, that $p_{k+1}(M)$ is $k+1$-solid over M, and that $t = p_{k+1}(\mathcal{C}(M))$. If M is $k(M)+1$ solid, then $\mathcal{C}(M)$ is the core of M. We say that M is k-*sound* iff $M = \mathcal{C}_k(M)$, and simply *sound* iff $M = \mathcal{C}(M)$. When we wish to consider $\mathcal{C}(M)$ as a premouse with degree of soundness attached, we set

$$k(\mathcal{C}(M)) = k(M) + 1.$$

If M is $k+1$ solid, then M^{k+1} exists. M^{k+1} is the reduct which codes $\mathcal{C}_{k+1}(M)$.

For the notion of *generalized solidity witness*, see [23]. Roughly speaking, a generalized solidity witness for $\alpha \in p_1(M)$ is transitive structure whose theory includes $\mathrm{Th}_1^M(\alpha \cup p_1(M) - (\alpha+1))$. Being a generalized witness for an $\alpha \in p_k(M)$ is a $r\Pi_k$ condition, hence preserved by $r\Sigma_k$ embeddings. Such embeddings may not preserve being a standard witness.

The extension-of-embeddings lemmas relate reducts to the structures they code. The downward extension of embeddings lemma tells us that if S is amenable and $\pi\colon S \to N^n$ is Σ_0, then there is a (unique) M such that $S = M^n$. The upward extension lemma tells us that if $\pi\colon M^n \to S$ is Σ_1 and preserves the wellfoundedness of certain relations (the important one being $\in^M$ as it is described in the predicate of M^n), then there is a unique N such that $S = N^n$. See 5.10 and 5.11 of [23].

Remark 1.6 We have defined cores here as they are defined in [23]. In [10] they are defined in slightly different fashion. First, [10] works directly with the $\mathcal{C}_{k+1}(M)$, rather than with the reducts which code them. The translations indicated above show that is not a real difference; see [10], page 40. Second, if $k \geq 1$, then [10] puts the standard solidity witnesses for $p_k(M)$ into the hull collapsing to $\mathcal{C}_{k+1}(M)$, and if $k \geq 2$, it also puts $\rho_{k-1}(M)$ into this hull if $\rho_{k-1}(M) < o(M)$. The definition from [23] used above does not do this directly. We are grateful to Schindler and Zeman for pointing out that nevertheless these objects do get into the cores as defined in [23], and therefore the two definitions of $\mathcal{C}_{k+1}(M)$ are equivalent. [For example, let $k = 2$ and let M be 1-sound, with $\alpha \in p_1(M)$. Let $r = p_1(M) \setminus (\alpha+1)$. Let $\pi\colon \mathcal{C}_2(M) \to M$ be the anticore map, and $\pi(\beta) = \alpha$ and $\pi(s) = r$. The relation "W is a generalized solidity witness for α, r" is Π_1 over M. (It is important to add *generalized* here. Being a standard witness is only Π_2.) Since π is Σ_2 elementary, there is a generalized solidity witness for β, s over $\mathcal{C}_2(M)$ in $\mathcal{C}_2(M)$. But any generalized witness generates the standard one ([23], 7.4), so the standard solidity witness U for β, s is in $\mathcal{C}_2(M)$. Being the standard witness is Π_2, so $\pi(U)$ is the standard witness for α, r, and this witness is in $\mathrm{ran}(\pi)$, as desired.]

1.4 Elementarity of maps

Given n-sound acceptable J-structures M and N, and $\pi\colon M^n \to N^n$ a Σ_0 elementary embedding on their n-th reducts, then by decoding the reducts we get a unique $\hat{\pi}\colon M \to N$ that is Σ_n elementary and is such that $\pi \subseteq \hat{\pi}$. If π is Σ_1 elementary, then $\hat{\pi}$ is Σ_{n+1} elementary. The decoding is done iteratively, and yields that for $k < n$, $\hat{\pi}\colon M^k \to N^k$ is Σ_{n-k} or Σ_{n-k+1}, respectively.$\hat{\pi}$ is called the *n-completion* of π. See lemmas 5.8 and 5.9 of [23]. These lemmas record additional elementarity properties of $\hat{\pi}$, codified in definition 5.12 as *$r\Sigma_{n+1}$-elementarity* if π is Σ_1, and *weak $r\Sigma_{n+1}$-elementarity* if π is only Σ_0. Such maps are *cardinal preserving* , in that $M \models$ "γ is a cardinal" iff $N \models$ "$\pi(\gamma)$ is a cardinal", except possibly the weakly $r\Sigma_0$ maps. In this case, we shall always just add cardinal preservation as an additional hypothesis. This leads us to:

Definition 1.7 *Let M and N be Jensen premice with $n = k(M) = k(N)$, and $\pi\colon M \to N$; then*

(a) π is weakly elementary *iff π is the n-completion of $\pi \restriction M^n$, and $\pi \restriction M^n\colon M^n \to N^n$ is Σ_0 and cardinal preserving.*

(b) π is elementary *iff π is the n-completion of $\pi \restriction M^n$, and $\pi \restriction M^n\colon M^n \to N^n$ is Σ_1.*

(c) π is an n-embedding *iff π is elementary and* cofinal*, in the sense that* $\sup \pi``\rho_n(M) = \rho_n(N)$.

The elementary maps are correspond to those which are near n-embeddings in the sense of [21]. The cofinal elementary maps correspond to the n-embeddings. When $n \geq 1$, the weakly elementary embeddings correspond to those that are n-apt in the sense of [21], $\Sigma_0^{(n)}$ is the sense of [40], or n-lifting in the sense of [25]. There are many other levels of elementarity isolated in these references, but for our purposes this is enough.

In particular, we shall not use the notion of *weak n-embedding* defined in [10]. In the end, that notion is not very natural, and in a number of places it does not do the work that the authors of [10] thought that it did. In particular, there are problems with how it was used in the Shift Lemma, the copying construction, and the Weak Dodd-Jensen Lemma. These problems are discussed in [25], and a varity of ways to repair the earlier proofs are given. The simplest of these is to use weakly elementary maps instead of of weak n-embeddings at the appropriate places.

The following is clear from the definition:

Proposition 1.8 *Let M and N be Jensen premice with $n = k(M) = k(N)$, and $\pi\colon M \to N$ be weakly elementary; then*

(1) π is Σ_n elementary,

(2) $\pi(p_k(M)) = p_k(N)$ for all $k \leq n$, and

(3) $\pi(\rho_k(M)) = \rho_k(N)$ for $k < n-1$, and $\sup \pi``\rho_n(M) \leq \rho_n(N)$, and

(4) for any $\alpha < \rho_n(M)$, $\pi(Th_n^M(\alpha \cup p_n(M))) = Th_n(\pi(\alpha) \cup p_n(N))$.

It is easy to see that if π is (weakly) elementary as a map from (M, n) to (N, n), and $k < n$, then π is (weakly) elementary as a map from $(M, k) \to (N, k)$. Indeed, $\pi \restriction M^k$ is a stage in the decoding of $\pi \restriction M^n$. If $k(M) \neq k(N)$, then we say $\pi\colon M \to N$ is (weakly) elementary iff it is (weakly) elementary as a map from (M, n) to $N, n)$, where $n = \inf(k(M), k(N))$.

Note that if $\pi\colon M \to N$ is weakly elementary, and $k = \inf k(M), k(N)$, then π moves generalized solidity witnesses for $p_k(M)$ to generalized solidity witnesses for $p_k(N)$. For example, being a generalized witness for $p_1(M)$ is a Π_1 fact, so preserved by Σ_1 embeddings. Even cofinal elementary maps may fail to move standard solidity witnesses to standard solidity witnesses.

Here are some natural contexts in which the levels of elementarity play a role.

(i) The natural map from the core of M to M is elementary and cofinal, that is, a full n-embedding.

(ii) The maps $\hat{i}^{\mathcal{T}}_{\alpha,\beta}$ along branches of iteration trees are elementary and cofinal (see below).

(iii) If $\pi \colon M \to N$ is weakly elementary, and $\mathcal{T}$ is a weakly normal tree on M, then $\pi\mathcal{T}$ is weakly normal, and the copy maps $\pi_\alpha \colon \mathcal{M}_\alpha^{\mathcal{T}} \to \mathcal{M}_\alpha^{\pi\mathcal{T}}$ are weakly elementary.

(iv) If π, M, N, and $\mathcal{T}$ are as in (iii), and in addition, $\rho_k(N) \leq \pi(\rho_k(M))$ for $k = k(M)$, then all the π_α satisfy the corresponding condition, and if $\mathcal{T}$ is normal, then so is $\pi\mathcal{T}$.

(v) By Lemma 1.3 of [21]), if $\pi \colon M \to N$ is elementary, and $\mathcal{T}$ is a weakly normal tree on M, then the copy maps $\pi_\alpha \colon \mathcal{M}_\alpha^{\mathcal{T}} \to \mathcal{M}_\alpha^{\pi\mathcal{T}}$ are elementary. (They are not necessarily cofinal.) If π is only weakly elementary, then the copy maps are weakly elementary. The Dodd-Jensen and weak Dodd-Jensen lemmas holds in the category of weakly elementary maps.

(vi) The maps $\pi_\tau^{\nu,\gamma}$ occuring in an embedding normalization are elementary. The maps σ_γ are weakly elementary, but may not be elementary, so far as we can see. (See section 1.)

(vii) The lifting maps that occur in the proof of iterability are only weakly elementary. They are not in general elementary. (See below.)

1.5 Iteration trees

If M is a premouse with $n = k(M)$, and E is a short extender over M with $\kappa_E < \rho_n(M)$ and $P(\kappa_E))^M \subseteq \mathrm{dom}(E)$, then we set

$$\begin{aligned} \mathrm{Ult}(M, E) &= \mathrm{Ult}_n(M, E) \\ &= \text{ decoding of } \mathrm{Ult}_0(M^n, E). \end{aligned}$$

The canonical embedding of M^n into $\mathrm{Ult}(M^n, E)$ is Σ_1 and cofinal. Its n-completion $i_E^M \colon M \to \mathrm{Ult}_n(M, E)$ is therefore an n-embedding. (We assume here that $\mathrm{Ult}_n(M, E)$ is wellfounded, though one could make sense of these statements even if it is not.) By convention,

$$k(M) = k(\mathrm{Ult}(M, E)).$$

Rather than coding and decoding, one can define $\mathrm{Ult}(M, E)$ directly, as in [10]:

$$\mathrm{Ult}(M, E) = \{[a, f_{\tau,q}]_E^M \mid a \in [\lambda]^{<\omega} \wedge q \in M \wedge \tau \in \mathrm{SK}_n\},$$

where $n = k(M)$ and SK_n is the set of $r\Sigma_n$ Skolem terms.

If in addition $\rho(M) \leq \kappa_E$, $p(M)$ is solid, and E is close to M, then $\rho(M) = \rho(\mathrm{Ult}(M,E))$, and $\pi(p(M)) = p(\mathrm{Ult}(M,E))$, and $p(\mathrm{Ult}(M,E))$ is also solid.

Our notation and terminology regarding iteration trees is essentially that of [29]. If $\mathcal{T}$ is a tree on M, then $\mathcal{M}_\alpha^{\mathcal{T}}$ is its α-th model, and $E_\alpha^{\mathcal{T}}$ is the *exit extender* taken from the sequence of $\mathcal{M}_\alpha^{\mathcal{T}}$ and used to form

$$\mathcal{M}_{\alpha+1}^{\mathcal{T}} = \mathrm{Ult}(\mathcal{M}_{\alpha+1}^{*,\mathcal{T}}, E_\alpha^{\mathcal{T}}).$$

where

$$\mathcal{M}_{\alpha+1}^{*,\mathcal{T}} = \mathcal{M}_\beta^{\mathcal{T}} | \langle \xi, k \rangle$$

for some $\beta = T\text{-pred}(\alpha+1)$, and some $\langle \xi, k \rangle \leq l(\mathcal{M}_\beta^{\mathcal{T}})$ such that $\mathrm{crit}(E_\alpha^{\mathcal{T}}) < \rho_k(\mathcal{M}_\beta^{\mathcal{T}}|\xi)$. We put $\alpha+1 \in D^{\mathcal{T}}$ iff $\mathcal{M}_{\alpha+1}^{\mathcal{T}} \lhd \mathcal{M}_\beta^{\mathcal{T}}$ iff $l(\mathcal{M}_{\alpha+1}^{*,\mathcal{T}}) < l(\mathcal{M}_\beta^{\mathcal{T}})$, and we say $\mathcal{T}$ drops at $\alpha+1$ in this case. So unlike [29], drops in degree yield elements of $D^{\mathcal{T}}$ too. If $\alpha \leq_T \beta$ and $(\alpha, \beta]_T \cap D^{\mathcal{T}} = \emptyset$, then the canonical embedding

$$i_{\alpha,\beta}^{\mathcal{T}} \colon \mathcal{M}_\alpha^{\mathcal{T}} \to \mathcal{M}_\beta^{\mathcal{T}}$$

is cofinal and elementary; that is, it is an n-embedding, where $n = k(\mathcal{M}_\alpha^{\mathcal{T}}) = k(\mathcal{M}_\beta^{\mathcal{T}})$. All extenders in $\mathcal{T}$ are close to the models to which they are applied, so if $\mathrm{crit}(i_{\alpha,\beta}^{\mathcal{T}}) \geq \rho(\mathcal{M}_\alpha^{\mathcal{T}})$, then $\rho(\mathcal{M}_\alpha^{\mathcal{T}}) = \rho(\mathcal{M}_\beta^{\mathcal{T}})$ and $i_{\alpha,\beta}^{\mathcal{T}}(p(\mathcal{M}_\alpha^{\mathcal{T}})) = p(\mathcal{M}_\beta^{\mathcal{T}})$.

We shall also have a use for the natural partial embeddings that exist along branches that have dropped.

Definition 1.9 *Let $\mathcal{U}$ be an iteration tree, and $\alpha <_U \beta$. Then $\hat{\imath}_{\alpha,\beta}^{\mathcal{U}}$ is the natural map from a (perhaps proper!) initial segment of $\mathcal{M}_\alpha^{\mathcal{U}}$ into $\mathcal{M}_\beta^{\mathcal{U}}$. More precisely*

$$\hat{\imath}_{\alpha,\beta+1}^{\mathcal{U}} = i_{\beta+1}^{*\mathcal{U}} \circ \hat{\imath}_{\alpha,\gamma}^{\mathcal{U}}$$

if $\gamma = U\text{-pred}(\beta+1)$, and

$$\hat{\imath}_{\alpha,\lambda}^{\mathcal{U}}(x) = i_{\beta,\lambda}^{\mathcal{U}}(\hat{\imath}_{\alpha,\beta}(x))$$

if β is past the last drop in $[0,\lambda)_U$.

It would have been more natural to have originally defined $i_{\alpha,\beta}^{\mathcal{U}}$ the way we just defined $\hat{\imath}_{\alpha,\beta}^{\mathcal{U}}$, but it is too late for that now. The difference between "$\hat{\imath}$" and "i" is barely visible anyway.

If $\mathcal{T}$ is an iteration tree, then $\mathrm{lh}(\mathcal{T})$ is the domain of its tree order, that is, $\mathrm{lh}(\mathcal{T}) = \{\alpha \mid \mathcal{M}_\alpha^{\mathcal{T}} \text{ exists }\}$. So if $\mathrm{lh}(\mathcal{T}) = \alpha+1$, then $\mathcal{M}_\alpha^{\mathcal{T}}$ exists, but $E_\alpha^{\mathcal{T}}$ does not. $\mathcal{T}\restriction\beta$ is the initial segment $\mathcal{U}$ of $\mathcal{T}$ such that $\mathrm{lh}(\mathcal{U}) = \beta$. So $\mathcal{M}_\alpha^{\mathcal{T}\restriction\alpha+1}$ exists, but there is no exit extender $E_\alpha^{\mathcal{T}\restriction\alpha+1}$.

By *normal* we shall mean "Jensen normal".

Definition 1.10 *Let $\mathcal{T}$ be an iteration tree on a premouse M; then $\mathcal{T}$ is* normal *iff*

(1) if $\beta+1 < \mathrm{lh}(\mathcal{T})$ and $\alpha < \beta$, then $\mathrm{lh}(E_\alpha^{\mathcal{T}}) < \mathrm{lh}(E_\beta^{\mathcal{T}})$, and

(2) if $\alpha+1 < \mathrm{lh}(\mathcal{T})$, then T-$\mathrm{pred}(\alpha+1)$ is the least β such that $\mathrm{crit}(E_\alpha^{\mathcal{T}}) < \lambda(E_\beta^{\mathcal{T}})$, and

(3) $\mathcal{M}_{\alpha+1}^{*,\mathcal{T}} = \mathcal{M}_\beta^{\mathcal{T}} | \langle \eta, k \rangle$, where $\langle \eta, k \rangle \leq l(\mathcal{M}_\beta^{\mathcal{T}})$ is largest so that $\mathrm{crit}(E) < \rho_k(\mathcal{M}_\beta^{\mathcal{T}} | \xi)$.

Definition 1.11 *Let $\mathcal{T}$ be a normal iteration tree on a Jensen premouse; then for and $\beta < \mathrm{lh}(\mathcal{T})$,*

$$\begin{aligned}\lambda_\beta^{\mathcal{T}} &= \sup\{\lambda_F \mid \exists \eta < \beta (F = E_\eta^{\mathcal{T}})\}\\ &= \sup\{\lambda_F \mid \exists \eta (\eta + 1 \leq_T \beta \wedge F = E_\eta^{\mathcal{T}})\}\end{aligned}$$

So $\lambda_\beta^{\mathcal{T}}$ is the sup of the "Jensen generators" of extenders used to produce $\mathcal{M}_\beta^{\mathcal{T}}$. For $k = k(\mathcal{M}_\beta^{\mathcal{T}})$, $\mathcal{M}_\beta^{\mathcal{T}} = h^{k+1}``(\mathrm{ran}(\hat{\imath}_{0,\beta}) \cup \lambda_\beta^{\mathcal{T}})$.

If $\mathcal{T}$ is normal, then T-$\mathrm{pred}(\beta+1)$ is the largest α such that $\lambda_\alpha^{\mathcal{T}} \leq \mathrm{crit}(E_\beta^{\mathcal{T}})$. Another useful characterization is the following. Let θ be $\mathrm{crit}(E_\beta^{\mathcal{T}})^+$, as computed in $\mathcal{M}_\beta^{\mathcal{T}} | \mathrm{lh}(E_\beta^{\mathcal{T}})$. Then

$$T\text{-pred}(\beta+1) = \text{ least } \alpha \text{ such that } \mathcal{M}_\alpha^{\mathcal{T}} | \theta = \mathcal{M}_\beta^{\mathcal{T}} | \theta.$$

Note here that θ is passive in $\mathcal{M}_\beta^{\mathcal{T}}$, so for α as on the right, θ is passive in $\mathcal{M}_\alpha^{\mathcal{T}}$. The formula may fail if we replace the $|$ by $||$, for when $\lambda_{E_\alpha^{\mathcal{T}}} = \mathrm{crit}(E_\beta^{\mathcal{T}})$, T-$\mathrm{pred}(\beta+1)$ is $\alpha+1$, not α.

Figure 1.2 shows how the agreement of models in a normal iteration tree is propagated when the tree is augmented by one new extender. (Figures like this were first drawn by Itay Neeman.)

If one replaces the condition $\mathrm{crit}(E_\alpha^{\mathcal{T}}) < \lambda(E_\beta^{\mathcal{T}})$ by the condition $\mathrm{crit}(E_\alpha^{\mathcal{T}}) < \nu(E_\beta^{\mathcal{T}})$ in the definition of (Jensen) normality, one obtains a definition of *ms-normality*. (This is called s-normality in [4, §5].) In fact, there are some advantages to working with ms-normal trees, even in the context of Jensen premice. One is that full background constructions of Jensen-normally iterable M seems to require superstrong extenders in V (but see [15]). On the other hand, one can show granted only a Woodin with a measurable above that there is a ms-normally iterable Jensen mouse with a Woodin cardinal, granted that there is in V a Woodin with a measurable above it. ([10] yields an ms-iterable ms-mouse with a Woodin, and [3] and [4] then translates it to an ms-normally iterable Jensen mouse with a Woodin.) Nevertheless, 1.10 is the more common notion of normality in the setting of Jensen premice,

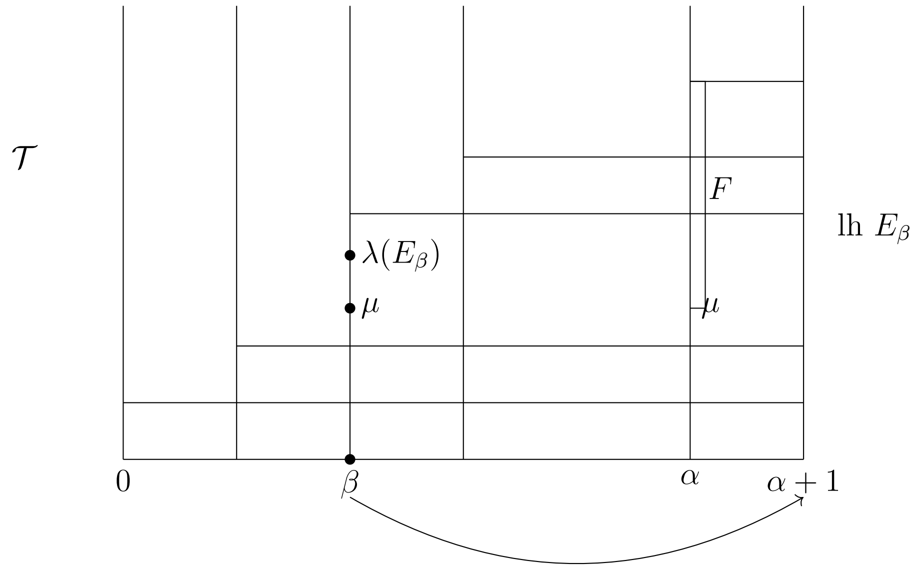


Figure 1.2: A normal tree $\mathcal{T}$, extended normally by F. The vertical lines represent the models, and the horizontal ones represent their levels of agreement. $\text{crit}(F) = \mu$, and β is least such that $\mu < \lambda(E_\beta^{\mathcal{T}})$. The arrow at the bottom represents the ultrapower embedding generated by F.

and it will serve our purposes. We believe that there are elementary simulations of Jensen normal trees by ms-normal trees, and vice-versa, but we have not verified this carefully.

Remark 1.12 ms-normal iterations preserve ms-solidity. As we remarked earlier, Jensen normal iterations may not.

We also need stacks of normal trees.

Definition 1.13 *Let M be a premouse; then s is a* normal M-stack *iff $s = \langle (\nu_\alpha, k_\alpha, \mathcal{T}_\alpha) \mid \alpha < \beta \rangle$, and there are premice M_α for $\alpha < \beta$ such that*

(1) $\mathcal{T}_\alpha$ is a normal tree on $M_\alpha | \langle \nu_\alpha, k_\alpha \rangle$,

(2) $M_0 = M$,

(3) if $\alpha < \beta$ and α is a limit ordinal, then M_α is the direct limit of the M_β for $\beta < \alpha$, and

(4) if $\gamma+1=\alpha<\beta$, then M_α is the last model of $\mathcal{T}_\alpha$

The definition allows a gratuitous drop at the beginning of each normal tree $\mathcal{T}_\alpha$. If $\langle \nu_\alpha, k_\alpha\rangle = l(M_\alpha)$ for all α, then we say s is *maximal.* We allow $k_\alpha = -1$, with the convention that $P|\langle \nu, -1\rangle = P||\nu$ as above.

In (3), the direct limit is under the obvious partial maps $\hat{\imath}^s_{\xi,\gamma}\colon M_\xi \to M_\gamma$, for $\xi < \gamma < \alpha$. We demand that for $\alpha<\beta$ a limit, there are only finitely many drops along the branches producing these maps, and that the direct limit is wellfounded. We write $M_\xi(s)$ and $\mathcal{T}_\xi(s)$ for M_ξ and $\mathcal{T}_\xi$. If $\mathrm{dom}(s) = \alpha+1$, then we write $\mathcal{U}(s) = \mathcal{T}_\alpha(s)$ for the last tree in the stack. $\mathcal{U}(s)$ could have no last model.

1.6 Jensen normal genericity iterations

Jensen normal genericity iterations must be allowed to drop, unless our identities are generated by superstrong extenders. However, this dropping will not occur along the main branch, so it is harmless. We explain this briefly now. The reader should see [29, §7] for more detail on the extender algebra and genericity iterations.

Let M be a premouse, and $\mu < \delta$ cardinals of M. We let $\mathbb{B} = \mathbb{B}^M_{\mu,\delta}$ be the ω-generator *extender algebra* determined by the extenders on the $M|\delta$-sequence with critical point $> \kappa$. . $\mathbb{B}$ is the Lindenbaum algebra of a certain infinitary theory T in the propositional language $\mathcal{L}_{\delta,0}$ generated by the sentence symbols A_n, for $n<\omega$. For $x \subset \omega$, $x \models A_n$ iff $n \in x$, and then $x \models \varphi$ for φ an arbitrary sentence of $\mathcal{L}_0$ has the natural meaning. The axioms of T are those sentences of the form

$$\bigvee_{\alpha<\kappa} \varphi_\alpha \longleftrightarrow \bigvee_{\alpha<\lambda} i_E(\langle \varphi_\xi \colon \xi<\kappa\rangle) \restriction \lambda,$$

whenever E is on the $M|\delta$-sequence, $\mathrm{crit}(E) = \kappa > \mu$, $i_E(\langle \varphi_\xi \colon \xi < \kappa\rangle) \restriction \lambda \in M|\eta$, for some cardinal η of M such that $\eta < \lambda_E$. Let us write $T = T(M|\delta, \mu)$.

The usual argument shows that if δ is Woodin in M, then $M \models$ "$\mathbb{B}$ is δ-c.c.". It is also clear that if M comes from a background construction in V, then every $x \in V$ satisfies all axioms of T. This is because if E generates an axiom as above, and E^* is its background extender, then $E \restriction \eta = E^* \restriction \eta \cap M$, for all M-cardinals η.

Given an iterable M as above, and an $x \subset \omega$, we form a Jensen normal tree $\mathcal{T}$ on M as follows: $E^{\mathcal{T}}_\alpha$ is the first extender on the sequence of $\mathcal{M}^{\mathcal{T}}_\alpha$ with critical point above $\hat{\imath}^{\mathcal{T}}_{0,\alpha}(\mu)$ that induces an axiom of $T(\mathcal{M}^{\mathcal{T}}_\alpha | \sup \hat{\imath}^{\mathcal{T}}_{0,\alpha}``\delta, \hat{\imath}^{\mathcal{T}}_{0,\alpha}(\mu))$ not satisfied by x. The rest is determined by the rules of Jensen normal trees. Note the hat above the i in the formula! $[0,\alpha)_T$ may have dropped, but it will never drop below the image of μ. It may happen that $\hat{\imath}^{\mathcal{T}}_{0,\alpha}(\delta)$ is undefined, however.

As usual, the construction of $\mathcal{T}$ terminates with a last model $\mathcal{M}_\alpha^\mathcal{T}$ such that x satisfies all the axioms of $T(\mathcal{M}_\alpha^\mathcal{T} | \sup \hat{\imath}_{0,\alpha}^\mathcal{T} “\delta, \hat{\imath}_{0,\alpha}^\mathcal{T}(\mu))$ are satisfied by x. We must see that in this case, $[0, \alpha)_T$ has not dropped. Suppose that is has, and let $\xi + 1 \leq_T \alpha$ be the site of the last drop, and $T - \text{-pred}(\xi + 1) = \gamma$. Let $E = E_\gamma^\mathcal{T}$, and let

$$\psi = \bigvee_{\alpha < \kappa} \varphi_\alpha \longleftrightarrow \bigvee_{\alpha < \lambda} i_E(\langle \varphi_\xi \colon \xi < \kappa \rangle) \upharpoonright \lambda$$

be the bad axiom induced by E, and η a cardinal of $\mathcal{M}_\gamma^\mathcal{T}$ such that $\psi \in \mathcal{M}_\gamma^\mathcal{T} | \eta$. Since we dropped when applying it, $\eta \leq \text{crit}(E_\xi^\mathcal{T})$, so $\hat{\imath}_{\gamma,\alpha}^\mathcal{T} \upharpoonright \eta$ is the identity. But also, $\mathcal{M}_\gamma^\mathcal{T} | \text{lh}(E) \trianglelefteq \mathcal{M}_{\xi+1}^*$, so $\hat{\imath}_{\gamma,\alpha}^\mathcal{T}(E)$ exists. Clearly, $\hat{\imath}_{\gamma,\alpha}^\mathcal{T}(E)$ still induces ψ as an axiom of $T(\mathcal{M}_\alpha^\mathcal{T} | \sup \hat{\imath}_{0,\alpha}^\mathcal{T} “\delta, \hat{\imath}_{0,\alpha}^\mathcal{T}(\mu))$. Since x does not satisfy ψ, the genericity iteration did not terminate at α, contradiction.

1.7 Iteration strategies

Let M be a premouse. $G(M, \theta)$ is the game of length θ in which I and II cooperate to produce a normal tree on M, with II picking branches at limit steps, and being obliged to stay in the category of wellfounded models. See [29], where the game is called $G_k(M, \theta)$, for $k = k(M)$. A *θ-iteration strategy* for M is a winning strategy for II in $G(M, \theta)$.

Similarly, in $G(M, \eta, \theta)$ the players produce a normal stack of length θ on M, with II picking branches at limit ordinals and I doing the rest, and II being obliged to insure all models are wellfounded. A *(η, θ)-iteration strategy* for M is a winning strategy for II in $G(M, \eta, \theta)$. See [29].

It is natural to generalize these standard iteration games so that player I has the freedom to “drop gratuitously” on any of his moves. For example, if M is premouse, we let $G^+(M, \theta)$ be the variant of $G_{k(M)}(M, \theta)$ in which player II must pick cofinal wellfounded branches at limit steps as before, and given that $\mathcal{T}$ with $\text{lh}(\mathcal{T}) = \alpha + 1$ is the play so far, I must pick E_α from the $\mathcal{M}_\alpha = \mathcal{M}_\alpha^\mathcal{T}$ sequence such that $\text{lh}(E_\beta) < \text{lh}(E_\alpha)$ for all $\beta < \alpha$. (Here $\mathcal{M}_0 = M$.) As before, we set

$$\xi = T\text{-pred}(\alpha + 1) = \text{ least } \beta \text{ s.t. } \text{crit}(E_\alpha) < \lambda(E_\beta).$$

Let $\langle \nu, k \rangle$ be least such that $\rho(\mathcal{M}_\xi^\mathcal{T}) \leq \text{crit}(E_\alpha)$, or $\langle \nu, k \rangle = l(\mathcal{M}_\xi)$. Let $\gamma = \text{crit}(E_\alpha)^+$ in the sense of $\mathcal{M}_\alpha | \text{lh}(E_\alpha)$, or equivalently, in the sense of $\mathcal{M}_\xi | \langle \nu, k \rangle$. We now allow I to pick *any* $\langle \eta, l \rangle$ such that

$$\langle \gamma, 0 \rangle \leq \eta, l \rangle \leq \langle \nu, k \rangle,$$

and we set

$$\mathcal{M}_{\alpha+1} = \text{Ult}(\mathcal{M}_\xi|\langle\eta, l\rangle, E_\alpha).$$

We write $\mathcal{M}_\xi|\langle\eta, l\rangle = \mathcal{T}\text{-pd}(\alpha+1)$.

Definition 1.14 *A* weakly normal *tree on an lpm M is a play of some $G^+(M,\theta)$ in which player* II *has not yet lost.*

In older terminology, a weakly normal tree is just one that is length-increasing and nonoverlapping.

We let $G^+(M,\lambda,\theta)$ be the variant of $G_{k(M)}(M,\lambda,\theta)$ in which I is allowed gratuitous dropping within each of the λ rounds. For notational reasons, we'll allow him to drop in the base model for the beginning of a round as well, though this is no extra generality in fact. We call a position in $G^+(M,\lambda,\theta)$ in which II has not yet lost an *M-stack*.

Definition 1.15 *An M-*stack *is a sequence $s = \langle(\nu_\alpha, k_\alpha, \mathcal{T}_\alpha) \mid \alpha < \beta\rangle$ with all the properties of normal M-stacks, save that the $\mathcal{T}_\alpha$ may be only weakly normal.*

We allow some or all of the weakly normal trees in our M-stack to be empty.

Given an an M-stack s as above, we write $(\nu_i(s), k_i(s), \mathcal{T}_i(s))$ for $s(i)$, $M_0(s) = M$, and $M_{i+1}(s)$ for the last model of $\mathcal{T}_i(s)$, when $i < \text{dom}(s) - 1$. We write $\mathcal{U}(s)$ for $\mathcal{T}_{\text{dom}(s)-1}(s)$, the last normal tree in s. We write $M_\infty(s)$ for the last model of $\mathcal{U}(s)$, if it has one.

We shall be most interested in M-stacks of finite length.

If s is a normal M-stack, then we identify s with its sequence of trees $\mathcal{T}_i(s)$, the $\nu_i(s)$ and $k_i(s)$ being determined by normality.

A complete strategy for M is an iteration strategy Σ that acts on all finite M-stacks that are according to Σ.

Definition 1.16 *Let $M \in H_\theta$; then a* complete strategy for M with scope H_θ *is a winning strategy for player* II *in the game $G^+_{k(M)}(M,\omega,\theta)$.*

Notice that a complete strategy Σ for M tells us how to iterate the full M at its level of soundness, as well as how to iterate initial segments of it. In practice, the iteration strategies for initial segments of M determined by Σ are consistent with one another; we spell this out in section 5.3. Although Σ is only required to act on finite stacks, whenever s is a run of $G^+_{k(M}(M,\omega,\theta)$ by Σ, then the direct limit $M_\omega(s)$ of the $M_i(s)$ for $i < \omega$ sufficiently large exists, and is wellfounded.

Given $\pi\colon M \to N$ weakly elementary, we can copy an M-stack s to an N-stack πs, until we reach an illfounded model on the πs side. Thus if Ω is a complete strategy for N, we have the complete pullback strategy Ω^π for M.

Remark 1.17 It is possible that $\pi\colon M \to N$ is weakly elementary, $\mathcal{T}$ is normal on M, and $\pi\mathcal{T}$ is not normal. $\pi\mathcal{T}$ will be weakly normal, however. In §5.2 we describe the natural normal tree on N into which $\pi\mathcal{T}$ embeds; this tree is called $(\pi\mathcal{T})^+$.

Definition 1.18 *[Pullback strategies] If Σ is a strategy for N, and $\pi\colon M \to N$ is weakly elementary, then Ω^π is the pullback strategy for M, given by*

$$\Omega^\pi(s) = \Omega(\pi s),$$

for all s such that $\pi s \in \mathrm{dom}(\Omega)$.

The copy maps are all weakly elementary, and if π is fully elementary, then the copy maps are all fully elementary. (Cf. 1.3 of [21].)

Tail strategies are defined by

Definition 1.19 *Let Ω be a complete strategy for M, and let s be an M-stack according to Ω such that $M_\infty(s)$ exists; then Ω_s is the complete strategy for $M_\infty(s)$ given by:*

$$\Omega_s(t) = \Omega(s^\frown t),$$

for all $M_\infty(s)$-stacks t.

The following notation will be useful:

Definition 1.20 *Let Ω be a complete strategy for M, and let s be an M-stack according to Ω such that $M_\infty(s)$ exists, and let $N = M_\infty(s)|\langle \nu, k\rangle$; then $\Omega_{s,N} = \Omega_{s^\frown\langle \nu,k,\emptyset\rangle}$. We also write $\Omega_{s,\langle \nu,k\rangle}$ for $\Omega_{s,N}$.*

When $N = M|\langle \nu, k\rangle$, we write Ω_N or $\Omega_{\langle \nu,k\rangle}$ for $\Omega_{\emptyset,N}$. It is also useful to have a notation for a join of strategies:

Definition 1.21 *Let Ω be a complete strategy for M, and s an M-stack by Ω; then $\Omega_{s,<\nu} = \bigcup\{\Omega_{s,\langle \eta,k\rangle} \mid \eta < \nu \wedge k \leq \omega\}$.*

Note that in general, $\Omega_{s,<\nu}$ is strictly weaker than $\Omega_{s,\langle \nu,0\rangle}$.

1.8 Coarse structure

One must consider also iteration trees on transitive models M that are not equipped with any distinguished fine structural hierarchy. In that case, we shall always assume $M \models \mathsf{ZFC}$, for simplicity. In general, V_α^M plays the role that $M|\alpha$ would in the fine structural case. All extenders are total on the models to which they are applied, and all embeddings are fully elementary in the $\in$-language. We shall sometimes call such M, and associated objects like iteration trees or embeddings acting on them, *coarse*, in order to distinguish them from their fine-structural cousins.

Definition 1.22 *Let E be an extender over V; then E is* nice *iff*

(a) E is strictly short, that is, $\mathrm{lh}(E) < \lambda(E)$,

(b) $\mathrm{lh}(E)$ *is strongly inaccessible, but not a measurable cardinal,*

(c) $V_{\mathrm{lh}(E)} \subseteq \mathrm{Ult}(V, E)$.

Nice E can be used to background extenders in a Jensen premouse, even though $\mathrm{lh}(E) < \lambda(E)$. In practice, our background extenders will be such that $\mathrm{lh}(E)$ is the least strongly inaccessible strictly above η, for some η, so that (b) holds. The requirements of (b) enable us to avoid a counterexample to UBH for stacks of normal trees due to Woodin. See 3.21 below.

Definition 1.23 *Let $\mathcal{T}$ be an iteration tree on a coarse M; then*

(a) $\mathcal{T}$ is nice *iff whenever $\alpha + 1 < \mathrm{lh}(\mathcal{T})$, then $\mathcal{M}_\alpha^{\mathcal{T}} \models$ "$E_\alpha^{\mathcal{T}}$ is nice".*

(b) $\mathcal{T}$ is normal *iff*

(i) if $\alpha < \beta$ and $\beta + 1 < \mathrm{lh}(\mathcal{T})$, then $\mathrm{lh}(E_\alpha^{\mathcal{T}}) < \mathrm{lh}(E_\beta^{\mathcal{T}})$, and

(ii) if $\alpha + 1 < \mathrm{lh}(\mathcal{T})$, then $T\text{-pred}(\alpha + 1)$ is the least β such that $\mathrm{crit}(E_\alpha^{\mathcal{T}}) < \mathrm{lh}(E_\beta^{\mathcal{T}})$.

This definition of normality is only appropriate for nice trees, but all our coarse iteration trees will be nice, so that is ok. In fact, we shall restrict the choice of extenders in $\mathcal{T}$ even further.

Definition 1.24 *A sequence $\vec{F} = \langle F_\alpha \mid \alpha < \mu \rangle$ is* coarsely coherent *iff each F_α is a nice extender over V, and*

(1) $\alpha < \beta \Rightarrow \mathrm{lh}(F_\alpha) < \mathrm{lh}(F_\beta)$, and

(2) if $i\colon V \to \mathrm{Ult}(V, F_\alpha)$ *is the canonical embedding, and* $\vec{E} = i(\vec{F})$*, then* $\vec{E}\restriction\alpha = \vec{F}\restriction\alpha$*, and* $\mathrm{lh}(F_\alpha)) < \mathrm{lh}(E_\alpha)$*.*

Given a coarsely coherent $\vec{F}$, an $\vec{F}$-iteration tree is one where all extenders used are taken from $\vec{F}$ and its images. Similarly for $\vec{F}$-stacks of normal trees. So the trees in an $\vec{F}$-stack are nice. $\vec{F}$-iteration strategies are defined in the obvious way. The following simple lemma uses only clause (1) of coarse coherence.

Lemma 1.25 *Let* $\vec{F}$ *be coarsely coherent, and let* Σ *be an* $\vec{F}$*-iteration strategy for* V*; them for any* N*, there is at most one normal* $\vec{F}$*-iteration tree played according to* Σ *whose last model is* N*.*

Proof. Let $\mathcal{T}$ and $\mathcal{U}$ be distinct such trees. Because both are played by Σ and normal, there must be a β such that $\mathcal{T}\restriction\beta + 1 = \mathcal{U}\restriction\beta + 1$, but $G \neq H$, where $G = E^{\mathcal{T}}_\beta$ and $H = E^{\mathcal{U}}_\beta$. Both G and H are taken from $i(\vec{F})$, where $i = i^{\mathcal{T}}_{0,\beta} = i^{\mathcal{U}}_{0,\beta}$. Say G occurs before H in $i(\vec{F})$. Then $G \in N$ because $\mathcal{U}$ is normal. But $G \notin N$ because $\mathcal{T}$ is normal. □

The iteration strategies for coarse M that we shall consider will choose unique cofinal wellfounded branches.

Definition 1.26 *Let* $M \models \mathsf{ZFC} +$ "$\vec{F}$ *is coarsely coherent"; then*

(a) M *is* uniquely $\theta, \vec{F}$-iterable for normal trees *iff whenever* $\mathcal{T}$ *is a normal* $\vec{F}$*-iteration tree on* M*, and* $\mathrm{lh}(\mathcal{T})$ *is a limit ordinal* $< \theta$*, then* $\mathcal{T}$ *has a unique cofinal wellfounded branch.* M *is* uniquely $\vec{F}$-iterable for normal trees *iff* M *is uniquely* $\theta, \vec{F}$*-iterable for normal trees, for all* θ*.*

(b) M *is* strongly uniquely $\theta, \vec{F}$-iterable (for finite stacks) *iff whenever* $\langle \mathcal{U}_1, \ldots, \mathcal{U}_n\rangle$ *is a finite stack of normal nice iteration trees, with* $\mathcal{U}_1$ *on* M*, and* $\mathrm{lh}(\mathcal{U}_i) < \theta$ *for all* i*, and* $\mathcal{U}_n$ *has limit length, then* $\mathcal{U}_n$ *has a unique cofinal wellfounded branch.* M *is* strongly uniquely $\vec{F}$-iterable *iff it is strongly uniquely* $\theta, \vec{F}$*-iterable for all* θ*.*

Assuming AD^+, we get such M and $\vec{F}$ via the Γ-Woodin construction due to Woodin. See [36][§3] and [30][§10]. These M also satisfy "I am strongly uniquely $\vec{F}$-iterable", and hence are suitable as background universes for a hod mouse construction. We say more about this in §3 (see 3.11), and in §5.5.

Woodin has shown that if κ is supercompact, $\vec{F}$ is coarsely coherent and such that $\kappa < \mathrm{crit}(E)$ for all E on $\vec{F}$, and, and UBH holds in $V^{\mathrm{Col}(\omega,<\kappa)}$ for normal $\vec{F}$-trees on V, then V is uniquely $\vec{F}$-iterable. See Theorem 3.10. We show in 3.21 below that this implies that V is strongly uniquely $\vec{F}$-iterable, via a strategy that normalizes and condenses well.

1.9 Full background extender constructions

In this paper, we shall be looking very carefully at full background extender constructions, and in particular at how an iteration strategy Σ^* for the background universe induces iteration strategies for the premice occurring in such a construction. In our applications, the background universes will satisfy "I am strongly uniquely $\vec{F}$-iterable", where $\vec{F}$ is the sequence of background extenders used in the construction, and Σ^* will be the corresponding $\vec{F}$-iteration strategy. In this section we look at the well known construction of pure extender premice. Section 5.5 lays out the obvious generalization to hod mice.

We shall use the notation of [15] in this context. The reader should look at [15], and at [1] on which it relies, for full definitions.

Let w be a wellorder of V_δ, and $\kappa < \delta$. A *w-construction above κ* is a full background construction in which the background extenders are nice, have critical points $> \kappa$, cohere with w, have strictly increasing strengths, and are minimal (first in Mitchell order, then in w).

More precisely, such a construction $\mathbb{C}$ consists of premice $M^{\mathbb{C}}_{\nu,k}$, with $k(M_{\nu,k}) = k$, and extenders $F^{\mathbb{C}}_\nu$ obtained as follows. (In the notation of [10], $M_{\nu,k} = \mathbb{C}_k(\mathcal{N}_\nu)$, and $F^{\mathbb{C}}_\nu$ is a choice of background extender for the last extender of $M_{\nu,0} = \mathcal{N}_\nu$.) We let $M_{0,0}$ be the passive premouse with universe V_ω. For any k, ν,

$$M_{\nu,k+1} = \text{core}(M_{\nu,k}) =_{\mathsf{def}} C(M_{\nu,k}).$$

We have an anti-core embedding $\pi : M_{\nu,k+1} \to M_{\nu,k}$ with $\text{crit}(\pi) \geq \rho(M_{\nu,k})$. For $k < \omega$ sufficiently large, $M_{\nu,k} = M_{\nu,k+1}$ (except of course that its associated k has changed), and we set

$$M_{\nu,\omega} = \text{eventual value of } M_{\nu,k} \text{ as } k \to \omega,$$

and

$$\begin{aligned} M_{\nu+1,0} = {} & \text{rud closure of } M_{\nu,\omega} \cup \{M_{\nu,\omega}\}, \\ & \text{arranged as a passive premouse.} \end{aligned}$$

Finally, if ν is a limit, put

$$\begin{aligned} \mathcal{M}^{<\nu} = {} & \text{unique passive } P \text{ such that for all premice } N, \\ & N \lhd P \text{ iff } N \lhd M_{\alpha,0} \text{ for all sufficiently large } \alpha < \nu. \end{aligned}$$

Case 1. There is an F such that $(M^{<\nu}, F)$ is a Jensen premouse, and F is certifiable, in the sense of Definition 2.1 of [15].

A bicephalus argument shows that F is unique, and we set

$$M_{\nu,0} = (M^{<\nu}, F).$$

Case 2. Otherwise.

Then we set

$$M_{\nu,0} = M^{<\nu}.$$

(Again, our convention is that in case 1, $M^{<\nu}$ is not an initial segment of $M_{\nu,0}$.) A certificate for F in the sense of 2.1 of [15] is a short extender F^*. Let us write $\kappa_F = \mathrm{crit}(F)$ and $\lambda_F = i_F(\kappa_F)$. F^* must have strength some inaccessible cardinal $\eta > \lambda_F$, and satisfy

$$F^* {\restriction} \lambda_F \cap M^{<\omega} = F {\restriction} \lambda_F.$$

Since F^* is short, $i_{F^*}(\kappa_F) \geq \eta > \lambda_F$, so we cannot replace λ_F by $\lambda_F + 1$ in this equation. We add here the demands that

(i) F^* is nice, i.e. $\mathrm{lh}\, F^* = \eta$,

(ii) $\forall \tau < \nu\ (\mathrm{lh}\, F^{\mathbb{C}}_\tau < \eta)$,

(iii) $i_{F^*}(w) \cap V_\eta = w \cap V_\eta$,

(iv) $F^* \in V_\delta$, and $\mathrm{crit}(F^*) > \kappa$.

We then choose $F^{\mathbb{C}}_\nu$ to be the unique certificate for F such that

$(*)$ $F^{\mathbb{C}}_\nu$ is a certificate for F, minimal in the Mitchell order among all certificates for F, and w-least among all Mitchell order minimal certificates for F.

This has the consequence that $\mathrm{lh}(F^{\mathbb{C}}_\nu)$ is the *least* strongly inaccessible η such that $\lambda_F < \eta$ and $\forall \tau < \nu\ (\mathrm{lh}\, F^{\mathbb{C}}_\tau < \eta)$. We also get that $F^{\mathbb{C}}_\nu$ "coheres with $\mathbb{C}$". That is, letting $\mathbb{C}{\restriction}\gamma = \langle (M_{\tau,k}, F_\tau) \mid \tau < \gamma \wedge k \leq \omega \rangle$,

1. $i_{F^{\mathbb{C}}_\nu}(\mathbb{C}){\restriction}\nu = \mathbb{C}{\restriction}\nu$,

2. $M^{i_{F^{\mathbb{C}}_\nu}(\mathbb{C})}_{\langle \nu,0 \rangle}$ is passive.

Thus the sequence $\vec{F}^{\mathbb{C}}$ of all $F_\nu^{\mathbb{C}}$ is coarsely coherent. By a $\mathbb{C}$-iteration, we mean a $\vec{F}^{\mathbb{C}}$-iteration in the sense explained above. The length of a construction $\mathbb{C}$ is the lexicographically least $\langle \mu, l\rangle$ such that $M^{\mathbb{C}}_{\langle \mu, l\rangle}$ does not exist.

Associated to a construction $\mathbb{C}$ we have *resurrection maps* $\mathrm{Res}_{\nu,k}[N] = \langle \eta, l\rangle$ for some $\langle \eta, l\rangle \leq_{\mathrm{lex}} \langle \nu, k\rangle$. The idea is that N traces back to $M_{\eta,l}$ by following anti-core maps. $\sigma_{\nu,k}[N]$ is the associated elementary (at level l) embedding of N into $M_{\eta,l}$. For example, suppose $\mathrm{Res}_{\nu,k}$ and $\sigma_{\nu,k}$ are defined. We define $\mathrm{Res}_{\nu,k+1}$, $\sigma_{\nu,k+1}$ by

A. If $N = M_{\nu,k+1}$, then $\mathrm{Res}_{\nu,k+1}[N] = \langle \nu, k+1\rangle$ and $\sigma_{\nu,k+1}[N] = \text{identity}$.

B. If $N \lhd M_{\nu,k+1}|(\rho^+)^{M_{\nu,k+1}}$, where $\rho = \rho(M_{\nu,k})$, then $\mathrm{Res}_{\nu,k+1}[N] = \mathrm{Res}_{\nu,k}[N]$ and $\sigma_{\nu,k+1}[N] = \sigma_{\nu,k}[N]$.

C. Otherwise, letting $\pi : M_{\nu,k+1} \to M_{\nu,k}$ be the anti-core map, $\mathrm{Res}_{\nu,k+1}[N] = \mathrm{Res}_{\nu,k}[\pi(N)]$ and $\sigma_{\nu,k+1}[N] = \sigma_{\nu,k}[\pi(N)] \circ \pi$.

The reader should see [1] for the remainder of the definition. Two points on agreeement of resurrection maps:

1. if $N \lhd M_{\nu,k}$ and $\forall N'\ (N \unlhd N' \unlhd M_{\nu,k} \Rightarrow \rho(N') \geq \gamma)$, then $\sigma_{\nu,k}[N] \restriction \gamma = \text{identity}$.
2. if $N \unlhd N^* \unlhd M_{\nu,k}$, and $\forall N'\ (N \unlhd N' \unlhd N^* \Rightarrow \rho(N') \geq \gamma)$, then $\sigma_{\nu,k}[N] \restriction \gamma = \sigma_{\nu,k}[N^*] \restriction \gamma$.

These of course just come from the fact that the anti-core map $\pi : C(M) \to M$ is the identity on $\rho(M)$.

Now let $\mathbb{C} = \langle (M_{\nu,k}, F_\nu^*) \mid \langle \nu, k\rangle <_{\mathrm{lex}} \langle \mu, l\rangle\rangle$ be a construction above κ. Take $\kappa = 0$ to save notation. Let Σ^* be an iteration strategy for nice trees on V. We wish to describe the induced complete strategy Σ for $M_{\nu,k}$. For $\mathcal{T}$ a weakly normal iteration tree played by Σ, we shall have a *conversion system for* $\mathcal{T}$ in the sense of Definition 2.2 of [15]. Such a conversion system converts trees on $M_{\nu,k}$ to trees on V. The particular conversion system we construct we call $\mathrm{lift}(\mathcal{T}, M_{\nu,k}, \mathbb{C}, \Sigma^*)$. In general, a $\mathbb{C}$-conversion system for a weakly normal tree $\mathcal{T}$ consists of

(i) an iteration tree $\mathcal{T}^*$ on V,

(ii) indices $\langle \eta_\xi, l_\xi\rangle$ for $\xi < \mathrm{lh}\,\mathcal{T}$,

(iii) maps π_ξ for $\xi < \mathrm{lh}\,\mathcal{T}$,

so that, using P_ξ, $i_{\xi,\nu}$, F_ξ, P^*_ξ, $i^*_{\xi,\nu}$, F^*_ξ for the models, embeddings, and exit extenders of $\mathcal{T}$ and $\mathcal{T}^*$

1. $\pi_\xi : P_\xi \to M^{P^*_\xi}_{\langle \eta_\xi, l_\xi\rangle}$ is weakly elementary (where $M^{P^*_\xi}_{\langle \eta_\xi, l_\xi\rangle}$ is $M_{\langle \eta_\xi, l_\xi\rangle}$ in $i^*_{0,\xi}(\mathbb{C})$),
2. $\mathcal{T}$ and $\mathcal{T}^*$ have the same tree order,
3. if $\xi <_T \nu$ and $(\xi, \nu]_T$ does not drop in model or degree, then $\langle \eta_\nu, l_\nu\rangle = i^*_{\xi,\nu}(\langle \eta_\xi, l_\xi\rangle)$ and $\pi_\nu \circ i_{\xi,\nu} = i^*_{\xi,\nu} \circ \pi_\xi$.
4. if $\xi = T\text{-pred}(\nu + 1)$ and this is a drop in model or degree to $\bar{P} \lhd P_\xi$, then $\langle \eta_{\nu+1}, l_{\nu+1}\rangle = i^*_{\xi,\nu+1}(\mathrm{Res}^{P^*_\xi}_{\eta_\xi,l_\xi}[\pi_\xi(\bar{P})])$.
5. Let $\lambda_\xi = i_{F_\xi}(\mathrm{crit}(F_\xi))$, and $\alpha_\xi = \mathrm{lh}\, F_\xi$ be the index of F_ξ in P_ξ, and σ_ξ be the resurrection map $\sigma^{i^*_{0,\xi}(\mathbb{C})}_{\eta_\xi,l_\xi}[\pi_\xi(P_\xi \parallel \langle \alpha_\xi, 0\rangle)]$. Then for $\xi < \nu$,

$$\pi_\nu \restriction \lambda_\xi = \sigma_\xi \circ \pi_\xi \restriction \lambda_\xi$$

and

$$P^*_\xi | \sup \sigma_\xi \circ \pi_\xi \text{``}\, \lambda_\xi = P^*_\nu | \sup \sigma_\xi \circ \pi_\xi \text{``}\, \lambda_\xi.$$

The particular conversion system $\mathrm{lift}(\mathcal{T}, M_{\nu,k}, \mathbb{C})$ is determined by these conditions and the fact that

(a) let $\xi = T\text{-pred}(\nu + 1)$, and $\alpha_\nu = \mathrm{lh}\, F_\nu$, so that F_ν is the last extender of $P_\nu|\langle \alpha_\nu, 0\rangle$. Let

$$G = \text{last extender of } \mathrm{Res}^{P^*_\nu}_{\eta_\nu,l_\nu}[\pi_\nu(P_\nu|\langle \alpha_\nu, 0\rangle)];$$

then

$$F^*_\nu = \text{background extender for } G \text{ provided by } i^*_{0,\nu}(\mathbb{C}).$$

(b) let ξ, ν etc. be as in (a). If $(\xi, \nu + 1]_T$ is not a drop in model or degree, then

$$\pi_{\nu+1}([a, f]^{P_\xi}_{F_\nu}) = [\sigma \circ \pi_\nu(a), \pi_\xi(f)]^{P^*_\xi}_{F^*_\nu},$$

where $\sigma = \sigma_{\eta_\nu,l_\nu}[\pi_\nu(P_\nu|\langle \alpha_\nu, 0\rangle)]$. If it is a drop, to $\bar{P} \lhd P_\xi$, then

$$\pi_{\nu+1}([a, f]^{\bar{P}}_{F_\nu}) = [\sigma \circ \pi_\nu(a), \tau \circ \pi_\xi(f)]^{P^*_\xi}_{F^*_\nu},$$

where σ is as above, and $\tau = \sigma_{\eta_\xi,l_\xi}[\bar{P}]^{P^*_\xi}$.

The strategy Σ induced by Σ^* is defined as follows: given $\mathcal{T}$ on $M_{\nu,k}$,

$$\mathcal{T} \text{ is by } \Sigma \implies \text{lift}(\mathcal{T}, M_{\nu,k}, \mathbb{C}) \text{ is by } \Sigma^*.$$

If Σ^* is a strategy for the background universe or even just a partial strategy defined on all trees of the form $\text{lift}(\mathcal{T}, M_{\nu,k}, \mathbb{C})$, then Σ is a strategy for $M_{\nu,k}$. (There may be $\mathcal{T}$ such that $\text{lift}(\mathcal{T}, M_{\nu,k}, \mathbb{C})$ does not exist, because it enters the realm of illfounded models. But these trees $\mathcal{T}$ are not according to Σ.)

We may occasionally use the notation $\text{lift}(\mathcal{T}, M_{\nu,k}, \mathbb{C}, \Sigma^*)$ for the largest initial segment of $\text{lift}(\mathcal{T}, M_{\nu,k}, \mathbb{C})$ that is by Σ^*. So $\mathcal{T}$ is by Σ iff $\text{lift}(\mathcal{T}, M_{\nu,k}, \mathbb{C}) = \text{lift}(\mathcal{T}, M_{\nu,k}, \mathbb{C}, \Sigma^*)$.

We need to see that the lifted tree $\mathcal{T}^*$ is normal. (This is true even if $\mathcal{T}$ itself is only weakly normal.)

Lemma 1.27 *Let $\mathcal{T}$ be weakly normal, and let $\text{lift}(\mathcal{T}, M_{\nu,k}, \mathbb{C}, \Sigma^*) = \langle \mathcal{T}^*, \langle (\eta_\xi, l_\xi) \mid \xi < \text{lh}\,\mathcal{T} \rangle, \langle \pi_\xi \mid \xi < \text{lh}\,\mathcal{T} \rangle \rangle$; then $\mathcal{T}^*$ is normal.*

Proof. Let P_ξ, $i_{\xi,\nu}$, F_ξ, P_ξ^*, $i_{\xi,\nu}^*$, F_ξ^* be the models, embeddings, and extenders of $\mathcal{T}$ and $\mathcal{T}^*$. Set

$$\begin{aligned} \kappa_\xi &= \text{crit}\, F_\xi, & \lambda_\xi &= i_{F_\xi}(\kappa_\xi), \\ \kappa_\xi^* &= \text{crit}\, F_\xi^*, & \lambda_\xi^* &= i_{F_\xi^*}(\kappa_\xi^*). \end{aligned}$$

Let

$$\sigma_\xi = \sigma_{\eta_\xi, l_\xi}^{i_{0,\xi}^*(\mathbb{C})}[\pi_\xi(P_\xi \parallel \langle \alpha_\xi, 0 \rangle)]$$

be the resurrection embedding, so that

$$F_\xi^* = \text{background extender for } \sigma_\xi \circ \pi_\xi(F_\xi) \text{ provided by } i_{0,\xi}^*(\mathbb{C}).$$

Recall that in Jensen indexing, F is indexed at $\text{lh}\, F = (\lambda_F^+)^{\text{Ult}(M,F)}$.

Sublemma 1.27.1 *Let $\xi + 1 < \text{lh}\,\mathcal{T}$; then*

(*a*) $\sigma_\xi \circ \pi_\xi(\lambda_\xi) < \lambda_\xi^* = \pi_{\xi+1}(\lambda_\xi)$,

(*b*) $\sigma_\beta \restriction \pi_{\xi+1}(\text{lh}\, F_\xi) = $ *identity, for all* $\beta \geq \xi + 1$,

(*c*) $\pi_\beta \restriction (\text{lh}\, F_\xi + 1) = \pi_{\xi+1} \restriction (\text{lh}\, F_\xi + 1)$, *for all* $\beta \geq \xi + 1$.

Proof. For (a): let $G = \sigma_\xi \circ \pi_\xi(F_\xi)$. Since F^*_ξ is the background in $i^*_{0,\xi}(\mathbb{C})$ for G, $\lambda^*_\xi > \lambda_G = \sigma_\xi \circ \pi_\xi(\lambda_\xi)$. But

$$\begin{aligned}\pi_{\xi+1}(\lambda_\xi) &= \pi_{\xi+1}([\varnothing, \text{contant } \kappa_\xi \text{ function}]^{\bar{P}_\tau}_{F_\xi}) \\ &= [\varnothing, \text{contant } \kappa^*_\xi \text{ function}]^{\bar{P}^*_\tau}_{F_\xi} \\ &= \lambda^*_\xi,\end{aligned}$$

where $\tau = T\text{-pred}(\xi+1)$ and $\bar{P}_\tau \trianglelefteq P_\tau$ is appropriate.

For (b), we have since $\mathcal{T}$ is weakly normal that for all $\beta \geq \xi+1$, $\operatorname{lh} F_\xi$ is a cardinal in P_β, and $\rho_{k(P_\beta)}(P_\beta) \geq \operatorname{lh} F_\xi$. We then get by induction on β that $\rho_{l_\beta}(M^{i^*_{0,\beta}(\mathbb{C})}_{\eta_\beta,l_\beta}) \geq \pi_{\xi+1}(\operatorname{lh} F_\xi)$, and $\pi_{\xi+1}(\operatorname{lh} F_\xi)$ is a cardinal in $M^{i^*_{0,\beta}(\mathbb{C})}_{\eta_\beta,l_\beta}$, for all $\beta \geq \xi+1$. This gives (b).

For (c), we have $\lambda_{\xi+1} > \operatorname{lh} F_\xi$, so

$$\begin{aligned}\pi_\beta \restriction (\operatorname{lh} F_\xi + 1) &= \sigma_{\xi+1} \circ \pi_{\xi+1} \restriction (\operatorname{lh} F_\xi + 1), \\ &= \pi_{\xi+1} \restriction (\operatorname{lh} F_\xi + 1),\end{aligned}$$

for all $\beta > \xi + 1$. □

Now we show $\mathcal{T}^*$ is normal. First, let $\alpha < \beta$, with $\beta + 1 < \operatorname{lh} \mathcal{T}^*$. Then

$$\begin{aligned}\operatorname{lh} F^*_\alpha < \lambda^*_\alpha &= \pi_{\alpha+1}(\lambda_\alpha) = \pi_\beta(\lambda_\alpha) \\ &= \sigma_\beta \circ \pi_\beta(\lambda_\alpha) < \sigma_\beta \circ \pi_\beta(\lambda_\beta) < \operatorname{lh} F^*_\beta,\end{aligned}$$

as desired.

For the rest, it is enough to show that whenever $\alpha < \beta$, then

$$\kappa_\beta < \lambda_\alpha \quad \text{iff} \quad \kappa^*_\beta < \operatorname{lh} F^*_\alpha.$$

Suppose first $\kappa_\beta < \lambda_\alpha$. Then

$$\begin{aligned}\kappa^*_\beta = \sigma_\beta \circ \pi_\beta(\kappa_\beta) = \pi_\beta(\kappa_\beta) &= \sigma_\alpha \circ \pi_\alpha(\kappa_\beta) \\ &< \sup \sigma_\alpha \circ \pi_\alpha ``\lambda_\alpha < \operatorname{lh} F^*_\alpha.\end{aligned}$$

Suppose next $\kappa_\beta \geq \lambda_\alpha$. Then

$$\kappa^*_\beta = \sigma_\beta \circ \pi_\beta(\kappa_\beta) \geq \sigma_\beta \circ \pi_\beta(\lambda_\alpha) = \pi_\beta(\lambda_\alpha) = \lambda^*_\alpha.$$

But $\lambda^*_\alpha > \operatorname{lh} F^*_\alpha$, so $\kappa^*_\beta > \operatorname{lh} F^*_\alpha$. □

If Σ^* is defined on stacks of normal trees, then we can extend the lifting process and the induced strategy Σ for $M_{\nu,k}$ so that it is defined on stacks of weakly normal trees. For example, if $\langle \mathcal{T}, \mathcal{U} \rangle$ is a stack on $M_{\nu,k}$, and $P_\xi = \mathcal{M}_\xi^{\mathcal{T}}$ is the last model of $\mathcal{T}$, and lift$(\mathcal{T}, M_{\nu,k}, \mathbb{C})$ has tree $\mathcal{T}^*$ by Σ^* with last model P_ξ^*, then we have

$$\pi_\xi : P_\xi \to M_{\eta_\xi, l_\xi}^{P_\xi^*}$$

from this lift. But $\Sigma^*_{\mathcal{T}^*, P_\xi^*}$ is a strategy for P_ξ^* on normal trees and by what we just said, it induces a strategy Ω on $M_{\eta_\xi, l_\xi}^{P_\xi^*}$. (We did not need that the background universe was V.) We let

$$\begin{aligned} \Sigma_{\mathcal{T}, P_\xi} &= \Omega^{\pi_\xi} \\ &= \pi_\xi\text{-pullback of } \Omega. \end{aligned}$$

2 Normalizing stacks of iteration trees

First, given $\mathcal{T}$ a normal tree on M, and $\mathcal{U}$ a normal tree on the last model of $\mathcal{T}$, we shall define the embedding normalization $W(\mathcal{T},\mathcal{U})$ of $\langle\mathcal{T},\mathcal{U}\rangle$. As we do so, we show that $\mathcal{T}$ embeds into $\mathcal{U}$ in a natural way, via what we call later a psuedo-hull embedding. We then describe how branches of $W(\mathcal{T},\mathcal{U})$ are generated by branches of $\mathcal{T}$ and $\mathcal{U}$. Finally, in the last subsection we describe the possible ways to normalize a finite stack of normal trees.

There are two sorts of base models M we are interested in:

1. $M \models \mathsf{ZFC}$, M transitive. This is the "coarse structural" case. Here we shall assume that $\mathcal{T}$ and $\mathcal{U}$ are "nice", in that in $M_\alpha^\mathcal{T}$, $E_\alpha^\mathcal{T}$ has length = strength an inaccessible cardinal, and similarly for $\mathcal{U}$. This simplifies various things.

2. M is a premouse. It may be an ordinary premouse, a hybrid premouse (such as those that provide examples of the N^*'s referred to in 0.4), or a hod premouse. In this case, we want to consider arbitrary fine-structural $\langle\mathcal{T},\mathcal{U}\rangle$, with dropping allowed.

The definition of $W(\mathcal{T},\mathcal{U})$ will make sense in both cases. In this section we shall focus on the case that M is a pure extender premouse, with Jensen indexing for its extender sequence. Until we get to section 6, this is what we shall mean by a premouse. We do need to define $W(\mathcal{T},\mathcal{U})$ in the coarse structural case as well, and we shall indicate how to do so as we proceed.

One important feature of the fine structural case is:

Fact 2.0 *Let M and N be premice, and Σ an iteration strategy for M; then there is at most one normal iteration tree $\mathcal{T}$ according to Σ having last model N.*

In the coarse structural case, this is not clear, even if Σ chooses unique cofinal wellfounded branches. We shall use this fact in an important way in the proof of Lemma 2.59 below. One could recover the fact in the coarse structural case by restricting to iterations where the extenders come from some coherent sequence. We shall essentially do that.

The definition of $W(\mathcal{T},\mathcal{U})$ does not require that any iteration strategy for M be fixed; however, it may break down by reaching illfounded models, even if the models of $\mathcal{T}^\frown\mathcal{U}$ are wellfounded. In the case we care about, M has an iteration strategy Σ, $\langle\mathcal{T},\mathcal{U}\rangle$ is played according to Σ, and the initial segment of $W(\mathcal{T},\mathcal{U})$ up to our point of interest is also played by Σ. We can then invoke Fact 2.0, relative to Σ, for the models in $W(\mathcal{T},\mathcal{U})$ up to our point of interest.

2.1 Normalizing trees of length 2

Let M be a premouse, E on the sequence of M, $\mathrm{crit}(E) < \rho_{k(M)}(M)$, and $N = \mathrm{Ult}(M, E)$. Let F be on the sequence of N, and $\mathrm{crit}(F) < \lambda(E)$. It follows that $\mathrm{Ult}(N, F)$ makes sense; let $Q = \mathrm{Ult}(N, F)$. So $k(M) = k(N)$, and both ultrapowers are $k(M)$-ultrapowers.

Let

$$\kappa = \mathrm{crit}(E), \quad \mu = \mathrm{crit}(F).$$

Let $\mathcal{T}$ be the iteration tree such that $E_0^{\mathcal{T}} = E$, $E_1^{\mathcal{T}} = F$, $\mathcal{M}_0^{\mathcal{T}} = M$, $\mathcal{M}_1^{\mathcal{T}} = N$, and $\mathcal{M}_2^{\mathcal{T}} = Q$. Since $\mu < \lambda(E)$, $\mathcal{T}$ is not normal. We show how to normalize it. There are two cases.

Case 1. $\mathrm{crit}(F) \leq \mathrm{crit}(E)$.

Since $\mu \leq \kappa$ and E is an extender over M (that is, over the reduct M^n, for $n = k(M)$), F is also an extender over M. Let $P = \mathrm{Ult}(M, F)$, and $i_F^M : M \to P$ be the canonical embedding. We have the diagram

$$\begin{tikzcd}
N \arrow[r, "F"] & Q \arrow[r, "\tau"] & i_F^M(N) = \mathrm{Ult}_0(P, i_F^M(E)) \\
M \arrow[u, "E"] \arrow[r, "F"] & P \arrow[ur, "i_F^M(E)"'] &
\end{tikzcd}$$

Suppose first that $M \models \mathsf{ZFC}$; then N is definable over M from E, and i_F^M moves the fact that $N = \mathrm{Ult}_0(M, E)$ over to the fact that $i_F^M(N) = \mathrm{Ult}_0(P, i_F^M(E))$. τ is the natural embedding from $i_F^N(N)$ to $i_F^M(N)$. That is,

$$\tau([a, g]_F^N) = [a, g]_F^M$$

for $g : [\mu]^{|a|} \to N$, with $g \in N$. The tree $\mathcal{U}$ with models

$$\mathcal{M}_0^{\mathcal{U}} = M,\ \mathcal{M}_1^{\mathcal{U}} = N,\ \mathcal{M}_2^{\mathcal{U}} = P,\ \mathcal{M}_3^{\mathcal{U}} = \mathrm{Ult}_0(\mathcal{P}, i_F^M(E))$$

and extenders

$$E_0^{\mathcal{U}} = E,\ E_1^{\mathcal{U}} = F,\ E_2^{\mathcal{U}} = i_F^M(E),$$

is normal. We call $\mathcal{U}$ the *embedding normalization* of $\mathcal{T}$.

Remark 2.1 This implicitly assumes $\mathrm{lh}\, E < \mathrm{lh}\, F$. If $\mathrm{lh}\, F < \mathrm{lh}\, E$, then F is already on the M-sequence, and the extenders of $\mathcal{U}$ would be $E_0^{\mathcal{U}} = F$, $E_1^{\mathcal{U}} = i_F^M(E)$. The diagrams and calculations above don't change, however.

The proof just given was based on N being definable over M as its E-ultrapower and i_F^M acting elementarily on this definition. But of course, $\mathrm{OR}^N > \mathrm{OR}^M$ is possible, and anyway, we need to know i_F^M has enough elementarity. If $M \models \mathsf{ZFC}$, all is fine. We now give a more careful proof that works in general.

Let us assume $k(M) = k(N) = 0$; otherwise we can replace M and N by their $k(M)$-reducts in the following argument. So every $x \in Q$ has the form $i_F^N(g)(b)$ for $g \in N$ and $b \in [\nu(F)]^{<\omega}$. We can write $g = i_E^M(h)(a)$, where $h \in M$ and $a \in [\nu(E)]^{<\omega}$. So

$$\begin{aligned} x &= i_F^N(i_E^M(h)(a))(b) \\ &= i_F^N \circ i_E^M(h)(i_F^N(a))(b), \end{aligned}$$

with b, $i_F^N(a) \in [\sup i_F^N``\nu(E)]^{<\omega}$. Let

$$G = (\text{extender of } i_F^N \circ i_E^M) \restriction \sup i_F^N``\nu(E),$$

so that

$$Q = \mathrm{Ult}(M, G).$$

The space of G is κ, and its critical point is μ. Let us write

$$\begin{aligned} R &= \mathrm{Ult}_0(P, i_F^M(E)) \\ H &= (\text{extender of } i^P_{i_F^M(E)} \circ i_F^M) \restriction \sup i_F^M``\nu(E). \end{aligned}$$

It is easy to see that

$$R = \mathrm{Ult}(M, H).$$

But then we can calculate that G is a subextender of H. For let $b \in [\nu(F)]^{<\omega}$ and $g : [\mu]^{|b|} \to [\nu(E)]^l$ with $g \in N$. Let $A \subseteq [\mathrm{crit}(E)]^l$ with $A \in N$. (Equivalently, $A \in M$.) We have

$$\begin{aligned} ([b,g]_F^N, A) \in G \quad & \text{iff} \quad [b,g]_F^N \in i_F^N \circ i_E^M(A) \\ & \text{iff} \quad \text{for } F_b \text{ a.e. } \bar{\mu},\ g(\bar{\mu}) \in i_E^M(A) \\ & \text{iff} \quad \text{for } F_b \text{ a.e. } \bar{\mu},\ (g(\bar{\mu}), A) \in E \\ & \text{iff} \quad ([b,g]_F^M, i_F^M(A)) \in i_F^M(E) \\ & \text{iff} \quad [b,g]_F^M \in i^P_{i_F^M(E)} \circ i_F^M(A) \\ & \text{iff} \quad ([b,g]_F^M, A) \in H. \end{aligned}$$

So letting $\sigma : \mathrm{lh}\, G \to \mathrm{lh}\, H$ be given by

$$\sigma([b,g]^N_F) = [b,g]^M_F,$$

we have

$$(a,A) \in G \quad \text{iff} \quad (\sigma(a),A) \in H,$$

so G is a subextender of H under σ. We can therefore define τ from $\mathcal{Q}$ into $\mathcal{R}$ by

$$\tau([a,f]^M_G) = [\sigma(a),f]^M_H.$$

Note $\tau \restriction \mathrm{lh}(F) = \sigma \restriction \mathrm{lh}(F) =$ identity. One can easily show that in the case $M \models \mathsf{ZFC}$, our current definition of τ coincides with the earlier one.

Remark 2.2 Another way to obtain τ is the following. Let $\psi \colon \mathrm{Ult}(M,E) \to \mathrm{Ult}(P,E^*)$ be the Shift Lemma map, where $E^* = i^M_F(E)$. That is, $\psi([a,f]^M_E) = [i^M_F(a), i^M_F(f)]^P_{E^*}$. By the Shift Lemma, ψ agrees with i^M_F on $\nu(E)$. It follows that F is an initial segment of E_ψ, the extender of ψ. The factor embedding from $\mathrm{Ult}(N,F)$ to $\mathrm{Ult}(N,E_\psi)$ is our τ. One can check that it is the same as the embedding we defined above.

We now digress a bit to discuss the full normalization of $\mathcal{T}$. Full normalization is not important for this paper, but it is very useful in its sequels [33], [34], and [35]. See [33] for a more complete discussion of full normalization.

To fully normalize $\mathcal{T}$, we must replace $i^M_F(E)$ by a subextender of itself. We can use condensation to show that the appropriate subextender is on the P-sequence. To see this, let $\langle (\beta_i,k_i) \mid 0 \le i < n \rangle$ be the $\mathrm{lh}(E)$-dropdown sequence of M. That is

$$(\beta_0,k_0) = (\mathrm{lh}\, E, 0)$$

and

$$\begin{aligned}(\beta_{i+1},k_{i+1}) = {} & \text{lexigraphically least } (\alpha,j) \text{ such that} \\ & \langle \alpha, l\rangle <_{\text{lex}} l(M) \text{ and } \rho(M|\langle \alpha,l\rangle < \rho(M|\langle \beta_i,k_i\rangle)).\end{aligned}$$

So long as they are defined, the ordinals

$$\rho^*_i = \rho(M|\langle \beta_i,k_i\rangle)$$

are strictly decreasing as i increases. The $\langle \beta_i,k_i\rangle$ increase, lexicographically. Note that ρ^*_i is a cardinal of $M|\beta_{i+1}$ with respect to $r\Sigma_{k_{i+1}}$ functions, and $\langle \beta_{i+1},k_{i+1}\rangle$ is lex-largest such that this is true.

Let n be least such that (β_n, k_n) cannot be defined this way, and set

$$(\beta_n, k_n) = l(M) = \langle \hat{o}(M), k(M) \rangle.$$

Notice that E was total on the reduct $M^{k(M)}$, so that $\mathrm{crit}(E) < \rho(M|\langle \beta_i, k_i \rangle)$ for all $i < n$, so by our case hypothesis, $\mathrm{crit}(F) < \rho(M|\langle \beta_i, k_i \rangle)$ for all $i < n$. Thus we have

$$\pi_i : M|\langle \beta_i, k_i \rangle \to \mathrm{Ult}(M|\langle \beta_i, k_i \rangle, F)$$

for all $i \le n$. We have

$$\pi_n = i_F^M$$

and $\mathrm{Ult}(M|\langle \beta_n, k_n \rangle, F) = \mathrm{Ult}_0(M, F) = P$. So $R = \mathrm{Ult}(P, \pi_n(E))$ was the last model of our embedding normalization.

Claim 2.3 $Q = \mathrm{Ult}(P, \pi_0(E))$.

Proof. $\mathrm{lh}(E)$ is a regular cardinal in N. So

$$\pi_0 = i_F^{M\|\mathrm{lh}(E)} = i_F^N \restriction N\|\mathrm{lh}(E),$$

and thus

$$\pi_0(\nu(E)) = i_F^N(\nu(E)).$$

Let

$$L = (\text{extender of } i^P_{\pi_0(E)} \circ i_F^M) \restriction i_F^N(\nu(E)),$$

then it is easy to see that

$$\mathrm{Ult}(P, \pi_0(E)) = \mathrm{Ult}(M, L).$$

Recall that G was the extender of length $i_F^N(\nu(E))$ given by $i_F^N \circ i_E^M$. As before, we get $\bar{\sigma} : \mathrm{lh}(G) \to \mathrm{lh}(L)$ by

$$\bar{\sigma}([b, g]_F^N) = [b, g]_F^{M\|\mathrm{lh}(E)},$$

defined for $b \in [\nu(E)]^{<\omega}$ and $g : [\mu]^{|b|} \to \nu(E)$ with $g \in N$. (We assume here $k(M) = k(N) = 0$; otherwise replace M and N by their $k(M)$-reducts.) But all such g are in $M\|\mathrm{lh}(E)$, so

$$\bar{\sigma} = \text{identity}.$$

As before, we get that G is a subextender of L under $\bar{\sigma}$, but this just means that $G = L$, proving Claim 2.3. $\square$

Claim 2.4 *For $0 \le i \le n$, $\mathrm{Ult}(M|\langle \beta_i, k_i \rangle, F)$ is an initial segment of P.*

Proof. $\mathrm{Ult}(M|\langle\beta_n, k_n\rangle, F) = P$. Now suppose $\mathrm{Ult}(M|\langle\beta_{i+1}, k_{i+1}\rangle, F)$ is an initial segment of P. So then $\pi_{i+1}(M|\langle\beta_i, k_i\rangle)$ is an initial segment of P. It will suffice to show $\mathrm{Ult}(M|\langle\beta_i, k_i\rangle, F) \trianglelefteq \pi_{i+1}(M|\langle\beta_i, k_i\rangle)$. But consider the factor map

$$\psi : \mathrm{Ult}(M|\langle\beta_i, k_i\rangle, F) \to \pi_{i+1}(M|\langle\beta_i, k_i >)$$

given by

$$\psi([a, f]_F^{M|\beta_i}) = [a, f]_F^{M|\beta_{i+1}}$$

for f a function given by a $r\Sigma_{k_i}$-Skolem term interpreted over $M|\beta_i$. For simplicity, let us assume $k_i = k_{i+1} = 0$, so this just amounts to $f \in M|\beta_i$. Let $\rho = \rho_i^*$; that is, assuming $k_i = 0$, let

$$\begin{aligned} \rho &= \rho_1(M|\beta_i), \\ p &= p_1(M|\beta_i), \\ S &= \mathrm{Ult}(M|\beta_i, F). \end{aligned}$$

So $\psi : S \to \pi_{i+1}(M|\beta_i)$. Now ρ is still a cardinal in $M|\beta_{i+1}$. So $({}^{\mu}\alpha)^{M|\beta_i} = ({}^{\mu}\alpha)^{M|\beta_{i+1}}$ for all $\alpha < \rho$. So

$$\mathrm{crit}(\psi) \geq \sup \pi_i``\rho.$$

Also,

$$S = \mathrm{Hull}_1^S(\sup \pi_i``\rho \cup \{\pi_i(p)\}),$$

as is easily checked. So $\rho_1(S) \leq \sup \pi_i``\rho$. Using the solidity witnesses, it is easy to see that

$$\rho_1(S) = \sup \pi_i``\rho \quad \text{and} \quad p_1(S) = \pi_i(p).$$

We can apply condensation to ψ to see that $S \trianglelefteq \pi_{i+1}(M|\beta_i)$ once we show that $\sup \pi_i``\rho$ is not an index of an extender on the $\pi_{i+1}(M|\beta_i)$-sequence.

Suppose it were. Then $\sup \pi_i``\rho$ is not a cardinal of $\pi_{i+1}(M|\beta_i)$, so $\mathrm{crit}(\psi) = \sup \pi_i``\rho$. This implies that π_{i+1} is discontinuous at ρ and that

$$M|\beta_{i+1} \models \mathrm{cof}(\rho) = \mu.$$

But then

$$\mathrm{Ult}(M|\beta_{i+1}, F) \models \mathrm{cof}(\sup \pi_i``\rho) = \mu.$$

But indices of extenders have successor cardinal cofinalities, and μ is a limit cardinal in $\mathrm{Ult}(M|\beta_{i+1}, F)$, so $\sup \pi_i``\rho$ is not an index in $\mathrm{Ult}(M|\beta_{i+1}, F)$-sequence. Therefore it is not an index in the $\pi_{i+1}(M|\beta_i)$-sequence. □

By Claim 2.4, $\pi_0(E)$ is on the sequence of P. Thus our full normalization of $\mathcal{T}$ is the tree $\mathcal{S}$, where

$$\mathcal{M}_0^{\mathcal{S}} = M,\ \mathcal{M}_1^{\mathcal{S}} = N,\ \mathcal{M}_2^{\mathcal{S}} = P,\ \mathcal{M}_3^{\mathcal{S}} = Q,$$

and

$$E_0^{\mathcal{S}} = E,\ E_1^{\mathcal{S}} = F,\ E_2^{\mathcal{S}} = \pi_0(E)).$$

Again, this assumes $\mathrm{lh}(F) \geq \mathrm{lh}(E)$. Otherwise it is $E_0^{\mathcal{S}} = F$ and $E_1^{\mathcal{S}} = \pi_0(E)$.

The following diagram summarizes Case 1.

$$\begin{tikzcd}
N \arrow[r, "i_F^N"] & Q \arrow[r, "\tau"] & R \\
M \arrow[u, "E"] \arrow[r, "i_F^M"] & P \arrow[u, "i_F^N(E)"'] \arrow[ur, "i_F^M(E)"'] &
\end{tikzcd}$$

Here $i_F^N(E) = \pi_0(E)$. The notation is justified because $(N|\,\mathrm{lh}(E), E) = M|\,\mathrm{lh}(E)$, so i_F^N moves F as an amenable predicate, and produces thereby what we called $\pi_0(E)$. The construction in Claim 2.4 shows that in fact $i_F^N(E)$ is a subextender of $i_F^M(E)$ under the map $\sigma : i_F^N(\nu(E)) \to i_F^M(\nu(E))$ we identified earlier, $\sigma([b, g]_F^N) = [b, g]_F^M$ for $g : [\mu]^{|b|} \to \nu(E)$ with g in N.

Remark 2.5 All embeddings in the diagram above are all elementary and cofinal. All but τ are ultrapower embeddings. τ is easily seen to be weakly elementary, and it is cofinal because all the other embeddings are cofinal.

Remark 2.6 If G is the extender of $i_F^N \circ i_E^M$, then in fact $\nu(G) = \sup i_F^N``\nu(E)$, as shown by our earlier calculation. So $\nu(i_F^N(E)) = \sup i_F^N``\nu(E)$.

Remark 2.7 Let us consider the case that $\nu(E)$ is a cardinal in M. Then $({}^{\mu}\alpha)^M = ({}^{\mu}\alpha)^N$ for all $\alpha < \nu(E)$, so for σ as above, $\sigma \restriction \sup i_F^N``\nu(E) = \text{identity}$. Thus $i_F^N(E)$ is the trivial completion of $i_F^M(E) \restriction \sup i_F^M``\nu(E)$. If i_F^M is continuous at $\nu(E)$ (i.e. $\mathrm{cof}^M(\nu(E)) \neq \mu$), then $i_F^N(E) = i_F^M(E)$ and $Q = R$. If i_F^M is discontinuous at $\nu(E)$ (i.e. $\mathrm{cof}^M(\nu(E)) = \mu$), then $Q \neq R$, and in fact $\mathrm{crit}(\tau) = \sup i_F^M``\nu(E)$.

So in this case, the embedding normalization of $\mathcal{T}$ uses $i_F^M(E)$ to continue from P, while the full normalization may use a proper initial segment of $i_F^M(E)$ to continue from P.

Case 2. $\mathrm{crit}(E) < \mathrm{crit}(F)$.

Let $\mu = \mathrm{crit}(F)$ and $\kappa = \mathrm{crit}(E)$. We have assumed $\mu < \lambda(E)$, as otherwise $\mathcal{T}$ is already normal. Let

$$P = \mathrm{Ult}(M|\langle \xi, k \rangle, F)$$

where $\langle \xi, k\rangle$ is lexicographically least such that $\rho(M|\langle \xi, k\rangle) \leq \mu$. Let

$$i : M|\langle \xi, k\rangle \to P$$

be the canonical embedding, $i = i_F^{M|\langle \xi,k\rangle}$. The embedding normalization of $\mathcal{T}$ continues from M, N (assuming $\mathrm{lh}(E) < \mathrm{lh}(F)$), and then P by using $i(E)$ now. Note $i(E)$ should be applied to M, not P, in a normal tree. So let

$$R = \mathrm{Ult}(M, i(E)).$$

Let G be the extender of $i_F^N \circ i_E^M$, and notice that G is short, with $\lambda(G) ==$ $i_F^N(\lambda(E)) = \sup i_F^N``\lambda(E)$. Let

$$\sigma : i_F^N(\lambda(E)) \to i_F^{M|\langle \xi,k\rangle}(\lambda(E))$$

be given by

$$\sigma([b, g]_F^N) = [b, g]_F^{M|\langle \xi,k\rangle},$$

for $g : [\mu]^{|b|} \to \lambda(E)$ with $g \in N$. (Note that for $n = k(M) = k(N)$, we have $\kappa < \rho_n(M)$, so $\lambda(E) < \rho_n(N)$, so every $r\Sigma_n^N$ such function g belongs to N.) We claim that

Claim 2.8 *G is a subextender of $i(E)$ under σ.*

Remark 2.9 *In this case, G and $i(E)$are short, and σ is the identity on their common domain.*

Proof. Let $a \subseteq \sup i_F^N``\lambda(E)$ be finite, and let $A \subseteq [\kappa]^{|a|}$ be in M. Let $a = [b, g]_F^N$, where $g \in N$ and $g : [\mu]^{|b|} \to [\nu(E)]^{|a|}$. Then

$$\begin{aligned}
(a, A) \in G \quad &\text{iff} \quad ([b, g]_F^N, A) \in G \\
&\text{iff} \quad [b, g]_F^N \in i_F^N \circ i_E^M(A) \\
&\text{iff} \quad \text{for } F_b \text{ a.e. } \bar{\mu},\ g(\bar{\mu}) \in i_E^M(A) \\
&\text{iff} \quad \text{for } F_b \text{ a.e. } \bar{\mu},\ (g(\bar{\mu}), A) \in E \\
&\text{iff} \quad ([b, g]_F^{M|\xi}, A) \in i(E) \\
&\text{iff} \quad (\sigma(a), A) \in i(E).
\end{aligned}$$

Thus we have a factor map $\tau : Q \to R$ from $\mathrm{Ult}(M, G)$ to $\mathrm{Ult}(M, i(E))$ given by

$$\tau([a, f]_G^M) = [\sigma(a), f]_{i(E)}^M.$$

Assuming $\mathrm{lh}(E) < \mathrm{lh}(F)$, the embedding normalization of $\mathcal{T}$ is then $\mathcal{U}$, where

$$E_0^{\mathcal{U}} = E,\ E_1^{\mathcal{U}} = F,\ E_2^{\mathcal{U}} = i(E).$$

If $\mathrm{lh}(F) < \mathrm{lh}(E)$, it is $E_0^{\mathcal{U}} = F$, $E_1^{\mathcal{U}} = i(E)$.

The full normalization is obtained as in Case 1. Let

$$\pi_0 : M||\,\mathrm{lh}(E) \to \mathrm{Ult}(M||\,\mathrm{lh}(E), F)$$

be the canonical embedding. Letting $\bar{\sigma}([b,g]_F^N) = [b,g]_F^{M||\,\mathrm{lh}(E)}$ for b, g as above, we have $\bar{\sigma} =$ identity, which yields $G = \pi_0(E)$. One can show that $\pi_0(E)$ is on the P-sequence by considering the $\mathrm{lh}(E)$ dropdown sequence of $M|\xi$ and using condensation, as in Case 1.

The situation in Case 2 is summarized by the diagram

$$\begin{tikzcd}
N \arrow[r, "i_F^N"] & Q \arrow[r, "\tau"] & R \\
M \arrow[u, "E"] \arrow[ur, "i_F^N(E)"] \arrow[urr, "i_F^{M|\xi}(E)"'] & & \\
M|\xi \arrow[u, "\trianglelefteq"] \arrow[r, "i_F^{M|\xi}"'] & P &
\end{tikzcd}$$

We have assumed here $k = 0$ to remove some clutter. Again, all the embeddings in the diagram are cofinal and elementary. In the case of τ, this is because it is weakly elementary, and it is cofinal because all the other embeddings are cofinal.

Remark 2.10 If $\langle \xi, k\rangle = \langle \mathrm{lh}(E), 0\rangle$, then $i_F^{M|\xi} = i_F^N \restriction N|\,\mathrm{lh}(E)$, so $i_F^N(E) = i_F^{M|\xi}(E)$, and $Q = R$. This is what happens if $\nu(E) \le \mathrm{crit}(F) < \lambda(E)$. The original $\mathcal{T}$ is ms-normal but not Jensen normal. Its embedding normalization is Jensen normal, and has the same last model as $\mathcal{T}$.

If $\langle \xi, k\rangle = l(M)$, then the diagram simplifies to

$$\begin{tikzcd}
N \arrow[r, "i_F^N"] & Q \arrow[r, "\tau"] & R \\
M \arrow[u, "E"] \arrow[ur, "i_F^N(E)"] \arrow[urr, "i_F^M(E)"'] \arrow[r, "i_F^M"'] & P &
\end{tikzcd}$$

If $\mu < \nu(E)$ and $\nu(E)$ is a cardinal of M and $\langle \xi, k\rangle = l(M)$, then $i_F^N(E)$ is the trivial completion of $i_F^M(E)\restriction \sup i_F^N``\nu(E)$. In this case, $Q = R$ iff $\mathrm{cof}^M(\nu(E)) \neq \mu$, and if $Q \neq R$, then $\mathrm{crit}(\tau) = \sup i_F^N``\nu(E)$. $\square$

Remark 2.11 In both cases, the embedding normalization of $\langle\langle E\rangle, \langle F\rangle\rangle$ may break down by reaching an illfounded model. Similarly for full normalization. (There we also used condensation, hence indirectly iterability.)

Again we are interested in the case M has an iteration strategy Σ. In that case, the models are all wellfounded, and things work out as above. It doesn't yet matter what Σ is, since the trees are finite.

2.2 Normalizing $\mathcal{T}^\frown\langle F\rangle$

Let M be a premouse, and $\mathcal{T}$ a normal tree on M having last model N. and let F be on the N-sequence. Let Q be the longest initial segment of N such that $\mathrm{Ult}(Q, F)$ makes sense, that is, such that F is total on Q and $\mathrm{crit}(F) < \rho_{k(Q)}(Q)$. We construct a normal tree $\mathcal{W}$ on M such that $\mathrm{Ult}(Q, F)$ embeds into the last model of $\mathcal{W}$ via a weakly elementary map. We call $\mathcal{W}$ the *embedding normalization of* $\mathcal{T}^\frown\langle F\rangle$, and write

$$\mathcal{W} = W(\mathcal{T}, F).$$

Let α be the least such that F is on the $\mathcal{M}_\alpha^\mathcal{T}$-sequence. Then $\mathcal{M}_\alpha^\mathcal{T}$ agrees with Q up to $\mathrm{lh}(F)+1$, and Q agrees with $\mathrm{Ult}(Q, F)$ up to $\mathrm{lh}(F)$, but not $\mathrm{lh}(F)+1$. By Fact 2.0, $\mathcal{W}$ must start out with $\mathcal{T}\restriction(\alpha+1)$, if it is being played by some iteration strategy Σ for M such that $\mathcal{T}\restriction(\alpha+1)$ is played by Σ. This is the context that is motivating our definition of $\mathcal{W}$, so we set

$$\mathcal{W}\restriction(\alpha+1) = \mathcal{T}\restriction(\alpha+1).$$

(This does *not* imply $E_\alpha^\mathcal{W} = E_\alpha^\mathcal{T}$, just $\mathcal{M}_\alpha^\mathcal{W} = \mathcal{M}_\alpha^\mathcal{T}$.)

Now let $\beta \leq \alpha$ be least such that $\mu < \lambda(E_\beta^\mathcal{T})$, or $\beta = \alpha$. F must be applied to an initial segment of $\mathcal{M}_\beta^\mathcal{W} = \mathcal{M}_\beta^\mathcal{T}$ in $\mathcal{W}$. That is

$$E_\alpha^\mathcal{W} = F,$$

and the rest is dictated by normality:

$$W\text{-pred}(\alpha+1) = \beta,$$

and

$$\mathcal{M}_{\alpha+1}^{*,\mathcal{W}} = \mathcal{M}_\beta^\mathcal{T}|\langle \xi_0, k_0\rangle$$

where $\langle \xi_0, k_0 \rangle$ is least such that $\rho_(\mathcal{M}_\beta | \langle \xi_0, k_0 \rangle) \leq \mu$ or $\langle \xi_0, k_0 \rangle = l(\mathcal{M}_\beta^{\mathcal{W}})$, and

$$\mathcal{M}_{\alpha+1}^{\mathcal{W}} = \mathrm{Ult}(\mathcal{M}_{\alpha+1}^{*,\mathcal{W}}, F).$$

This gives us $\mathcal{W}\restriction(\alpha+2)$.

Case 1. $Q \neq N$.

In this case Q is a proper initial segment of $M_\beta^{\mathcal{T}} | \mathrm{lh}(E_\beta^{\mathcal{T}})$, by the following claim.

Claim 2.12 *Let $\mathcal{T}$ be a normal iteration tree, $\beta+1 < \mathrm{lh}(\mathcal{T})$, and $\mathcal{M}_\beta^{\mathcal{T}} | \mathrm{lh}(E_\beta^{\mathcal{T}} \trianglelefteq R \trianglelefteq$ $\mathcal{M}_\theta^{\mathcal{T}}$ for some $\theta \geq \beta+1$; then $\mathrm{lh}(E_\beta^{\mathcal{T}}) \leq \rho_{k(R)}(R)$.*

Proof. Let $S = \mathcal{M}_\theta^{\mathcal{T}}$. It is easy to see that $\rho_{k(S)}(S) \geq lh(G)$ for all extenders G used in the branch $[0,\theta)_T$. Since some G with $\mathrm{lh}(G) \geq \mathrm{lh}(E_\beta^{\mathcal{T}}$ was used in $[0,\theta)_T$, we are done if $R = S$. If $\hat{o}(R) = \hat{o}(S)$ but $k(R) < k(S)$, then $\rho_{k(S)}(S) \leq \rho_{k(R)}(R)$, so again we are done. Finally, if $\hat{o}(R) < \hat{o}(S)$, then $R \in S$, so $\rho_{k(R)}(R) < \mathrm{lh}(E_\beta^{\mathcal{T}}) \leq o(R)$ implies that $\mathrm{lh}(E_\beta^{\mathcal{T}})$ is not a cardinal in S. This is a contradiction. □

Let $N = \mathcal{M}_\theta^{\mathcal{T}}$ and $Q = N|\langle \xi, k \rangle$. We apply the claim to $R = N|\langle \xi, k+1 \rangle$. We have $Q \triangleleft N$, so this makes sense. We have $\rho(Q) = \rho_{k(R)}(R) \leq \mu < \mathrm{lh}(E_\beta^{\mathcal{T}})$. It follows from the claim that $R \triangleleft \mathcal{M}_\beta^{\mathcal{T}} | \mathrm{lh}(E_\beta^{\mathcal{T}})$. But $Q \triangleleft R$. Thus Q is a proper initial segment of $M_\beta^{\mathcal{T}} | \mathrm{lh}(E_\beta^{\mathcal{T}})$.

Thus $Q = \mathcal{M}_\beta^{\mathcal{T}} | \langle \xi_0, k_0 \rangle$, $\alpha = \beta$, and $\mathcal{M}_{\alpha+1}^{\mathcal{W}} = \mathrm{Ult}(Q, F)$. So we set

$$\begin{aligned} W(\mathcal{T}, F) &= \mathcal{W}\restriction(\alpha+2) \\ &= \mathcal{T}\restriction(\beta+1)^\frown\langle F \rangle. \end{aligned}$$

We call this the *dropping case* in the definition of $W(\mathcal{T}, F)$. In this case, $\mathrm{Ult}(Q, F)$ is actually equal to the last model of $W(\mathcal{T}, F)$.

Case 2. $Q = N$, and $\mathrm{lh}(\mathcal{T}) = \beta + 1$.

Since $\mathrm{lh}(\mathcal{T}) = \beta+1$, then $Q = N = \mathcal{M}_\beta^{\mathcal{T}}$. Thus $\alpha = \beta$, and again

$$\begin{aligned} W(\mathcal{T}, F) &= \mathcal{W}\restriction(\alpha+2) \\ &= \mathcal{T}\restriction(\beta+1)^\frown\langle F \rangle. \end{aligned}$$

Again, $\mathrm{Ult}(Q, F)$ is actually equal to the last model of $W(\mathcal{T}, F)$. The difference between this and the previous case is just that we did not drop when we applied F to $\mathcal{T}$.

Case 3. $Q = N$, and $lh(\mathcal{T}) > \beta + 1$.

In this case, $\mathrm{Ult}(N, F)$ makes sense, so $\langle \mathrm{lh}(E_\beta^\mathcal{T}), 0\rangle \leq \langle \xi_0, k_0\rangle$, and in fact that $\mathrm{Ult}(\mathcal{M}_\eta^\mathcal{T}, F)$ makes sense for all η such that $\beta < \eta < \mathrm{lh}(\mathcal{T})$.

For $\eta < \mathrm{lh}(\mathcal{T})$, set

$$\phi(\eta) = \begin{cases} \eta, & \text{if } \eta < \beta; \\ (\alpha+1) + (\eta - \beta), & \text{if } \eta \geq \beta. \end{cases}$$

So $\phi : [0, \mathrm{lh}(\mathcal{T})) \cong [0, \beta) \cup [\alpha+1, (\alpha+1) + (\mathrm{lh}(\mathcal{T}) - \beta))$ order-preservingly. We define $\mathcal{M}_{\phi(\eta)}^W$, and

$$\pi_\eta : \mathcal{M}_\eta^\mathcal{T} \to \mathcal{M}_{\phi(\eta)}^\mathcal{W}.$$

For $\eta < \beta$, $\phi(\eta) = \eta$ and $\mathcal{M}_\eta^\mathcal{T} = \mathcal{M}_\eta^\mathcal{W}$ and $\pi_\eta = \text{identity}$. We let

$$\pi_\beta = \text{canonical embedding of } \mathcal{M}_\beta^\mathcal{T}|\langle \xi_0, k_0\rangle \text{ into } \mathrm{Ult}(\mathcal{M}_\beta^\mathcal{T}|\langle \xi_0, k_0\rangle, F).$$

(So the display above is a bit off; for $\eta = \beta$, π_η may not act on all of $\mathcal{M}_\eta^\mathcal{T}$. For $\eta \neq \beta$, π_η will act on all of $\mathcal{M}_\eta^\mathcal{T}$.) Note that F is close to $\mathcal{M}_\beta^\mathcal{T}|\langle \xi_0, k_0\rangle$ because it arose in a later model of $\mathcal{T}$, so that π_β is cofinal and elementary.

We define π_η and $\mathcal{M}_{\phi(\eta)}^\mathcal{W}$ for $\eta \geq \beta + 1$ by induction.

For $\eta = \beta + 1$, we let

$$E_{\phi(\beta)}^\mathcal{W} = \pi_\beta(E_\beta^\mathcal{T}),$$

and let $\tau \leq \beta$ be least such that $\mathrm{crit}(E_{\phi(\beta)}^\mathcal{W}) < \lambda(E_\tau^\mathcal{W})$, and $\langle \gamma, k\rangle$ be least such that $\mathrm{crit}(E_{\phi(\beta)}^\mathcal{W}) \geq \rho_{k+1}(M_\tau^\mathcal{W}|\gamma)$, and set

$$\mathcal{M}_{\phi(\beta+1)}^\mathcal{W} = \mathrm{Ult}(\mathcal{M}_\tau^\mathcal{W}|\langle \gamma, k\rangle, E_{\phi(\beta)}^\mathcal{W}),$$

as required by normality. We get $\pi_{\beta+1}$ from the Shift Lemma. There are two cases.

Case A. $\mathrm{crit}(E_\beta^\mathcal{T}) \geq \mu$.

Since $\pi_\beta = i_F^{M_\beta|\langle \xi_0, k_0\rangle}$, $\mathrm{crit}(\pi_\beta(E_\beta^\mathcal{T})) > \mathrm{lh}(F)$. But $F = E_\alpha^\mathcal{W}$. Thus $\pi_\beta(E_\beta^\mathcal{T}) = E_{\phi(\beta)}^\mathcal{W}$ is applied to $\mathcal{M}_{\alpha+1}^\mathcal{W} = \mathcal{M}_{\phi(\beta)}^\mathcal{W}$, or an initial segment of it. That is

$$\tau = \phi(\beta) = \alpha + 1$$

in this case. In $\mathcal{T}$, we must have

$$T\text{-pred}(\beta + 1) = \beta,$$

because β was least such that $\mu < \nu(E^{\mathcal{T}}_\beta)$. Similarily, the case hypothesis implies that

$$\mathcal{M}^{\mathcal{T}}_{\beta+1} = \mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_\beta|\langle\xi_1, k_1\rangle, E^{\mathcal{T}}_\beta)$$

where $\langle\xi_1, k_1\rangle \leq_{\mathrm{lex}} \langle\xi_0, k_0\rangle$. We have that $\pi_\beta : M_\beta|\langle\xi_1, k_1\rangle \to \pi_\beta(M_\beta|\langle\xi_1, k_1\rangle)$ is elementary, so we can set

$$\pi_{\beta+1}([a, f]^{M^{\mathcal{T}}_\beta|\xi_1}_{E^{\mathcal{T}}_\beta}) = [\pi_\beta(a), \pi_\beta(f)]^{\mathcal{M}^{\mathcal{W}}_{\phi(\beta)}|\pi_\beta(\xi_1)}_{E^{\mathcal{W}}_{\phi(\beta)}}$$

as in the Shift Lemma. (If $k_1 > 0$, $\pi_\beta(f^{\mathcal{M}^{\mathcal{T}}_\beta|\xi_1}_{\tau,q}) = f^{\mathcal{M}^{\mathcal{W}}_{\phi(\beta)}|\pi_\beta(\xi_1)}_{\tau,\pi_\beta(q)}$.) We have that $\pi_{\beta+1}$ is elementary (a near k_1-embedding) by [21], and $\pi_{\beta+1}\restriction \mathrm{lh}(E^{\mathcal{T}}_\beta + 1) = \pi_\beta\restriction \mathrm{lh}(E^{\mathcal{T}}_\beta)$.

Case B. $\mathrm{crit}(E^{\mathcal{T}}_\beta) < \mu$.

Then $\mathrm{crit}(\pi_\beta(E^{\mathcal{T}}_\beta)) = \mathrm{crit}(E^{\mathcal{T}}_\beta)$, so $\tau = T\text{-pred}(\beta+1) = W\text{-pred}(\phi(\beta+1))$. It is clear that $E^{\mathcal{T}}_\beta$ and $\pi_\beta(E^{\mathcal{T}}_\beta)$ are applied to the same initial segment of $\mathcal{M}^{\mathcal{T}}_\tau = \mathcal{M}^{\mathcal{W}}_\tau$. Letting this be $\mathcal{M}^{\mathcal{T}}_\tau|\langle\gamma, k\rangle$, we get

$$\pi_{\beta+1} : \mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_\tau|\langle\gamma, k\rangle, E^{\mathcal{T}}_\beta) \to \mathrm{Ult}(\mathcal{M}^{\mathcal{W}}_\tau|\langle\gamma, k\rangle, \pi_\beta(E^{\mathcal{T}}_\beta))$$

from

$$\pi_{\beta+1}([a, f]^{M_\tau|\gamma}_{E^{\mathcal{T}}_\beta}) = [\pi_\beta(a), f]^{M_\tau|\gamma}_{E^{\mathcal{W}}_{\phi(\beta)}}.$$

Again, $\pi_{\beta+1}$ is elementary, and $\pi_{\beta+1}$ agrees with π_β on $\mathrm{lh}(E^{\mathcal{T}}_\beta) + 1$.

Remark 2.13 In Case A, $\phi(T\text{-pred}(\beta + 1)) = W\text{-pred}(\phi(\beta + 1))$, while in Case B, this fails, and in fact $T\text{-pred}(\beta + 1) = W\text{-pred}(\beta + 1)$. It is because ϕ may not preserve point-of-application for extenders that $\mathcal{T}$ may not be a hull of $\mathcal{W}$, under ϕ and the π_η's, in the sense of Sargsyan's thesis [16]. In fact, $\mathcal{T}$ will be such a hull iff $\mathrm{crit}(E^{\mathcal{T}}_\eta) \geq \mu$ for all $\eta \geq_{\mathcal{T}} \beta$. For example, this happens when $\mathcal{T}$ factors as $\mathcal{T}\restriction(\beta + 1)^\frown\mathcal{S}$, where $\mathcal{S}$ is a tree on $\mathcal{M}^{\mathcal{T}}_\beta$ with all critical points $\geq \mu$.

The successor case when $\eta > \beta$ is similar. Suppose by induction that whenever $\xi, \delta \leq \eta$:

(1) $E^{\mathcal{W}}_{\phi(\delta)} = \pi_\delta(E^{\mathcal{T}}_\delta)$.

(2) if $\delta \neq \beta$, then π_δ is an elementary embedding from $\mathcal{M}^{\mathcal{T}}_\delta$ to $\mathcal{M}^{\mathcal{W}}_{\phi(\delta)}$. ($\pi_\beta$ is cofinal elementary from $\mathcal{M}^{\mathcal{T}}_\beta|\langle\xi_0, k_0\rangle$ to $\mathcal{M}^{\mathcal{W}}_{\phi(\beta)}$.)

(3) if $\xi < \delta$, then π_δ agrees with π_ξ on $\mathrm{lh}(E^{\mathcal{T}}_\xi) + 1$.

(4) (a) if T-pred$(\delta) \neq \beta$ then $\phi(T\text{-pred}(\delta)) = W\text{-pred}(\phi(\delta))$

(b) if T-pred$(\delta) = \beta$, then

i. $\mathrm{crit}(E^{\mathcal{T}}_{\delta-1}) \geq \mu \Rightarrow \phi(T\text{-pred}(\delta)) = W\text{-pred}(\phi(\delta))$

ii. $\mathrm{crit}(E^{\mathcal{T}}_{\delta-1}) < \mu \Rightarrow W\text{-pred}(\phi(\delta)) = \beta$

(c) i. if $\delta \neq \beta$, then ($\delta \, T \, \xi$ iff $\phi(\delta) \, W \, \phi(\xi)$)

ii. $\beta \, T \, \xi \Rightarrow (\phi(\beta) \, W \, \phi(\xi))$ iff the first extender used in $(\beta, \xi]_T$ has critical point $\geq \mu$).

(5) (a) if $\delta \neq \beta$, then $\delta \in D^{\mathcal{T}}$ iff $\phi(\delta) \in D^{\mathcal{W}}$, and $\deg^{\mathcal{T}}(\delta) = \deg^{\mathcal{W}}(\phi(\delta))$

(b) if $\delta \neq \beta$, $\delta \, T \, \xi$, and $D^{\mathcal{T}} \cap (\xi, \delta]_T = \varnothing$, then $\pi_\xi \circ i^{\mathcal{T}}_{\delta,\xi} = i^{\mathcal{W}}_{\phi(\delta),\phi(\xi)} \circ \pi_\delta$

we then define $\pi_{\eta+1} : \mathcal{M}^{\mathcal{T}}_{\eta+1} \to \mathcal{M}^{\mathcal{W}}_{\phi(\eta+1)}$ so as to maintain those conditions. Namely,

$$E^{\mathcal{W}}_{\phi(\eta)} = \pi_\eta(E^{\mathcal{T}}_\eta),$$

and letting τ be least such that $\mathrm{crit}(E^{\mathcal{W}}_{\phi(\eta)}) < \lambda(E^{\mathcal{W}}_\tau)$, and $\langle \gamma, k \rangle$ be appropriate for normal trees,

$$\mathcal{M}^{\mathcal{W}}_{\phi(\eta+1)} = \mathrm{Ult}(\mathcal{M}^{\mathcal{W}}_\tau | \langle \gamma, k \rangle, E^{W}_{\phi(\eta)}).$$

We get $\pi_{\eta+1}$ from the Shift Lemma, with two cases, as before.

Case A. $\mathrm{crit}(E^{\mathcal{T}}_\eta) \geq \mu$.

Let $\sigma = T$-pred$(\eta+1)$, i.e. σ is least such that $\mathrm{crit}(E^{\mathcal{T}}_\eta) < \lambda(E^{\mathcal{T}}_\sigma)$. Clauses (1) and (3) above tell us that $\phi(\sigma)$ is the least θ in $\mathrm{ran}(\phi)$ such that $\mathrm{crit}(E^{\mathcal{W}}_{\phi(\eta)}) < \lambda(E^{\mathcal{W}}_\theta)$. But $\tau \geq \phi(\beta)$ by our case hypotheses, so $\tau \in \mathrm{ran}(\phi)$, so $\tau = \phi(\sigma)$. We leave it to the reader to show that if

$$\mathcal{M}^{\mathcal{T}}_{\eta+1} = \mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_\sigma | \langle \lambda, i \rangle, E^{\mathcal{T}}_\eta),$$

then in fact $i = k$, and $\pi_\sigma(\lambda) = \gamma$. Thus we set

$$\pi_{\eta+1}([a, f]^{\mathcal{M}^{\mathcal{T}}_\sigma | \lambda}_{E^{\mathcal{T}}_\eta}) = [\pi_\eta(a), \pi_\sigma(f)]^{\mathcal{M}^{\mathcal{W}}_\tau | \gamma}_{E^{\mathcal{W}}_{\phi(\eta)}},$$

and everything works out so that (1)-(5) still hold.

Case B. $\mathrm{crit}(E^{\mathcal{T}}_\eta) < \mu$.

Again, let $\sigma = T$-pred$(\eta+1)$. So $\sigma \leq \beta$. Since $\pi_\eta \restriction \mathrm{lh}(E^{\mathcal{T}}_\beta) = \pi_\beta \restriction \mathrm{lh}(E^{\mathcal{T}}_\beta)$, $\pi_\eta \restriction \mu =$ identity, so $\mathrm{crit}(E^{\mathcal{T}}_\eta) = \mathrm{crit}(E^{\mathcal{W}}_{\phi(\eta)})$. Thus $\sigma = \tau$. One can show that $E^{\mathcal{T}}_\eta$ and $E^{\mathcal{W}}_{\phi(\eta)}$

are applied to the same initial segment of $\mathcal{M}_\tau^\mathcal{T} = \mathcal{M}_\tau^\mathcal{W}$, via ultrapowers of the same degree. So we have

$$\pi_{\eta+1} : \mathrm{Ult}(\mathcal{M}_\tau^\mathcal{T}|\langle\gamma, k\rangle, E_\eta^\mathcal{T}) \to \mathrm{Ult}(\mathcal{M}_\tau^\mathcal{W}|\langle\gamma, k\rangle, E_{\phi(\eta)}^\mathcal{W})$$

given by

$$\pi_{\eta+1}([a, f]_{E_\eta^\mathcal{T}}^{\mathcal{M}_\tau^\mathcal{T}|\gamma}) = [\pi_\eta(a), f]_{E_{\phi(\eta)}^\mathcal{W}}^{\mathcal{M}_\tau^\mathcal{W}|\gamma}.$$

The reader can check (1)-(5) still hold.

This finishes the definition of $\pi_{\eta+1}$. For λ a limit, $\mathcal{M}_{\phi(\lambda)}^\mathcal{W}$ and $\pi_\lambda : \mathcal{M}_\lambda^\mathcal{T} \to \mathcal{M}_{\phi(\lambda)}^\mathcal{W}$ are defined by

$$\mathcal{M}_{\phi(\lambda)}^\mathcal{W} = \mathrm{dir\,lim\ of\ } \mathcal{M}_{\phi(\alpha)}^\mathcal{W} \text{ for } \alpha\, T\, \lambda \text{ sufficiently large},$$
$$\pi_\lambda(i_{\alpha\lambda}^\mathcal{T}(x)) = i_{\phi(\alpha),\phi(\lambda)}^\mathcal{W}, \text{ for } \alpha\, T\, \lambda \text{ sufficiently large}.$$

(1)-(5) imply this makes sense, and that (1)-(5) continue to hold. This completes our description of the embedding-normalization of $\mathcal{T}^\frown\langle F\rangle$.

We must see that for N the last model of $\mathcal{T}$ and R the last model of $\mathcal{W}$, $\mathrm{Ult}(N, F)$ embeds elementarily into R. But

Lemma 2.14 *For any $\gamma \geq \beta$, F is an initial segment of the extender of π_γ.*

Proof. F is the extender of π_β. Since $\pi_\beta \restriction (\mu^+)^{M_\beta|\xi} = \pi_\gamma \restriction (\mu^+)^{M_\beta|\xi}$ (because $(\mu^+)^{M_\beta|\xi} < \mathrm{lh}(E_\beta^\mathcal{T})$), we are done. □

Thus there is a natural factor embedding τ from $\mathrm{Ult}(N, F)$ into R, given by $\tau([a, f]_F^N) = \pi_\gamma(f)(a)$, where $N = M_\gamma^\mathcal{T}$.

Lemma 2.15 *τ is weakly elementary.*

Proof. Let $n = k(N)$. Let G be the shortest initial segment of the extender of π_γ such that $\pi_\gamma(N^n) = \mathrm{Ult}_0(N^n, G)$. Then F is an initial segment of G, and $\tau \restriction \mathrm{Ult}_0(N^n, F)$ is Σ_0 elementary from $\mathrm{Ult}_0(N^n, F)$ to $\mathrm{Ult}_0(N^n, G)$, and Σ_1 elementary on $\mathrm{ran}(i_F^{N^n})$, which is cofinal in $\mathrm{Ult}_0(N^n, F)$.. This implies that τ is $r\Sigma_n$ elementary, and $r\Sigma_{n+1}$ elementary on a set cofinal in $\rho_n(\mathrm{Ult}(N, F)$.

The remaining clauses in definition 1.7, concerning the preservation of parameters and projecta, follow from the fact that i_F^N and π_γ are weakly elementary, and $\tau \circ i_F^N = \pi_\gamma$.

□

Remark 2.16 We do not know whether τ must be fully elementary. The problem is that $\pi_\gamma``\rho_n(N)$ may not be cofinal in $\rho_n(R)$. If M-to-N does not drop in $\mathcal{T}$, then M-to-R does not drop in $\mathcal{W}$, and therefore π_γ is cofinal and elementary, so τ is cofinal and elementary. When M-to-N drops, τ may fail to be elementary, so far as we can see.

In a sufficiently coarse case, $\mathcal{W}$ is also the full normalization of $\langle \mathcal{T}, F\rangle$..

Remark 2.17 There is an analogous construction that starts with an ms-normal tree $\mathcal{T}$ on M, and an extender F on the sequence of its last model N, and produces an ms-normal tree $\mathcal{W}^{\mathrm{ms}}(\mathcal{T}, F)$ such that $\mathrm{Ult}(N, F)$ embeds into its last model.

We shall write $X(\mathcal{T}, F)$ for the full normalization of $\langle \mathcal{T}, F\rangle$. In a sufficiently coarse case, $X(\mathcal{T}, F) = W(\mathcal{T}, F)$.

Proposition 2.18 *Let M, $\mathcal{T}$, F, and β be as above. Suppose also that $\mathcal{T}$ is ms-normal, and that $k(M) = \omega$ and $\rho_\omega(M) = o(M)$. Let $\mu = \mathrm{crit}(F)$, and suppose that for all $\gamma + 1 < \mathrm{lh}(\mathcal{T})$,*

$$\mathcal{M}^{\mathcal{T}}_\gamma \models \nu(E^{\mathcal{T}}_\gamma) \text{ is a cardinal of } \mathrm{cof} \neq \mu.$$

(So $\mathcal{T}$ does not drop anywhere, and all models have degree ω.) Then for all $\gamma < \mathrm{lh}\,\mathcal{T}$ such that $\gamma \geq \beta$

$$\mathcal{M}^{\mathcal{W}}_{\phi(\gamma)} = \mathrm{Ult}_\omega(M^{\mathcal{T}}_\gamma, F),$$

and the embedding normalization map π_γ is the same as the F-ultrapower map.

Proof. We show this by induction on γ. For $\gamma = \beta$, this is the definition of $\mathcal{M}^{\mathcal{W}}_{\phi(\beta)}$ and π_β. Suppose it holds for all $\gamma \leq \eta$, we must show it holds at $\eta + 1$. Let $E = E^{\mathcal{T}}_\eta$ and $E^* = \pi_\eta(E) = E^{\mathcal{W}}_{\phi(\eta)}$. Let $\sigma = T\text{-pred}(\eta + 1)$.

Case 1. $\mu \leq \mathrm{crit}(E)$.

Then $\sigma \geq \beta$, and $\phi(\sigma) = W\text{-pred}(\phi(\eta + 1))$. Let $S = \mathrm{Ult}_\omega(M^{\mathcal{T}}_{\eta+1}, F)$, and let $i_F^{M^{\mathcal{T}}_{\eta+1}}$ be the canonical embedding. We have the diagram

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{T}}_{\eta+1} \arrow[r, "i_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}}"] & S \arrow[r, "\tau"] & \mathcal{M}^{\mathcal{W}}_{\phi(\eta+1)} \\
\mathcal{M}^{\mathcal{T}}_\sigma \arrow[u, "E"] \arrow[r, "i_F^{\mathcal{M}^{\mathcal{T}}_\sigma} = \pi_\sigma"'] & \mathcal{M}^{\mathcal{W}}_{\phi(\sigma)} \arrow[ur, "E^*"'] &
\end{tikzcd}$$

Here τ comes from the argument in Case 1 of two-step normalization. Namely, let G be the extender of $i_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}} \circ i_E^{\mathcal{M}^{\mathcal{T}}_{\sigma}}$, and H be the extender of $i_{E^*}^{\mathcal{M}^{\mathcal{W}}_{\phi(\sigma)}} \circ i_F^{\mathcal{M}^{\mathcal{T}}_{\sigma}}$. Note $\nu(G) = \sup i_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}}``\nu(E)$ and $\nu(H) = \sup i_F^{\mathcal{M}^{\mathcal{T}}_{\eta}}``\nu(E)$, by our cofinality assumption.

Claim 2.19 *G is a subextender of H under the map ψ, where*

$$\psi([b,g]_F^{M^{\mathcal{T}}_{\eta+1}}) = [b,g]_F^{M^{\mathcal{T}}_{\eta}},$$

for $b \in [\nu(F)]^{<\omega}$ and $g : [\mu]^{|b|} \to \nu(E)$, $g \in \mathcal{M}^{\mathcal{T}}_{\eta+1}$.

Proof. We calculate as before: for b, g as above and $A \subseteq [\mathrm{crit}(E)]^{<\omega}$ with $A \in \mathcal{M}^{\mathcal{T}}_{\sigma}$,

$$\begin{array}{rll} ([b,g]_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}}, A) \in G & \text{iff} & [b,g]_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}} \in i_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}} \circ i_E^{\mathcal{M}^{\mathcal{T}}_{\sigma}}(A) \\ & \text{iff} & \text{for } F_b \text{ a.e. } \mu,\ g(\mu) \in i_E^{\mathcal{M}^{\mathcal{T}}_{\sigma}}(A) \end{array}$$

(by Łos for $\mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_{\eta+1}, F)$)

$$\begin{array}{ll} \text{iff} & \text{for } F_b \text{ a.e. } \mu,\ (g(\mu), A) \in E \\ \text{iff} & ([b,g]_F^{M^{\mathcal{T}}_{\eta}}, i_F^{M^{\mathcal{T}}_{\eta}}(A)) \in E^* \end{array}$$

(by Łos for $\mathrm{Ult}(M^{\mathcal{T}}_{\eta}, F)$)

$$\begin{array}{ll} \text{iff} & [b,g]_F^{\mathcal{M}^{\mathcal{T}}_{\eta}} \in i_{E^*}^{\mathcal{M}^{\mathcal{T}}_{\sigma}}(i_F^{\mathcal{M}^{\mathcal{T}}_{\eta}}(A)) \end{array}$$

(since $i_{E^*}^{\mathcal{M}^{\mathcal{T}}_{\sigma}}$ and $i_{E^*}^{\mathcal{M}^{\mathcal{T}}_{\eta}}$ agree on subsets of $\mathrm{crit}(E^*)$)

$$\begin{array}{ll} \text{iff} & [b,g]_F^{\mathcal{M}^{\mathcal{T}}_{\eta}} \in i_{E^*}^{\mathcal{M}^{\mathcal{T}}_{\sigma}}(i_F^{\mathcal{M}^{\mathcal{T}}_{\sigma}}(A)) \end{array}$$

(since $i_F^{\mathcal{M}^{\mathcal{T}}_{\eta}}$ agrees with π_η, hence π_γ, hence $i_F^{\mathcal{M}^{\mathcal{T}}_{\sigma}}$ on subsets of $\mathrm{crit}(E)$)

$$\begin{array}{ll} \text{iff} & ([b,g]_F^{\mathcal{M}^{\mathcal{T}}_{\eta}}, A) \in H. \end{array}$$

□

But now $\mathcal{M}^{\mathcal{T}}_{\eta}$ and $\mathcal{M}^{\mathcal{T}}_{\eta+1}$ have the same functions $g : [\mu]^{<\omega} \to \nu(E)$, by our "coarseness" assumptions. So $\psi =$ identity, and $G = H$, and $S = \mathcal{M}^{\mathcal{W}}_{\phi(\eta+1)}$. So our diagram is

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{T}}_{\eta+1} \arrow[r, "i_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}}"] \arrow[r, bend right, "\pi_{\eta+1}"'] & \mathcal{M}^{\mathcal{W}}_{\phi(\eta+1)} \\
\mathcal{M}^{\mathcal{T}}_{\sigma} \arrow[u, "E"] \arrow[r, "\pi_\sigma = i_F^{\mathcal{M}^{\mathcal{T}}_{\sigma}}"'] & \mathcal{M}^{\mathcal{W}}_{\phi(\sigma)} \arrow[u, "E^*"']
\end{tikzcd}$$

It remains to show $i_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}} = \pi_{\eta+1}$. Since both maps make the diagram commute, it is enough to show $i_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}} \restriction \nu(E) = \pi_{\eta+1} \restriction \nu(E)$. But $\pi_{\eta+1} \restriction \nu(E) = \pi_\eta \restriction \nu(E)$ by the Shift Lemma, and $\pi_\eta \restriction \nu(E) = i_F^{\mathcal{M}^{\mathcal{T}}_\eta} \restriction \nu(E)$ by induction, and $i_F^{\mathcal{M}^{\mathcal{T}}_\eta} \restriction \nu(E) = i_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}} \restriction \nu(E)$ because $\mathcal{M}^{\mathcal{T}}_\eta$ and $\mathcal{M}^{\mathcal{T}}_{\eta+1}$ have the same functions $g : [\mu]^{<\omega} \to \nu(E)$.

Case 2. $\mathrm{crit}(E) < \mu$.

Let $\sigma = T\text{-pred}(\eta+1)$. Then in this case, $\sigma = W\text{-pred}(\eta+1)$. Let $S = \mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_{\eta+1}, F)$. We have the diagram

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{T}}_{\eta+1} \arrow[r, "i_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}}"] & S \arrow[r, "\tau"] & \mathcal{M}^{\mathcal{W}}_{\phi(\eta+1)} \\
\mathcal{M}^{\mathcal{T}}_\sigma = \mathcal{M}^{\mathcal{W}}_\sigma \arrow[u, "E"] \arrow[urr, "E^*"'] & &
\end{tikzcd}$$

We show that $S = \mathcal{M}^{\mathcal{W}}_{\phi(\eta+1)}$ and $i_F^{\mathcal{M}^{\mathcal{T}}_{\eta+1}} = \pi_{\eta+1}$ by the calculations in Case 2 of two-step normalization. $\square$

Definition 2.20 *For $\mathcal{U}$ a normal iteration tree on M, let*

$$\mathcal{U}^{<\gamma} = \mathcal{U}\restriction(\alpha+1), \text{ where } \alpha \text{ is least such that } \mathrm{lh}\, E^{\mathcal{U}}_\alpha \geq \gamma,$$

and $\mathcal{U}^{<\gamma} = \mathcal{U}$ if there is no such α. Let

$$\mathcal{U}^{>\gamma} = \langle \mathcal{M}^{\mathcal{U}}_\eta \mid E^{\mathcal{U}}_\eta \text{ exists} \wedge \gamma < \lambda(E^{\mathcal{U}}_\eta)\rangle.$$

Definition 2.21 *Let M, $\mathcal{T}$, F and $\mathcal{W}$ be as above, then we write*

$$W(\mathcal{T}, F) = \mathcal{T}^{<\mathrm{lh}\, F} {}^\frown \langle F \rangle^\frown i_F \text{“} \mathcal{T}^{>\mathrm{crit}(F)}$$

for the embedding normalization of $\mathcal{T}^\frown\langle F\rangle$ just defined. We write $\alpha^{\mathcal{T},F}$, $\beta^{\mathcal{T},F}$, $\phi^{\mathcal{T},F}$, and $\pi^{\mathcal{T},F}_\xi$ for the auxiliary objects $\alpha, \beta, \phi, \pi_\xi$ that we defined above.

The full normalization $X(\mathcal{T}, F)$ of $\mathcal{T}^\frown\langle F\rangle$ can be obtained as follows. We assume that $\mathcal{T}$ is normal on M, N is the last model of $\mathcal{T}$, F is on the N sequence, and $\mathrm{crit}(F) < \rho_n(N)$, for $n = k(N)$. Let

$$\mathcal{W} = \mathcal{T}^{<\mathrm{lh}\, F} {}^\frown \langle F \rangle^\frown i_F \text{“} \mathcal{T}^{>\mathrm{crit}(F)}$$

be the embedding normalization. Let $\mathcal{T}^{<\operatorname{lh} F} = \mathcal{T}\restriction(\alpha+1)$, $\beta = W\text{-pred}(\alpha+1)$, and $\phi : \operatorname{lh}\mathcal{T} \to \operatorname{lh} W$ be as above. The full normalization is $\mathcal{X}$, where

$$\mathcal{X}\restriction(\alpha+2) = \mathcal{W}\restriction(\alpha+2)$$

and

$$\mathcal{M}^{\mathcal{X}}_{\phi(\eta)} = \operatorname{Ult}(\mathcal{M}^{\mathcal{T}}_{\eta}, F) \text{ for } \eta > \beta.$$

(Note that if $\eta > \beta$, then some G such that $\operatorname{crit}(F) = \mu < \lambda(G)$ was used on the branch to $M^{\mathcal{T}}_{\eta}$, so for $k = k(\mathcal{M}^{\mathcal{T}}_{\eta})$, $\mu < \rho_k(M^{\mathcal{T}}_{\eta})$.) The tree order of $\mathcal{X}$ is the same as that of $\mathcal{W}$. We have

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{T}}_{\eta} \arrow[r, "i_F^{\mathcal{M}^{\mathcal{T}}_{\eta}}"] \arrow[rr, bend right, "\pi_\eta"'] & \mathcal{M}^{\mathcal{X}}_{\phi(\eta)} \arrow[r, "\tau"] & \mathcal{M}^{\mathcal{W}}_{\phi(\eta)}
\end{tikzcd}$$

where τ is the natural factor map. What remains is to find the extenders $E^{\mathcal{X}}_{\phi(\eta)}$ that make $\mathcal{X}$ into a normal iteration tree. For this, let $E = E^{\mathcal{T}}_{\eta}$, and

$$\pi : M^{\mathcal{T}}_{\eta}|\langle \operatorname{lh}(E), 0\rangle \to \operatorname{Ult}(M^{\mathcal{T}}_{\eta}|\langle \operatorname{lh}(E), 0\rangle, F)$$

be the canonical embedding. One can show using condensation that $\pi(E)$ is on the sequence of $\mathcal{M}^{\mathcal{X}}_{\phi(\eta)}$. Moreover, for $\sigma = W\text{-pred}(\eta+1)$,

$$\mathcal{M}^{\mathcal{X}}_{\phi(\eta+1)} = \operatorname{Ult}(\mathcal{M}^{\mathcal{W}^-}_{\sigma}|\langle \xi, n\rangle, \pi(E)),$$

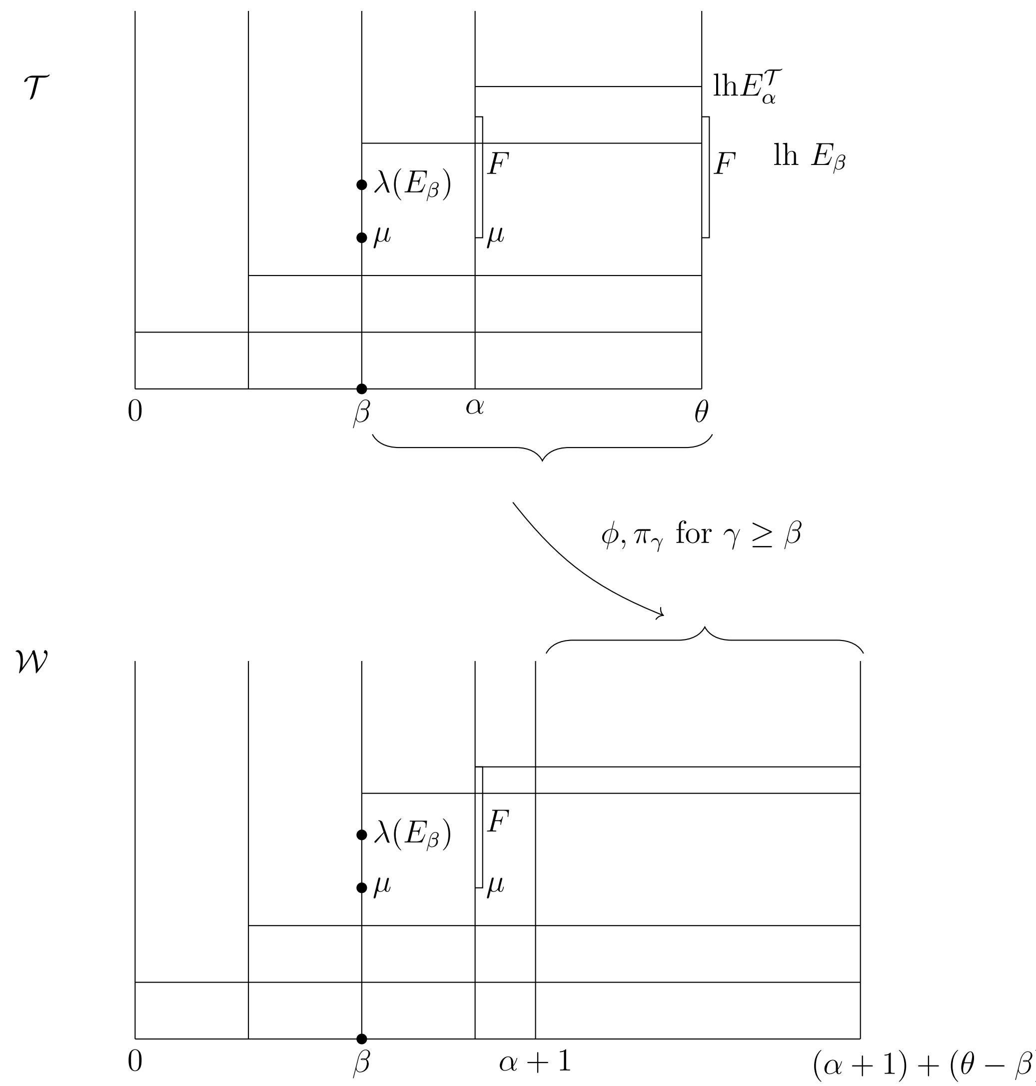
where $n = k(\mathcal{M}^{\mathcal{W}}_{\eta+1}) = k(\mathcal{M}^{\mathcal{T}}_{\eta+1})$ and ξ is appropriate. The details here are like those in the two-step case. Since we don't actually need full normalization in comparing iteration strategies, we give no further detail here. There is a much more careful discussion in [33]. Here is a diagram of the situation.

$$\begin{tikzcd}
N \arrow[r, "i_F^N"] & \operatorname{Ult}(N, F) \arrow[r, "\tau"] & R \\
M \arrow[u, "\hat{\imath}^{\mathcal{T}}"] \arrow[ur, "\hat{\imath}^{\mathcal{X}}"] \arrow[urr, "\hat{\imath}^{\mathcal{W}}"'] & &
\end{tikzcd}$$

Each $\mathcal{M}^{\mathcal{T}}_\eta$ is mapped into $\mathcal{M}^{\mathcal{X}}_{\phi(\eta)}$, and that in turn is mapped into $\mathcal{M}^{\mathcal{W}}_{\phi(\eta)}$.

Returning to $W(\mathcal{T}, F)$, here are a few illustrations that the reader may or may not find helpful. Let $\mathcal{T}$ be normal on M of length $\theta+1$, F on the sequence of $\mathcal{M}^{\mathcal{T}}_\theta$, $\mu = \mathrm{crit}(F)$, β least such that $\mu < \lambda(E^{\mathcal{T}}_\beta)$, and α least such that F is on the sequence of $\mathcal{M}^{\mathcal{T}}_\alpha$, as above. We assume in the diagram that $\beta < \theta$, and that $\mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_\theta, F)$ makes sense. Let $\phi : \theta \cong [0, \beta) \cup [\alpha+1, (\alpha+1)+(\theta-\beta)]$ be the order-isomorphism as above.

We illustrate first the embedding of $\mathcal{T}$ into $\mathcal{W}(\mathcal{T}, F)$, as it appears in the agreement diagrams. We draw them as if $\beta < \alpha$, although $\beta = \alpha$ is possible.

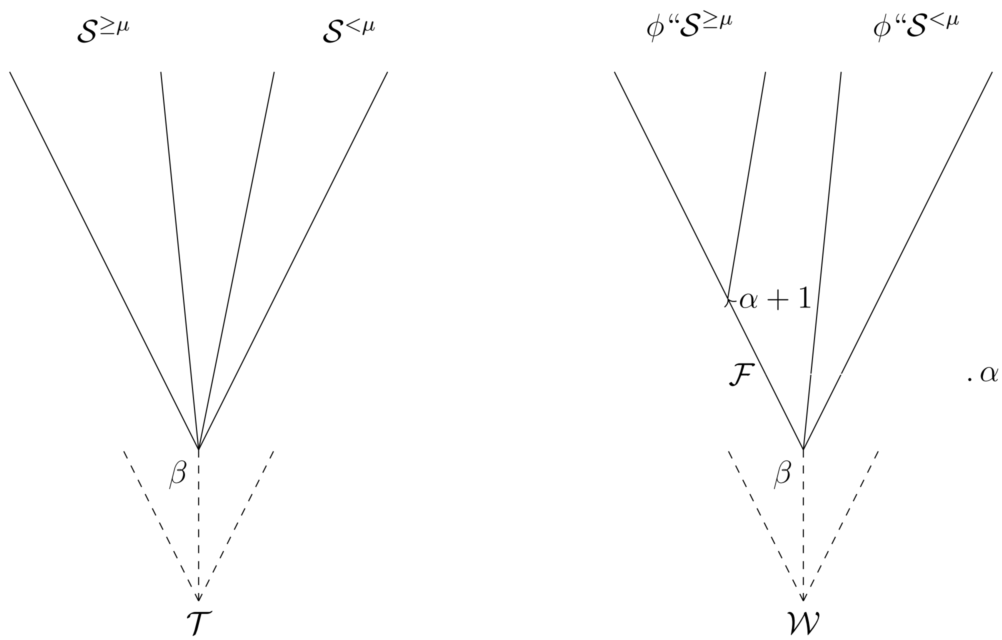


We

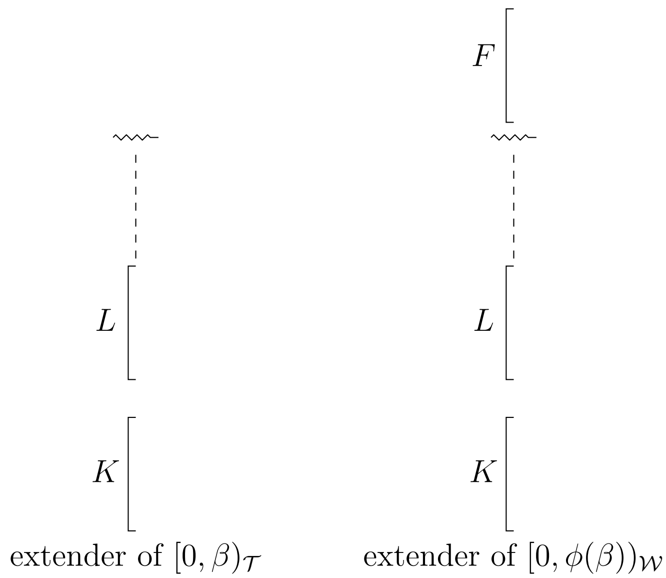
have

$$\mathcal{T}\restriction(\alpha+1) = \mathcal{W}\restriction(\alpha+1),$$
$$F = E^{\mathcal{W}}_\alpha,$$
$$\text{and } i_F\text{“}\mathcal{T}^{>\mu} = \text{ remainder of } \mathcal{W}.$$

The next diagram show how ϕ may fail to preserve tree order. By (4)(c) above, we can have $\delta \leq_T \xi$ but $\phi(\delta) \not\leq_W \phi(\xi)$ iff $\delta = \beta$, and the first extender G used in $(0,\xi)_T$ such that G is applied to an initial segment of $\mathcal{M}^{\mathcal{T}}_\beta$ satisfies $\mathrm{crit}(G) < \mu$. Let $S^{<\mu}$ be the set of such $\xi >_{\mathcal{T}} \beta$, and $S^{\geq\mu}$ the remaining $\xi >_{\mathcal{T}} \beta$. The picture is

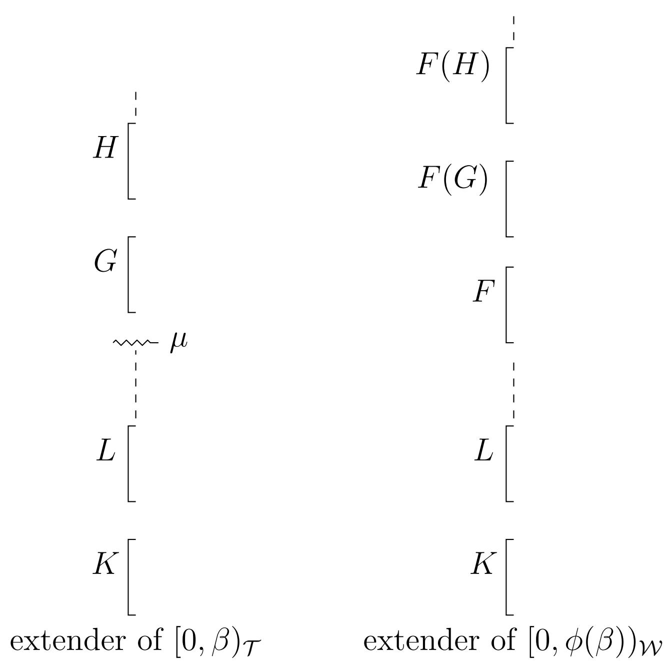


Finally, we illustrate the relationship between the branch extenders of $[0,\xi)_T$ and $[0,\phi(\xi))_W$. If $\xi < \beta$, they are equal. For $\xi = \beta$, the picture is



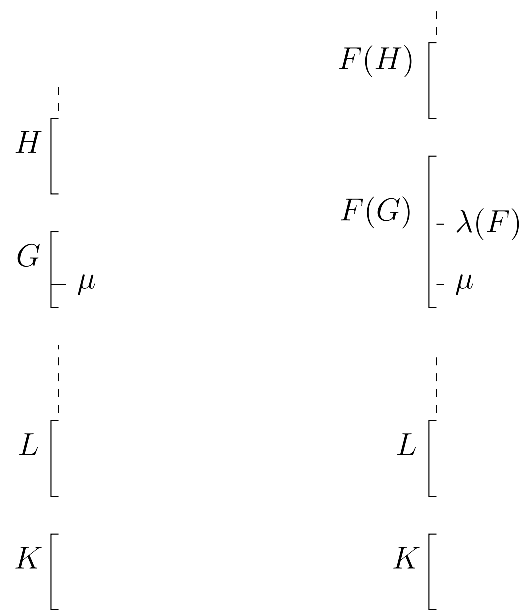
because $[0,\beta)_T \subseteq [0,\phi(\beta))_W$, and just the one additional extender F is used.

For $\xi > \beta$, let G be the first extender used in $[0,\xi)_T$ such that $\lambda(G) \geq \lambda(E^{\mathcal{T}}_{\beta})$. The picture depends on whether $\mu \leq \mathrm{crit}(G)$. If $\mu \leq \mathrm{crit}(G)$, it is



In this case, F is used on $[0, \phi(\xi))_W$, and the remaining extender used are the images of old ones under copy maps.

If $\mathrm{crit}(G) < \mu < \lambda(G)$, the picture is



extender of $[0, \beta)_{\mathcal{T}}$ extender of $[0, \phi(\beta))_{\mathcal{W}}$

In this case, the two branches use the same extenders until G is used on $[0, \xi)_T$. At that point and after, $[0, \phi(\xi))_W$ uses the images of extenders under the copy maps.

Notice that in either case, there is an L used in $[0, \phi(\xi))_W$ such that $\mathrm{crit}(L) \leq \mathrm{crit}(F) < \lambda(F) \leq \lambda(L)$. This will be important later.

Remark 2.22 There is nothing guaranteeing that the models of $W(\mathcal{T}, F)$ are well-founded. In our context of interest, $\mathcal{T}$ is played according to an iteration strategy Σ. Part of "normalizing well" for Σ will then be that $W(\mathcal{T}, F)$ is according to Σ.

2.3 The extender tree $\mathcal{V}^{\mathrm{ext}}$

The fact that $\phi^{\mathcal{T},F}$ does not fully preserve tree order or tree predecessor is awkward. Here is another way to visualize our embedding of $\mathcal{T}$ into $W(\mathcal{T}, F)$ given by $\phi^{\mathcal{T},F}$ and the $\pi_\xi^{\mathcal{T},F}$'s.

For $\mathcal{V}$ a normal tree, let

$$\mathrm{Ext}(\mathcal{V}) = \{E_\alpha^{\mathcal{V}} \mid \alpha + 1 < \mathrm{lh}\,\mathcal{V}\}$$

be the set of extenders used. Note $\mathrm{Ext}(\mathcal{V})$ determines $\mathcal{V}$ modulo a strategy Σ for the base model of $\mathcal{V}$, by normality. For $\gamma < \mathrm{lh}(\mathcal{V})$,

$$s_\gamma^{\mathcal{V}} = \text{increasing enumeration of } \{E_\alpha^{\mathcal{V}} \mid \alpha + 1 \leq_{\mathcal{V}} \gamma\},$$

increasing in order of use (index, length).

Note that of $s_\gamma^{\mathcal{V}}$, $\mathcal{M}_\gamma^{\mathcal{V}}$ and $\mathcal{V}{\restriction}(\alpha+1)$ each determines the others, by normality. Set

$$\mathcal{V}^{\mathrm{ext}} = \{s_\gamma^{\mathcal{V}} \mid \gamma < \mathrm{lh}\,\mathcal{V}\}.$$

$\mathcal{V}^{\mathrm{ext}}$ determines $\mathcal{V}$. The structure $(\mathcal{V}^{\mathrm{ext}}, \subseteq)$ is the *extender-tree* of $\mathcal{V}$.

Here are two simple facts. If F and G are extenders, then F and G *overlap* iff $[\mathrm{crit}(F), \lambda(F)) \cap [\mathrm{crit}(G), \lambda(G)) \neq \varnothing$. We say F and G are *compatible* iff $\exists\alpha(F = G{\restriction}\alpha$ or $G = F{\restriction}\alpha)$. Then for normal $\mathcal{V}$:

1. If $s^\frown\langle F\rangle \in \mathcal{V}^{\mathrm{ext}}$ and $s^\frown\langle G\rangle \in \mathcal{V}^{\mathrm{ext}}$, then F and G overlap.
2. If $s, t \in \mathcal{V}^{\mathrm{ext}}$ and $s(i)$ is compatible with $t(k)$, then $i = k$ and $s{\restriction}(i+1) = t{\restriction}(i+1)$.

Now let $\mathcal{T}$ be normal on M, and $\mathcal{W} = W(\mathcal{T}, F)$. Let $\phi = \phi^{\mathcal{T},F}$, $\pi_\xi = \pi_\xi^{\mathcal{T},F}$, etc. We define a partial map

$$p_{\mathcal{T},F} : \mathrm{Ext}(\mathcal{T}) \to \mathrm{Ext}(\mathcal{W})$$

by

$$p_{\mathcal{T},F}(E_\xi^{\mathcal{T}}) = \pi_\xi(E_\xi^{\mathcal{T}}) = E_{\phi(\xi)}^{\mathcal{W}}.$$

So $p_{\mathcal{T},F}(E_\xi^{\mathcal{T}}){\downarrow}$ iff $\xi \in \mathrm{dom}\,\phi$, and either $\xi \neq \beta$, or $\xi = \beta$ and $\mathcal{M}_\beta^{\mathcal{T}} | \mathrm{lh}(E_\beta^{\mathcal{T}}) \trianglelefteq \mathcal{M}_{\alpha+1}^{*,\mathcal{W}}$.

We can view p as acting on branch extenders. For $s \in \mathcal{T}^{\mathrm{ext}}$, let

$$i_s^\gamma = i_s = \begin{cases} \text{least } i \text{ such that } \mathrm{crit}(F) < \lambda(s(i)), & \text{if this exists;} \\ \text{undefined,} & \text{otherwise.} \end{cases}$$

Let $\xi \in \mathrm{dom}\,\phi$ and $s = s_\xi^{\mathcal{T}}$. Then if $\mathrm{dom}(\phi) = \beta + 1$, we have

$$s_{\phi(\xi)}^{\mathcal{W}} = \begin{cases} s, & \text{if } \xi < \beta; \\ s^\frown\langle F\rangle, & \text{if } \xi = \beta. \end{cases}$$

If $\mathrm{dom}(\phi) > \beta + 1$, then i_s exists precisely when $s = s_\xi^{\mathcal{T}}$ for some $\xi \geq \beta + 1$, and

$$s_{\phi(\xi)}^{\mathcal{W}} = \begin{cases} s, & \text{if } \xi < \beta; \\ s^\frown\langle F\rangle, & \text{if } \xi = \beta; \\ s{\restriction}i_s{}^\frown\langle F\rangle^\frown\langle \psi^{\mathcal{T},F}(s(i)) \mid i \geq i_s\rangle, & \text{if } \mathrm{crit}(F) \leq \mathrm{crit}(s(i_s)); \\ s{\restriction}i_s{}^\frown\langle \psi^{\mathcal{T},F}(s(i)) \mid i \geq i_s\rangle, & \text{if } \mathrm{crit}(s(i_s)) < \mathrm{crit}(F). \end{cases}$$

So if E is used before H in $s_\xi^{\mathcal{T}}$, then $p_{\mathcal{T},F}(E)$ is used before $p_{\mathcal{T},F}(H)$ in $s_{\phi(\xi)}^{\mathcal{W}}$.

Definition 2.23 *Let $\mathcal{W} = W(,F)$, and suppose $s \in \mathcal{T}^{\text{ext}}$ is such that $\forall \mu \in \text{dom}(s)$, $p_{\mathcal{T},F}(s(\mu))\downarrow$; then*

$$\hat{p}_{\mathcal{T},F}(s) = \textit{unique shortest } t \in \mathcal{W}^{\text{ext}} \textit{ such that}$$
$$\forall \mu \in \text{dom}(s),\ p_{\mathcal{T},F}(s(\mu)) \in \text{ran}(t).$$

For $\hat{p} = \hat{p}_{,F}$, we have that $\hat{p}(s^{\mathcal{T}}_{\xi}) = s^{\mathcal{W}}_{\phi(\xi)}$, except when $\xi = \beta$. At β, we have $s^{\mathcal{W}}_{\phi(\beta)} = \hat{p}(s^{\mathcal{T}}_{\beta})^\frown\langle F\rangle$. The map $\hat{p}\colon \mathcal{T}^{\text{ext}} \to W(\mathcal{T},F)^{\text{ext}}$ does preserve $\subseteq$.

Proposition 2.24 *Let $s,t \in \text{dom}(\hat{\psi}^{\mathcal{T},F})$; then*

(1) *$s \subseteq t \Rightarrow \hat{p}(s) \subseteq \hat{p}(t)$, and*

(2) *$s \perp t \Rightarrow \hat{p}(s) \perp \hat{p}(t)$.*

2.4 Psuedo-hull embeddings

An iteration strategy Σ for M *condenses well* iff whenever $\mathcal{U}$ is by Σ, and π is a sufficiently elementary embedding from $\mathcal{T}$ into $\mathcal{U}$ such that $\pi\restriction M \cup \{M\}$ is the identity, then $\mathcal{T}$ is by Σ. By weakening the elementarity required of π, we obtain stronger condensation properties.

In the Hull Condensation property of [16], one is given an embedding $\sigma : \text{lh}\,\mathcal{T} \to \text{lh}\,\mathcal{U}$ and embeddings $\tau_\alpha : \mathcal{M}^{\mathcal{T}}_\alpha \to \mathcal{M}^{\mathcal{U}}_{\sigma(\alpha)}$. σ preserves tree order and tree-predecessor. The τ_α's have the agreement one would get from a copying construction, and they commute with the branch embeddings of $\mathcal{T}$ and $\mathcal{U}$. Moreover, $\tau_\alpha(E^{\mathcal{T}}_\alpha) = E^{\mathcal{U}}_{\sigma(\alpha)}$. A simple example in the way $\mathcal{T} = \pi\mathcal{W}$ sits inside $\mathcal{U} = \pi(\mathcal{W})$, in the case $\pi\colon H \to V$ is elementary and $\pi\restriction M \cup \{M\} =$ id .

A hull embedding $(\sigma, \vec{\tau})$ as above induces a map $\psi : \text{Ext}(\mathcal{T}) \to \text{Ext}(\mathcal{U})$ by

$$\psi(E^{\mathcal{T}}_\alpha) = \tau_\alpha(E^{\mathcal{T}}_\alpha).$$

We then get $\hat{\psi} : \mathcal{T}^{\text{ext}} \to \mathcal{U}^{\text{ext}}$ by

Definition 2.25 *$\hat{\psi}(s) =$ unique shortest $t \in \mathcal{U}^{\text{ext}}$ such that $\forall \mu \in \text{dom}(s)\ \psi(s(\mu)) \in \text{ran}\, t$.*

$\hat{\psi}$ preserves $\subseteq$ and incompatibility in the extender trees. $\hat{\psi}$ is related to σ by

$$\hat{\psi}(s^{\mathcal{T}}_{\alpha+1}) = s^{\mathcal{U}}_{\sigma(\alpha+1)}.$$

But for λ a limit, $\hat{\psi}(s^{\mathcal{T}}_\lambda)$ may be a proper initial segment of $s^{\mathcal{U}}_{\sigma(\lambda)}$.

We now define the notion of a pseudo-hull embedding from $\mathcal{T}$ into $\mathcal{U}$. This will be a triple $(u, \vec{t}, p)$ with most of the properties of σ, $\vec{\tau}$, ψ above. The main thing we drop is the requirement that $u(T\text{-pred}(\gamma+1)) = U\text{-pred}(u(\gamma+1))$. We shall also allow the t_α's to be partial, in a controlled way. Recall here the partial branch embeddings $\hat{\imath}^{\mathcal{U}}_{\alpha,\beta}$. (Cf. 1.9.)

Definition 2.26 *Let $\mathcal{T}$ and $\mathcal{U}$ be normal iteration trees on a premouse M. A* pseudo-hull embedding *of $\mathcal{T}$ into $\mathcal{U}$ is a system*

$$\langle u, \langle t^0_\beta \mid \beta < \operatorname{lh}\mathcal{T}\rangle, \langle t^1_\beta \mid \beta + 1 < \operatorname{lh}\mathcal{T}\rangle, p\rangle$$

such that

(a) $u : \{\alpha \mid \alpha + 1 < \operatorname{lh}\mathcal{T}\} \to \{\alpha \mid \alpha + 1 < \operatorname{lh}\mathcal{U}\}$, $\alpha < \beta \Rightarrow u(\alpha) < u(\beta)$, *and λ is limit iff $u(\lambda)$ is limit.*

(b) $p : \operatorname{Ext}(\mathcal{T}) \to \operatorname{Ext}(\mathcal{U})$ *is such that E is used before F on the same branch of $\mathcal{T}$ iff $p(E)$ is used before $p(F)$ on the same branch of $\mathcal{U}$. Thus p induces $\hat{p} : \mathcal{T}^{\text{ext}} \to \mathcal{U}^{\text{ext}}$ as in Definition 2.25.*

(c) *Let $v : \operatorname{lh}\mathcal{T} \to \operatorname{lh}\mathcal{U}$ be given by*

$$s^{\mathcal{U}}_{v(\beta)} = \hat{p}(s^{\mathcal{T}}_\beta)$$

Then

$$t^0_\beta : M^{\mathcal{T}}_\beta \to M^{\mathcal{U}}_{v(\beta)}$$

is total and elementary. Moreover, for $\alpha <_T \beta$,

$$t^0_\beta \circ \hat{\imath}^{\mathcal{T}}_{\alpha,\beta} = \hat{\imath}^{\mathcal{U}}_{v(\alpha),v(\beta)} \circ t^0_\alpha.$$

In particular, the two sides have the same domain.

(d) *For $\alpha + 1 < \operatorname{lh}\mathcal{T}$, $v(\alpha) \le_U u(\alpha)$, and*

$$t^1_\alpha = \hat{\imath}^{\mathcal{U}}_{v(\alpha),u(\alpha)} \circ t^0_\alpha.$$

Moreover,

$$\begin{aligned} p(E^{\mathcal{T}}_\alpha) &= t^1_\alpha(E^{\mathcal{T}}_\alpha) \\ &= E^{\mathcal{U}}_{u(\alpha)}. \end{aligned}$$

Moreover, for $\alpha < \beta < \operatorname{lh}\mathcal{T}$,

$$t^0_\beta \restriction \operatorname{lh}(E^{\mathcal{T}}_\alpha) + 1 = t^1_\alpha \restriction \operatorname{lh}(E^{\mathcal{T}}_\alpha) + 1.$$

(e) *If* $\beta = T\text{-pred}(\alpha+1)$, *then* $U\text{-pred}(u(\alpha)+1) \in [v(\beta), u(\beta)]_U$, *and setting* $\beta^* = U\text{-pred}(u(\alpha)+1)$,

$$t^0_{\alpha+1}([a,f]^P_{E^{\mathcal{T}}_\alpha}) = [t^1_\alpha(a), \hat{\imath}^{\mathcal{U}}_{v(\beta),\beta^*} \circ t^0_\beta(f)]^{P^*}_{E^{\mathcal{U}}_{u(\alpha)}},$$

where $P \trianglelefteq M^{\mathcal{T}}_\beta$ *is what* $E^{\mathcal{T}}_\alpha$ *is applied to, and* $P^* \trianglelefteq M^{\mathcal{U}}_{\beta^*}$ *is what* $E^{\mathcal{U}}_{u(\alpha)}$ *is applied to.*

The appropriate diagram to go with (e) of Definition 2.26 (for the non-dropping case is)

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{T}}_{\alpha+1} \arrow[rr, "t^0_{\alpha+1}"] & & \mathcal{M}^{\mathcal{U}}_{v(\alpha+1)} \\
& \mathcal{M}^{\mathcal{U}}_{u(\beta)} & \\
& & \mathcal{M}^{\mathcal{U}}_{\beta^*} \arrow[ul] \arrow[uu, "E^{\mathcal{U}}_{u(\alpha)}"'] \\
\mathcal{M}^{\mathcal{T}}_\beta \arrow[uuu, "i^{\mathcal{T}}_{\beta,\alpha+1}"] \arrow[uur, "t^1_\beta"] \arrow[urr, "\rho"] \arrow[rr, "t^0_\beta"] & & \mathcal{M}^{\mathcal{U}}_{v(\beta)} \arrow[u] \\
\mathcal{M}^{\mathcal{T}}_\alpha \arrow[rr, "t^1_\alpha"] & & \mathcal{M}^{\mathcal{U}}_{u(\alpha)}
\end{tikzcd}$$

Here $\hat{\imath}^{\mathcal{U}}_{v(\beta),\beta^*} \circ t^0_\beta = \rho$ is a possibly partial map, defined and elementary on P.

Remark 2.27 By clause (c), $v(0) = 0$ and $t^0_0 = \text{id}$. It is possible that $u(0) > 0$. By clause (d), $v(\alpha+1) = u(\alpha)+1$. Clause (b) implies that $\alpha+1 \leq_T \beta+1$ iff $v(\alpha+1) \leq_U v(\beta+1)$. If $\lambda < \text{lh}(\mathcal{T})$ is a limit ordinal, then $v(\lambda) = \sup\{v(\xi) \mid \xi <_T \lambda\}$. So v preserves tree order, and is continuous at limits. The map u may not preserve tree order.

Remark 2.28 Given $u(\alpha)$ and t^1_α, we can characterize $v(\alpha)$ as the least $\xi \leq_U u(\alpha)$ such that $\operatorname{ran}(t^1_\alpha) \subseteq \operatorname{ran}(\hat{\imath}^{\mathcal{U}}_{\xi,u(\alpha)})$.

In the context of Jensen premice, embeddings that agree on $\operatorname{lh}(E)$ will generally be forced to agree on $\operatorname{lh}(E)+1$. For example, in clause (e) of 2.26, $t^0_{\alpha+1}$ agrees with t^1_α on $lh(E^{\mathcal{T}}_\alpha)+1$, because the Shift Lemma produces this kind of agreement. One does encounter embeddings that agree on λ_E, but not on λ_E+1.

The agreement of maps in a psuedo-hull embedding is given by

Lemma 2.29 *Let* $< u, \langle t^0_\beta \mid \beta < \operatorname{lh}\mathcal{T}\rangle, \langle t^1_\beta \mid \beta+1 < \operatorname{lh}\mathcal{T}\rangle, p\rangle$ *be a psuedo-hull embedding of* $\mathcal{T}$ *into* $\mathcal{U}$*; then*

(a) if $\alpha+1 < \operatorname{lh}(\mathcal{T})$*, then* t^1_α *agrees with* t^0_α *on* $\lambda^{\mathcal{T}}_\alpha$*, and*

(b) if $\beta < \alpha < \operatorname{lh}(\mathcal{T})$*, then* t^0_α *agrees with* t^0_β *on* $\lambda^{\mathcal{T}}_\beta$

Proof. For (a), notice that if F is used in $s^{\mathcal{T}}_\alpha$, then $p(F)$ is used in $s^{\mathcal{U}}_{v(\alpha)}$, and so $\lambda_{p(F)} \leq \operatorname{crit}(\hat{\imath}^{\mathcal{U}}_{v(\alpha),u(\alpha)})$. Thus $\sup t^0_\alpha``\lambda^{\mathcal{T}}_\alpha \leq \operatorname{crit}(\hat{\imath}^{\mathcal{U}}_{v(\alpha),u(\alpha)})$. But $t^1_\alpha = \hat{\imath}^{\mathcal{U}}_{v(\alpha),u(\alpha)} \circ t^0_\alpha$, so we have (a).

For (b): we have that t^1_β agrees with t^0_α on $\operatorname{lh}(E^{\mathcal{T}}_\beta)+1$ by the definition, and t^1_β agrees with t^0_β on $\lambda^{\mathcal{T}}_\beta$ by (a). Since $\lambda^{\mathcal{T}}_\beta < \operatorname{lh}(E^{\mathcal{T}}_\beta)$, we are done. □

One could not replace $\lambda^{\mathcal{T}}_\alpha$ by $\sup\{\operatorname{lh}(F) \mid F \in \operatorname{ran}(s^{\mathcal{T}}_\alpha)\}$ in the lemma above. The reason is that there could be a last extender F used in $s^{\mathcal{T}}_\alpha$. (So $F = E^{\mathcal{T}}_\beta$ where $\alpha = \beta+1$.) Then $p(F)$ is the last extender used in $s^{\mathcal{U}}_{v(\alpha)}$. It could be that $\operatorname{crit}(\hat{\imath}^{\mathcal{U}}_{v(\alpha),u(\alpha)}) = \lambda_{p(F)}$, and thus t^1_α and $t^0_{\alpha+1}$ both disagree with t^0_α at λ_F. (This is the only way the stronger agreement lemma can fail.)

Remark 2.30 The proof of 4.2 gives a formula for the point of application of $E^{\mathcal{U}}_{u(\alpha)}$ under a psuedo hull embedding of $\mathcal{T}$ into $\mathcal{U}$, namely

$$U\text{-pred}(u(\alpha)+1) = \text{least } \eta \in [v(\beta), u(\beta)]_U \text{ such that } \operatorname{crit} \hat{\imath}^{\mathcal{U}}_{\eta,u(\beta)} > \hat{\imath}^{\mathcal{U}}_{v(\beta),\eta} \circ t^0_\beta(\mu),$$

where

$$\beta = T\text{-pred}(\alpha+1) \text{ and } \mu = \operatorname{crit}(E^{\mathcal{T}}_\alpha).$$

Remark 2.31 It is easy to see that $\mathcal{T}, \mathcal{U}$, and u determine the rest of the psuedo-hull embedding. For p is given by $p(E^{\mathcal{T}}_\alpha) = E^{\mathcal{U}}_{u(\alpha)}$, and p determines $\hat{p}$ and v. We then determine the copy maps t^0_α and t^1_α by induction on α. t^1_α is determined by t^0_α by $t^1_\alpha = \hat{\imath}^{\mathcal{U}}_{v(\alpha),u(\alpha)} \circ t^0_\alpha$. If α is a limit, we easily get t^0_α from $v(\alpha)$ and the fact that $t^0_\alpha \circ \hat{\imath}^{\mathcal{T}}_{\beta,\alpha} = \hat{\imath}^{\mathcal{U}}_{v(\beta),v(\alpha)} \circ t^0_\beta$ holds whenever $\beta <_T \alpha$. Clause (e) determines $t^0_{\alpha+1}$ from earlier t^i's.

p determines u, hence p determines the whole of the psuedo-embedding as well.

Remark 2.32 $\mathcal{T}$ is a pseudo-hull of $W(\mathcal{T}, F)$. In our embedding normalization notation, $u = \phi^{\mathcal{T},F}$, and $t^1_\beta = \pi_\beta^{\mathcal{T},F}$, and $p(E_\xi^{\mathcal{T}}) = E_{u(\xi)}^{W(\mathcal{T},F)}$, which determines $\hat{p}$ and v. u agrees with v except at $\beta = \beta^{\mathcal{T},F}$, where we have $v(\beta) = \beta$ and $u(\beta) = \alpha^{\mathcal{T},F} + 1$.

Definition 2.33 *Let Φ be a psuedo-hull embedding from $\mathcal{T}$ into $\mathcal{U}$, and Ψ be a psuedo-hull embedding from $\mathcal{U}$ into $\mathcal{V}$; then $\Psi \circ \Phi$ is the psuedo-hull embedding from $\mathcal{T}$ into $\mathcal{V}$ obtained by composing the corresponding component maps of Φ and Ψ.*

It is easy to check that composing corresponding maps does indeed produce a psuedo-hull embedding.

2.5 Normalizing $\mathcal{T}^\frown\mathcal{U}$

First, note that $W(\mathcal{T}, F)$ makes sense in somewhat greater generality. Let $\mathcal{T}$ be a normal tree on the premouse M. Let $\mathcal{S}$ be another normal tree on M, and F be on the sequence of the last model of $\mathcal{S}$. Let α be least such that F is on the sequence of $M_\alpha^{\mathcal{S}}$, so that $\mathcal{S}\restriction(\alpha+1) = \mathcal{S}^{<\mathrm{lh}(F)}$. Let β be such that $\beta = \mathcal{S}\text{-pred}(\alpha+1)$ would hold in any normal $\mathcal{S}'$ extending $\mathcal{S}\restriction(\alpha+1)$ such that $F = E_\alpha^{\mathcal{S}'}$. That is, $\mathcal{S}\restriction\beta+1 = \mathcal{S}^{<\mathrm{crit}(F)}$. Suppose that

$$\mathcal{T}\restriction\beta + 1 = \mathcal{S}\restriction\beta + 1.$$

Suppose also that if $\beta + 1 < \mathrm{lh}(\mathcal{T})$, then $\mathrm{dom}(F) < \lambda(E_\beta^{\mathcal{T}})$, that is,

$$\mathcal{T}\restriction\beta + 1 = \mathcal{T}^{<\mathrm{crit}(F)}.$$

We define a normal tree $W(\mathcal{T}, \mathcal{S}, F)$.

Remark 2.34 The last supposition holds if either $\alpha = \beta$ and $\mathrm{lh}(F) < \mathrm{lh}(E_\beta^{\mathcal{T}})$, or $\alpha > \beta$, and $\mathrm{lh}(E_\beta^{\mathcal{S}}) \leq \mathrm{lh}(E_\beta^{\mathcal{T}})$. This will be the case when we use $W(\mathcal{T}, \mathcal{S}, F)$ to define $W(\mathcal{T},\mathcal{U})$.

Let $Q \trianglelefteq N = M_\theta^{\mathcal{T}}$, where $\theta + 1 = \mathrm{lh}(\mathcal{T})$, and let

$$\mu = \mathrm{crit}(F).$$

Suppose that $\mathrm{Ult}(Q, F)$ makes sense, that is, $\mathrm{dom}(F) \leq \rho_{k(Q)}(Q)$. Suppose also that Q is the longest initial segment of N to which F applies, that is, either $Q = N$, or $\rho(Q) \leq \mu < \rho_{k(Q)}(Q)$. We want to define $W(\mathcal{T}, \mathcal{S}, F)$ so that $\mathrm{Ult}(Q, F)$ embeds weakly elementarily into the last model of $W(\mathcal{T}, \mathcal{S}, F)$.

There are three cases.

Case 1. $Q \neq N$.

In this case Q is a proper initial segment of $M_\beta^{\mathcal{T}}|\operatorname{lh}(E_\beta^{\mathcal{T}})$, by the argument given in the dropping case of the definition of $W(\mathcal{T}, F)$.

$$W(\mathcal{T}, \mathcal{S}, F) = \mathcal{S}\restriction(\alpha+1)^\frown\langle F\rangle$$

is the unique normal continuation $\mathcal{W}$ of $\mathcal{S}\restriction(\alpha+1)$ of length $\alpha+2$ such that $E_\alpha^{\mathcal{W}} = F$. Note here that $\mathcal{M}_\beta^{\mathcal{T}} = \mathcal{M}_\beta^{\mathcal{S}}$, and Q is what F would be applied to in a normal continuation of $\mathcal{S}\restriction\alpha+1$. (Unlike the case $\mathcal{T} = \mathcal{S}$ we discussed before, it is possible that $Q \neq N$ and $\alpha > \beta$.) In this dropping case, the last model of $W(\mathcal{T}, \mathcal{S}, F)$ is equal to $\operatorname{Ult}(Q, F)$, and doesn't just embed it.

Case 2. $Q = N$, and $\operatorname{lh}(\mathcal{T}) = \beta + 1$.

Again

$$W(\mathcal{T}, \mathcal{S}, F) = \mathcal{S}\restriction(\alpha+1)^\frown\langle F\rangle$$

is the unique normal $\mathcal{S}'$ of length $\alpha + 2$ extending $\mathcal{S}$ such that $E_\alpha^{\mathcal{S}'} = F$. $Q = N = \mathcal{M}_\beta^{\mathcal{T}}$, and so $\operatorname{Ult}(Q, F)$ is equal to the last model of $W(\mathcal{T}, \mathcal{S}, F)$.

Case 3. $\operatorname{lh}\mathcal{T} > \beta + 1$, and $Q = N$.

In this case, we construct $\mathcal{W} = W(\mathcal{T}, \mathcal{S}, F)$ just as before. We set

$$\mathcal{W}\restriction(\alpha+1) = \mathcal{S}\restriction(\alpha+1),$$

and

$$\mathcal{M}_{\alpha+1}^{\mathcal{W}} = \operatorname{Ult}(\mathcal{M}_\beta^{\mathcal{T}}|\langle\gamma, k\rangle, F),$$

where k, γ are appropriate for normality. (Note $M_\beta^{\mathcal{T}} = M_\beta^{\mathcal{S}} = M_\beta^{\mathcal{W}}$.) Let $\phi(\xi) = \xi$ for $\xi < \beta$, and $\phi(\xi) = (\alpha+1) + (\xi - \beta)$ for $\xi \geq \beta$. Let $\pi_\xi = \mathrm{id}$ for $\xi < \beta$, and $\pi_\beta : M_\beta^{\mathcal{T}}|\langle\gamma, k\rangle \to M_{\alpha+1}^{W}$ be the canonical embedding. Note that by our case hypothesis, F applies to $\mathcal{M}_\theta^{\mathcal{T}}$, and hence to $\mathcal{M}_\beta^{\mathcal{T}}|\langle\operatorname{lh}(E_\beta^{\mathcal{T}})$, so $\langle\operatorname{lh}(E_\beta^{\mathcal{T}}), 0\rangle \leq \langle\gamma, k\rangle$. Thus π_β moves $E_\beta^{\mathcal{T}}$. So we can use the Shift lemma to lift the rest of $\mathcal{T}$, defining an elementary

$$\pi_\xi : M_\xi^{\mathcal{T}} \to M_{\phi(\xi)}^{W}$$

for $\xi > \beta$, by induction on ξ. If $\sigma = T\text{-pred}(\xi)$, then $\phi(\sigma) = W\text{-pred}(\phi(\xi))$, unless $\sigma = \beta$ and $\operatorname{crit}(E_{\xi-1}^{\mathcal{T}}) < \mu$. In this case, $\operatorname{crit}(E_{\phi(\xi)-1}^{W}) = \operatorname{crit}(E_{\xi-1}^{\mathcal{T}}) < \mu$, so $W\text{-pred}(\phi(\xi)) = \beta$, rather than $\phi(\beta)$. We write

$$W(\mathcal{T}, \mathcal{S}, F) = \mathcal{S}^{<\operatorname{lh}F\frown}\langle F\rangle^\frown i_F\text{``}\mathcal{T}^{>\operatorname{crit}(F)}$$

in this case.

Remark 2.35 Recall that $\mathcal{T}$ and $\mathcal{S}$ were normal on M. Let Σ be an iteration strategy according to which both $\mathcal{T}$ and $\mathcal{S}$ are played. F and Σ determine $\mathcal{S}\restriction(\alpha+1)$, because F determines $M_\alpha^{\mathcal{S}}|\operatorname{lh} F$, and thus $\mathcal{S}\restriction(\alpha+1)$ as the unique normal tree on M by Σ leading to a model having F on its sequence, and using only extenders of length $\operatorname{lh} F$. $\mathcal{S}\restriction(\alpha+1)$ is all we need of $\mathcal{S}$ to determine $W(\mathcal{T},\mathcal{S},F)$. So we could write $W(\mathcal{T},\Sigma,F)$ for $W(\mathcal{T},\mathcal{S},F)$, or if Σ is understood, write $W(\mathcal{T},F) = W(\mathcal{T},\mathcal{S},F)$.

Notation 2.35.1 Let $\alpha^{\mathcal{T},\mathcal{S},F}$ and $\beta^{\mathcal{T},\mathcal{S},F}$ be the α and β described above. In Case 3, let $\phi^{\mathcal{T},\mathcal{S},F}$ and $\pi_\xi^{\mathcal{T},\mathcal{S},F}$ for $\xi < \operatorname{lh}\mathcal{T}$ be the maps ϕ and π_ξ described there. In Cases 1 and 2, let $\operatorname{dom}(\phi^{\mathcal{T},\mathcal{S},F}) = \beta+1$, with $\phi^{\mathcal{T},\mathcal{S},F}(\xi) = \xi$ if $\xi < \beta$, and $\phi^{\mathcal{T},\mathcal{S},F}(\beta) = \alpha+1$. (Where $\alpha = \alpha^{\mathcal{T},\mathcal{S},F}$ and $\beta = \beta^{\mathcal{T},\mathcal{S},F}$.) Let $\pi_\xi^{\mathcal{T},\mathcal{S},F} = \mathrm{id}$ if $\xi < \beta$, and $\pi_\beta^{\mathcal{T},\mathcal{S},F} : \mathcal{M}_{\alpha+1}^{*,\mathcal{W}} : \mathcal{M}_\beta^{\mathcal{T}}|\xi \to \mathcal{M}_{\alpha+1}^{\mathcal{W}}$ be the canonical embedding in those cases.

In cases 2 and 3, we have a psuedo-hull embedding $\Phi_{\mathcal{T},\mathcal{S},F} = \langle u, \langle t_\xi^0 \mid \xi < \operatorname{lh}\mathcal{T}\rangle, \langle t_\xi^1 \mid \xi+1 < \operatorname{lh}\mathcal{T}\rangle$ from $\mathcal{T}$ into $W(\mathcal{T},\mathcal{S},F)$. It is determined by setting

$$u = \phi_{\mathcal{T},\mathcal{S},F}.$$

Some of its other maps are given by

$$t_\xi^1 = \pi_\xi^{\mathcal{T},\mathcal{S},F}$$

and

$$p(E_\xi^{\mathcal{T}}) = \pi_\xi^{\mathcal{T},\mathcal{S},F}(E_\xi^{\mathcal{T}}).$$

In case 1, these objects determine a *partial psuedo-hull embedding* from $\mathcal{T}\restriction\beta+1$ into $\mathcal{W}(\mathcal{T},\mathcal{S},F)$. This is a system with all the properties of a psuedo-hull embedding, except that its last map t_β^1 may only be defined on some $Q \trianglelefteq \mathcal{M}_\beta^{\mathcal{T}}$.

The ilustrations associated to $W(\mathcal{T},\mathcal{S},F)$ are pretty much the same as before, allowing for the possibility that $\mathcal{S} \neq \mathcal{T}$. In particular, if $\xi \geq \beta^{\mathcal{T},\mathcal{S},F}$, then F either appears directly as one of the extenders used in $[0,\phi(\xi))_W$, or appears indirectly via some extender $F(G)$ used in $[0,\phi(\xi))_W$, where $\operatorname{crit}(G) < \mu < \lambda(G)$ and G is used in $[0,\xi)_{\mathcal{T}}$.

Now let $\mathcal{T}$ be a normal tree on a premouse M, with last model Q, and let $\mathcal{U}$ be a normal tree on Q. We do not assume that $\mathcal{U}$ has a last model. We shall define $W(\mathcal{T},\mathcal{U}) = \mathcal{W}$, the embedding normalization of $\mathcal{T}^\frown\mathcal{U}$. For this, we define

$$\mathcal{W}_\gamma = W(\mathcal{T},\mathcal{U}|(\gamma+1)),$$

the embedding normalization of $\mathcal{T}^\frown\mathcal{U}|(\gamma+1)$, by induction on γ. Let us write

$$Q_\gamma = \mathcal{M}_\gamma^{\mathcal{U}} = \text{last model of } \mathcal{U}|(\gamma+1).$$

We shall maintain that each $\mathcal{W}_\gamma$ has a last model

$$\begin{aligned} R_\gamma &= \text{last model of } \mathcal{W}_\gamma \\ &= \mathcal{M}^{\mathcal{W}_\gamma}_{z(\gamma)}, \end{aligned}$$

and that there is an elementary embedding

$$\sigma_\gamma : Q_\gamma \to R_\gamma.$$

As we go we construct psuedo-hull embeddings $\Phi_{\eta,\gamma}$, for $\eta <_U \gamma$, from an appropriate initial segment of $\mathcal{W}_\eta$ to $\mathcal{W}_\gamma$. $\Phi_{\eta,\gamma}$ is determined by its u-map $\phi_{\eta,\gamma}$ acting on an initial segment of $\mathrm{lh}(\mathcal{W}_\eta)$, and its t^1-maps we call

$$\pi^{\eta,\gamma}_\tau \colon \mathcal{M}^{\mathcal{W}_\eta}_\tau \to \mathcal{M}^{\mathcal{W}_\gamma}_{\phi_{\eta,\gamma}(\tau)},$$

defined when $\tau \in \mathrm{dom}(\phi_{\eta,\gamma})$. (There is the possibility that $\pi^{\eta,\gamma}_\tau$ acts only on some proper initial segment of $\mathcal{M}^{\mathcal{W}_\eta}_\tau$. That happens iff $(\eta, \gamma]_U$ has a drop.) Roughly, the system $(\langle \mathcal{W}_\gamma \mid \gamma < \mathrm{lh}(\mathcal{U})\rangle, \langle \Phi_{\eta,\gamma} \mid \eta <_U \gamma\rangle)$ is an iteration tree of iteration trees, whose base node is $\mathcal{W}_0 = \mathcal{T}$, and whose overall structure is induced by $\mathcal{U}$. The $\Phi_{\eta,\gamma}$ are the branch embeddings of this tree.

We set

$$\mathcal{W}_0 = \mathcal{T},$$

and let σ_0 be the identity. Now suppose everything is given up to γ. We let

$$F_\gamma = \sigma_\gamma(E^{\mathcal{U}}_\gamma)).$$

Let α_γ be the least ξ such that F_γ is on the sequence of $\mathcal{M}^{\mathcal{W}_\gamma}_\xi$. So So F_γ is on the sequence of $\mathcal{M}^{\mathcal{W}_\gamma}_\xi$ for all ξ such that $\alpha_\gamma \le \xi \le z(\gamma)$. We assume the following agreement hypotheses:

$(*)_\gamma$

(i) For $\eta \le \xi \le \gamma$, $\sigma_\eta \restriction (\mathrm{lh}(E^{\mathcal{U}}_\eta + 1) = \sigma_\xi \restriction (E^{\mathcal{U}}_\eta + 1)$.

(ii) For $\eta < \xi < \gamma$, $\alpha_\eta < \alpha_\xi$ and $\mathrm{lh}(F_\eta) < \mathrm{lh}(F_\xi)$.

(iii) For $\eta < \xi \le \gamma$, R_η agrees with R_ξ up to $\mathrm{lh}(F_\eta)$, but $\mathrm{lh}(F_\eta)$ is a cardinal of R_ξ, so they disagree at $\mathrm{lh}(F_\eta)$.

(iv) For $\eta < \xi \le \gamma$, $\mathcal{W}_\eta \restriction (\alpha_\eta + 1) = \mathcal{W}_\xi \restriction (\alpha_\eta + 1)$, and $E^{\mathcal{W}_\xi}_{\alpha_\eta} = F_\eta$.

(v) For $\eta < \gamma$,

(a) for all $\xi < \alpha_\eta$, $\mathrm{lh}(E_\xi^{\mathcal{W}_\eta}) < \mathrm{lh}(F_\eta)$, and

(b) if $\alpha_\eta < z(\eta)$, then $\mathrm{lh}(F_\eta) < \mathrm{lh}(E_{\alpha_\eta}^{\mathcal{W}_\eta})$.

Claim 2.36 *(ii) and (v) of* $(*)_{\gamma+1}$ *hold.*

Proof. For (ii), if $\eta < \gamma$, then $\mathrm{lh}(E_\eta^{\mathcal{U}}) < \mathrm{lh}(E_\gamma^{\mathcal{U}})$, so $\mathrm{lh}(F_\eta) < \mathrm{lh}(F_\gamma)$ by (i) at γ. Moreover, if $\alpha_\gamma \leq \alpha_\eta$, then by (iv), F_γ is on the sequence of $\mathcal{M}_{\alpha_\eta}^{\mathcal{W}_\gamma} = \mathcal{M}_{\alpha_\eta}^{\mathcal{W}_\eta}$. So F_η is also on the $\mathcal{M}_{\alpha_\eta}^{\mathcal{W}_\gamma}$ sequence. Since $\mathrm{lh}(F_\eta) < \mathrm{lh}(F_\gamma)$ and F_γ is on the R_γ sequence, we get that F_η is on the R_γ sequence. This contradicts (iv) at γ.

(v)(a) holds because otherwise F_γ would be on the sequence of some $\mathcal{M}_\xi^{itW_\gamma}$ for $\xi < \alpha_\gamma$. For (v)(b), suppose $\alpha_\gamma < z(\gamma)$. Since F_γ is on the sequences of $\mathcal{M}_{\alpha_\gamma}^{\mathcal{W}_\gamma}$ and of $\mathcal{M}_{\alpha_\gamma+1}^{\mathcal{W}_\gamma}$, we must have $\mathrm{lh}(F_\gamma) < \mathrm{lh}(E_{\alpha_\gamma}^{\mathcal{W}_\gamma})$. □

Now suppose $\eta = U\text{-pred}(\gamma+1)$. We set

$$\mathcal{W}_{\gamma+1} = W(\mathcal{W}_\eta, \mathcal{W}_\gamma, F_\gamma).$$

Let us check that this makes sense. Let us write $F = F_\gamma$ and $\alpha = \alpha_\gamma$. Clearly $\alpha = \alpha^{\mathcal{W}_\eta, \mathcal{W}_\gamma, F}$. Let

$$\bar{\mu} = \mathrm{crit}(E_\gamma^{\mathcal{U}}),$$

and

$$\mu = \sigma_\gamma(\bar{\mu}) = \mathrm{crit}(F).$$

Let

$$\begin{aligned} \beta &= \beta^{\mathcal{W}_\eta, \mathcal{W}_\gamma, F} \\ &= \text{ least } \xi \text{ such that } \mu < \lambda(E_\xi^{\mathcal{W}_\gamma}) \text{ or } \xi = z(\gamma) \end{aligned}$$

be the tree predecessor of $\alpha+1$ in any normal continuation $\mathcal{S}$ of $\mathcal{W}_\gamma \upharpoonright (\alpha+1)$ that uses F. Since η is the least ξ such that $\bar{\mu} < \lambda(E_\xi^{\mathcal{U}})$, we have by (i) of $(*)_\gamma$ that

$$\eta = \text{ the least } \xi \text{ such that } \mu < \lambda(F_\eta).$$

But $\mathcal{W}_\eta \upharpoonright (\alpha_\eta + 1) = \mathcal{W}_\gamma \upharpoonright (\alpha_\eta + 1)$, and $E_{\alpha_\eta}^{\mathcal{W}_\gamma} = F_\eta$ or else $\eta = \gamma$. In either case, $\beta \leq \alpha_\eta$, so

$$\mathcal{W}_\eta \upharpoonright (\beta+1) = \mathcal{W}_\gamma \upharpoonright (\beta+1).$$

Moreover, since $\beta \leq \alpha_\eta$, if $\beta < z(\eta)$ then

$$\mathrm{lh}(E_\beta^{\mathcal{W}_\gamma}) \leq \mathrm{lh}(E_\beta^{\mathcal{W}_\eta}),$$

with equality holding iff $\beta < \alpha_\eta$. These are the conditions we needed to check, so $W(\mathcal{W}_\eta, \mathcal{W}_\gamma, F)$ makes sense.

Let $\Phi_{\eta,\gamma+1}$ be the (possibly partial) psuedo-hull embedding $\Phi_{\mathcal{W}_\eta,\mathcal{W}_\gamma,F}$. Its u-map is

$$\phi_{\eta,\gamma+1} = \phi_{\mathcal{W}_\eta,\mathcal{W}_\gamma,F},$$

and its t^1 maps are

$$\pi_\tau^{\eta,\gamma+1} = \pi_\tau^{\mathcal{W}_\eta,\mathcal{W}_\gamma,F}.$$

For $\delta <_U \eta$,

$$\Phi_{\delta,\gamma+1} = \Phi_{\eta,\gamma+1} \circ \Phi_{\delta,\eta}.$$

This of course means that $\phi_{\delta,\gamma+1} = \phi_{\eta,\gamma+1} \circ \phi_{\delta,\gamma+1}$, and $\pi_\tau^{\delta,\gamma+1} = \pi_{\phi_{\delta,\eta}(\tau)}^{\eta,\gamma+1} \circ \pi_\tau^{\delta,\eta}$. Here the compositions are condidered as defined wherever they make sense.

Note that $\Phi_{\eta,\gamma+1}$ is partial iff $\gamma+1 \in D^{\mathcal{U}}$. If $\gamma+1 \in D^{\mathcal{U}}$, then $\mathrm{dom}(\phi_{\eta,\gamma+1}) = \beta+1$, and $\pi_\beta^{\eta,\gamma+1}$ acts on a proper initial segment of $\mathcal{M}_\beta^{\mathcal{W}_\eta}$.

$\sigma_{\gamma+1}$ is determined as follows. Let

$$Q_{\gamma+1} = \mathrm{Ult}(Q^*, E_\gamma^{\mathcal{U}}),$$

where $Q^* \trianglelefteq Q_\eta$.

Let $R^* = R_\eta$ if $Q^* = Q_\eta$, and $R^* = \sigma_\eta(Q^*)$ otherwise. $\sigma_\eta \restriction Q^*$ is elementary from Q^* to R^*.

Suppose first that we drop in $\mathcal{U}$, i.e. $Q^* \neq Q_\eta$. Then $\rho(Q^*) \leq \bar{\mu}$, and σ_η is a near $k(Q^*)+1$ embedding, so

$$\mu = \sigma_\gamma(\bar{\mu}) = \sigma_\eta(\bar{\mu}) \leq \rho(R^*),$$

while $\rho_{k(R^*)}(R^*) = \sigma_\eta(\rho_{k(Q)}(Q)) > \mu$. so R^* is what we would apply F to in a normal continuation of $\mathcal{W}_\gamma \restriction (\alpha+1)$. Moreover,

$$\mathcal{W}_{\gamma+1} = \mathcal{W}_\gamma^{< \mathrm{lh}\, F} {}^\frown \langle F \rangle {}^\frown \mathrm{Ult}(R^*, F,)$$

because we are in ether case 1 of the definition of $W(\mathcal{W}_\eta, \mathcal{W}_\gamma, F)$. So $R_{\gamma+1} = \mathrm{Ult}(R^*, F)$, and we can take $\sigma_{\gamma+1}$ to be the Shift Lemma map.

Suppose next that $Q^* = Q_\eta$, so that we are in case 2 or case 3, and

$$\mathcal{W}_{\gamma+1} = \mathcal{W}_\gamma^{< \mathrm{lh}\, F} {}^\frown \langle F \rangle {}^\frown i_F``\mathcal{W}_\eta^{> \mathrm{crit}(F)}.$$

For $\tau \leq z(\eta)$, we have an elementary $\pi_\tau^{\eta,\gamma+1} : \mathcal{M}_\tau^{\mathcal{W}_\eta} \to \mathcal{M}_{\phi_{\eta,\gamma+1}(\tau)}^{W_{\gamma+1}}$. Since we are not dropping in $\mathcal{U}$,

$$Q_{\gamma+1}^{\mathcal{U}} = \mathrm{Ult}(Q_\eta^{\mathcal{U}}, E_\gamma^{\mathcal{U}}).$$

and

$$\phi_{\eta,\gamma+1}(z(\eta)) = z(\gamma+1).$$

We have then the diagram

$$\begin{tikzcd}
Q_{\gamma+1} \arrow[r, "\theta"] & \mathrm{Ult}(R_\eta, F) \arrow[r, "\psi"] & R_{\gamma+1} = \mathcal{M}_{z(\gamma+1}^{\mathcal{W}_{\gamma+1}} \\
Q_\eta \arrow[u, "i_{\eta,\gamma+1}^{\mathcal{U}}"] \arrow[r, "\sigma_\eta"'] & R_\eta = \mathcal{M}_{z(\eta)}^{\mathcal{W}_\eta} \arrow[u] \arrow[ur, "\pi_\tau^{\eta,\gamma+1} i"'] &
\end{tikzcd}$$

Here θ is given by the Shift Lemma, and ψ comes from the fact that F is an initial segment of the extender of $\pi_{z(\eta)}^{\eta,\gamma+1}$, as we remarked before. (So $\psi \restriction \mathrm{lh}\, F = \mathrm{id}$.) We then set

$$\sigma_{\gamma+1} = \psi \circ \theta.$$

So when $\gamma + 1 \notin D^{\mathcal{U}}$, we have the diagram

$$\begin{tikzcd}
\mathcal{M}_{\gamma+1}^{\mathcal{U}} \arrow[r, "\sigma_{\gamma+1}"] & R_{\gamma+1} \\
\mathcal{M}_\eta^{\mathcal{U}} \arrow[u, "i_{\eta,\gamma+1}^{\mathcal{U}}"] \arrow[r, "\sigma_\eta"'] & R_\eta \arrow[u, "\pi_{z(\eta)}^{\eta,\gamma+1}"']
\end{tikzcd}$$

When $\gamma + 1 \in D^{\mathcal{U}}$, we have the diagram

$$\begin{tikzcd}
\mathcal{M}_{\gamma+1}^{\mathcal{U}} \arrow[r, "\sigma_{\gamma+1}"] & R_{\gamma+1} \\
\mathcal{M}_{\gamma+1}^{*,\mathcal{U}} \arrow[u, "i_{\gamma+1}^{*,\mathcal{U}}"] \arrow[r, "\sigma_\eta"'] & \sigma_\eta(\mathcal{M}_{\gamma+1}^{*,\mathcal{U}}) \arrow[u, "\pi_\beta^{\eta,\gamma+1}"']
\end{tikzcd}$$

where $\beta = \beta^{W_\eta, \mathcal{W}_\gamma, F}$.

Claim 2.37 $(*)_{\gamma+1}$ *holds.*

Proof. Left to the reader. □

We have completed the definition of $W_{\gamma+1}$.

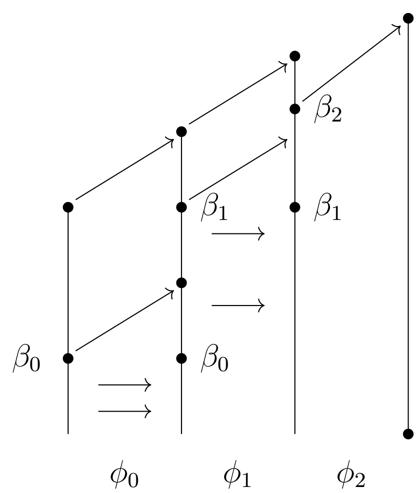
If $\lambda < \mathrm{lh}(\mathcal{U})$ is a limit ordinal, then

$$\mathcal{W}_\lambda = \lim_{\alpha <_U \lambda} \mathcal{W}_\alpha,$$

where we make sense of the direct limit using the psuedo hull embeddings $\Phi_{\eta,\gamma}$ for $\eta <_U \gamma <_U \lambda$. We give a little more detail on this below.

In our context of interest, $\langle \mathcal{T}, \mathcal{U} \rangle$ is played by a background-induced iteration strategy Σ for M, and we shall show that all $\mathcal{W}_\alpha$ are by Σ. So in our context of interest, all models above are wellfounded.

Here are a couple illustrations that the reader may or may not find helpful. Let $\gamma_0 U \gamma_1 U \gamma_2 U \gamma_3$ be successive elements of a branch of U,. Write $\phi_i = \phi_{\gamma_i,\gamma_{i+1}}$. Let $\beta_i = \beta^{W_{\gamma_i}, \mathcal{W}_{\tau_i}, F_i}$, where $\tau_i = \gamma_{i+1} - 1$ and $F_i = \sigma_{\tau_i}(E^{\mathcal{U}}_{\tau_i})$. Thus $\mathcal{W}_{\gamma_{i+1}} = W(\mathcal{W}_{\gamma_i}, \mathcal{W}_{\tau_i}, F_i)$, and $\beta_i = \mathrm{crit}(\phi_i)$. The ϕ_i might look like:

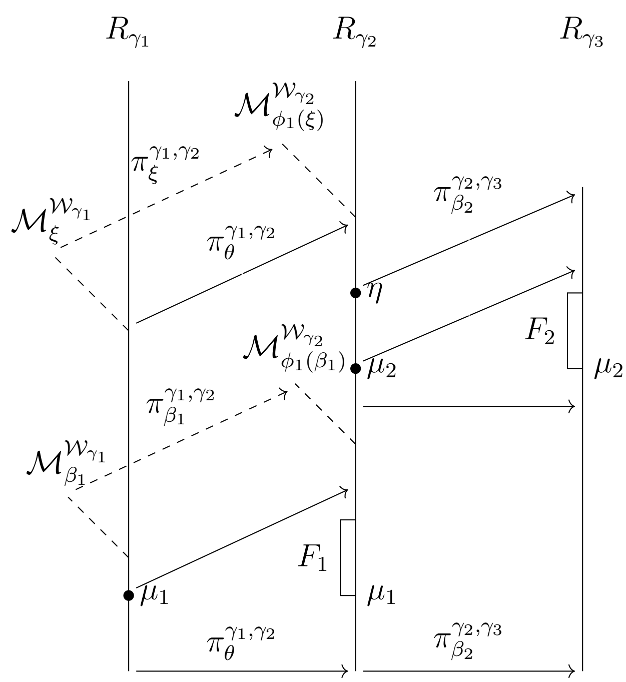


The last step pictured involves a drop. Notice that $\beta_{i+1} \geq \phi_i(\beta_i)$. (equality is possible.) This is because $\mathcal{U}$ is normal. In $\mathcal{W}_{\gamma_{i+1}}$, $\mathcal{M}^{\mathcal{W}_{\gamma_{i+1}}}_{\phi_i(\beta_i)}$ is immediately above $\mathcal{M}^{\mathcal{W}_{\gamma_{i+1}}}_{\beta_i}$ via an F_i-ultrapower. Moreover, $\mathcal{W}_{\gamma_{i+1}} \restriction (\alpha+1) = \mathcal{W}_{\tau_i} \restriction (\alpha+1)$, where $\alpha + 1 = \phi_i(\beta_i)$. By our choice of α, $\lambda(E^{\mathcal{W}_{\tau_i}}_\xi) \leq \lambda(F_i)$ for all $\xi < \alpha$. But $\lambda(F_i) \leq \mathrm{crit}(F_{i+1})$, since $\mathcal{U}$ is normal, so F_{i+1} cannot be applied to any $\mathcal{M}^{\mathcal{W}_{\gamma_{i+1}}}_\xi$ for $\xi < \phi_i(\beta_i)$.

Because $\beta_{i+1} \geq \phi_i(\beta_i)$, and above $\phi_i(\beta_i)$, $\mathrm{ran}(\phi_i)$ is an initial segment of ORD $-$ $\phi(\beta_i)$, we see that along any branch b of $\mathcal{U}$, the direct liimit of the $\phi_{\gamma,\eta}$ for $\gamma, \eta \in b$ is wellfounded.

In fact, the direct limit has order type $\lambda + \theta$, where $\lambda = \sup_{\eta \in b} \operatorname{crit}(\phi_{\eta,b})$, and $\theta = \operatorname{lh} \mathcal{T} - \beta$, where β is least such that $\phi_{0,b}(\beta) \geq \lambda$.

In addition to the ϕ-maps on indices of models, we have the π-maps on the models. Let $\mu_i = \operatorname{crit}(F_i)$, and let $\operatorname{lh}(\mathcal{W}_{\gamma_1}) = \theta + 1$. Let η be the level of R_{γ_2}, or equivalently $\mathcal{M}_{\beta_2}^{\mathcal{W}_{\gamma_2}}$, that we drop to when we apply F_2. The picture is



One can look at $\Phi_{\eta,\gamma}$, for $\eta <_U \gamma$, as a map on the extender trees. Let

$$\psi_{\eta,\gamma} : \operatorname{Ext}(W_\eta) \to \operatorname{Ext}(W_\gamma)$$

by

$$\psi_{\eta,\gamma}(E_\xi^{W_\eta}) = \pi_\xi^{\eta,\gamma}(E_\xi^{W_\eta}) = E_{\phi_{\eta,\gamma}(\xi)}^{W_\gamma}.$$

So $\psi_{\eta,\gamma}(E_\xi^{W_\eta})\downarrow$ iff $\xi \in \operatorname{dom} \phi_{\eta,\gamma}$. Let

$$\hat{\psi}(s) = \text{ least } t \in \mathcal{W}_\gamma^{\text{ext}} \text{ such that } \psi``\operatorname{ran}(s) \subseteq \operatorname{ran}(t).$$

Then

Proposition 2.38 *Let $\eta <_U \gamma$ and $\phi_{\eta,\gamma}(\alpha)\downarrow$, and suppose whenever $\eta \leq_U \xi <_U \gamma$, then $\phi_{\eta,\xi}(\alpha) \geq \mathrm{crit}(\phi_{\xi,\gamma})$. Then for $s = s_\alpha^{\mathcal{W}_\eta}$,*

$$s^{W_\gamma}_{\phi_{\eta,\gamma}(\alpha)} = \hat{psi}_{\eta,\gamma}(s)^\frown\langle F_\tau \mid \tau + 1 \leq_U \gamma \textit{ and for all } i \in \mathrm{dom}\,\psi^0_{\eta,\gamma}(s),$$
$$\lambda(\hat{\psi}_{\eta,\gamma}(s)(i)) \leq \mathrm{crit}(F_\tau)\rangle$$

We omit the simple proof. The proposition says that the branch extender to $\mathcal{M}^{\mathcal{W}_\gamma}_{\phi_{\eta,\gamma}(\alpha)}$ consists of blow-ups by $\psi_{\eta,\gamma}$ of extenders used in the branch to $\mathcal{M}^{\mathcal{W}_\eta}_\alpha$, together with certain F_τ's used in $\mathcal{U}$ from η to γ. It generalizes our pictures on 63 and before.

The map $\hat{\psi}_{\eta,\gamma} : W^{\mathrm{ext}}_\eta \to W^{\mathrm{ext}}_\gamma$ does preserve $\subseteq$.

Proposition 2.39 *Let $s, t \in \mathrm{dom}(\hat{\psi}_{\eta,\gamma})$; then*

(1) *$s \subseteq t \Rightarrow \hat{\psi}_{\eta,\gamma}(s) \subseteq \hat{\psi}_{\eta,\gamma}(t)$, and*

(2) *$s \perp t \Rightarrow \hat{\psi}_{\eta,\gamma}(s) \perp \hat{\psi}_{\eta,\gamma}(t)$.*

Suppose now that $\lambda \leq \mathrm{lh}(\mathcal{U})$ is a limit ordinal, and we have defined $\mathcal{W}_\gamma, \sigma_\gamma$, and the $\Phi_{\eta,\gamma}$ for $\eta, \gamma < \lambda$. We let $W(\mathcal{T}, \mathcal{U}\restriction\lambda)$ be the lim inf of the $\mathcal{W}_\gamma$ for $\gamma < \mathrm{lh}\,\mathcal{U}$. More precisely, let

$$F_\gamma = \sigma_\gamma(E^{\mathcal{U}}_\gamma)$$

and

$$\begin{aligned}\alpha_\gamma = &\text{ least } \alpha \text{ such that } F_\gamma \text{ is on the sequence of } \mathcal{M}^{W_\gamma}_\alpha\\ &= \text{largest } \alpha \text{ such that } W_{\gamma+1}\restriction(\alpha+1) = W_\gamma\restriction(\alpha+1).\end{aligned}$$

We put

$$W(\mathcal{T}, \mathcal{U}\restriction\lambda) = \bigcup_{\gamma < \mathrm{lh}\,\mathcal{U}} W_\gamma\restriction(\alpha_\gamma + 1).$$

Since $\gamma < \eta \Rightarrow \alpha_\gamma < \alpha_\eta$, $W(\mathcal{T}, \mathcal{U})$ has limit length. There are no new σ's or Φ's to be defined at this stage.

Now let $\lambda \leq \mathrm{lh}\,\mathcal{U}$ be a limit, and let b be a cofinal branch of $\mathcal{U}\restriction\lambda$ (not necessarily a wellfounded one). We define the embedding normalization

$$\mathcal{W}_b = W(\mathcal{T}, \mathcal{U}^\frown b)$$

by forming the direct limit of the $\mathcal{W}_\gamma$, for $\gamma \in b$, under the $\Phi_{\eta,\gamma}$ for $\eta <_U \gamma$ in b.

We begin with $\mathrm{lh}(\mathcal{W}_b)$. Let us put

$$\langle \eta, \xi \rangle \in I \quad \text{iff} \quad \eta \in b, \text{ and for all sufficiently large } \gamma \in b,\ \phi^{\eta,\gamma}(\xi)\downarrow.$$

Put

$$\langle \eta, \xi \rangle \leq_I \langle \delta, \theta \rangle \quad \text{iff} \quad \text{for all sufficiently large } \gamma \in b,\ \phi_{\eta,\gamma}(\xi) \leq \phi_{\delta,\gamma}(\theta).$$

It is easy to see that $\leq_I$ is a prewellorder (even if b is illfounded, or drops infinitely often). We set

$$\mathrm{lh}(\mathcal{W}_b) = \mathrm{otp}(I, \leq_I).$$

For $\eta \in b$, we let $\phi_{\eta,b}(\xi)\downarrow$ iff $\langle \eta, \xi \rangle \in I$, and in that case, set

$$\phi_{\eta,b}(\xi) = \text{rank of } \langle \eta, \xi \rangle \text{ in } (I, \leq_I).$$

We define the tree order $\leq_{W_b}$ by: given $\langle \eta, \xi \rangle$ and $\langle \delta, \theta \rangle \in I$

$$\phi_{\eta,b}(\xi) \leq_{W_b} \phi^{\delta,b}(\theta) \quad \text{iff} \quad \text{for all sufficiently large } \gamma \in b,\ \phi_{\eta,\gamma}(\xi) \leq_{W_\gamma} \phi_{\delta,\gamma}(\theta).$$

Although the $\phi_{\eta,\gamma}$ do not completely preserve tree order, they almost do so. See clause (4) on p.54 and the illustration on p.61. Using this, we can show $\leq_{W_b}$ is a tree order. $\phi_{\eta,b}$ may fail to preserve tree order, but again, this can only happen in a way similar to (4) on p.54. We record this in a proposition.

Proposition 2.40 *Let $\langle \eta, \xi \rangle$, $\langle \eta, \delta \rangle \in I$, and suppose $\xi \leq_{W_\eta} \delta$ but $\phi_{\eta,b}(\xi) \not\leq_{W_b} \phi_{\eta,b}(\delta)$. Then there is a unique $\gamma \geq \eta$ in b such that letting $U\text{-pred}(\theta + 1) = \gamma$ with $\theta + 1 \in b$, $F = F_\theta$, and and $\beta = \beta^{\mathcal{W}_\gamma, \mathcal{W}_\theta, F}$, we have*

1. *$\beta = \phi_{\eta,\gamma}(\xi) \leq_{W_\gamma} \phi_{\eta,\gamma}(\delta)$, and*
2. *letting G be the first extender used in $[0, \phi_{\eta,\gamma}(\delta))$ such that $\lambda(G) \geq \lambda(E_\beta^{\mathcal{W}_\gamma})$, we have $\mathrm{crit}(G) < \mathrm{crit}(F) < \lambda(G)$.*

Moreover, in this case, if $\xi = W_\eta\text{-pred}(\delta)$, $\beta = \phi_{\eta,\gamma}(\xi) = \mathcal{W}_\gamma\text{-pred}(\phi_{\eta,\gamma}(\delta))$, and

$$W_{\theta+1}\text{-pred}(\phi_{\eta,\theta+1}(\delta)) = \beta = W_{\theta+1}\text{-pred}(\phi_{\eta,\theta+1}(\xi)).$$

We omit the easy proof. Using such arguments, we can show $\leq_{W_b}$ is a tree order, and

Proposition 2.41 *Let $\langle \eta, \xi \rangle$ and $\langle \delta, \theta \rangle \in I$. Then $\phi_{\eta,b}(\xi) = W_b\text{-pred}(\phi_{\delta,b}(\theta))$ iff for all sufficiently large $\gamma \in b$, $\phi^{\eta,\gamma}(\xi) = W_\gamma\text{-pred}(\phi_{\delta,\gamma}(\theta))$.*

Here is a more concrete description of $\mathrm{lh}(\mathcal{W}_b)$ and $\phi_{\eta,b}$. Let

$$\begin{aligned}\delta &= \mathrm{lh}\, W(\mathcal{T},\mathcal{U}{\restriction}\lambda)\\ &= \sup_{\gamma<\lambda} \alpha_\gamma\\ &= \sup\{\mathrm{crit}\, \phi_{\eta,\gamma} \mid \eta <_U \gamma \wedge \gamma \in b\}.\end{aligned}$$

(The last equality holds because if $\eta = U\text{-pred}(\gamma+1)$ and $\gamma+1 \leq_U \tau$ where $\tau \in b$, then $\mathrm{crit}(\phi_{\eta,\gamma+1}) \leq \alpha_\gamma < \mathrm{crit}(\phi_{\gamma+1,\tau})$.)

Case 1. b drops somewhere.

Let $\gamma+1$ be least in $b \cap D^{\mathcal{U}}$, and $\eta = U\text{-pred}(\gamma+1)$, and $\beta = \beta^{\mathcal{W}_\eta,\mathcal{W}_\gamma,F_\gamma} = \mathrm{crit}(\phi_{\eta,\gamma+1})$. Let $\beta = \phi_{0,\eta}(\tau)$. Then for all $\gamma+1 \leq_U \theta <_U \rho$, with $\rho \in b$,

$$\begin{aligned}\mathrm{crit}(\phi_{\theta,\rho}) &= \phi_{\eta,\theta}(\beta)\\ &= \mathrm{lh}(\mathcal{W}_\theta) - 1.\end{aligned}$$

(Further dropping cuts down on the domains of the π-maps, not on that of the ϕ-maps.) Thus

$$\begin{aligned}\mathrm{lh}(\mathcal{W}_b) &= \delta + 1\\ &= \phi_{\eta,b}(\beta) + 1 = \phi_{0,b}(\tau) + 1.\end{aligned}$$

Case 2. b does not drop.

Let

$$\begin{aligned}\tau = \tau_b = \text{least } \alpha < \mathrm{lh}\,\mathcal{T} \text{ such that for all } \gamma <_U \xi\\ \text{with } \xi \in b,\ \phi_{0,\gamma}(\alpha) \geq \mathrm{crit}(\phi_{\gamma,\xi}).\end{aligned}$$

Then

$$\begin{aligned}\phi_{0,b}(\tau) &= \delta,\\ \mathrm{lh}(\mathcal{W}_b) &= \delta + (\mathrm{lh}\,\mathcal{T} - \tau),\end{aligned}$$

and for $\xi \geq \tau$ with $\xi < \mathrm{lh}(\mathcal{T})$,

$$\phi_{0,b}(\xi) = \delta + (\xi - \tau).$$

This case can happen in two ways: it can be that $\phi_{0,\eta}(\tau) = \mathrm{crit}(\phi_{\eta,\gamma})$ for some $\eta <_U \gamma$ with $\gamma \in b$, in which case that is true for all sufficiently large such η, γ. Or it can happen that $\phi_{0,\eta}(\tau) > \mathrm{crit}(\phi_{\eta,\gamma})$, for all $\eta <_U \gamma$ with $\gamma \in b$. In that case, τ is a limit ordinal, and the extenders in b are being inserted cofinally into the branch extender of $[0, \tau)_T$.

It can happen in Case 2 that τ is a limit ordinal, but some $\phi_{0,\eta}(\tau)$ and its images are in the "eventual critical points" along b. In that case, some tail of the extenders used in b are being inserted after the blow-ups of all those in $[0, \tau)_T$.

Now we define the models and extenders of $\mathcal{W}_b$. Suppose $\alpha = \phi_{\eta,b}(\gamma) < \mathrm{lh}(\mathcal{W}_b)$. Suppose $\eta \leq \xi < \delta \in b$. Then we have the map $\pi^{\xi,\delta}_{\phi_{\eta,\xi}(\gamma)}$ acting on either $\mathcal{M}^{\mathcal{W}_\xi}_{\phi_{\eta,\xi}(\gamma)}$ or an initial segment thereof. We let

$$\mathcal{M}^{\mathcal{W}_\gamma}_\alpha = \mathrm{dir\,lim} \text{ of the } \mathcal{M}^{\mathcal{W}_\xi}_{\phi_{\eta,\xi}(\gamma)} \text{ under the } \pi^{\xi,\delta}_{\phi_{\eta,\xi}(\gamma)}\text{'s.}$$

If b does not drop after η, then we have

$$\pi^{\eta,b}_\gamma : \mathcal{M}^{\mathcal{W}_\eta}_\gamma \to \mathcal{M}^{\mathcal{W}_b}_{\phi_{\eta,b}(\gamma)}$$

as the direct limit map. Otherwise $\pi^{\eta,b}_\gamma$ may (or may not) act on a proper initial segment of $\mathcal{M}^{\mathcal{W}_\eta}_\gamma$.

Finally, if $\alpha = \phi_{\eta,b}(\gamma) < \mathrm{lh}(\mathcal{W}_b)$ and $\alpha + 1 < \mathrm{lh}(\mathcal{W}_\gamma)$, then

$$E^{\mathcal{W}_b}_\alpha = \pi^{\eta,b}_\gamma(E^{\mathcal{W}_\eta}_\gamma).$$

One can check that with this choice of extenders, $\mathcal{W}_b$ is a normal iteration tree on M. For example, suppose that $\eta \in b$ and that for all $\xi \geq \eta$ in b, $W_\xi\text{-pred}(\phi_{\eta,\xi}(\gamma + 1)) = \phi_{\eta,\xi}(\theta)$, and we aren't dropping, so

$$\mathcal{M}^{\mathcal{W}_\xi}_{\phi_{\eta,\xi}(\gamma+1)} = \mathrm{Ult}(\mathcal{M}^{\mathcal{W}_\xi}_{\phi_{\eta,\xi}(\theta)}, E^{\mathcal{W}_\xi}_{\phi_{\eta,\xi}(\gamma)}).$$

Then

$$\mathcal{M}^{\mathcal{W}_b}_{\phi^{\eta,b}(\gamma+1)} = \mathrm{Ult}(\mathcal{M}^{\mathcal{W}_b}_{\phi^{\eta,b}(\theta)}, E^{\mathcal{W}_b}_{\phi^{\eta,b}(\gamma)}).$$

because each of the three objects in this equation is a direct limit of its ξ-approximations, for $\xi \in b$, and the maps commute appropriately. We omit further detail.

Now we also have the natural map

$$\sigma_b : M^{\mathcal{U}}_b \to R_b,$$

where R_b is the last model of W_b, given by

$$\sigma_b(i^{\mathcal{U}}_{\gamma,b}(x)) = \pi^{\gamma,b}_{z(\gamma)}(\sigma_\gamma(x)).$$

In the abstract, it may happen that not all models of $\mathcal{W}_b$ are wellfounded. In our context of interest, $\langle \mathcal{T}, \mathcal{U}^\frown b\rangle$ is played according to an iteration strategy Σ for M, and we show that Σ is sufficiently good that $\mathcal{W}_b$ is also played by Σ.

Now suppose $\lambda < \mathrm{lh}\,\mathcal{U}$ and $b = [0, \lambda)_U$, and all models of $\mathcal{W}_b$ are wellfounded. Then we set

$$\begin{aligned}
\mathcal{W}_\lambda &= \mathcal{W}_b,\\
\phi_{\eta,\lambda} &= \phi_{\eta,b},\\
\pi_\gamma^{\eta,\lambda} &= \pi_\gamma^{\eta,b},\\
\sigma_\lambda &= \sigma_b,
\end{aligned}$$

and continue with the inductive construction of $W(\mathcal{T}, \mathcal{U})$. If some model of W_b is illfounded, we stop the construction, and say that $W(\mathcal{T}, \mathcal{U})$ is undefined.

Finally, if $\mathcal{U}$ has a last model, we set $W(\mathcal{T}, \mathcal{U}) = W_\gamma$, where $\mathrm{lh}\,\mathcal{U} = \gamma + 1$. If $\mathcal{U}$ has limit length λ, then $W(\mathcal{T}, \mathcal{U}) = W(\mathcal{T}, \mathcal{U}\restriction\lambda)$ has already been defined.

To summarize our notation associated to $W(\mathcal{T}, \mathcal{U})$: for $\gamma < \mathrm{lh}\,\mathcal{U}$,

$$F_\gamma = \sigma_\gamma(E_\gamma^{\mathcal{U}})$$

where $\sigma_\gamma : \mathcal{M}_\gamma^{\mathcal{U}} \to R_\gamma =$ last model of $\mathcal{W}_\gamma$, and

$$\mathcal{W}_{\gamma+1} = W(\mathcal{W}_\eta, \mathcal{W}_\gamma, F_\gamma)$$

where $\eta = U\text{-pred}(\gamma + 1)$. By normality, modulo an iteration strategy according to which all $\mathcal{W}_\gamma$ are played, R_γ and $\mathcal{W}_\gamma$ determine each other, while F_γ and $\mathcal{W}_\gamma\restriction(\alpha_\gamma + 1)$ determine each other. The R_γ's are not the models of a single iteration tree, but we do have

Proposition 2.42 *Let $\gamma < \eta < \mathrm{lh}\,\mathcal{U}$. Then*

(*a*) *R_γ agrees with R_η below $\mathrm{lh}\, F_\gamma$, and*

(*b*) *F_γ is on the sequence of R_γ, but not that of R_η. In fact, $\mathrm{lh}(F_\gamma)$ is a cardinal of R_η.*

The following diagram summarizes the situation. We draw the diagram as if the maps in question exist, although sometimes they may not, because of dropping. Let $z(\eta) + 1 = \mathrm{lh}(W_\eta)$, and let $i^{\mathcal{W}_\eta} : M \to R_\eta$ be the canonical embedding (assuming M-to-R_η does not drop).

$$
\begin{tikzcd}
R_0 = \mathcal{M}_0^{\mathcal{U}} \arrow[rr] \arrow[drr] & & \mathcal{M}_\eta^{\mathcal{U}} \arrow[r, "i^{\mathcal{U}}_{\eta,\gamma}"] \arrow[d, "\sigma_\eta"] & \mathcal{M}_\gamma^{\mathcal{U}} \arrow[dd, "\sigma_\gamma"] \\
& & R_\eta \arrow[dr, "\pi^{\eta,\gamma}_{z(\eta)}"] & \\
& \mathcal{M}_\sigma^{\mathcal{W}_\eta} \arrow[dr, "\pi^{\eta,\gamma}_\sigma"] & & R_\gamma \\
& & \mathcal{M}^{\mathcal{W}_\gamma}_{\phi_{\eta,\gamma}(\sigma)} & \\
M \arrow[uuuu, "i^{\mathcal{W}_0}"] \arrow[uur, "i^{\mathcal{W}_\eta}_{0\sigma}"] \arrow[uuurr, "i^{\mathcal{W}_\eta}"] \arrow[urr] \arrow[uurrr, bend right, "i^{\mathcal{W}_\gamma}"'] & & &
\end{tikzcd}
$$

The various embeddings all commute:

(i) $i^{\mathcal{W}_\gamma} = \pi^{\eta,\gamma}_{z(\eta)} \circ i^{\mathcal{W}_\eta}$

(ii) $\pi^{\eta,\gamma}_\sigma \circ i^{\mathcal{W}_\eta}_{\xi,\sigma} = \pi^{\eta,\gamma}_\xi \circ i^{\mathcal{W}_\gamma}_{\phi_{\eta,\gamma}(\xi),\phi_{\eta,\gamma}(\sigma)}$ (general version of (i))

(iii) $\sigma_\gamma \circ i^{\mathcal{U}}_{\eta,\gamma} = \pi^{\eta,\gamma}_{z(\eta)} \circ \sigma_\eta$.

In a sufficiently coarse case, the upper triangle in the diagram above collapses.

Proposition 2.43 *Let $\mathcal{T}$ be normal on M, and $\mathcal{U}$ normal on the last model $\mathcal{T}$. Suppose also that $\mathcal{T}$ and $\mathcal{U}$ are ms-normal Suppose that whenever $\alpha + 1 < \operatorname{lh} \mathcal{T}$,*

$$\mathcal{M}^{\mathcal{T}}_\alpha \models \nu(E^{\mathcal{T}}_\alpha) \text{ is strongly inaccessible.}$$

Let $\mathcal{W}_\eta$, $\sigma_\eta : \mathcal{M}^{\mathcal{U}}_\eta \to R_\eta$, $R_\eta = \mathcal{M}^{\mathcal{W}_\eta}_{z(\eta)}$ etc., be as above. Then

(1) *$R_\eta = \mathcal{M}^{\mathcal{U}}_\eta$, and $\sigma_\eta = \mathrm{id}$, for all $\eta < \operatorname{lh}(\mathcal{U})$;*

(2) *if $\eta <_U \gamma$, then $i^{\mathcal{U}}_{\eta,\gamma} = \pi^{\eta,\gamma}_{z(\eta)}$.*

Proof. Proposition 2.18 generalizes to $W(\mathcal{W}_\eta, \mathcal{W}_\gamma F)$, where F comes from $\mathcal{W}_\gamma$. We use that repeatedly. $\square$

Remark 2.44 There is a tacit hypothesis in 2.43 that all models in W_γ are well-founded. The ms-normality hypothesis is there because if we replace $\nu(E_\alpha^{\mathcal{T}})$ by $\lambda(E_\alpha^{\mathcal{T}})$ above, then the hypothesis implies that $M \models$ " there is a superstrong cardinal".

Remark 2.45 We shall need also to consider $W(\mathcal{T}, \mathcal{U})$ when $\langle \mathcal{T}, \mathcal{U} \rangle$ is a stack on some M that is not a premouse of any kind. In that case we shall assume that $M \models \mathsf{ZFC}$, and M is the background universe for some construction for a fine-structural object. The background extenders used in this construction will constitute a *coarsely coherent* sequence $\vec{F} \in M$. (Cf. p.34 and following.) Each $F \in \vec{F}$ will be "nice", in that

$$M \models \mathrm{lh}(F) = \nu(F) = \mathrm{strength}(E_\alpha^{\mathcal{S}}) \text{ is strongly inaccessible.}$$

We shall only be interested in trees $\mathcal{S}$ on M such that $E_\alpha^{\mathcal{S}} \in i_{0,\alpha}^{\mathcal{S}}(\vec{F})$, for all α. Normality for such $\mathcal{S}$ on M means

1. $\alpha < \beta \Rightarrow \mathrm{lh}\, E_\alpha^{\mathcal{S}} < \mathrm{lh}\, E_\beta^{\mathcal{S}}$, and
2. S-pred$(\gamma + 1) =$ least β such that $\mathrm{crit}(E_\alpha^{\mathcal{T}}) < \mathrm{lh}(E_\beta^{\mathcal{T}})$.

Given $\langle \mathcal{T}, \mathcal{U} \rangle$ a normal stack on M, with all extenders taken from images of $\vec{F}$ as above, we can define $W(\mathcal{T}, \mathcal{U})$ as above. In this coarse case we shall have $\sigma_\gamma =$ id for all γ, and hence $F_\gamma = E_\gamma^{\mathcal{U}}$ for all γ. Having defined $\mathcal{W}_\eta$ for $\eta \leq \gamma$, and with $R_\gamma = M_\gamma^{\mathcal{U}}$, we let

$$\alpha = \text{least } \tau \text{ such that for} \eta = \mathrm{lh}(E_\gamma^{\mathcal{U}}), V_\eta^{\mathcal{M}_\tau^{\mathcal{U}}} = V_\eta^{\mathcal{M}_\gamma^{\mathcal{U}}}.$$

It is easy to see that α is the least τ such that $E_\gamma^{\mathcal{U}} \in i_{0,\tau}^{\mathcal{U}} \circ i^{\mathcal{T}}(\vec{F})$. We define

$$\begin{aligned} \mathcal{W}_{\gamma+1} &= W(\mathcal{W}_\eta, \mathcal{W}_\gamma, E_\gamma^{\mathcal{U}}) \\ &= \mathcal{W}_\gamma \restriction (\alpha + 1)^\frown \langle E_\gamma^{\mathcal{U}} \rangle^\frown i_{E_\beta^{\mathcal{U}}} \text{``} W_\eta^{> \mathrm{crit}(E_\gamma^{\mathcal{U}})}. \end{aligned}$$

The coherence of $\vec{F}$ implies that if $\sigma < \alpha$, then $\mathrm{lh}(E_\sigma^{\mathcal{W}_\gamma}) < \mathrm{lh}(E_\gamma^{\mathcal{U}})$, so that $\mathcal{W}_\gamma \restriction (\alpha + 1)^\frown \langle E_\gamma^{\mathcal{U}} \rangle$ is normal, so $\mathcal{W}_{\gamma+1}$ is normal.

This completes our definition of embedding normalization. Since we do not need full normalization in this paper, we shall not discuss it further here.

2.6 Normalization commutes with copying

We prove that normalization commutes with copying. The proof is completely straightforward, but takes a while to put on paper, because of the many embeddings involved. We shall use this fact to show that the pullback of a strategy that normalizes well also normalizes well. The proof also serves as an introduction to our proof that normalization commutes with lifting to a background universe. That in turn is used in the proof that if a strategy for the background universe normalizes well, then so do the strategies on premice that it induces. (See 3.26.)

Theorem 2.46 *Let $\langle\mathcal{T},\mathcal{U}\rangle$ be a stack on a premouse M, and let $\psi\colon M \to N$ be elementary. Let $\langle\mathcal{T}^*,\mathcal{U}^*\rangle = \psi\langle\mathcal{T},\mathcal{U}\rangle$ be the stack on N obtained by copying. Suppose that $W(\mathcal{T}^*,\mathcal{U}^*)$ exists; then*

(1) $W(\mathcal{T},\mathcal{U})$ exists, and $\psi W(\mathcal{T},\mathcal{U}) = W(\mathcal{T}^*,\mathcal{U}^*)$, and

(2) let $\mathcal{U}$ and $\mathcal{U}^*$ have last models Q and Q^* respectively, and $W(\mathcal{T},\mathcal{U})$ and $W(\mathcal{T}^*,\mathcal{U}^*)$ have last model R and R^* respectively, and let

(i) $\rho\colon Q \to Q^*$ be the map from copying $\langle\mathcal{T},\mathcal{U}\rangle$ to $\langle\mathcal{T}^*,\mathcal{U}^*\rangle$,

(ii) $\sigma\colon Q \to R$ be the normalization map associated to $W(\mathcal{T},\mathcal{U})$,

(iii) $\theta\colon R \to R^*$ be the map from copying $W(\mathcal{T},\mathcal{U})$ to $W(\mathcal{T}^*,\mathcal{U}^*)$, and

(iv) $\sigma^*\colon Q^* \to R^*$ be the normalization map associated to $W(\mathcal{T}^*,\mathcal{U}^*)$;

then $\theta\circ\sigma = \sigma^*\circ\rho$.

Proof.

The embedding normalization $W(\mathcal{T},\mathcal{U})$ has associated to it normal trees $\mathcal{W}_\gamma$ on M, for $\gamma < \mathrm{lh}\,\mathcal{U}$. We also have partial maps $\phi_{\eta,\gamma} : \mathrm{lh}\,\mathcal{W}_\eta \to \mathrm{lh}\,\mathcal{W}_\gamma$ for $\eta <_{\mathcal{U}} \gamma$, and for $\tau \in \mathrm{dom}\,\phi_{\eta,\gamma}$, a map $\pi^{\eta,\gamma}_\tau : \mathcal{M}^{\mathcal{W}_\eta}_\tau \to \mathcal{M}^{\mathcal{W}_\gamma}_{\phi_{\eta,\gamma}(\tau)}$. We have $R_\gamma =$ last model of $\mathcal{W}_\gamma$, $\sigma_\gamma : \mathcal{M}^{\mathcal{U}}_\gamma \to R_\gamma$, and $F_\gamma = \sigma_\gamma(E^{\mathcal{U}}_\gamma)$. $W(\mathcal{W}_\eta, F_\gamma) = \mathcal{W}_{\gamma+1}$, when $\eta = U\text{-pred}(\gamma+1)$.

Similarly, $W(\mathcal{T}^*,\mathcal{U}^*)$ has associated trees $\mathcal{W}^*_\gamma$ on N for $\gamma < \mathrm{lh}\,\mathcal{U}^* = \mathrm{lh}\,\mathcal{U}$, together with partial maps $\phi^*_{\eta,\gamma} : \mathrm{lh}\,\mathcal{W}^*_\eta \to \mathrm{lh}\,\mathcal{W}^*_\gamma$ for $\eta <_{U^*} \gamma$ (equivalently, $\eta <_U \gamma$), and for $\tau \in \mathrm{dom}\,\phi^*_{\eta,\gamma}$, a map $\overset{*}{\pi}{}^{\eta,\gamma}_\gamma$. We have $R^*_\gamma =$ last model of $\mathcal{W}^*_\gamma$, $\sigma^*_\gamma : \mathcal{M}^{\mathcal{U}^*}_\gamma \to R^*_\gamma$, and $F^*_\gamma = \sigma^*_\gamma(E^{\mathcal{U}^*}_\gamma)$. We have that $\mathcal{W}^*_{\gamma+1} = W(\mathcal{W}^*_\eta, E^{\mathcal{U}^*}_\gamma)$ when $\eta = U^*\text{-pred}(\gamma+1)$ (equivalently, $\eta = U\text{-pred}(\gamma+1)$).

We shall prove that for all γ,

$$\psi\mathcal{W}_\gamma = \mathcal{W}^*_\gamma.$$

The proof is by induction on γ, with a subinduction on initial segments of $\mathcal{W}_\gamma$. Given that we know this holds for $\mathcal{W}_\gamma \restriction \eta$, we have copy maps

$$\psi^\gamma_\tau \colon \mathcal{M}^{\mathcal{W}_\gamma}_\tau \to \mathcal{M}^{\mathcal{W}^*_\gamma}_\tau$$

defined for all $\tau < \eta$. $\psi^\gamma_0 = \psi$ for all γ.

For $\gamma < \mathrm{lh}\,\mathcal{U}$, let

$$\psi^{\mathcal{U}}_\gamma \colon \mathcal{M}^{\mathcal{U}}_\gamma \to \mathcal{M}^{\mathcal{U}^*}_\gamma$$

be the copy map. So $\psi^{\mathcal{U}}_0$ is the copy map given by the fact that $\mathcal{T}^* = \psi\mathcal{T}$, and the remaining $\psi^{\mathcal{U}}_\gamma$ come from the fact that $\mathcal{U}^* = (\psi^{\mathcal{U}}_0)\mathcal{U}$.

We write $z(\nu)$ for $\mathrm{lh}\,\mathcal{W}_\nu - 1$ and $z^*(\nu)$ for $\mathrm{lh}\,\mathcal{W}^*_\nu - 1$. We may use ∞ for $z(\nu)$ or $z^*(\nu)$ when context permits. So $R_\nu = \mathcal{M}^{\mathcal{W}_\nu}_{z(\nu)} = \mathcal{M}^{\mathcal{W}_\nu}_\infty$. If $(\nu, \gamma]_U$ does not drop, then $\phi_{\nu,\gamma}(z(\nu)) = z(\gamma)$, and $\pi^{\nu,\gamma}_{z(\nu)} = \pi^{\nu,\gamma}_\infty \colon R_\nu \to R_\gamma$.

Lemma 2.47 *Let $\gamma < \mathrm{lh}\,\mathcal{U}$. Then*

(1) $\mathcal{W}^*_\gamma = \psi\mathcal{W}_\gamma$.

(2) *Whenever $\nu <_U \gamma$ and $(\nu, \gamma]_U$ does not drop in model or degree, then for all $\tau < \mathrm{lh}\,\mathcal{W}_\nu$, $\psi^\gamma_{\phi_{\nu,\gamma}(\tau)} \circ \pi^{\nu,\gamma}_\tau = \overset{*}{\pi}{}^{\nu,\gamma}_\tau \circ \psi^\nu_\tau$.*

(3) $\phi_{\eta,\nu} = \phi^*_{\eta,\nu}$, *if $\eta, \nu \leq \gamma$ and $\eta \leq_U \nu$.*

(4) $\psi^\gamma_{z(\gamma)} \circ \sigma_\gamma = \sigma^*_\gamma \circ \psi^{\mathcal{U}}_\gamma$.

Here is a diagram of (2):

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{W}_\gamma}_{\phi_{\nu,\gamma}(\tau)} \arrow[r, "\psi^\gamma_{\phi_{\nu,\gamma}(\tau)}"] & \mathcal{M}^{\mathcal{W}^*_\gamma}_{\phi^*_{\nu,\gamma}(\tau)} \\
\mathcal{M}^{\mathcal{W}_\nu}_\tau \arrow[u, "\pi^{\nu,\gamma}_\tau"] \arrow[r, "\psi^\nu_\tau"] & \mathcal{M}^{\mathcal{W}^*_\nu}_\tau \arrow[u, "\overset{*}{\pi}{}^{\nu,\gamma}_\tau"']
\end{tikzcd}$$

There is a diagram related to (4) and the case $\tau = z(\nu)$ of (2) near the end of the proof.

Proof. We prove 2.47 by induction. Suppose that it is true at all $\nu \leq \gamma$. We show it at $\gamma + 1$. Let $\nu = U\text{-pred}(\gamma + 1)$, and

$$F = F_\gamma = \sigma_\gamma(E^{\mathcal{U}}_\gamma),$$

and

$$\begin{aligned}\alpha &= \alpha_\gamma^{\mathcal{T},\mathcal{U}}\\ &= \alpha(\mathcal{W}_\nu, \mathcal{W}_\gamma, F)\\ &= \text{least } \tau \text{ such that } F \text{ is on the } M_\tau^{\mathcal{W}_\gamma}\text{-sequence.}\end{aligned}$$

So

$$\begin{aligned}\mathcal{W}_{\gamma+1} &= W(\mathcal{W}_\nu, \mathcal{W}_\gamma, F)\\ &= \mathcal{W}_\gamma{\restriction}(\alpha+1)^\frown\langle F\rangle^\frown i_F\text{``}\mathcal{W}_\nu^{>\mathrm{crit}(F)}.\end{aligned}$$

Let also

$$F^* = F_\gamma^* = \sigma_\gamma^*(E_\gamma^{\mathcal{U}^*}).$$

Since $\mathcal{U}^*$ is a copy of $\mathcal{U}$, $\nu = U^*\text{-pred}(\gamma+1)$, so

$$\mathcal{W}_{\gamma+1}^* = W(\mathcal{W}_\nu^*, \mathcal{W}_\gamma^*, F^*).$$

Claim 2.48 *(1)* $\psi_{z(\gamma)}^\gamma(F) = F^*$,

(2) $\alpha = \alpha(\mathcal{W}_\gamma^*, F^*)$, *and*

(3) $\beta(\mathcal{W}_\nu, \mathcal{W}_\gamma, F) = \beta(\mathcal{W}_\nu^*, \mathcal{W}_\gamma^*, F^*)$.

Proof. For (1), we have

$$\begin{aligned}\psi_{z(\gamma)}^\gamma(F) &= \psi_{z(\gamma)}^\gamma \circ \sigma_\gamma(E_\gamma^{\mathcal{U}})\\ &= \sigma_\gamma^* \circ \psi_\gamma^{\mathcal{U}}(E_\gamma^{\mathcal{U}})\\ &= \sigma_\gamma^*(E_\gamma^{\mathcal{U}^*})\\ &= F^*.\end{aligned}$$

For (2), it is enough to show that $\mathrm{lh}(F) < \mathrm{lh}(E_\tau^{\mathcal{W}_\gamma})$ if and only if $\mathrm{lh}(F^*) < \mathrm{lh}(E_\tau^{\mathcal{W}_\gamma^*})$. But if $\mathrm{lh}(F) < \mathrm{lh}(E_\tau^{\mathcal{W}_\gamma})$, then applying the copy maps ψ^γ, we have

$$\begin{aligned}\mathrm{lh}(F^*) = \mathrm{lh}(\psi_{z(\gamma)}^\gamma(F)) &= \mathrm{lh}(\psi_\tau^\gamma(F))\\ &< \mathrm{lh}(\psi_\tau^\gamma(E_\tau^{\mathcal{W}_\gamma}))\\ &= \mathrm{lh}(E_\tau^{\mathcal{W}_\gamma^*}).\end{aligned}$$

The first line holds because $\psi^\gamma_{z(\gamma)}$ agrees with ψ^γ_τ on $\mathrm{lh}(E_\tau^{\mathcal{W}_\gamma})$. Conversely, if if $\mathrm{lh}(F) > \mathrm{lh}(E_\tau^{\mathcal{W}_\gamma})$, then if $\mathrm{lh}(F^*) > \mathrm{lh}(E_\tau^{\mathcal{W}^*_\gamma})$ by the same calculation.

For (3), we must show that $\mathrm{crit}(F) < \lambda(E_\tau^{\mathcal{W}_\gamma})$ if and only if $\mathrm{crit}(F^*) < \lambda(E_\tau^{\mathcal{W}^*_\gamma})$. But this follows from the agreement of the copy maps ψ^γ in exactly the same way. $\square$

The claim easily implies that $\phi_{\nu,\gamma+1} = \phi^*_{\nu,\gamma+1}$, which then gives us (3) of 2.47 at $\gamma+1$.

We now define the copy maps $\psi^{\gamma+1}_\tau \colon \mathcal{M}_\tau^{\mathcal{W}_{\gamma+1}} \to \mathcal{M}_\tau^{\mathcal{W}^*_{\gamma+1}}$ that witness $\mathcal{W}^*_{\gamma+1} = \psi\mathcal{W}_{\gamma+1}$. As we do so, we show that (2) of 2.47 holds, that is, the ψ^ν and $\psi^{\gamma+1}$ maps commute with the embedding normalization maps of models of $\mathcal{W}_\nu$ into models of $\mathcal{W}_{\gamma+1}$ and models of $\mathcal{W}^*_\nu$ into models of $\mathcal{W}^*_{\gamma+1}$.

We have $\mathcal{W}_{\gamma+1}\restriction(\alpha+1) = \mathcal{W}_\gamma\restriction(\alpha+1)$ and $\mathcal{W}^*_{\gamma+1}\restriction(\alpha+1) = \mathcal{W}^*_\gamma\restriction(\alpha+1)$, so can set

$$\psi^{\gamma+1}_\tau = \psi^\gamma_\tau, \text{ for all } \tau \leq \alpha.$$

Now $F = E_\alpha^{\mathcal{W}_{\gamma+1}}$ and $F^* = E_\alpha^{\mathcal{W}^*_{\gamma+1}}$, moreover $\psi^\gamma_\alpha(F) = \psi^\gamma_{z(\gamma)}(F) = F^*$ because $\mathrm{lh}(F) < \mathrm{lh}(E_\alpha^{\mathcal{W}_\gamma})$ if $\alpha < z(\gamma)$. Letting $P = \mathcal{M}_\beta^{\mathcal{W}_\nu}|\langle \eta, k\rangle$ be such that

$$\mathcal{M}_{\alpha+1}^{\mathcal{W}_{\gamma+1}} = \mathrm{Ult}(P, F),$$

we have

$$\mathcal{M}_{\alpha+1}^{\mathcal{W}^*_{\gamma+1}} = \mathrm{Ult}(P^*, F^*),$$

where $P^* = \mathcal{M}_\beta^{\mathcal{W}^*_\nu}|\langle \psi^\nu_\beta(\eta), k\rangle$. (Here we make the usual convention if $\eta = o(\mathcal{M}_\beta^{\mathcal{W}_\nu})$.) This is because $\mathcal{W}_\nu\restriction(\beta+1) = \mathcal{W}_\gamma\restriction(\beta+1)$, and similarly at the (*) level, by the properties of embedding normalization. So $\psi^\nu_\beta = \psi^\gamma_\beta$, and thus agrees with $\psi^\gamma_{z(\gamma)}$ to $\lambda(E_\beta^{\mathcal{W}_\gamma})$, hence past $\mathrm{crit}(F)$. So we can let

$$\psi^{\gamma+1}_{\alpha+1}([a,f]^P_F) = [\psi^{\gamma+1}_\alpha(a), \psi^{\gamma+1}_\beta(f)]^{P^*}_{F^*},$$

by the Shift lemma, and we have $\psi W_{\gamma+1}\restriction(\alpha+2) = W^*_{\gamma+1}\restriction(\alpha+2)$. Note that $\alpha+1 = \phi_{\nu,\gamma+1}(\beta)$, so $\psi^{\gamma+1}_{\phi_{\nu,\gamma+1}(\beta)} \circ \pi^{\nu,\gamma+1}_\beta = {}^*\!\pi^{\nu,\gamma+1}_\beta \circ \psi^\nu_\beta$ by the Shift lemma, and this gives us the new instance of (2) of 2.47.

The general successor case above $\alpha+1$ is similar. Suppose we have $\psi\mathcal{W}_{\gamma+1}\restriction(\eta+1) = \mathcal{W}^*_{\gamma+1}\restriction(\eta+1)$ as witnessed by $\psi^{\gamma+1}_\tau$ for $\tau \leq \eta$. Suppose $\eta > \alpha$. Let

$$\eta = \phi_{\nu,\gamma+1}(\xi) = \phi^*_{\nu,\gamma+1}(\xi),$$

$$G = E_\eta^{\mathcal{W}_{\gamma+1}},$$

and

$$G^* = E_\eta^{\mathcal{W}^*_{\gamma+1}}.$$

Then

$$\begin{aligned}
\psi_\eta^{\gamma+1}(G) &= \psi^{\gamma+1}_{\phi_{\nu,\gamma+1}(\xi)}(\pi_\xi^{\nu,\gamma+1}(E_\xi^{\mathcal{W}_\nu})) \\
&= \overset{*}{\pi}{}_\xi^{\nu,\gamma+1}(\psi_\xi^\nu(E_\xi^{\mathcal{W}_\nu})) \\
&= \overset{*}{\pi}{}_\xi^{\nu,\gamma+1}(E_\xi^{\mathcal{W}^*_{\gamma+1}}) \\
&= E_\eta^{\mathcal{W}^*_{\gamma+1}} = G^*.
\end{aligned}$$

The Shift lemma now gives us $\psi^{\gamma+1}_{\eta+1}$ as above, and we have $\psi \mathcal{W}_{\gamma+1}\restriction(\eta+2) = \mathcal{W}^*_{\gamma+1}\restriction(\eta+2)$.

We leave the limit case of the subinduction to the reader. This finishes the subinduction proving (1), (2), and (3) of 2.47 at step $\gamma+1$. For (4), let us set $\tau = \gamma+1$. To simplify things, let us assume that $(\nu, \gamma+1]_U$ is not a drop. Consider the diagram

$$\begin{tikzcd}
 & R_\tau \arrow[rr, "\psi^\nu_\infty"] & & R^*_\tau \\
\mathcal{M}^{\mathcal{U}}_\tau \arrow[ur, "\sigma_\tau"] \arrow[rr] & & \mathcal{M}^{\mathcal{U}^*}_\tau \arrow[ur, "\sigma^*_\tau"'] & \\
 & R_\nu \arrow[uu, "\pi^{\nu,\tau}_\infty"'] \arrow[rr, "\psi^\nu_\infty"] & & R^*_\nu \arrow[uu, "\overset{*}{\pi}{}^{\nu,\tau}_\infty"'] \\
\mathcal{M}^{\mathcal{U}}_\nu \arrow[uu] \arrow[ur, "\sigma_\nu"] \arrow[rr, "\psi^{\mathcal{U}}_\nu"'] & & \mathcal{M}^{\mathcal{U}^*}_\nu \arrow[uu] \arrow[ur, "\sigma^*_\nu"'] &
\end{tikzcd}$$

We are asked to show that $\sigma^*_\tau \circ \psi^{\mathcal{U}}_\tau = \psi^\tau_\infty \circ \sigma_\tau$, in other words, that the square on the top face of the cube commutes. The square on the bottom commutes by our induction hypothesis. The square in front commutes because $\mathcal{U}^*$ is a copy of $\mathcal{U}$. That the square in back commutes is clause (2) of our lemma at $\gamma+1$, which we just proved. The squares on the left and right faces commute by the properties of embedding normalization.

It is clear from these facts that the top square commutes on $\mathrm{ran}(i^{\mathcal{U}}_{\nu,\tau})$. Since $\mathcal{M}^{\mathcal{U}}_{\tau}$ is generated by $\mathrm{ran}(i^{\mathcal{U}}_{\nu,\tau}) \cup \lambda(E^{\mathcal{U}}_{\tau})$, it is enough to see that the top square commutes on $\lambda(E^{\mathcal{U}}_{\tau})$.

Let $a \in [\lambda(E^{\mathcal{U}}_{\tau})]^{<\omega}$. So $\sigma_\tau(a) \in [\lambda(F)]^{<\omega}$, and $\sigma_\tau(a) = \pi^{\nu,\tau}_{\alpha}(a)$ by the agreement properties of embedding normalization maps. Thus

$$\begin{aligned}\psi^{\tau}_{\infty}(\sigma_\tau(a)) &= \psi^{\tau}_{\infty}(\pi^{\nu,\tau}_{\alpha}(a)) \\ &= \psi^{\tau}_{\alpha}(\pi^{\nu,\tau}_{\alpha}(a)),\end{aligned}$$

using the agreement properties of the ψ^τ maps. On the other hand, $\psi^{\mathcal{U}}_{\tau}(a) \in [\lambda(E^{\mathcal{U}^*}_{\tau}]^{<\omega}$, so

$$\sigma^*_\tau(\psi^{\mathcal{U}}_{\tau}(a)) = \overset{*}{\pi}{}^{\nu,\tau}_{\alpha}(\psi^{\mathcal{U}}_{\tau}(a))$$

by the agreement in normalization maps on the $\mathcal{W}^*$ side. But

$$\psi^{\tau}_{\alpha}(\pi^{\nu,\tau}_{\alpha}(a)) = \overset{*}{\pi}{}^{\nu,\tau}_{\alpha}(\psi^{\mathcal{U}}_{\tau}(a))$$

by clause (2) of 2.47 at τ. Thus $\psi^{\tau}_{\infty} \circ \sigma_\tau(a) = \sigma^*_\tau \circ \psi^{\mathcal{U}}_{\tau}(a)$, as desired.

This finishes the step from γ to $\gamma+1$ in the inductive proof of 2.47. We leave the limit step to the reader. $\square$

It is easy to see that Theorem 2.46 follows from Lemma 2.47.

$\square$

2.7 The branches of $W(\mathcal{T},\mathcal{U})$

Let $\mathcal{T}$ be normal on M, and $\mathcal{U}$ be normal on the last model of $\mathcal{T}$. Let us adopt the notation of the last section, so that we have $\mathcal{W}_\gamma$, F_γ, α_γ, β_γ, $\phi_{\eta,\gamma}$, $\pi^{\eta,\gamma}_{\tau}$, and so on. Suppose $\mathrm{lh}\,\mathcal{U}$ is a limit ordinal θ, and let

$$\lambda = \mathrm{lh}\, W(\mathcal{T},\mathcal{U}) = \sup_{\gamma<\theta} \alpha^{\mathcal{T},\mathcal{U}}_{\gamma}.$$

Here we assume $W(\mathcal{T},\mathcal{U})$ exists, i.e. embedding normalization has so far produced only wellfounded models. Let b be a cofinal branch of $\mathcal{U}$. We do not assume $\mathcal{M}^{\mathcal{U}}_{b}$ is wellfounded. Note that $\mathcal{W}_b$ still makes sense, as defined above.

Proposition 2.49 *$\lambda = \phi_{0,b}(\tau)$, where τ is least such that whenever $\eta, \gamma \in b$ and $\eta <_U \gamma$, then* $\mathrm{crit}\,\phi_{\eta,\gamma} \le \phi_{0,\eta}(\tau)$.

Proof. Let $\eta+1\in b$, and $\sigma\in U\text{-pred}(\eta+1)$. Then $\phi_{\sigma,\eta+1}(\mathrm{crit}(\phi_{\sigma,\eta+1}))=\alpha_\eta+1$, so $\alpha_\eta+1\le\mathrm{crit}(\phi_{\eta+1,\xi})$ for all $\xi\in b$. It follows that $\phi_{0,b}(\tau)\ge\lambda$. But if $\sigma<\tau$, we can find $\gamma+1\in b$ with $\eta=U\text{-pred}(\gamma+1)$ such that $\phi_{0,\eta}(\sigma)<\mathrm{crit}(\phi_{\eta,\gamma+1})$. Then $\phi_{0,\eta}(\sigma)=\phi_{0,b}(\sigma)<\alpha_\gamma<\lambda$. Finally, $\lambda\in\mathrm{ran}\,\phi_{0,b}$ (because any $\xi<\mathrm{lh}(\mathcal{W}_\gamma)$ not in $\mathrm{ran}\,\phi_{0,\gamma}$ is fixed by $\phi_{\gamma,b}$), so $\lambda=\phi_{0,b}(\tau)$. $\square$

Proposition 2.50 *Let $a=[0,\lambda)_{W_b}$. Then*

$$\xi\in a \quad\text{iff}\quad \exists\eta\in b\,(\xi\le\mathrm{crit}(\phi_{\eta,b})\wedge\xi\le_{W_\eta}\phi_{0,\eta}(\tau)).$$

We omit the easy proof.

Remark 2.51 We don't get a "continuously" from b. If τ is fixed in advance, then continuously in those b such that $\tau=\tau_b$, we can produce the corresponding a's.

Definition 2.52 *In the situation above, we write*

$$a=\mathrm{br}(b,\mathcal{T},\mathcal{U})$$

and

$$\tau=m(b,\mathcal{T},\mathcal{U})$$

for the branch of $W(\mathcal{T},\mathcal{U})$ and model of $\mathcal{T}$ determined by b.

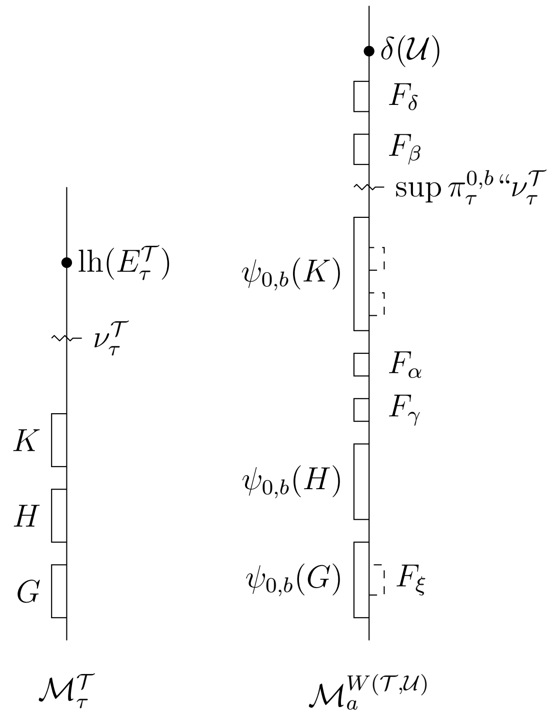
Remark 2.53 Let E_b be the extender of $i_b^{\mathcal{U}}$. It is an extender over the model $\mathcal{M}_\xi^{\mathcal{T}}$, where $\xi+1=\mathrm{lh}\,\mathcal{T}$. One can show that τ is the least α such that either E_b is an extender over $\mathcal{M}_\alpha^{\mathcal{T}}|\,\mathrm{lh}\,E_\alpha^{\mathcal{T}}$, or $\alpha=\xi$.

The branch extender of a is given by

Proposition 2.54 *Let $a=\mathrm{br}(b,\mathcal{T},\mathcal{U})$ and $\tau=m(b,\mathcal{T},\mathcal{U})$ be as above. Then*

$$s_a^{W(\mathcal{T},\mathcal{U})}=\hat{\psi}_{0,b}(s_\tau^{\mathcal{T}})^\frown\langle F_\sigma\mid\sigma+1\in b\wedge\forall i\in\mathrm{dom}(\hat{\psi}_{0,b}(s_\tau^{\mathcal{T}})\;\lambda(\hat{\psi}_{0,b}(s_\tau^{\mathcal{T}}(i)))\le\mathrm{crit}(F_\sigma)\rangle.$$

Here we are writing $s_a^{W(\mathcal{T},\mathcal{U})}$ for $s_\lambda^{\mathcal{W}_b}$, because $s_a^{W(\mathcal{T},\mathcal{U})}$ really only depends on a and $W(\mathcal{T},\mathcal{U})$. We omit the proof of 2.54. For what it's worth, here is a picture



Note $\delta(\mathcal{U}) = \delta(W(\mathcal{T},\mathcal{U}))$. The F's in the picture were all used in b. Some got put directly into $s_a^{W(\mathcal{T},\mathcal{U})}$, others indirectly via some $\psi_{0,b}(G)$.

Branches of $W(\mathcal{T},\mathcal{U})$ of the form $\mathrm{br}(b,\mathcal{T},\mathcal{U})$ come from cofinal branches of $\mathcal{U}$ and *models* of $\mathcal{T}$. There may also be cofinal branches of $W(\mathcal{T},\mathcal{U})$ coming from cofinal branches of $\mathcal{U}$ and *maximal* (perhaps not cofinal) *branches* of $\mathcal{T}$. So we extend our definitions.

Definition 2.55 *Let $\mathcal{W} = W(\mathcal{T},\mathcal{U})$, where $\mathcal{T}$ is normal on M and $\mathcal{U}$ is normal on the last model of $\mathcal{T}$. For $\xi < \mathrm{lh}\,\mathcal{T}$,*

(*a*) *for $\gamma + 1 < \mathrm{lh}\,\mathcal{U}$, letting $\eta = U\text{-pred}(\gamma+1)$, we set*

$$\mathrm{nd}_{\mathcal{W}}(\xi,\gamma+1) = \begin{cases} \phi_{0,\eta}(\xi), & \text{if } \phi_{0,\eta}(\xi)\!\downarrow \text{ and } \phi_{0,\eta}(\xi) \leq_{W_\eta} \mathrm{crit}(\phi_{\eta,\gamma+1}); \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

(*b*) *For any $\gamma < \mathrm{lh}\,\mathcal{U}$,*

$$\tau \in \mathrm{br}_{\mathcal{W}}(\xi,\gamma) \quad \text{iff} \quad \tau = \mathrm{nd}(\xi_0,\gamma_0+1), \quad \text{for some } \xi_0 \leq_T \xi \text{ and } \gamma_0 + 1 \leq_U \gamma.$$

We shall drop the subscript and write $\mathrm{nd}(\xi,\gamma)$ and $\mathrm{br}(\xi,\gamma)$ when context permits. Notice that if $\tau = \mathrm{nd}(\xi,\gamma+1)$, then whenever $\gamma+1 \leq_U \delta$, then $\phi_{0,\delta}(\xi)\downarrow$, and $\tau \leq_{W_\delta} \phi_{0,\delta}(\xi)$. This is true even if $\tau = \mathrm{crit}(\phi_{\eta,\gamma+1})$ holds in the definition of nd_W, because $\mathrm{crit}(\phi_{\eta,\gamma+1}) \leq_{W_\delta} \phi_{\eta,\delta}(\mathrm{crit}(\phi_{\eta,\gamma+1}))$. This gives

Proposition 2.56 *1. Let $\xi_0 \leq_T \xi_1$ and $\gamma_0+1 \leq_U \gamma_1+1$. Then*

$$\mathrm{nd}(\xi_0,\gamma_0+1) \leq_{W(\mathcal{T},\mathcal{U})} \mathrm{nd}(\xi,\gamma_1+1)$$

if both are defined,

2. $\mathrm{br}(\xi,\gamma)$ *is a branch of* $W(\mathcal{T},\mathcal{U})$ *(not cofinal),*

3. $\xi_0 \leq_T \xi_1$ *and* $\gamma_0 \leq_U \gamma_1 \Rightarrow \mathrm{br}(\xi_0,\gamma_0)$ *is an initial segment of* $\mathrm{br}(\xi_1,\gamma_1)$.

Proof. Routine. □

Definition 2.57 *Let c be a branch of $\mathcal{T}$ and b be a branch of $\mathcal{U}$. Then*

1. $\mathrm{br}_{\mathcal{W}}(c,b) = \bigcup_{\xi\in c,\gamma\in b} \mathrm{br}_{\mathcal{W}}(\xi,\gamma)$,

2. *c is b-*minimal *iff for any $\xi \in c$, $\mathrm{br}_{\mathcal{W}}(c\cap\xi,b) \neq \mathrm{br}_{\mathcal{W}}(c,b)$.*

Again we omit the subscript $\mathcal{W}$ when possible.

Remark 2.58 1. If b is cofinal in $\mathrm{lh}(\mathcal{U})$, then $\mathrm{br}(c,b)$ is the $\leq_{W(\mathcal{T},\mathcal{U})}$-downward closure of $\phi_{0,b}$“$c \cap \mathrm{lh}(W(\mathcal{T},\mathcal{U}))$.

2. Equivalent are: (1) c is b-minimal, (2) for cofinally many $\xi \in c$, $\exists\gamma+1 \in b$ such that $\mathrm{nd}(\xi,\gamma+1)\downarrow$, (3) for all $\xi \in c$, $\exists\gamma+1 \in b$, $\mathrm{nd}(\xi,\gamma+1)\downarrow$.

We do not assume in Definition 2.57 that b and c are maximal branches. So for example $\mathrm{br}([0,\xi]_T,[0,\gamma]_U) = \mathrm{br}(\xi,\gamma)$.

We shall show that if a is a cofinal branch of $W(\mathcal{T},\mathcal{U})$, then $a = \mathrm{br}(c,b)$ for some cofinal branch b of $\mathcal{U}$ and some c; moreover, there is a unique such b, and a unique such b-minimal c. For this, we must assume that all $\mathcal{W}_\gamma$ are played according to a common iteration strategy. The following is the key lemma.

Lemma 2.59 *Let $\mathcal{T}$, $\mathcal{U}$ be as above, and suppose there is an iteration strategy Σ for M such that all $\mathcal{W}_\gamma$, $\gamma < \mathrm{lh}\,\mathcal{U}$, are according to Σ. Let γ and δ be $\leq_U$-incomparable, and let η be largest such that $\eta <_U \gamma$ and $\eta <_U \delta$. Let $\alpha = \phi_{\eta,\gamma}(\bar\alpha)$ and $\varepsilon = \phi_{\eta,\delta}(\bar\varepsilon)$, where $\bar\alpha \geq \mathrm{crit}(\phi_{\eta,\gamma})$ and $\bar\varepsilon \geq \mathrm{crit}(\phi_{\eta,\delta})$; then $s_\alpha^{\mathcal{W}_\gamma}$ is incompatible with $s_\varepsilon^{\mathcal{W}_\delta}$.*

Proof. Let $u = s^{\mathcal{W}_\gamma}_\alpha$, $\bar{u} = s^{\mathcal{W}_\eta}_{\bar{\alpha}}$, $v = s^{\mathcal{W}_\delta}_\varepsilon$ and $\bar{v} = s^{\mathcal{W}_\eta}_{\bar{\varepsilon}}$. Assume toward contradiction that either $u \subseteq v$, or $v \subseteq u$.

Let

$$\gamma_0 + 1 = \text{least } \xi \in (\eta, \gamma]_U,$$
$$\delta_0 + 1 = \text{least } \xi \in (\eta, \delta]_U,$$

so that $E^{\mathcal{U}}_{\gamma_0}$ and $E^{\mathcal{U}}_{\delta_0}$ are the extenders used in $\mathcal{U}$ along the two branches, and F_{γ_0} and F_{δ_0} stretch $\mathcal{W}_\eta$ into $\mathcal{W}_{\gamma_0+1}$ and $\mathcal{W}_{\delta_0+1}$. Let

$$k(\bar{u}) = \begin{cases} \text{least } i \text{ such that } \mathrm{crit}(F_{\gamma_0}) < \lambda(\bar{u}(i)), & \text{if this exists;} \\ \mathrm{dom}(\bar{u}), & \text{otherwise,} \end{cases}$$

and

$$k(\bar{v}) = \begin{cases} \text{least } i \text{ such that } \mathrm{crit}(F_{\delta_0}) < \lambda(\bar{v}(i)), & \text{if this exists;} \\ \mathrm{dom}(\bar{v}), & \text{otherwise.} \end{cases}$$

Claim 2.60 *$k(\bar{u}) = k(\bar{v})$, and for $k = k(\bar{u})$, $\bar{u}\restriction k = \bar{v}\restriction k = u\restriction k = v\restriction k$.*

Proof. Let $k = k(\bar{u})$. If $k < k(\bar{v})$, then $v(k) = \bar{v}(k)$, so $\lambda(v(k)) \leq \mathrm{crit}(F_{\delta_0})$. But $\lambda(u(k)) \geq \lambda(F_{\gamma_0})$. [$s^{W_{\gamma_0+1}}_{\phi_{\eta,\gamma_0+1}(\bar{\alpha})}(k) = H$ is defined because $\bar{\alpha} \geq \mathrm{crit}(\phi_{\eta,\gamma_0+1})$, and $\lambda(H) \geq \nu(F_{\gamma_0})$ in all cases of its definition. $u(k) = \psi_{\gamma_0+1,\gamma}(H)$, so $\lambda(u(k)) \geq \lambda(H)$.] Since $u(k) = v(k)$, we have $\lambda(F_{\gamma_0}) \leq \mathrm{crit}(F_{\delta_0})$, so F_{γ_0} and F_{δ_0} do not overlap, contradiction. $k(\bar{v}) < k(\bar{u})$ leads to a parallel contradiction. So we have $k(\bar{u}) = k(\bar{v}) = k$.

For $i < k$, $u(i) = \bar{u}(i)$ and $v(i) = \bar{v}(i)$. So $\bar{u}\restriction k = \bar{v}\restriction k = u\restriction k = v\restriction k$. $\square$

Fix $k = k(\bar{u})$. We may assume by symmetry that $\gamma_0 < \delta_0$.

Claim 2.61 *$k \in \mathrm{dom}(\bar{u})$, and moreover, $\mathrm{crit}(\bar{u}(k)) < \mathrm{crit}(F_{\gamma_0})$.*

Proof. If either statement fails, then

$$s^{\mathcal{W}_{\gamma_0+1}}_{\phi_{\eta,\gamma_0+1}(\bar{\alpha})}(k) = F_{\gamma_0}.$$

Since the $E^{\mathcal{U}}_\tau$ used in $(\gamma_0+1, \gamma]_U$ have crit $\geq \lambda(E^{\mathcal{U}}_{\gamma_0})$, we get

$$\psi_{\gamma_0+1,\gamma}(F_{\gamma_0}) = F_{\gamma_0}.$$

(In fact, $\phi_{\gamma_0+1,\gamma}\restriction(\gamma_0+1) = \text{identity}$, and $\pi^{\gamma_0+1,\gamma}_{\gamma_0} = \text{identity}$.) So

$$u(k) = F_{\gamma_0}.$$

But $k = k(\bar{v})$, and from this we get

$$\lambda(F_{\delta_0}) \leq \lambda(v(k))$$

as in Claim 2.60. Since $\lambda(F_{\gamma_0}) < \lambda(F_{\delta_0})$, we have a contradiction. $\square$

Let $G = \bar{u}(k)$ and $H = u(k)$. By Claim 2.61, along the branch from η to γ, G is being stretched above its critical point into H, by the copy maps corresponding to the F_τ for $\tau + 1 \leq_U \gamma$ and $\eta \leq \tau$. Let $\gamma_1 \leq \gamma$ be least such that the stretching is over with at γ_1. That is, setting

$$\begin{aligned} G &= E_\xi^{\mathcal{W}_\eta} \\ \gamma_1 &= \text{least } \tau \leq \gamma \text{ such that } \mathrm{crit}(\phi_{\tau,\gamma}) > \phi_{\eta,\tau}(\xi) \\ &= \text{least } \tau \leq \gamma \text{ such that } \pi_\xi^{\eta,\tau}(G) = H. \end{aligned}$$

If $\eta <_U \tau + 1 \leq_U \gamma_1$, so that F_τ was used in producing $\mathcal{W}_{\gamma_1}$ from $\mathcal{W}_\eta$, then F_τ is an initial segment of all the extenders of copy maps $\pi_\rho^{\mu,\tau+1}$, where $\mu = U\text{-pred}(\tau + 1)$, and $\rho \geq \mathrm{crit}(\phi_{\mu,\tau+1})$. From this we get

Claim 2.62 *For $\eta <_U \tau + 1 \leq_U \gamma_1$, $\lambda(F_\tau) < \lambda(H)$.*

Proof. Just given. $\square$

Claim 2.63 *$H \neq F_{\delta_0}$.*

Proof. Suppose $H = F_{\delta_0}$. We claim that $\gamma_1 \leq \delta_0$. If γ_1 is a limit ordinal, then $\gamma_1 = \sup\{\tau + 1 \mid \eta <_U \tau + 1 <_U \gamma_1\}$, so by Claim 2.62, $\lambda(F_{\delta_0}) > \lambda(F_\tau)$ for cofinally many τ in γ_1, which implies $\delta_0 \geq \gamma_1$. If γ_1 is not a limit ordinal, we have $\gamma_1 = \tau + 1$ where F_τ is used, so that $\lambda(F_\tau) < \lambda(H) = \lambda(F_{\delta_0})$. Thus $\tau < \delta_0$, so $\gamma_1 = \tau + 1 \leq \delta_0$.

On the other hand, H is used in W_{γ_1} on the way to R_{γ_1}. Thus R_{γ_1} and R_{δ_0} agree below $\mathrm{lh}(H)$, while $H = F_{\delta_0}$ is on the R_{δ_0}-sequence, but not on the R_{γ_1}-sequence. This implies $\delta_0 < \gamma_1$, a contradiction. $\square$

By Claim 2.63, $k \in \mathrm{dom}(\bar{v})$, and letting $L = \bar{v}(k)$, $\mathrm{crit}(L) < \mathrm{crit}(F_{\delta_0})$. So L is being stretched above its critical point into H along the branch from η to δ. Let $\delta_1 \leq \delta$ be least such that the stretching is over with at δ_1; that is, setting

$$\begin{aligned} L &= E_\mu^{\mathcal{W}_\eta} \\ \delta_1 &= \text{least } \tau \leq_U \delta \text{ such that } \mathrm{crit}(\phi_{\tau,\delta}) > \phi_{\eta,\tau}(\mu) \\ &= \text{least } \tau \leq_U \delta \text{ such that } \pi_\mu^{\eta,\tau}(L) = H. \end{aligned}$$

Since $\gamma_1 \neq \delta_1$, we have $\lambda_{\gamma_1}^{\mathcal{U}} \neq \lambda_{\delta_1}^{\mathcal{U}}$. Assume $\lambda_{\gamma_1}^{\mathcal{U}} < \lambda_{\delta_1}^{\mathcal{U}}$. (It no longer matters whether $\gamma_0 < \delta_0$, so this is not a loss of generality.) That is, we have a $\tau + 1 \leq_U \delta_1$ such that for all $\sigma + 1 \leq_U \gamma_1$, $\lambda(E_\sigma^{\mathcal{U}}) < \lambda(E_\tau^{\mathcal{U}})$. This yields:

(∗) $\tau \leq \delta_1$, and whenever $\sigma + 1 \leq_U \gamma_1$, then $\lambda(F_\sigma) < \lambda(F_\tau)$.

Thus $\tau > \sigma$, whenever $\sigma + 1 \leq_U \gamma_1$. So $\tau \geq \gamma_1$. We have that H is used in both $\mathcal{W}_{\gamma_1}$ and $\mathcal{W}_{\delta_1}$, so R_{γ_1} agrees with R_{δ_1} below $\mathrm{lh}(H)$, which is a cardinal in both models. But F_τ is used in $\mathcal{W}_\delta$, before H, so $\mathrm{lh}(F_\tau)$ is a cardinal in both R_{γ_1} and R_{δ_1}.

But then R_{γ_1} and R_τ agree up to $\mathrm{lh}(F_\tau)$, since $R_\tau \parallel \mathrm{lh}(F_\tau) = R_{\delta_1} \parallel \mathrm{lh}(F_\tau)$. F_τ is on the R_τ-sequence, and not the R_{γ_1}-sequence, so $\tau < \gamma_1$. Contradiction. □

Corollary 2.64 *Let $\tau = \mathrm{nd}(\xi, \gamma_0 + 1)$ and $\sigma = \mathrm{nd}(\rho, \delta_0 + 1)$, where $\gamma_0 + 1$ and $\delta_0 + 1$ are $\leq_U$-minimal. (I.e. $\gamma_0' + 1 <_U \gamma_0 + 1 \Rightarrow \tau \neq \mathrm{nd}(\xi, \gamma_0' + 1)$, and similarly for $\delta_0 + 1$, σ, and ρ.) Suppose that for $\eta_0 = U\text{-pred}(\gamma_0 + 1)$ and $\eta_1 = U\text{-pred}(\delta_0 + 1)$, we have that η_0 and η_1 are $\leq_U$-incomparable. Then τ and σ are $\leq_{W(\mathcal{T},\mathcal{U})}$-incomparable.*

Proof. Let η be largest such that $\eta <_U \eta_0$ and $\eta <_U \eta_1$. By the minimality of γ_0 and γ_1,

$$\mathrm{crit}(\phi_{\eta,\gamma_0+1}) \leq \phi_{0,\eta}(\xi)$$

and

$$\mathrm{crit}(\phi_{\eta,\gamma_1+1}) \leq \phi_{0,\eta}(\rho).$$

By Lemma 2.59, $s_{\tau_0}^{\mathcal{W}_{\gamma_0+1}} \perp s_{\tau_1}^{\mathcal{W}_{\gamma_1+1}}$. But $s_{\tau_0}^{\mathcal{W}_{\gamma_0+1}} = s_{\tau_1}^{W(\mathcal{T},\mathcal{U})}$ and $s_{\tau_1}^{\mathcal{W}_{\gamma_1+1}} = s_{\tau_1}^{W(\mathcal{T},\mathcal{U})}$, so we are done. □

Corollary 2.65 *Let a be a cofinal branch of $W(\mathcal{T},\mathcal{U})$, and suppose $a = \mathrm{br}(c_0, b_0) = \mathrm{br}(c_1, b_1)$. Then $b_0 = b_1$, and b_0 is cofinal in $\mathcal{U}$. Moreover, if c_0 and c_1 are b_0-minimal, then $c_0 = c_1$.*

Proof. We show first that b_0 is cofinal. Let $\mu < \mathrm{lh}\,\mathcal{U}$, and let $\tau \in a$ with $\tau > \alpha_\mu$, and

$$\tau = \mathrm{nd}(\xi, \gamma + 1),$$

for $\xi \in c_0$ and $\gamma + 1 \in b_0$. Let $\eta = U\text{-pred}(\gamma + 1)$. Then

$$\tau = \phi_{0,\eta}(\xi) \leq \mathrm{crit}(\phi_{\eta,\gamma+1}) < \alpha_\gamma + 1,$$

so $\alpha_\mu < \alpha_\gamma + 1$, so $\mu \leq \gamma$. Hence b_0 is cofinal. Similarly for b_1.

Remark 2.66 The proof showed that if $\mathrm{nd}(\xi, \gamma + 1)\downarrow$ and $\mathrm{nd}(\xi, \gamma + 1) > \alpha_\mu$, then $\gamma \geq \mu$.

Suppose toward contradiction that $b_0 \neq b_1$. Let $\eta_0 \in b_0$ and $\eta_1 \in b_1$ be $\leq_U$-incomparable. $\tau_0, \tau_1 \in a$ with $\tau_0 > \alpha_{\eta_0}$ and $\tau_1 > \alpha_{\eta_1}$ and $\tau_0 = \mathrm{nd}(\xi, \gamma_0 + 1)$, $\tau_1 = \mathrm{nd}(\rho, \gamma_1 + 1)$ for some $\gamma_0 + 1 \in b_0$ and $\gamma_1 + 1 \in b_1$. Then $\eta_0 \leq_U \gamma_0 + 1$ and $\eta_1 \leq_U \gamma_1 + 1$ by the remark above. By Corollary 2.64, τ_0 is $\leq_T$-incomparable with τ_1. Since $\tau_0, \tau_1 \in a$, this is a contradiction.

Finally, suppose c_0 and c_1 are b_0-minimal. We claim $c_0 = c_1$. For that it suffices to show

Claim 2.66.1 *Suppose* $\mathrm{nd}(\xi, \gamma+1)$ *and* $\mathrm{nd}(\rho, \delta+1)$ *are defined and* $\leq_{W(\mathcal{T},\mathcal{U})}$*-comparable. Suppose* $\gamma + 1$ *and* $\delta + 1$ *are* $\leq_U$*-comparable. Then* ξ *and* ρ *are* $\leq_T$*-comparable.*

Proof. Although the ϕ-maps do not fully preserve tree order, we do have

(*i*) $\phi_{\eta,\gamma}(\xi) \leq \phi_{\eta,\gamma}(\rho) \Rightarrow \xi \leq_{W_\eta} \rho$

(*ii*) ξ, ρ are $\leq_{W_\eta}$-incomparable and $\phi_{\eta,\gamma}(\xi)\downarrow$ and $\phi_{\eta,\gamma}(\rho)\downarrow$ implies $\phi_{\eta,\gamma}(\xi)$ and $\phi_{\eta,\gamma}(\rho)$ are $\leq_{W_\gamma}$-incomparable.

Now let $\xi, \gamma + 1, \rho, \delta + 1$ be as in our hypotheses, and suppose ξ and ρ are $\leq_T$-incomparable. By (*ii*), we cannot have $\gamma + 1 = \delta + 1$. Suppose without loss of generality $\gamma + 1 <_U \delta + 1$. Let

$$\eta = U\text{-pred}(\gamma + 1)$$

and

$$\mu = U\text{-pred}(\delta + 1).$$

Then $\phi_{0,\eta}(\xi)$ is $\leq_{W_\eta}$-incomparable with $\phi_{0,\eta}(\rho)$. Since $\phi_{0,\eta}(\xi) \leq \mathrm{crit}(\phi_{\eta,\gamma+1})$ we see that $\phi_{0,\eta}(\eta)$ is incomparable in $\mathcal{W}_{\gamma+1}$ with $\phi_{0,\gamma+1}(\rho)$. (If $\phi_{0,\eta}(\xi) < \mathrm{crit}(\phi_{\eta,\gamma+1})$, this follows from (*ii*). If $\phi_{0,\eta}(\xi) = \mathrm{crit}(\phi_{\eta,\gamma+1})$, it follows from the definition of $\mathcal{W}_{\gamma+1}$.) Since $\phi_{0,\eta}(\xi) < \mathrm{crit}(\phi_{\gamma+1,\mu})$, $\phi_{0,\eta}(\xi)$ is $\mathcal{W}_\mu$-incomparable with $\phi_{0,\mu}(\rho)$, contradiction. □

□

Finally, we show (assuming still that all $\mathcal{W}_\gamma$, $\gamma < \mathrm{lh}\,\mathcal{U}$, are by a common Σ.)

Lemma 2.67 *For any cofinal branch* a *of* $W(\mathcal{T},\mathcal{U})$*, there is a cofinal branch* b *of* $\mathcal{U}$ *and a branch* c *of* $\mathcal{T}$ *such that* $\mathrm{br}_{\mathcal{W}}(c, b) = a$*.*

Proof. We begin by decoding notes of $\mathcal{U}$ from nodes of $W(\mathcal{T},\mathcal{U})$. For $\xi < \mathrm{lh}(W(\mathcal{T},\mathcal{U}))$, set

$$d(\xi) = \text{least } \gamma \text{ such that } \xi \leq \alpha_\gamma.$$

Claim 2.67.1

$$\begin{aligned} d(\xi) &= \textit{least } \gamma \textit{ such that } s_\xi^{\mathcal{W}_\gamma} = s_\xi^{W(\mathcal{T},\mathcal{U})} \\ &= \textit{least } \gamma \textit{ such that } \mathcal{M}_\xi^{\mathcal{W}_\gamma} = \mathcal{M}_\xi^{W(\mathcal{T},\mathcal{U})}. \end{aligned}$$

Proof. The two characterization are clearly equivalent. So it is enough to show that $\xi \leq \alpha_\gamma \Leftrightarrow \mathcal{M}_\xi^{\mathcal{W}_\gamma} = \mathcal{M}_\xi^{W(\mathcal{T},\mathcal{U})}$. The $\Rightarrow$ direction is trivial. But if $\mathcal{M}_\xi^{\mathcal{W}_\gamma} = \mathcal{M}_\xi^{W(\mathcal{T},\mathcal{U})}$, then $\mathcal{W}_\gamma{\restriction}(\xi{+}1) = W(\mathcal{T},\mathcal{U}){\restriction}(\xi{+}1)$ by normality. Since $\mathcal{W}_\gamma{\restriction}(\alpha_\gamma{+}2) = W(\mathcal{T},\mathcal{U}){\restriction}(\alpha_\gamma{+}2)$ (because F_γ was used in the latter, and not the former), $\xi \leq \alpha_\gamma$. □

Claim 2.67.2 $\xi_0 \leq_{W(\mathcal{T},\mathcal{U})} \xi_1 \Rightarrow d(\xi_0) \leq_U d(\xi_1)$.

Proof. Let $\gamma_0 = d(\xi_0)$ and $\gamma_1 = d(\xi_1)$. We claim that $\xi_0 \in \operatorname{ran} \phi_{0,\gamma_0}$. For let τ be least such that $\phi_{0,\gamma_0}(\tau) \geq \xi_0$. If $\phi_{0,\gamma_0}(\tau)\xi_0$, then there must be $0 \leq_U \eta <_U \sigma + 1 \leq_U \gamma_0$ such that

$$\operatorname{crit}(\phi_{\eta,\sigma+1}) \leq \xi < \phi_{\eta,\sigma+1}(\operatorname{crit}(\phi_{\eta,\sigma+1}))$$

and $\eta = U\text{-pred}(\sigma + 1)$. (All discontinuities in ϕ_{0,γ_0} arise this way.) But then $\xi < \alpha_\sigma + 1$, so $\xi \leq \alpha_\sigma$, and $\sigma < \gamma_0$, contradiction.

Similarly, $\xi_1 \in \operatorname{ran} \phi_{0,\gamma_1}$.

We claim that γ_0 and γ_1 are comparable in $\mathcal{U}$. Suppose not, and let η be largest such that $\eta <_U \gamma_0$ and $\eta <_U \gamma_1$. Let

$$\xi_0 = \phi_{\eta,\gamma_0}(\bar{\xi}_0)$$

and

$$\xi_1 = \phi_{\eta,\gamma_1}(\bar{\xi}_1).$$

The hypotheses of 2.59 are satisfied, noting that $\bar{\xi}_0 \geq \operatorname{crit}(\phi_{\eta,\gamma_0})$ because otherwise $s_{\xi_0}^{\mathcal{W}_{\gamma_0}} = s_{\xi_0}^{\mathcal{W}_\eta}$, whilst γ_0 was least such that $s_{\xi_0}^{\mathcal{W}_{\gamma_0}}$ appears as a branch extender. Similarly, $\bar{\xi}_1 \geq \operatorname{crit}(\phi_{\eta,\gamma_1})$. The other hypotheses of 2.59 hold, so we conclude $s_{\xi_0}^{\mathcal{W}_{\gamma_0}}$ is compatible with $s_{\xi_1}^{\mathcal{W}_{\gamma_1}}$. This implies ξ_0 and ξ_1 are comparable in $W(\mathcal{T},\mathcal{U})$. Finally, $\xi_0 \leq_{W(\mathcal{T},\mathcal{U})} \xi_1 \Rightarrow \xi_0 \leq \xi_1$, and trivially $\xi_0 \leq \xi_1 \Rightarrow d(\xi_0) \leq d(\xi_1)$. Since $d(\xi_0)$ and $d(\xi_1)$ are $\leq_U$-comparable, $d(\xi_0) \leq_U d(\xi_1)$, as desired. □

Claim 2.67.3 $d : \operatorname{lh}(W(\mathcal{T},\mathcal{U})) \to \operatorname{lh}\mathcal{U}$ *is an order-homomorphism, and* $\operatorname{ran}(d)$ *is cofinal in* $\operatorname{lh}(\mathcal{U})$.

Proof. As we remarked, $\xi_0 \leq \xi_1 \Rightarrow d(\xi_0) \leq d(\xi_1)$ is trivial. Pick any $\gamma < \mathrm{lh}\,\mathcal{U}$, and $\xi < \mathrm{lh}\,W(\mathcal{T},\mathcal{U})$ with $\xi > \alpha_\gamma$. (The α_γ's are strictly increasing.) Then $d(\xi) > \gamma$. □

It follows that for any branch a of $W(\mathcal{T},\mathcal{U})$, we can set

$$d(a) = \{\gamma \mid \exists \xi \in a\,(\gamma \leq_U d(\xi))\},$$

and $d(a)$ is a branch of $\mathcal{U}$. If a is cofinal in $W(\mathcal{T},\mathcal{U})$, then $d(a)$ is cofinal in $\mathcal{U}$.

Next we decode nodes of $\mathcal{T}$. For any $\xi < \mathrm{lh}(W(\mathcal{T},\mathcal{U}))$, set

$$e(\xi) = \text{unique } \alpha < \mathrm{lh}\,\mathcal{T} \text{ such that } \phi_{0,d(\xi)}(\alpha) = \xi.$$

We showed in the proof of Claim 2.67.2 that $\xi \in \mathrm{ran}(\phi_{0,d(\xi)})$.

Claim 2.67.4 $\xi_0 \leq_{W(\mathcal{T},\mathcal{U})} \xi_1 \Rightarrow e(\xi_0) \leq_T e(\xi_1)$.

Proof. Let $\gamma_i = d(\xi_i)$ and $\bar{\xi}_i = e(\xi_i)$. As we noted above, the σ maps do not introduce new tree-order relationships in $\mathrm{ran}\,\phi$.

Subclaim 2.67.1 *If $\phi_{\eta,\gamma}(\mu) \leq_{W_\gamma} \phi_{\eta,\gamma}(\nu)$, then $\mu \leq_{W_\eta} \nu$.*

Proof. Easy induction on γ. □

So if $\bar{\xi}_0 \not\leq_T \bar{\xi}_1$, then $\phi_{0,\gamma_0}(\bar{\xi}_0) \not\leq_{W_{\gamma_0}} \phi_{0,\gamma_0}(\bar{\xi}_1)$. That is, $\xi_0 \not\leq_{W_{\gamma_0}} \phi_{0,\gamma_0}(\bar{\xi}_1)$. If $\mathrm{crit}(\phi_{\gamma_0,\gamma_1}) > \xi_0$, then we get $\xi_0 \not\leq_{W_{\gamma_1}} \xi_1$, and since $\xi_1 \leq \alpha_{\gamma_1}$, $\xi_0 \not\leq_{W(\mathcal{T},\mathcal{U})} \xi_1$, as desired. So assume $\xi_0 \geq \mathrm{crit}(\phi_{\gamma_0,\gamma_1})$.

If $\xi_0 = \mathrm{crit}(\phi_{\gamma_0,\gamma_1})$, then $\xi_0 \leq_{W_{\gamma_1}} \phi_{\gamma_0,\gamma_1}(\sigma)$ iff $\xi_0 \leq_{W_{\gamma_0}} \sigma$ for all σ. Since $\xi_0 \not\leq_{W_{\gamma_0}} \phi_{0,\gamma_0}(\bar{\xi}_1)$, this yields $\xi_0 \not\leq_{W_{\gamma_1}} \xi_1$, so $\xi_0 \not\leq_{W(\mathcal{T},\mathcal{U})} \xi_1$, as desired.

Finally, suppose $\xi_0 > \mathrm{crit}(\phi_{\gamma_0,\gamma_1})$. So letting $\tau + 1 \leq \gamma_1$ be least such that $\gamma_0 < \tau + 1$, and

$$\beta = \beta(\mathcal{W}_{\gamma_0}, \mathcal{W}_\tau, F_\tau),$$

we have

$$\beta < \xi_0 \leq \alpha_{\gamma_0} < \alpha_\tau.$$

No extender in $\mathrm{ran}\,\psi_{\gamma_0,\gamma_1}$ can have critical point in the interval $[\mathrm{crit}(F_\tau), \lambda(F_\tau)]$. This implies that if $\tau+1 \leq_U \gamma$ and $\beta < \xi \leq \alpha_\tau$, then for all $\sigma \in \mathrm{dom}\,\phi_{\gamma_0,\gamma}$, $\xi \not\leq_{W_\gamma} \phi_{\gamma_0,\gamma}(\sigma)$. In particular, $\xi_0 \not\leq_{W_{\gamma_1}} \xi_1$, so $\xi_0 \not\leq_{W(\mathcal{T},\mathcal{U})} \xi_1$, as desired. □

For a branch a of $W(\mathcal{T},\mathcal{U})$, we set

$$e(a) = \{\beta \mid \exists \xi \in a\,(\beta \leq_T e(\xi))\}.$$

So $e(a)$ is a branch of $\mathcal{T}$. Even if a is cofinal in $W(\mathcal{T},\mathcal{U})$, $e(a)$ may not be cofinal in $\mathcal{T}$. $e(a)$ may have a largest element, or be a maximal branch of $\mathcal{T}$ not chosen by $\mathcal{T}$.

Claim 2.67.5 *Let a be cofinal in $W(\mathcal{T},\mathcal{U})$. Then $a = \mathrm{br}_{\mathcal{W}}(e(a), d(a))$, and $e(a)$ is $d(a)$-minimal.*

Proof. Let $b = d(a)$ and $c = e(a)$. Let $\xi \in a$, we wish to show $\xi \in \mathrm{br}(c, b)$. Let η be least such that $\xi \leq \alpha_\eta$, so that $\eta \in b$. Let $\phi_{0,\eta}(\bar{\xi}) = \xi$, so that $\bar{\xi} \in c$. Let $\gamma+1 \in b$ be such that $\eta = U\text{-pred}(\gamma+1)$. It will be enough to show that $\xi = \mathrm{nd}(\bar{\xi}, \gamma+1)$. For that, it is enough to show that $\xi \leq \mathrm{crit}(\phi_{\eta,\gamma+1})$.

Let $\rho \in a$ be such that $\alpha_\eta < \rho$. Let σ be least such that $\rho \leq \alpha_\sigma$, so that $\sigma \in b$ and $\gamma + 1 \leq_U \sigma$. Let $\phi_{0,\sigma}(\bar{\rho}) = \rho$. If $\xi > \mathrm{crit}(\phi_{\eta,\gamma+1})$, then $\xi \in (\mathrm{crit}(\phi_{\eta,\gamma+1}), \alpha_\eta]$. But we observed above that ξ is "dead" along branches containing $\gamma+1$ for extensions in $\mathrm{ran}\,\phi_{\eta,\sigma}$, so since ρ is in $\mathrm{ran}\,\phi_{\eta,\sigma}$, $\xi \not\leq_{W_\sigma} \rho$. But $\mathcal{W}_\sigma \restriction (\alpha_\sigma + 1) = W(\mathcal{T},\mathcal{U}) \restriction (\alpha_\sigma + 1)$, so $\xi \not\leq_{W(\mathcal{T},\mathcal{U})} \rho$, contrary to $\rho \in a$.

It is easy to see that $e(a)$ is $d(a)$-minimal. □
□

Definition 2.68 *Given $\mathcal{T}$ normal on M, and $\mathcal{U}$ normal on the last model of $\mathcal{T}$, we write $\mathrm{br}_{\mathcal{W}}(\mathcal{T},\mathcal{U})$ for the function $\mathrm{br}_{\mathcal{W}}$ (defined on pairs of nodes and pairs of branches) defined above. We write $\mathrm{br}^{\mathcal{W}}_{\mathcal{U}}$ for the function d and $\mathrm{br}^{\mathcal{W}}_{\mathcal{T}}$ for the function e defined above.*

Notation 2.68.1 To reconcile with our previous notation: if b is cofinal in $\mathcal{U}$, there is exactly one branch c of $\mathcal{T}$ such that

(*i*) $c = [0, \tau]_T$ or $c = [0, \tau)_T$ for some $\tau < \mathrm{lh}\,\mathcal{T}$, and

(*ii*) $\mathrm{br}_{\mathcal{W}}(c, b)$ is cofinal in $W(\mathcal{T},\mathcal{U})$.

This uses that $\mathcal{T}$ has a last model. We defined $\mathrm{br}(b, \mathcal{T}, \mathcal{U})$ to be $\mathrm{br}_{\mathcal{W}}(c, b)$, for the unique such c. We define $m(b, \mathcal{T}, b)$ to be the unique τ as in (*i*). We probably won't use that earlier notation much.

For τ in (*i*) a limit ordinal, the earlier notation does not distinguish between $c = [0, \tau)_T$ and $c = [0, \tau]_T$, whereas the current one does. $c = [0, \tau)_T$ is the case where, roughly speaking, the measures in E_b concentrate on proper initial segments of $\mathcal{M}^{\mathcal{T}}_c | \delta(\mathcal{T} \restriction \sup c) = \mathcal{M}^{\mathcal{T}}_\tau | \lambda^{\mathcal{T}}_\tau$.

Remark 2.69 We assumed $\mathcal{T}$ has a last model, but one could generalize some of this by dropping that, and assuming that $\mathcal{U}$ is on $\mathcal{M}(\mathcal{T})$.

Remark 2.70 There are two special cases worth mentioning.

(*a*) $\mathcal{T}^\frown\mathcal{U}$ is already normal. Then $W(\mathcal{T}^\frown\mathcal{U}) = \mathcal{T}^\frown\mathcal{U}$, and $\mathrm{br}_W(c, b) = c^\frown b$.

(b) $\mathcal{U}$ is a tree on $\mathcal{M}|\kappa$, where $\kappa = \inf\{\text{crit}(E^{\mathcal{T}}_\eta) \mid \eta + 1 < \text{lh}\,\mathcal{T}\}$. Then if $\mathcal{U}$ has limit length, then $W(\mathcal{T},\mathcal{U}) = \mathcal{U}$-on-$\mathcal{M}$, i.e. $\mathcal{U}$ regarded as a tree on $\mathcal{M}$. For b a cofinal branch of $\mathcal{U}$, $\mathcal{W}_b = W(\mathcal{T},\mathcal{U}^\frown b) = \mathcal{U}^\frown b^\frown(i^{\mathcal{U}}_b)\mathcal{T}$, and $\text{br}_W(c,b) = b^\frown\phi``c$, where $\phi(\eta) = \text{lh}\,\mathcal{U} + \xi$.

In our application, however, $\mathcal{T}$ and $\mathcal{U}$ will definitely not be separated this way.

Remark 2.71 $\text{br}^{\mathcal{T},\mathcal{U}}_W$ makes sense in the coarse structural case. Our proof that it is 1-1 and onto used fine structure (via 2.59), as well as the hypothesis that all $\mathcal{W}_\gamma$ are by some fixed Σ. So that part is limited to the fine structural case. But not much fine structure was used, and we shall adapt the proof to the coarse structural case later.

2.8 Normalizing longer stacks

There seem to be in the abstract many different ways to normalize a stack $\langle \mathcal{U}_1, ..., \mathcal{U}_n\rangle$, one for each way of associating the $\mathcal{U}_i$. If we are in the case that embedding normalization coincides with full normalization, and there is a fixed strategy Σ for M according to which all these normalizations are played, such that for any N there is at most one normal Σ-iteration from M, then clearly all these normalizations are the same. They are just the unique normal tree by Σ from M to the last model of $\vec{\mathcal{U}}$. We shall be in that situation below when we deal with coarse iterations of a background universe. But in general, it seems that the various normalizations of $\vec{\mathcal{U}}$ might all be different from one another.

We shall define Σ *normalizes well* by demanding that whenever $\vec{\mathcal{U}}$ is a finite stack by Σ, then all normalizations of $\vec{\mathcal{U}}$ are by Σ. In addition, we demand that Σ pull back to itself under normalization maps.

Definition 2.72 *Let $\vec{\mathcal{U}} = \langle \mathcal{U}_1, \ldots, \mathcal{U}_n\rangle$ be a finite stack of normal trees on M, where $n > 1$. Let $M_0 = M$, and M_i be the last model of $\mathcal{U}_i$ for $1 \le i \le n$. A* 1-step normalization *of $\vec{\mathcal{U}}$ is a triple $\langle k, \vec{\mathcal{V}}, \vec{\pi}\rangle$ such that $\vec{\mathcal{V}}$ is a stack of length $n-1$ on $M = M_0$, and*

(1) $1 \le k < n$,

(2) $\mathcal{V}_m = \mathcal{U}_m$ for all $m < k$, and $\mathcal{V}_k = W(\mathcal{U}_k, \mathcal{U}_{k+1})$,

(3) Letting $N_0 = M$ and N_i be the last model of $\mathcal{V}_i$ for $i < n$, we have that

(a) $\pi_i \colon M_i \to N_i$ is the identity for $i < k$,

(b) $\pi_k\colon M_{k+1} \to N_k$ *is the map given by embedding normalization, and*

(c) for $k < i < n$*,* $\mathcal{V}_i = \pi_i \mathcal{U}_{i+1}$*, and* $\pi_{i+1}\colon M_{i+1} \to N_i$ *is the copy map.*

Clearly, $\vec{\mathcal{U}}$ and k determine the rest of the normalization.

Definition 2.73 *Let* $\vec{\mathcal{U}} = \langle \mathcal{U}_1, \dots, \mathcal{U}_n \rangle$ *be a finite stack of normal trees on* M*, where* $n > 1$*. Let* $1 \leq t < n$*; then a* t-step normalization *of* $\vec{\mathcal{U}}$ *is a sequence* s *with domain* $t+1$ *such that* $s(0) = \vec{\mathcal{U}}$*, and whenever* $0 \leq i < t$*,* $s(i+1)$ *is a 1-step normalization of* $\vec{\mathcal{V}}$*, where* $\vec{\mathcal{V}}$ *is the second coordinate of* $s(i)$*.*

A *complete normalization* of $\langle \mathcal{U}_1, \dots, \mathcal{U}_n \rangle$ is an $n-1$ step normalization of $\langle \mathcal{U}_1, \dots, \mathcal{U}_n \rangle$. We shall sometimes identify a t-step normalization s of $\vec{\mathcal{U}}$ with the stack of trees in the second coordinate of $s(t)$. If $t = n-1$, then this is a single normal tree on M.

Remark 2.74 We have no example of a stack $\vec{\mathcal{U}}$ on a premouse M having complete normalizations which produce distinct normal trees on M. If $\mathrm{lh}(\vec{\mathcal{U}}) = 3$, then there are two possible ways to normalize $\vec{\mathcal{U}}$. Must they always produce the same normal tree?

For $m \geq 1$, and $i \geq 0$, let us write $\mathcal{V}_m^{s(i)}$ for the m-th tree in $s(i)$ (or in its third coordinate, if $i > 0$), and $N_m^{s(i)}$ for the last model of $\mathcal{V}_m^{s(i)}$. Let $N_0^{s(i)} = M$, for all i. For any $e < i < n$, and any m such that $N_m^{s(i)}$ exists, there is a unique l such that $N_m^{s(i)}$ comes from $N_l^{s(e)}$, in the sense that $s(e)\restriction(l+1)$ is normalized to $s(i)\restriction(m+1)$ by $s\restriction(e, i]$. Let us write

$$l = o^{s,i,e}(m)$$

in this case. Composing normalization maps and copy maps given by $s\restriction(e, i]$ yields a canonical

$$\pi_{l,m}^{s,i,e}\colon N_l^{s(e)} \to N_m^{s(i)},$$

where $l = o^{s,i,e}(m)$. So if s is a normalization of $\langle \mathcal{U}_1, \dots, \mathcal{U}_n \rangle$ with $\mathrm{dom}(s) = i+1$, then the stack $\vec{\mathcal{V}}^{s(i)}$ has last model $N_m^{s(i)}$, where $m = n - i$, and $n = o^{s,i,0}(m)$, and $\pi_{n,m}^{s,i,0}$ is the natural map from the last model of $\vec{\mathcal{U}}$ to the last model of $\vec{\mathcal{V}}$. Let us write

$$\pi^s = \pi_{n,m}^{s,i,0}$$

in this case. So π^s is the natural map from the last model of $s(0)$ to the last model of the stack in $s(\mathrm{dom}(s) - 1)$ that is given by s. All $\pi_{l,m}^{s,i,e}$ have the form π^u, for u obtained from s in a simple way.

Probably the most natural order in which to normalize a stack is bottom-up.

Definition 2.75 *Let $\vec{\mathcal{U}} = \langle \mathcal{U}_1, \ldots, \mathcal{U}_n \rangle$ be a finite stack of normal trees on M; then the* bottom-up normalization *of $\vec{\mathcal{U}}$ is the complete normalization s of $\vec{\mathcal{U}}$ such that for each $i \geq 1$ in $\mathrm{dom}(s)$, $s(i)$ has first coordinate 1. We write $W(\vec{\mathcal{U}})$ for the normal tree on M in the second coordinate of $s(\mathrm{dom}(s) - 1)$, and also call $W(\vec{\mathcal{U}})$ the bottom-up normalization of $\vec{\mathcal{U}}$.*

The definitions above extend to stacks $\vec{\mathcal{U}}$ on M of infinite length. Again, it seems to makes sense to normalize in any order, but the most natural way is bottom-up. Suppose for example that $\vec{\mathcal{U}} = \langle \mathcal{U}_n \mid n < \omega \rangle$. Let $\mathcal{W}_0 = \mathcal{U}_0$, and for $n \geq 1$ let

$$\mathcal{W}_n = W(\langle \mathcal{U}_i \mid i \leq n \rangle).$$

For $n \geq 0$, let

$$\Phi_n \colon \mathcal{W}_n \to \mathcal{W}_{n+1}$$

be the psuedo-hull embedding given by the fact that $\mathcal{W}_{n+1} = W(\mathcal{W}_n, \pi\mathcal{U}_{n+1})$ for the appropriate π. (Φ_n is partial iff $\mathcal{U}_{n+1}$ drops along its main branch.) Then we set

$$W(\vec{\mathcal{U}}) = \lim_n \mathcal{W}_n,$$

where the limit is taken using the Φ_n. Clearly, we could continue further into the transfinite, and so $W(\vec{\mathcal{U}})$ makes sense for stacks $\vec{\mathcal{U}}$ of normal trees of any length. See [27].

In fact, one could go beyond linear stacks of normal trees, and consider normalizing arbitrary trees on M. See [27] for a discussion. In this paper we shall not need more than normalization for finite stacks of normal trees.

3 Strategies that condense and normalize well

In this section we define what it is for an iteration strategy to normalize well, and to have strong hull condensation. We prove some elementary facts related to these definitions. We then show that under an appropriate form of UBH, there is a unique iteration strategy Σ^* for V that normalizes well. UBH easily implies that Σ^* also has strong hull condensation. Finally, we show that, via full background constructions, Σ^* induces iteration strategies for premice that normalize well and have strong hull condensation.

The version of UBH we shall use in this section is open. However, assuming AD^+, it does hold in the coarse Γ-Woodin models constructed by Woodin (see for example [30]). So working in such a model, one can use the results of this section to construct strategies that normalize and condense well.

3.1 The definitions

Definition 3.1 *Let Σ be an iteration strategy for M defined on finite stacks of normal trees. We say that Σ* normalizes well *iff whenever $\vec{\mathcal{U}}$ is a finite stack by Σ, and s is a t-step normalization of $\vec{\mathcal{U}}$, and $\vec{\mathcal{V}} = \vec{\mathcal{V}}^{s(t)}$ is the stack in $s(t)$, then*

(1) $\vec{\mathcal{V}}$ is by Σ, and

(2) if $\pi = \pi^s$ is the natural map from the last model Q of $\vec{\mathcal{U}}$ to the last model R of $\vec{\mathcal{V}}$, then $\Sigma_{\vec{\mathcal{U}},Q} = (\Sigma_{\vec{\mathcal{V}},R})^\pi$.

Clearly, if Σ normalizes well, then so do all its tail strategies.

Let us say that Σ *2-normalizes well* iff the conclusions of 3.1 hold for stacks $\vec{\mathcal{U}}$ of length 2. So if Σ 2-normalizes well, then whenever $\langle \mathcal{T}, \mathcal{U}\rangle$ is by Σ, then $W(\mathcal{T}, \mathcal{U})$ is defined. That is, the definition never produces illfounded models, because it is producing a tree by Σ. Moreover, Σ pulls back to itself under the normalization map of $W(\mathcal{T}, \mathcal{U})$.

Suppose Σ normalizes well, and $\mathcal{T}$ is a normal tree on M with last model Q that is according to Σ. Let $\mathcal{U}$ on Q be normal and by $\Sigma_{\mathcal{T},Q}$ and of limit length, and let

$$b = \Sigma_{\mathcal{T},Q}(\mathcal{U}) = \Sigma(\langle \mathcal{T}, \mathcal{U}\rangle),$$

and

$$a = \Sigma(W(\mathcal{T}, \mathcal{U})).$$

Then

$$a = \mathrm{br}_W^{\mathcal{T},\mathcal{U}}(c, b)$$

where c is some branch $[0,\tau)_{\mathcal{T}}$ or $[0,\tau]_{\mathcal{T}}$ of $\mathcal{T}$ that is chosen by Σ. (I.e. $\mathcal{T}\restriction(\tau+1)$ is by Σ.) Moreover,

$$b = \mathrm{br}_{\mathcal{U}}^{\mathcal{T},\mathcal{U}}(a).$$

In other words, $\Sigma(\langle\mathcal{T},\mathcal{U}\rangle)$ and $\Sigma(W(\mathcal{T},\mathcal{U}))$ determine each other, modulo $\mathcal{T}$. (This "moreover" part applies in the fine-structural case, with all $\mathcal{W}_\gamma$ by a fixed Σ.)

Proposition 3.2 *Let Σ be an iteration strategy for M defined on finite stacks of normal trees, and suppose that whenever $\mathcal{V}$ is a normal tree by Σ with last model R, then the tail strategy $\Sigma_{\mathcal{V},R}$ 2-normalizes well. Then Σ normalizes well.*

Proof. We show by induction on n that Σ normalizes well for stacks of length n. For $n = 2$ this is true by hypothesis. Let $\vec{\mathcal{T}}^\frown\langle\mathcal{U}_1,\mathcal{U}_2\rangle^\frown\vec{\mathcal{S}}$ be a stack of length $n+1$ by Σ. We want to see that the 1-step normalization obtained by replacing $\langle\mathcal{U}_1,\mathcal{U}_2\rangle$ by $W(\mathcal{U}_1,\mathcal{U}_2)$, and $\vec{\mathcal{S}}$ by $\pi\vec{\mathcal{S}}$ for π the normalization map, behaves well. It is clear that this implies t-step normalizations behave well, for all t. The proof is by induction on the length of $\mathcal{U}_2$.

Let $\mathcal{V}$ be a complete normalization of $\vec{\mathcal{T}}$, with θ the normalization map from $N = \mathcal{M}_\infty^{\vec{\mathcal{T}}}$ to $N^* = \mathcal{M}_\infty^{\mathcal{V}}$. θ lifts $\mathcal{U}_1$ to $\theta\mathcal{U}_1$; let $\rho\colon \mathcal{M}_\infty^{\mathcal{U}_1} \to \mathcal{M}_\infty^{\theta\mathcal{U}_1}$ be the copy map. Note that $\langle\mathcal{V},\theta\mathcal{U}_1,\rho\mathcal{U}_2\rangle$ is a stack by Σ, because $\Sigma_{\mathcal{V},N^*}$ pulls back under θ to $\Sigma_{\vec{\mathcal{T}},N}$ by our induction hypothesis. Let Q^* be its last model. Let

$$\mathcal{W}^* = W(\theta\mathcal{U}_1,\rho\mathcal{U}_2),$$

and let R^* be the last model of $\mathcal{W}^*$, and $\sigma^*\colon Q^* \to R^*$ the normalization map. The hypothesis of our proposition tells us that $\langle\mathcal{V},\mathcal{W}^*\rangle$ is by Σ, and that

$$\Sigma_{\langle\mathcal{V},\theta\mathcal{U}_1,\rho\mathcal{U}_2\rangle,Q^*} = (\Sigma_{\langle\mathcal{V},\mathcal{W}^*\rangle,R^*})^{\sigma^*}.$$

Let Q be the last model of $\vec{\mathcal{T}}^\frown\langle\mathcal{U}_1,\mathcal{U}_2\rangle$, let

$$\mathcal{W} = W(\mathcal{U}_1,\mathcal{U}_2),$$

and let R be the last model of $\mathcal{W}$. Let $\sigma\colon Q \to R$ be the normalization map. The situation can be encapsulated in the following diagram.

$$
\begin{tikzcd}
 & & & R^* \\
 & N^* \arrow[r, "\theta\mathcal{U}_1"'] \arrow[urr, "\mathcal{W}^*"] & P^* \arrow[r, "\rho\mathcal{U}_2"'] & Q^* \arrow[u, "\sigma^*"] \\
 & & & R \arrow[uu, bend right, "\phi"'] \\
M \arrow[r, "\vec{\mathcal{T}}"'] \arrow[uur, "\mathcal{V}"] & N \arrow[r, "\mathcal{U}_1"'] \arrow[uu, "\theta"] \arrow[urr, "\mathcal{W}"] & P \arrow[r, "\mathcal{U}_2"'] & Q \arrow[u, "\sigma"'] \arrow[uu, bend left, "\psi"]
\end{tikzcd}
$$

Here $P = \mathcal{M}_\infty^{\mathcal{U}_1}$, and $P^* = \mathcal{M}_\infty^{\rho\mathcal{U}_1}$. The maps $\psi\colon Q \to Q^*$ and $\phi\colon R \to R^*$ are copy maps. We get ϕ from Theorem 2.46; in this case, copying $\langle \mathcal{U}_1, \mathcal{U}_2\rangle$ via θ commutes with normalizing $\langle \mathcal{U}_1, \mathcal{U}_2\rangle$. We have

$$\phi \circ \sigma = \sigma^* \circ \psi$$

from 2.46.

Since $\theta\mathcal{W} = \mathcal{W}^*$, and Σ pulls back to itself under θ by induction, we have that $\vec{\mathcal{T}}{}^\frown\langle\mathcal{W}\rangle$ is by Σ, and $\Sigma_{\vec{\mathcal{T}}{}^\frown\langle\mathcal{W}\rangle,R} = (\Sigma_{\langle\mathcal{V},\mathcal{W}^*\rangle,R^*})^\phi$. It follows that

$$
\begin{aligned}
(\Sigma_{\vec{\mathcal{T}}{}^\frown\langle\mathcal{W}\rangle,R})^\sigma &= (\Sigma_{\langle\mathcal{V},\mathcal{W}^*\rangle,R^*})^{\phi\circ\sigma} \\
&= (\Sigma_{\langle\mathcal{V},\mathcal{W}^*\rangle,R^*})^{\sigma^*\circ\psi} \\
&= ((\Sigma_{\langle\mathcal{V},\mathcal{W}^*\rangle,R^*})^{\sigma^*})^{\psi} \\
&= ((\Sigma_{\langle\mathcal{V},\theta\mathcal{U}_1,\rho\mathcal{U}_2\rangle,Q^*})^{\psi} \\
&= \Sigma_{\vec{\mathcal{T}}{}^\frown\langle\mathcal{U}_1,\mathcal{U}_2\rangle,Q}.
\end{aligned}
$$

Line 1 holds because Σ normalizes well for $\vec{\mathcal{T}}$, line 2 comes from 2.46, line 4 holds because $\Sigma_{\mathcal{V},N^*}$ 2-normalizes well, and line 5 holds because Σ normalizes well for $\vec{\mathcal{T}}$.

This takes care of the case $\vec{\mathcal{S}} = \emptyset$. The general case follows easily. Since $\Sigma_{\vec{\mathcal{T}}{}^\frown\langle\mathcal{W}\rangle,R})^\sigma = \Sigma_{\vec{\mathcal{T}}{}^\frown\langle\mathcal{U}_1,\mathcal{U}_2\rangle,Q}$ and $\vec{\mathcal{S}}$ is by $\Sigma_{\vec{\mathcal{T}}{}^\frown\langle\mathcal{U}_1,\mathcal{U}_2\rangle,Q}$, we have that $\sigma\vec{\mathcal{S}}$ is by $\Sigma_{\vec{\mathcal{T}}{}^\frown\langle\mathcal{W}\rangle,R}$, and moreover the $\vec{\mathcal{T}}{}^\frown\langle\mathcal{W}\rangle{}^\frown\sigma\vec{\mathcal{S}}$-tail of Σ pulls back under the relevant copy map to the $\vec{\mathcal{T}}{}^\frown\langle\mathcal{U}_1,\mathcal{U}_2\rangle{}^\frown\vec{\mathcal{S}}$-tail of Σ. $\square$

A very similar argument shows that the property of normalizing well passes to pullback strategies.

Theorem 3.3 *Let Σ be an iteration strategy for N that normalizes well, and let $\pi\colon M \to N$ be sufficiently elementary that the pullback strategy Σ^π exists; then Σ^π normalizes well.*

Proof. Let $\langle \mathcal{V}, \mathcal{U}_1, \mathcal{U}_2 \rangle$ be a stack by Σ^π, with last model Q. Let $\mathcal{W} = W(\mathcal{U}_1, \mathcal{U}_2)$ have last model R, and $\sigma\colon Q \to R$ be the normalization map. We want to see that $\langle \mathcal{V}, \mathcal{W} \rangle$ is by Σ^π, and that the $\langle \mathcal{V}, \mathcal{W} \rangle$-tail of Σ^π pulls back under σ to the $\langle \mathcal{V}, \mathcal{U}_1, \mathcal{U}_2 \rangle$-tail of Σ^π.

We have the diagram

$$\begin{tikzcd}
 & & & R^* \\
N \arrow[r, "\pi\mathcal{V}"] & K^* \arrow[r, "\theta\mathcal{U}_1"'] \arrow[urr, "\mathcal{W}^*"] & P^* \arrow[r, "\rho\mathcal{U}_2"'] & Q^* \arrow[u, "\sigma^*"] \\
 & & & R \arrow[uu, bend right, "\phi"'] \\
M \arrow[r, "\mathcal{V}"'] \arrow[uu, "\pi"] & K \arrow[r, "\mathcal{U}_1"'] \arrow[uu, "\theta"] \arrow[urr, "\mathcal{W}"] & P \arrow[r, "\mathcal{U}_2"'] & Q \arrow[u, "\sigma"'] \arrow[uu, bend left, "\psi"]
\end{tikzcd}$$

Here θ and ρ are copy maps generated by π, and $\mathcal{W}^*$ is the normalization of $\langle \theta\mathcal{U}_1, \rho\mathcal{U}_2 \rangle$. σ^* is the associated normalization map. ψ and ϕ are copy maps, which we have because copying commutes with normalization. $\phi \circ \sigma = \sigma^* \circ \psi$ by 2.46.

The copy map ϕ tells us that $\langle \mathcal{V}, \mathcal{W} \rangle$ is by Σ^π. The rest is given by

$$\begin{aligned}
(\Sigma^\pi_{\langle \mathcal{V}, \mathcal{W} \rangle, R})^\sigma &= (\Sigma_{\langle \pi\mathcal{V}, \mathcal{W}^* \rangle, R^*})^{\phi \circ \sigma} \\
&= (\Sigma_{\langle \pi\mathcal{V}, \mathcal{W}^* \rangle, R^*})^{\sigma^* \circ \psi} \\
&= ((\Sigma_{\langle \pi\mathcal{V}, \mathcal{W}^* \rangle, R^*})^{\sigma^*})^{\psi} \\
&= ((\Sigma_{\langle \pi\mathcal{V}, \theta\mathcal{U}_1, \rho\mathcal{U}_2 \rangle, Q^*})^{\psi} \\
&= \Sigma^\pi_{\vec{\mathcal{T}}^\frown \langle \mathcal{U}_1, \mathcal{U}_2 \rangle, Q}.
\end{aligned}$$

This is what we want. $\square$

We turn to strong hull condensation.

Definition 3.4 *Let Σ be an iteration strategy for a premouse M. Then Σ has* strong hull condensation *iff whenever s is a stack of weakly normal trees by Σ with last model N, and $\mathcal{T}$ and $\mathcal{U}$ are normal trees on N such that $\mathcal{U}$ is by $\Sigma_{s,N}$, and $\mathcal{T}$ is a pseudo-hull of $\mathcal{U}$, then $\mathcal{T}$ is by $\Sigma_{s,N}$.*

Because less is required of a psuedo-hull embedding than is required of a hull embedding in [16], the property is stronger than the corresponding one in [16], hence the name.

Remark 3.5 In [33] we introduce a still weaker sort of embedding of iteration trees, and make use of the resulting "very strong hull condensation". It turns out that strategies for premice that normalize well and have strong hull condensation also have very strong hull condensation, and this implies that they fully normalize well. However, the proof of this requires a strategy-comparison argument. Strong hull condensation has the virtue that we can verify it directly for background-induced strategies, so we can use it in proving a comparison theorem.

Strong hull condensation is preserved by pullbacks:

Proposition 3.6 *Let $\pi\colon M \to P$ be weakly elementary, and let Σ be a strategy for P having strong hull condensation; then Σ^π has strong hull condensation.*

Proof.(Sketch.) Let s be a stack on M with last model N, and $\mathcal{U}$ be on N and by $(\Sigma^\pi)_s$. Let $\mathcal{T}$ be a psuedo-hull of $\mathcal{U}$. Let Q be the last model of πs, and $\psi\colon N \to Q$ the copy map. It is not hard to see that $\psi\mathcal{T}$ is a psuedo-hull of $\psi\mathcal{U}$. Since $\psi\mathcal{U}$ is by $\Sigma_{\pi s,Q}$, $\psi\mathcal{U}$ is by $\Sigma_{\pi s,Q}$, so $\mathcal{T}$ is by $(\Sigma^\pi)_s$, as desired. $\square$

3.2 Coarse strategies that condense and normalize well

In the context of coarse iteration trees, we shall restrict ourselves to the nice ones. (Cf. 1.23.)One reason is that UBH fails in general (Woodin, cf. [39]), but may well hold for nice trees on V. In fact, countable closure is enough to avoid the counterexamples for normal trees, but we shall stick with niceness.

Definition 3.7 *(a) M is* uniquely θ-iterable for normal trees *iff whenever $\mathcal{T}$ is a normal, nice iteration tree on M, and $\mathrm{lh}(\mathcal{T})$ is a limit ordinal $< \theta$, then $\mathcal{T}$ has a unique cofinal wellfounded branch.*

(b) *M is* strongly uniquely θ-iterable for finite stacks *iff whenever $\langle \mathcal{U}_1, \ldots, \mathcal{U}_n \rangle$ is a finite stack of normal nice iteration trees, with $\mathcal{U}_1$ on M, and $\mathrm{lh}(\mathcal{U}_i) < \theta$ for all i, and $\mathcal{U}_n$ has limit length, then $\mathcal{U}_n$ has a unique cofinal wellfounded branch.*

(c) *We say that M is* uniquely θ-iterable above κ *for normal trees (respectively, strongly uniquely θ iterable above κ for finite stacks) if (a) (respectively (b)) holds for trees with all critical points $> \kappa$.*

The unique ω_1 iterability of V, in either sense, follows from UBH for the associated class of trees, by [8]. For iterations of uncountable length, we need UBH in the appropriate collapse extension.

Theorem 3.8 (Folk.) *Let $\theta < \kappa$, and suppose that* UBH *holds in $V[G]$, where G is $\mathrm{Col}(\omega, \theta)$ generic over V, when restricted to normal nice iteration trees above κ; then V is uniquely θ^+ iterable for normal trees above κ.*

Proof.[*Sketch.*] Given $\mathcal{T}$ in V of limit length $< \theta^+$, we can regard $\mathcal{T}$ as a tree on $V[G]$ because $\theta < \kappa$. In $V[G]$, $\mathcal{T}$ is countable, so by UBH in $V[G]$ and [8] in $V[G]$, it has a unique cofinal, wellfounded branch. Because the collapse is homogeneous, this branch is in V. □

In one situation, UBH in V implies instances of UBH in $V[G]$:

Theorem 3.9 (Woodin) *Let δ be Woodin, and let $\mathcal{T}$ be a nice tree on V that is above δ. Suppose $|\mathcal{T}| < \delta$, and let G be V-generic for a poset of size $< \delta$; then in $V[G]$, there is at most one cofinal branch of $\mathcal{T}$.*

Proof.Sketch. We may assume G is countable in $V[H]$, where H is V-generic for the countable stationary tower $\mathbb{Q}_{<\delta}$. Suppose toward contradiction that b and c are distinct cofinal branches of $\mathcal{T}$ in $V[G]$. $\mathcal{T}$ can be regarded as a tree on $V[H]$, and b and c are still wellfounded when it is regarded this way.

But let $\pi \colon V \to M = \mathrm{Ult}(V, H)$ be the generic elementary embedding. Since M is closed under countable sequences in $V[H]$, $\pi\mathcal{T} \in M$, and one can check that b and c are wellfounded as branches of $\pi\mathcal{T}$. (Essentially the same functions into the ordinals are used in forming $\mathcal{M}_b^{\mathcal{T}}$ and $\mathcal{M}_b^{\pi\mathcal{T}}$, for example.) Thus UBH fails in M for nice trees above $\pi(\delta)$, contrary to the elementarity of π. □

At supercompacts, we catch our tail:

Theorem 3.10 (Woodin) *Suppose that κ is supercompact, and that* UBH *holds for nice iteration trees on V above κ. Then for all θ, V is uniquely θ-iterable for normal trees above κ.*

Proof. Given $\mathcal{T}$ above κ on V, with $\mathcal{T} \in V_\theta$, let $j : V \to M$, $\text{crit}(j) = \kappa$, $j{\restriction}V_\theta \in M$. In M, the lifted tree $j\mathcal{T}$ has size $< \inf\{\text{crit}(E_\alpha^{j\mathcal{T}}) \mid \alpha + 1 < \text{lh}\,\mathcal{T}\}$, so by 3.10 and 3.9, $j\mathcal{T}$ has a cofinal wellfounded branch b in M. (Note $j(\kappa)$ is a limit of Woodin cardinal in M.) The copy map $\sigma : M_b^{\mathcal{T}} \to M_b^{j\mathcal{T}}$ witnesses that b is wellfounded branch of $\mathcal{T}$. $\square$

We shall see in a moment that there is a reason to distinguish between UBH for normal trees and UBH for stacks, in that UBH for stacks of length 2 fails granted sufficiently large cardinals.

One way to get strong unique iterabilty is to work with M that have no Woodin cardinals. Such M may have Γ-Woodin cardinals, for some large pointclass Γ, and be interesting for that reason. Consider for example the case $\Gamma = \Sigma_2^1$: if δ is least such that $L(V_\delta) \models \delta$ is Woodin, then $\forall \eta < \delta$, $(L(V_\eta)$ is uniquely δ-iterable). If in addition every set has a sharp, then $L(V_\delta)$ is θ-iterable for all θ. Under AD^+, we have *coarse* Γ*-Woodin mice* at all scaled pointclasses Γ, as we now describe.

There are a number of variants on the notion of a coarse Γ-Woodin mouse. The following is good enough for our purposes here. Assume AD^+, and let Γ_0, Γ_1 be a good (i.e. closed under $\exists^{\mathbb{R}}$) lightface pointclasses with the scale property such that $\Gamma_0 \subseteq \Delta_1$. Let A be a universal Γ_1 set, and let U code the theory (with parameters) of $(V_{\omega+1}, \in, A)$. Let S and T be trees on some $\omega \times \kappa$ that project to U and $\neg U$. Using his work in [6], Woodin has shown ([36]) that there is a pair $N^* \in \text{HC}$, a wellorder $\lhd$ of N^*, and an iteration strategy Σ^* for N^* such that for $\delta = o(N^*)$,

(a) (fullness) $N^* = V_\delta^{L(N^* \cup \{S,T,\lhd\})}$,

(b) N^* is f-Woodin, for all $f\colon \delta \to \delta$ such that $f \in C_{\Gamma_0}(N^*, \lhd)$,

(c) for all $\eta \leq \delta$, there is an $f\colon \eta \to \eta$ such that $f \in C_{\Gamma_1}(V_\eta^{N^*}, \lhd \cap V_\eta^{N^*})$ and $V_\eta^{N^*}$ is not f-Woodin, and

(d) Σ^* is an (ω_1, ω_1)-iteration strategy $L(N^*, S, T, \lhd)$, with respect to nice trees based on N^*.

Definition 3.11 *Assume* AD^+*, and let* Γ_0 *be a good pointclass with the scale property. A* coarse Γ_0-Woodin mouse *is a tuple* $(N^*, \lhd, S, T, \Sigma^*)$ *as described above.*

Of course, S and T determine U, and hence Γ_1.

Let $M = L(N^*, S, T, \lhd)$, where $(N^*, \lhd, S, T, \Sigma^*)$ is a coarse Γ_0 mouse. If g is M-generic over some countable-in-V poset $\mathbb{P} \in M$, then $M[g]$ knows projective-in-U truth about the reals it sees. Moreover, if $i\colon M \to R$ is elementary and R is wellfounded, then as usual $i(S)$ and $i(T)$ can be used to compute projective-in-U

truth in $R[h]$, for any h that is R-generic over some countable-in-V poset. So letting C_0 and C_1 be the C_{Γ_0} and C_{Γ_1} operators (defined on HC^V), M can define $C_0 \restriction M$ and $C_1 \restriction M$, and if $i\colon M \to R$ is elementary and R is wellfounded, then $i(C_k \restriction M) = C_k \restriction R$ for $k = 0, 1$. Thus the C_{Γ_0} and C_{Γ_1} operators can be defined over M and its iterates by Σ^*.

It follows that M and its iterates are C_{Γ_1}-full, and Σ^* is guided by C_{Γ_1} Q-structures. More precisely,

Lemma 3.12 *Assume* AD^+, *and let* $(N^*, \lhd, S, T, \Sigma^*)$ *be a coarse* Γ-*Woodin mouse. Let* $\vec{\mathcal{T}}, \mathcal{U}$ *be a stack of nice normal trees played by* Σ^**; then the following are equivalent*

(1) $\Sigma^*_{\vec{\mathcal{T}}}(\mathcal{U}) = b$,

(2) $C_{\Gamma_1}(\mathcal{M}(\mathcal{U})) \subseteq \mathcal{M}_b^{\mathcal{T}}$,

(3) $\mathcal{M}_b^{\mathcal{U}}$ *is wellfounded.*

Proof. Just outlined. □

It follows that Q is an iterate of M, then Q satisfies "I am strongly uniquely θ-iterable for stacks of normal trees, for all $\theta < \omega_1^V$". The strategy witnessing this is $\Sigma^* \restriction Q$. Moreover, Σ^* is definable from U, so Q and its generic extensions are correct for the theory of $(\mathrm{HC}, \in, \Sigma^*)$. In particular

Corollary 3.13 *Assume* AD^+, *and let* $(N^*, \lhd, S, T, \Sigma^*)$ *be a coarse* Γ-*Woodin mouse; then*

$$N^* \models \text{“I am strongly uniquely iterable for stacks of normal trees”}.$$

It is easy to see that the C_Γ-guided strategy occurring in the proof of 3.12 normalizes well.

Strong unique iterability for stacks is too much to ask if V has extenders overlapping Woodin cardinals. (There are no such extenders in the Γ-Woodin models of 3.11.) The problem is that UBH is false in such models. In [14], the authors construct a stack $\vec{\mathcal{U}} = \langle \mathcal{U}_0, \mathcal{U}_1 \rangle$ of normal iteration trees on V such that for some strong limit cardinal δ of cofinality ω,

(i) $\mathcal{U}_0 = \langle F \rangle$, where $\mathrm{lh}\, F =$ strength $(F) = \delta$,

(ii) $\mathcal{U}_1$ is an alternating chain on $V_\delta = V_\delta^{\mathrm{Ult}(V,F)}$, with distinct branches b and c, and

(iii) both $\mathcal{M}_b^{\mathcal{U}_1}$ and $\mathcal{M}_c^{\mathcal{U}_1}$ are wellfounded.

The key here is that because $V_\delta = V_\delta^{\mathrm{Ult}(V,F)}$, both $i_b^{\mathcal{U}_1}$ and $i_c^{\mathcal{U}_1}$ can be extended so as to act on V, and the construction arranges that $i_b(F) = i_c(F)$. But then $\mathcal{M}_b^{\mathcal{U}_1} = \mathrm{Ult}(V, i_b(F)) = \mathrm{Ult}(V, i_c(F) = \mathcal{M}_c^{\mathcal{U}_1}$. So not only are b and c both wellfounded as branches of $\vec{\mathcal{U}}$, in fact $\mathcal{M}_b^{\mathcal{U}_1} = \mathcal{M}_c^{\mathcal{U}_1}$!

In the example above, $\mathrm{Ult}(V,F)$ is not closed under ω-sequences, so $\mathcal{U}_0$ is far from nice. However, W.H. Woodin has shown that under stronger large cardinal assumptions, we can modify the example so as to get a stack of length 2 of nice trees on V. Namely, suppose we start with μ a normal measure on δ_0, where δ_0 is Woodin, and F_0 an extender with length = strength equal to δ_0. Let $\mathcal{I}$ be a linear iteration of μ of length ω, with direct limit model N. Let F and δ be the images in N of F_0 and δ_0. Then let $\mathcal{U}_0$ be the normal tree determined by $\mathcal{I}^\frown\langle F\rangle$, so that the last model of $\mathcal{U}_0$ is $M = \mathrm{Ult}(V,F)$. and let $\mathcal{U}_1$ be an alternating chain on M with branches b and c which, when acting on N, satisfy $i_b(F) = i_c(F)$. The construction of [14] gives us this $\mathcal{U}_2$; we only need $\mathrm{cof}(\delta) = \omega$ to hold in V, it need not hold in N. Again we have $\mathcal{M}_b^{\vec{\mathcal{U}}} = \mathcal{M}_c^{\vec{\mathcal{U}}}$, so both branches are wellfounded. But now $\vec{\mathcal{U}}$ is nice.

Remark 3.14 Woodin's counterexample uses that $\mathrm{lh}(F)$ has measurable cofinality. We shall see in 3.21 that this is essential. That gives us a way of avoiding the counterexample in applications.

In both examples, the branches b and c are not equally good. For example, consider the first example. Let E_b and E_c be the two branch extenders. Since our chain was constructed by the one-step method, exactly one of $\mathrm{Ult}(V, E_b)$ and $\mathrm{Ult}(V, E_c)$ is wellfounded. But in $\langle\mathcal{U}_0, \mathcal{U}_1{}^\frown b\rangle$ and $\langle\mathcal{U}_0, \mathcal{U}_1{}^\frown c\rangle$, these branch extenders are applied to $\mathrm{Ult}(V,F)$ rather than V. We have taken advantage of non-normality to hide the difference between b and c. If we normalize, the difference shows up:

$$W(\mathcal{U}_0, \mathcal{U}_1{}^\frown b) = \mathcal{U}_1{}^\frown b^\frown i_b^{\mathcal{U}_1}(F)$$

and

$$W(\mathcal{U}_0, \mathcal{U}_1{}^\frown c) = \mathcal{U}_1{}^\frown c^\frown i_c^{\mathcal{U}_1}(F).$$

Here $\mathcal{U}_1{}^\frown b$ and $\mathcal{U}_1{}^\frown c$ are acting on V, where only one of the two is actually an iteration tree, in that all its models are wellfounded. This leads us to the following definition.

Definition 3.15 *We say that M is* uniquely θ-iterable for finite stacks *iff*

(a) M is uniquely θ-iterable for normal trees, and

(b) *letting Σ_0 be the unique θ-iteration strategy for M on normal trees, there is an iteration strategy Σ witnessing θ-iterability for finite stacks such that $\Sigma_0 \subseteq \Sigma$, and Σ normalizes well.*

If M is uniquely θ-iterable above κ for finite stacks, and Σ_0 is its unique strategy for normal trees, then there is a unique extension Σ of Σ_0 such that Σ normalizes well. $\Sigma(\langle \mathcal{U}_1, ..., \mathcal{U}_n \rangle)$ is the unique cofinal branch b of $\mathcal{U}_n$ such that some, equivalently all, normalizations of $\langle \mathcal{U}_1, ..., \mathcal{U}_n {}^\frown b \rangle$ are by Σ_0. So Σ witnesses that M is uniquely θ-iterable above κ for finite stacks iff

(i) Σ witnesses that M is θ-iterable above κ for finite stacks,

(ii) Σ normalizes well, and

(iii) the restriction of Σ to normal trees chooses unique cofinal wellfounded branches.

There is at most one such strategy Σ for M.

We conjecture that if V is uniquely θ-iterable above κ for normal trees, then it is uniquely θ-iterable above κ for finite stacks of normal trees. We shall prove below that this is true if we restrict our trees so that the extenders used all come from a fixed coarsely coherent sequence $\vec{F}$ and its images. That restriction seems mild so far as applications go. In this context, $\theta, \vec{F}$-iterability, unique $\theta, \vec{F}$-iterability, and so on, have the obvious meanings.

Strong hull condensation for $\vec{F}$ iteration strategies choosing unique wellfounded branches is immediate.

Lemma 3.16 *Suppose that $M \models$ "$\vec{F}$ is coarsely coherent", and that Σ witnesses that M is uniquely $\theta, \vec{F}$-iterable for normal trees; then Σ has strong hull condensation.*

Proof.. Suppose $\mathcal{U}$ is a normal $\vec{F}$-trees on M is by Σ, and Φ is a psuedo-hull embedding of $\mathcal{T}$ into $\mathcal{U}$. For $\alpha < \mathrm{lh}(\mathcal{T})$, $t^0_\alpha \colon \mathcal{M}^{\mathcal{T}}_\alpha \to \mathcal{M}^{\mathcal{U}}_{v(\alpha)}$ is elementary, where these are the maps of Φ. Thus $\mathcal{M}^{\mathcal{T}}_\alpha$ is wellfounded. Since all its models are wellfounded, $\mathcal{T}$ is by Σ. $\square$

Theorem 3.17 *Let $\vec{F}$ be coarsely coherent, and suppose that V is uniquely $\theta, \vec{F}$-iterable for normal trees; then V is uniquely $\theta, \vec{F}$-iterable.*

Proof. Let Σ_1 be the strategy witnessing that V is uniquely $\theta, \vec{F}$-iterable for normal trees. We must extend Σ_1 to a strategy Σ acting on finite stacks.

We first extend Σ_1 to Σ_2, acting on stacks of length ≤ 2. Let $\langle \mathcal{T}, \mathcal{U} \rangle$ be a 2-stack of $\vec{F}$-trees, with $\mathcal{T}$ by Σ_1. We define $\Sigma_2(\langle \mathcal{T}, \mathcal{U} \rangle)$ by induction on $\mathrm{lh}(\mathcal{U})$, maintaining by induction that $W(\mathcal{T}, \mathcal{U})$ is by Σ_1. The copy maps guarantee that so long as $W(\mathcal{T}, \mathcal{U})$ is defined, it is an $\vec{F}$-tree. It is not hard to show that $W(\mathcal{T}, \mathcal{U} \restriction (\gamma + 1))$ is by Σ_1, then so is $W(\mathcal{T}, \mathcal{U} \restriction (\gamma + 2))$. So suppose $\mathcal{U}$ of limit length λ. It is enough show that there is a unique cofinal branch b of $\mathcal{U}$ such that setting

$$\mathcal{W}_b = W(\mathcal{T}, \mathcal{U}^\frown b),$$

$\mathcal{W}_b$ is by Σ_1. For then we can set

$$\Sigma(\langle \mathcal{T}, \mathcal{U} \rangle) = b,$$

and our induction hypothesis remains true at $\lambda + 1$. To show this, let $\mathcal{W} = W(\mathcal{T}, \mathcal{U})$ and let $a = \Sigma_1(\mathcal{W})$ be the unique cofinal, wellfounded branch of $\mathcal{W}$. The results of section 1.5 go through for $\vec{F}$-iteration trees on V, because of 1.25. Adopting the notation of 1.5, let

$$b = \mathrm{br}^{\mathcal{W}}_{\mathcal{U}}(a)$$

be the cofinal branch of $\mathcal{U}$ determined by a. We claim that all models of $\mathcal{W}_b$ are wellfounded. Let us adopt our usual embedding normalization notation: Let $\eta = \mathrm{lh}(W(\mathcal{T}, \mathcal{U}))$. So

$$\eta = \sup_{\gamma < \lambda} \alpha_\gamma = \phi_{0,b}(\tau),$$

where $\tau < \mathrm{lh}(\mathcal{T})$, and $\tau = m(b, \mathcal{T}, \mathcal{U})$. So $\mathcal{M}^{\mathcal{W}_b}_\eta = \mathcal{M}^{\mathcal{W}}_a$ is wellfounded. We show by induction on ξ that if $\eta \leq \xi < \mathrm{lh}(\mathcal{W}_b)$, then $\mathcal{M}^{\mathcal{W}_b}_\xi$ is wellfounded, and hence $\mathcal{W}_b \restriction (\xi + 1)$ is by Σ_0. This is trivial if ξ is a successor ordinal, because Σ_1 cannot lose at a successor step. But if ξ is a limit, then we have

$$\xi = \phi_{0,b}(\bar{\xi})$$

for some limit ordinal $\bar{\xi} < \mathrm{lh}(\mathcal{T})$. If c is a branch of $\mathcal{W}_b$ that is cofinal in ξ, then we have a unique branch $\bar{c}$ of $\mathcal{T}$ that is cofinal in $\bar{\xi}$ such that

$$\phi_{0,b}\text{``}\bar{c} \text{ is cofinal in } c\,.$$

If $\bar{c} \neq [0, \bar{\xi}]_T$, then $\bar{c}$ is illfounded in $\mathcal{T}$, so c is illfounded in $\mathcal{W}_b$. So all cofinal in ξ branches of $\mathcal{W}_b$ are illfounded, except possibly for $[0, \xi]_{W_b}$. Thus $[0, \xi]_{W_b}$ is wellfounded, as desired.

Since $\mathcal{M}^{\mathcal{U}}_b = \mathcal{M}^{\mathcal{W}_b}_\infty$, we have that $\mathcal{M}^{\mathcal{U}}_b$ is wellfounded, and II does not lose if he sets $\Sigma_2(\langle \mathcal{T}, \mathcal{U} \rangle) = b$.

This completes the definition of Σ_2 on stacks of length ≤ 2. Clearly, normalizations of stacks by Σ_2 are by Σ_1. Suppose now we have Σ_n where $n \geq 2$, and

$(*)_n$ whenever $\vec{\mathcal{T}}$ is an $\vec{F}$-stack of length $\leq n$ played by Σ_n, and having last model R, then there is a normal $\vec{F}$-iteration tree on V with last model R.

There is then exactly one such $\mathcal{T}$ by 1.25, and we write

$$\mathcal{T} = W(\vec{\mathcal{T}}).$$

We define Σ_{n+1} as follows: if $\vec{\mathcal{T}}^\frown\langle\mathcal{U}\rangle$ is a stack of length $\leq n+1$ played by Σ_{n+1},

$$\Sigma_{n+1}(\vec{\mathcal{T}}^\frown\langle\mathcal{U}\rangle) = \Sigma_2(\langle W(\vec{\mathcal{T}}),\mathcal{U}\rangle)).$$

Clearly, Σ_{n+1} is an $\vec{F}$-iteration strategy defined on stacks of length at most $n+1$, extending Σ_n. If $\vec{\mathcal{T}}^\frown\langle U\rangle$ is a stack on V by Σ_{n+1} with last model R, then $\langle W(\vec{\mathcal{T}}),\mathcal{U}\rangle$ is a 2-stack by Σ_2 with last model R, so $W(W(\vec{\mathcal{T}}),\mathcal{U})$ is a normal tree with last model R. Thus $(*)_{n+1}$ holds, and we can go on.

Let

$$\Sigma = \bigcup_n \Sigma_n.$$

We must show that Σ normalizes well. For this, the following notation is useful.

Definition 3.18 *(1) Let $\mathcal{W}$ be a normal iteration tree, and δ a limit ordinal. We say that b is a* δ-branch *of $\mathcal{W}$ iff $\delta = \sup\{lh(E_\alpha^{\mathcal{W}}) \mid \alpha+1 \in b\}$.*

(2) Let $\mathcal{W}$ and $\mathcal{U}$ be normal iteration trees, let b be a branch of $\mathcal{U}$ of limit order type (perhaps maximal), and let c be a branch of $\mathcal{W}$ (perhaps maximal). We say that b fits into *c iff for any extender F used in b, there is an extender G used in c such that* $\mathrm{crit}(G) \leq \mathrm{crit}(F) \leq \mathrm{lh}(F) \leq \mathrm{lh}(G)$.

Lemma 3.19 *Let $\mathcal{W}$ and $\mathcal{U}$ be normal iteration trees, and let δ be a limit ordinal; then*

(1) for any δ-branch c of $\mathcal{W}$, there is at most one δ-branch b of $\mathcal{U}$ such that b fits into c, and

(2) for any δ branch b of $\mathcal{U}$, there is at most one δ-branch c of $\mathcal{W}$ such that b fits into c.

Proof. Routine. □

Lemma 3.20 *Let $\langle\mathcal{T},\mathcal{U}\rangle$ be a stack of nice iteration trees on M, and b a cofinal branch of $\mathcal{U}$; then b fits into* $\mathrm{br}(b,\mathcal{T},\mathcal{U})$.

Proof. This is clear from the construction, and the fact that the copy maps are the identity in this coarse case. See the earlier diagrams of the extender tree of $W(\mathcal{T}, \mathcal{U})$. $\square$

By proposition 3.2, it is enough to show that all tails of Σ 2-normalize well. So let $\vec{\mathcal{S}}$ be a stack by Σ with last model Q, and let $\langle \mathcal{T}, \mathcal{U} \rangle$ be by $\Sigma_{\vec{\mathcal{S}},Q}$ with last model R. We must see that $W(\mathcal{T}, \mathcal{U})$ is by $\Sigma_{\vec{\mathcal{S}},Q}$, and that $\Sigma_{\vec{\mathcal{S}}^\frown \langle \mathcal{T}, \mathcal{U} \rangle, R} = \Sigma_{\vec{\mathcal{S}}^\frown \langle W(\mathcal{T}, \mathcal{U}) \rangle, R}$. Here we are making use of the fact that the normalization maps in this coarse case are all the identity.

The proof is by induction on $\mathrm{lh}(\mathcal{U})$, and the harder case is $\mathrm{lh}(\mathcal{U}) = \lambda + 1$ for some limit ordinal λ, so let us just handle that case. Let $b = [0, \lambda)_U$, and $\delta = \delta(\mathcal{U})$. Since $\vec{\mathcal{S}}^\frown \langle \mathcal{T}, \mathcal{U} \rangle$ is by Σ, we see from the definition of Σ that

$$\mathcal{W}_0 = W(W(\vec{\mathcal{S}}^\frown \langle \mathcal{T} \rangle), \mathcal{U})$$

is the unique normal $\vec{F}$-tree on V with last model $R = \mathcal{M}^{\mathcal{U}}_\lambda$. Moreover $\mathcal{W}_0$ chooses the δ-branch

$$a = \mathrm{br}(b, W(\vec{\mathcal{S}}^\frown \langle \mathcal{T} \rangle), \mathcal{U}),$$

and a is the unique δ-branch of $\mathcal{W}_0$ into which b fits. Let

$$c = \mathrm{br}(b, \mathcal{T}, \mathcal{U})$$

be the unique δ-branch of $W(\mathcal{T}, \mathcal{U} \restriction \lambda)$ such that b fits into c, and

$$d = \mathrm{br}(c, W(\vec{\mathcal{S}}), W(\mathcal{T}, \mathcal{U} \restriction \lambda))).$$

d is a δ-branch of

$$\mathcal{W}_1 = W(W(\vec{\mathcal{S}}), W(\mathcal{T}, \mathcal{U} \restriction \lambda)),$$

and c fits into d, so b fits into d. By our induction hypothesis, $\mathcal{W}_1$ is according to Σ_1. Because the copy maps are the identity, the common part model $\mathcal{M}(\mathcal{W}_1) = V_\delta^{W(\mathcal{T}, \mathcal{U} \restriction \lambda)} = V_\delta^R$. By our uniqueness lemma for normal $\vec{F}$-iterations, $\mathcal{W}_1$ is an initial segment of $\mathcal{W}_0$, so a is a δ-branch of $\mathcal{W}_1$. Since b fits into both a and d,

$$a = d.$$

But $\Sigma_{\vec{\mathcal{S}},Q}$ chose a branch of $W(\mathcal{T}, \mathcal{U} \restriction \lambda)$ that fit into a, so it chose c. Thus $W(\mathcal{T}, \mathcal{U})$ is by $\Sigma_{\vec{\mathcal{S}},Q}$.

Finally, let $\Phi = \Sigma_{\vec{\mathcal{S}}^\frown \langle \mathcal{T}, \mathcal{U} \rangle, R}$ and $\Psi = \Sigma_{\vec{\mathcal{S}}^\frown \langle W(\mathcal{T}, \mathcal{U}) \rangle, R}$. We must see that $\Phi = \Psi$. Let $\vec{\mathcal{V}}^\frown \langle \mathcal{Y} \rangle$ be played by both strategies, with $\mathcal{Y}$ of limit length. Let M be the last model of $\vec{\mathcal{V}}$, and let $\mathcal{W}$ be the normal tree on V with last model M. Let $\delta = \delta(\mathcal{Y})$. Let a be the δ-branch of $W(\mathcal{W}, \mathcal{Y})$. Both Φ and Ψ choose branches of $\mathcal{Y}$ that fit into a. So they agree on $\vec{\mathcal{V}}^\frown\langle\mathcal{Y}\rangle$.

□

We now show how to avoid Woodin's counterexample. Recall that if $\vec{F}$ is coarsely coherent, then each E in $\vec{F}$ is nice, so that $\mathrm{lh}(E)$ is inaccessible, but not measurable.

Theorem 3.21 *Let $\vec{F}$ be coarsely coherent, and suppose that V is uniquely $\theta, \vec{F}$-iterable; then V is strongly uniquely $\theta, \vec{F}$-iterable.*

Proof. Let Σ be the unique iteration strategy witnessing that V is uniquely $\theta, \vec{F}$-iterable. We must show that it witnesses strong uniqueness. Suppose not. We then have a stack $\langle \vec{\mathcal{T}}, \mathcal{U}\rangle$ by Σ, and a cofinal wellfounded branch b of $\mathcal{U}$ such that $\mathcal{M}_b^{\mathcal{U}}$ is wellfounded, but $\Sigma(\langle \vec{\mathcal{T}}, \mathcal{U}\rangle) \neq b$. By replacing $\vec{\mathcal{T}}$ with its normalization, we may assume that $\vec{\mathcal{T}} = \mathcal{T}$ is a single normal tree.

Let $\mathcal{W} = W(\mathcal{T}, \mathcal{U})$, and let

$$a = \mathrm{br}(b, \mathcal{T}, \mathcal{U}).$$

Since $\Sigma(\langle \mathcal{T}, \mathcal{U}\rangle \neq b$, $\Sigma(\mathcal{W}) \neq a$, and unique normal iterability then implies

$$\mathcal{M}_a^{\mathcal{W}} \text{ is illfounded.}$$

Let $\phi_{0,b}(\tau) = \mathrm{lh}(W(\mathcal{T}, \mathcal{U})$. We see then from the normalization construction that

$$\mathcal{M}_a^{\mathcal{W}} = \mathrm{Ult}(\mathcal{M}_\tau^{\mathcal{T}}, E_b),$$

where E_b is the extender of b.

We need some elementary covering properties of the models in $\mathcal{T}$. For $\eta < \mathrm{lh}(\mathcal{T})$, let

$$\nu_\eta = \sup(\{\mathrm{lh}(G) \mid G \text{ is used in } [0, \eta)_T\}).$$

It is clear that ν_η is either inaccessible or a limit of inaccessibles in $\mathcal{M}_\eta^{\mathcal{T}}$.

Claim 3.22 *Let $X \subseteq \mathcal{M}_\eta^{\mathcal{T}}$ be countable in V; then there is a $Y \supseteq X$ such that $Y \in \mathcal{M}_\eta^{\mathcal{T}}$ and $\mathcal{M}_\eta^{\mathcal{T}} \models |Y| \leq \nu_\eta$.*

Proof. There are $f_n \in V$, for $n < \omega$, such that every $x \in X$ is of the form $i_{0,\eta}(f_n)(a)$, for some $a \in [\nu_\eta]^{<\omega}$. So we can take $Y = \{i_{0,\eta}(f_n)(a) \mid n < \omega \text{ and } a \in [\nu_\eta]^{<\omega}\}$. □

Claim 3.23 *Suppose $\mathcal{M}_\eta \models$ "θ is regular but not measurable"; then θ has uncountable cofinality in V.*

Proof. We prove this by induction on η. It is trivial for $\eta = 0$. Suppose we have it for $\eta < \lambda$, where λ is a limit ordinal. Let θ be regular but not measurable in $\mathcal{M}_\lambda$, and let $\theta = i_{\alpha,\lambda}(\beta)$. By induction, $\mathrm{cof}^V(\beta) > \omega$. But $i_{\alpha,\lambda}$ is continuous at β, because β is regular but not measurable in $\mathcal{M}_\alpha$. Thus $\mathrm{cof}^V(\theta) > \omega$.

Finally, suppose the claim holds at η, and let θ be regular but not measurable in $\mathcal{M}_{\eta+1}$. Let $\nu = lh(E^{\mathcal{T}}_\eta) = \nu_{\eta+1}$. If $\theta < \nu$, then the agreement between $\mathcal{M}_\eta$ and $\mathcal{M}_{\eta+1}$ implies θ is regular but not measurable in $\mathcal{M}_\eta$, so $\mathrm{cof}^V(\theta) > \omega$ by induction. If $\theta = \nu$, then θ is regular but not measurable in $\mathcal{M}_\eta$ by our hypothesis on the extenders in $\vec{F}$, so again $\mathrm{cof}^V(\theta) > \omega$. Finally, if $\theta > \nu$ and $\mathrm{cof}^V(\theta) = \omega$, then θ is singular in $\mathcal{M}_{\eta+1}$ by claim 3.22, contradiction. □

Now let $\nu = \nu_{\tau+1} = \mathrm{lh}(E^{\mathcal{T}}_\tau)$. We have that $i^{\mathcal{U}}_b(\nu) \geq \delta(\mathcal{U})$, for if not, then $\phi_{0,b}(\tau) < \lambda$. (See 2.49, and the discussion near it.) But ν is regular and not measurable in $\mathcal{M}^{\mathcal{U}}_0 = \mathcal{M}^{\mathcal{T}}_\infty$, so $i^{\mathcal{U}}_b$ is continuous at ν. Moreover, $\mathrm{cof}^V(\nu) > \omega$, while $\mathrm{cof}^V(\delta(U)) = \omega$ because b is not the only cofinal branch of $\mathcal{U}$. Thus we can fix ρ such that

$$\rho < \nu \text{ and } i^{\mathcal{U}}_b(\rho) > \delta(\mathcal{U}).$$

Since the measures in E_b all concentrate on bounded subsets of ρ, we also have

$$\nu_\tau \leq \rho.$$

Let us fix a witness to the illfoundedness of $\mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_\tau, E_b)$, namely $f_n \in \mathcal{M}_\tau$ and $a_n \in [\delta(\mathcal{U})]^{<\omega}$ such that $\pi(f_{n+1})(a_{n+1}) \in \pi(f_n)(a_n)$ for all n, where

$$\pi\colon \mathcal{M}_\tau \to \mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_\tau, E_b)$$

is the canonical embedding. By 3.22, we can cover $\{f_n \mid n < \omega\}$ by a set $Y \in \mathcal{M}^{\mathcal{T}}_\tau$ such that $|Y| \leq \rho$ in $\mathcal{M}^{\mathcal{T}}_\tau$. Let $Y \subseteq N$, where N is a rank initial segment of $\mathcal{M}^{\mathcal{T}}_\tau$, and let P be the transitive collapse of $\mathrm{Hull}^N(Y \cup \rho)$. Letting g_n be the collapse of f_n, we see that

$$\mathrm{Ult}(P, E_b) \text{ is illfounded,}$$

as witnessed by the g_n's and a_n's. But $\mathcal{M}^{\mathcal{U}}_0$ agrees with $\mathcal{M}^{\mathcal{T}}_\tau$ up to ν, so

$$P \in \mathcal{M}^{\mathcal{U}}_0.$$

Further, $\mathrm{Ult}(P, E_b)$ embeds into $i^{\mathcal{U}}_b(P)$, so $i^{\mathcal{U}}_b(P)$ is wellfounded. But $i^{\mathcal{U}}_b(P)$ is wellfounded in $\mathcal{M}^{\mathcal{U}}_b$, so $\mathcal{M}^{\mathcal{U}}_b$ is illfounded, contradiction. □

Putting the last two theorems together, we get

Corollary 3.24 *Let $\vec{F}$ be coarsely coherent, and suppose that V is uniquely $\theta, \vec{F}$-iterable for normal trees; then V is strongly uniquely $\theta, \vec{F}$-iterable.*

In the theory of hod mice, it is important that strategies be moved to themselves by their own iteration maps. More precisely, we would like to know that if $i\colon M \to N$ comes from a stack of trees $\vec{\mathcal{T}}$ by Σ, then $i(\Sigma \cap M) = \Sigma_{\vec{\mathcal{T}},N} \cap N$. We shall obtain this from the corresponding property of coarse strategies Σ such that Σ witnesses that V is strongly uniquely $\theta, \vec{F}$-iterable.

Lemma 3.25 *Let $\vec{F}$ be coarsely coherent, and let Σ witness that V is strongly uniquely $\theta, \vec{F}$-iterable. Suppose that $i\colon V \to N$ comes from a stack of trees $\vec{\mathcal{T}}$ by Σ; then $i(\Sigma) = \Sigma_{\vec{\mathcal{T}},N} \cap N$.*

Proof. Both $i(\Sigma)$ and $\Sigma_{\vec{\mathcal{T}},N}$ choose wellfounded branches. Since these are unique (in V!), the two strategies cannot disagree. □

3.3 Fine strategies that normalize well

Next, we show that if Σ^* is an iteration strategy for a coarse N^* that normalizes well, then the strategies for premice induced by Σ^* via a full background extender construction also normalize well.

The reader should see the preliminaries section for our definitions and notation related to background constructions, and to the conversion of iteration strategies they mediate.

Theorem 3.26 *Let $\mathbb{C}$ be a w-construction done in some universe $N^* \models \mathsf{ZFC}$, and let Σ^* be a $\vec{F}^{\mathbb{C}}$-iteration strategy for finite stacks on N^*. Suppose that Σ^*normalizes well. Let M be a model of $\mathbb{C}$, and Σ its induced strategy; then Σ normalizes well.*

Remark 3.27 We believe that the proof of 3.26 works even if the construction $\mathbb{C}$ is allowed to use extenders that are not nice, so that embedding normalization does not coincide with full normalization at the background level. This just means that certain embeddings are no longer the identity, and hence must be given names in the proof to follow.

Proof. By 3.2, it is enough to show that all tails of Σ 2-normalize well. We consider first a 2-stack on $M^{\mathbb{C}}_{\nu_0,k_0}$ itself.

Let $\mathcal{T}$ be normal on $M^{\mathbb{C}}_{\nu_0,k_0}$, and $\mathcal{U}$ normal on the last model of $\mathcal{T}$, with $\langle \mathcal{T}, \mathcal{U} \rangle$ by Σ. Let $\langle \mathcal{T}^*, \mathcal{U}^* \rangle$ come from lifting $\langle \mathcal{T}, \mathcal{U} \rangle$ as above. We shall show that $W(\mathcal{T}, \mathcal{U})$

lifts to an initial segment of $W(\mathcal{T}^*,\mathcal{U}^*)$. (If $\mathcal{U}$ has limit length, $W(\mathcal{T},\mathcal{U})$ lifts to $W(\mathcal{T}^*,\mathcal{U}^*)$. If it has successor length, then dropping along the main branch of $\mathcal{U}$ can cause $W(\mathcal{T},\mathcal{U})$ to lift to a proper initial segment of $W(\mathcal{T}^*,\mathcal{U}^*)$.) Since $W(\mathcal{T}^*,\mathcal{U}^*)$ is by Σ^*, we get that $W(\mathcal{T},\mathcal{U})$ is by Σ.

More precisely, let

$$\text{lift}(\mathcal{T}, M_{\nu_0,k_0}, \mathbb{C}) = \langle \mathcal{T}^*, \langle \eta_\xi^{\mathcal{T}}, l_\xi^{\mathcal{T}} \mid \xi \leq \xi_0\rangle, \langle \psi_\xi^{\mathcal{T}} \mid \xi \leq \xi_0\rangle\rangle.$$

We are using "ψ" rather than "π" for the maps so as not to clash with our notation for embedding normalization.

Let

$$\text{lift}(\psi_{\xi_0}^{\mathcal{T}}\mathcal{U}, M_{\eta_{\xi_0}^{\mathcal{T}}, l_{\xi_0}^{\mathcal{T}}}^{i_{0,\xi_0}^{\mathcal{T}^*}(\mathbb{C})}, i_{0,\xi_0}^{\mathcal{T}^*}(\mathbb{C})) = \langle \mathcal{U}^*, \langle\langle \eta_\xi^{\mathcal{U}}, l_\xi^{\mathcal{U}}\rangle \mid \xi < \text{lh}\,\mathcal{U}\rangle, \langle \rho_\xi \mid \xi < \text{lh}\,\mathcal{U}\rangle\rangle.$$

Let $\tau_\xi : \mathcal{M}_\xi^{\mathcal{U}} \to \mathcal{M}_\xi^{(\theta_{\xi_0})\mathcal{U}}$ be the copy map, and

$$\psi_\xi^{\mathcal{U}} = \rho_\xi \circ \tau_\xi,$$

so that

$$\psi_\xi^{\mathcal{U}} : \mathcal{M}_\xi^{\mathcal{U}} \to Q_\xi,$$

where

$$Q_\xi = M_{\eta_\xi^{\mathcal{U}}, l_\xi^{\mathcal{U}}}^{i_{0,\xi}^{\mathcal{U}^*} \circ i_{0,\xi_0}^{\mathcal{T}^*}(\mathbb{C})}.$$

So $\psi_\xi^{\mathcal{U}}$ is the lifting map on $\mathcal{M}_\xi^{\mathcal{U}}$ given by our conversion of $\langle \mathcal{T},\mathcal{U}\rangle$ to $\langle \mathcal{T}^*,\mathcal{U}^*\rangle$.

The embedding normalization $W(\mathcal{T},\mathcal{U})$ has associated to it normal trees $\mathcal{W}_\gamma$ on $M_{\nu_0,k_0}^{\mathbb{C}}$, for $\gamma < \text{lh}\,\mathcal{U}$. We also have partial maps $\phi_{\eta,\gamma} : \text{lh}\,\mathcal{W}_\eta \to \text{lh}\,\mathcal{W}_\gamma$ for $\eta <_{\mathcal{U}} \gamma$, and for $\tau \in \text{dom}\,\phi_{\eta,\gamma}$, a map $\pi_\tau^{\eta,\gamma} : \mathcal{M}_\tau^{\mathcal{W}_\eta} \to \mathcal{M}_{\phi_{\eta,\gamma}(\tau)}^{\mathcal{W}_\gamma}$. We have $R_\gamma =$ last model of $\mathcal{W}_\gamma$, $\sigma_\gamma : \mathcal{M}_\gamma^{\mathcal{U}} \to R_\gamma$, and $F_\gamma = \sigma_\gamma(E_\gamma^{\mathcal{U}})$. $W(\mathcal{W}_\eta, F_\gamma) = \mathcal{W}_{\gamma+1}$, when $\eta = U\text{-pred}(\gamma+1)$.

Similarly, $W(\mathcal{T}^*,\mathcal{U}^*)$ has associated trees $\mathcal{W}_\gamma^*$ on N^* for $\gamma < \text{lh}\,\mathcal{U}^* = \text{lh}\,\mathcal{U}$, together with partial maps $\phi_{\eta,\gamma}^* : \text{lh}\,\mathcal{W}_\eta^* \to \text{lh}\,\mathcal{W}_\gamma^*$ for $\eta <_{U^*} \gamma$ (equivalently, $\eta <_U \gamma$), and for $\tau \in \text{dom}\,\phi_{\eta,\gamma}^*$, a map $\overset{*}{\pi}{}_\gamma^{\eta,\gamma}$. Since Σ^* normalizes well, the $\mathcal{W}_\gamma^*$ are by Σ^*; moreover, by 2.43, the last model of $\mathcal{W}_\gamma^*$ is $\mathcal{M}_\gamma^{\mathcal{U}^*}$. We have that $\mathcal{W}_{\gamma+1}^* = W(\mathcal{W}_\eta^*, E_\gamma^{\mathcal{U}^*})$ when $\eta = U^*\text{-pred}(\gamma+1)$ (equivalently, $\eta = U\text{-pred}(\gamma+1)$).

We shall prove that each $\mathcal{W}_\gamma$ lifts into $\mathcal{W}_\gamma^* \restriction \text{lh}\,\mathcal{W}_\gamma$, and hence is by Σ. The proof is by induction on γ, with a subinduction on initial segments of $\mathcal{W}_\gamma$. Basically, we are just showing that embedding normalization commutes with our conversion method. The proof is like the proof that embedding normalization commutes with copying given in 2.46, but there is more to it because in addition to copying, we are passing to resurrected background extenders. Nevertheless, the main quality required to put such a proof on paper is sufficient patience.

For $\gamma < \mathrm{lh}\,\mathcal{U}$, set

$$\mathrm{lift}(\mathcal{W}_\gamma, M_{\nu_0,k_0}, \mathbb{C}) = \langle \mathcal{S}^*_\gamma, \langle\langle \eta^\gamma_\xi, l^\gamma_\xi\rangle \mid \xi < \mathrm{lh}\,\mathcal{W}_\gamma\rangle, \langle \psi^\gamma_\xi \mid \xi < \mathrm{lh}\,\mathcal{W}_\gamma\rangle\rangle.$$

We shall show, among other things, that $\mathcal{S}^*_\gamma = \mathcal{W}^*_\gamma \restriction \mathrm{lh}\,\mathcal{W}_\gamma$, so that $\mathcal{W}_\gamma$ is by Σ.

As before, we write $z(\nu)$ for $\mathrm{lh}\,\mathcal{W}_\nu - 1$ and $z^*(\nu)$ for $\mathrm{lh}\,\mathcal{W}^*_\nu - 1$. We write ∞ for $z(\nu)$ or $z^*(\nu)$ when context permits. So $R_\nu = \mathcal{M}^{\mathcal{W}_\nu}_{z(\nu)} = \mathcal{M}^{\mathcal{W}_\nu}_\infty$, and if $(\nu,\gamma]_U$ does not drop, then $\phi_{\nu,\gamma}(z(\nu)) = z(\gamma)$, and $\pi^{\nu,\gamma}_{z(\nu)} = \pi^{\nu,\gamma}_\infty \colon R_\nu \to R_\gamma$.

Lemma 3.28 *Let $\gamma < \mathrm{lh}\,\mathcal{U}$. Then*

(1) $\mathcal{S}^*_\gamma = \mathcal{W}^*_\gamma \restriction \mathrm{lh}\,\mathcal{W}_\gamma$.

(2) *Whenever $\nu <_U \gamma$ and $(\nu,\gamma]_U$ does not drop in model or degree, then for all $\tau < \mathrm{lh}\,\mathcal{W}_\nu$,*

  (i) $\langle \eta^\gamma_{\phi_{\nu,\gamma}(\tau)}, l^\gamma_{\phi_{\nu,\gamma}(\tau)}\rangle = \overset{*}{\pi}{}^{\nu,\gamma}_\tau(\langle \eta^\nu_\tau, l^\nu_\tau\rangle)$, *and*

  (ii) $\psi^\gamma_{\phi_{\nu,\gamma}(\tau)} \circ \pi^{\nu,\gamma}_\tau = \overset{*}{\pi}{}^{\nu,\gamma}_\tau \circ \psi^\nu_\tau$.

(3) $\phi_{\eta,\nu} \subseteq \phi^*_{\eta,\nu}$, *if $\eta, \nu \le \gamma$ and $\eta \le_U \nu$.*

(4) (i) $\langle \eta^\gamma_{z(\gamma)}, l^\gamma_{z(\gamma)}\rangle = \langle \eta^{\mathcal{U}}_\gamma, l^{\mathcal{U}}_\gamma\rangle$, *and $i^{\mathcal{W}^*_\gamma}_{0,\infty_\gamma}(\mathbb{C})$ agrees with $i^{\mathcal{W}^*_\gamma}_{0,z^*(\gamma)}(\mathbb{C})$ at and below this point,*

  (ii) $\psi^\gamma_{z(\gamma)} \circ \sigma_\gamma = \psi^{\mathcal{U}}_\gamma$.

Proof.

Here is a diagram related to 3.28:

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{U}}_\gamma \arrow[r, "\sigma_\gamma"] \arrow[rr, bend left, "\psi^{\mathcal{U}}_\gamma"] & R_\gamma \arrow[r, "\psi^\gamma_\infty"] & Q_\gamma \in \mathcal{M}^{\mathcal{S}^*_\gamma}_\infty \\
\mathcal{M}^{\mathcal{U}}_\nu \arrow[u, "i^{\mathcal{U}}_{\nu,\gamma}"] \arrow[r, "\sigma_\nu"] \arrow[rr, bend right, "\psi^{\mathcal{U}}_\nu"'] & R_\nu \arrow[u, "\pi^{\nu,\gamma}_\infty"] \arrow[r, "\psi^\nu_\infty"] & Q_\nu \in \mathcal{M}^{\mathcal{S}^*_\nu}_\infty \arrow[u, "\overset{*}{\pi}{}^{\nu,\gamma}_\infty"]
\end{tikzcd}$$

The fact that ψ^γ_∞ maps to Q_γ is (i). The fact that the triangle on the top commutes is (ii). That the square on the right commutes is (2), in the case $\tau = z(\nu)$. We of course need (2) at other τ as well. That square on the left commutes is a basic fact about embedding normalization.

The reader might look back at the diagram near the end of the proof of 2.47. $\mathcal{M}^{\mathcal{U}^*}_\nu$ in that diagram corresponds to Q_ν in the present one. We can take R^*_ν of that diagram to also be Q_ν in the present one, because our tree on the background universe is nice. We don't actually need that; if the background extenders were not nice, then in the present case we would be introducing some $\sigma^*_\nu\colon Q_\nu \to R^*_\nu$ via the embedding normalization of $\langle \mathcal{T}^*, \mathcal{U}^*\rangle$. ψ^ν_∞ would map into R^*_ν, rather than Q_ν, and the present diagram would transform into the previous one. (See remark 3.27 above.)

We prove 3.28 by induction on γ. For $\gamma = 0$, $\mathcal{W}_0 = \mathcal{T}$ and $\mathcal{W}^*_0 = \mathcal{T}^*$, so (1) holds; moreover, $\langle \eta^0_\xi, l^0_\xi\rangle = \langle \eta^{\mathcal{T}}_\xi, l^{\mathcal{T}}_\xi\rangle$ and $\psi^0_\xi = \psi^{\mathcal{T}}_\xi$. (2) and (3) are vacuous. (4) holds: in this case, $z(0) = z^*(0) = \mathrm{lh}(\mathcal{T}) - 1$, and $\langle \eta^0_{z(0)} l^0_{z(0)}\rangle = \langle \eta^{\mathcal{U}}_0, l^{\mathcal{U}}_0\rangle$ because $\mathcal{U}$ is on the last model of $\mathcal{T}$. That gives (i). For (ii), $\psi^{\mathcal{U}}_0 = \rho_0 \circ \tau_0 = \psi^{\mathcal{T}}_{\xi_0}$, since $\rho_0 =$ identity and $\tau_0 = \psi^{\mathcal{T}}_{\xi_0}$. But $\sigma_0 =$ identity, so $\psi^{\mathcal{U}}_0 = \psi^0_{\xi_0} \circ \sigma_0$, as desired.

Now suppose Lemma 3.28 is true at all $\nu \leq \gamma$. We show it at $\gamma + 1$. Let $\nu = U\text{-pred}(\gamma + 1)$, and

$$\begin{aligned}\alpha &= \alpha^{\mathcal{T},\mathcal{U}}_\gamma \\ &= \text{least } \tau \text{ such that } F_\gamma \text{ is on the } \mathcal{M}^{\mathcal{W}_\gamma}_\tau\text{-sequence.}\end{aligned}$$

Set $F = F_\gamma$. So

$$\begin{aligned}\mathcal{W}_{\gamma+1} &= W(\mathcal{W}_\nu, F) \\ &= \mathcal{W}_\gamma \restriction (\alpha+1)^\frown \langle F\rangle^\frown i_F\text{``}\mathcal{W}^{>\mathrm{crit}(F)}_\nu.\end{aligned}$$

Then $\nu = U^*\text{-pred}(\gamma + 1)$, and

$$\mathcal{W}^*_{\gamma+1} = W(\mathcal{W}^*_\nu, E^{\mathcal{U}^*}_\gamma).$$

$E^{\mathcal{U}^*}_\gamma$ came from lifting $E^{\mathcal{U}}_\gamma$ by $\psi^{\mathcal{U}}_\gamma$, and then resurrecting it, and using the background extender for that. More precisely, let $\psi^{\mathcal{U}}_\gamma(E^{\mathcal{U}}_\gamma)$ be the last extender of

$$Q_\gamma|\langle\theta, 0\rangle =_{\mathsf{def}} \bar{P}$$

and

$$G = \sigma_{\langle \eta^{\mathcal{U}}_\gamma, l^{\mathcal{U}}_\gamma\rangle}[\bar{P}](\psi^{\mathcal{U}}_\gamma(E^{\mathcal{U}}_\gamma)).$$

Set

$$G^* = \text{background extender for } G \text{ provided by } i^{\mathcal{U}^*}_{0,\gamma} \circ i^{\mathcal{T}^*}_{0,\xi_0}(\mathbb{C}) = \mathbb{C}^{M^{\mathcal{U}^*}_\gamma}.$$

Then $E_\gamma^{\mathcal{U}^*} = G^*$, and

$$\mathcal{W}^*_{\gamma+1} = W(\mathcal{W}^*_\nu, G^*).$$

Recall that $\alpha = \alpha(\mathcal{W}_\gamma, F)$.

Claim 3.29 *$\alpha = \alpha(\mathcal{W}^*_\gamma, G^*)$, and G^* is the background extender for $\sigma \circ \psi^\gamma_\alpha(F)$ provided by $i_{0\alpha}^{\mathcal{W}^*_\gamma}(\mathbb{C})$, where σ is the resurrection map $\sigma_{\langle \eta^\gamma_\alpha, l^\gamma_\alpha\rangle}[M_{\langle \eta^\gamma_\alpha, l^\gamma_\alpha\rangle} \parallel \langle \mathrm{lh}\, \psi^\gamma_\alpha(F), 0\rangle]$ of $i_{0\alpha}^{\mathcal{W}^*_\gamma}(\mathbb{C})$.*

Proof. F is on the $\mathcal{M}_\alpha^{\mathcal{W}_\gamma}$-sequence, so there is a background extender H^* for $\sigma \circ \psi^\gamma_\alpha(F)$ provided by $i_{0,\alpha}^{\mathcal{W}^*_\gamma}(\mathbb{C})$. The extender $E_\alpha^{\mathcal{W}^*_\gamma}$ used to exit $\mathcal{M}_\alpha^{\mathcal{W}^*_\gamma}$ comes from lifting and resurrecting $E_\alpha^{\mathcal{W}_\gamma}$. But F comes before $E_\alpha^{\mathcal{W}_\gamma}$, so H^* comes before $E_\alpha^{\mathcal{W}^*_\gamma}$ in $i_{0,\alpha}^{\mathcal{W}^*_\gamma}(\mathbb{C})$. But letting $E_\alpha^{\mathcal{W}^*_\gamma} = F_\theta^{i_{0,\alpha}^{\mathcal{W}^*_\gamma}(\mathbb{C})}$, we then have

$$i_{0,\alpha}^{\mathcal{W}^*_\gamma}(\mathbb{C}) \restriction \theta = i_{0,\tau}^{\mathcal{W}^*_\gamma}(\mathbb{C}) \restriction \theta$$

for all $\tau \geq \alpha$, and in particular, for $\tau + 1 = \mathrm{lh}\, \mathcal{W}_\gamma$. Moreover, the part of the lifting and resurrecting maps acting on F does not change from α to τ:

$$\sigma \circ \psi^\gamma_\alpha(F) = \sigma' \circ \psi^\gamma_\tau(F),$$

where σ' is appropriate for resurrecting $\psi^\gamma_\tau(F)$ in $\mathcal{M}_\tau^{\mathcal{W}^*_\gamma}$, and hence also in $\mathcal{M}_{\tau^*}^{W^*_\gamma} = \mathcal{M}_\gamma^{\mathcal{U}^*}$. But our inductive hypothesis says

$$\begin{aligned} \psi^\gamma_\tau(F) &= \psi^\gamma_\tau \circ \sigma_\gamma(E_\gamma^{\mathcal{U}}) \\ &= \psi_\gamma^{\mathcal{U}}(E_\gamma^{\mathcal{U}}), \end{aligned}$$

so $\sigma \circ \psi^\gamma_\alpha(F) = \sigma' \circ \psi^\gamma_\tau(F) = G$. Thus $H^* = G^*$. Hence $\alpha(W^*_\gamma, G^*) \leq \alpha$.

But suppose $G^* \in i_{0\xi}^{\mathcal{W}^*_\gamma}(\mathbb{C})$ for some $\xi < \alpha$. Since $\mathrm{lh}\, E_\xi^{\mathcal{W}_\gamma} < \mathrm{lh}\, F$, $\mathrm{lh}(E_\xi^{\mathcal{W}^*_\gamma}) < \mathrm{lh}\, G^*$, and so G^* occurs after $E_\xi^{W^*_\gamma}$ in $i_{0,\xi}^{\mathcal{W}^*_\gamma}(\mathbb{C})$. So $\mathcal{M}_\beta^{\mathcal{W}^*_\gamma}$ does not compute $V_{\mathrm{lh}\, G^*}$ the same way that $\mathcal{M}_\xi^{\mathcal{W}^*_\gamma}$ does, for all $\beta > \xi$. This implies $G^* \notin i_{0\beta}^{\mathcal{W}^*_\gamma}(\mathbb{C})$, for all $\beta > \xi$, contrary to $G^* \in i_{0,\tau}^{\mathcal{W}^*_\gamma}(\mathbb{C})$ for $\tau + 1 = \mathrm{lh}\, \mathcal{W}^*_\gamma$.

This shows $\alpha = \alpha(W^*_\gamma, G^*)$. In the course of the proof we also showed the rest of Claim 3.29. □

Claim 3.30 *1. The iteration tree in* $\mathrm{lift}(\mathcal{W}_\gamma \restriction (\alpha+1)^\frown \langle F\rangle, M_{\nu_0,k_0}, \mathbb{C})$ *is* $\mathcal{W}^*_\gamma \restriction (\alpha + 1)^\frown \langle G^*\rangle$.

2. $\beta = \beta^{\mathcal{W}^*_\gamma, G^*}$.

Proof. Part 1 is just Claim 3.30 restated. Part 2 follows at once from the fact that the lifted tree is normal; cf. 1.27. $\square$

Since $\alpha(W_\gamma, F) = \alpha(\mathcal{W}^*_\gamma, G)$ and $\beta^{\mathcal{W}_\gamma, F} = \beta^{\mathcal{W}^*_\gamma, G^*}$, we have $\phi_{\nu,\gamma+1} \subseteq \phi^*_{\nu,\gamma+1}$.

Remark 3.31 *If* $D^{\mathcal{U}} \cap [0, \gamma+1]_U = \varnothing$, *then* $\mathrm{lh}\, \mathcal{W}_{\gamma+1} = \mathrm{lh}\, W^*_{\gamma+1}$, *and* $\phi_{\nu,\gamma+1} = \phi^*_{\nu,\gamma+1}$.

We now show that (1) and (2) of Lemma 3.28 hold at $\gamma + 1$. For this, we show by induction on ξ that for $]\ \xi \leq \mathrm{lh}\, \mathcal{W}_\gamma$, letting $\mathcal{S}^* = \mathcal{S}^*_{\gamma+1}$,

Induction Hypothesis $(\dagger)_\xi$**:**

(1) $\mathcal{S}^* \restriction \xi = \mathcal{W}^*_{\gamma+1}$

(2) if $(\nu, \gamma+1]_U$ does not drop in model or degree, and $\phi_{0,\gamma+1}(\tau) < \xi$, then

  (a) $\langle \eta^{\gamma+1}_{\phi_{0,\gamma+1}(\tau)}, l^{\gamma+1}_{\phi_{0,\gamma+1}(\tau)} \rangle = \overset{*}{\pi}{}^{\nu,\gamma+1}_\tau(\langle \eta^\nu_\tau, l^\nu_\tau \rangle)$, and

  (b) $\psi^{\gamma+1}_{\phi_{0,\gamma+1}(\tau)} \circ \pi^{\nu,\gamma+1}_\gamma = \overset{*}{\pi}{}^{\nu,\gamma+1}_\tau \circ \psi^\nu_\tau$.

Note that the limit step in the inductive proof of $(\dagger)_\xi$ is trivial.

Base Case 1. $\xi = \alpha + 1$.

We have $\mathcal{W}_{\gamma+1} \restriction (\alpha+1) = \mathcal{W}_\gamma \restriction (\alpha+1)$ and $\mathcal{W}^*_{\gamma+1} \restriction (\alpha+1) = \mathcal{W}^*_\gamma \restriction (\alpha+1)$. Since Lemma 3.28 holds at γ, we get $(\dagger)_\xi(1)$. For $(\dagger)_\xi(2)$, let $\phi_{\nu,\gamma+1}(\tau) < \alpha + 1$. Then $\tau < \beta$ and $\phi_{\nu,\gamma+1}(\tau) = \tau$. Moreover $\pi^{\nu,\gamma+1}_\tau$ and $\overset{*}{\pi}{}^{\nu,\gamma+1}_\tau$ are the identity. So $(\dagger)_\xi(2)$ boils down to $\langle \eta^{\gamma+1}_\tau, l^{\gamma+1}_\tau \rangle = \langle \eta^\gamma_\tau, l^\gamma_\tau \rangle$, and $\psi^{\gamma+1}_\tau = \psi^\gamma_\tau$. This holds because $\mathcal{W}_\nu \restriction (\tau+1) = \mathcal{W}_{\gamma+1} \restriction (\tau+1)$, so their lifts are equal.

Base Case 2. $\xi = \alpha + 2$.

We have

$$\mathcal{W}_{\gamma+1} \restriction (\alpha + 2) = \mathcal{W}_{\gamma+1} \restriction (\alpha + 1)^\frown \langle F \rangle$$

and

$$\mathcal{W}^*_{\gamma+1} \restriction (\alpha + 2) = \mathcal{W}^*_{\gamma+1} \restriction (\alpha + 1)^\frown \langle G^* \rangle.$$

By Claim 3.29, G^* is the background extender for $\sigma \circ \psi^{\gamma+1}_\alpha(F)$ provided by $i^{\mathcal{W}^*_{\gamma+1}}_{0,\alpha}(\mathbb{C})$. So

$$\begin{aligned} \mathcal{S}^* \restriction (\alpha + 2) &= \mathcal{S}^* \restriction (\alpha + 1)^\frown \langle G^* \rangle \\ &= \mathcal{W}^*_{\gamma+1} \restriction (\alpha + 2), \end{aligned}$$

and we have $(\dagger)_\xi(1)$. (Note that G^* is applied to $\mathcal{M}_\beta^{\mathcal{S}^*}$ in $\mathcal{S}^*$, because lifting produces normal trees.)

For $(\dagger)_\xi(2)$, the new case to consider is $\tau = \beta$. Note that

$$\psi_\beta^\nu = \psi_\beta^{\nu+1},$$
$$\pi_\beta^{\nu,\gamma+1} = i_{\beta,\alpha+1}^{\mathcal{W}_{\gamma+1}}$$

and

$$\pi_\beta^{*\nu,\gamma+1} = i_{\beta,\alpha+1}^{\mathcal{W}^*_{\gamma+1}}.$$

The first because $\mathcal{W}_{\gamma+1}\restriction(\beta+1) = \mathcal{W}_\nu\restriction(\beta+1)$, and the second two by our definition of embedding normalization. (Note we are in the case that $(\beta, \alpha+1]_{\mathcal{W}_{\gamma+1}}$ is not a drop in model or degree.) But

$$\psi_{\alpha+1}^{\gamma+1} \circ i_{\beta,\alpha+1}^{\mathcal{W}_{\gamma+1}} = i_{\beta,\alpha+1}^{\mathcal{W}^*_{\gamma+1}} \circ \psi_\beta^{\gamma+1}$$

holds because lifting maps commute with the tree embedding in a conversion system. This gives

$$\psi_{\alpha+1}^{\gamma+1} \circ \pi_\beta^{\nu,\gamma+1} = \pi_\beta^{*\nu,\gamma+1} \circ \psi_\beta^\nu$$

as desired.

If $\mathrm{lh}\,\mathcal{W}_\nu = \beta+1$ or $\gamma+1 \in D^{\mathcal{U}}$ or $\deg^{\mathcal{U}}(\gamma+1) < \deg^{\mathcal{U}}(\nu)$, then $\mathrm{lh}\,\mathcal{W}_{\gamma+1} = \alpha+2$, so we are done. So suppose $\mathrm{lh}\,\mathcal{W}_\nu > \beta+1$, and $(\nu, \gamma+1]_U$ is not a drop of any kind in $\mathcal{U}$.

Inductive Case 1. $(\dagger)_{\xi+1}$ holds, and $\xi \geq \alpha+1$.

We must prove $(\dagger)$ at $\xi+2$. We are assuming $\xi+1 < \mathrm{lh}\,\mathcal{W}_{\gamma+1}$. Let

$$E = E_\xi^{\mathcal{W}_{\gamma+1}}.$$

Let σ be the resurrection map for $\psi_\xi^{\gamma+1}(E)$ in the construction of $\mathcal{M}_\xi^{\mathcal{S}^*} = \mathcal{M}_\xi^{\mathcal{W}^*_{\gamma+1}}$, namely

$$\sigma = \sigma^{i_{0\xi}^{\mathcal{S}^*}(\mathbb{C})}_{\langle \eta_\xi^{\gamma+1}, l_\xi^{\gamma+1}\rangle}[M_{\langle \eta_\xi^{\gamma+1}, l_\xi^{\gamma+1}\rangle}|\langle \mathrm{lh}\,\psi_\xi^{\gamma+1}(E), 0\rangle].$$

Let

$$E^* = \text{background extender for } \sigma \circ \psi_\xi^{\gamma+1}(E) \text{ provided by } i_{0\xi}^{\mathcal{S}^*}(\mathbb{C}).$$

So

$$\mathcal{S}^*\restriction(\xi+2) = \mathcal{S}^*\restriction(\xi+1)^\frown\langle E^*\rangle.$$

Claim 3.32 $E^* = E_\xi^{\mathcal{W}^*_{\gamma+1}}$.

Proof. Since $\xi \geq \alpha + 1$, we can write

$$\xi = \phi_{\nu,\gamma+1}(\bar{\xi}), \quad \bar{\xi} \geq \beta$$

Let

$$\bar{E} = E_{\bar{\xi}}^{\mathcal{W}_\nu},$$

so that

$$E = \pi_{\bar{\xi}}^{\nu,\gamma+1}(\bar{E}).$$

Letting $H = \sigma \circ \psi_\xi^{\gamma+1}(E)$, we have

$$\begin{aligned} H &= \sigma \circ (\psi_\xi^{\gamma+1} \circ \pi_{\bar{\xi}}^{\nu,\gamma+1}(\bar{E})) \\ &= \sigma \circ (\overset{*}{\pi}{}_{\bar{\xi}}^{\nu,\gamma+1} \circ \psi_{\bar{\xi}}^{\nu}(\bar{E})) \end{aligned}$$

by induction. Let $\bar{\sigma}$ be the resurrection map for $\psi_{\bar{\xi}}^\nu(\bar{E})$ in the $\mathcal{M}_{\bar{\xi}}^{\mathcal{S}^*} = \mathcal{M}_{\bar{\xi}}^{\mathcal{W}^*_\nu}$ construction, i.e.

$$\bar{\sigma} = \sigma_{\langle \eta_{\bar{\xi}}^\nu, l_{\bar{\xi}}^\nu \rangle}^{i_{0\nu}^{\mathcal{S}^*}(\mathbb{C})}[M_{\langle \eta_{\bar{\xi}}^\nu, l_{\bar{\xi}}^\nu \rangle} | \langle \mathrm{lh}\, \psi_{\bar{\xi}}^\nu(\bar{E}), 0 \rangle].$$

It is not hard to see that

$$\overset{*}{\pi}{}_{\bar{\xi}}^{\nu,\gamma+1}(\bar{\sigma}) = \sigma.$$

This is because $\overset{*}{\pi}{}_{\bar{\xi}}^{\nu,\gamma+1}(\langle \eta_{\bar{\xi}}^\nu, l_{\bar{\xi}}^\nu \rangle) = \langle \eta_\xi^{\gamma+1}, l_\xi^{\gamma+1} \rangle$ by induction hypothesis (2)(a), and similarly $\overset{*}{\pi}{}_{\bar{\xi}}^{\nu,\gamma+1}(\psi_{\bar{\xi}}^\nu(\bar{E})) = \psi_\xi^{\gamma+1}(\pi_{\bar{\xi}}^{\nu,\gamma+1}(\bar{E})) = \psi_\xi^{\gamma+1}(E)$. But then

$$\begin{aligned} E_\xi^{\mathcal{W}^*_{\gamma+1}} &= \overset{*}{\pi}{}_{\bar{\xi}}^{\nu,\gamma+1}(E_{\bar{\xi}}^{\mathcal{W}^*_\nu}) \\ &= \overset{*}{\pi}{}_{\bar{\xi}}^{\nu,\gamma+1}(\text{background for } \bar{\sigma}(\psi_{\bar{\xi}}^\nu(\bar{E})) \text{ in } i_{0,\bar{\xi}}^{\mathcal{W}^*_\nu}(\mathbb{C})) \\ &= \text{background for } \overset{*}{\pi}{}_{\bar{\xi}}^{\nu,\gamma+1}(\bar{\sigma}(\psi_{\bar{\xi}(\bar{E})}^\nu)) \text{ in } i_{0\xi}^{\mathcal{W}^*_{\gamma+1}}(\mathbb{C}) \\ &= \text{background for } \sigma(\overset{*}{\pi}{}_{\bar{\xi}}^{\nu,\gamma+1}(\psi_{\bar{\xi}}^\nu(\bar{E}))) \text{ in } i_{0\xi}^{\mathcal{W}^*_{\gamma+1}}(\mathbb{C}) \\ &= \text{background for } H \text{ in } i_{0\xi}^{\mathcal{W}^*_{\gamma+1}}(\mathbb{C}) \\ &= E^* \end{aligned}$$

as desired. □

From Claim 3.32, we have that $\mathcal{S}^* \restriction (\xi + 2)$ is the unique normal continuation of $\mathcal{S}^* \restriction (\xi + 1) = \mathcal{W}^*_{\gamma+1} \restriction (\xi + 1)$ via $E_\xi^{\mathcal{W}^*_{\gamma+1}}$. That is, $\mathcal{S}^* \restriction (\xi + 2) = \mathcal{W}^*_{\gamma+1} \restriction (\xi + 2)$.

It remains to show, keeping our previous notation:

Claim 3.33 $\psi^{\gamma+1}_{\xi+1} \circ \pi^{\nu,\gamma+1}_{\bar{\xi}+1} = \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\xi}+1} \circ \psi^{\gamma}_{\bar{\xi}+1}$.

Proof. Both maps act on $\mathcal{M}^{\mathcal{W}_\nu}_{\bar{\xi}+1}$. The right hand side embeds it elementarily into $M_{\eta',l'}$ of $i^{\nu,\gamma+1}_{0,\xi+1}(\mathbb{C})$, where

$$\langle \eta', l' \rangle = \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\xi}+1}(\langle \eta^{\nu}_{\bar{\xi}+1}, l^{\nu}_{\bar{\xi}+1} \rangle)$$

The right hand side embeds $M^{W_\nu}_{\bar{\xi}+1}$ elementarily into $M_{\langle \eta^{\gamma+1}_{\xi+1}, l^{\gamma+1}_{\xi+1} \rangle}$ of $i^{\mathcal{W}^*_{\gamma+1}}_{0,\xi+1}(\mathbb{C})$. So first we show $(\dagger)_{\xi+1}(2)(a)$:

Subclaim 3.33.1 $\langle \eta^{\gamma+1}_{\xi+1}, l^{\gamma+1}_{\xi+1} \rangle = \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\xi}+1}(\langle \eta^{\nu}_{\bar{\xi}+1}, l^{\nu}_{\bar{\xi}+1} \rangle)$.

Proof. Let

$$\begin{aligned} \theta &= \mathcal{W}_{\gamma+1}\text{-pred}(\xi+1) \\ &= \mathcal{W}^*_{\gamma+1}\text{-pred}(\xi+1) \\ &= \mathcal{S}^*_{\gamma+1}\text{-pred}(\xi+1). \end{aligned}$$

<u>Case 1</u>. $\mathrm{crit}(\bar{E}) \geq \mathrm{crit}(F_\gamma)$, or $\theta < \beta$.

In this case, $\theta = \phi_{\nu,\gamma+1}(\bar{\theta}) = \phi^*_{\nu,\gamma+1}(\bar{\theta})$ for $\bar{\theta} = \mathcal{W}_\nu\text{-pred}(\bar{\xi}+1)$. We have

$$\mathcal{M}^{\mathcal{W}_\nu}_{\bar{\xi}+1} = \mathrm{Ult}(\bar{P}, \bar{E}),$$

where $\bar{P} \trianglelefteq \mathcal{M}^{\mathcal{W}_\nu}_{\bar{\theta}}$. Let

$$P = \pi^{\nu,\gamma+1}_{\bar{\theta}}(\bar{P}).$$

Embedding normalization leads to

$$\mathcal{M}^{\mathcal{W}_{\gamma+1}}_{\xi+1} = \mathrm{Ult}(P, E),$$

where recall $E = \pi^{\nu,\gamma+1}_{\bar{\xi}}(\bar{E})$. Letting ρ be the resurrection map for P in $\mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{\theta}$, i.e.

$$\rho = \sigma_{\langle \eta^{\gamma+1}_{\theta}, l^{\gamma+1}_{\theta} \rangle}[\psi^{\gamma+1}_{\theta}(P)]^{i^{W^*_{\gamma+1}}_{0,\theta}(\mathbb{C})},$$

mapping $\psi^{\gamma+1}_{\theta}(P)$ into $M_{\eta,l}$ of $i^{\mathcal{W}^*_{\gamma+1}}_{0,\theta}(\mathbb{C})$, where

$$\langle \eta, l \rangle = \mathrm{Res}_{\langle \eta^{\gamma+1}_{\theta}, l^{\gamma+1}_{\theta} \rangle}[\psi^{\gamma+1}_{\theta}(P)],$$

we have

$$\langle \eta^{\gamma+1}_{\xi+1}, l^{\gamma+1}_{\xi+1} \rangle = i^{\mathcal{W}^*_{\gamma+1}}_{\theta,\xi+1}(\langle \eta, l \rangle),$$

because $\mathcal{W}^*_{\gamma+1}|(\xi{+}2) = \mathcal{S}^*|(\xi{+}2)$ is a conversion system. Note that $\overset{*}{\pi}^{\nu,\gamma+1}_{\bar{\theta}}(\langle \eta^{\gamma+1}_{\bar{\theta}}, l^{\gamma+1}_{\bar{\theta}} \rangle) = \langle \eta^{\gamma+1}_{\theta}, l^{\gamma+1}_{\theta} \rangle$ by induction. (I.e. Subclaim 3.33.1 at $\bar{\theta}$ instead of $\bar{\xi}$.) Also, $\overset{*}{\pi}^{\nu,\gamma+1}_{\bar{\theta}}(\psi^{\nu}_{\bar{\theta}}(\bar{P})) = \psi^{\gamma+1}_{\theta}(\pi^{\nu,\gamma+1}_{\bar{\theta}}(\bar{P})) = \psi^{\gamma+1}_{\theta}(P)$. It follows that

$$\langle \eta, l \rangle = \overset{*}{\pi}^{\nu,\gamma+1}_{\bar{\theta}}(\mathrm{Res}_{\eta^{\nu}_{\bar{\theta}}, l^{\nu}_{\bar{\theta}}}[\psi^{\nu}_{\bar{\theta}}(\bar{P})]^{i^{W^*_{\nu}}_{0,\bar{\theta}}(\mathbb{C})}).$$

Thus

$$\begin{aligned}
\langle \eta^{\gamma+1}_{\xi+1}, l^{\gamma+1}_{\xi+1} \rangle &= i^{\mathcal{W}^*_{\gamma+1}}_{\theta,\xi+1}(\langle \eta, l \rangle) \\
&= i^{W^*_{\gamma+1}}_{\theta,\xi+1} \circ \overset{*}{\pi}^{\nu,\gamma+1}_{\bar{\theta}}(\mathrm{Res}_{\eta^{\nu}_{\bar{\theta}}, l^{\nu}_{\bar{\theta}}}[\psi^{\nu}_{\bar{\theta}}(\bar{P})]) \\
&= \overset{*}{\pi}^{\nu,\gamma+1}_{\bar{\xi}+1} \circ i^{\mathcal{W}^*_{\nu}}_{\bar{\theta},\bar{\xi}+1}(\mathrm{Res}_{\eta^{\nu}_{\bar{\theta}}, l^{\nu}_{\bar{\theta}}}[\psi^{\nu}_{\bar{\theta}}(\bar{P})]) \\
&= \overset{*}{\pi}^{\nu,\gamma+1}_{\bar{\xi}+1}(\langle \eta^{\nu}_{\bar{\xi}+1}, l^{\nu}_{\bar{\xi}+1} \rangle),
\end{aligned}$$

as desired.

Case 2. $\theta = \beta$, and $\mathrm{crit}(\bar{E}) < \mathrm{crit}(F)$.

In this case, $\mathcal{W}_{\nu}$-pred$(\bar{\xi}{+}1) = \mathcal{W}_{\gamma+1}$-pred$(\bar{\xi}{+}1) = \beta$. The argument above works, with $\bar{\theta} = \theta = \beta$ and $\bar{P} = P$, and $\pi^{\nu,\gamma+1}_{\bar{\theta}}$ and $\overset{*}{\pi}^{\nu,\gamma+1}_{\bar{\theta}}$ replaced by the identity map. (As they are if $\bar{\theta} < \beta$, this case is like the case $\bar{\theta} < \beta$.) The relevant calculation is

$$\begin{aligned}
\langle \eta^{\gamma+1}_{\xi+1}, l^{\gamma+1}_{\xi+1} \rangle &= i^{\mathcal{W}^*_{\gamma+1}}_{\beta,\xi+1}(\mathrm{Res}_{\eta^{\gamma+1}_{\beta}, l^{\gamma+1}_{\beta}}[\psi^{\gamma+1}_{\beta}(P)]) \\
&= i^{\mathcal{W}^*_{\gamma+1}}_{\beta,\xi+1}(\mathrm{Res}_{\eta^{\nu}_{\beta}, l^{\nu}_{\beta}}[\psi^{\nu}_{\beta}(P)]) \\
&= \overset{*}{\pi}^{\nu,\gamma+1}_{\bar{\xi}+1} \circ i^{\mathcal{W}^*_{\nu}}_{\beta,\bar{\xi}+1}(\mathrm{Res}_{\eta^{\nu}_{\beta}, l^{\nu}_{\beta}}[\psi^{\nu}_{\beta}(P)]) \\
&= \overset{*}{\pi}^{\nu,\gamma+1}_{\bar{\xi}+1}(\langle \eta^{\nu}_{\bar{\xi}+1}, l^{\nu}_{\bar{\xi}+1} \rangle).
\end{aligned}$$

The first equation holds because $\mathcal{W}^*_{\gamma+1}\restriction(\xi+2) = \mathcal{S}^*\restriction(\xi+2)$ is a conversion system. The second comes from $\mathcal{W}^*_{\gamma+1}\restriction(\beta+1) = \mathcal{S}^*\restriction(\beta+1)$. The third comes from properties of embedding normalization. The last comes from $\mathcal{W}^*_{\nu}$ being a conversion system. $\square$

We now finish proving Claim 3.33. We keep the notation above. Let us assume that we in Case 1. Let $x \in \mathcal{M}^{\mathcal{W}_{\nu}}_{\bar{\xi}+1}$ be arbitrary, and let

$$x = [a, f]^{\bar{P}}_{\bar{E}},$$

where $a \subseteq h_{\bar{E}}$ is finite and $f \in \bar{P}$. (We assume $k(\bar{P}) = 0$ for simplicity.) Then

$$\begin{aligned}\psi^{\gamma+1}_{\xi+1} \circ \pi^{\nu,\gamma+1}_{\bar{\xi}+1}(x) &= \psi^{\gamma+1}_{\xi+1}(\pi^{\nu,\gamma+1}_{\bar{\xi}+1}([a, f]^{\bar{P}}_{\bar{E}})) \\ &= \psi^{\gamma+1}_{\xi+1}([\pi^{\nu,\gamma+1}_{\bar{\xi}}(a), \pi^{\nu,\gamma+1}_{\bar{\theta}}(f)]^{P}_{E})\end{aligned}$$

(by the properties of embedding normalization, and the fact $\pi^{\nu,\gamma+1}_{\bar{\theta}}(\bar{P}) = P$ and $\pi^{\nu,\gamma+1}_{\bar{\xi}}(\bar{E}) = E$)

$$= [\sigma \circ \psi^{\gamma+1}_{\xi} \circ \pi^{\nu,\gamma+1}_{\bar{\xi}}(a), \rho \circ \psi^{\gamma+1}_{\theta} \circ \pi^{\nu,\gamma+1}_{\bar{\theta}}(f)]^{\mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{\theta}}_{E^*},$$

where σ resurrects $\psi^{\gamma+1}_{\xi}(E)$ and ρ resurrects $\psi^{\gamma+1}_{\theta}(P)$, as defined above. We have

$$\sigma = \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\xi}}(\bar{\sigma}), \quad \text{and} \quad \rho = \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\theta}}(\bar{\rho}).$$

Further

$$\begin{aligned}\overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\xi}+1} \circ \psi^{\nu}_{\bar{\xi}+1}(x) &= \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\xi}+1}(\psi^{\nu}_{\bar{\xi}+1}([a, f]^{\bar{P}}_{\bar{E}})) \\ &= \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\xi}+1}([\bar{\sigma} \circ \psi^{\nu}_{\bar{\xi}}(a), \bar{\rho} \circ \psi^{\nu}_{\bar{\theta}}(f)]^{\mathcal{M}^{\mathcal{W}^*_{\nu}}_{\bar{\theta}}}_{E^{\mathcal{W}^*_{\nu}}_{\bar{\xi}}}) \\ &= [\overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\xi}} \circ \bar{\sigma} \circ \psi^{\nu}_{\bar{\xi}}(a), \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\theta}} \circ \bar{\rho} \circ \psi^{\nu}_{\bar{\theta}}(f)]^{\mathcal{M}^{\mathcal{W}^*_{\nu+1}}_{\theta}}_{E^{\mathcal{W}^*_{\nu+1}}_{\xi}} \\ &= [\sigma \circ \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\xi}} \circ \psi^{\nu}_{\bar{\xi}}(a), \rho \circ \overset{*}{\pi}{}^{\nu,\gamma+1}_{\bar{\theta}} \circ \psi^{\nu}_{\bar{\theta}}(f)]^{\mathcal{M}^{\mathcal{W}^*_{\nu+1}}_{\theta}}_{E^*} \\ &= [\sigma \circ \psi^{\gamma+1}_{\xi} \circ \pi^{\nu,\gamma+1}_{\bar{\xi}}(a), \rho \circ \psi^{\gamma+1}_{\theta} \circ \pi^{\nu,\gamma+1}_{\bar{\theta}}(f)]^{\mathcal{M}^{\mathcal{W}^*_{\nu+1}}_{\theta}}_{E^*}.\end{aligned}$$

The first 4 lines come from the way embedding normalization and lifting work. The last line comes from our induction hypothesis.

This proves Claim 3.33. (We leave Case 2 to the reader.) □

Returning to the inductive proof of $(\dagger)_\xi$, we see that the limit case is trivial. We are left with

Inductive Case 2. ξ is a limit ordinal, and $(\dagger)_\xi$).

We must prove $(\dagger)_{\xi+1}$. We have $\mathcal{S}^*{\restriction}\xi = \mathcal{W}^*_{\gamma+1}{\restriction}\xi$. Since Σ^* normalizes well, the branch $[0, \xi]_{\mathcal{W}^*_{\gamma+1}}$ of $\mathcal{W}^*_{\gamma+1}$ produced by embedding normalization is equal to $\Sigma^*(\mathcal{S}^*{\restriction}\xi)$. Thus $\mathcal{S}^*{\restriction}(\xi+1) = \mathcal{W}^*_{\gamma+1}{\restriction}(\xi+1)$. One can then prove $(\dagger)_{\xi+1}$ by looking at how the objects it deals with come from the $\mathcal{M}^{\mathcal{W}_\nu}_\tau$ and $\mathcal{M}^{\mathcal{W}^*_\nu}_\tau$ for $\tau <_{W_\gamma} \phi^{-1}_{\nu,\gamma+1}(\xi)$, and using our induction hypothesis $(\dagger)_\xi$. We omit further detail.

This completes our inductive proof of (1) and (2) of Lemma 3.28. We have already proved (3) of Lemma 3.28. We now prove (4).

Recall that $z(\eta) = \mathrm{lh}\,\mathcal{W}_\eta - 1$. The following diagram summarizes the proof of (4).

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{U}}_{\gamma+1} \arrow[r, "\sigma_{\gamma+1}"] & \mathcal{M}^{\mathcal{W}_{\gamma+1}}_{z(\gamma+1)} \arrow[r, "\psi^{\gamma+1}_{z(\gamma+1)}"] & M_{\eta_{z(\gamma+1)}, l_{z(\gamma+1)}} & \in & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{z(\gamma+1)} \\
\mathcal{M}^{\mathcal{U}}_{\nu} \arrow[u, "E^{\mathcal{U}}_{\gamma}"] \arrow[r, "\sigma_\nu"'] & \mathcal{M}^{\mathcal{W}_{\nu}}_{z(\nu)} = R_\nu \arrow[u, "\pi^{\nu,\gamma+1}_{z(\nu)}"'] \arrow[r, "\psi^{\nu}_{z(\nu)}"'] & M_{\eta_{z(\nu)}, l_{z(\nu)}} \arrow[u, "{}^*\pi^{\nu,\gamma+1}_{z(\nu)}"'] & \in & \mathcal{M}^{\mathcal{W}^*_{\nu}}_{z(\nu)} \arrow[u]
\end{tikzcd}$$

That the square on the right commutes is $(\dagger)_{z(\gamma+1)}$. We have shown already that the square on the left commutes. We have that $\psi^{\mathcal{U}}_{\nu} = \psi^{\mathcal{W}_\nu}_{z(\nu)} \circ \sigma_\nu$ by induction. Further, the diagram

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{U}}_{\gamma+1} \arrow[r, "\psi^{\mathcal{U}}_{\gamma+1}"] & M_{\eta_{z(\gamma+1)}, l_{z(\gamma+1)}} \in \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{z(\gamma+1)} = \mathcal{M}^{\mathcal{U}^*_{\gamma+1}} \\
\mathcal{M}^{\mathcal{U}}_{\nu} \arrow[u, "i^{\mathcal{U}}_{\nu,\gamma+1}"] \arrow[r, "\psi^{\mathcal{U}}_{\nu}"'] & M_{\eta_{z(\nu)}, l_{z(\nu)}} \in \mathcal{M}^{\mathcal{W}^*_{\nu}}_{z(\nu)} = M^{\mathcal{U}^*_\nu} \arrow[u, "i^{\mathcal{U}^*}_{\nu,\gamma+1}"]
\end{tikzcd}$$

commutes, since it is part of the copy and conversion of $\mathcal{U}$ to $\mathcal{U}^*$. So $\psi^{\mathcal{U}}_{\gamma+1}$ agrees with $\psi^{\gamma+1}_{z(\gamma+1)} \circ \sigma_{\gamma+1}$ on $\mathrm{ran}\, i^{\mathcal{U}}_{\nu,\gamma+1}$. But $\mathcal{M}^{\mathcal{U}}_{\gamma+1}$ is generated by $\mathrm{ran}\, i^{\mathcal{U}}_{0,\gamma+1}$ union $\lambda_{E^{\mathcal{U}}_\gamma}$. For $a \in [\lambda_{E^{\mathcal{U}}_\gamma}]^{<\omega}$,

$$\psi^{\gamma+1}_{z(\gamma+1)} \circ \sigma_{\gamma+1}(a) = \psi^{\gamma}_{z(\gamma)} \circ \sigma_\gamma(a). \tag{$*$}$$

To see ($*$), note first $\sigma_\gamma \restriction \lambda_{E^{\mathcal{U}}_\gamma} = \sigma_{\gamma+1} \restriction \lambda_{E^{\mathcal{U}}_\gamma}$ by facts about embedding normalization. (See e.g. p.58) So it is enough to show that $\psi^{\gamma+1}_{z(\gamma+1)}$ agrees with $\psi^{\gamma}_{z(\gamma)}$ on λ_{F_γ}. But for $\alpha = \alpha^{\mathcal{T},\mathcal{U}}_{\gamma}$ as before, $W_\gamma \restriction (\alpha+1) = W_{\gamma+1} \restriction (\alpha+1)$. Also, $\lambda_{F_\gamma} < \lambda_{E^{W_\gamma}_\alpha}$. Thus for $\lambda = \lambda_{F_\gamma}$,

$$\begin{aligned}
\psi^{\gamma}_{z(\gamma)} \restriction \lambda &= \psi^{\gamma}_{\alpha} \restriction \lambda \\
&= \psi^{\gamma+1}_{\alpha} \restriction \lambda \\
&= \psi^{\gamma+1}_{z(\gamma+1)} \restriction \lambda.
\end{aligned}$$

This completes the proof of ($*$).

But $\psi_\gamma^{\mathcal{U}} = \psi_{z(\gamma)}^\gamma \circ \sigma_\gamma$ by induction, and $\psi_\gamma^{\mathcal{U}}$ agrees with $\psi_{\gamma+1}^{\mathcal{U}}$ on $\lambda_{E_\gamma^{\mathcal{U}}}$, by the properties of conversion systems. So $\psi_{\gamma+1}^{\mathcal{U}}$ agrees with $\psi_{z(\gamma+1)}^{\gamma+1} \circ \sigma_{\gamma+1}$ on $\lambda_{E_\gamma^{\mathcal{U}}}$, as desired.

This completes the proof of (4) in Lemma 3.28 in the case that $[0, \gamma+1]_U$ does not drop in model or degree, so that we have $z(\gamma) = \mathrm{lh}\, \mathcal{W}_{\gamma+1}^* - 1$ as well, and $\mathcal{M}_{\gamma+1}^{\mathcal{U}^*} = \mathcal{M}_{z(\gamma+1)}^{\mathcal{W}_{\gamma+1}^*}$. We leave the dropping case to the reader.

This completes the proof that if Lemma 3.28 holds at γ, then it holds at $\gamma+1$.

Now suppose γ is a limit ordinal. Let

$$\lambda = \sup\{\alpha^{\mathcal{T},\mathcal{U}} \mid \xi < \gamma\}.$$

So $W(\mathcal{T},\mathcal{U}\restriction\gamma) = \mathcal{W}_\gamma\restriction\lambda$, and $W(\mathcal{T}^*,\mathcal{U}^*\restriction\gamma) = \mathcal{W}_\gamma^*\restriction\lambda$. Also

$$\mathcal{S}_\gamma^*\restriction\lambda = \mathcal{W}_\gamma^*\restriction\lambda,$$

because $\mathcal{S}_\xi^*\restriction\alpha_\xi = \mathcal{W}_\xi^*\restriction\alpha_\xi = \mathcal{W}_\gamma^*\restriction\alpha_\xi$ for $\xi < \lambda$. Since Σ^* normalizes well, $[0,\lambda)_{W_\gamma^*} = \Sigma^*(\mathcal{W}_\gamma^*\restriction\lambda)$. Thus

$$\mathcal{S}_\gamma^*\restriction(\lambda+1) = \mathcal{W}_\gamma^*\restriction(\lambda+1).$$

We now go on to prove $(\dagger)_\xi$, for $\xi \geq \lambda$, by induction. The proof is similar to the one above. Having $(\dagger)_\xi$ for $\xi = \mathrm{lh}\,\mathcal{W}_\gamma$, we go on to prove (4) as above. We omit further detail.

This proves Lemma 3.28. □

Now let $lh(\mathcal{U}) = \gamma+1$. So $W(\mathcal{T},\mathcal{U}) = \mathcal{W}_\gamma$ and $W(\mathcal{T}^*,\mathcal{U}^*) = \mathcal{W}_\gamma^*$. By Lemma 3.28, $\mathcal{W}_\gamma$ lifts to $\mathcal{W}_\gamma^*$, so $\mathcal{W}_\gamma$ is by Σ. Let $\tau = z(\gamma)$. Let $P = \mathcal{M}_\gamma^{\mathcal{U}}$, $R = \mathcal{M}_\tau^{\mathcal{W}_\gamma}$, and $S = \mathcal{M}_\tau^{\mathcal{W}_\gamma^*}$. We have $N = M_{\eta_\gamma^{\mathcal{U}}, l_\gamma^{\mathcal{U}}} = M_{\eta_\tau^\gamma, l_\tau^\gamma}$ in the construction of $\mathcal{M}_\gamma^{\mathcal{U}^*} = \mathcal{M}_\tau^{\mathcal{W}_\gamma^*}$, by Lemma 3.28. Moreover, the lemma tells us that $\psi_\gamma^{\mathcal{U}} = \psi_\tau^\gamma \circ \sigma_\gamma$. Let then Ω be the strategy for N induced by the construction of $\mathcal{M}_\gamma^{\mathcal{U}^*}$. Then

$$\begin{aligned}\Sigma_{\langle\mathcal{T},\mathcal{U}\rangle,P} &= \Omega^{\psi_\gamma^{\mathcal{U}}}\\ &= \Omega^{\psi_\tau^\gamma\circ\sigma_\gamma}\\ &= (\Omega^{\psi_\tau^\gamma})^{\sigma_\gamma}\\ &= (\Sigma_{\mathcal{W}_\gamma,R})^{\sigma_\gamma}.\end{aligned}$$

Thus Σ 2-normalizes well.

Finally, we must show that all tails of Σ 2-normalize well. It is enough to consider tails of the form $\Sigma_{\mathcal{T},Q}$, where $\mathcal{T}$ is normal on $M_{\nu_0,k_0}^{\mathbb{C}}$. Let

$$\mathrm{lift}(\mathcal{T}, M_{\nu_0,k_0}, \mathbb{C}) = \langle \mathcal{T}^*, \langle \eta_\xi^{\mathcal{T}}, l_\xi^{\mathcal{T}} \mid \xi \leq \xi_0\rangle, \langle \psi_\xi^{\mathcal{T}} \mid \xi \leq \xi_0\rangle\rangle.$$

Let Ω be the iteration strategy for

$$Q^* = M^{i^{\mathcal{T}^*}_{0,\xi_0}(\mathbb{C})}_{\eta^{\mathcal{T}}_{\xi_0}, l^{\mathcal{T}}_{\xi_0}}, i^{\mathcal{T}^*}_{0,\xi_0}(\mathbb{C}))$$

that is induced by $\Sigma^*_{\mathcal{T}^*, \mathcal{M}^{\mathcal{T}^*}_{\xi_0}}$. The argument we have just given shows that Ω 2-normalizes well. But $\Sigma_{\mathcal{T},Q}$ is by definition the pullback of Ω via $\psi^{\mathcal{T}}_{\xi_0}$. So by 3.3, $\Sigma_{\mathcal{T},Q}$ 2-normalizes well.

This finishes our proof of Theorem 3.26. □

Iteration strategies that normalize well are also coherent, in the following sense.

Definition 3.34 *Let Σ be an iteration strategy for a premouse P, defined on finite stacks of normal trees.*

(1) We say that Σ is coherent for normal trees *iff whenever $\mathcal{T}$ is a normal tree by Σ, and $N \trianglelefteq \mathcal{M}^{\mathcal{T}}_\alpha$ and $N \trianglelefteq \mathcal{M}^{\mathcal{T}}_\beta$, then $\Sigma_{\mathcal{T}\restriction(\alpha+1),N} = \Sigma_{\mathcal{T}\restriction(\beta+1),N}$.*

(2) Σ is coherent *iff every tail Σ_s of Σ is coherent for normal trees. In this case, we say (P, Σ) is* strategy coherent.

Lemma 3.35 *Suppose Σ is a strategy for a premouse P, and Σ normalizes well; then Σ is coherent.*

Proof. Since all tails of Σ normalize well, it is enough to show that Σ is coherent for normal trees. Let $\mathcal{T}$ be normal and by Σ, and let $N \triangleleft \mathcal{M}^{\mathcal{T}}_\alpha$ and $N \triangleleft \mathcal{M}^{\mathcal{T}}_\beta$. Let $\Psi_0 = \Sigma_{\mathcal{T}\restriction(\alpha+1),N}$ and $\Psi_1 = \Sigma_{\mathcal{T}\restriction(\beta+1),N}$, and let $\mathcal{U}$ be a normal tree of limit length on N that is by both Ψ_0 and Ψ_1. Then

$$W(\mathcal{T} \restriction (\alpha+1), \mathcal{U}) = W(\mathcal{T} \restriction (\beta+1), \mathcal{U}) = W(\mathcal{T} \restriction (\gamma+1), \mathcal{U}),$$

where γ is least such that $N \trianglelefteq \mathcal{M}^{\mathcal{T}}_\gamma$. Let $b_0 = \Psi_0(\mathcal{U})$ and $b_1 = \Psi_1(\mathcal{U})$, and let

$$a_i = \mathrm{br}(b_i, \mathcal{T} \restriction (\gamma+1), \mathcal{U})).$$

Since Σ normalizes well, $\Sigma(W(\mathcal{T} \restriction (\gamma+1)) = a_i$, for $i = 0, 1$. Thus $a_0 = a_1$. By 2.65, $b_0 = b_1$, as desired. □

Remark 3.36 We say that Σ is *positional* iff whenever s and t are stacks by Σ, and N is an initial segment of the last model of each, then $\Sigma_{s,N} = \Sigma_{t,N}$. Positionality implies strategy coherence. The techniques of [33] show that normalizing well and strong hull condensation together imply positionality, but the proof is not an elementary combinatorial one like that above.

3.4 Fine strategies that condense well

We show that if Σ^* is an iteration strategy for V that has strong hull condensation, then the strategies for premice induced by Σ^* via a full background extender construction also have strong hull condensation. The proof is routine, but we include it for the sake of completeness. The corresponding result for ordinary hull condensation was proved by Sargsyan in [16].

Theorem 3.37 *Let $N^* \models \mathsf{ZFC}+$ "$\mathbb{C}$ is a w construction". Let Σ^* be a $\theta, \vec{F}^{\mathbb{C}}$-iteration strategy for N^*. Suppose that $\langle \nu, k\rangle < \mathrm{lh}(\mathbb{C})$, and Σ is the iteration strategy for $M^{\mathbb{C}}_{\nu,k}$ induced by Σ^*. Suppose finally that Σ^* has strong hull condensation; then Σ has strong hull condensation.*

Proof. We show that Σ condenses properly on normal trees. The proof that all its tails Σ_s do so as well is similar.

Let $\mathcal{U}$ be a normal iteration tree on $M = M^{\mathbb{C}}_{\nu_0,k_0}$ that is by Σ, and let $\Phi\colon \mathcal{T} \to \mathcal{U}$ be a psuedo-hull embedding, with

$$\Phi = \langle u, \langle t^0_\beta \mid \beta < \mathrm{lh}(\mathcal{T})\rangle, \langle t^1_\beta \mid \beta < \mathrm{lh}(\mathcal{T})\rangle, p\rangle.$$

We must see that $\mathcal{T}$ lifts to a tree by Σ^*. Let

$$\mathrm{lift}(\mathcal{U}, M_{\nu_0,k_0}, \mathbb{C}) = \langle \mathcal{U}^*, \langle \theta_\xi, m_\xi \mid \xi < \mathrm{lh}(\mathcal{U})\rangle, \langle \psi_\xi \mid \xi < \mathrm{lh}(\mathcal{U})\rangle\rangle.$$

It is enough to show that $\mathcal{T}$ lifts to a psuedo-hull of $\mathcal{U}^*$. For this, let

$$\mathrm{lift}(\mathcal{T}, M_{\nu,k}, \mathbb{C}) = \langle \mathcal{T}^*, \langle \eta_\xi, l_\xi \mid \xi < \mathrm{lh}(\mathcal{T})\rangle, \langle \varphi_\xi \mid \xi < \mathrm{lh}(\mathcal{T})\rangle\rangle.$$

We shall construct a psuedo-hull embedding $\Phi^*\colon \mathcal{T}^* \to \mathcal{U}^*$ by induction, with

$$\Phi^* = \langle u, \langle w^0_\beta \mid \beta < \mathrm{lh}(\mathcal{T})\rangle, \langle w^1_\beta \mid \beta < \mathrm{lh}(\mathcal{T})\rangle, q\rangle.$$

Notice here that $u^{\Phi^*} = u = u^{\Phi}$. Because Φ^* is to be a psuedo-hull embedding, this completely determines the putative Φ^*, and what we have to show is just what we get is indeed a psuedo-hull embedding of $\mathcal{T}^*$ into $\mathcal{U}^*$.

For $\gamma \leq \mathrm{lh}(\mathcal{T})$, let

$$\Phi^*_\gamma = \Phi^* \restriction \gamma = \langle u \restriction \{\xi \mid \xi + 1 < \gamma, \langle w^0_\beta \mid \beta < \gamma\rangle, \langle w^1_\beta \mid \beta < \alpha\rangle, q_\gamma\rangle.$$

Let v be the "minimal realization" map of Φ and Φ^*, given by $v(0) = 0$, $v(\alpha+1) = u(\alpha) + 1$, and $v(\lambda) = \sup_{\alpha<\lambda} v(\alpha)$ for λ a limit ordinal. Let $Q_\alpha = M^{i^{\mathcal{T}^*}_{0,\alpha}(\mathbb{C})}_{\eta_\alpha, l_\alpha}$, and $X_\alpha = M^{i^{\mathcal{U}^*}_{0,\alpha}(\mathbb{C})}_{\theta_\alpha, m_\alpha}$. Thus $\varphi_\alpha\colon \mathcal{M}^{\mathcal{T}}_\alpha \to Q_\alpha$ and $\psi_\alpha\colon \mathcal{M}^{\mathcal{U}}_\alpha \to X_\alpha$ are the liftup maps of the two conversion systems. We show by induction on γ that

(1) $\Phi^* \upharpoonright \gamma$ is a psuedo-hull embedding of $\mathcal{T}^* \upharpoonright \gamma$ into $\mathcal{U}^*$,

(2) for $\alpha < \gamma$, $\psi_{v(\alpha)} \circ t^0_\alpha = w^0_\alpha \circ \varphi_\alpha$, and

(3) for $\alpha < \gamma$, $w^0_\alpha(Q_\alpha) = X_{v(\alpha)}$.

Let $(*)_\gamma$ be the conjunction of (1)-(3). The following diagram illustrates the situation:

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{U}}_{u(\alpha)} \arrow[r, "\psi_{u(\alpha)}"] & X_{u(\alpha)} \in \mathcal{M}^{\mathcal{U}^*}_{u(\alpha)} \\
\mathcal{M}^{\mathcal{U}}_{v(\alpha)} \arrow[u, "i^{\mathcal{U}}_{v(\alpha),u(\alpha)}"] \arrow[r, "\psi_{v(\alpha)}"] & X_{v(\alpha)} \in \mathcal{M}^{\mathcal{U}^*}_{v(\alpha)} \arrow[u, "i^{\mathcal{U}^*}_{v(\alpha),u(\alpha)}"'] \\
\mathcal{M}^{\mathcal{T}}_{\alpha} \arrow[u, "t^0_\alpha"] \arrow[uu, bend left=60, "t^1_\alpha"] \arrow[r, "\varphi_\alpha"'] & Q_\alpha \in \mathcal{M}^{\mathcal{T}^*}_{\alpha} \arrow[u, "w^0_\alpha"'] \arrow[uu, bend right=60, "w^1_\alpha"']
\end{tikzcd}$$

Some care is needed in reading this diagram. The bottom rectangle is just (2) and (3) of our induction hypotheses, and is always valid. The top rectangle involves only the conversion of $\mathcal{U}$ to $\mathcal{U}^*$. It is valid if and only if $(v(\alpha), u(\alpha)]_U$ does not drop (in model or degree), so that $i^{\mathcal{U}^*}_{v(\alpha),u(\alpha)}(X_{v(\alpha)}) = X_{u(\alpha)}$. In the case that $(v(\alpha), u(\alpha]_U$ drops, something like it is valid. We discuss that below.

To start with, Φ^*_1 is given by setting $v(0) = 0$ and $w^0_0 =$ identity map from $N^* = \mathcal{M}^{\mathcal{T}^*}_0$ to $N^* = \mathcal{M}^{\mathcal{U}^*}_0$.

If λ is a limit, and $(*)_\alpha$ for $\alpha < \lambda$, then

$$\Phi^*_\lambda = \bigcup_{\alpha<\lambda} \Phi^*_\alpha$$

in the obvious componentwise sense. It is clear that $(*)_\lambda$ holds.

If $\gamma = \lambda + 1$ for $\lambda < \mathrm{lh}(\mathcal{T})$ a limit such that $(*)_\lambda$, then $\Phi^*_{\lambda+1}$ is just Φ^*_λ together with the map w^0_λ, defined as follows. Recall that v preserves tree order, and

$$v(\lambda) = \sup_{\alpha<\lambda} v(\alpha).$$

For $\alpha <_T \lambda$ and $x \in \mathcal{M}^{\mathcal{T}^*}_\alpha$, we set

$$w^0_\lambda(i^{\mathcal{T}^*}_{\alpha,\lambda}(x)) = i^{\mathcal{U}^*}_{v(\alpha),v(\lambda)}(w^0_\alpha(x)).$$

Using (1) at $\gamma < \lambda$, we see that w^0_λ is well defined, elementary, and as required for $(*)_{\lambda+1}$.

Finally, suppose we have $\Phi^*_{\alpha+1}$ satisfying $(*)_{\alpha+1}$. The whole of $\Phi^*_{\alpha+2}$ is determined by $u(\alpha)$, which is already given to us, but we must see this choice works; that is, that $(*)_{\alpha+2}$ holds for the system it determines.

The following notation is useful. Let $\mathbb{D}$ be any background construction, and let F be an extender on the sequence of $M^{\mathbb{D}}_{\eta,j}$. We let

$$\mathrm{res}^{\mathbb{D}}(F) = \sigma^{\mathbb{D}}_{\eta,j}[M_{\eta,j}|\langle \mathrm{lh}(F), 0\rangle](F)$$

be the complete resurrection of F in $\mathbb{D}$. If G is the last extender of $M^{\mathbb{D}}_{\nu,0}$, and we let

$$B^{\mathbb{D}}(G) = F^{\mathbb{D}}_\nu,$$

for the unique such ν, be the associated background extender. So $(B \circ \mathrm{res})^{\mathbb{D}}(F)$ is the background extender for F given by $\mathbb{D}$.

Set

$$w^1_\alpha = i^{\mathcal{U}^*}_{v(\alpha),u(\alpha)} \circ w^0_\alpha,$$

as we are forced to do. Let $\mathbb{C}_\alpha = i^{\mathcal{T}^*}_{0,\alpha}(\mathbb{C})$ and $\mathbb{D}_\alpha = i^{\mathcal{U}^*}_{0,\alpha}(\mathbb{C})$. Note that $w^1_\alpha(\mathbb{C}_\alpha) = \mathbb{D}_{u(\alpha)}$.

Let

$$\begin{aligned}
G &= E^{\mathcal{T}}_\alpha,\\
G^* &= E^{\mathcal{T}^*}_\alpha = (B \circ \mathrm{res})^{\mathbb{C}_\alpha}(\varphi_\alpha(G)),\\
H &= E^{\mathcal{U}}_{u(\alpha)},\\
H^* &= E^{\mathcal{U}^*}_{u(\alpha)} = (B \circ \mathrm{res})^{\mathbb{D}_{u(\alpha)}}(\psi_{u(\alpha)}(H)).
\end{aligned}$$

Lemma 4.2 below tells us that the following claim is what we need.

Claim 3.38 *$w^1_\alpha(G^*) = H^*$.*

Proof. Suppose first that $(v(\alpha, u(\alpha]_U$ does not drop. In that case, $i^{\mathcal{U}^*}_{v(\alpha),u(\alpha)}(X_{v(\alpha)}) = X_{u(\alpha)}$, so the top rectangle in the diagram above is valid. Because $t^1_\alpha(G) = H$, we get $w^1_\alpha(\varphi_\alpha(G)) = \psi_{u(\alpha)}(H)$. But then the elementarity of w^1_α implies that $w^1_\alpha(G^*) = H^*$.

Suppose now that $(v(\alpha), u(\alpha]_U$ drops. Let $I = t^0_\alpha(G)$. Since $H = \hat{\imath}^{\mathcal{U}}_{v(\alpha),u(\alpha)}(I)$, all extenders used along $(v(\alpha), u(\alpha]_U$ have critical points below the current image of λ_I. This implies that the drops are to levels that are in the image of the $\langle \mathrm{lh}(I), 0\rangle$

dropdown sequence of $\mathcal{M}^{\mathcal{U}}_{v(\alpha)}$. Let Z_n be the n-th level of the $\langle \mathrm{lh}(I), 0\rangle$ dropdown sequence of $\mathcal{M}^{\mathcal{U}}_{v(\alpha)}$, starting with $Z_0 = \mathcal{M}^{\mathcal{U}}_{v(\alpha)}|\langle \mathrm{lh}(I), 0\rangle$. We then have a fixed n such that

$$\hat{\imath}^{\mathcal{U}}_{v(\alpha),u(\alpha)} \colon Z_n \to \mathcal{M}^{\mathcal{U}}_{u(\alpha)}$$

is elementary.

Now let us move over to $\mathcal{U}^*$. We have that $\psi_{v(\alpha)}(Z_n) = Z^*_n$ is a level of $X_{v(\alpha)}$. By looking at how resurrection works in the dropping case, one can see

$$X_{u(\alpha)} = \text{ resurrection of } i^{\mathcal{U}^*}_{v(\alpha,u(\alpha)}(Z^*_n) \text{ in } \mathbb{D}_{u(\alpha)}.$$

That gives us $j \colon Z^*_n \to X_{u(\alpha)}$ obtained by composing $i^{\mathcal{U}^*}_{v(\alpha),u(\alpha)}$ with the resurrection map for $i^{\mathcal{U}^*}_{v(\alpha,u(\alpha)}(Z^*_n)$ of $\mathcal{M}^{\mathcal{U}^*}_{u(\alpha)}$. Note

$$j \circ \psi_{v(\alpha)} = \psi_{u(\alpha)} \circ \hat{\imath}^{\mathcal{U}}_{v(\alpha),u(\alpha)},$$

by the way dropping and resurrection interact in our conversion system. But now $\hat{\imath}^{\mathcal{U}}_{v(\alpha),u(\alpha)}(I) = H$, so

$$\begin{aligned}
H^* &= (B \circ \mathrm{res})^{\mathbb{D}_{u(\alpha)}}(\psi_{u(\alpha)}(H)) \\
&= (B \circ \mathrm{res})^{\mathbb{D}_{u(\alpha)}}(j \circ \psi_{v(\alpha}(I)) \\
&= (B \circ \mathrm{res})^{D_{u(\alpha)}}(i^{\mathcal{U}^*}_{v(\alpha,u(\alpha)} \circ \psi_{v(\alpha)}(I)) \\
&= i^{\mathcal{U}^*}_{v(\alpha),u(\alpha)} \circ w^0_\alpha((B \circ \mathrm{res})^{\mathbb{C}_\alpha}(\varphi_\alpha(G))) \\
&= w^1_\alpha(G^*).
\end{aligned}$$

The third line holds because resurrecting $i^{\mathcal{U}^*}_{v(\alpha,u(\alpha)} \circ \psi_{v(\alpha)}(I)$ in $\mathcal{M}^{\mathcal{U}^*}_{u(\alpha)}$ can be thought of as first doing a partial resurrection of that reaches $j \circ \psi_{v(\alpha}(I)$, then doing the rest. The second to last line holds because $\psi_{v(\alpha)}(I) = \psi_{v(\alpha)} \circ t^0_\alpha(G) = w^0_\alpha \circ \varphi_\alpha(G)$, and w^0_α is elementary, and moves $\mathbb{C}_\alpha$ to $\mathbb{D}_{v(\alpha)}$.

This proves the claim. $\square$

By Lemma 4.2, there is a unique psuedo-hull embedding Ψ from $\mathcal{T}^* \restriction (\alpha+2)$ to $\mathcal{U}^*$ that extends $\Phi^*_{\alpha+1}$ and satisfies $u^\Psi(\alpha) = u(\alpha)$. Let $\Phi^*_{\alpha+2}$ be this Ψ. We claim that $(*)_{\alpha+2}$ holds.

Let $\beta = T\text{-pred}(\alpha+1)$, and let $\tau = U\text{-pred}(u(\alpha)+1)$. Because Φ is a psuedo-hull embedding, $\tau \in [v(\beta), u(\beta)]_U$. Let us assume for simplicity that there is no relevant dropping, that is,

(a) $(\alpha + 1) \notin D^{\mathcal{T}}$, and

(b) $D^{\mathcal{U}} \cap [v(\beta), v(\alpha+1)] = \emptyset$.

So $\mathcal{M}^{\mathcal{T}}_{\alpha+1} = \mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_{\beta}, G))$ and $\mathcal{M}^{\mathcal{U}}_{v(\alpha+1)} = \mathrm{Ult}(\mathcal{M}^{\mathcal{U}}_{\tau}, H))$. Let $\rho = i^{\mathcal{U}}_{v(\beta),\tau)} \circ t^0_\beta$ and $\rho^* = i^{\mathcal{U}^*}_{v(\beta),\tau} \circ w^0_{v(\beta)}$. The lifting construction yields $\mathcal{M}^{\mathcal{T}^*}_{\alpha+1} = \mathrm{Ult}(\mathcal{M}^{\mathcal{T}^*}_{\beta}, G*))$ and $\mathcal{M}^{\mathcal{U}^*}_{v(\alpha+1)} = \mathrm{Ult}(\mathcal{M}^{\mathcal{U}^*}_{\tau}, H^*))$, moreover

$$X_{v(\alpha+1)} = i^{\mathcal{U}^*}_{v(\beta),v(\alpha+1)}(v(\beta)).$$

$w^0_{v(\alpha)+1}$ is given by the Shift Lemma:

$$w^0_{v(\alpha+1)}([a, f]^{\mathcal{M}^{\mathcal{T}^*}_{\beta}}_{G^*}) = [w^1_{u(\alpha)}(a), \rho^*(f)]^{\mathcal{M}^{\mathcal{U}^*}_{\tau}}_{H^*}.$$

One can calculate that $\psi_{v(\alpha+1)} \circ t^0_{\alpha+1} = w^0_{\alpha+1} \circ \varphi_{\alpha+1}$, and $w^0_{\alpha+1}(Q_{\alpha+1})) = X_{v(\alpha+1)}$.

We leave the case that one of our no-dropping hypotheses (a) and (b) above fails to the reader. $\square$

4 Comparing iteration strategies

We shall prove the main Comparison Theorem for pure extender mice, in Jensen indexing. The proof adapts easily to ms-indexing, and to hod mice. We shall discuss hod mice in the next section.

The first two subsections contain some preliminary lemmas. The last contains the comparison argument.

4.1 Extending psuedo-hull embeddings

We shall prove an elementary lemma on the extendibility of psuedo-hull embeddings. Its proof uses

Proposition 4.1 *Let $\mathcal{S}$ be a normal tree, let $\delta \leq_S \eta$, and suppose that $P \trianglelefteq \mathcal{M}^{\mathcal{S}}_\eta$, but $P \ntrianglelefteq \mathcal{M}^{\mathcal{S}}_\sigma$ whenever $\sigma <_S \delta$. Suppose also that $P \in \mathrm{ran}(\hat{\imath}^{\mathcal{S}}_{\delta,\eta})$. Let*

$$\begin{aligned} \alpha = & \text{ least } \gamma \text{ such that } P \trianglelefteq \mathcal{M}^{\mathcal{S}}_\gamma \\ = & \text{ least } \gamma \text{ such that } o(P) < \mathrm{lh}(E^{\mathcal{S}}_\gamma) \text{ or } \gamma = \eta, \end{aligned}$$

and

$$\beta = \text{least } \gamma \in [0,\eta]_S \text{ such that } \mathrm{crit}(\hat{\imath}^{\mathcal{S}}_{\gamma,\eta}) > o(P) \text{ or } \gamma = \eta.$$

Then $\beta \in [\delta,\eta]_S$, and

(a) either $\beta = \alpha$, or $\beta = \alpha + 1$, and $\lambda(E^{\mathcal{S}}_\alpha) \leq o(P) < \mathrm{lh}(E^{\mathcal{S}}_\alpha)$;

(b) if $P = \mathrm{dom}(E^{\mathcal{S}}_\xi)$, then S-pred$(\xi + 1) = \alpha = \beta$.

(We allow $\delta = \eta$, with the understanding $\hat{\imath}_{\delta,\delta}$ is the identity.)

Proof. By normality, for any $\gamma < \eta$, $P \trianglelefteq \mathcal{M}^{\mathcal{S}}_\gamma$ iff $\mathrm{lh}(E^{\mathcal{S}}_\gamma) > o(P)$. So the two characterizations of α are equivalent. Clearly, $P \trianglelefteq \mathcal{M}^{\mathcal{S}}_\beta$, and thus $\alpha \leq \beta$. We have that $o(P) \geq \mathrm{lh}(E^{\mathcal{S}}_\sigma)$ for all $\sigma <_S \delta$, and hence by normality, for all $\sigma <_S \delta$ whatsoever. So $\delta \leq \alpha$, and $\beta \in [\delta,\eta]_S$.

Suppose $\alpha < \beta$; then $o(P) < \mathrm{lh}(E^{\mathcal{S}}_\alpha)$, so $o(P) < \mathrm{lh}(E^{\mathcal{S}}_\sigma)$ where σ is least such that $\alpha \leq \sigma$ and $\sigma + 1 \leq_S \beta$. If $o(P) < \lambda(E^{\mathcal{S}}_\sigma)$, then because $\delta \leq \sigma$ and $P \in \mathrm{ran}(\hat{\imath}^{\mathcal{S}}_{\delta,\eta})$, we have $o(P) < \mathrm{crit}(E^{\mathcal{S}}_\sigma)$, which contradicts our definition of β. So $\lambda(E^{\mathcal{S}}_\sigma) < o(P) < \mathrm{lh}(E^{\mathcal{S}}_\sigma)$. If $\mathrm{crit}(\hat{\imath}^{\mathcal{S}}_{\sigma+1,\eta}) = \lambda(E^{\mathcal{S}}_\sigma)$, then P is not in $\mathrm{ran}(\hat{\imath}^{\mathcal{S}}_{\delta,\eta})$, so $\mathrm{crit}(\hat{\imath}^{\mathcal{S}}_{\sigma+1,\eta}) > o(P)$, and thus $\beta = \sigma + 1$.

This yields (a). For (b), note that if $\lambda(E^{\mathcal{S}}_\sigma) \le o(P) < \mathrm{lh}(E^{\mathcal{S}}_\sigma)$, then P cannot be the domain of an extender used in $\mathcal{S}$. So we have $\alpha = \beta$. We have already observed that S-pred$(\xi+1) = \alpha$.

$\square$

On extending psuedo-hull embeddings, we have

Lemma 4.2 *Let $\Phi =< u, \langle t^0_\beta \mid \beta \le \alpha\rangle, \langle t^1_\beta \mid \beta < \alpha\rangle, p\rangle$ be a psuedo-hull embedding of $\mathcal{T}$ into $\mathcal{U}$, and let F be an extender on the $\mathcal{M}^{\mathcal{T}}_\alpha$-sequence such that $\mathrm{lh}(F) > \mathrm{lh}(E^{\mathcal{T}}_\beta)$ for all $\beta < \alpha$. Let $\mathcal{T}^\frown\langle F\rangle$ be the unique putative normal tree $\mathcal{S}$ extending $\mathcal{T}$ such that $F = E^{\mathcal{S}}_\alpha$. Let $\xi < \mathrm{lh}(\mathcal{U})$; then the following are equivalent:*

(1) There is a psuedo-hull embedding Ψ of $\mathcal{T}^\frown\langle F\rangle$ into $\mathcal{U}$ such that $\Phi \subseteq \Psi$ and $u^\Psi(\alpha) = \xi$,

(2) $v(\alpha) \le_U \xi$, and $E^{\mathcal{U}}_\xi = \hat{\imath}^{\mathcal{U}}_{v(\alpha),\xi} \circ t^0_\alpha(F)$.

Moreover, there is at most one such Ψ.

Proof. It is easy to see from definition 2.26 that (1) implies (2).

Suppose that ξ witnesses that (2) holds. Set $u(\alpha) = \xi$ and $t^1_\alpha = \hat{\imath}^{\mathcal{U}}_{v(\alpha),\xi} \circ t^0_\alpha$. Clearly,

$$t^1_\alpha \restriction \lambda^{\mathcal{T}}_\alpha = t^0_\alpha \restriction \lambda^{\mathcal{T}}_\alpha,$$

and

$$\mathrm{crit}(\hat{\imath}^{\mathcal{U}}_{v(\alpha),\xi}) \ge \lambda^{\mathcal{U}}_{v(\alpha)}.$$

Let $p(F) = G = E^{\mathcal{U}}_\xi$. We shall find $t^0_{\alpha+1}$ such that $\Psi =< u, \langle t^0_\beta \mid \beta \le \alpha+1\rangle, \langle t^1_\beta \mid \beta \le \alpha\rangle, p\rangle$ is a psuedo-hull embedding of $\mathcal{S} = \mathcal{T}^\frown\langle F\rangle$ into $\mathcal{U}$.

Let $\mu = \mathrm{crit}(F)$ and $\mu^* = \mathrm{crit}(G)$. Let

$$\beta = S\text{-pred}(\alpha+1) = \text{ least } \eta \text{ s.t. } \mu < \lambda^{\mathcal{T}}_{\eta+1},$$

and

$$\beta^* = U\text{-pred}(\xi+1) = \text{ least } \eta \text{ s.t. } \mu^* < \lambda^{\mathcal{U}}_{\eta+1}.$$

Let $\gamma = (\mu^+)^{\mathcal{M}^{\mathcal{T}}_\alpha|\mathrm{lh}(F)}$ and $P = \mathcal{M}^{\mathcal{T}}_\alpha|\gamma$. Similarly, let $\gamma^* = ((\mu^{*,+})^{\mathcal{M}^{\mathcal{U}}_\xi|\mathrm{lh}(G)}$ and $P^* = \mathcal{M}^{\mathcal{U}}_\xi|\gamma^*$. So P is the domain of F (the sets measured by it), P^* is the domain of G, and $t^1_\alpha(P) = P^*$. The rules of normality tell us that

$$\beta = \text{ least } \eta \text{ s.t. } P = \mathcal{M}^{\mathcal{T}}_\eta|\gamma,$$

and

$$\beta^* = \text{ least } \eta \text{ s.t. } P^* = \mathcal{M}^{\mathcal{U}}_\eta | \gamma^*.$$

(P and P^* are passive, so these identities imply that γ and γ^* are passive stages in $\mathcal{M}^{\mathcal{T}}_\beta$ and $\mathcal{M}^{\mathcal{U}}_{\beta^*}$.) Suppose first that $\beta < \alpha$. We then have that $\mu < \lambda^{\mathcal{T}}_\alpha$, so

$$\begin{aligned} P^* &= t^1_\alpha(P) \\ &= t^0_\alpha(P) \\ &= t^1_\beta(P) \\ &= \hat{\imath}^{\mathcal{U}}_{v(\beta),u(\beta)} \circ t^0_\beta(P), \end{aligned}$$

where the last equalities hold because $\mu < \lambda_{E^{\mathcal{T}}_\beta}$. Thus P^* is in the range of $\hat{\imath}^{\mathcal{U}}_{v(\beta),u(\beta)}$. Proposition 4.1, with $\delta = v(\beta)$, $\eta = u(\beta)$, and P^* as its P then tells us that

$$\beta^* = \text{least } \eta \in [v(\beta), u(\beta)]_U \text{ such that crit } \hat{\imath}^{\mathcal{U}}_{\eta,u(\beta)} > \hat{\imath}^{\mathcal{U}}_{v(\beta),\eta} \circ t^0_\beta(\mu).$$

Let Q be the first level of $\mathcal{M}^{\mathcal{T}}_\beta$ beyond P that projects to or below μ, and let Q^* be the first level of $\mathcal{M}^{\mathcal{T}}_{\beta^*}$ beyond P^* that projects to or below μ^*. So $\mathcal{M}^{\mathcal{T}}_{\alpha+1} = \text{Ult}(Q, F)$ and $\mathcal{M}^{\mathcal{U}}_{\xi+1} = \text{Ult}(Q^*, G)$. Let

$$\rho = (\hat{\imath}^{\mathcal{U}}_{v(\beta),\beta^*} \circ t^0_\beta) \restriction Q.$$

We have that

$$\rho \restriction P = t^1_\beta \restriction P = t^0_\alpha \restriction P = t^1_\alpha \restriction P.$$

We can then set

$$t^0_{\alpha+1}([a, f]^Q_F) = [t^1_\alpha(a), \hat{\imath}^{\mathcal{U}}_{v(\beta),\beta^*} \circ t^0_\beta(f)]^{Q^*}_G,$$

as we are required to do by definition 2.26, and the Shift Lemma tells us that $t^0_{\alpha+1}$ as defined is indeed well-defined, elementary, and agrees with t^1_α as required in a psuedo hull embedding.

We must check clause (b) of definition 2.26. The new case involves F and G; we must see that $E \in \text{ran}(s^{\mathcal{S}}_\beta)$ iff $p(E) \in s^{\mathcal{U}}_{\beta^*}$. But for $E \in \text{Ext}(\mathcal{T})$,

$$\begin{aligned} E \in \text{ran}(s^{\mathcal{T}}_\beta) &\Leftrightarrow p(E) \in s^{\mathcal{U}}_{v(\beta)} \\ &\Leftrightarrow p(E) \in \text{ran}(s^{\mathcal{U}}_{\beta^*}). \end{aligned}$$

The right-to-left implication in line 2 holds because if $E \notin \operatorname{ran}(s^{\mathcal{T}}_\beta)$ and $\operatorname{lh}(E) < \operatorname{lh}(E^{\mathcal{T}}_\beta)$, then E is incompatible with some $H \in \operatorname{ran}(s^{\mathcal{T}}_\beta)$, so $p(E)$ is incompatible with $p(H) \in s^{\mathcal{U}}_{v(\beta)}$, so the right hand side of line 2 fails. On the other hand, if $\operatorname{lh}(E) \geq \operatorname{lh}(E^{\mathcal{T}}_\beta)$, then $\operatorname{lh}(p(E)) \geq \operatorname{lh}(p(E^{\mathcal{T}}_\beta)) = \operatorname{lh}(E^{\mathcal{U}}_{u(\beta)})$, and since $\beta^* \leq u(\beta)$, again the right hand side of line 2 fails.

The case that $\alpha = \beta$ is similar. In this case, we apply the proposition to P^* with $\delta = v(\beta)$ and $\eta = \xi$. This gives us that

$$\beta^* = \text{least } \eta \in [v(\beta), \xi]_U \text{ such that } \operatorname{crit} \hat{\imath}^{\mathcal{U}}_{\eta,\xi} > \hat{\imath}^{\mathcal{U}}_{v(\beta),\eta} \circ t^0_\beta(\mu).$$

We leave the remaining details to the reader. $\square$

Remark 4.3 The proof gives a formula for the point of application of $E^{\mathcal{U}}_{u(\alpha)}$ under a psuedo hull embedding of $\mathcal{T}$ into $\mathcal{U}$, namely

$$U\text{-pred}(u(\alpha)+1) = \text{least } \eta \in [v(\beta), u(\beta)]_U \text{ such that } \operatorname{crit} \hat{\imath}^{\mathcal{U}}_{\eta,u(\beta)} > \hat{\imath}^{\mathcal{U}}_{v(\beta),\eta} \circ t^0_\beta(\mu),$$

where

$$\beta = T\text{-pred}(\alpha+1) \text{ and } \mu = \operatorname{crit}(E^{\mathcal{T}}_\alpha).$$

Remark 4.4 One can have the following situation, for $F = E^{\mathcal{T}}_\alpha$:

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{T}}_\alpha \arrow[d, "t^0_\alpha"'] \arrow[dr, "\rho"] \arrow[drr, "t^1_\alpha"] & & \\
\mathcal{M}^{\mathcal{U}}_{v(\alpha)} \arrow[r] & \mathcal{M}^{\mathcal{U}}_\gamma \arrow[r, "\hat{\imath}_{\gamma,u(\alpha)}"'] & \mathcal{M}^{\mathcal{U}}_{u(\alpha)}
\end{tikzcd}$$

It can happen that $\operatorname{dom} \rho = \mathcal{M}^{\mathcal{T}}_\alpha$, but $\operatorname{dom} t^1_\alpha = \mathcal{M}^{\mathcal{T}}_\alpha \restriction \operatorname{lh} F$, so $t^1_\alpha(F)$ is the last extender of $\mathcal{M}^{\mathcal{U}}_{u(\alpha)}$. In this case, $\hat{\imath}_{\gamma,u(\alpha)}$ is acting like a resurrection embedding, resurrecting $\rho(F)$, and $(\gamma, u(\alpha)]_U$ drops.

4.2 Resurrection embeddings as branch embeddings

We prove a technical lemma on normal iterations past levels of a background construction.

Let Σ be an iteration strategy for the premouse P_0, for finite stacks of normal trees, that normalizes well and has strong hull condensation. Suppose that Σ is universally Baire. Let $\mathbb{C}$ be a w-construction above $|P_0|^+$, and $\langle \nu_0, k_0 \rangle <$ length($\mathbb{C}$). Let us write $M_{\nu,k} = M^{\mathbb{C}}_{\nu,k}$. Suppose that whenever $\langle \nu, k \rangle <_{\text{lex}} \langle \nu_0, k_0 \rangle$, $M^{\mathbb{C}}_{\nu,k}$ is not a Σ-iterate of P_0. It has been known since the mid-80s that whenever $\langle \nu, k \rangle \leq_{\text{lex}} \langle \nu_0, k_0 \rangle$, only the P_0 side moves if we compare it with $M_{\nu,k}$ by least disagreement, using Σ to pick branches.

Thus for $\langle \nu, k \rangle \leq_{\text{lex}} \langle \nu_0, k_0 \rangle$, we have

$$\begin{aligned} \mathcal{W}^*_{\nu,k} = &\text{ unique shortest normal tree on } P_0 \text{ by } \Sigma \\ &\text{ with last model } Q \trianglerighteq M_{\nu,k}. \end{aligned}$$

Our technical lemma says that below $\langle \nu_0, k_0 \rangle$, the resurrection embeddings of $\mathbb{C}$ are captured by branch embeddings of the $\mathcal{W}^*_{\nu,k}$.

Lemma 4.5 *Let $\langle \theta, j \rangle \leq \langle \nu_0, k_0 \rangle$, and let $P \trianglelefteq M^{\mathbb{C}}_{\theta,j}$. Let $\tau = \sigma_{\theta,j}[P]^{\mathbb{C}}$, so that $\tau : P \to M^{\mathbb{C}}_{\theta_0,j_0}$, where $\langle \theta_0, j_0 \rangle = \text{Res}_{\theta,j}[P]$. Let*

$$\mathcal{T} = \mathcal{W}^*_{\theta,j} \restriction (\alpha + 1), \textit{ where } \alpha \textit{ is least such that } \mathcal{M}^{\mathcal{W}^*_{\theta,j}}_\alpha \trianglerighteq P.$$

Then $\mathcal{T} = \mathcal{W}^*_{\theta_0,j_0} \restriction (\alpha + 1)$, $\mathcal{W}^*_{\theta_0,j_0}$ has last model $\mathcal{M}^{\mathcal{W}^*_{\theta_0,j_0}}_\xi = M^{\mathbb{C}}_{\theta_0,j_0}$, and $\alpha \leq_{\mathcal{W}^*_{\theta_0,j_0}} \xi$, and $\tau = \hat{\imath}^{\mathcal{W}^*_{\theta_0,j_0}}_{\alpha,\xi}$.

We remark that our convention that $P \ntrianglelefteq Q$ when Q is active and $P = Q||o(Q)$ matters here. It could be that for α as in the lemma, $E = E^{\mathcal{W}^*_{\theta,j}}_{\alpha-1}$ is such that $\text{lh}(E) = o(P)$. The resurrection embedding τ is given by a branch of $\mathcal{W}^*_{\theta_0,j_0}$ that has α in it, and may not have $\alpha - 1$ in it, even though P is an initial segment of $\mathcal{M}^{\mathcal{W}^*_{\theta,j}}_{\alpha-1}$ in a weaker sense.

Recall that M^- is the premouse that is equal to M, except that $k(M^-) = k(M) - 1$.

Sublemma 4.5.1 *Suppose that $M_{\nu,k}$ is not $k+1$-sound. Let $\pi : M^-_{\nu,k+1} \to M_{\nu,k}$ be the anticore embedding. Let $\xi_0 + 1 = \text{lh}\, \mathcal{W}^*_{\nu,k+1}$; then*

(a) *$\mathcal{W}^*_{\nu,k}$ has last model $M_{\nu,k}$,*

(b) *$\mathcal{W}^*_{\nu,k+1} = \mathcal{W}^*_{\nu,k} \restriction (\xi_0 + 1)$,*

(c) *ξ_0 is the least η such that $\text{lh}\, E^{\mathcal{W}^*_{\nu,k}}_\eta > \rho(M_{\nu,k})$, and*

(d) *letting $lh(W^*_{\nu,k}) = \xi_1 + 1$, we have $\xi_0 <_{\mathcal{W}^*_{\nu,k}} \xi_1$, and $\hat{\imath}^{\mathcal{W}^*_{\nu,k}}_{\xi_0,\xi_1} = \pi$.*

Proof.

By definition, $\mathcal{M}_{\xi_1}^{\mathcal{W}^*_{\nu,k}} \trianglerighteq M_{\nu,k}$. But $M_{\nu,k}$ is not sound ($= k+1$-sound), so $\mathcal{M}_{\xi_1}^{\mathcal{W}^*_{\nu,k}} = M_{\nu,k}$. This gives (*a*).

The iteration $\mathcal{W}^*_{\nu,k}$ from P_0 to $M_{\nu,k}$ must have dropped. The last drop had to be to $M_{\nu,k+1}$, and it lies on the branch to $M_{\nu,k}$. So we can fix η such that $M_{\nu,k+1} = \operatorname{dom} \hat{\imath}_{\eta,\xi_1}^{W^*_{\nu,k}}$, and $\hat{\imath}_{\eta,\xi_1}^{\mathcal{W}^*_{\nu,k}} = \pi$. We have that $M_{\nu,k+1} \trianglelefteq M_\eta^{W^*_{\nu,k}}$.

Remark 4.6 Here and elsewhere, we are allowing the convention that a normal tree W may replace its last model $\langle Q, i\rangle$ with $\langle Q, n\rangle$ for any $n < i$. Otherwise, if $M_{\nu,k}$ and $M_{\nu,i+1}$ have the same universe, we couldn't possibly have both be normal iterates of P_0!

Letting $\rho = \rho(M_{\nu,k})$, we have that $M_{\nu,k+1}$ agrees with $M_{\nu,k}$ to $\rho^{+M_{\nu,k}} = \rho^{+M_{\nu,k+1}}$. Thus $\mathcal{W}^*_{\nu,k+1}$ and $\mathcal{W}^*_{\nu,k}$ use the same extenders E such that $\operatorname{lh} E \leq \rho$.

We claim that $\mathcal{W}^*_{\nu,k+1}$ uses no extenders E such that $\operatorname{lh}(E) > \rho$. For if $\mathcal{W}^*_{\nu,k+1}$ uses E such that $\operatorname{lh} E > \rho$, then the branch P_0-to-$\mathcal{M}_{\xi_0}^{\mathcal{W}^*_{\nu,k+1}}$ uses such an E, since $\xi_0 + 1 = \operatorname{lh} W^*_{\nu,k+1}$. $\operatorname{lh}(E) \leq o(M_{\nu,k+1})$ because $\mathcal{W}^*_{\nu,k+1}$ was of minimal length. But then $\rho \leq \operatorname{crit}(E)$ is impossible, because $\operatorname{dom}(E) \subseteq M_{\nu,k+1}$, and $M_{\nu,k+1}$ is sound. However, $\operatorname{crit}(E) < \rho$ is also impossible, since no model on the branch $[0, \xi_0]$ after E can project into $(\operatorname{crit}(E), \operatorname{lh} E)$.

So we have that $\mathcal{W}^*_{\nu,k+1} = \mathcal{W}^*_{\nu,k} \restriction \xi_0 + 1$. We have (a)-(c) of the sublemma already. For (d), we need to see $\xi_0 = \eta$. Since $M_{\nu,k+1} \trianglelefteq \mathcal{M}_\eta^{\mathcal{W}^*_{\nu,k}}$, $\xi_0 \leq \eta$. Suppose $\xi_0 < \eta$. Then $\operatorname{lh}(E_{\xi_0}^{\mathcal{W}^*_{\nu,k}}) > o(M_{\nu,k+1})$ because $M_{\nu,k+1} \trianglelefteq \mathcal{M}_\eta^{\mathcal{W}^*_{\nu,k}}$. But let K be the extender applied to $M_{\nu,k+1}$ in the branch of $\mathcal{W}^*_{\nu,k}$ leading to $M_{\nu,k}$, i.e. $K = E_\theta^{\mathcal{W}^*_{\nu,k}}$, where $W^*_{\nu,k}$-pred$(\theta+1) = \eta$ and $\theta + 1 \leq_{W^*_{\nu,k}} \xi_1$. $\rho \leq \operatorname{crit}(K) < o(M_{\nu,k+1}$, and $\operatorname{dom}(K) \subseteq M_{\nu,k+1}$. It follows that $\operatorname{lh}(E_{\xi_0}^{\mathcal{W}^*_{\nu,k}} = o(M_{\nu,k+1})$, as otherwise $\operatorname{dom}(K)$ is larger than that. But then $E_{\xi_0}^{\mathcal{W}^*_{\nu,k}}$ is on the sequence of $M_{\nu,k+1}$, but not that of $\mathcal{M}_\eta^{\mathcal{W}^*_{\nu,k}}$, contrary to our choice of η.

$\square$

Proof. [Proof of Lemma 4.5] We go by induction on $\langle \theta, j\rangle$. Suppose Lemma 4.5 holds for $\langle \theta', j'\rangle <_{\text{lex}} \langle \theta, j\rangle$, as well as for all $Q \triangleleft P$, where $P \trianglelefteq M_{\theta,j}$. Let

$$\begin{aligned}\rho = {} & \text{least } \kappa \text{ such that } \kappa = \rho_n(S) \text{ for some } S \trianglelefteq M_{\theta,j} \\ & \text{such that } P \trianglelefteq S, \text{ and } n = k(S).\end{aligned}$$

(Here we do *not* mean $\kappa = \rho(S) = \rho_{n+1}(S)$, where $n = k(S)$.) Pick S to be the first such. We can assume that $\rho < o(P)$, as otherwise $\tau =$ identity, and all is trivial. Thus $k(S) > 0$.

The reader can check that $\sigma_{\theta,j}[S]\restriction P = \sigma_{\theta,j}[P] = \tau$. If $S \lhd M_{\theta,j}$, then we can find some $\langle \theta', j' \rangle <_{\text{lex}} \langle \theta, j \rangle$ such that $S = M_{\theta',j'}$. [Let $\langle \nu, k \rangle$ be least such that $S \unlhd M_{\nu,k}$. If $S \neq M_{\nu,k}$, then $M_{\nu,k} = \text{core}(M_{\nu,k-1}) \neq M_{\nu,k-1}$. S projects to ρ, so $\rho(M_{\nu,k-1}) < \rho$. But this leads to a Q such that $S \unlhd Q \unlhd M_{\theta,j}$ and $\rho(Q) < \rho$.]

The argument above also shows that $\sigma_{\theta,j}[S] = \sigma_{\theta',j'}[S]$. So we can apply our induction hypothesis at θ', j'. Note that $\mathcal{W}^*_{\theta,j}\restriction(\alpha+1) = \mathcal{W}^*_{\theta',j'}\restriction(\alpha+1)$.

Thus we may assume $S = M_{\theta,j}$. So $j = k(S)$ and $j > 0$. If $\sigma_{\theta,j}[S] = \sigma_{\theta,j-1}[S]$, then as $\langle \theta, j-1 \rangle <_{\text{lex}} \langle \theta, j \rangle$, our induction hypothesis carries the day. Otherwise, we have that $M_{\theta,j-1}$ is not sound. Moreover

$$\sigma_{\theta,j}[S] = \pi \circ \sigma_{\theta,j-1}[S],$$

where $\pi : M^-_{\theta,j} \to M_{\theta,j-1}$ is the anticore embedding.

Let $\alpha + 1 = \text{lh}\,\mathcal{W}^*_{\theta,j}$ and $\beta + 1 = \text{lh}\,\mathcal{W}^*_{\theta,j-1}$. By the sublemma, $S \unlhd \mathcal{M}_\alpha^{\mathcal{W}^*_{\theta,j}}$ and $M_{\theta,j-1} = \mathcal{M}_\beta^{W^*_{\theta,j-1}}$, $\alpha \leq_{W^*_{\theta,j-1}} \beta$, and

$$\pi = \hat{\imath}^{\mathcal{W}^*_{\theta,j}}_{\alpha,\beta}.$$

Also, $\mathcal{W}^*_{\theta,j}$ uses only extenders of lh $\leq \rho$, so α is the least γ such that $P \unlhd \mathcal{M}_\gamma^{W^*_{\theta,j}}$.

Remark 4.7 The reason that the statement of Lemma 4.5 does not have $\alpha + 1 = \text{lh}\,\mathcal{W}^*_{\theta,j}$ is that that is clearly not always true. It becomes true when we reduce $\langle \theta, j \rangle$ to a $\langle \theta', j' \rangle$ with $S = M_{\theta',j'}$.

Let $P_1 = \pi(P)$. Let

$$\alpha_1 = \text{least } \gamma \text{ such that } P_1 \unlhd \mathcal{M}_\gamma^{\mathcal{W}^*_{\theta,j-1}}.$$

We can assume $\text{crit}(\pi) \leq o(P)$, as otherwise $P \unlhd M_{\theta,j-1}$ and $\tau = \sigma_{\theta,j-1}[P]$, so we are done by induction.

Claim 4.7.1 *$\alpha <_{\mathcal{W}^*_{\theta,j-1}} \alpha_1 \leq_{W^*_{\theta,j-1}} \beta$.*

Proof. Let $\gamma \in (\alpha, \beta]_{\mathcal{W}^*_{\theta,j-1}}$ be least such that $o(P_1) < \text{crit}(\hat{\imath}^{\mathcal{W}^*_{\theta,j-1}}_{\gamma,\beta})$. We claim that $\alpha_1 = \gamma$. Certainly, $P_1 \unlhd \mathcal{M}_\gamma^{\mathcal{W}^*_{\theta,j-1}}$. Also, $P_1 \ntrianglelefteq \mathcal{M}_\alpha^{\mathcal{W}^*_{\theta,j-1}}$. Since P_1 is in the range of $\hat{\imath}^{\mathcal{W}^*_{\theta,j-1}}_{\alpha,\beta}$, we get $\alpha_1 = \gamma$. See the proof of Proposition 4.1. □

The claim also showed that

$$\pi\restriction P = \hat{\imath}_{\theta,\alpha_1}\restriction P.$$

Now we apply our induction hypothesis to $P_1 \unlhd M_{\theta,j-1}$. We get θ_0, j_0 such that

1. $\mathcal{W}^*_{\theta_0,j_0}\restriction(\alpha_1+1) = W^*_{\theta,j-1}\restriction(\alpha_1+1)$.

2. $\mathcal{W}^*_{\theta_0,j_0}$ has last model $M_{\theta_0,j_0} = \mathcal{M}_\xi^{\mathcal{W}^*_{\theta_0,j_0}}$, and

3. $\alpha_1 \le_{W^*_{\theta_0,j_0}} \xi$, and $\sigma_{\theta,j-1}[P_1] = \hat{\imath}^{\mathcal{W}^*_{\theta_0,j_0}}_{\alpha_1,\xi}$.

But $\sigma_{\theta,j}[P] = \sigma_{\theta,j-1}[P_1] \circ \pi$. This yields $\sigma_{\theta,j}[P] = \hat{\imath}^{\mathcal{W}^*_{\theta_0,j_0}}_{\alpha_1,\xi} \circ \hat{\imath}^{\mathcal{W}^*_{\theta_0,j_0}}_{\alpha,\alpha_1} = \hat{\imath}^{\mathcal{W}^*_{\theta_0,j_0}}_{\alpha,\xi}$, as desired. □

4.3 Iterating into a backgrounded strategy

Let $\vec{F}$ be coarsely coherent, and $\vec{\mathcal{T}}$ a finite stack of normal $\vec{F}$-trees on V with last model R. We have shown in section 3 that there is at most one normal $\vec{F}$-tree $\mathcal{W}$ on V with last model R; we write $W(\vec{\mathcal{T}}) = \mathcal{W}$ in this case.

Definition 4.8 *(1) $\Omega^{\mathsf{UBH}}_{n,\vec{F}}$ is the partial iteration strategy for V: if $\mathcal{T}$ is a normal $\vec{F}$-tree by $\Omega^{\mathsf{UBH}}_{n,\vec{F}}$ of limit length, then*

$$\Omega^{\mathsf{UBH}}_{n,\vec{F}}(\mathcal{T}) = b \quad \text{iff} \quad b \text{ is the unique cofinal wellfounded branch of } \mathcal{T}.$$

(2) $\Omega^{\mathsf{UBH}}_{\vec{F}}$ is the partial iteration strategy for V: if $\vec{\mathcal{T}}^\frown\langle\mathcal{U}\rangle$ is a finite stack of normal $\vec{F}$-trees by $\Omega^{\mathsf{UBH}}_{\vec{F}}$ such that $\mathcal{U}$ has limit length, then

$$\Omega^{\mathsf{UBH}}_{\vec{F}}(\vec{\mathcal{T}}^\frown\langle\mathcal{U}\rangle) = b \quad \text{iff} \quad b \text{ is the unique cofinal branch of } \mathcal{U} \text{ such that } W(\vec{\mathcal{T}}^\frown\langle\mathcal{U}^\frown b\rangle) \text{ is by } \Omega^{\mathsf{UBH}}_{n,\vec{F}}.$$

So if V is uniquely iterable for $\vec{F}$-trees, then $\Omega^{\mathsf{UBH}}_{\vec{F}}$ is total, and it is the unique iteration strategy witnessing this. Moreover, $\Omega^{\mathsf{UBH}}_{\vec{F}}$ normalizes well, and has strong hull condensation. But our notation allows the case that $\Omega^{\mathsf{UBH}}_{\vec{F}}$ is partial.

Definition 4.9 *Let $\mathbb{C}$ be a w-construction above κ. Suppose $M_{\nu,k}$ exists. Then $\Omega^{\mathbb{C}}_{\nu,k}$ is the partial strategy for $M_{\nu,k}$ induced by $\Omega^{\mathsf{UBH}}_{\vec{F}^{\mathbb{C}}}$, i.e.*

$$\vec{\mathcal{T}} \text{ is by } \Omega^{\mathbb{C}}_{\nu,k} \quad \text{iff} \quad \text{lift}(\vec{\mathcal{T}}, M_{\nu,k}, \mathbb{C}) \text{ is by } \Omega^{\mathsf{UBH}}_{\vec{F}^{\mathbb{C}}}.$$

So if V is uniquely $\vec{F}^{\mathbb{C}}$-iterable above κ, then $\Omega^{\mathbb{C}}_{\nu,k}$ is total, normalizes well, and has strong hull condensation.

The following is essentially Theorem 0.5, but in the pure extender model case.

Theorem 4.10 *Let Σ be an iteration strategy for the premouse P_0, for finite stacks of normal trees, that normalizes well and has strong hull condensation. Suppose that Σ is universally Baire. Let $\mathbb{C}$ be a w-construction above $|P_0|^+$, and $\langle \nu, k\rangle <$ length$(\mathbb{C})$. Then either*

1. *there is a (unique) normal tree $\mathcal{T}$ by Σ on P_0 with last model $Q \trianglerighteq M_{\nu,k}$, and $\Sigma_{\mathcal{T},M_{\nu,k}} = \Omega^{\mathbb{C}}_{\nu,k}$, or*
2. *there is an $\langle \eta, l\rangle <_{\text{lex}} \langle \nu, k\rangle$ and a normal $\mathcal{T}$ on P_0 by Σ with last model $M_{\eta,l}$, and $\Sigma_{\mathcal{T},M_{\eta,l}} = \Omega^{\mathbb{C}}_{\eta,l}$.*

Remark 4.11 We did not assume unique iterability in the hypothesis of Theorem 4.10, but we did get the $\Omega^{\mathbb{C}}_{\eta,l}$ are total, until we reach on $M_{\nu,k}$ that is beyond Σ. Before that point, $\mathbb{C}$-lifted trees have unique cofinal wellfounded branches.

Proof. [Proof of Theorem 4.10] The proof is by induction on $\langle \nu, k\rangle$. Suppose that Theorem 4.10 holds at all $\langle \nu, k\rangle <_{\text{lex}} \langle \nu_0, k_0\rangle$. For $\langle \nu, k\rangle \leq_{\text{lex}} \langle \nu_0, k_0\rangle$, let

$$\begin{aligned}\mathcal{W}^*_{\nu,k} = &\text{ unique shortest normal tree on } P_0 \text{ by } \Sigma\\ &\text{ with last model } Q \trianglerighteq M_{\nu,k}.\end{aligned}$$

For $\langle \nu, k\rangle <_{\text{lex}} \langle \nu_0, k_0\rangle$, $\mathcal{W}^*_{\nu,k}$ exists by induction hypothesis. But in fact, $\mathcal{W}^*_{\nu,k}$ always exists because we are in the pure extender case, and this was proved long ago. (Cf. [20].) In the hod mouse case, we would have to proceed inductively on the construction of $\mathcal{W}^*_{\nu_0,k_0}$ at this point.

Let $M = M_{\nu_0,k_0}$, and let $\mathcal{U}$ be a normal tree on M that is of limit length, and is by both $\Sigma_{\mathcal{W}^*_{\nu_0,k_0},M}$ and $\Omega^{\mathbb{C}}_{\nu_0,k_0}$. Let

$$\text{lift}(\mathcal{U}, M, \mathbb{C}) = \langle \mathcal{U}^*, \langle \eta_\tau, l_\tau \mid \tau < \text{lh}\,\mathcal{U}\rangle, \langle \psi^{\mathcal{U}}_\tau \mid \tau < \text{lh}\,\mathcal{U}\rangle\rangle.$$

Lemma 4.12 *If b is a cofinal, wellfounded branch of $\mathcal{U}^*$, then $\Sigma_{\mathcal{W}^*_{\nu_0,k_0},M}(\mathcal{U}) = b$.*

Lemma 4.12 implies that $\mathcal{U}^*$ has at most one cofinal wellfounded branch. Moreover, that branch is identified by Σ, if it exists, and Σ is universally Baire. So a simple reflection argument will then give that $\mathcal{U}^*$ has a cofinal, wellfounded branch.

From this we get that $\Sigma_{\mathcal{W}^*_{\nu_0,k_0},M}$ and $\Omega^{\mathbb{C}}_{\nu_0,k_0}$ agree on normal trees, and then it is easy to see that they must agree on finite stacks of normal trees.

Proof. [Proof of Lemma 4.12] Let

$$S_\gamma = \mathcal{M}^{\mathcal{U}^*}_\gamma$$
$$N_\gamma = M^{S_\gamma}_{\eta_\gamma,l_\gamma} = M^{i^{\mathcal{U}^*}_{0,\gamma}(\mathbb{C})}_{\eta_\gamma,l_\gamma},$$

so that

$$\psi^{\mathcal{U}}_\gamma : \mathcal{M}^{\mathcal{U}}_\gamma \to N_\gamma$$

is elementary. We have $M = \mathcal{M}^{\mathcal{U}}_0 = N_0$, and $\psi^{\mathcal{U}}_0 =$ identity. We write $(\mathcal{W}^*_{\nu,k})^{S_\gamma}$ for $\langle \nu, k\rangle \leq_{\text{lex}} i^{\mathcal{U}^*}_{0,\gamma}(\langle \nu_0, k_0\rangle)$ to stand for $i^{\mathcal{U}^*}_{0,\gamma}(\langle \eta, l\rangle \mapsto \mathcal{W}^*_{\eta,l})_{\nu,k}$. Note that $i^{\mathcal{U}^*}_{0,\tau}(\Sigma) \cap S_\gamma = \Sigma \cap S_\gamma$, because Σ is universally Baire. Also $i^{\mathcal{U}^*}_{0,\gamma}(P_0) = P_0$. Thus $(\mathcal{W}^*_{\nu,k})^{S_\gamma}$ is by Σ.

The statement above also make sense for b replacing γ. So $S_b = \mathcal{M}^{\mathcal{U}^*}_b$, $N_b = M^{S_b}_{\eta_b,l_b}$, $\psi^{\mathcal{U}}_b : \mathcal{M}^{\mathcal{U}}_b \to N_b$, etc. Set

$$\mathcal{W}^*_\gamma = {\mathcal{W}^*_{\eta_\gamma,l_\gamma}}^{S_\gamma}$$

for $\gamma < \mathrm{lh}\,\mathcal{U}$ or $\gamma = b$. So $\mathcal{W}^*_0$ is our normal tree from P_0 to M that is by Σ. The last model of $\mathcal{W}^*_\gamma$ is N_γ. If $\nu <_U \gamma$ and $(\nu, \gamma]_U$ does not drop, then $i^{\mathcal{U}^*}_{\nu,\gamma}(\mathcal{W}^*_\nu) = \mathcal{W}^*_\gamma$. (This is not the case if we have a drop.)

Now let's look at the embedding normalization of $\langle \mathcal{W}^*_0, \mathcal{U}\rangle$. Set

$$\mathcal{W}_\gamma = W(\mathcal{W}^*_0, \mathcal{U}\restriction(\gamma + 1))$$

for $\gamma < \mathrm{lh}\,\mathcal{U}$, and

$$\mathcal{W}_b = W(\mathcal{W}^*_0, \mathcal{U}^\frown b).$$

So $\mathcal{W}_0 = \mathcal{W}^*_0$. The $\mathcal{W}_\gamma$'s are all by Σ, because Σ normalizes well and $\mathcal{U}\restriction(\gamma + 1)$ is by Σ. It will suffice to show that $\mathcal{W}_b$ is by Σ. That is because if $\Sigma(\langle \mathcal{W}_0, \mathcal{U}\rangle) = c$, then $\mathcal{W}_c$ is by Σ because Σ normalizes well, so $\mathrm{br}(b, W_0, \mathcal{U}) = \mathrm{br}(c, W_0, \mathcal{U})$, so $b = c$.

We shall show

Sublemma 4.12.1 *$\mathcal{W}_b$ is pseudo-hull of $\mathcal{W}^*_b$.*

That is enough to yield Lemma 4.12, since $\mathcal{W}^*_b$ is by Σ, and Σ has strong hull condensation.

Proof. [Proof of Sublemma 4.12.1] We construct by induction on γ a pseudo-hull embedding Φ_γ from $\mathcal{W}_\gamma$ into $\mathcal{W}^*_\gamma$. We write $z(\gamma) = \mathrm{lh}\,\mathcal{W}_\gamma - 1$, $z^*(\gamma) = \mathrm{lh}\,\mathcal{W}^*_\gamma - 1$, and

$$\Phi_\gamma = \langle u^\gamma, \langle t^{0,\gamma}_\beta \mid \beta \leq z(\gamma)\rangle, \langle t^{1,\gamma}_\beta \mid \beta < z(\gamma)\rangle, p^\gamma\rangle.$$

We also use $\Phi_\gamma \restriction \xi$ for the "initial segment" of Φ_γ that is a pseudo-hull embedding of $\mathcal{W}_\gamma \restriction \xi$ into $\mathcal{W}^*_\gamma$.

Remark 4.13 $\operatorname{dom} u^\gamma = z(\gamma)$. Let v^γ be as in Definition 2.26, i.e. $\hat{p}^\gamma(s_\alpha^{\mathcal{W}_\gamma}) = s_{v^\gamma(\alpha)}^{\mathcal{W}^*_\gamma}$. Then $\operatorname{dom} v^\gamma = z(\gamma)+1$. We shall maintain by induction that $v^\gamma(z(\gamma)) \leq_{W^*_\gamma} z^*(\gamma)$.

Let's recall the rest of our notation related to embedding normalization. We have partial maps $\phi_{\nu,\gamma} : \operatorname{lh} \mathcal{W}_\nu \to \operatorname{lh} \mathcal{W}_\gamma$ for $\nu <_U \gamma$, the maps being total if $(\nu, \gamma]_U$ does not drop in model or degree. We have also

$$\pi_\tau^{\nu,\gamma} : \mathcal{M}_\tau^{\mathcal{W}_\nu} \to \mathcal{M}_{\phi_{\nu,\gamma}(\tau)}^{\mathcal{W}_\gamma}$$

elementarily, for $\nu <_U \gamma$ and $\tau \in \operatorname{dom} \phi_{\nu,\gamma}$. Let also

$$e_{\nu,\gamma}(E_\alpha^{\mathcal{W}_\nu}) = E_{\phi_{\nu,\gamma}(\alpha)}^{\mathcal{W}_\gamma},$$

so that $e_{\nu,\gamma}$ is the natural partial map from $\operatorname{Ext}(\mathcal{W}_\nu)$ to $\operatorname{Ext}(\mathcal{W}_\gamma)$. (This map was called $\psi_{\nu,\gamma}$ in section 2.) $R_\eta = \mathcal{M}_{z(\eta)}^{\mathcal{W}_\eta}$ is the last model of $\mathcal{W}_\eta$. $\sigma_\eta : \mathcal{M}_\eta^{\mathcal{U}} \to R_\eta$, and

$$F_\eta = \sigma_\eta(E_\eta^{\mathcal{U}})$$

and

$$\mathcal{W}_{\eta+1} = W(\mathcal{W}_\xi, \mathcal{W}_\eta, F_\eta)$$

where $\xi = U\text{-pred}(\eta+1)$. Finally,

$$\alpha_\eta = \text{least } \alpha \text{ such that } F_\eta \text{ is on the } \mathcal{M}_\alpha^{\mathcal{W}_\eta} \text{ sequence.}$$

Since $\mathcal{W}_0 = \mathcal{W}^*_0$, Φ_0 is trivial, consisting of identity embeddings.

Remark 4.14 Let us look at the definition of Φ_1 in a simple case. Let $F = E_0^{\mathcal{U}} = \psi_0^{\mathcal{U}}(E_0^{\mathcal{U}})$. Let G be the resurrection of F in $\mathbb{C}$, and suppose $G = F$ for simplicity. Let F^* be the background extender for F given by $\mathbb{C}$. Then $\mathcal{W}_1 = W(\mathcal{W}_0, F)$ and $\mathcal{W}^*_1 = i_{F^*}(\mathcal{W}_0)$. Let $\alpha = \alpha(\mathcal{W}_0, F)$. The last model of $\mathcal{W}^*_1$ is $i_{F^*}(M)$, and $i_{F^*}(M)$ agrees with $\operatorname{Ult}_0(M, F)$ up to $\operatorname{lh}(F)+1$. (The "plus 1" part is important, and it is why we were careful about choosing our background extenders.) It follows that $\mathcal{W}^*_1$ uses F; in fact $\mathcal{W}_1 \restriction (\alpha+2) = \mathcal{W}^*_1 \restriction (\alpha+2)$, with $F = E_{\alpha+1}^{\mathcal{W}_1} = E_{\alpha+1}^{\mathcal{W}^*_1}$. This gives us the desired psuedo-hull embedding from $\mathcal{W}_1$ to $\mathcal{W}^*_1$. For example, the map $p^1\colon \operatorname{Ext}(\mathcal{W}_1) \to \operatorname{Ext}(\mathcal{W}^*_1)$ is given by:

$$p^1(E) = E, \text{ if } E = E_\xi^{\mathcal{W}_1} \text{ for some } \xi \leq \alpha + 1,$$

and if there is no dropping at $\alpha + 1$,

$$p^1(e_{0,1}(E)) = i_{F^*}(E).$$

This is typical of the general successor step. Various maps that are the identity in this special case are no longer so in the general case. In particular, the resurrection maps may not be the identity. But the key is still that if $\mathcal{W}_{\gamma+1} = W(\mathcal{W}_\nu, \mathcal{W}_\gamma, F)$, and $H = \psi_\gamma^{\mathcal{U}}(E_\gamma^{\mathcal{U}})$ is the blow up of F in the last model of $\mathcal{W}_\gamma^*$, and G is the resurrection of H inside S_γ, then $\mathcal{W}_{\gamma+1}^* = i_{G^*}(\mathcal{W}_\nu^*)$, and G is used in $\mathcal{W}_{\gamma+1}^*$. [There is a small revision to the first part of the conclusion in the dropping case.] In showing this, we shall need to know that the map resurrecting H to G appears as a branch embedding inside a certain normal tree $\mathcal{W}_\gamma^{**}$ extending $\mathcal{W}_\gamma^*$.

Setting $p^{\gamma+1}(F) = G$ determines everything. For we certainly want $p^{\gamma+1}$ to agree with p^γ on the extenders used before F in $\mathcal{W}_{\gamma+1}$. Moreover, we need to take a limit of the Φ_η's along branches of $\mathcal{U}$ in order to get past limit ordinals, and this requires that $p^{\gamma+1} \circ e_{\nu,\gamma+1} = i_{\nu,\gamma+1}^{\mathcal{U}^*} \circ p^\nu$. But this accounts for all the extenders in $\mathrm{dom}(p^{\gamma+1})$, so we have completely determined $p^{\gamma+1}$, and hence $\Phi_{\gamma+1}$, from Φ_ν.

The following little lemma says something about how $i_{\nu,\gamma}^{\mathcal{U}^*}(\mathcal{W}_\nu^*)$ sits inside $\mathcal{W}_\gamma^*$.

Lemma 4.15 *Suppose $\nu <_U \gamma$, and $(\nu, \gamma]_U$ does not drop. Let $\beta \leq z(\nu)$; then*

$$\sup i_{\nu,\gamma}^{\mathcal{U}^*} \text{“}\beta \leq_{W_\gamma^*} i_{\nu,\gamma}^{\mathcal{U}^*}(\beta).$$

Moreover, setting $\theta = \sup i_{\nu,\gamma}^{\mathcal{U}^*}$“$\beta$, we have that $(\theta, i_{\nu,\gamma}^{\mathcal{U}^*}(\beta)]_{W_\gamma^*}$ does not drop, and there is a unique embedding $l\colon \mathcal{M}_\beta^{\mathcal{W}_\nu^*} \to \mathcal{M}_\theta^{\mathcal{W}_\gamma^*}$ such that

$$i^{\mathcal{W}_\gamma^*}_{\theta, i_{\nu,\gamma}^{\mathcal{U}^*}(\beta)} \circ l = i_{\nu,\gamma}^{\mathcal{U}^*} \restriction \mathcal{M}_\beta^{\mathcal{W}_\nu^*}.$$

Proof. We have

$$i_{\nu,\gamma}^{\mathcal{U}^*}(\mathcal{W}_\nu^*) = \mathcal{W}_\gamma^*$$

because $(\nu, \gamma]_U$ did not drop. If β is a successor ordinal, or $i_{\nu,\gamma}^{\mathcal{U}^*}$ is continuous at β, then $\theta = i_{\nu,\gamma}^{\mathcal{U}^*}(\beta)$ and all is trivial. Otherwise, let $\tau <_{W_\nu^*} \beta$ be the site of the last drop; then $i_{\nu,\gamma}^{\mathcal{U}^*}(\tau)$ is the site of the last drop in $[0, i_{\nu,\gamma}^{\mathcal{U}^*}(\beta)]_{W_\nu^*}$, and $i_{\nu,\gamma}^{\mathcal{U}^*}(\tau) <_{\mathcal{W}_\gamma^*} \theta$. Finally, we can define l by: if $\eta \in (\tau, \beta)_{W_\nu^*}$ and

$$\mu = i_{\nu,\gamma}^{\mathcal{U}^*}(\eta),$$

then
$$l(i^{\mathcal{W}^*_\nu}_{\eta,\beta}(x)) = i^{\mathcal{W}^*_\gamma}_{\mu,\theta}(i^{\mathcal{U}^*}_{\nu,\gamma}(x)).$$
It is easy to see that this works. $\square$

The following diagram illustrates the lemma.

$$\begin{tikzcd}
& \mathcal{M}^{\mathcal{W}^*_\gamma}_\theta \arrow[r, "j_1"] & \mathcal{M}^{\mathcal{W}^*_\gamma}_{i^{\mathcal{U}^*}_{\nu,\gamma}(\beta)} \\
P_0 \arrow[ur, "j_0"] \arrow[r, "j"'] & \mathcal{M}^{\mathcal{W}^*_\nu}_\beta \arrow[u, "l"'] \arrow[ur, "i^{\mathcal{U}^*}_{\nu,\gamma}"'] &
\end{tikzcd}$$

Here $j_1 \circ j_0 = i^{\mathcal{U}^*}_{\nu,\gamma}(j)$. (The diagram assumes j exists, which is of course not the general case.) j_0 is given by the downward closure of $\{i^{\mathcal{U}^*}_{\nu,\gamma}(E) \mid E \text{ is used in } [0,\beta)_{W^*_\nu}\}$.

We proceed to the general successor step. Suppose we are give Φ_η for $\eta \leq \gamma$, and let us define $\Phi_{\gamma+1}$.

For any $\gamma + 1 < \mathrm{lh}\,\mathcal{U}$, let res_γ be the map resurrecting $\psi^{\mathcal{U}}_\gamma(E^{\mathcal{U}}_\gamma)$ inside S_γ. That is
$$\mathrm{res}_\gamma = (\sigma_{\eta_\gamma,l_\gamma}[M_{\eta_\gamma,l_\gamma}|\langle \mathrm{lh}\,\psi^{\mathcal{U}}_\gamma(E^{\mathcal{U}}_\gamma), 0\rangle])^{S_\gamma}.$$
Recall that $v^\gamma(z(\gamma)) \leq_{W^*_\gamma} z^*(\gamma)$ by induction. Let
$$t^\gamma = i^{\mathcal{W}^*_\gamma}_{v^\gamma(z(\gamma)),z^*(\gamma)} \circ t^{0,\gamma}_{z(\gamma)},$$
so that
$$t^\gamma : R_\gamma \to N_\gamma.$$

Induction Hypothesis †.

$(\dagger)_\gamma$ (*a*) For $\xi < \eta \leq \gamma$, $\Phi_\xi \restriction (\alpha_\xi + 1) = \Phi_\eta \restriction (\alpha_\xi + 1)$.

(*b*) For all $\eta \leq \gamma$, t^η is well defined; that is, $v^\eta(z(\eta)) \leq_{W^*_\eta} z^*(\eta)$.

(*c*) Let $\nu < \eta \leq \gamma$, and $\nu <_U \eta$, and suppose that $(\nu, \eta]_U$ does not drop. Let $i^* = i^{\mathcal{U}^*}_{\nu,\eta}$, and let $\tau = \phi_{\nu,\eta}(\xi)$; then

(i) if $\xi < z(\nu)$, then $u^\eta(\tau) = i^*(u^\nu(\xi))$,

(ii) if $\xi < z(\nu)$, setting $j = i^{\mathcal{W}^*_\nu}_{v^\nu(\xi),u^\nu(\xi)}$ and $k = i^{\mathcal{W}^*_\eta}_{v^\eta(\tau),u^\eta(\tau)}$, there is an embedding $l \colon \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(\xi)} \to \mathcal{M}^{\mathcal{W}^*_\eta}_{v^\eta(\tau)}$ such that $k \circ l = i^* \circ j$, and $t^{0,\eta}_\tau \circ \pi^{\nu,\eta}_\xi = l \circ t^{0,\nu}_\xi$, and

(iii) if $\xi = z(\nu)$, then setting $j = i^{\mathcal{W}_\nu^*}_{v^\nu(\xi), z^*(\nu)}$ and $k = i^{\mathcal{W}_\eta^*}_{v^\eta(\tau), z^*(\eta)}$, there is an embedding $l\colon \mathcal{M}^{\mathcal{W}_\nu^*}_{v^\nu(\xi)} \to \mathcal{M}^{\mathcal{W}_\eta^*}_{v^\eta(\tau)}$ such that $k \circ l = i^* \circ j$, and $t^{0,\eta}_\tau \circ \pi^{\nu,\eta}_\xi = l \circ t^{0,\nu}_\xi$.

(d) For $\nu < \eta \leq \gamma$, $t^{0,\eta}_{z(\eta)} \restriction (\operatorname{lh} F_\nu + 1) = \operatorname{res}_\nu \circ t^\nu \restriction (\operatorname{lh} F_\nu + 1)$.

(e) For $\xi \leq \gamma$, $\psi^{\mathcal{U}}_\xi = t^\xi \circ \sigma_\xi$.

There will be one additional induction hypothesis, but we must develop some notation before stating it.

Remark 4.16 Literally speaking, $(\dagger)_\gamma.(d)$ does not make sense, because $t^\nu(\operatorname{lh} F_\nu) \notin \operatorname{dom}(\operatorname{res}_\nu)$. Here and below, we are declaring that if $\sigma\colon P \to Q$ is a resurrection map, then $\sigma(o(P)) = o(Q)$.

$(\dagger)_\gamma.(c)$(i) says that $p^\eta(e_{\nu,\eta}(E)) = i^{\mathcal{U}^*}_{\nu,\eta}(p^\nu(E))$. Here is a diagram to go with the rest of this clause. In the diagram, $\tau = \phi_{\nu,\eta}(\xi)$. The far right assumes $u^\nu(\xi)$ exists, that is, $\xi < z(\nu)$.

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{W}_\eta}_\tau \arrow[r, "t^{0,\eta}_\tau"] & \mathcal{M}^{\mathcal{W}_\eta^*}_{v^\eta(\tau)} \arrow[r, "k"] & \mathcal{M}^{\mathcal{W}_\eta^*}_{u^\eta(\tau)} \\
\mathcal{M}^{\mathcal{W}_\nu}_\xi \arrow[u, "\pi^{\nu,\eta}_\xi"] \arrow[r, "t^{0,\nu}_\xi"] & \mathcal{M}^{\mathcal{W}_\nu^*}_{v^\nu(\xi)} \arrow[u, "l"] \arrow[r, "j"] & \mathcal{M}^{\mathcal{W}_\nu^*}_{u^\nu(\xi)} \arrow[u, "i^{\mathcal{U}^*}_{\nu,\eta}"]
\end{tikzcd}$$

Here j and k are the branch embeddings of $\mathcal{W}_\nu^*$ and $\mathcal{W}_\eta^*$. There is a similar diagram when $\xi = z(\nu)$, with $z^*(\nu)$ and $z^*(\eta)$ replacing $u^\nu(\xi)$ and $u^\eta(\tau)$.

Remark 4.17 One can think of $(\dagger)_\gamma.(c)$ as follows. The normalization process yields a natural psuedo-hull embedding Ψ from $\mathcal{W}_\nu$ into $\mathcal{W}_\eta$. The map $i^{\mathcal{U}^*}_{\nu,\eta}$ yields a full hull embedding Ω from $\mathcal{W}_\nu^*$ into $\mathcal{W}_\eta^*$. We are keeping track of the sense in which $\Phi_\eta \circ \Psi = \Omega \circ \Phi_\nu$.

Remark 4.18 The embedding along the bottom row of the diagram above is either $t^{1,\nu}_\xi$ or t^ν, depending on whether $\xi < z(\nu)$. The embedding along the top is either $t^{1,\eta}_\tau$ or t^η. So $(\dagger)_\gamma.(c)$ implies that

$$t^{1,\eta}_{\phi_{\nu,\eta}(\xi)} \circ \pi^{\nu,\eta}_\xi = i^{\mathcal{U}^*}_{\nu,\eta} \circ t^{1,\nu}_\xi$$

if $\xi < z(\nu)$, and

$$t^\eta \circ \pi^{\nu,\eta}_{z(\nu)} = i^{\mathcal{U}^*}_{\nu,\eta} \circ t^\nu.$$

Remark 4.19 $(\dagger)_\gamma$ implies that for $\nu < \eta \leq \gamma$,

$$t^{1,\eta}_{\alpha_\nu}\restriction(\mathrm{lh}\, F_\nu + 1) = \mathrm{res}_\nu \circ t^\nu \restriction(\mathrm{lh}\, F_\nu + 1).$$

This is because $\alpha_\nu < z(\eta)$, and $F_\nu = E^{\mathcal{W}_\eta}_{\alpha_\nu}$. So on $\mathrm{lh}(F_\nu)+1$, $t^{1,\eta}_{\alpha_\nu}$ agrees with $t^{0,\eta}_{z(\eta)}$ by the agreement properties of psuedo-hulls (2.29), and hence with $\mathrm{res}_\nu \circ t^\nu$ by $(\dagger)_\gamma.(d)$.

If $\alpha_\nu < z(\nu)$, then since Φ_ν is a pseudo-hull embedding, $t^\nu\restriction(\mathrm{lh}\, E^{\mathcal{W}_\nu}_{\alpha_\nu} + 1) = t^{1,\nu}_{\alpha_\nu}(\restriction \mathrm{lh}\, E^{W_\nu}_{\alpha_\nu} + 1)$. But $\mathrm{lh}\, F_\nu < \mathrm{lh}\, E^{\mathcal{W}_\nu}_{\alpha_\nu}$, so t^ν and $t^{1,\nu}_{\alpha_\nu}$ agree on $\mathrm{lh}(F_\nu)+1$.

Thus $t^{1,\nu}_{\alpha_\nu} \neq t^{1,\nu+1}_{\alpha_\nu}$ in general. (In fact, always.) If $\alpha_\nu = z(\nu)$, $t^{1,\nu}_{\alpha_\nu}$ is not defined, but $t^{1,\nu+1}_{\alpha_\nu}$ is. If $\alpha_\nu < z(\nu)$, they only agree up to $\mathrm{lh}\, F_\nu$ if res_ν is the identity on $t^{1,\nu}_{\alpha_\nu}(\mathrm{lh}\, F_\nu)$.

This is all consistent with $(\dagger)_\gamma.(a)$, because $t^{1,\nu}_{\alpha_\nu}$ is not part of $\Phi_\nu\restriction(\alpha_\nu+1)$. The map $t^{1,\xi}_\eta$ is recording how the extender $E^{\mathcal{W}_\xi}_\eta$ is blown up into $\mathcal{W}^*_\xi$. As we go from ν to $\nu+1$, $E^{\mathcal{W}_\nu}_{\alpha_\nu}$ is replaced by $F_\nu = E^{\mathcal{W}_{\nu+1}}_{\alpha_\nu}$. So the map blowing it up must be changed somewhat — even below $\mathrm{lh}\, F_\nu$, if there is resurrection going on in S_ν. But $E^{\mathcal{W}_\nu}_{\alpha_\nu}$ is not part of $\mathcal{W}_\nu\restriction(\alpha_\nu+1)$, so this does not affect (a).

Remark 4.20 In most cases, $(\dagger)_\gamma.(d)$ implies that t^η agrees with $\mathrm{res}_\nu \circ t^\nu$ on $\mathrm{lh}(F_\nu)+1$. For letting $G_\nu = t^{1,\eta}_{\alpha_\nu}(F_\nu)$, we have that

$$\mathrm{crit}(\hat{\imath}^{\mathcal{W}^*_\eta}_{v^\eta(z(\eta)),z^*(\eta)}) \geq \lambda_{G_\nu}.$$

Thus in any case, t^η agrees with $\mathrm{res}_\nu \circ t^\nu$ on λ_{F_ν}. The stronger agreement will fail iff $\mathrm{crit}(\hat{\imath}^{\mathcal{W}^*_\eta}_{v^\eta(z(\eta)),z^*(\eta)}) = \lambda_{G_\nu}$. The reader can check that for this to happen, F_ν must be the last extender used in $\mathcal{W}_\eta$, so that $\eta = \nu+1$, and $z(\eta) = \alpha_\nu + 1$.

In defining $\Phi_{\gamma+1}$, we shall make use of 4.5, which implies that res_γ is present in a branch embedding of some $(\mathcal{W}^*_{\nu,k})^{S_\gamma}$.

Recall that $t^\gamma = \hat{\imath}^{\mathcal{W}^*_\gamma}_{v^\gamma(z(\gamma)),z^*(\gamma)} \circ t^{0,\gamma}_{z(\gamma)}$, and $t^\gamma \colon R_\gamma \to N_\gamma$. Let

$$H = H_\gamma = \psi^{\mathcal{U}_\gamma}(E^{\mathcal{U}_\gamma}) = t^\gamma(F),$$

where $F = F_\gamma$. We use here $(\dagger)_\gamma.(e)$. Let

$$G = G_\gamma = \mathrm{res}_\gamma(H_\gamma),$$

and

$$G^* = E^{\mathcal{U}^*}_\gamma,$$

so that G^* is the background for G provided by $i^{\mathcal{U}^*}_{0\gamma}(\mathbb{C})$. G comes from resurrecting P inside S_γ, where

$$P = N_\gamma | \operatorname{lh} H_\gamma.$$

We apply our lemma on absorbing resurrection maps into the $\mathcal{W}^*$'s. Setting

$$\tau = \tau_\gamma = \text{least } \xi \text{ such that } P \trianglelefteq \mathcal{M}_\xi^{\mathcal{W}^*_\gamma},$$

we have that

Claim 4.21 *1. If $\alpha_\gamma = z(\gamma)$, then $\tau_\gamma \in [v^\gamma(\alpha_\gamma), z^*(\gamma)]_{W^*_\gamma}$,*

2. If $\alpha_\gamma < z(\gamma)$, then $\tau_\gamma \in [v^\gamma(\alpha_\gamma), u^\gamma(\alpha_\gamma)]_{W^*_\gamma}$.

Proof.

1. If $\alpha_\gamma = z(\gamma)$, then $v^\gamma(\alpha_\gamma) \leq_{W^*_\gamma} z^*(\gamma)$. $t^\gamma(F_\gamma) = \hat{i}^{\mathcal{W}^*_\gamma}_{v(\alpha_\gamma), z^*(\gamma)} \circ t^{0,\gamma}_{z(\gamma)}(F_\gamma)$ is on the sequence of $\mathcal{M}^{\mathcal{W}^*_\gamma}_{z^*(\gamma)}$. Since $\operatorname{lh} E^{\mathcal{W}_\gamma}_\xi < \operatorname{lh} F$ for all $\xi < \alpha_\gamma$, $\operatorname{lh}(p^\gamma(E^{\mathcal{W}_\gamma}_\xi)) < \operatorname{lh} t^\gamma(F)$ for all $\xi < \alpha_\gamma$. Cofinally many extenders used in $[0, v(\alpha_\gamma))_{\mathcal{W}^*_\gamma}$ are in $\operatorname{ran} p^\gamma$, which gives $\operatorname{lh} t^{0,\gamma}_{z(\gamma)}(F) > \operatorname{lh} E^{\mathcal{W}^*_\gamma}_\xi$ for all $\xi < v^\gamma(\alpha_\gamma)$. So $v^\gamma(\alpha_\gamma)$ is less than or equal to the least τ such that $t^\gamma(F)$ is on the $M^{\mathcal{W}^*_\gamma}_\tau$ sequence. That τ is the least η such that $t^\gamma(F) = \hat{i}^{\mathcal{W}^*_\gamma}_{v(\alpha_\gamma),\eta} \circ t^{0,\gamma}_{z(\gamma)}(F)$, so that $\tau \in [v^\gamma(\alpha_\gamma), z^*(\gamma)]_{W^*_\gamma}$. (See proposition 4.1.)

2. If $\alpha_\gamma < z(\gamma)$, then $t^\gamma(F) = t^{1,\gamma}_{\alpha_\gamma}(F) = \hat{i}^{W^*_\gamma}_{v^\gamma(\alpha_\gamma), u^\gamma(\alpha_\gamma)} \circ t^{0,\gamma}_{\alpha_\gamma}(F)$. In this case

$$\tau = \text{least } \beta \in [v(\alpha_\gamma), u(\alpha_\gamma)]_{W^*_\gamma} \text{ such that } \operatorname{crit}(\hat{i}_{\tau, u(\alpha_\gamma)}) > \hat{i}_{v(\alpha_\gamma),\tau}(\operatorname{lh} F).$$

   This can be shown as in 1. We omit the details.

$\square$

By Lemma 4.5, there is a normal tree $\mathcal{W}^{**}_\gamma$ such that

(i) $\mathcal{W}^{**}_\gamma$ is by Σ, and extends $\mathcal{W}^*_\gamma \restriction (\tau_\gamma + 1)$,

(ii) letting $\xi_\gamma = \operatorname{lh} \mathcal{W}^{**}_\gamma - 1$, G_γ is on the $\mathcal{M}^{\mathcal{W}^{**}_\gamma}_{\xi_\gamma}$ sequence, and not on the $\mathcal{M}^{\mathcal{W}^{**}_\gamma}_\alpha$ sequence for any $\alpha < \xi_\gamma$,

(iii) $\tau_\gamma \leq_{\mathcal{W}^{**}_\gamma} \xi_\gamma$, and $\hat{i}^{\mathcal{W}^{**}_\gamma}_{\tau_\gamma, \xi_\gamma} \restriction (\operatorname{lh} H_\gamma + 1) = \operatorname{res}_\gamma \restriction (\operatorname{lh} H_\gamma + 1)$.

Let

$$N^*_\gamma = \mathcal{M}^{\mathcal{W}^{**}_\gamma}_{\xi_\gamma}$$
$$= (M_{\theta,j})^{S_\gamma},$$

where $\langle \theta, j\rangle = (\mathrm{Res}_{\eta_\gamma,l_\gamma}[M_{\eta_\gamma,l_\gamma}|\langle \mathrm{lh}\, H_\gamma, 0\rangle])^{S_\gamma}$. We shall show that W^{**}_γ is an initial segment of $W^*_{\gamma+1}$, and that G_γ is used in $W^*_{\gamma+1}$. (So $G_\gamma = E^{W^*_{\gamma+1}}_{\xi_\gamma}$.) By induction, the same has been true at all $\nu < \gamma$. That is, we have

Induction Hypothesis $(\dagger)_\gamma$**.**

$(\dagger)_\gamma$ (f) . For all $\nu < \gamma$, N^*_ν agrees with N_γ strictly below $\mathrm{lh}\, G_\nu$. G_ν is on the N^*_ν-sequence, but $\mathrm{lh}\, G_\nu$ is a cardinal of N_γ. W^{**}_ν is an initial segment of $W^*_\gamma \restriction (v^\gamma(\alpha_\gamma)+1)$.

This gives more meaning to $(\dagger)_\gamma.(d)$. $\mathrm{res}_\nu \circ t^\nu$ maps $R_\nu \parallel \mathrm{lh}\, F_\nu$ elementarily into $N^*_\nu \parallel \mathrm{lh}\, G_\nu$, and t^γ maps $R_\gamma \parallel \mathrm{lh}\, F_\nu$ elementarily into $N_\gamma \parallel \mathrm{lh}\, G_\nu$. But the domain and range models here are the same, and the maps agree on the ordinals. So $\mathrm{res}_\nu \circ t^\nu \restriction (R_\nu \parallel \mathrm{lh}\, F_\nu) = t^\gamma \restriction (R_\gamma \parallel \mathrm{lh}\, F_\nu)$.

Here is a diagram showing where G came from, in the case that $\alpha_\gamma = z(\gamma)$.

$$\begin{tikzcd}
 & & (N_\gamma, H) & \\
 & & & (\mathcal{M}^{\mathcal{W}^{**}_\gamma}_{\xi_\gamma}, G) \\
 & & (\mathcal{M}^{\mathcal{W}^*_\gamma}_{\tau_\gamma}, H) \arrow[uu, "k"] \arrow[ur, "l"] & \\
\mathcal{M}^{\mathcal{U}}_\gamma \arrow[r, "\sigma_\gamma"] & (R_\gamma, F) \arrow[uuur, "t^\gamma"] \arrow[r, "t^{0,\gamma}_{\alpha_\gamma}"] & \mathcal{M}^{\mathcal{W}^*_\gamma}_{v^\gamma(\alpha_\gamma)} \arrow[u] & \\
 & & P_0 \arrow[ull, "\mathcal{W}_\gamma"] \arrow[ul] \arrow[u, "\mathcal{W}^*_\gamma"] \arrow[uuur, "\mathcal{W}^{**}_\gamma"'] &
\end{tikzcd}$$

Here k is the branch embedding of $\mathcal{W}_\gamma^*$, and it is the identity on $\mathrm{lh}(H)+1$. l is the branch embedding of $\mathcal{W}_\gamma^{**}$, and it agrees with res_γ on $\mathrm{lh}(H)+1$.

If $\alpha_\gamma < z(\gamma)$, then the corresponding diagram is:

$$\begin{tikzcd}
 & & (N_\gamma, H) & & (\mathcal{M}^{\mathcal{W}_\gamma^*}_{u^\gamma(\alpha_\gamma)}, H) & \\
 & & & & & (\mathcal{M}^{\mathcal{W}_\gamma^{**}}_{\xi_\gamma}, G) \\
 & & & & (\mathcal{M}^{\mathcal{W}_\gamma^*}_{\tau_\gamma}, H) \arrow[uu, "k"] \arrow[ur, "l"] & \\
\mathcal{M}^{\mathcal{U}}_\gamma \arrow[r, "\sigma_\gamma"] & (R_\gamma, F) \arrow[uuur, "t^\gamma"] & & (\mathcal{M}^{\mathcal{W}_\gamma}_{\alpha_\gamma}, F) \arrow[uuur, "t^{1,\gamma}_{\alpha_\gamma}"] \arrow[r, "t^{0,\gamma}_{\alpha_\gamma}"] & \mathcal{M}^{\mathcal{W}_\gamma^*}_{v^\gamma(\alpha_\gamma)} \arrow[u] & \\
 & & & & P_0 \arrow[u, "\mathcal{W}_\gamma^*"] \arrow[uuur, "\mathcal{W}_\gamma^{**}"'] \arrow[ul] \arrow[ulll, "\mathcal{W}_\gamma"] \arrow[ullll, "\mathcal{W}_\gamma"] &
\end{tikzcd}$$

Here again, k is the branch embedding of $\mathcal{W}_\gamma^*$, and it is the identity on $\mathrm{lh}(H)+1$. l is the branch embedding of $\mathcal{W}_\gamma^{**}$, and it agrees with res_γ on $\mathrm{lh}(H)+1$. R_γ and $\mathcal{M}^{\mathcal{W}_\gamma}_{\alpha_\gamma}$ agree up to $\mathrm{lh}(F)+1$, and t^γ agrees with $t^{1,\gamma}_{\alpha_\gamma}$ on $\mathrm{lh}(F)+1$. (In fact, on $\lambda_{E^{\mathcal{W}_\gamma}_{\alpha_\gamma}}$.)

In either case, we get

Claim 4.22 $\mathrm{res}_\gamma \circ t^\gamma$ *agrees with* $\hat{\imath}^{\mathcal{W}_\gamma^{**}}_{v^\gamma(\alpha_\gamma),\xi_\gamma} \circ t^{0,\gamma}_{\alpha_\gamma}$ *on* $\mathrm{lh}(F)+1$.

Proof. Suppose $\alpha_\gamma < z(\gamma)$. Let k and l be as in the diagram above. Then for $\eta \le \mathrm{lh}(F)$,

$$
\begin{aligned}
\mathrm{res}_\gamma \circ t^\gamma(\eta) &= \mathrm{res}_\gamma \circ t^{1,\gamma}_{\alpha_\gamma}(\eta) \\
&= \mathrm{res}_\gamma \circ (k \circ \hat{\imath}^{\mathcal{W}^*_\gamma}_{v^\gamma(\alpha_\gamma),\tau_\gamma} \circ t^{0,\gamma}_{\alpha_\gamma})(\eta) \\
&= \mathrm{res}_\gamma \circ (\hat{\imath}^{\mathcal{W}^*_\gamma}_{v^\gamma(\alpha_\gamma),\tau_\gamma} \circ t^{0,\gamma}_{\alpha_\gamma})(\eta) \\
&= l \circ (\hat{\imath}^{\mathcal{W}^*_\gamma}_{v^\gamma(\alpha_\gamma),\tau_\gamma} \circ t^{0,\gamma}_{\alpha_\gamma})(\eta) \\
&= \hat{\imath}^{\mathcal{W}^{**}_\gamma}_{v^\gamma(\alpha_\gamma),\xi_\gamma} \circ t^{0,\gamma}_{\alpha_\gamma}(\eta),
\end{aligned}
$$

as desired. The calculation when $\alpha_\gamma = z(\gamma)$ is similar. $\square$

Now let

$$\nu = U\text{-pred}(\gamma + 1).$$

Thus we have

$$S_{\gamma+1} = \mathrm{Ult}(S_\nu, G^*),$$

where G^* is the background extender for $G = G_\gamma$ provided by $i^{\mathcal{U}^*}_{0\gamma}(\mathbb{C})$. We write

$$i_{G^*} = i^{\mathcal{U}^*}_{\nu,\gamma+1}$$

for the canonical embedding.

Case 1. $(\nu, \gamma + 1]_U$ does not drop in model or degree.

In this case, we have

$$
\begin{aligned}
\langle \eta_{\gamma+1}, l_{\gamma+1}\rangle &= i_{G^*}(\langle \eta_\nu, l_\nu\rangle) \\
N_{\gamma+1} &= i_{G^*}(N_\nu)
\end{aligned}
$$

and

$$\mathcal{W}^*_{\gamma+1} = i_{G^*}(\mathcal{W}^*_\nu).$$

Our goal is to define $\Phi_{\gamma+1}$, and with it $t^{\gamma+1}$, so that the following diagram is realized (among other things).

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{U}}_{\gamma+1} \arrow[rr, "\sigma_{\gamma+1}"] \arrow[rrrr, bend left, "\psi^{\mathcal{U}}_{\gamma+1}"] & & R_{\gamma+1} \arrow[rr, "t^{\gamma+1}"] & & N_{\gamma+1} & S_{\gamma+1} \\
\mathcal{M}^{\mathcal{U}}_{\nu} \arrow[u] \arrow[r, "\sigma_\nu"] & R_\nu \arrow[ur, "\pi^{\nu,\gamma+1}_{z(\nu)}"] \arrow[rr, "t^\nu"] & & N_\nu \arrow[ur, "i_{G^*}"] & & S_\nu \arrow[u, "i_{G^*}"'] \\
& & P_0 \arrow[ull] \arrow[ul, "\mathcal{W}_\nu"] \arrow[uu, "\mathcal{W}_{\gamma+1}"'] & & P_0 \arrow[ul, "\mathcal{W}^*_\nu"] \arrow[uu, "\mathcal{W}^*_{\gamma+1}"'] &
\end{tikzcd}$$

As we remarked in the case $\gamma+1=1$, it is important to see that the resurrection of the blowup of F, which is in our case G, is used in $\mathcal{W}^*_{\gamma+1}$.

Claim 4.23 *(a)* $\mathcal{W}^*_{\gamma+1}\restriction\xi_\gamma = \mathcal{W}^{**}_\gamma\restriction\xi_\gamma$.

(b) $G = E^{\mathcal{W}^*_{\gamma+1}}_{\xi_\gamma}$.

Proof. Let $\mu = \mathrm{crit}(F)$, where $F = F_\gamma$. Let $\sigma_\gamma(\bar{\mu}) = \mu$, where $\bar{\mu} = \mathrm{crit}(E^{\mathcal{U}}_\gamma)$. Since $\mathcal{U}$ does not drop at $\gamma+1$, no level of $M^{\mathcal{U}}_\nu$ beyond $\mathrm{lh}\, E^{\mathcal{U}}_\nu$ projects to or below $\bar{\mu}$. So no level of R_ν beyond $\mathrm{lh}\, F_\nu$ projects to or below μ. So no level of N_ν beyond $\mathrm{lh}\, H_\nu$ projects to or below $t^\nu(\mu)$. Thus res_ν is the identity on $t^\nu(\mu)^{+N_\nu}$, and $N^*_\nu\restriction(t^\nu(\mu)^+)^{N^*_\nu} = N_\nu\restriction(t^\nu(\mu)^+)^{N_\nu}$. Also, $(t^\nu(\mu)^+)^{N^*_\nu} < \lambda_{G_\nu}$. Thus

$$N_\nu|t^\nu(\mu)^{+N_\nu} = N^*_\nu|t^\nu(\mu)^{+N^*_\nu} = N_\gamma|t^\nu(\mu)^{+N_\gamma}.$$

But also, if $\nu < \gamma$, then no proper initial segment of $\mathcal{M}^{\mathcal{U}}_\gamma$ projects to or below $\mathrm{lh}\, E^{\mathcal{U}}_\nu$, so no proper initial segment of N_γ projects to or below $\mathrm{lh}\, G_\nu$, so $\mathrm{res}_\gamma = \mathrm{id}$ on $\mathrm{lh}\, G_\nu$, and $N_\gamma|t^\gamma(\mu^+)^{N_\gamma} = N^*_\gamma|t^\gamma(\mu^+)^{N^*_\gamma}$.. Thus in both cases ($\nu < \gamma$ and $\nu = \gamma$),

$$N_\gamma|t^\gamma(\mu^+)^{N_\gamma} = N^*_\gamma|t^\gamma(\mu^+)^{N^*_\gamma}.$$

Letting $\lambda = t^\gamma(\mu^+)^{N_\gamma^*}$, we have then that $i_{G^*}(N_\gamma|\lambda) = i_{G^*}(N_\gamma^*|\lambda)$. But $\mathrm{Ult}(N_\gamma^*, G)$ agrees with $i_{G^*}(N_\gamma^*|\lambda)$ up to $\mathrm{lh}\, G + 1$. (We chose G^* so that they would agree *at* $\mathrm{lh}\, G$.) Thus

$$N_{\gamma+1} \parallel \mathrm{lh}\, G = N_\gamma^* \parallel \mathrm{lh}\, G$$

and $\mathrm{lh}\, G$ is a cardinal in $N_{\gamma+1}$. Since $W_{\gamma+1}^*$ and W_γ^{**} are normal trees by the same strategy Σ, we get Claim 4.23. □

By lemma 4.2, there is a unique psuedo-hull embedding Ψ of $\mathcal{W}_{\gamma+1}\restriction(\alpha_\gamma + 2)$ into $\mathcal{W}_{\gamma+1}^*$ such that Ψ extends $\Phi_\gamma\restriction(\alpha_\gamma+1)$, and $u^\Psi(\alpha_\gamma) = \xi_\gamma$, or equivalently, $p^\Psi(F) = G$. We let $\Phi_{\gamma+1}\restriction(\alpha_\gamma + 2)$ be the unique such Ψ.

In order to establish the proper notation related to $\Phi_{\gamma+1}\restriction(\alpha_\gamma + 2)$, as well as its relationship to Φ_ν, we shall now just run through the proof of lemma 4.2 again.

Let's keep our notation $\mu = \mathrm{crit}(F)$, and write

$$\mu^* = t^\nu(\mu) = t^\gamma(\mu) = \mathrm{crit}(G).$$

Let

$$\beta = \beta^{\mathcal{W}_\nu, F},$$

so that F is applied to $\mathcal{M}_\beta^{\mathcal{W}_\nu} = \mathcal{M}_\beta^{\mathcal{W}_{\gamma+1}}$ in $\mathcal{W}_{\gamma+1}$. Let

$$\beta^* = W_{\gamma+1}^*\text{-pred}(\xi_\gamma + 1),$$

so that G is applied to $\mathcal{M}_{\beta^*}^{\mathcal{W}_{\gamma+1}^*} = \mathcal{M}_{\beta^*}^{\mathcal{W}_\gamma^{**}}$ in $\mathcal{W}_{\gamma+1}^*$.

Claim 4.24 *(a) $\beta^* \le \tau_\nu$, and $\mathcal{M}_{\beta^*}^{\mathcal{W}_\nu^*} = \mathcal{M}_{\beta^*}^{\mathcal{W}_\nu^{**}} = \mathcal{M}_{\beta^*}^{\mathcal{W}_\gamma^*} = \mathcal{M}_{\beta^*}^{\mathcal{W}_\gamma^{**}} = \mathcal{M}_{\beta^*}^{\mathcal{W}_{\gamma+1}^*}$.*

(b) $\beta^* = \mu^*$.

(c) If $\beta < z(\nu)$, then $\beta^* \in [v^\nu(\beta), u^\nu(\beta)]_{W_\nu^*}$.

(d) If $\beta = z(\nu)$, then $\beta^* \in [v^\nu(\beta), z^*(\nu)]_{W_\nu^*}$.

Proof. Let P be the domain of F and P^* the domain of G; that is,

$$P = R_\gamma|(\mu^+)^{R_\gamma}$$

and

$$P^* = N_\gamma|(t^\gamma(\mu)^+)^{N_\gamma} = N_\gamma^*|(t^\gamma(\mu)^+)^{N_\gamma^*}.$$

(N_γ agrees with N_γ^* this far because we are not dropping when we apply F.) By the rules of normality,

$$\beta^* = \text{least } \alpha \text{ such that } P^* = \mathcal{M}_\alpha^{\mathcal{W}_\gamma^{**}}|o(P^*).$$

Put another way, $\mathcal{W}_\gamma^{**}\restriction\beta^*+1$ is unique shortest normal tree on P_0 by Σ such that P^* is an initial segment of its last model, and $o(P^*)$ is passive in its last model. But we showed in the proof of Claim 4.23 that $P^* = N_\nu^*|o(P^*)$, and $o(P^*) < \lambda_{G_\nu}$. We also showed that $(\mathrm{res}_\nu)\restriction P^* =$ identity. Thus $P^* = N_\nu|o(P^*)$, and $o(P^*) < \lambda_{H_\nu}$. So P^* is a passive initial segment of the last models of $\mathcal{W}_\nu^*, \mathcal{W}_\nu^{**}, \mathcal{W}_\gamma^*, \mathcal{W}_\gamma^{**}$, and $\mathcal{W}_{\gamma+1}^*$. Thus all these trees agree up to β^*+1. As $o(P^*) < \mathrm{lh}(H_\nu)$, $\beta^* \le \tau_\nu$. This yields (a).

For (b), note that μ^* is a cardinal of S_γ, so $|\mathcal{M}_\alpha^{\mathcal{W}_\gamma^*}| < \mu^*$ in S_γ, for all $\alpha < \mu^*$. It follows that $\mu^* \le \beta^*$, and if $s = s_{\mu^*}^{\mathcal{W}_\gamma^*}$ is the branch extender, then $s\colon \mu^* \to V_{\mu^*}$. If $\mu^*+1 = \mathrm{lh}(W_\gamma^*)$ or $\lambda(E_{\mu^*})^{\mathcal{W}_\gamma^*)} > \mu^*$, then $\beta^* \le \mu^*$. So we may assume that $E = E_{\mu^*}^{\mathcal{W}_\gamma^*}$ exists, and $\lambda_E = \mu^*$. This implies $P^* = \mathcal{M}_{\mu^*}^{\mathcal{W}_\gamma^*}||\,\mathrm{lh}(E)$.

Working in S_γ, let

$$\mathcal{T} = i_{G^*}(\mathcal{W}_\gamma^*).$$

and

$$Q = i_{G^*}(\mathcal{M}_{\mu^*}^{\mathcal{W}_\gamma^*}) = \mathcal{M}_\theta^{\mathcal{T}},$$

where $\theta = i_{G^*}(\mu^*)$. Since $s = i_{G^*}(s)\restriction\mu^*$, we have that $\mathcal{M}_{\mu^*}^{\mathcal{W}_\gamma^*} = \mathcal{M}_{\mu^*}^{\mathcal{T}}$, $\mu^* \in [0,\theta)_T$, and $[\mu^*,\theta)_T$ has no drops. Thus $\mathcal{M}_{\mu^*}^{\mathcal{W}_\gamma^*}$ agrees with Q up to their common value of $\mu^{*,+}$, and in particular, E is on the Q-sequence. It follows that E is on the sequence of $i_{G^*}(P^*)$. But now let

$$k\colon\ \mathrm{Ult}(P^*, G) \to i_{G^*}(P^*)$$

be the canonical factor map. We have that $\mathrm{crit}(k) = \lambda_G$, and in particular, $\mathrm{crit}(k) > o(P^*)$. Since $o(P^*)$ is passive in $\mathrm{Ult}(P^*, G)$, it must be passive in $i_{G^*}(P^*)$, contrary to our assumption that E is indexed there. This proves (b).

For (c): if $\beta < z(\nu)$, then $\mu < \lambda_{E_\beta^{\mathcal{W}_\nu}}$, so

$$\begin{aligned}\mu^* &= t^\nu(\mu) = t_\beta^{1,\nu}(\mu)\\ &= \hat{\imath}^{\mathcal{W}_\nu^*}_{v^\nu(\beta),u^\nu(\beta)} \circ t_\beta^{0,\nu}(\mu).\end{aligned}$$

Also, $\mu^* < \lambda(E_{u^\nu(\beta)}^{\mathcal{W}_\nu^*})$, so $\beta^* \le u^\nu(\beta)$ and $P^* \lhd M_{u^\nu(\beta)}^{\mathcal{W}_\nu^*} \parallel \lambda(E_{u^\nu(\beta)}^{\mathcal{W}_\nu^*})$. But since

$$P^*,\ \mu^* \in \mathrm{ran}\, i^{\mathcal{W}_\nu^*}_{v^\nu(\beta),u^\nu(\beta)}$$

(we don't actually need $\hat{\imath}$ because in this case $[v^\nu(\beta), u^\nu(\beta)]_{W_\nu^*}$ does not drop), we get

$$\beta^* = \text{least } \alpha \in [v^\nu(\beta), u^\nu(\beta)]_{\mathcal{W}_\nu^*} \text{ such that } \mathrm{crit}(i^{\mathcal{W}_\nu^*}_{\alpha,u^\nu(\beta)}) > i^{\mathcal{W}_\nu^*}_{v^\nu(\beta),\alpha}(t_\beta^{0,\nu}(\mu)) \text{ or } \alpha = u^\nu(\beta)..$$

Proposition 4.1 essentially proves this, but the situation is not quite the same, so we repeat the argument.

First, note that $v^\nu(\beta) \leq \beta^*$. For if $E = E_\eta^{\mathcal{W}_\nu}$ is used in $[0,\beta)_{W_\nu}$, then $\lambda_E \leq \mu$, and thus $\lambda_{p^\nu(E)} = t_\eta^{1,\nu}(\lambda_E) \leq t^\nu(\lambda_E) \leq t^\nu(\mu) = \mu^*$. This implies $v^\nu(\beta) \leq \beta^*$.

We have by the agreement of $\mathcal{W}_\gamma^{**}$ with $\mathcal{W}_\nu^*$ up to β^*+1 that

$$\beta^* = \text{least } \alpha \text{ such that } P^* = \mathcal{M}_\alpha^{\mathcal{W}_\nu^*}|o(P^*).$$

Let α be least such that $\alpha \in [v^\nu(\beta), u^\nu(\beta)]_{\mathcal{W}_\nu^*}$ and $\mathrm{crit}(i_{\alpha,u^\nu(\beta)}^{\mathcal{W}_\nu^*}) > i_{v^\nu(\beta),\alpha}^{\mathcal{W}_\nu^*}(t_\beta^{0,\nu}(\mu))$ or $\alpha = u^\nu(\beta)$. We want to see $\beta^* = \alpha$. Since $P^* = \mathcal{M}_{u^\nu(\beta)}^{\mathcal{W}_\nu^*}|o(P^*)$, we have $\beta^* \leq \alpha$. We must see $\alpha \leq \beta^*$. If $\alpha = v^\nu(\beta)$, this holds, so assume $\alpha > v^\nu(\beta)$.

If $\sigma < \alpha$ and $E_\sigma^{\mathcal{W}_\nu^*}$ is used in $[0,\alpha)_{W_\nu^*}$, then $\lambda(E_\sigma^{\mathcal{W}_\nu^*}) \leq o(P^*)$. This is true if $\sigma+1 \leq v^\nu(\beta)$ because $v^\nu(\beta) \leq \beta^*$. If $v^\nu(\beta) < \sigma+1$, then E_σ is used in $(v^\nu(\beta),\alpha]_{W_\nu^*}$, and since $P^* \in \mathrm{ran}\, i_{v^\nu(\beta),u^\nu(\beta)}^{\mathcal{W}_\nu^*}$, $o(P^*) < \mathrm{crit}(E_\sigma)$, and α was not least.

It follows that $\mathrm{lh}(E_\sigma^{\mathcal{W}_\nu^*}) \leq o(P^*)$ for all $\sigma < \alpha$ such that $E_\sigma^{\mathcal{W}_\nu^*}$ is used in $[0,\alpha)_{W_\nu^*}$, and hence for all $\sigma < \alpha$ whatsoever. So if $\sigma < \alpha$, $P^* \neq \mathcal{M}_\sigma^{\mathcal{W}_\nu^*}|o(P^*)$, as E_σ is on the sequence of the latter model, but not of the former. Thus $\alpha \leq \beta^*$, as desired.

This gives *(c)*. The proof of *(d)* is similar. □

With regard to part (b) of the claim: it is perfectly possible that β is a successor ordinal. We can even have $\beta = \alpha+1$, where $\lambda_{E_\alpha} = \mu$. In this case $v^\nu(\beta) < \beta^* = \mu^*$, and $t_\beta^{0,\nu}(\mu) < \mu^*$ as well. So $\beta^* = \mu^*$ is strictly between $v^\nu(\beta)$ and either $u^\nu(\beta)$ or $z^*(\nu)$, as the case may be. This is a manifestation of the fact that the psuedo-hull embeddings Φ_ν are very far from being onto, when $\nu > 0$.

Claim 4.25 *1. If $\beta < z(v)$, then $\beta^* =$ least $\alpha \in [v^\nu(\beta), u^\nu(\beta)]_{W_\nu^*}$ such that $\mathrm{crit}(i_{\alpha,u^\nu(\beta)}^{\mathcal{W}_\nu^*}) > i_{v^\nu(\beta),\alpha}^{\mathcal{W}_\nu^*}(t_\beta^{0,\nu}(\mu))$.*

2. If $\beta = z(v)$, then $\beta^* =$ least $\alpha \in [v^\nu(\beta), z^*(\nu)]_{W_\nu^*}$ such that $\mathrm{crit}(i_{\alpha,z^*(\nu)}^{\mathcal{W}_\nu^*}) > i_{v^\nu(\beta),\alpha}^{\mathcal{W}_\nu^*}(t_\beta^{0,\nu}(\mu))$.

3. In either case, the embeddings t^ν, $\mathrm{res}_\nu \circ t^\nu$, and $i_{v^\nu(\beta),\beta^*}^{\mathcal{W}_\nu^*} \circ t_\beta^{0,\nu}$ all agree on the domain of F.

Proof. This is what we actually showed in Claim 4.24. The following diagram illustrates the situation when $\beta < z(\nu)$.

$$\begin{tikzcd}
 & & N_\nu & \mathcal{M}^{\mathcal{W}_\nu^*}_{u^\nu(\beta)} \\
 & & & \mathcal{M}^{\mathcal{W}_\nu^*}_{\beta^*} \arrow[u, "k"'] \\
\mathcal{M}^{\mathcal{U}}_\nu \arrow[r, "\sigma_\nu"] & R_\nu \arrow[uur, "t^\nu"] & \mathcal{M}^{\mathcal{W}_\nu}_\beta \arrow[r, "t^{0,\nu}_\beta"] \arrow[uur, "t^{1,\nu}_\beta"] & \mathcal{M}^{\mathcal{W}_\nu^*}_{v^\nu(\beta)} \arrow[u]
\end{tikzcd}$$

We have shown that both k and res_ν are the identity on the domain of G, that is, on $t^\nu(\mu)^+$ of $\mathcal{M}^{\mathcal{W}_\nu^*}_{\beta^*}$. The agreement of t^ν with $t^{1,\nu}_\beta$ on $\mathrm{lh}(E^{\mathcal{W}_\nu}_\beta)$, which is strictly greater than $(\mu^+)^{R_\gamma}$, completes the proof. The case that $\beta = z(\nu)$ is similar. $\square$

Now let
$$\rho = i^{\mathcal{W}_\nu^*}_{v^\nu(\beta),\beta^*} \circ t^{0,\nu}_\beta,$$
so that $\rho : \mathcal{M}^{\mathcal{W}_\nu}_\beta \to \mathcal{M}^{\mathcal{W}_\nu^*}_{\beta^*}$. On the domain of F, ρ agrees with t^ν and with $\mathrm{res}_\nu \circ t^\nu$. We can then define $\Phi_{\gamma+1}$ at $\alpha_\gamma + 1$. That is, we set

$$\begin{aligned}
u^{\gamma+1}\restriction\alpha_\gamma &= u^\gamma\restriction\alpha_\gamma, \\
p^{\gamma+1}\restriction \mathrm{Ext}(W_\gamma\restriction\alpha_\gamma) &= p^\gamma\restriction \mathrm{Ext}(W_\gamma\restriction\alpha_\gamma), \\
t^{0,\gamma+1}_\eta &= t^{0,\gamma}_\eta \qquad \text{for } \eta \le \alpha_\gamma,
\end{aligned}$$

and
$$t^{1,\gamma+1}_\eta = t^{1,\gamma}_\eta \qquad \text{for } \eta < \alpha_\gamma.$$

Then we set

$$\begin{aligned}
u^{\gamma+1}(\alpha_\gamma) &= \xi_\gamma, \\
p^{\gamma+1}(F) &= G,
\end{aligned}$$

and let $t^{0,\gamma+1}_{\alpha_\gamma+1}$ be given by the Shift Lemma,

$$t^{0,\gamma+1}_{\alpha_\gamma+1}([a,f]^{\mathcal{M}^{\mathcal{W}_\nu}_\beta}_F) = [\mathrm{res}_\gamma \circ t^\gamma(a), \rho(f)]^{\mathcal{M}^{\mathcal{W}_\nu^*}_{\beta^*}}_G.$$

We have shown that ρ agrees with $\mathrm{res}_\nu \circ t^\nu$ on the domain of F. By $(\dagger)_\gamma$, ρ agrees with t^γ on the domain of F. Since res_γ is the identity on the domain of H (cf. 4.23), ρ agrees with $\mathrm{res}_\gamma \circ t^\gamma$ on the domain of F, and we can apply the Shift Lemma here. Let us also set

$$t^{1,\gamma+1}_{\alpha_\gamma} = \hat{\imath}^{\mathcal{W}^{**}_\gamma}_{v^\gamma(\alpha_\gamma),\xi_\gamma} \circ t^{0,\gamma}_{\alpha_\gamma}.$$

Then $t^{1,\gamma+1}\colon \mathcal{M}^{\mathcal{W}_{\gamma+1}}_{\alpha_\gamma} \to \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{u^{\gamma+1}(\alpha_\gamma)} = \mathcal{M}^{\mathcal{W}^{**}_\gamma}_{\xi_\gamma}$, and $t^{1,\gamma+1}$ agrees with $\mathrm{res}_\gamma \circ t^\gamma$ on $\mathrm{lh}(F)+1$, by claim 4.22.

This gives us $\Phi_{\gamma+1}\restriction(\alpha_\gamma+2)$.

Claim 4.26 *$\Phi_{\gamma+1}\restriction(\alpha_\gamma+2)$ is a pseudo-hull embedding of $\mathcal{W}_{\gamma+1}\restriction(\alpha_\gamma+2)$ into $\mathcal{W}^*_{\gamma+1}\restriction(\xi_\gamma+2)$, and extends $\Phi_\gamma\restriction(\alpha_\gamma+1)$.*

Proof. We checked some of the psuedo-hull properties as we defined $\Phi_{\gamma+1}$. We must still check that $t^{1,\gamma+1}_{\alpha_\gamma}$ satisfies properties (d) and (e) of definition 2.26. Noting that $E^{\mathcal{W}_\gamma}_{\alpha_\gamma} = F$ and that $t^{1,\gamma+1}_{\alpha_\gamma}$ agrees with $\mathrm{res}_\gamma \circ t^\gamma$ on $\mathrm{lh}(F)+1$, this is easy to do. See the proof of lemma 4.2. □

We can define the remainder of the maps $u^{\gamma+1}$ and $p^{\gamma+1}$ of $\Phi_{\gamma+1}$ right now. If $\alpha_\gamma + 1 \leq \alpha < z(\nu)$, then we set

$$u^{\gamma+1}(\phi_{\nu,\gamma+1}(\alpha)) = i_{G^*}(u^\nu(\alpha)),$$

and

$$p^{\gamma+1}(e_{\nu,\gamma+1}(E)) = i_{G^*}(p^\nu(E)),$$

for $E = E^{\mathcal{W}_\nu}_\alpha$. Note that this then holds true for any E, since if $E = E^{\mathcal{W}_\nu}_\xi$ for some $\xi < \beta$, then $p^{\gamma+1}(e_{\nu,\gamma+1}(E)) = p^{\gamma+1}(E) = p^\nu(E) = i^{G^*}(p^\nu(E))$.

The definition of the t-maps of $\Phi_{\gamma+1}$, and the proof that everything fits together properly, must be done by induction.

As we define $\Phi_{\gamma+1}$, we shall also check the applicable parts of $(\dagger)_{\gamma+1}$. We begin with

Claim 4.27 *$\Phi_{\gamma+1}\restriction(\alpha_\gamma+2)$ satisfies the applicable clauses of $(\dagger)_{\gamma+1}$.*

Proof. We have $\Phi_{\gamma+1}\restriction(\alpha_\gamma+1) = \Phi_\gamma\restriction(\alpha_\gamma+1)$ by construction, which yields $(\dagger)_{\gamma+1}$(a).

Suppose that $(\dagger)_{\gamma+1}$(b) is applicable, that is, that $z(\gamma+1) = \alpha_\gamma+1$. So $z(\nu) = \beta$. We have $v^{\gamma+1}(\alpha_\gamma+1) = \xi_\gamma+1$. So what we must see is that $\xi_\gamma + 1 \leq_{W^*_{\gamma+1}} z^*(\gamma+1)$. That is, we must see that G is used on the branch to $z^*(\gamma+1)$. We are in the non-dropping case, so $z^*(\gamma+1) = i_{G^*}(z^*(\nu))$. The relevant diagram here is

$$\begin{tikzcd}
 & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{i_{G^*}(\beta^*)} \arrow[r] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{z^*(\gamma+1)} \\
\mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{\xi_\gamma+1} \arrow[ur, "\sigma"] & & \\
 & \mathcal{M}^{\mathcal{W}^*_\nu}_{\beta^*} \arrow[ul, "i_G"] \arrow[uu, "i_{G^*}"'] \arrow[r] & \mathcal{M}^{\mathcal{W}^*_\nu}_{z^*(\nu)} \\
 & \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(\beta)} \arrow[u] &
\end{tikzcd}$$

If s is the branch extender $s = s^{\mathcal{W}^*_\nu}_{\beta^*}$, then $i_{G^*}(s(i)) = s(i)$ for all $i \in \mathrm{dom}(s)$, and thus $s \subseteq s^{\mathcal{W}^*_{\gamma+1}}_{i_{G^*}(\beta^*)}$. It follows that

$$\mathcal{M}^{\mathcal{W}^*_\nu}_{\beta^*} = \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{\beta^*},$$

and that

$$i_{G^*} \restriction \mathcal{M}^{\mathcal{W}^*_\nu}_{\beta^*} = i^{\mathcal{W}^*_{\gamma+1}}_{\beta^*, i_{G^*}(\beta^*)}.$$

The factor map σ in our diagram is the identity on the generators of G. It follows that G is compatible with the first extender used in $i^{\mathcal{W}^*_{\gamma+1}}_{\beta^*, i_{G^*}(\beta^*)}$, and thus G is that extender, as desired.

Turning to $(\dagger)_{\gamma+1}$(c), the new applicable cases are (ii) and (iii), when $\xi = \beta$ and $\tau = \alpha_\gamma + 1$. Let us suppose that it is (ii) that applies, that is, that $\beta < z(\nu)$. The last paragraph showed that G is used on the branch to $i_{G^*}(\beta^*)$ in this case as well. We have the diagram

$$\begin{tikzcd}
 & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{i_{G^*}(\beta^*)} \arrow[r, "\varphi"] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{u^{\gamma+1}(\alpha_\gamma+1} \\
\mathcal{M}^{\mathcal{W}_{\gamma+1}}_{\alpha_\gamma+1} \arrow[r, "t^{0,\gamma+1}_{\alpha_\gamma+1}"] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{\xi_\gamma+1} \arrow[u, "\sigma"'] & \\
 & \mathcal{M}^{\mathcal{W}^*_\nu}_{\beta^*} \arrow[u, "i_G"'] \arrow[r, "h"] & \mathcal{M}^{\mathcal{W}^*_\nu}_{u^\nu(\beta)} \arrow[uu, "i^{\mathcal{U}^*}_{\nu,\gamma+1}"'] \\
\mathcal{M}^{\mathcal{W}_\nu}_\beta \arrow[uu, "i_F"'] \arrow[ur, "\rho"] \arrow[r, "t^{0,\nu}_\beta"] & \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(\beta)} \arrow[u, "f"'] &
\end{tikzcd}$$

Here $\pi^{\nu,\gamma+1}_\beta = i_F^{\mathcal{M}^{\mathcal{W}_\nu}_\beta}$. The branch embeddings $\varphi \circ \sigma$ of $\mathcal{W}^*_{\gamma+1}$ and $h \circ f$ of $\mathcal{W}^*_\nu$ play the roles of k and j in $(\dagger)_\gamma.(c)$. The role of l is played by $i_G \circ f$. The diagram commutes, so we are done. The case $\beta = z(\nu)$ is similar.

Turning to $(\dagger)_\gamma.(d)$, it is enough to show that $t^{0,\gamma+1}_{\alpha_\gamma+1}$ agrees with $\mathrm{res}_\gamma \circ t^\gamma$ on $\mathrm{lh}(F)+1$. But this follows from the Shift Lemma.

We turn to $(\dagger)_\gamma.(e)$, that $\psi^{\mathcal{U}}_{\gamma+1} = t^{\gamma+1} \circ \sigma_{\gamma+1}$. This is applicable when $z(\gamma+1) = \alpha_\gamma + 1$, and hence since we didn't drop, $z(\nu) = \beta$. So $\mathcal{M}^{\mathcal{W}_\nu}_\beta = R_\nu$, $\mathcal{M}^{\mathcal{W}_{\gamma+1}}_{\alpha_\gamma+1} = R_{\gamma+1}$, $\mathcal{M}^{\mathcal{W}^*_\nu}_{z^*(\nu)} = N_\nu$, and $\mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{z^*(\gamma+1)} = N_{\gamma+1}$. Expanding the diagram immediately above a little, while making these substitutions, we get

$$
\begin{tikzcd}
 & & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{i_{G^*}(\beta^*)} \arrow[r, "\varphi"] & N_{\gamma+1} \\
\mathcal{M}^{\mathcal{U}}_{\gamma+1} \arrow[r, "\sigma_{\gamma+1}"] \arrow[urrr, bend left, "\psi^{\mathcal{U}}_{\gamma+1}"] & R_{\gamma+1} \arrow[r, "t^{0,\gamma+1}_{\alpha_\gamma+1}"] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{\xi_\gamma+1} \arrow[u, "\sigma"'] & \\
 & & \mathcal{M}^{\mathcal{W}^*_{\nu}}_{\beta^*} \arrow[u, "i_G"'] \arrow[r, "h"] & N_{\nu} \arrow[uu, "i^{\mathcal{U}^*}_{\nu,\gamma+1}"'] \\
\mathcal{M}^{\mathcal{U}}_{\nu} \arrow[r, "\sigma_\nu"] \arrow[uu, "i^{\mathcal{U}}_{\nu,\gamma+1}"] \arrow[urrr, bend right, "\psi^{\mathcal{U}}_{\nu}"'] & R_{\nu} \arrow[r, "t^{0,\nu}_{\beta}"] \arrow[uu, "i_F"'] \arrow[ur, "\rho"] & \mathcal{M}^{\mathcal{W}^*_{\nu}}_{\upsilon^\nu(\beta)} \arrow[u] &
\end{tikzcd}
$$

We have $t^{\gamma+1} = \varphi \circ \sigma \circ t^{0,\gamma+1}_{\alpha_\gamma+1}$ and $t^\nu = h \circ \rho$.

Note first that $\psi^{\mathcal{U}}_{\gamma+1}$ agrees with $t^{\gamma+1} \circ \sigma_{\gamma+1}$ on $\operatorname{ran}(i^{\mathcal{U}}_{\nu,\gamma+1})$. This is because

$$
\begin{aligned}
\psi^{\mathcal{U}}_{\gamma+1} \circ i^{\mathcal{U}}_{\nu,\gamma+1} &= i^{\mathcal{U}^*}_{\nu,\gamma+1} \circ \psi^{\mathcal{U}}_{\nu} \\
&= i^{\mathcal{U}^*}_{\nu,\gamma+1} \circ (h \circ \rho \circ \sigma_\nu)
\end{aligned}
$$

(by $(\dagger)_\nu$)

$$
= t^{\gamma+1} \circ \sigma_{\gamma+1} \circ i^{\mathcal{U}}_{\nu,\gamma+1}.
$$

The last equality holds because of the commutativity of the non-ψ part of the diagram.

$\mathcal{M}^{\mathcal{U}}_{\gamma+1}$ is generated by $\operatorname{ran}(i^{\mathcal{U}}_{\nu,\gamma+1}) \cup \lambda$, where $\lambda = \lambda_{E^{\mathcal{U}}_\gamma}$. So it is now enough to show that $\psi^{\mathcal{U}}_{\gamma+1}$ agrees with $t^{\gamma+1} \circ \sigma_{\gamma+1}$ on λ. But note

$$
\begin{aligned}
\psi^{\mathcal{U}}_{\gamma+1} \restriction \lambda &= \operatorname{res}_\gamma \circ \psi^{\mathcal{U}}_{\gamma} \restriction \lambda \\
&= \operatorname{res}_\gamma \circ t^\gamma \circ \sigma_\gamma \restriction \lambda
\end{aligned}
$$

(by $(\dagger)_\gamma$)

$$= t^{\gamma+1} \circ \sigma_\gamma \restriction \lambda$$

(because $t^{\gamma+1}$ agrees with $\mathrm{res}_\gamma \circ t^\gamma$ on λ_F)

$$= t^{\gamma+1} \circ \sigma_{\gamma+1} \restriction \lambda.$$

The last equality holds because σ_γ agrees with $\sigma_{\gamma+1}$ on $\mathrm{lh}(F)+1$, by our earlier work on normalization. This proves $(\dagger)_{\gamma+1}$(e).

For $(\dagger)_{\gamma+1}$(f), note that $N_{\gamma+1}$ agrees with $N^*_\gamma = \mathcal{M}^{\mathcal{W}^{**}_\gamma}_{\xi_\gamma}$ below $\mathrm{lh}(G)$, and the latter is a cardinal in $N_{\gamma+1}$. This and $(\dagger)_\gamma$(f) give us what we want.

This proves Claim 4.27.

$\square$

For the rest, we define $\Phi_{\gamma+1}\restriction\eta+1$, for $\alpha_\gamma+1 < \eta \le z(\gamma+1)$, by induction on η, and verify that it is a psuedo-hull embedding. At the same time, we prove those clauses in $(\dagger)_{\gamma+1}$ that make sense by stage η. The agreement clauses (a), (d), and (f) already make sense once we have $\Phi_{\gamma+1}\restriction(\alpha_\gamma+2)$, and we have already verified them. So we must consider clauses (b), (c), and (e).

First, suppose we are given $\Phi_{\gamma+1}\restriction(\eta+1)$, where $\alpha_\gamma+2 \le \eta+1 < z(\gamma+1)$. We must define $\Phi_{\gamma+1}\restriction(\eta+2)$. Let

$$\phi_{\nu,\gamma+1}(\tau) = \eta,$$
$$E = E^{\mathcal{W}_{\gamma+1}}_\eta,$$

and

$$K = E^{\mathcal{W}_\nu}_\tau.$$

Let

$$E^* = p^{\gamma+1}(E) \text{ and } K^* = p^\nu(K).$$

We have already defined $p^{\gamma+1}$ so that $i_{G^*}(K^*) = E^*$, and $u^{\gamma+1}(\eta) = i_{G^*}(u^\nu(\tau))$. We can simply apply lemma 4.2 to obtain $\Phi_{\gamma+1}\restriction(\eta+2)$ from $\Phi_{\gamma+1}\restriction(\eta+1)$. For we have the diagram from $(\dagger)_{\gamma+1}$(c).

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{W}_{\gamma+1}}_\eta \arrow[r, "t^{0,\gamma+1}_\eta"] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\eta)} \arrow[r] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{u^{\gamma+1}(\eta)} \\
\mathcal{M}^{\mathcal{W}_\nu}_\tau \arrow[u, "\pi^{\nu,\gamma+1}_\tau"'] \arrow[r, "t^{0,\nu}_\tau"] & \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(\tau)} \arrow[u, "l"] \arrow[r] & \mathcal{M}^{\mathcal{W}^*_\nu}_{u^\nu(\tau)} \arrow[u, "i^{\mathcal{U}^*}_{\nu,\gamma+1}"']
\end{tikzcd}$$

Taking $\xi = u^{\gamma+1}(\eta)$, we see from the commutativity of this diagram that $E_\xi^{\mathcal{W}^*_{\gamma+1}} = i^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\eta),\xi} \circ t^{0,\gamma+1}_\eta(E^{\mathcal{W}_{\gamma+1}}_\eta)$. Thus the condition (2) in 4.2 is fulfilled, and we can let $\Phi_{\gamma+1}\restriction(\eta+2)$ be the unique psuedo-hull embedding of $\mathcal{W}_{\gamma+1}\restriction(\eta+2)$ into $\mathcal{W}^*_{\gamma+1}$ that extends $\Phi_{\gamma+1}\restriction(\eta+1)$, and maps E to $i_{G^*}(p^\nu(K))$.

We now verify the applicable parts of $(\dagger)_{\gamma+1}$. The proofs are like the successor case $\eta = \alpha_\gamma$ that we have already done. We consider first clause (c). The new case to consider is $\xi = \tau + 1$. We have $\phi_{\nu,\gamma+1}(\tau+1) = \eta+1$. Let $\sigma = W_\nu$-pred$(\tau+1)$ and $\theta = W_{\gamma+1}$-pred$(\eta+1)$ index the places K and E are applied. Let σ^* and θ^* index the models in $\mathcal{W}^*_\nu$ and $\mathcal{W}^*_{\gamma+1}$ to which K^* and E^* are applied. Let us write $i^* = i_{G^*}$. We have $i^*(K^*) = E^*$ and $i^*(\sigma^*) = \theta^*$.

For purposes of drawing the following diagram, we assume $\tau + 1 < z(\nu)$. The situation is

$$\begin{tikzcd}
 & \mathcal{M}^{\mathcal{W}_{\gamma+1}}_{\eta+1} \arrow[r, "t^{0,\gamma+1}_{\eta+1}"] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\eta+1)} \arrow[r, dashed] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{u^{\gamma+1}(\eta+1)} \\
 & \mathcal{M}^{\mathcal{W}_\nu}_{\tau+1} \arrow[u, "\pi^{\nu,\gamma+1}_{\tau+1}"'] \arrow[r, "t^{0,\nu}_{\tau+1}"] & \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(\tau+1)} \arrow[u, "h"] \arrow[r, dashed] & \mathcal{M}^{\mathcal{W}^*_\nu}_{u^\nu(\tau+1)} \arrow[u, "i^*"'] \\
\mathcal{M}^{\mathcal{W}_{\gamma+1}}_\theta \arrow[uur, "E"] \arrow[rr, "t^{0,\gamma+1}_\theta"'] & & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\theta)} \arrow[r, dashed] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{\theta^*} \arrow[uul, "E^*"'] \\
\mathcal{M}^{\mathcal{W}_\nu}_\sigma \arrow[u, "\pi"] \arrow[uur, "K"'] \arrow[rr, "t^{0,\nu}_\sigma"'] & & \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(\sigma)} \arrow[u, "l"'] \arrow[r, dashed] & \mathcal{M}^{\mathcal{W}^*_\nu}_{\sigma^*} \arrow[u, "i^*"'] \arrow[uul, "K^*"] \\
P_0 \arrow[u] \arrow[uu, bend left] & & & P_0 \arrow[u] \arrow[uu, bend right]
\end{tikzcd}$$

There are two cases being covered in this diagram:

(Case A.) $\text{crit}(F) \leq \text{crit}(K)$. In this case, $\theta = \phi_{\nu,\gamma+1}(\sigma)$, and $\pi = \pi^{\nu,\gamma+1}_\sigma$. The map l in our diagram is given by the part of $(\dagger)_\gamma.(c)$ we have already verified.

(Case B.) $\mathrm{crit}(K) < \mathrm{crit}(F)$. In this case, $\theta = \sigma \leq \beta$, where $\beta = \beta^{\mathcal{W}_\nu, F}$. Moreover, $\mathcal{W}_\nu \restriction (\sigma + 1) = \mathcal{W}_{\gamma+1} \restriction (\theta + 1)$, and π is the identity. Moreover, $\beta \leq \alpha_\nu$ by the way normalization works, so the part of $(\dagger)_\gamma.(a)$ we have already verified tells us that $t_\sigma^{0,\nu} = t_\theta^{0,\gamma+1}$, and $\mathcal{M}_{v^\nu(\sigma}^{\mathcal{W}_\nu^*} = \mathcal{M}_{v^{\gamma+1}(\theta)}^{\mathcal{W}_{\gamma+1}^*}$. We take l to be the identity as well. In other words, the bottom left rectangle in the diagram above consists of identity embeddings.

We also have $\mathrm{dom}(E) = \mathrm{dom}(K) < \mathrm{crit}(i^*)$ in this case (though $E \neq K$ is perfectly possible). So then $\mathrm{dom}(E^*) = \mathrm{dom}(K^*)$, which implies that $\mathcal{M}_{\sigma^*}^{\mathcal{W}_\nu^*} = \mathcal{M}_{\theta^*}^{\mathcal{W}_{\gamma+1}^*}$, and $i^* \restriction \mathcal{M}_{\sigma^*}^{\mathcal{W}_\nu^*}$ is the identity. Thus the bottom right rectangle also consists of identity embeddings. (It is however possible that $u^\nu(\sigma) \neq u^{\gamma+1}(\sigma)$ in this case.)

In both cases, our job is to define h so that it fits into the diagram as shown. Using the notation just established, we can handle the cases in parallel.

We define h using the Shift Lemma:

$$h([a, f]_{K^*}^{\mathcal{M}_{\sigma^*}^{\mathcal{W}_\nu^*}}) = [i^*(a), i^*(f)]_{E^*}^{\mathcal{M}_{\theta^*}^{\mathcal{W}_{\gamma+1}^*}}.$$

Note here that $i^*(u^\nu(\tau)) = u^{\gamma+1}(\eta)$ by our induction hypotheses, so i^* maps $\mathcal{M}_{u^\nu(\tau)}^{\mathcal{W}_\nu^*}$, the model where we found K^*, elementarily into $\mathcal{M}_{u^{\gamma+1}(\eta)}^{\mathcal{W}_{\gamma+1}^*}$, the model that had E^*. So the Shift Lemma gives us h, and that $h \circ i_{K^*} = i_{E^*} \circ i^*$.

We shall leave it to the reader to show that the rectangle on the upper right of our diagram commutes. If s is the branch extender of $[0, u^\nu(\tau + 1)]_{W_\nu^*}$ and t is the branch extender of $[0, u^{\gamma+1}(\eta + 1)]_{W_{\gamma+1}^*}$, then $i^*(s) = t$. Moreover, if $s(a) = K^*$ and $t(b) = E^*$, then $i^*(s \restriction (a+1)) = t \restriction (b+1)$. This implies that the upper right rectangle commutes.

So we are left to show that $h \circ t_{\tau+1}^{0,\nu} = t_{\eta+1}^{0,\gamma+1} \circ \pi_{\tau+1}^{\nu,\gamma+1}$. Let $x = [b, f]_K^{\mathcal{M}_\sigma^{\mathcal{W}_\nu}}$ be in $\mathcal{M}_{\tau+1}^{\mathcal{W}_\nu}$. Then

$$\begin{aligned} h \circ t_{\tau+1}^{0,\nu}(x)) &= h(t_{\tau+1}^{0,\nu}([b, f]_K^{\mathcal{M}_\sigma^{\mathcal{W}_\nu}})) \\ &= h([t_\tau^{1,\nu}(b), i_{v^\nu(\sigma),\sigma^*}^{\mathcal{W}_\nu^*} \circ t_\sigma^{0,\nu}(f)]_{K^*}^{\mathcal{M}_{\sigma^*}^{\mathcal{W}_\nu^*}} \\ &= [i^* \circ t_\tau^{1,\nu}(b)), i^* \circ i_{v^\nu(\sigma),\sigma^*}^{\mathcal{W}_\nu^*} \circ t_\sigma^{0,\nu}(f)]_{E^*}^{\mathcal{M}_{\theta^*}^{\mathcal{W}_{\gamma+1}^*}}. \end{aligned}$$

The second step uses our definition of $t_{\tau+1}^{0,\nu}$. On the other hand,

$$
\begin{aligned}
t^{0,\gamma+1}_{\eta+1} \circ \pi^{\nu,\gamma+1}_{\tau+1}(x) &= t^{0,\gamma+1}_{\eta+1}(\pi^{\nu,\gamma+1}_{\tau+1}([b,f]^{\mathcal{M}^{\mathcal{W}_\nu}_\sigma}_K)) \\
&= t^{0,\gamma+1}_{\eta+1}([\pi^{\nu,\gamma+1}_\tau(b), \pi(f)]^{\mathcal{M}^{\mathcal{W}_{\gamma+1}}_\theta}_E) \\
&= [t^{1,\gamma+1}_\eta \circ \pi^{\nu,\gamma+1}_\tau(b), i^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\theta),\theta^*} \circ t^{0,\gamma+1}_\theta \circ \pi(f)]^{\mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{\theta^*}}_{E^*}.
\end{aligned}
$$

Now let's compare the two expressions above. The function f is moved the same way in both cases because the bottom rectangles in the diagram above commute. That is,

$$i^* \circ i^{\mathcal{W}^*_\nu}_{v^\nu(\sigma),\sigma^*} \circ t^{0,\nu}_\sigma = i^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\theta),\theta^*} \circ t^{0,\gamma+1}_\theta \circ \pi.$$

So we just need to see that

$$t^{1,\gamma+1}_\eta \circ \pi^{\nu,\gamma+1}_\tau = i^* \circ t^{1,\nu}_\tau.$$

But this follows from the part of $(\dagger)_{\gamma+1}$(c) that we have already verified. The relevant diagram is

$$
\begin{tikzcd}
\mathcal{M}^{\mathcal{W}_{\gamma+1}}_\eta \arrow[r, "t^{0,\gamma+1}_\eta"] \arrow[rr, bend left, "t^{1,\gamma+1}_\eta"] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\eta)} \arrow[r, dashed] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{u^{\gamma+1}(\eta)} \\
\mathcal{M}^{\mathcal{W}_\nu}_\tau \arrow[u, "\pi^{\nu,\gamma+1}_\tau"] \arrow[r, "t^{0,\nu}_\tau"] \arrow[rr, bend right, "t^{1,\nu}_\tau"'] & \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(\tau)} \arrow[u] \arrow[r, dashed] & \mathcal{M}^{\mathcal{W}^*_\nu}_{u^\nu(\tau)} \arrow[u, "i^*"']
\end{tikzcd}
$$

Thus we have verified the new case of $(\dagger)_{\gamma+1}$(c) that is applicable to $\Phi_{\gamma+1}\restriction(\eta+2)$.

We turn to $(\dagger)_{\gamma+1}$(e). If it is applicable, then $z(\gamma+1) = \eta+1$, and because we did not drop, $z(\nu) = \tau+1$. We must show that $\psi^{\mathcal{U}}_{\gamma+1} = t^{\gamma+1} \circ \sigma_{\gamma+1}$. We have $R_{\gamma+1} = \mathcal{M}^{\mathcal{W}_{\gamma+1}}_{\eta+1}$, and $R_\nu = \mathcal{M}^{\mathcal{W}_\nu}_{\tau+1}$. Making these substitutions and expanding the upper part of the diagram above, we get

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{U}}_{\gamma+1} \arrow[r, "\sigma_{\gamma+1}"] & R_{\gamma+1} \arrow[r, "t^{0,\gamma+1}_{z(\gamma+1)}"] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(z(\gamma+1))} \arrow[r] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{z^*(\gamma+1)} = N_{\gamma+1} \\
\mathcal{M}^{\mathcal{W}_\nu}_{\nu} \arrow[u] \arrow[r, "\sigma_\nu"] & R_\nu \arrow[u, "\pi^{\nu,\gamma+1}_{z(\nu)}"] \arrow[r, "t^{0,\nu}_{z(\nu)}"] & \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(z(\nu))} \arrow[u, "h"] \arrow[r] & \mathcal{M}^{\mathcal{W}^*_\nu}_{z^*(\nu)} = N_\nu \arrow[u, "i^*"]
\end{tikzcd}$$

The embedding across the bottom row is $t^\nu \circ \sigma_\nu$, and hence by induction, it is $\psi^{\mathcal{U}}_\nu$. The embedding across the top row is $t^{\gamma+1} \circ \sigma_{\gamma+1}$. The diagram commutes, so

$$\begin{aligned}
\psi^{\mathcal{U}}_{\gamma+1} \circ i^{\mathcal{U}}_{\nu,\gamma+1} &= i^{\mathcal{U}^*}_{\nu,\gamma} \circ \psi^{\mathcal{U}}_\nu \\
&= i^* \circ t^\nu \circ \sigma_\nu. \\
&= t^{\gamma+1} \circ \sigma_{\gamma+1} \circ i^{\mathcal{U}}_{\nu,\gamma+1}.
\end{aligned}$$

Thus $t^{\gamma+1} \circ \sigma_{\gamma+1}$ agrees with $\psi^{\mathcal{U}}_{\gamma+1}$ on $\mathrm{ran}(i^{\mathcal{U}}_{\nu,\gamma+1})$. So it will be enough to show the two embeddings agree on $\lambda = \lambda_{E^{\mathcal{U}}_\gamma}$. For that, we calculate exactly as we did in the case $\eta = \alpha_\gamma + 1$:

$$\begin{aligned}
\psi^{\mathcal{U}}_{\gamma+1} \restriction \lambda &= \mathrm{res}_\gamma \circ \psi^{\mathcal{U}}_\gamma \restriction \lambda \\
&= \mathrm{res}_\gamma \circ t^\gamma \circ \sigma_\gamma \restriction \lambda \\
&= t^{\gamma+1} \circ \sigma_\gamma \restriction \lambda \\
&= t^{\gamma+1} \circ \sigma_{\gamma+1} \restriction \lambda.
\end{aligned}$$

The last equality holds because σ_γ agrees with $\sigma_{\gamma+1}$ on $\mathrm{lh}(F)+1$, by our earlier work on normalization. This proves $(\dagger)_\gamma.(e)$.

Finally, suppose that λ is a limit ordinal, and we have defined $\Phi_{\gamma+1}\restriction\eta$ for all $\eta < \lambda$. Then we set

$$\Phi_{\gamma+1}\restriction\lambda = \bigcup_{\eta<\lambda} \Phi_{\gamma+1}\restriction\eta.$$

We are of course assuming $\Phi_{\gamma+1}\restriction\eta$ is a subsystem of $\Phi_{\gamma+1}\restriction\beta$ whenever $\eta < \beta$, and the psuedo-hull properties clearly pass through limits, so this gives us a psuedo-hull embedding of $\mathcal{W}_{\gamma+1}\restriction\lambda$ into $\mathcal{W}^*_{\gamma+1}\restriction\lambda$.

In order to define $\Phi_{\gamma+1}\restriction(\lambda+1)$, for $\lambda \leq z(\gamma+1)$ a limit ordinal, let τ be such that

$$\lambda = \phi_{\nu,\gamma+1}(\tau).$$

Consider $r = \hat{p}^{\gamma+1}(s^{\mathcal{W}_{\gamma+1}}_\lambda)$. Since $\Phi_{\gamma+1}\restriction\lambda$ is a psuedo-hull embedding, $\hat{p}^{\gamma+1}$ is $\subseteq$-preserving on $\mathcal{W}^{\mathrm{ext}}_{\gamma+1}$. Thus r is the extender of some branch b of $\mathcal{W}^*_{\gamma+1}$. In fact, b is the downward closure of $\{i_{G^*}(v^\nu(\xi)) \mid \xi <_{W_\nu} \tau\}$. Recall that the v-maps preserve tree order, so that $\{i_{G^*}(v^\nu(\xi)) \mid \xi <_{W_\nu} \tau\}$ is contained in the branch $[0, i_{G^*}(v^\nu(\tau)]_{W^*_{\gamma+1}}$ of $\mathcal{W}^*_{\gamma+1}$. So

$$v^{\gamma+1}(\lambda) = \sup\{i_{G^*}(v^\nu(\xi)) \mid \xi <_{W_\nu} \tau\}.$$

Moreover, we can define $t_\lambda^{0,\gamma+1} : \mathcal{M}_\lambda^{\mathcal{W}_{\gamma+1}} \to \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\lambda)}$ using the commutativity given by (c) of definition 2.26:

$$t_\lambda^{0,\gamma+1}(i^{\mathcal{W}_{\gamma+1}}_{\theta,\lambda}(x)) = i^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\theta),v^{\gamma+1}(\lambda)}(t_\theta^{0,\gamma+1}(x).$$

It is easy to verify the agreement of $t_\lambda^{0,\gamma+1}$ with earlier embeddings specified in clause (d) of 2.26. Thus $\Phi_{\gamma+1}\restriction(\lambda+1)$ is a psuedo-hull embedding.

We must check that the applicable parts of $(\dagger)_{\gamma+1}$ hold. Let us keep the notation of the last paragraph. For part (b), we must consider the case $z(\gamma+1) = \lambda$. We have not dropped in $(\nu, \gamma+1]_U$, so $z(\nu) = \tau$, and $v^\nu(\tau) \leq_{W^*_\nu} z^*(\nu)$ by $(\dagger)_\nu$. We showed that $v^{\gamma+1}(\lambda) \leq_{W^*_{\gamma+1}} i_{G^*}(v^\nu(\tau))$ in the last paragraph. So $v^{\gamma+1}(\lambda) \leq_{W^*_{\gamma+1}} i_{G^*}(z^*(\nu)) = z^*(\gamma+1)$, as desired.

For (c), the new case is $\xi = \tau$, and $\lambda = \phi_{\nu,\gamma+1}(\tau)$. Everything in sight commutes, so things work out. Let's work them out. Setting $i^* = i^{\mathcal{U}^*}_{\nu,\gamma+1}$, and letting k be the branch embedding from $\mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\lambda)}$ to $\mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{i^*(v^\nu(\tau))}$, the relevant diagram is

$$\begin{tikzcd}
 & & \mathcal{M}_\lambda^{\mathcal{W}_{\gamma+1}} \arrow[r, "t_\lambda^{0,\gamma+1}"] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\lambda)} \arrow[r, "k"] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{i^*(v^\nu(\tau))} & \\
 & & \mathcal{M}_\tau^{\mathcal{W}_\nu} \arrow[u, "\pi_\tau^{\nu,\gamma+1}"'] \arrow[r, "t_\tau^{0,\nu}"] & \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(\tau)} \arrow[u, "l"] & & \\
 & \mathcal{M}_\theta^{\mathcal{W}_{\gamma+1}} \arrow[uur] \arrow[rrr, "t_\theta^{0,\gamma+1}"'] & & & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\theta)} \arrow[uul] & \\
 & \mathcal{M}_\sigma^{\mathcal{W}_\nu} \arrow[u, "\pi_\sigma^{\nu,\gamma+1}"'] \arrow[ur] \arrow[rrr, "t_\sigma^{0,\nu}"'] & & & \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(\sigma)} \arrow[u, "i^*"'] \arrow[uul] & \\
P_0 \arrow[uur] \arrow[ur] & & & & & P_0 \arrow[uul] \arrow[ul]
\end{tikzcd}$$

Here we are taking $\theta = \phi_{\nu,\gamma+1}(\sigma)$, where $\sigma <_{W_\nu} \tau$, and σ is sufficiently large that $\phi_{\nu,\gamma+1}$ preserves tree order above σ. We also take σ to be a successor ordinal, so that $i^*(v^\nu(\sigma)) = v^{\gamma+1}(\tau)$. The map l is defined by

$$l(i^{\mathcal{W}^*_\nu}_{v^\nu(\sigma),v^\nu(\tau)}(x)) = i^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\theta),v^{\gamma+1}(\lambda)}(i^*(x)).$$

(Where of course we are taking the union over all such successor ordinals σ.) If we draw the same diagram with τ replaced by some sufficiently large $\tau_0 <_{W_\nu} \tau$ and λ replaced by $\lambda_0 = \phi_{\nu,\gamma+1}(\tau_0)$, then all parts of our diagram commute, because we have verified $(\dagger)_{\gamma+1}$ that far already. Since all these approximating diagrams commute, l is well-defined, and the diagram displayed commutes. Moreover, it is easy to check that $k \circ l = i^* \restriction \mathcal{M}^{\mathcal{W}^*_\nu}_{v^\nu(\tau)}$. Thus we have $(\dagger)_{\gamma+1}$(c).

The proof of $(\dagger)_{\gamma+1}$(e) is exactly the same as it was in the successor case, so we omit it.

Remark 4.28 Actually, that proof seems to show that $(\dagger)_\gamma.(e)$ is redundant, in that it follows from the other clauses.

Thus $t^{\gamma+1} \circ \sigma_{\gamma+1}$ agrees with $\psi^{\mathcal{U}}_{\gamma+1}$ on $\mathrm{ran}(i^{\mathcal{U}}_{\nu,\gamma})$. So it will be enough to show the two embeddings agree on $\lambda_{E^{\mathcal{U}}_\gamma}$. For that, it is enough to see $t^{\gamma+1}$ agrees with t^ν on λ_F. But in fact, $t^{\gamma+1}$ agrees with t^ν on $\mathrm{lh}(F_\xi)$, for all $\xi < \nu$, so we are done.

This completes our work associated to the definition of $\Phi_{\gamma+1} \restriction \lambda + 1$, for $\lambda > \alpha_\gamma$ a limit. Thus we have completed the definition of $\Phi_{\gamma+1}$, and the verification of $(\dagger)_{\gamma+1}$, in Case 1.

Case 2. $(\nu, \gamma+1]_U$ drops, in either model or degree.

Let

$$\begin{aligned}
\bar{\mu} &= \mathrm{crit}(E^{\mathcal{U}}_\gamma), \\
\bar{P} &= \mathrm{dom}(E^{\mathcal{U}}_\gamma), \\
\bar{Q} &= \text{ first level of } \mathcal{M}^{\mathcal{U}}_\nu \text{ beyond } \bar{P} \\
&\quad \text{that projects to or below } \bar{\mu}.
\end{aligned}$$

We have that

$$\bar{P} = \mathcal{M}^{\mathcal{U}}_\nu | (\bar{\mu}^+)^{\mathcal{M}^{\mathcal{U}}_\nu | \mathrm{lh}(E^{\mathcal{U}}_\nu)} = \mathcal{M}^{\mathcal{U}}_\gamma | (\bar{\mu}^+)^{\mathcal{M}^{\mathcal{U}}_\gamma | \mathrm{lh}(E^{\mathcal{U}}_\gamma)}.$$

Let

$$\begin{aligned} \mu &= \sigma_\nu(\bar{\mu}) &&= \mathrm{crit}(F), \\ P &= \sigma_\nu(\bar{P}) &&= \mathrm{dom}(F), \\ Q &= \sigma_\nu(\bar{Q}) &&= \text{ first level of } R_\nu \text{ beyond } P \\ & && \quad \text{that projects to or below } \mu. \end{aligned}$$

Since σ_ν agrees with σ_γ on $\mathrm{lh}(F_\nu)$, we can replace σ_ν by σ_γ in the first two equations. (But if $\nu < \gamma$, then $\bar{Q} \notin \mathrm{dom}(\sigma_\gamma)$.) We have that

$$P = R_\nu|(\mu^+)^{R_\nu|\mathrm{lh}(F_\nu)} = R_\gamma|(\mu^+)^{R_\gamma|\mathrm{lh}(F)}.$$

In this case, $z(\gamma+1) = \alpha_\gamma + 1$, and

$$\mathcal{W}_{\gamma+1} = \mathcal{W}_\gamma \restriction (\alpha_\gamma + 1)^\frown \langle \mathrm{Ult}(Q, F)\rangle.$$

Claim A. $\mathrm{res}_\gamma \circ t^\gamma$ agrees with $\mathrm{res}_\nu \circ t^\nu$ on λ_{F_ν}.

Proof. This is clear if $\nu = \gamma$. But if $\nu < \gamma$, then t^γ agrees with $\mathrm{res}_\nu \circ t^\nu$ on λ_{F_ν} by $(\dagger)_\gamma$(d). (See the remarks after the statement of $(\dagger)_\gamma$.) But also, res_γ is the identity on $\mathrm{res}_\nu \circ t^\nu(\lambda_{F_\nu})$, because $\nu < \gamma$. This yields the claim. □

We have $H = t^\gamma(F)$ and $G = \mathrm{res}_\gamma(G)$. We have that $\mathrm{res}_\gamma \colon N_\gamma|\mathrm{lh}(H) \to N^*_\gamma|\mathrm{lh}(G)$, and that res_γ agrees with $\hat{\imath}^{\mathcal{W}^{**}_\gamma}_{\tau_\gamma, \xi_\gamma}$ on $\mathrm{lh}(H)$. Let

$$\begin{aligned} Q^* &= M^{S_\nu}_{\eta,l}, \text{ where } \langle \eta, l\rangle = \mathrm{Res}_{\eta_\nu, l_\nu}[t^\nu(Q)]^{S_\nu}, \\ \sigma^* &= \sigma_{\eta_\nu, l_\nu}[t^\nu(Q)]^{S_\nu}, \\ \mu^* &= \sigma^*(t^\nu(\mu)), \text{ and} \\ P^* &= \sigma^*(t^\nu(P)). \end{aligned}$$

σ^* is a partial resurrection map at stage ν. We had $\mathrm{res}_\nu \colon N_\nu|\mathrm{lh}(H_\nu) \to N^*_\nu|\mathrm{lh}(G_\nu)$. σ^* resurrects more, namely $t^\nu(Q)$, but doesn't trace it as far back in $i^{\mathcal{U}^*}_{0,\nu}(\mathbb{C})$. Because no proper level of $t^\nu(Q)$ projects to $t^\nu(\mu)$, σ^* agrees with res_ν on $t^\nu(P)$. So

$$\sigma^* \circ t^\nu \restriction P = \mathrm{res}_\nu \circ t^\nu \restriction P = \mathrm{res}_\gamma \circ t^\gamma \restriction P,$$

the last equality being Claim A. The embeddings displayed also agree at P, where they have value P^*. Note that $P = \mathrm{dom}(F)$ and $P^* = \mathrm{dom}(G)$.

We have that Q^* is the last model of $(W^*_{\eta,l})^{S_\nu}$. Set

$$\mathcal{T}^* = (\mathcal{W}^*_{\eta,l})^{S_\nu}.$$

Lemma 4.5 tells us that $\mathcal{T}^*$ has the following form. Let ξ be least such that $t^\nu(Q) \trianglelefteq \mathcal{M}_\xi^{\mathcal{W}_\nu^*}$. Then $\mathcal{T}^* \restriction \xi+1 = \mathcal{W}_\nu^* \restriction \xi+1$, and letting $\mathrm{lh}(\mathcal{T}^*) = \eta+1$, $\xi \leq_{T^*} \eta$ and $\sigma^* = \hat{\imath}_{\xi,\eta}^{\mathcal{T}^*}$.

We have that

$$\mathcal{W}_{\gamma+1}^* = i_{G^*}(\mathcal{T}^*), \text{ and}$$
$$N_{\gamma+1} = i_{G^*}(Q^*),$$

by the way that lifting to the background universe works in the dropping case. As in the non-dropping case, the key is

Claim B.

(i) $\mathcal{W}_{\gamma+1}^* \restriction \xi_\gamma + 1 = \mathcal{W}_\gamma^{**} \restriction \xi_\gamma + 1$, and

(ii) $G = E_{\xi_\gamma}^{\mathcal{W}_{\gamma+1}^*}$.

Proof. We have that $\mathrm{dom}(G) = \mathrm{res}_\gamma \circ t^\gamma(P) = \mathrm{res}_\nu \circ t^\nu(P)$ by claim A, so $\mathrm{dom}(G) = \sigma^* \circ t^\nu(P) = P^* = Q^*|(\mu^{*+})^{Q^*}$. P is $\mathcal{M}_{\alpha_\gamma}^{\mathcal{W}_\gamma}|\mathrm{lh}(F)$ cut off at its μ^+. So P^* is $\mathrm{res}_\gamma \circ t^\gamma(\mathcal{M}_{\alpha_\gamma}^{\mathcal{W}_\gamma}|\mathrm{lh}(F))$, cut off at its $(\mu^*)^+$, that is, P^* is $\mathcal{M}_{\xi_\gamma}^{\mathcal{W}_\gamma^{**}}|\mathrm{lh}(G)$, cut off at $(\mu^*)^+$.

Thus Q^* agrees with $\mathcal{M}_{\xi_\gamma}^{\mathcal{W}_\gamma^{**}}|\mathrm{lh}(G)$ up to their common value for $(\mu^*)^+$. It follows that $i_{G^*}(Q^*)$ agrees with $\mathrm{Ult}(\mathcal{M}_{\xi_\gamma}^{\mathcal{W}_\gamma^{**}}|\mathrm{lh}(G), G)$ up to $\mathrm{lh}(G)+1$, with the agreement at $\mathrm{lh}(G)$ holding by our having chosen a minimal G^* for G. Claim B now follows from the fact that $\mathcal{W}_\gamma^{**}$ and $\mathcal{W}_{\gamma+1}^*$ are normal trees by the same strategy. $\square$

We now get $\Phi_{\gamma+1}$ by setting $p^{\gamma+1}(F) = G$, and applying Lemma 4.2. We must see that $(\dagger)_{\gamma+1}$ holds. Part (a) is clear.

Let $\beta^* = W_{\gamma+1}\text{-pred}(\xi_\gamma)$. Claim C.

(1) $\mathrm{lh}(\mathcal{T}^*) = \beta^* + 1$, and $Q^* = \mathcal{M}_{\beta^*}^{\mathcal{W}_{\gamma+1}^*}$.

(2) $\beta^* = \mu^*$, and if $s = s_{\mu^*}^{\mathcal{T}^*}$, then $s\colon \mu^* \to V_{\mu^*}$.

Proof. By definition, β^* is the least α such that $\mathcal{M}_\alpha^{\mathcal{W}_{\gamma+1}^*}|o(P^*) = P^*$. But Q^* is the last model of $\mathcal{T}^*$, and $P^* = Q^*|o(P^*)$, so since $\mathcal{T}^*$ and $\mathcal{W}_{\gamma+1}^*$ are normal trees by the same strategy, $\beta^* < \mathrm{lh}(\mathcal{T}^*)$ and $\mathcal{M}_{\beta^*}^{\mathcal{T}^*} = \mathcal{M}_{\beta^*}^{\mathcal{W}_{\gamma+1}^*}$. This gives (1).

Part (2) is proved exactly as in case 1. $\square$

Now consider $(\dagger)_{\gamma+1}$(b). We have $v^{\gamma+1}(\alpha_\gamma+1) = \xi_\gamma+1$, and $z^*(\gamma+1) = i_{G^*}(\mu^*)$. So we must see that $\xi_\gamma + 1 \leq_{W_{\gamma+1}^*} i_{G^*}(\mu^*)$, that is, that G is used on the branch of

$\mathcal{W}^*_{\gamma+1}$ to $i_{G^*}(\mu^*)$. But if $s = s^{\mathcal{T}^*}_{\mu^*}$, then $s = i_{G^*}(s)\restriction\mu^*$, so μ^* is on the branch of $W^*_{\gamma+1}$ to $i_{G^*}(\mu^*)$. Moreover, $i_{G^*}(s)(\mu^*)$ is compatible with G, so it is equal to G, as desired.

$(\dagger)_{\gamma+1}$(c) is vacuous, because we have dropped. We shall leave the agreement conditions (d) and (f) to the reader, and consider (e). That is, we show $\psi^{\mathcal{U}}_{\gamma+1} = t^{\gamma+1} \circ \sigma_{\gamma+1}$. The relevant diagram is

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{U}}_{\gamma+1} \arrow[r, "\sigma_{\gamma+1}"] & R_{\gamma+1} \arrow[rr, "t^{0,\gamma+1}_{\alpha_\gamma+1}"] & & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{\xi_{\gamma+1}} \arrow[r, "k"] & \mathcal{M}^{\mathcal{W}^*_{\gamma+1}}_{i_{G^*}(\mu^*)} \\
\bar{Q} \arrow[u, "\hat{\imath}^{\mathcal{U}}_{\nu,\gamma+1}"'] \arrow[r, "\sigma_\nu"] & Q \arrow[u, "i_F"'] \arrow[r, "t^\nu"] & t^\nu(Q) \arrow[r, "\sigma^*"] & Q^* \arrow[u, "i_G"'] \arrow[ur, "i_{G^*}"'] & \\
 & & P_0 \arrow[u, "\mathcal{W}^*_\nu"] \arrow[ur, "\mathcal{T}^*"'] & &
\end{tikzcd}$$

Here $k = i^{\mathcal{W}^*_{\gamma+1}}_{v^{\gamma+1}(\alpha_\gamma+1),z^*(\gamma+1)}$. Thus the embedding along the top row is $t^{\gamma+1} \circ \sigma_{\gamma+1}$. The lifting process defines $\psi^{\mathcal{U}}_{\gamma+1}$ by

$$\psi_{\gamma+1}([a,f]^{\bar{Q}}_{E^{\mathcal{U}}_\gamma}) = [res_\gamma \circ \psi_\gamma(a), \sigma^* \circ \psi_\nu(f)]^{Q^*}_{G^*},$$

where we have dropped a few superscripts for readability. Let us write $\hat{\imath}$ for $\hat{\imath}^{\mathcal{U}}_{\nu,\gamma+1}$. Then $\psi_{\gamma+1}$ agrees with $t^{\gamma+1} \circ \sigma_{\gamma+1}$ on $\mathrm{ran}(\hat{\imath})$, because

$$\begin{aligned}
t^{\gamma+1} \circ \sigma_{\gamma+1} \circ \hat{\imath} &= i_{G^*} \circ \sigma^* \circ t^\nu \circ \sigma_\nu \\
&= i_{G^*} \circ \sigma^* \circ \psi_\nu \\
&= \psi_{\gamma+1} \circ \hat{\imath}.
\end{aligned}$$

The first line comes from the commutativity of the diagram, the second from $(\dagger)_\nu$(e), and the last from the definition of $\psi_{\gamma+1}$.

So it is enough to see that $\psi_{\gamma+1}$ agrees with $t^{\gamma+1} \circ \sigma_{\gamma+1}$ on λ, where $\lambda = \lambda(E^{\mathcal{U}}_\gamma)$. But note that $t^{\gamma+1} = k \circ t^{0,\gamma+1}_{\alpha_\gamma+1}$, and $\mathrm{crit}(k) \geq \lambda_G$. So $t^{\gamma+1}$ agrees with the Shift Lemma map $t^{0,\gamma+1}_{\alpha_\gamma+1}$ on λ_F. Thus $t^{\gamma+1}$ agrees with $\mathrm{res}_\gamma \circ t^\gamma$ on λ_F. So we can calculate

$$\begin{aligned}
\psi_{\gamma+1}\restriction\lambda &= \mathrm{res}_\gamma \circ \psi_\gamma\restriction\lambda \\
&= res_\gamma \circ t^\gamma \circ \sigma_\gamma\restriction\lambda \\
&= t^{\gamma+1} \circ \sigma_{\gamma+1}\restriction\lambda.
\end{aligned}$$

The second line comes from $(\dagger)_\gamma$)(e), and the third from our argument above, together with the fact $\sigma_\gamma \restriction \lambda = \sigma_{\gamma+1} \restriction \lambda$.

This finishes case 2, and hence the defintion of $\Phi_{\gamma+1}$ and verification of $(\dagger)_{\gamma+1}$.

We leave the detailed definition of Φ_λ and verification of $(\dagger)_\lambda$, for λ a limit ordinal or $\lambda = b$, to the reader. The normalization $\mathcal{W}_\lambda$ is a direct limit of the $\mathcal{W}_\nu$ for $\nu \in [0, \lambda)_U$. The tree $\mathcal{W}^*_\lambda$ is $i^{\mathcal{U}^*}_{\nu,\lambda}(\mathcal{W}^*_\nu)$, for ν past the last drop. So it is a direct limit too. We define Φ_λ to be the direct limit of the Φ_ν for $\nu \in [0, \lambda)_U$ past the last drop. Part (c) of $(\dagger)$ tells us we can do that. We omit further detail.

This finishes our proof of Sublemma 4.12.1, that $\mathcal{W}_b$ is a psuedo-hull of $\mathcal{W}^*_b$. $\square$

That in turn proves Lemma 4.12 $\square$

Lemma 4.29 *Let $M = M_{\nu_0,k_0}$, and let $\mathcal{U}$ be a normal tree on M that is of limit length, and is by both $\Sigma_{\mathcal{W}^*_{\nu_0,k_0},M}$ and $\Omega^{\mathbb{C}}_{\nu_0,k_0}$. Let*

$$\text{lift}(\mathcal{U}, M, \mathbb{C}) = \langle \mathcal{U}^*, \langle \eta_\tau, l_\tau \mid \tau < \text{lh}\,\mathcal{U}\rangle, \langle \psi^{\mathcal{U}}_\tau \mid \tau < \text{lh}\,\mathcal{U}\rangle\rangle;$$

then $\mathcal{U}^*$ has a cofinal, wellfounded branch.

Proof. Let $\pi \colon H \to V_\theta$ be elementary, where H is countable and transitive, and θ is sufficiently large, and everything relevant is in $\text{ran}(\pi)$. Let $\mathcal{S} = \pi^{-1}(\mathcal{U})$, $\mathcal{S}^* = \pi^{-1}(\mathcal{U}^*)$, and $\mathcal{T} = \pi^{-1}(\mathcal{W}^*_{\nu_0,k_0})$.

Because Σ is universally Baire, $\pi^{-1}(\Sigma) = \Sigma \cap H$, so $\langle \mathcal{T}, \mathcal{S}\rangle$ is by Σ. Moreover, letting

$$b = \Sigma(\langle \mathcal{T}, \mathcal{S}\rangle),$$

we have that $b \in H$. (Because $b \in H[g]$ for all g on $\text{Col}(\omega, \tau)$, for $\tau \in H$ sufficiently large.) It will be enough to see that $\mathcal{M}^{\mathcal{S}^*}_b$ is wellfounded, as then the elementarity of π yields a cofinal wellfounded branch of $\mathcal{U}^*$.

By [8], $\mathcal{S}^*$ has a cofinal, wellfounded branch c. The proof of Sublemma 4.12.1 shows that $\mathcal{W}_c$ is a psuedo-hull of $\mathcal{W}^*_c$, where $\mathcal{W}_c = W(\mathcal{T}, \mathcal{S}^\frown c)$ and $\mathcal{W}^*_c = i^{\mathcal{S}^*}_c(\mathcal{T})$. That is because we can run the construction of Φ_c in H; we don't need $c \in H$ to do that. But then $\mathcal{W}^*_c$ is by Σ, so $\mathcal{W}_c$ is by Σ by strong hull condensation, and $c = \Sigma(\langle \mathcal{T}, \mathcal{S}\rangle)$ since Σ normalizes well. Thus $c = b$, and $\mathcal{M}^{\mathcal{S}^*}_b$ is wellfounded, as desired. $\square$

We can now finish the proof of Theorem 4.10. We have just shown that $\Sigma_{\mathcal{W}^*_{\nu_0,k_0},M}$ agrees with $\Omega^{\mathbb{C}}_{\nu_0,k_0}$ on normal trees. We must see that they agree on finite stacks $\vec{\mathcal{T}}$

of normal trees. But for such $\vec{\mathcal{T}}$,

$$\begin{aligned}\vec{\mathcal{T}} \text{ is by } \Omega^{\mathbb{C}}_{\nu_0,k_0} &\Leftrightarrow \text{lift}(\vec{\mathcal{T}}) \text{ is by } \Omega^{\mathsf{UBH}}_{F^{\mathbb{C}}}\\ &\Leftrightarrow W(\text{lift}(\vec{\mathcal{T}})) \text{ is by } \Omega^{\mathsf{UBH}}_{n,F^{\mathbb{C}}}\\ &\Leftrightarrow \text{lift}(W(\vec{\mathcal{T}})) \text{ is by } \Omega^{\mathsf{UBH}}_{n,F^{\mathbb{C}}}\\ &\Leftrightarrow W(\vec{\mathcal{T}}) \text{ is by } \Sigma.\end{aligned}$$

The first equivalence is our definition of $\Omega^{\mathbb{C}}_{\nu_0,k_0}$. The second comes from our definition of $\Omega^{\mathsf{UBH}}_{F^{\mathbb{C}}}$. The third comes from the fact that embedding normalization commutes with lifting to the background universe, which we proved in the proof of Theorem 3.26. The last comes from the agreement of Σ with $\Omega^{\mathbb{C}}_{\nu_0,k_0}$ on normal trees.

This finishes the proof of Theorem 4.10. □

5 Fine structure for the least-branch hierarchy

We now adapt the definitions and results of the previous sections to mice that are being told their own background-induced iteration strategy. This leads to the basic solidity and universality theorems for such mice.

The main new problem is that in the solidity/universality proof, when we compare (M, H, ρ) with M, we must do so by iterating them into some background construction $\mathbb{C}$. Thus disagreements will very often happen when the two sides agree with each other, but not with $\mathbb{C}$. If we proceed naively, this renders invalid the usual argument that we can't end up above M on both sides. Our solution is to modify the way the phalanx is iterated, so that sometimes we move the whole phalanx up, including its exchange ordinal. Schlutzenberg has, independently and earlier, developed this idea much more thoroughly in another context.

This section is organized as follows. First, we define *least branch premice*, and the background constructions that produce such objects. We then more or less wave our hands over the assertion that everything in the previous sections generalizes routinely, so long as these background constructions do not break down by reaching some level at which solidity or universality of the standard parameter fails. (We are of course simply assuming unique iterability for V, as we did in the previous sections!) We then use the comparison process we get from this hand-waving to prove solidity and universality.

5.1 Least branch premice

A *least branch premouse* (lpm) is a variety of acceptable J-structure. Acceptable J-structures are structures of the form $(J^A_\alpha, \in, A \cap J^A_\alpha)$ that are amenable, and satisfy a local form of GCH. The basic fine structural notions, like projecta, standard parameters, and solidity witnesses, can be defined at this level of generality, and various elementary facts involving them proved. This is done in [23], and we assume familiarity with that material here. See the preliminaries section for more.

The language $\mathcal{L}_0$ of least branch premice should therefore have symbols $\in$ and $\dot{A}$. It is more convenient in our situation to have $\in$, predicate symbols $\dot{E}, \dot{F}, \dot{\Sigma}, \dot{B}$, and constant symbol $\dot{\gamma}$. If M is an lpm, then $M = (N, k)$, where N is an amenable structure for $\mathcal{L}_0$, and $k = k(M)$. We often identify M with N. The predicates and constant of N can be amalgamated in some fixed way into a single amenable $\dot{A}^{\mathcal{M}}$. So we are within the framework of [23]. $o(M)$ is of course the ordinal height of M. We let $\hat{o}(M)$ be the α such that $o(M) = \omega\alpha$. The index of M is

$$l(M) = \langle \hat{o}(M), k(M) \rangle.$$

If $\langle \nu, l\rangle \leq_{\text{lex}} l(M)$, then $M|\langle \nu, l\rangle$ is the initial segment N of M with index $l(N) = \langle \nu, l\rangle$. (So $\dot{E}^N = \dot{E}^M \cap N$, $\dot{F}^N = \dot{E}^M_\nu$, $\dot{\Sigma}^N = \dot{\Sigma}^M \cap N$, and $\dot{B}^N$ is determined by $\dot{\Sigma}^M$ is a way that will become clear shortly.) In order that M be an lpm, all its initial segments N must be $k(N)$-sound. If $\nu \leq \hat{o}(M)$, then we write $M|\nu$ for $M|\langle \nu, 0\rangle$.

As with ordinary premice, if M is an lpm, then $\dot{E}^M$ is the sequence of extenders that go into constructing M, and $\dot{F}^M$ is either empty, or codes a new extender being added to our model by M. $\dot{F}^M$ must satisfy the Jensen conditions; that is, if $F = \dot{F}^M$ is nonempty (i.e., M is *extender-active*), then $M \models \text{crit}(F)^+$ exists, and for $\mu = \text{crit}(F)^{+M}$, $o(M) = i^M_F(\mu)$. $\dot{F}^M$ is just the graph of $i^M_F\restriction(M|\mu)$. M must satisfy the Jensen initial segment condition (ISC). That is, the *whole* initial segments of $\dot{F}^M$ must appear in $\dot{E}^M$. If there is a largest whole proper initial segment, then $\dot{\gamma}^M$ is its index in $\dot{E}^M$. Otherwise, $\dot{\gamma}^M = 0$. Finally, an lpm M must be *coherent*, in that $i^M_F(\dot{E}^M)\restriction o(M) + 1 = \dot{E}^M{}^\frown\langle\emptyset\rangle$.

In other words, the conditions for adding extenders to M are just as in Jensen.

The predicates $\dot{\Sigma}^M$ and $\dot{B}^M$ are used to record information about an iteration strategy Ω for M. The strategy Ω will be determined by its action on normal trees, in an absolute way, so that we need only tell the model we are building how Ω acts on normal trees, and then the model itself can recover the action of Ω on the various non-normal trees it sees. Since this simplifies the notation, it is what we shall do.

Let us write $M|\langle \nu, -1\rangle$ for $(M|\langle \nu, 0\rangle)^-$; that is, for $M|\langle \nu, 0\rangle$ with its last extender predicate set to $\emptyset$.

Definition 5.1 *An M-*tree *is a triple $s = \langle \nu, k, \mathcal{T}\rangle$ such that*

(1) $\langle \nu, k\rangle \leq_{\text{lex}} l(M)$, and

(2) $\mathcal{T}$ is a normal iteration tree on $M|\langle \nu, k\rangle$.

We allow here $\mathcal{T}$ to be empty. The case $k = -1$ allows us to drop by throwing away a last extender predicate. Given an M-tree s we write $s = \langle \nu(s), k(s), \mathcal{T}(s)\rangle$. We write $M_\infty(s)$ for the last model of $\mathcal{T}(s)$, if it has one. We say $\text{lh}(\mathcal{T}(s))$ is the length of s.

What we shall feed into an lpm M is information about how its iteration strategy acts on M-trees.

$\dot{\Sigma}^M$ is a predicate that codes the strategy information added at earlier stages, with $\dot{\Sigma}^M(s, b)$ meaning that $\mathcal{T}(s)$ is a normal tree on $M|\langle \nu(s), k(s)\rangle$ of limit length, and $\mathcal{T}(s)^\frown b$ is according to the strategy. We write $\Sigma^M_{\nu,k}$ for the partial iteration strategy for $M|\langle \nu, k\rangle$ determined by $\dot{\Sigma}^M$. We write

$$\begin{aligned}\Sigma^M(s) = b &\text{ iff } \dot{\Sigma}^M(s, b)\\ &\text{ iff } \Sigma^M_{\nu(s),k(s)}(\mathcal{T}(s)) = b.\end{aligned}$$

We say that s is according to Σ^M iff $\mathcal{T}(s)$ is according to $\Sigma^M_{\nu(s),k(s)}$.

We now describe how strategy information is coded into the $\dot{B}^M$ predicate. Here we use the $\mathfrak{B}$-operator discovered by Schlutzenberg and Trang in [28]. In the original version of this paper, we made use of a different coding, one that has fine-structural problems. The authors of [37] discovered those problems. The discussion to follow is taken from [37].

Definition 5.2 *M is branch-active (or just B-active) iff*

(a) there is a largest $\eta < o(M)$ such that $M|\eta \models \mathsf{KP}$, and letting $N = M|\eta$,

(b) there is a $<_N$-least N-tree s such that s is by Σ^N, $\mathcal{T}(s)$ has limit length, and $\Sigma^N(s)$ is undefined.

(c) for N and s as above, $o(M) \leq o(N) + lh(\mathcal{T}(s))$.

Note that being branch-active can be expressed by a Σ_2 sentence in $\mathcal{L}_0 - \{\dot{B}\}$. This contrasts with being extender-active, which is not a property of the premouse with its top extender removed. In contrast with extenders, we know when branches must be added before we do so.

Definition 5.3 *Suppose that M is branch-active. We set*

$$\begin{aligned}b^M &= \{\alpha \mid \eta + \alpha \in \dot{B}^M\},\\ \eta^M &= \textit{the largest } \eta \textit{ such that } M|\eta \models \mathsf{KP},\\ s^M &= \textit{least } M|\eta^M\textit{-tree such that } \dot{\Sigma}^{M|\eta^M} \textit{ is undefined, and}\\ \nu^M &= \textit{unique } \nu \textit{ such that } \eta^M + \nu = o(M).\end{aligned}$$

Moreover, for $s = s^M$,

(1) M is a potential lpm *iff b^M is a cofinal branch of $\mathcal{T}(s)\restriction\nu^M$.*

(2) M is honest *iff $\nu^M = \mathrm{lh}(\mathcal{T}(s))$, or $\nu^M < lh(\mathcal{T}(s))$ and $b^M = [0, \nu^M)_{T(s)}$.*

(3) M is an lpm iff M is an honest potential lpm.

(4) M is strategy-active *iff* $\nu^M = \mathrm{lh}(\mathcal{T}(s))$.

We demand of an lpm M that if M is not $\dot{B}$-active, then $\dot{B}^M = \emptyset$.

The $\dot{\Sigma}$ predicate of an lpm grows at strategy-active stages. More precisely, suppose that $\hat{o}(Q)$ is a successor ordinal, and $M = Q|(\hat{o}(Q)-1)$. If M is strategy-active, then in order for Q to be an lpm, we must have

$$\dot{\Sigma}^Q = \dot{\Sigma}^M \cup \{\langle s, b^M\rangle\},$$

while if M is not strategy-active, we must have $\dot{\Sigma}^Q = \dot{\Sigma}^M$. If $\hat{o}(Q)$ is a limit ordinal, then we require that $\dot{\Sigma}^Q = \bigcup_{\eta<\hat{o}(Q)} \dot{\Sigma}^{Q|\eta}$. We see then that if M is an lpm and $\nu < \hat{o}(M)$, then $\dot{\Sigma}^{M|\nu} \subseteq \dot{\Sigma}^M$, and $M|\nu$ is strategy-active iff $\dot{\Sigma}^{M|\nu} \neq \dot{\Sigma}^M$.

This completes our definition of what it is for M to be a least-branch premouse, the definition being by induction on the hierarchy of M.

Definition 5.4 *M is a least branch premouse (lpm) iff M is an acceptable J structure meeting the requirements stated above.*

Notice that if M is an lpm, then no level of M is both $\dot{B}$-active and extender-active, because $\dot{B}$-active stages are additively decomposable.

Returning to the case that M is branch-active, note that η^M is a Σ_0^M singleton, because it is the least ordinal in $\dot{B}^M$ (because 0 is in every branch of every iteration tree), and thus s^M is also a Σ_0^M singleton. We have separated honesty from the other conditions because it is not expressible by a Q-sentence, whereas the rest is. Honesty is expressible by a Boolean combination of Σ_2 sentences. See 5.9 below.

The original version of this monograph required that when $o(M) < \eta^M + lh(\mathcal{T}(s))$, $\dot{B}^M$ is empty, whereas here we require that it code $[0, o(M))_{T(s)}$, in the same way that $\dot{B}^M$ will have to code a new branch when $o(M) = \eta^M + lh(\mathcal{T}(s)$. Of course, $[0, \nu^M)_{T(s)} \in M$ when $o(M) < \eta^M + lh(\mathcal{T}(s))$ and M is honest, so the current $\dot{B}^M$ seems equivalent to the original $\dot{B}^M = \emptyset$. However, $\dot{B}^M = \emptyset$ leads to Σ_1^M being too weak, with the consequence that a Σ_1 hull of M might collapse to something that is not an lpm. (The hull could satisfy $o(H) = \eta^H + lh(\mathcal{T}(s^H))$, even though $o(M) < \eta^M + lh(\mathcal{T}(s^M))$. But then being an lpm requires $\dot{B}^H \neq \emptyset$.) Our current choice for $\dot{B}^M$ solves that problem.

Remark 5.5 Suppose N is an lpm, and $N \models \mathsf{KP}$. It is very easy to see that $\dot{\Sigma}^N$ is defined on all N-trees s that are by $\dot{\Sigma}^N$ iff there are arbitrarily large $\xi < o(N)$ such that $N|\xi \models \mathsf{KP}$. If M is branch-active, then η^M is a successor admissible; moreover, we do add branch information, related to exactly one tree, at each successor admissible. Waiting until the next admissible to add branch information is just a convenient way to make sure we are done coding in the branch information for a given tree before we move on to the next one. One could go faster.

We say that an lpm M is (fully) passive if $\dot{F}^M = \emptyset$ and $\dot{B}^M = \emptyset$.

We would like to see that being an lpm is preserved by the appropriate embeddings. Q-formulae are useful for that.

Definition 5.6 *A rQ-*formula *of $\mathcal{L}_0$ is a conjunction of formulae of the form*

(a) $\forall u \exists v (u \subseteq v \wedge \varphi)$, where φ is a Σ_1 formula of $\mathcal{L}_0$ such that u does not occur free in φ,

or of the form

(b) "$\dot{F} \neq \emptyset$, and for $\mu = \mathrm{crit}(\dot{F})^+$, there are cofinally many $\xi < \mu$ such that ψ", where ψ is Σ_1.

Formulae of type (a) are usually called Q-formulae. Being a passive lpm can be expressed by a Q-sentence, but in order to express being an extender-active lpm, we need type (b) clauses, in order to say that the last extender is total. rQ formulae are π_2, and hence preserved downward under Σ_1-elementary maps. They are preserved upward under Σ_0 maps that are *strongly cofinal*.

Definition 5.7 *Let M and N be $\mathcal{L}_0$-structures and $\pi \colon M \to N$ be Σ_0 and cofinal. We say that π is* strongly cofinal *iff M and N are not extender active, or M and N are extender active, and letting $\pi``(\mathrm{crit}(\dot{F})^+)^M$ is cofinal in $(\mathrm{crit}(\dot{F})^+)^N$.*

It is easy to see that

Lemma 5.8 *rQ formulae are preserved downward under Σ_1-elementary maps, and upward under strongly cofinal Σ_0-elementary maps.*

Lemma 5.9 *(a) There is a Q-sentence φ of $\mathcal{L}_0$ such that for all transitive $\mathcal{L}_0$ structures M, $M \models \varphi$ iff M is a passive lpm.*

(b) There is a rQ-sentence φ of $\mathcal{L}_0$ such that for all transitive $\mathcal{L}_0$ structures M, $M \models \varphi$ iff M is an extender-active lpm.

(c) There is a Q-sentence φ of $\mathcal{L}_0$ such that for all transitive $\mathcal{L}_0$ structures M, $M \models \varphi$ iff M is a potential branch-active lpm.

Proof. (Sketch.) We omit the proofs of (a) and (b). For (c), note that "$\dot{B} \neq \emptyset$" is Σ_1. One can go on then to say with a Σ_1 sentence that if η is least in $\dot{B}$, then $M|\eta$ is admissible, and s^M exists. One can say with a Π_1 sentence that $\{\alpha \mid \dot{B}(\eta+\alpha)\}$ is a branch of $\mathcal{T}(s)$, perhaps of successor order type. One can say that $\dot{B}$ is cofinal in the ordinals with a Q-sentence. Collectively, these sentences express the conditions on potential lpm-hood related to $\dot{B}$. That the rest of M constitutes an extender-passive lpm can be expressed by a Π_1 sentence. $\square$

Corollary 5.10 *(a) If M is a passive (resp. extender-active, potential branch-active) lpm, and* $\mathrm{Ult}_0(M, E)$ *is wellfounded, then* $\mathrm{Ult}_0(M, E)$ *is a passive (resp.extender-active, potential branch-active) lpm.*

(b) Suppose that M is a passive (resp. extender-active, potential branch-active) lpm, and $\pi\colon H \to M$ is Σ_1-elementary; then H is a passive (resp. potential branch-active) lpm.

(c) Let $k(M) = k(H) = 0$, and $\pi\colon H \to M$ be Σ_2 elementary; then H is a branch-active lpm iff M is a branch-active lpm.

Proof. rQ-sentences are preserved upward by strongly cofinal Σ_0 embeddings, so we have (a). They are Π_2, hence preserved downward by Σ_1- elementary embeddings, so we have (b).

It is easy to see that honesty is expressible by a Boolean combination of Σ_2 sentences, so we get (c).

$\square$

Part (c) of Corollary 5.10 is not particularly useful. In general, our embeddings will preserve honesty of a potential branch active lpm M because $\dot{\Sigma}^M$ and $\dot{B}^M$ are determined by a complete iteration strategy for M that has strong hull condensation. So the more useful preservation theorem in the branch-active case applies to *hod pairs*, rather than to hod premice. See 5.21 below.

Remark 5.11 The following examples show that the preservation reults of 5.10 are optimal in certain respects.

(1) Let M be an extender-active lpm, and $N = \mathrm{Ult}_0(M, E)$, where E is a long extender over M whose space is $(\mathrm{crit}(\dot{F})^+)^M$, so that the canonical embedding $\pi\colon M \to N$ is discontinuous at $(\mathrm{crit}(\dot{F})^+)^M$. Then π is cofinal and Σ_0, so that M and N satisfy the same Q-sentences, but N is not an lpm, because its last extender is not total. π is not strongly cofinal, of course.

(2) The interpolation arguments in [19] yield examples of $\pi\colon M \to N$ being a weakly elementary (with $k(M) = k(N) = 0$), and N being an extender-active lpm, but M not being an lpm. Again, M falls short in that its last extender is not total.

The copying construction, and the lifting argument in the iterability proof, do give rise to maps that are only weakly elementary. However, in those cases we know the structures on both sides are lpms for other reasons. On the other hand, core maps and ultrapower maps are fully elementary, so we can apply (a) and (b) of Corollary 5.10 to them. We do need to do this.

5.2 Copying

Given $\pi\colon M \to N$ weakly elementary, we can copy an M-stack s to an N-stack πs, until we reach an illfounded model on the πs side. Thus if Ω is a complete strategy for N, we have the complete pullback strategy Ω^π for M. We extend the copying slightly, to incorporate some lifting. This will let us lift weakly normal trees to fully normal ones.

Let $\mathcal{T}$ be a weakly normal tree on the lpm M, and let $k = k(M)$. Let

$$\pi\colon M \to N|\langle \nu, k\rangle$$

be weakly elementary (that is, a weak k-embedding); then we can copy $\mathcal{T}$ to a fully normal tree $\mathcal{U}$ on N as follows. $\mathcal{U}$ has the same tree order as $\mathcal{T}$, so long as it is defined. Let M_α and N_α be the α-th models, and E_α and F_α the α-th extenders, of $\mathcal{T}$ and $\mathcal{U}$. We shall have a weakly elementary

$$\pi_\alpha\colon M_\alpha \to N_\alpha|\langle \nu_\alpha, k_\alpha\rangle.$$

Here $\pi_0 = \pi$, $\nu_0 = \nu$, and $k_0 = k$. We have the usual agreement and commutativity conditions:

(1) Whenever $\beta \leq \alpha$, $\pi_\beta \restriction \lambda(E_\beta)$ and $N_\alpha|\lambda(F_\beta) = N_\beta|\lambda(F_\beta)$, and

(2) whenever $\beta \leq_T \alpha$, then $\pi_\alpha \circ \hat{\imath}^{\mathcal{T}}_{\beta,\alpha} = \hat{\imath}^{\mathcal{U}}_{\beta,\alpha} \circ \pi_\beta$.

(We do not demand any further coordination of the points at which the two trees drop. $\mathcal{T}$ may drop gratuitously where $\mathcal{U}$ does not, and $\mathcal{U}$ may drop where $\mathcal{T}$ does not because the dropping point is above some $\langle \nu_\alpha, k_\alpha\rangle$.) The successor step is the following. We are given E_α on M_α; set

$$F_\alpha = \pi_\alpha(E_\alpha),$$

or $F_\alpha = \dot{F}^{N_\alpha|\langle\nu_\alpha,k_\alpha\rangle}$ if $E_\alpha = \dot{F}^{M_\alpha}$. Let $\beta = T\text{-pred}(\alpha+1) =$ least ξ such that $\kappa < \lambda(E_\xi)$, where $\kappa = \text{crit}(E_\alpha)$. By (1) above, $\beta = U\text{-pred}(\alpha+1)$ according to the rules of normality for $\mathcal{U}$. Let

$$M_{\alpha+1} = \text{Ult}(M_\beta|\langle\eta, l\rangle, E_\alpha),$$

and

$$N_{\alpha+1} = \text{Ult}(N_\beta|\langle\gamma, n\rangle, F_\alpha),$$

where $\langle\eta, l\rangle$ is chosen by I in $\mathcal{T}$, and $\langle\gamma, n\rangle$ is determined by normality. It is easy to see that

$$\langle\pi_\beta(\eta), l\rangle \leq_{\text{lex}} \langle\gamma, n\rangle.$$

(If $\langle\eta, l\rangle = l(M_\beta)$, we understand $\pi_\beta(\eta) = \nu_\beta$ here, and we have $l = k_\beta$. Since π_β is elementary, and no proper initial segment of M_β projects $\leq \kappa$, no proper initial segment of $N_\beta|\langle\nu_\beta, l\rangle$ projects $\leq \pi_\beta(\kappa)$. But $\pi_\beta(\kappa) = \text{crit}(F_\alpha)$, so $\langle\nu_\beta, l\rangle \leq_{\text{lex}} \langle\gamma, n\rangle$. If $\langle\eta, l\rangle <_{\text{lex}} l(M_\beta)$, a similar argument works.) We then set

$$\langle\nu_{\alpha+1}, k_{\alpha+1}\rangle = \hat{\imath}^{\mathcal{U}}_{\beta,\alpha+1}(\langle\pi_\beta(\eta), l\rangle)$$

and we have $\langle\nu_{\alpha+1}, k_{\alpha+1}\rangle \leq_{\text{lex}} l(N_{\alpha+1})$. $\pi_{\alpha+1}$ comes from the Shift Lemma definition:

$$\pi_{\alpha+1}([a, f]) = [\pi_\alpha(a), \pi_\beta(f)],$$

where the equivalence classes are in $\text{Ult}(M_\beta|\langle\eta, l\rangle, E_\alpha)$ and $\text{Ult}(N_\beta|\langle\gamma, n\rangle, F_\alpha)$ respectively. The proof of the Shift Lemma tells us that $\pi_{\alpha+1}$ is weakly elementary. (Even if we had started with elementary maps, the case that $\langle\pi_\beta(\eta), l\rangle <_{\text{lex}} \langle\gamma, n\rangle$ could lead to $\pi_{\alpha+1}$ not being fully elementary.)

Of course, at limit steps $\lambda < \text{lh}(\mathcal{T})$, we stop unless $[0, \lambda]_T$ is a wellfounded branch of $\mathcal{U}$. If it is, we get π_λ, ν_λ and k_λ from commutativity, and continue.

Definition 5.12 *Given $\pi\colon M \to N|\langle\nu, k\rangle$ weakly elementary and $\mathcal{T}$ on M weakly normal, $(\pi\mathcal{T})^+$ is the normal tree on N defined above. We call it the* (π, ν, k)-lift *of $\mathcal{T}$. If Ω is a strategy for N defined on normal trees, then $\Omega^{(\pi,\nu,k)}$ is the strategy on weakly normal trees given by pulling back: $\Omega^{(\pi,\nu,k)}(\mathcal{T}) = \Omega((\pi\mathcal{T})^+)$.*

We omit ν and k from the notation when no confusion can arise.

Similarly, we can π-lift M-stacks, and thus if Ω is a strategy for N defined on finite stacks of normal trees, then Ω^π exists, and is a complete strategy for M.

With $\pi =$ identity, we get

Lemma 5.13 *Let M be an lpm that is θ-iterable for normal trees; then M is θ iterable for weakly normal trees. If M is θ-iterable for finite stacks of normal trees, then M has a complete θ-iteration strategy.*

5.3 Least branch hod pairs

Suppose we reach the lpm M in a reasonable background construction $\mathbb{C}$. The strategies $\Omega_{\nu,k}$ for $M|\langle \nu, k\rangle$ that we get from $\mathbb{C}$ all come from a single strategy for V, and are therefore consistent with one another in the following sense.

Definition 5.14 *Let M be an lpm, and let $\Omega_{\nu,k}$ be an iteration strategy for $M|\langle \nu, k\rangle$, for each $\langle \nu, k\rangle \leq_{\text{lex}} l(M)$. We say $\langle \Omega_{\nu,k} \mid \langle \nu, k\rangle \leq_{\text{lex}} l(M)\rangle$ is* self-consistent *iff whenever $\langle \nu, k\rangle \leq_{\text{lex}} \langle \eta, l\rangle$, then*

$$\Omega_{\nu,k} = (\Omega_{\eta,l})^{(\ id_{\nu,k})}.$$

Here the Ω's may be a sequence of strategies acting on normal trees, or a sequence of complete strategies. They should all be θ-strategies, for some fixed θ.

Notice that if the $\Omega_{\nu,k}$ constitute a self-consistent system of strategies for M, then whenever ν is a cardinal of non-measurable cofinality in M, all $\Omega_{\eta,l}$ with $\nu \leq \eta < \hat{o}(M)$ agree on (stacks of) trees belonging to $M|\nu$. This is also true if $\eta = \hat{o}(M)$ and $\nu < \rho_l(M)$. Recall from 1.19 that tail strategies are defined by

$$\Omega_s(t) = \Omega(s^\frown t),$$

for all $M_\infty(s)$-stacks t. Also, for $N = M_\infty(s)|\langle \nu, k\rangle$, we set $\Omega_{s,N} = \Omega_{s,\langle \nu,k\rangle} = \Omega_{s^\frown\langle \nu,k,\emptyset\rangle}$. write $\Omega_{s,\langle \nu,k\rangle}$ for $\Omega_{s,N}$. When $N = M|\langle \nu, k\rangle$, we write Ω_N or $\Omega_{\langle \nu,k\rangle}$ for $\Omega_{\emptyset,N}$. Finally,

$$\Omega_{s,<\nu} = \bigcup\{\Omega_{s,\langle \eta,k\rangle} \mid \eta < \nu \wedge k \leq \omega\}$$

is our notation for a join of strategies.

Definition 5.15 *Let Ω be a complete strategy for M; then Ω is* self-consistent *iff whenever s is an M-stack by Ω such that $M_\infty(s)$ exists, then the family $\langle \Omega_{s,N} \mid N \trianglelefteq M_\infty(s)\rangle$ is self-consistent.*

Definition 5.16 *(M, Ω) is a* least branch hod pair *(lbr hod-pair) with scope H_δ iff*

(1) M is a least branch premouse,

(2) Ω is a complete iteration strategy for M, with scope H_δ,

(3) Ω is self-consistent, normalizes well, and has strong hull condensation, and

(4) If s is by Ω and has last model N, then $\dot{\Sigma}^N \subseteq \Omega_s$.

Of course, δ as in (2) is determined by Ω.

We say that (M, Ω) is *self-aware* just in case it has property (4).

Remark 5.17 This definition records the properties of the hod pairs we construct needed to prove the Comparison Theorem and the existence of cores. The other properties one might hope for seem to follow from these.

For example, if (M, Ω) is an lbr hod pair, then (M, Ω) is strategy coherent, as remarked above. Its iteration maps are minimal in the appropriate category of embeddings (5.27 and 5.28). Ω is pullback consistent by 5.25. More elaborate arguments involving phalanx comparisons show that Ω is positional, and fully normalizes well.

Remark 5.18 One can show that if Σ is a complete strategy for a premouse M that condenses well for normal trees, then Σ has a unique extension to a complete strategy for M that normalizes well. The proof follows the lines of our proof of 0.18. Thus, despite the title of this monograph, strong hull condensation is the key property that makes an iteration strategy well-behaved.

While we are at it, let us formally define pure extender pairs.

Definition 5.19 *(M, Ω) is a* pure extender pair *($L[E]$-pair) with scope H_δ iff*

(1) M is a pure extender premouse,

(2) Ω is a complete iteration strategy for M, with scope H_δ, and

(3) Ω is self-consistent, normalizes well, and has strong hull condensation.

Definition 5.20 *(M, Ω) is a* mouse pair *iff it is either a pure extender pair, or an lbr hod pair.*

One very useful elementary fact is

Lemma 5.21 *Let (M, Ω) be a mouse pair with scope H_δ, let $\pi \colon N \to M$ be weakly elementary, and suppose that if $\dot{F}^N \neq \emptyset$, then $\dot{F}^N$ is total over N; then (N, Ω^π) is a mouse pair with scope H_δ.*

Proof. Let us just consider the case that M is an lpm. In this case, N is an lpm by 5.10, except perhaps when M is branch-active. In this case, N is a potential branch-active lpm, and we must see that N is honest.

So let $\nu = \nu^N$, $b = b^N$, and $\mathcal{T} = \mathcal{T}(s^N)$. If $\nu = \mathrm{lh}(\mathcal{T})$, there is nothing to show, so assume $\nu < \mathrm{lh}(\mathcal{T})$. We must show that $b = [0, \nu)_T$. We have by induction that for $Q = N|\eta^N$, (Q, Ω^π_Q) is an lbr hod pair, and in particular, that it is self-aware. Thus $\mathcal{T}$ is by Ω^π, and so we just need to see that for $\mathcal{U} = \mathcal{T}\restriction\nu$, $\mathcal{U}^\frown b$ is by Ω^π, or equivalently, that $\pi\mathcal{U}^\frown b$ is by Ω. But it is easy to see that $\pi\mathcal{U}^\frown b$ is a psuedo-hull of $\pi(\mathcal{U})^\frown b^M$, and Ω has strong hull condensation, so we are done.

Thus N is an lpm. Ω^π is a complete iteration strategy defined on all N-stacks in H_δ, where H_δ is the scope of (M, Ω). Ω^π normalizes well by the the proof of 3.3, and has strong hull condensation by the proof of 3.6.

Self-consistency is straightforward; it's an instance of lifting commuting with copying. Let $\Psi = \Omega^\pi$. Let s be an N-stack such that $M_\infty(s)$ exists. Let $\langle \nu, k\rangle \leq_{\mathrm{lex}} \langle \eta, l\rangle \leq_{\mathrm{lex}} l(M_\infty(s))$. We must show that $\Psi_{s,\langle\nu,k\rangle} = (\Psi_{s,\langle\eta,l\rangle})^{\mathrm{id},\nu,k\rangle}$. We assume for notational simplicity that $s = \emptyset$, so $M_\infty(s) = N$. Let $\mathcal{U}$ be a normal tree on $N|\langle\nu,k\rangle$. The relevant diagram is:

$$\begin{tikzcd}
 & R \arrow[rr, "\tau^*"] & & S \\
M|\langle\pi(\nu),k\rangle \arrow[ur, "\pi\mathcal{U}"] \arrow[rr] & & M|\langle\pi(\eta),l\rangle \arrow[ur, "(\pi\mathcal{U})^+"'] & \\
 & P \arrow[uu] \arrow[rr, "\tau"] & & Q \arrow[uu] \\
N|\langle\nu,k\rangle \arrow[uu, "\pi"] \arrow[ur, "\mathcal{U}"] \arrow[rr, "\mathrm{id}"'] & & N|\langle\eta,l\rangle \arrow[uu, "\pi"'] \arrow[ur, "\mathcal{U}^+"'] &
\end{tikzcd}$$

Here $\mathcal{U}^+$ is the lift of $\mathcal{U}$ to $N|\langle\eta,l\rangle$ under the identity map. The bottom square represents that process, and τ is one of its maps. The top square represents the lifting of $\pi\mathcal{U}$ from $M|\langle\pi(\nu,k\rangle$ to $M|\langle\pi(\eta),l\rangle$, and τ^* is one of its maps. Because everything commutes, we get

$$\pi(\mathcal{U}^+) = (\pi\mathcal{U})^+,$$

i.e. lifting commutes with copying. Therefore

$$\begin{aligned}\mathcal{U} \text{ is by } (\Omega^\pi)_{\langle \nu,k\rangle} &\Leftrightarrow \pi\mathcal{U} \text{ is by } \Omega_{\langle \pi(\nu),k\rangle}\\ &\Leftrightarrow (\pi\mathcal{U})^+ \text{ is by } \Omega_{\langle \pi(\eta),l\rangle}\\ &\Leftrightarrow \pi(\mathcal{U}^+) \text{ is by } \Omega_{\langle \pi(\eta),l\rangle}\\ &\Leftrightarrow \mathcal{U}^+ \text{ is by } (\Omega^\pi)_{\langle \eta,l\rangle}.\end{aligned}$$

This is what we want. The case that $s \neq \emptyset$ or $\mathcal{U}$ is replaced by a stack is no different.

Finally, we must show that (N, Ω^π) is self-aware. Let P be a Ω^π iterate of N, via the stack s. Let Q be the corresponding iterate of M via πs, and let $\tau\colon P \to Q$ be the copy map. Then

$$\begin{aligned}\mathcal{U} \text{ is by } \dot{\Sigma}^P &\Rightarrow \tau(\mathcal{U}) \text{ is by } \dot{\Sigma}^Q\\ &\Rightarrow \tau(\mathcal{U}) \text{ is by } \Omega_{\pi s,Q}\\ &\Rightarrow \tau\mathcal{U} \text{ is by } \Omega_{\pi s,Q}\\ &\Rightarrow \mathcal{U} \text{ is by } (\Omega^\pi)_{s,P},\end{aligned}$$

as desired. □

5.4 Mouse pairs and the Dodd-Jensen Lemma

Mouse is generally taken to mean *iterable premouse*, and the Comparison Lemma is taken to say that any two mice M and N can be compared as to how much information they contain. But in fact, how M and N are compared depends on which iteration strategies witnessing their iterability are chosen. There is no mouse order on iterable premice, even of the pure extender variety, unless we make restrictive assumptions which imply that the iteration strategy is unique. The canonical information levels of the mouse order are occupied not by mice, but by mouse pairs. These pairs are the objects to which the Comparison Lemma, the Dodd-Jensen Lemma, and the other basic results of inner model theory apply. In the special case that M can have at most one strategy, we don't need to make the pair explicit, but in general, we do.

Let us introduce some terminology that reflects this conceptual adjustment.

Definition 5.22 *Let (P, Σ) and (Q, Ω) be mouse pairs, and $\pi\colon P \to Q$; then*

(a) $\pi\colon (P,\Sigma) \to (Q,\Omega)$ *is* elementary *(resp.* weakly elementary*) iff π is elementary (resp. weakly elementary) as a map from P to Q, and $\Sigma = \Omega^\pi$,*

(b) *(Q,Ω) is an* iterate of *(P,Σ) iff there is a stack s on P by Σ with last model Q such that $\Omega = \Sigma_{s,Q}$. If s can to be taken to be a single normal tree, then (Q,Ω) is a* normal iterate *of (P,Σ). If s can be taken so that P-to-Q in s does not drop, then (Q,Ω) is a* non-dropping iterate *of (P,Σ).*

(c) *$(P,\Sigma) \leq^* (Q,\Omega)$ iff there is an iterate (R,Ψ) of $Q,\Omega)$ and an elementary $\pi\colon (P,\Sigma) \to (R,\Psi)$. We call $\leq^*$ the* mouse pair order.

Here are some elementary facts stated in this language.

Lemma 5.23 *Let (P,Σ) be a mouse pair with scope H_δ, and let (Q,Ω) be an iterate of (P,Σ); then (Q,Ω) is a mouse pair with scope H_δ.*

Proof. If M is an lpm, then N is an lpm by 5.10. The properties in (3) and (4) of 5.16 clearly pass to tail strategies. □

Definition 5.24 *Let Ω be a complete iteration strategy for M. We say that Ω is* pullback consistent *iff whenever s is an M-stack by Ω, and $\pi\colon \mathcal{M}_\alpha^{\mathcal{T}_m(s)}|\langle\nu,k\rangle \to \mathcal{M}_\infty(s)$ is an iteration map of s, then for $t = s\restriction(m-1)^\frown\langle(\nu_m(s), k_m(s), \mathcal{T}_m(s)\restriction(\alpha+1))\rangle$,*

$$\Omega_{t,\langle\nu,k\rangle} = (\Omega_s)^\pi.$$

A pullback consistent strategy pulls back to itself under its own iteration maps, where by "iteration map" we mean any map of a branch segment generated somewhere in a stack s by the strategy, from one model to a later one. This is a strengthening of the pullback consistency condition from [16].

Lemma 5.25 *Let (M,Ω) be a mouse pair; then Ω is pullback consistent.*

Proof.(Sketch.) For example, suppose s consists of one normal tree $\mathcal{T}$, and that $\pi = i^{\mathcal{T}}_{\alpha,\beta}$, where $\beta+1 = \mathrm{lh}(\mathcal{T})$. Let $\mathcal{U}$ be a normal tree on $\mathcal{M}^{\mathcal{T}}_\alpha$. We want to see that $\langle \mathcal{T}\restriction(\alpha+1), \mathcal{U}\rangle$ is by Ω iff $\langle \mathcal{T}, \pi\mathcal{U}\rangle$ is by Ω.

Let $\mathcal{W}_\gamma = W(\mathcal{T}\restriction(\alpha+1), \mathcal{U}\restriction(\gamma+1))$ and $\mathcal{W}^*_\gamma = W(\mathcal{T}, \pi\mathcal{U}\restriction(\gamma+1))$. By induction on γ, we construct psuedo-hull embeddings Φ_γ from $\mathcal{W}_\gamma$ into $\mathcal{W}^*_\gamma$. The construction is pretty much the same as that in the proof of the Comparison Lemma, theorem 4.10. It works also for $\gamma = b$, where b is a branch of $\mathcal{U}$.

We have then that

$$\begin{aligned}\langle \mathcal{T}, \pi\mathcal{U}\restriction(\gamma+1)\rangle \text{ is by } \Omega &\Rightarrow W(\mathcal{T}, \pi\mathcal{U}\restriction(\gamma+1)) \text{ is by } \Omega\\ &\Rightarrow W(\mathcal{T}\restriction(\alpha+1), \mathcal{U}) \text{ is by } \Omega\\ &\Rightarrow \langle \mathcal{T}\restriction(\alpha+1), \mathcal{U}\restriction(\gamma+1)\rangle \text{ is by } \Omega,\end{aligned}$$

as desired. □

Corollary 5.26 *Let (P, Σ) be a mouse pair, and (Q, Ω) be a non-dropping iterate of (P, Σ), with iteration map π; then $\pi\colon (P, \Sigma) \to (Q, \Omega)$ is elementary (in the category of mouse pairs).*

The appropriate statement of the Dodd-Jensen Lemma on the minimality of iteration maps is:

Theorem 5.27 *(Dodd-Jensen Lemma) Let (P, Σ) be an mouse pair, let (Q, Ω) be an iterate of (P, Σ) via the stack s, and let $\pi\colon (P, \Sigma) \to (Q, \Omega)$ be weakly elementary; then*

(a) the branch P-to-Q of s does not drop, and

(b) letting $i_s\colon P \to Q$ be the iteration map, for all $\eta < o(M)$, $i_s(\eta) \leq \pi(\eta)$.

We omit the well known proof. Notice that it requires the assumption that $\Sigma^{\pi}_{s,N} = \Sigma$. This was at one time a nontrivial restriction on the applicability of the Dodd-Jensen Lemma, and led to the Weak Dodd-Jensen Lemma of [13]. Now that we can compare iteration strategies, the restriction is less important.

We get the Dodd-Jensen corollary on the uniqueness of iteration maps.

Corollary 5.28 *Let (P, Σ) be a mouse pair, (Q, Ω) a non-dropping iterate of (P, Σ) via the stack s, and suppose $(Q, \Omega) \trianglelefteq (R, \Psi)$, where (R, Ψ) is an iterate of (P, Σ) via the stack t; then*

(a) $(Q, \Omega) = (R, \Psi)$, and the branch P-to-R of t does not drop, and

(b) letting i_s and i_t be the two iteration maps, $i_s = i_t$.

In the language of mouse pairs, the Comparison Lemma reads

Theorem 5.29 *(Comparison Lemma) Assume* AD^+*, and let* (P, Σ) *and* (Q, Ψ) *be mouse pairs with scope HC of the same type; then* (P, Σ) *and* (Q, Ψ) *have a common iterate* (R, Ω)*, obtained via normal trees* $\mathcal{T}$ *on* P *and* $\mathcal{U}$ *on* Q *such that at least one of* P*-to-*R *and* Q*-to-*R *does not drop.*

This is a mild re-statement of Theorem 5.54, and we shall finish proving it in the section after next. For now let us assume it. We get

Corollary 5.30 *Assume* AD^+*; then*

(a) For (P, Σ) *and* (Q, Ψ) *mouse pairs with scope HC of the same type,*

$$\begin{aligned}(P, \Sigma) <^* (Q, \Psi) \Leftrightarrow &\exists (R, \Omega) \exists \pi [(R, \Omega) \text{ is a dropping iterate of } (Q, \Psi) \\ &\text{and } \pi \colon (P, \Sigma) \to (R, \Omega) \text{ is weakly elementary}].\end{aligned}$$

(b) When restricted to a fixed type, $\leq^*$ *is a prewellorder of mouse pairs with scope HC.*

Proof. The left-to-right direction of (a) follows from the Comparison Lemma. The right-to-left direction follows from Dodd-Jensen. For (b), the Comparison Lemma implies that $\leq^*$ is linear. That it is wellfounded follows from (a), using the proof of the Dodd-Jensen Lemma. $\square$

Strategy coherence is defined for mouse pairs just as it was for pure extender pairs

Definition 5.31 *A mouse pair* (P, Σ) *is* strategy coherent *iff whenever* (Q, Ψ) *is an iterate of* (P, Σ)*, and* $\mathcal{T}$ *is a normal tree by* Ψ*, and* $N \trianglelefteq \mathcal{M}^{\mathcal{T}}_\alpha$ *and* $N \trianglelefteq \mathcal{M}^{\mathcal{T}}_\beta$*, then* $\Psi_{\mathcal{T}\restriction(\alpha+1), N} = \Psi_{\mathcal{T}\restriction(\beta+1), N}$*.*

The proof of Lemma 3.35 gives

Lemma 5.32 *Let* (P, Σ) *be a mouse pair; then* (P, Σ) *is strategy coherent.*

This lemma is needed in the comparison proof for strategies, just as the usual extender coherence is needed in the comparison proof for pure extender mice.

Let (P, Σ) be a mouse pair. Recall that Σ is *positional* iff whenever (Q, Ψ) and (R, Ω) are iterates of (P, Σ), and $Q = R$, then $\Psi = \Omega$. The property is clearly related to what is called being positional in [16]. In the present context, with gratuitous dropping allowed, it implies strategy coherence.

[33] proves

Lemma 5.33 *Assume* AD^+*, and let* (P, Σ) *be a mouse pair with scope HC; then* Σ *is positional.*

Fortunately, this Lemma is not needed in the proof of the Comparison Lemma 5.29. Its proof instead relies on a comparison argument.

Here are two propositions that explain the relationship between pure extender mice and pure extender pairs.

Proposition 5.34 *Assume* AD^+*, and let* P *be a countable,* ω_1*-iterable pure extender premouse; then there is a* Σ *such that* (P, Σ) *is a pure extender pair.*

Proof. Let Ψ be an arbitrary ω_1 iteration strategy for P. We may assume Ψ is Suslin and co-Suslin by Woodin's Basis Theorem. Thus there is a coarse Γ-Woodin pair (N^*, Σ^*) that captures Ψ. Working in N^*, we get that P iterates by Ψ to a level (Q, Ψ) of the pure extender pair construction of N^*. Let $\pi \colon P \to Q$ be the iteration map; then (P, Ψ^π) is a pure extender pair. $\square$

Proposition 5.35 *Assume* AD^+*,* LEC*, and* $\theta_0 < \theta$*; then there are pure extender pairs* (P, Σ) *and* (P, Ω) *such that* $(P, \Sigma) <^* (P, \Omega)$*.*

Proof.(Sketch.) By LEC, there is a pure extender pair (P, Ω) such that Ω is not ordinal definable from a real. Fix such a pair. By the Basis theorem, there is a Σ such that (P, Σ) is a pure extender pair, and Σ is ordinal definable from a real. Suppose toward contradiction that $(P, \Omega) \leq^* (P, \Sigma)$; then

$$\Omega = (\Sigma_s)^\pi$$

for some stack s and iteration map π. Thus Ω is ordinal definable from a real, contradiction. $\square$

It follows that under the hypotheses of 5.35, there are pure extender pairs (P, Σ) and (P, Ω) such that for some R, P iterates normally by Σ to a proper initial segment of R, and normally by Ω to a proper extension of R.

The Dodd-Jensen Lemma hypothesis that $\Sigma^\pi_{s,P} = \Sigma$ is too restrictive for use in the proof of solidity and universality of standard parameters. For that proof, we need the Weak Dodd-Jensen Lemma of [13].

Note that the proofs we have given that background induced strategies normalize well and have strong hull condensation actually yield (ω_1, ω_1) strategies Ω such that each Ω_s, for $\mathrm{lh}(s) < \omega_1$, normalizes well and has strong hull condensation. We need this in the weak Dodd-Jensen argument to come.

Let N be a countable pure extender premouse or lpm, and $\langle e_i \mid i < \omega\rangle$ enumerate the universe of N. A map $\pi\colon N \to M$ is *$\vec{e}$-minimal* just in case π is elementary, and whenever $\sigma\colon N \to M|\langle \eta, k\rangle$ is elementary, then $\langle \eta, k\rangle = l(M)$, and if $\sigma \neq \pi$, then for i least such that $\sigma(e_i) \neq \pi(e_i)$, we have $\pi(e_i) < \sigma(e_i)$ (in the order of construction). A complete strategy Ω for N has the *weak Dodd-Jensen property relative to $\vec{e}$* iff whenever $M = M_\infty(s)$ for some stack s by Ω, and there is some elementary embedding fom N to an initial segment of M, then the branch N-to-M of s does not drop, and the iteration map i^s is $\vec{e}$-minimal.

Lemma 5.36 *(Weak Dodd-Jensen) Let (M, Ω) be a mouse pair with scope H_δ, and let $\vec{e}$ be an enumeration of the universe of M in order type ω. Suppose that Ω is defined on all countable M-stacks s from H_δ, and that for any such s having a last model, $(M_\infty(s), \Omega_s)$ is an lbr hod pair. Then there is a countable M-stack s by Ω having last model $N = M_\infty(s)$, and an elementary $\pi\colon M \to N$, such that*

(1) $(N, (\Omega_s)^\pi)$ is a mouse pair, and

(2) $(\Omega_s)^\pi$ has the weak Dodd-Jensen property relative to $\vec{e}$.

Proof. The proof from [13] goes over verbatim. Notice here that any such $(N, (\Omega_s)^\pi)$ is an lbr hod pair, by 5.23 and 5.21. □

We have stated the elementary results about lbr hod pairs of the last two sections as results about mouse pairs, because that is their natural context. We are mainly interested in lbr hod pairs for the rest of this paper, so we shall return to that level of generality.

5.5 Background constructions

It is easy to modify the background constructions of pure extender premice described in section 2 so that they produce least branch hod pairs. The background conditions for adding an extender are unchanged. If we have reached the stage at which $M_{\nu,0}$ is to be defined, then our construction, together with an iteration strategy for the background universe, will have provided us with complete iteration strategies $\Omega_{\eta,l}$ for $M_{\eta,l}$, for all $\eta < \nu$. Each $(M_{\eta,l}, \Omega_{\eta,l})$ will be a least branch hod pair. If $M_{\nu,0}$ is to be branch-active according to the lpm requirements, then we use the appropriate $\Omega_{\eta,l}$ to determine $\dot{B}^{M_{\nu,0}}$.

The strategies $\Omega_{\eta,l}$ for $M_{\eta,l}$ are determined by lifting to V, just as before. The additional strategy predicates in our structures affect what we mean by cores and resurrection, but otherwise, nothing changes.

As before, $M_{\nu,k+1}$ is the core of $M_{\nu,k}$. We shall need to show that the standard parameter of $M_{\nu,k}$ behaves well, so that this core is sound, and agrees with $M_{\nu,k}$ up to ξ, where $\xi = (\rho(M_{\nu,k})^+)^{M_{\nu,k}}$. Letting $\pi \colon M_{\nu,k+1} \to M_{\nu,k}$ be the uncoring map, and $\gamma < \xi$, this requires that $(\Omega_{\nu,k})_{\langle \gamma,l \rangle}$ agree with the π-pullback of $(\Omega_{\nu,k})_{\langle \pi(\gamma),l>}$ on all stacks belonging to $M_{\nu,k}|\xi$. We shall show this, but we shall not show that these two strategies agree on all $M_{\nu,k}|\langle \gamma, l\rangle$-stacks in V. We doubt that is true in general, but we do not have a counterexample.

Let w be a wellorder of V_δ, and $\kappa < \delta$. Let us assume

Iterability Hypothesis $\mathsf{IH}_{\kappa,\delta}$ For any coarsely coherent $\vec{F}$ such that all F_ν have critical point $> \kappa$, and belong to V_δ, V is uniquely $\vec{F}$- iterable for normal trees in V_δ.

We shall add an assumption regarding the existence of Γ-Woodin cardinals later, in order to have an environment in which we can apply $(*)(P, \Sigma)$.

A *least branch* w*- construction above* κ is a full background construction in which, as before, the background extenders are nice, have critical points $> \kappa$, cohere with w, have strictly increasing strengths, and are minimal (first in Mitchell order, then in w).

More precisely, such a construction $\mathbb{C}$ consists of least branch premice $M^{\mathbb{C}}_{\nu,k}$ and extenders $F^{\mathbb{C}}_\nu$. The length $\mathrm{lh}(\mathbb{C})$ of $\mathbb{C}$ is the least $\langle \nu, k\rangle$ such that $M^{\mathbb{C}}_{\nu,k}$ is not defined. $M_{0,0}$ is the passive premouse with universe V_ω, and $\Omega_{0,0}$ is its unique iteration strategy. The indices are pairs $\langle \nu, k\rangle$ such that $-1 \le k \le \omega$.

$\mathbb{C}$ determines resurrection maps $\mathrm{Res}_{\nu,k}$ and $\sigma_{\nu,k}$ for $\langle \nu, k\rangle <_{\mathrm{lex}} \mathrm{lh}(\mathbb{C})$, in the same way as before: we define $\mathrm{Res}_{\nu,k+1}$, $\sigma_{\nu,k+1}$ by

1. If $N = M_{\nu,k+1}$, then $\mathrm{Res}_{\nu,k+1}[N] = \langle \nu, k+1\rangle$ and $\sigma_{\nu,k+1}[N] = \text{identity}$.
2. If $N \lhd M_{\nu,k+1}|(\rho^+)^{M_{\nu,k+1}}$, where $\rho = \rho(M_{\nu,k})$, then $\mathrm{Res}_{\nu,k+1}[N] = \mathrm{Res}_{\nu,k}[N]$ and $\sigma_{\nu,k+1}[N] = \sigma_{\nu,k}[N]$.
3. Otherwise, letting $\pi : M_{\nu,k+1} \to M_{\nu,k}$ be the anti-core map, $\mathrm{Res}_{\nu,k+1}[N] = \mathrm{Res}_{\nu,k}[\pi(N)]$ and $\sigma_{\nu,k+1} = \sigma_{\nu,k}[\pi(N)] \circ \pi$.

For the definition of $\mathrm{Res}_{\nu,0}$ and $\sigma_{\nu,0}$ see [1]. The resurrection maps are fully elementary, and their agreement properties are the same as before.

The sequence $\langle F^{\mathbb{C}}_\nu \mid \dot{F}^{M_{\nu,0}} \neq \emptyset\rangle$ of background extenders will be coarsely coherent. Thus V is uniquely $\vec{F}$-iterable for stacks of normal trees above κ in V_δ. Let Σ^* be the iteration strategy witnessing this. Σ^* then induces complete strategies $\Omega^{\mathbb{C}}_{\nu,k}$ for $M^{\mathbb{C}}_{\nu,k}$, for each $\langle \nu, k\rangle < \mathrm{lh}(\mathbb{C})$. These are obtained by lifting, as before, with the additional feature that gratuitous dropping is treated like ordinary dropping in the definition of lift$(s, M_{\nu,k}, \mathbb{C})$, for s an $M_{\nu,k}$-stack. (The lift of a stack of weakly normal trees is a stack of normal trees on V.) That is,

$$s \text{ is by } \Omega^{\mathbb{C}}_{\nu,k} \text{ iff } \mathrm{lift}(s, M_{\nu,k}, \mathbb{C}) \text{ is by } \Sigma^*.$$

Remark 5.37 For example, let $s = \langle \beta, l, \mathcal{T} \rangle$ be an $M_{\nu,k}$-stack of length one, and let $N = M_{\nu,k}|\langle \beta, l \rangle$. Let

$$\langle \eta, l \rangle = \mathrm{Res}_{\nu,k}[N], \text{ and } \sigma = \sigma_{\nu,k}[N].$$

So σ is elementary from N to $M_{\eta,l}$. Then letting

$$\mathrm{lift}(\sigma\mathcal{T}, M_{\eta,l}, \mathbb{C}, \Sigma^*) = \langle \mathcal{T}^*, \langle (\eta_\xi, l_\xi) \mid \xi < \mathrm{lh}\,\mathcal{T} \rangle, \langle \pi_\xi \mid \xi < \mathrm{lh}\,\mathcal{T} \rangle \rangle,$$

we have that

$$\langle \beta, l, \mathcal{T} \rangle \text{ is by } \Omega_{\nu,k} \text{ iff } \mathcal{T}^* \text{ is by } \Sigma^*.$$

If $Q = \mathcal{M}^{\mathcal{T}}_\xi$ is the last model of $\mathcal{T}$, and $\tau : Q \to M^{\sigma\mathcal{T}}_\xi$ is the copy map, then $\pi_\xi \circ \tau$ maps Q into a model of the construction $i^{\mathcal{T}^*}_{0,\xi}(\mathbb{C})$. This enables us to define $\Omega_{\nu,k}$ on stacks extending s; for example, if $t = s^\frown \langle \gamma, n, \mathcal{U} \rangle$, then we handle the possibly gratuitous drop in Q by resurrecting $\pi_\xi(Q|\langle \gamma, n \rangle)$ from the stage $\pi_\xi(Q)$ inside $i^{\mathcal{T}^*}_{0,\xi}(\mathbb{C})$, just as above. Etc.

Our construction determines in this way complete iteration strategies $\Omega^{\mathbb{C}}_{\nu,k}$ for $M^{\mathbb{C}}_{\nu,k}$, defined on stacks in V_δ, for each $\langle \nu, k \rangle < \mathrm{lh}(\mathbb{C})$. We demand that $(M_{\nu,k}, \Omega_{\nu,k})$ be a least branch hod pair; otherwise we stop the construction and leave $M_{\nu,k}$ undefined.

Suppose now we have $M_{\nu,k}$ and $\Omega_{\nu,k}$. Let $\rho = \rho(M_{\nu,k})$ and $p = p(M_{\nu,k})$ be the $k+1$-st projectum and parameter. Let u be either the sequence of solidity witnesses for $p_k(M_{\nu,k})$, or that sequence together with $\rho_{k-1}(M_{\nu,k})$ if the latter is $< o(M_{\nu,k})$. Let

$$\pi \colon N \to M_{\nu,k}$$

where N is transitive and

$$\mathrm{ran}(\pi) = \mathrm{Hull}^{M_{\nu,k}}_{k+1}(\rho \cup \{p, u\}).$$

We shall prove

$(\dagger))_{\nu,k}$ **for** $k \geq 0$**.**

(a) $M_{\nu,k}|(\rho^+)^{M_{\nu,k}} = N|(\rho^+)^N$, and

(b) $\pi^{-1}(p)$ is solid over N.

Items (a) and (b) of (†) are the universality and solidity of the standard parameter. They are needed to see that the iteration maps of $\Omega_{\nu,k+1}$ are elementary, which goes into the proof that the lifting maps in the construction of $\Omega_{\nu,k+1}$ are weakly elementary. So we need (a) and (b) before we can define $\Omega_{\nu,k+1}$.

Corollary 5.65 below proves $(\dagger)_{\nu,k}$ under the assumption that for every countable M and $\pi\colon M \to M_{\nu,k}$ elementary, letting $\Psi = (\Omega_{\nu,k})^\pi \restriction \mathrm{HC}$, $L(\Psi, \mathbb{R}) \models \mathsf{AD}^+$. Note here that Ψ is $(<\kappa)$-Universally Baire, where κ is our lower bound on the critical points of background extenders, by the uniqueness implicit in $\mathsf{IH}_{\kappa,\delta}$. So $L(\Psi, \mathbb{R}) \models \mathsf{AD}^+$ follows from there being infinitely many Woodin cardinals below κ.

If $(M_{\nu,k}, \Omega_{\nu,k})$ satisfies $(\dagger)_{\nu,k}$, then we let $M_{\nu,k+1}$ be the transitive collapse of $\mathrm{Hull}_{k+1}^{M_{\nu,k}}(\rho \cup \{p, u\})$, with $k(M_{\nu,k+1}) = k+1$. The lifting procedure and our iterability hypothesis $\mathsf{IH}_{\kappa,\delta}$ yield a complete iteration strategy $\Omega_{\nu,k+1}$ for $M_{\nu,k+1}$ on stacks in V_δ. The proofs of theorems 3.26 and 3.37 show that $\Omega_{\nu,k+1}$ normalizes well, and has strong hull condensation. In fact, $(M_{\nu,k+1}, \Omega_{\nu,k+1})$ is a least branch hod pair.

Lemma 5.38 *Assume $\mathbb{C}$ satisfies $(\dagger)_{\nu,k}$; then*

(1) $(N, \Omega_{\nu,k+1})$ is a least branch hod pair, and

(2) setting $\gamma = (\rho^+)^{M_{\nu,k}}$, $(\Omega_{\nu,k})_{\langle\gamma,0\rangle} = (\Omega_{\nu,k+1})_{\langle\gamma,0\rangle}$.

Proof. Part (2) is an immediate consequence of the fact that for $\xi < (\rho^+)^{M_{\nu,k}}$ and $Q = M_{\nu,k}|\langle\xi, l\rangle$, $\mathrm{Res}_{\nu,k}[Q] = \mathrm{Res}_{\nu,k+1}[Q]$ and $\sigma_{\nu,k}[Q] = \sigma_{\nu,k+1}[Q]$.

For part (1), we repeat the proofs that background induced strategies normalize well and have strong hull condensation (3.26 and 3.37) that we gave in the pure extender model case. What is left is to show that (N, Ω) is self-aware, where $\Omega = \Omega_{\nu,k+1}$.

For this, let s be a stack on N by Ω, with last model P. Let $\mathcal{T} \in \dot{\Sigma}^P$. We must see that $\mathcal{T}$ is by Ω_s. Let

$$s^* = \mathrm{lift}(s, N, \mathbb{C}),$$

and let S^* be the last model of s^*. Let Σ^* be the unique $\vec{F}^{\mathbb{C}}$-iteration strategy for V, so that $\Sigma^*_{s^*}$ is the unique $\vec{F}^{\mathbb{D}}$ strategy for S^*, where $\mathbb{D}$ is the image of $\mathbb{C}$ in S^*. We have

$$\pi\colon N \to Q$$

where Q is a model of the construction of S^*. Let Ψ be the strategy for Q induced by the construction of S^*. We have that

$$\Omega_s = \Psi^\pi,$$

because this is how Ω is induced by Σ^*. So we are done if we show that $\pi\mathcal{T}$ is by Ψ.

But $\pi(\mathcal{T}) \in \dot{\Sigma}^Q$, so $\pi(\mathcal{T})$ is by Ψ because (Q, Ψ) is an lbr hod pair in S^*. Moreover, Ψ has strong hull condensation, not just in S^*, but in V. (That is because a psuedo-hull $\mathcal{W}$ of some $\mathcal{U}$ by Ψ lifts to a psuedo-hull $\mathcal{W}^*$, of some $\mathcal{U}^*$ by $\Sigma^* s^*$, and even if $\mathcal{W}$ and $\mathcal{W}^*$ are not in S^*, $\Sigma^*_{s^*}$ chooses unique-in-V cofinal wellfounded branches, so $\mathcal{W}^*$ is by $\Sigma^*_{s^*}$, and hence $\mathcal{W}$ is by Ψ.) Since $\pi\mathcal{T}$ is a hull of $\pi(\mathcal{T})$, $\pi\mathcal{T}$n is by Ψ, as desired.

□

If $(\dagger)_{\nu,k}$ is not the case, then we stop the construction, leaving $M_{\nu,k+1}$ undefined. If $(\dagger)_{\nu,k}$ holds, then we set

$$M_{\nu,k+1} = N,$$

let $\Omega_{\nu,k+1}$ be its $\mathbb{C}$-induced strategy,and continue.

For $k < \omega$ sufficiently large, $M_{\nu,k} = M_{\nu,k+1}$, and we set

$$M_{\nu,\omega} = \text{eventual value of } M_{\nu,k} \text{ as } k \to \omega,$$

and

$$\begin{aligned} M_{\nu+1,0} = &\text{ rud closure of } M_{\nu,\omega} \cup \{M_{\nu,\omega}\}, \\ &\text{arranged as a fully passive premouse.} \end{aligned}$$

Finally, if ν is a limit, put

$$\begin{aligned} \mathcal{M}^{<\nu} = &\text{ unique fully passive structure } P \text{ such that for all premice } N, \\ &N \lhd P \text{ iff } N \lhd M_{\alpha,l} \text{ for all sufficiently large } \langle \alpha, l \rangle < \langle \nu, 0 \rangle. \end{aligned}$$

If s is an $M^{<\nu}$-stack, then for all sufficiently large $\langle \alpha, l \rangle < \langle \nu, 0 \rangle$, s is an $M_{\alpha,l}$-stack. Moreover, $\Omega_{\alpha,l}(s) = \Omega_{\beta,n}(s)$ for $\langle \alpha, l \rangle \leq \langle \beta, n \rangle < \langle \nu, 0 \rangle$. Thus we can define

$$\begin{aligned} \Omega^{<\nu}(s) = &\text{ eventual value of } \Omega_{\alpha,l}(s), \\ &\text{for all sufficiently large } \langle \alpha, l \rangle < \langle \nu, 0 \rangle. \end{aligned}$$

Case 1. $M^{<\nu}$ is branch active.

In this case, we have a unique $M^{<\nu}$-critical $\langle \alpha, l, \mathcal{T} \rangle$, and this triple is not anomalous. Let $b = \Omega^{<\nu}(\langle \alpha, l, \mathcal{T} \rangle)$; then

$$M_{\nu,0} = (M^{<\nu}, \emptyset, B),$$

where $B = \{\eta + \gamma \mid \gamma \in b\}$, and η is the largest admissible level of $M^{<\nu}$.

Case 2. There is an F such that $(M^{<\nu}, F, \emptyset)$ is an lpm, $\mathrm{crit}(F) \geq \kappa$, and F is certifiable, in the sense of Definition 2.1 of [15].

As we remarked, cases 1 and 2 are mutually exclusive. We shall prove

$(\dagger)_{\nu,\mathbf{-1}}$. There is at most one F such that $(M^{<\nu}, F, \emptyset)$ is an lpm, $\mathrm{crit}(F) \geq \kappa$, and F is certifiable, in the sense of Definition 2.1 of [15].

See Corollary 6.5. This is the bicephalus lemma. We are now allowed either to set

$$M_{\nu,0} = (M^{<\nu}, \emptyset, \emptyset),$$

that is, to pass on the opportunity to add F, or to set

$$M_{\nu,0} = (M^{<\nu}, \emptyset, F).$$

In the latter case, we add the same demands of our certificate as we had in section 3, and again choose $F_\nu^{\mathbb{C}}$ to be the unique certificate for F such that

(∗) $F_\nu^{\mathbb{C}}$ is a certificate for F, minimal in the Mitchell order among all certificates for F, and w-least among all Mitchell order minimal certificates for F.

Thus the sequence of all $\vec{F}^{\mathbb{C}}$ of all $F_\nu^{\mathbb{C}}$ is coarsely coherent. By a $\mathbb{C}$-iteration, we mean a $\vec{F}^{\mathbb{C}}$-iteration in the sense explained above.

Case 3 Otherwise.

Then we set

$$M_{\nu,0} = (M^{<\nu}, \emptyset, \emptyset).$$

In any case, $\Omega_{\nu,0}$ is the $\mathbb{C}$-induced strategy for $M_{\nu,0}$.

This finishes the definition of what it is for $\mathbb{C}$ to be a least branch w- construction above κ.

Definition 5.39 *The length* $\mathrm{lh}(\mathbb{C})$ *of a construction* $\mathbb{C}$ *is the lexicographically least* $\langle \nu, k\rangle$ *such that either* $M^{\mathbb{C}}_{\langle \nu,k\rangle}$ *does not exist, or* $M^{\mathbb{C}}_{\nu,k}$ *exists, but* $(\dagger)_{\nu,k}$ *is false. In the latter case, we call* $\mathbb{C}$ pathological.

Remark 5.40 Clearly we must stop $\mathbb{C}$ if $(\dagger)_{\nu,k}$ fails for some $k \geq 0$, that is, some parameter is ill-behaved. It is not clear that we need to stop if $(\dagger)_{\nu,-1}$ fails. We might continue by not adding any extenders to $M^{<\nu}$, or by picking one of the certified extenders and adding it. However, such a bicephalus pathology would cause problems later, in the argument that a certified extender that coheres with $M^{<\mu}$ must satisfy the Jensen initial segment condition. Without this, we can't show the model we construct reaches even a Woodin cardinal, or, in the Γ-Woodin background model case, is universal. At any rate, we shall show in the next section if $\mathsf{IH}_{\kappa,\delta}$ holds, w is a wellorder of V_δ, and $\mathbb{C}$ is a w-construction above κ, then $\mathbb{C}$ is not pathological.

Definition 5.41 *A least branch w-construction above κ is* maximal *iff it never passes on an opportunity to add an extender.*

In addition to the $M_{\nu,k}^{\mathbb{C}}$ and $F_\nu^{\mathbb{C}}$, we also have complete strategies $\Omega_{\nu,k}^{\mathbb{C}}$ for $\langle \nu, k\rangle <$ lh$(\mathbb{C})$. These are induced by the unique $\vec{F}^{\mathbb{C}}$ iteration strategy for V above κ we have assumed exists. That strategy is defined on normal trees, but then has a unique extension to stacks of normal trees that normalizes well. So it induces complete strategies on the $M_{\nu,k}$ by the lifting procedure.

Remark 5.42 Suppose $\mathsf{IH}_{\kappa,\delta}$, w is a wellorder of V_δ, and $\mathbb{C}$ is a maximal, non-pathological w-construction above κ. It is tempting to conclude that each $M_{\nu,k}^{\mathbb{C}}$ is ordinal definable, but in fact this is not at all clear. The problem lies in the use of w to pick background extenders. Although our strategy for V is unique, different choices for the $F_\nu^{\mathbb{C}}$ lead to different ways of lifting trees on $M_{\nu,k}^{\mathbb{C}}$ to V, and hence possibly different candidates for $\Omega_{\nu,k}^{\mathbb{C}}$.

Remark 5.43 Let $M = M_{\nu,k}^{\mathbb{C}}$ and $\Omega = \Omega_{\nu,k}^{\mathbb{C}}$, and suppose $M \models \mathsf{ZFC}$. Then $\Omega\restriction M$ is definable over M, by a definition that is uniform in $\langle \nu, k\rangle$. That is because the restriction of Ω to normal trees in M is given by $\dot{\Sigma}^M$, and that determines its restriction to stacks of normal trees because Ω normalizes well, and that determines its restriction to stacks of weakly normal trees in M because Ω is self-consistent.

5.6 Comparison and the hod pair order

We can adapt Theorem 4.10 to hod pairs.

Definition 5.44 *Let (M, Σ) and (N, Ω) be mouse pairs; then*

(a) (M, Σ) iterates past (N, Ω) *iff there is a normal iteration tree $\mathcal{T}$ by Σ on M with last model Q such that $N \trianglelefteq Q$, and $\Sigma_{\mathcal{T},N} = \Omega$.*

(b) (M, Σ) iterates to (N, Ω) *iff there are $\mathcal{T}$ and Q as in (a), and moreover, $N = Q$, and the branch M-to-Q of $\mathcal{T}$ does not drop.*

(c) (M,Σ) iterates strictly past (N,Ω) *iff it iterates past* (N,Ω)*, but not to* (N,Ω)*.*

(d) (N,Ω) absorbs (M,Σ) *iff for some* $Q \trianglelefteq N$*,* (M,Σ) *iterates to* (Q,Ω_Q)*.*

The normal tree $\mathcal{T}$ above is completely determined by N and Σ; it must come by iterating away the least extender disagreement. We are interested in the case that (M,Σ) and (N,Ω) are strategy coherent and self-consistent. In that case, (M,Σ) iterates past (N,Ω) iff no strategy disagreements show up as we iterate, and no non-empty extenders from N participate in least disagreements, so that N does not move, and N is an initial segment of the final model on the M-side.

The following notation is convenient: let $\mathbb{C}$ be a construction such that $M^{\mathbb{C}}_{\nu,0}$ is extender-active; then

$$(M^{\mathbb{C}}_{\nu,-1},\Omega^{\mathbb{C}}_{\nu,-1}) = (M^{<\nu},\Omega^{<\nu}).$$

Setting $\gamma = \hat{o}(M^{\mathbb{C}}_{\nu,0})$, we can write this $(M^{\mathbb{C}}_{\nu,-1},\Omega^{\mathbb{C}}_{\nu,-1}) = (M^{\mathbb{C}}_{\nu,0}|\langle\gamma,-1\rangle),(\Omega^{\mathbb{C}}_{\nu,0})_{\langle\gamma,-1\rangle}$.

Adapting the proof of Theorem 4.10, we get

Theorem 5.45 *Assume* ZFC *plus* $\mathsf{IH}_{\kappa,\delta}$*, and let let* $\mathbb{C}$ *be a* w*-construction above* κ*, where* w *is a wellorder of* V_δ*. Suppose that* $M^{\mathbb{C}}_{\nu,k}$ *exists. Let* (P,Σ) *be a least branch hod pair, with* P *countable and* Σ *being* $<\delta$*-universally Baire. Suppose that* (P,Σ) *iterates strictly past* $(M^{\mathbb{C}}_{\eta,l},\Omega^{\mathbb{C}}_{\eta,l})$*, for all* $\langle\eta,l\rangle <_{\text{lex}} \langle\nu,k\rangle$*; then* (P,Σ) *iterates past* $(M^{\mathbb{C}}_{\nu,k},\Omega^{\mathbb{C}}_{\nu,k})$*.*

Remark 5.46 It is not possible that (P,Σ) iterates to $(M^{\mathbb{C}}_{\nu,-1},\Omega^{\mathbb{C}}_{\nu,-1})$, for some ν such that $F^{\mathbb{C}}_\nu \neq \emptyset$. For if so, then in $\mathrm{Ult}(V,F^{\mathbb{C}}_\nu)$, (P,Σ) would iterate strictly past $(M^{\mathbb{C}}_{\nu,-1},\Omega^{\mathbb{C}}_{\nu,-1})$, contradiction.

Remark 5.47 It follows by our work realizing resurrection embeddings as branch embeddings that if M iterates to $M^{\mathbb{C}}_{\nu,l+1}$, then it iterates strictly past $M^{\mathbb{C}}_{\nu,l}$. This terminology might be a bit confusing at first, because the iteration tree $\mathcal{T}$ from M to $M^{\mathbb{C}}_{\nu,l+1}$ is an initial segment of the tree $\mathcal{U}$ from M to $M^{\mathbb{C}}_{\nu,l}$. Along the branch of $\mathcal{U}$ from M to $M^{\mathbb{C}}_{\nu,l}$ we dropped once, at $M^{\mathbb{C}}_{\nu,l+1}$, from degree $l+1$ to degree l. That drop meant that M iterates past, but not to, $M_{\nu,l}$. This is the case even if $M^{\mathbb{C}}_{\nu,l} = M^{\mathbb{C}}_{\nu,l+1}$ as an lpm, with only the attached soundness level changing. Then $\mathcal{U}$ would be $\mathcal{T}$, together with one gratuitous drop in degree at the end.

Remark 5.48 We do not know whether there can be more than one $\langle\nu,k\rangle$ such that (P,Σ) iterates to $(M^{\mathbb{C}}_{\nu,k},\Omega^{\mathbb{C}}_{\nu,k})$.

The theorem easily implies theorem 0.5 of the introduction:

Theorem 5.49 *Assume* AD^+*, and let* (P, Σ) *be a least branch hod pair; then* $(^*)(P, \Sigma)$ *holds.*

Proof. Let N^* be a coarse Γ-Woodin model that Suslin-co-Suslin captures Σ, as in the hypothesis of $(^*)(P, \Sigma)$. We can then simply apply 5.45 inside N^*. $\square$

In order to apply $(^*)(P, \Sigma)$, we need to know that there are coarse Γ-Woodin models whose maximal hod-pair construction does not break down before they absorb (P, Σ). The following lemma will help with that.

Lemma 5.50 *Assume* $\mathsf{IH}_{\kappa,\delta}$*, and let let* $\mathbb{C}$ *be a* w*-construction above* κ*, where* w *is a wellorder of* V_δ*. Suppose that* $M^{\mathbb{C}}_{\nu,k}$ *exists. Let* (P, Σ) *be a least branch hod pair, with* P *countable and* Σ *being* $< \delta$*-universally Baire; then for any* ν, k*:*

(a) if (P, Σ) *iterates strictly past all* $(M^{\mathbb{C}}_{\mu,l}, \Omega^{\mathbb{C}}_{\mu,l})$ *such that* $\mu < \nu$*, then* $\mathbb{C}$ *satisfies* $(\dagger)_{\nu,-1}$*, and*

(b) if (P, Σ) *iterates strictly past* $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$*, then* $\mathbb{C}$ *satisfies* $(\dagger)_{\nu,k}$*.*

Proof. For (a), suppose toward contradiction that $F_0 \neq F_1$, and for $i \in \{0, 1\}$, $(M^{<\nu}, F_i, \emptyset)$ is an lpm, $\mathrm{crit}(F_i) \geq \kappa$, and F_i is certifiable, in the sense of Definition 2.1 of [15]. It follows that for $i \in \{0, 1\}$ there is a construction $\mathbb{C}_i$ such that $M^{\mathbb{C}_i}_{\nu,0} = (M^{<\nu}, F_i, \emptyset)$, and for all $\mu < \nu$ and k, $(M^{\mathbb{C}_i}_{\mu,k}, \Omega^{\mathbb{C}_i}_{\mu,k}) = (M^{\mathbb{C}}_{\mu,k}, \Omega^{\mathbb{C}}_{\mu,k})$. It follows from Theorem 5.45 that (P, Σ) iterates past both $(M^{\mathbb{C}_0}_{\nu,0}, \Omega^{\mathbb{C}_0}_{\nu,0})$ and $(M^{\mathbb{C}_1}_{\nu,0}, \Omega^{\mathbb{C}_1}_{\nu,0})$. This is impossible; it has to be the same iteration, but $F_0 \neq F_1$.

For (b), we have a normal tree $\mathcal{T}$ on P by Σ, with last model $N = \mathcal{M}^{\mathcal{T}}_\gamma$, such that either

(i) $M^{\mathbb{C}}_{\nu,k}$ is a proper initial segment of N, or

(ii) $M^{\mathbb{C}}_{\nu,k} = N$, and $[0, \gamma]_T$ drops (in model or degree).

We claim that in either case, $\mathbb{C}$ satisfies $(\dagger)_{\nu,k}$, a contradiction.

Let μ and s be the projectum and standard parameter of $M_{\nu,k}$. (That is, the $k+1$-st.) In case (i), $M_{\nu,k}$ is sound, so (a) and (b) of $(\dagger)_{\nu,k}$ hold trivially.

Suppose we are in case (ii), and let $Q = \mathcal{M}^{\mathcal{T}}_\theta | \langle \hat{o}(Q), k \rangle$ be the last structure we drop to in $[0, \gamma]_T$. So $k(Q) = k$, and Q is sound (i.e. $k+1$ sound), and setting

$$i = \hat{\imath}^{\mathcal{T}}_{\theta,\gamma},$$

we have that $i \colon Q \to N$ is elementary, and

$$\rho(Q) = \rho(N) = \mu \leq \mathrm{crit}(i).$$

Since there was no further dropping, Q and N agree to their common value for μ^+. Also, i maps $p(Q)$ to s, so s is solid. This gives us (a) and (b) of $(\dagger)_{\nu,k}$. $\square$

From this we get

Theorem 5.51 *Assume AD^+, and let (P, Σ) be an lbr hod pair with scope HC. Let (N^*, Ψ, δ^*) be a coarse Γ-Woodin that Suslin-co-Suslin captures Σ, in the sense of theorem 10.1 of [30], and let $\mathbb{C}$ be the maximal least branch construction of N^*; then there is an $\langle \nu, k\rangle$ such that*

(i) $\nu < \delta^*$,

(ii) $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$ exists (that is, the construction has not broken down yet), and

(iii) there is a normal $\mathcal{T}$ such that (P, Σ) iterates via $\mathcal{T}$ to $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$.

Remark 5.52 Clause (iii) of the conclusion can be understood as a truth in N^* about $\Sigma \cap N^*$. But letting $(\Omega^{\mathbb{C}}_{\nu,k})^*$ be the strategy on all stacks in V that is induced by $\mathbb{C}$ and Ψ, (iii) implies that in V, $\Sigma_{\mathcal{T},M_{\nu,k}} = (\Omega^{\mathbb{C}}_{\nu,k})^*$.

Proof. If not, then by applying 5.45 and 5.50 in N^*, we have that $\mathbb{C}$ does not break down at all, and P iterates past $M^{\mathbb{C}}_{\delta^*,0}$ in N^*. This contradicts universality at a Woodin cardinal. $\square$

We can now prove Theorem 0.2 of the introduction. First, some notation for cutpoint initial segments:

Definition 5.53 *For M and N lpms, we write $M \trianglelefteq^{\mathrm{ct}} N$ iff $M \trianglelefteq N$, and whenever E is on the N-sequence and $\mathrm{lh}(E) \geq o(M)$, then $\mathrm{crit}(E) > o(M)$.*

Theorem 5.54 *Assume AD^+, and let (P, Σ) and (Q, Ψ) be lbr hod pairs with scope HC; then there are normal trees $\mathcal{T}$ and $\mathcal{U}$ by Σ and Ψ respectively, with last models R and S respectively, such that either*

(a) $R \trianglelefteq^{\mathrm{ct}} S$, $\Sigma_{\mathcal{T},R} = \Psi_{\mathcal{U},R}$, and the branch P-to-R of $\mathcal{T}$ does not drop, or

(b) $S \trianglelefteq^{\mathrm{ct}} R$, $\Psi_{\mathcal{U},S} = \Sigma_{\mathcal{T},S}$, and the branch Q-to-S of $\mathcal{U}$ does not drop.

Proof. We find Γ-Woodin background universe N^* having universally Baire representations for both strategies. Letting $\mathbb{C}$ be the maximal least branch construction of N^*, we have that there are $\langle \nu, k\rangle$ and $\langle \mu, l\rangle$ such that (P, Σ) normally iterates to $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$, and (Q, Ψ) normally iterates to $(M^{\mathbb{C}}_{\mu,l}, \Omega^{\mathbb{C}}_{\mu,l})$. If say $\langle \nu, k\rangle \leq_{\mathrm{lex}} \langle \mu, l\rangle$, then

$(Q, \Psi))$ normally iterates past $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$, and the latter is a normal, nondropping iterate of (P, Σ). By perhaps using one more extender on the Q-side, we can arrange that $M^{\mathbb{C}}_{\nu,k}$ is a cutpoint of the last model. This yields a successful comparison of type (a). If $\langle \mu, l\rangle \leq_{\text{lex}} \langle \nu, k\rangle$, then we have a successful comparison of type (b).

$\square$

Theorem 5.54 was phrased in the language of mouse pairs in 5.29. We get at once

Corollary 5.55 *Assume* AD^+*, and let* (M, Ω) *be an lbr hod pair with scope HC; then every real in* M *is ordinal definable.*

It is natural to ask whether M satisfies "every real is ordinal definable". Borrowing Lemma 7.3 from the future, we have

Theorem 5.56 *Assume* AD^+*, and let* $M, \Omega)$ *be an lbr hod pair with scope HC. Suppose* $M \models \mathsf{ZFC} +$ *"δ is Woodin". Working in* M*, let* UB *be the collection of* $< \delta$*-universally Baire sets; then*

$$M \models \text{ there is a } (\Sigma^2_1)^{\mathsf{UB}} \text{ wellorder of } \mathbb{R}.$$

Proof.. Working in M, let $N \in C$ iff $N \trianglelefteq M$ and $\rho(N) = \omega$. We claim that N is in C if and only if there is a Ψ such that (N, Ψ) is an lbr hod pair, and Ψ is $< \delta$-universally Baire.

For let $N \in C$. By Lemma 7.3, Ω_N is $< \delta$-universally Baire in M. Clearly, (N, Ω_N) is an lbr hod pair in M.

Conversely, let (N, Ψ) be an lbr hod pair in M such that $\rho(N) = \omega$, and Ψ is $< \delta$ universally Baire in M. Let S be the first initial segment of M that projects to ω and is such that $S \notin N$. We apply Theorem 5.45 in M. Letting $\mathbb{C}$ be the maximal construction below δ in M, neither side can iterate past $M_{\langle \delta, 0\rangle}$ because δ is Woodin. It is easy to see then that there must be a $\langle \nu, k\rangle$ such that both (N, Ψ) and $S, \Omega_S)$ iterate to $M^{\mathbb{C}}_{\langle \nu, k\rangle}$; otherwise we would get $N \in S$ or $S \in N$. This then implies $S = N$, as desired. (It also implies $\Psi = \Omega_S$, by pullback consistency.)

This easily yields the theorem. $\square$

Theorem 5.56 stands in contrast to the situation with pure extender mice, as described for example in [22].

One feature of our comparison process is that we may often use the same extender on both sides. That does not happen in an ordinary comparison of premice by iterating least disagreements. This feature can be awkward. What we gain is that we never encounter strategy disagreements in our comparison process. A comparison process that involves iterating away strategy disagreements as we encounter them (such as the process of [16]) will also often use the same extender on both sides. But such a process (if we knew one in general) might have some advantages. In particular, it might be possible to get by without assuming the existence of a Γ-Woodin background universe, where Σ_0 and Σ_1 are in Γ.

5.7 The existence of cores

As in the case of ordinary premice, we can formulate our solidity and universality results abstractly, in a theorem about least branch premice having sufficiently good iteration strategies.

Theorem 5.57 *(The existence of cores.) Let M be a countable lpm, and let Ψ be a complete iteration strategy for M defined on all countable M-stacks by Σ. Suppose that whenever s is a countable M-stack by Ψ having last model N, then (N, Ψ_s) is a least branch hod pair. Suppose that Ψ is coded by a set of reals that is Suslin and co-Suslin in some $L(\Gamma, \mathbb{R})$, where $L(\Gamma, \mathbb{R}) \models \mathsf{AD}^+$. Let $\rho = \rho(M)$ and $r = p(M)$ be the projectum and standard parameter of M, and let*

$$H = \text{ transitive collapse of } Hull^M(\rho \cup r);$$

then

(1) r is solid, and

(2) $H|(\rho^+)^H = M|(\rho^+)^M$.

Remark 5.58 We don't need the full strength of a model of AD^+ with Ψ in it.

Proof. Let q be the longest solid initial segment of r. Let $r = q \cup s$, where either $s = \emptyset$ or $\min(q) > \max(s)$. Let

$$\alpha_0 = \text{ least } \beta \text{ such that } \mathrm{Th}^M_{k+1}(\beta \cup q) \notin M.$$

Here $k = k(M)$. We may assume $\alpha_0 \in M$, as otherwise $r = \emptyset$ and $\alpha_0 = \rho(M) = o(M)$, in which case the theorem is trivially true. Let

$$K = \text{ transitive collapse of } \mathrm{Hull}^M(\alpha_0 \cup q),$$

and let $\pi \colon K \to M$ be the collapse map. We may assume that $\alpha_0 \in K$, as otherwise $K \lhd M$, so $\mathrm{Th}^M_{k+1}(\alpha_0 \cup q) \in M$.

Claim 0.

(a) If $q = r$, then $\rho = \alpha_0$.

(b) If $q \neq r$, then $\rho < \alpha_0 \leq \max(s)$.

(c) $K \models \alpha_0$ is a cardinal.

Proof. (a) is clear. For (b), let W be the solidity witness for $q \cup \{\max(s)\}$, that is, the transitive collapse of $\mathrm{Hull}^M(\max(s) \cup q)$. We are assuming $W \notin M$. This implies that $\mathrm{Th}^M_{k+1}(\max(s) \cup q) \notin M$. [Proof: Suppose $T = \mathrm{Th}^M_{k+1}(\max(s) \cup q)$ is in M. Note $\max(s)$ is a cardinal of W, and $\max(s) = \mathit{crit}(\pi)$, where $\pi \colon W \to M$ is the uncollapse. So $T \in M|\pi(\alpha)$, and $M|\pi(\alpha) \models \mathsf{KP}$. So $W \in M|\pi(\alpha)$.] Thus $\alpha_0 \leq \max(s)$.

We have $\rho < \alpha_0$ because otherwise $p(M) = q$.

(c) is clear if $\alpha_0 = \rho$. So we may assume $\pi \neq \mathrm{id}$. (c) is clear if $\alpha_0 = \mathrm{crit}(\pi)$, so we may assume $\alpha_0 < \mathrm{crit}(\pi)$. Suppose $f \colon \beta \to \alpha_0$ is a surjection, with $\beta < \alpha_0$ and $f \in K$. Let $\pi(f)$ be $r\Sigma^M_{k+1}$ definable from parameters in $\gamma \cup q$, where $\beta < \gamma < \alpha_0$. Then from $\mathrm{Th}^M_{k+1}(\gamma \cup q)$ one can easily compute $\mathrm{Th}^M_{k+1}(\alpha_0 \cup q)$, so $\mathrm{Th}^M_{k+1}(\gamma \cup q) \notin M$, contrary to the minmality of α_0.

□

We shall show that if $q \neq r$, then $\mathrm{Th}^M_{k+1}(\alpha_0 \cup q) \in M$. This implies $q = r$, so r is solid. We then show that K satisfies conclusion (2). The argument is based on comparing the phalanx (M, K, α_0) with M, as usual.

Let $M = \{e_i \mid i < \omega\}$ be an enumeration of M in which for some n, $r = \langle e_0, ..., e_n \rangle$ (in descending order, so $e_0 = \max(r)$). By Lemma 5.36, we may assume that Ψ has the weak Dodd-Jensen property relative to $\vec{e}$. This involves replacing Ψ by a pullback of one of its tails, but we stay with the same M, and it is the first order theory of M that matters in (1) and (2).

Remark 5.59 Under the additional hypothesis that Ψ has the weak Dodd-Jensen property relative to some $\vec{e}$, we can strengthen the strategy agreement part of (2) to: for $\gamma = (\rho^+)^M$, $\Psi_{\langle \gamma, 0 \rangle} = (\Psi^\pi)_{\langle \gamma, 0 \rangle}$.

In the comparison argument, we iterate both M and (M, K, α_0) into the models of a common background construction. Additional phalanxes (N, L, β) may appear above (M, K, α_0) in its tree.

The background construction is the following. Working in our model of AD^+ having Ψ in it, let (N^*, Σ^*) be a witness to $(*)(M, \Psi)$. That is, we fix an inductive-like pointclass Γ_0 with the scale property such that Ψ is coded by a set of reals in $\Gamma_0 \cap \check{\Gamma_0}$. We then fix a "coarse Γ_0-Woodin" tuple $(N^*, \Sigma^*, \delta^*, \tau)$, as in theorem 10.1

of [30]. So $N^* \models \delta^*$ is Woodin, and Σ^* is an (ω_1, ω_1) iteration strategy for $N^*|\delta^*$, and fixing a universal Γ_0 set U, $i(\tau)^g = U \cap i(N^*)[g]$ for all g on $\mathrm{Col}(\omega, i(\delta^*))$, whenever i is an iteration map by Σ^*. We also have that the restriction of Σ^* to trees that are definable over $N^*|\delta^*$ is in N^*. We can assume that there is an $\vec{F}$ such that

(a) $N^* \models \vec{F}$ is coarsely coherent,

(b) δ^* is Woodin in N^* via extenders from $\vec{F}$, and

(c) $N^* \models$ "I am strongly uniquely $\vec{F}$-iterable for stacks of trees in V_{δ^*}."

Now we work in N^*. Let $\mathbb{C}$ be the $\vec{F}$-maximal least branch hod pair construction done in N^*. That is, we only use background extenders from $\vec{F}$, and we add extenders whenever possible subject to this proviso. The construction lasts until we reach some $\langle \nu, k\rangle < \langle \delta^*, 0\rangle$ such that $(\dagger)_{\nu,k}$ fails, or until we reach $\langle \nu, k\rangle = \langle \delta^*, 0\rangle$. Let $l(\mathbb{C})$ be this $\langle \nu, k\rangle$. We write

$$M_{\eta,l} = M^{\mathbb{C}}_{\eta,l} \text{ and } \Omega_{\eta,l} = \Omega^{\mathbb{C}}_{\eta,l}.$$

Claim 1. Let $k = k(M)$. There is an $\eta < \delta^*$ such that $\langle \eta, k\rangle \leq l(\mathbb{C})$, and (M, Ψ) iterates to $(M_{\eta,k}, \Omega_{\eta,k})$.

Proof.

Suppose first that $l(\mathbb{C}) = \langle \delta^*, 0\rangle$. Suppose also there is no $\langle \eta, k\rangle$ as in the claim. By theorem 5.45, (M, Ψ) iterates strictly past every $(M_{\eta,k}, \Omega_{\eta,k})$ such that $\eta < \delta^*$. It follows that there is a normal tree $\mathcal{T}$ on M by Ψ with last model N such that $M_{\delta^*,0} \trianglelefteq N$. We have that $\mathcal{T} \in N^*$. But δ^* is Woodin via $\vec{F}$, and $\mathbb{C}$ is $\vec{F}$-maximal. Moreover, $\mathbb{C}$ satisfies the uniqueness-of-extenders condition $(\dagger)\nu, -1$ for all $\nu < \delta^*$. So the usual universality argument shows that M cannot iterate past $M_{\delta^*,0}$ via a tree in N^*, a contradiction. Thus (M, Ψ) iterates to some $(M_{\eta,k}, \Omega_{\eta,k})$ with $\eta < \delta^*$.

Suppose next that $l(\mathbb{C}) = \langle \beta, l\rangle$, where $\beta < \delta^*$. So $M^{\mathbb{C}}_{\beta,l}$ exists, but $(\dagger)_{\beta,l}$ fails. It follows then from Theorem 5.45 and Lemma 5.50 that (M, Ψ) iterates to some $(M_{\eta,k}, \Omega_{\eta,k})$ with $\langle \eta, k\rangle \leq \langle \beta, l\rangle$.

This proves Claim 1. $\square$

Let us fix $k_0 = k(M)$, and $\eta_0 < \delta^*$ and $\mathcal{U}$ a normal tree on M with last model M_{η_0,k_0} witnessing Claim 1. For each $\langle \nu, l\rangle \leq_{\text{lex}} \langle \eta_0, k_0\rangle$, let $\mathcal{U}_{\nu,l}$ be the unique normal tree on M witnessing that (M, Ψ) iterates past $(M_{\nu,l}, \Omega_{\nu,l})$.

We now want to compare (M, K, α_0) with the $M_{\nu,l}$ for $\langle \nu, l\rangle \leq_{\text{lex}} \langle \eta_0, k_0\rangle$. For each such $\langle \nu, l\rangle$ we shall define a "psuedo iteration tree" $\mathcal{S}_{\nu,l}$ on (M, K, α_0). We shall have complete strategies attached to the models of $\mathcal{S}_{\nu,l}$, and as before, the key will be that no strategy disagreements with $\Omega_{\nu,l}$ show up, and that $M_{\nu,l}$ does not move.

The rules for forming $\mathcal{S}_{\nu,l}$ will be the usual ones for iterating a phalanx, with the exception that at certain steps we are allowed to move the whole phalanx up. (We don't throw away the phalanxes we had before, we just create a new one.) Whenever we introduce a new phalanx, we continue the construction of $\mathcal{S}$ by looking at the least disagreement between its second model and $M_{\nu,l}$.

Fix ν and l. Let us write $\mathcal{U} = \mathcal{U}_{\nu,l}$. At the same time that we define $\mathcal{S} = \mathcal{S}_{\nu,l}$, we shall copy it to a normal tree $\mathcal{T} = \mathcal{T}_{\nu,l}$ on M that is by Ψ. We allow a bit of padding in $\mathcal{T}$; that is, occasionally $\mathcal{M}_\theta^\mathcal{T} = \mathcal{M}_{\theta+1}^\mathcal{T}$. We shall have copy maps

$$\pi_\theta \colon \mathcal{M}_\theta^\mathcal{S} \to \mathcal{M}_\theta^\mathcal{T}$$

with the usual commutativity and agreement properties. We should write $\pi_\theta^{\nu,l}$ here, but will omit the superscripts when we can. The strategy we attach to $\mathcal{M}_\theta^\mathcal{S}$ is

$$\Sigma_\theta = (\Psi_{\mathcal{T}\restriction(\theta+1)})^{\pi_\theta}.$$

We shall have that $(\mathcal{M}_\theta^\mathcal{S}, \Sigma_\theta)$ is an lbr hod-pair. Finally, we have ordinals $\lambda_\theta^\mathcal{S}$ for each $\theta < \mathrm{lh}(\mathcal{S})$ that measure agreement between the models of $\mathcal{S}$, and tell us which one we should apply the next extender to.

We start with

$$\mathcal{M}_0^\mathcal{S} = M, \mathcal{M}_1^\mathcal{S} = K, \text{ and } \lambda_0^\mathcal{S} = \alpha_0,$$

and

$$\mathcal{M}_0^\mathcal{T} = \mathcal{M}_1^\mathcal{T} = M.$$

We let $\pi_0 =$ identity, and let $\pi_1 \colon K \to M$ be the uncollapse map. Since $\mathrm{crit}(\pi_1) \geq \alpha_0 = \lambda_0^\mathcal{S}$, π_0 and π_1 agree up to the relevant exchange ordinal. We think of 0 and 1 as distinct roots of $\mathcal{S}$. One additional root will be created each time we move a phalanx up, and only then.

As we proceed, we define what it is for a node θ of $\mathcal{S}$ to be *unstable*. We shall have that if θ is unstable, then $0 \leq_S \theta$ and $[0, \theta]_S$ does not drop. We then set

$$\alpha_\theta = \sup i_{0,\theta}^\mathcal{S} ``\alpha_0.$$

The idea is that θ is unstable iff $(\mathcal{M}_\theta^\mathcal{S}, \mathcal{M}_{\theta+1}^\mathcal{S}, \alpha_\theta)$ is a phalanx that we are allowed to move up. If θ is unstable, then $\theta+1$ is stable, and a new root in $\mathcal{S}$, that is, there are no $\xi <_S \theta+1$. These are the only roots, except for 0. Our first unstable node is 0, and 1 is stable.

The padding in $\mathcal{T}$ corresponds exactly to the unstable nodes of $\mathcal{S}$, in that θ is unstable iff $\mathcal{M}_\theta^\mathcal{T} = \mathcal{M}_{\theta+1}^\mathcal{T}$.

We maintain by induction on the construction of $\mathcal{S}$ that the current last model is stable, and conversely, every stable model is the last model at some stage. So really, we are defining $\mathcal{S}^\eta$, which has a stable last model, by induction on η, sometimes adding two models at once, and taking $\mathcal{S} = \bigcup_\eta \mathcal{S}^\eta$. We shall suppress the superscript η, however. All extenders used in $\mathcal{S}$ will be taken from stable nodes. We also maintain that if $\mathcal{M}_\theta^\mathcal{S}$ has been defined, then

Induction hypotheses. If θ is unstable, then

(i) $0 \leq_S \theta$, the branch $[0, \theta]_S$ does not drop in model or degree,

(ii) $\lambda_\theta^\mathcal{S} \leq \alpha_\theta \leq \rho_k(\mathcal{M}_\theta^\mathcal{S})$, where $k = k(M)$,

(iii) every $\tau \leq_S \theta$ is unstable,

(iv) there is a ξ such that $\mathcal{M}_\theta^\mathcal{S} = \mathcal{M}_\xi^\mathcal{U}$,

(v) $\rho(\mathcal{M}_\theta^\mathcal{S}) = \sup i_{0,\theta}^\mathcal{S}``\rho$,

(vi) $\alpha_\theta =$ least β such that $\mathrm{Th}_{k+1}^{\mathcal{M}_\theta^\mathcal{S}}(\beta \cup i_{0,\theta}^\mathcal{S}(q)) \notin \mathcal{M}_\theta^\mathcal{S}$.

Item (ii) explains why $[0, \theta]_S$ does not drop in model or degree, for an extender applied to $\mathcal{M}_\theta^\mathcal{S}$ must have critical point $< \lambda_\theta^\mathcal{S}$. Concerning item (iv), notice

Claim 2. If $0 \leq_S \theta$, and $[0, \theta]_S$ does not drop in model or degree, and $\mathcal{M}_\theta^\mathcal{S} = \mathcal{M}_\xi^\mathcal{U}$, then then $[0, \xi]_U$ does not drop in model or degree; moreover $i_{0,\theta}^\mathcal{S} = i_{0,\xi}^\mathcal{U}$.

Proof. This follows as usual the weak Dodd-Jensen property of Ψ. If for example that $[0, \xi]_U$ drops, then $i_{0,\theta}^\mathcal{S}$ maps M elementarily into a dropping Ψ-iterate of M, contradiction. Similarly, $i_{0,xi}^\mathcal{U}$ must be "to the left of" $i_{0,\theta}^\mathcal{S}$ with respect to $\vec{e}$. But also, $\pi_\theta \circ i_{0,\xi}^\mathcal{U}$ is an elementary map from M to $\mathcal{M}_\theta^\mathcal{T}$, so $i_{0,\theta}^\mathcal{T} = \pi_\theta \circ i_{0,\theta}^\mathcal{S}$ is to its left. So $i_{0,\theta}^\mathcal{S}$ is to the left of $i_{0,\xi}^\mathcal{U}$, so $i_{o,\theta}^\mathcal{S} = i_{0,\xi}^\mathcal{U}$. □

The following notation will be useful. For any node γ of $\mathcal{S}$, let

$$\mathrm{st}(\gamma) = \text{ least stable } \theta \text{ such that } \theta \leq_S \gamma,$$

and

$$\mathrm{rt}(\gamma) = \begin{cases} S\text{-pred}(\mathrm{st}(\gamma)) & \text{if } S\text{-pred}(\mathrm{st}(\gamma)) \text{ exists} \\ \mathrm{st}(\gamma) & \text{otherwise.} \end{cases}$$

Note that if θ is unstable and $\theta + 1 \leq_S \gamma$, then $\mathrm{rt}(\gamma) = \theta + 1$. If θ is the largest unstable ordinal $\leq_S \gamma$, then $\mathrm{rt}(\gamma) = \theta$. Finally, if there are unstable ordinals $\leq_S \gamma$, but no largest one, then $\mathrm{rt}(\gamma) = \sup\{\theta \mid \theta \leq_S \gamma \text{ and } \theta \text{ is unstable }\}$.

The construction of $\mathcal{S}$ can end in one of two ways:

(1) We reach a stable θ such that either

   (a) $M_{\nu,l} \lhd \mathcal{M}^{\mathcal{S}}_{\theta}$ or

   (b) $\mathcal{M}^{\mathcal{S}}_{\theta} \unlhd M_{\nu,l}$, and $[\mathrm{rt}(\theta), \theta]_S$ does not drop in model or degree.

In both cases, the full external strategies will be lined up, by Lemma 5.64 below. Case 1(b) constitutes a successful comparison of (M, K, α_0) with M, which iterated past $M_{\nu,l}$ via $\mathcal{U}$. So in case 1(b), we leave $\mathcal{S}_{\eta,m}$ undefined for all $\langle \eta, m\rangle >_{\mathrm{lex}} \langle \nu, l\rangle$. In case 1(a) our phalanx has iterated strictly past $M_{\nu,l}$, and so we go one to define $\mathcal{S}_{\nu,l+1}$.

There is a second way the construction of $\mathcal{S}$ can end.

(2) We reach a stable θ such that for some ξ, $\mathcal{M}^{\mathcal{S}}_{\theta} = \mathcal{M}^{\mathcal{U}}_{\xi}$, and neither $[rt(\theta), \theta]_S$ nor $[0, \xi]_U$ has dropped in model or degree. Moreover, letting $Q = \mathcal{M}^{\mathcal{S}}_{\theta} | \langle \hat{o}(\mathcal{M}^{\mathcal{S}}_{\theta}), -1\rangle$ be the result of removing the last extender predicate, we have that $Q \unlhd M_{\nu,l}$.

If $\mathcal{M}^{\mathcal{S}}_{\theta}$ is not extender-active, then this is the same as case 1(b) above (and we must have $\langle \nu, l\rangle = \langle \eta_0, k_0\rangle$). But if $\mathcal{M}^{\mathcal{S}}_{\theta}$ is extender-active, it is a new way to end. We think of it as a successful comparison, and leave $\mathcal{S}_{\eta,m}$ undefined for all $\langle \eta, m\rangle >_{\mathrm{lex}} \langle \nu, l\rangle$. Note that in the extender-active case, we have not actually lined up the strategies of $\mathcal{M}^{\mathcal{S}}_{\theta}$ and $\mathcal{M}^{\mathcal{U}}_{\xi}$. We've lined up the part of them that acts on Q, and we've lined up the last extender predicates themselves, but not how the strategies act on trees involving the last extender.

In both case (1) and case (2), the last model of $\mathcal{S}$ is $\mathcal{M}^{\mathcal{S}}_{\theta}$.

Claim 3. Induction hypotheses (i)-(vi) hold for $\theta = 0$ and $\theta = 1$.

Proof.. (i)-(vi) are trivial for $\theta = 0$, and vacuous for $\theta = 1$. □

The rules for extending $\mathcal{S}$ at successor steps are the following. Suppose $\mathcal{M}^{\mathcal{S}}_{\gamma}$ is the current last model, so that γ is stable, and suppose the construction is not required to stop by (1) or (2) above. So we have a least disagreement between $\mathcal{M}^{\mathcal{S}}_{\gamma}$ and $M_{\nu,l}$. Suppose the least disagreement involves only an extender E from the $\mathcal{M}^{\mathcal{S}}_{\gamma}$ sequence. By this we mean: letting $\tau = \mathrm{lh}(E)$,

- $M_{\nu,l} | \langle \tau, 0\rangle = \mathcal{M}^{\mathcal{S}}_{\gamma} | \langle \tau, -1\rangle$, and
- $(\Omega_{\nu,l})_{\langle \tau, 0\rangle} = (\Sigma_{\gamma})_{\langle \tau, -1\rangle}$.

Lemma 5.64 below proves that this is the case. Set

$$\lambda^{\mathcal{S}}_{\gamma} = \lambda_E.$$

Let ξ be least such that $\text{crit}(E) < \lambda_\xi^{\mathcal{S}}$. We declare that $S\text{-pred}(\gamma+1) = \xi$. Let $\langle \beta, n\rangle$ be lex least such that either $\rho(\mathcal{M}_\xi^{\mathcal{S}}|\langle \beta, n\rangle) \leq \text{crit}(E)$, or $\langle \beta, n\rangle = \langle \hat{o}(\mathcal{M}_\xi^{\mathcal{S}}), k(\mathcal{M}_\xi^{\mathcal{S}})\rangle$. We set

$$\mathcal{M}_{\gamma+1}^{\mathcal{S}} = \text{Ult}(\mathcal{M}_\xi^{\mathcal{S}}|\langle \beta, n\rangle, E),$$

and let $\hat{\imath}_{\xi,\gamma+1}^{\mathcal{S}}$ be the canonical embedding. We let

$$\mathcal{M}_{\gamma+1}^{\mathcal{T}} = \text{Ult}(\mathcal{M}_\xi^{\mathcal{T}}|\langle \pi_\xi(\beta), n\rangle, \pi_\gamma(E)),$$

and let $\pi_{\gamma+1}$ be given by the Shift Lemma, as usual. If ξ is stable, or if $\langle \beta, n\rangle <_{\text{lex}} \langle \hat{o}(\mathcal{M}_\xi^{\mathcal{S}}), k(\mathcal{M}_\xi^{\mathcal{S}})\rangle$, then we declare $\gamma+1$ to be stable, and we just go on now to look at least disagreement between $\mathcal{M}_{\gamma+1}^{\mathcal{S}}$ and $M_{\nu,l}$. Nothing unusual has happened.

Induction hypotheses (i)-(vi) concern only unstable nodes, so they are vacuously true at $\theta = \gamma+1$.

Remark 5.60 There is an anomalous case to consider here. It occurs also in the solidity proof for ordinary premice, where Schindler and Zeman found the arguments that take care of it. (See [24].) This case only occurs when $\alpha_0 = \text{lh}(F)$, for some extender F from the M-sequence. Equivalently, for some (all) unstable ξ, $\alpha_\xi = \text{lh}(F)$ for some F from the M-sequence. Then we could have an unstable ξ and a γ such that $S\text{-pred}(\gamma+1) = \xi$, and $\text{crit}(E_\gamma^{\mathcal{S}}) = \lambda(F)$, where F is the last extender of $\mathcal{M}_\xi^{\mathcal{S}}|\alpha_\xi$. Thus $\langle \beta, n\rangle = \langle \alpha_\xi, 0\rangle$, and $\mathcal{M}_{\gamma+1}^{\mathcal{S}} = \text{Ult}(\mathcal{M}_\xi^{\mathcal{S}}|\langle \alpha_\xi, 0\rangle)$ is not an lpm, because F is a missing whole initial segment of $i_{\xi,\gamma+1}^{\mathcal{S}}(F)$. But this is ok. The next disagreement will force us to apply $i_{\xi,\gamma+1}^{\mathcal{S}}(F)$ to $\mathcal{M}_\xi^{\mathcal{S}}$, and that will produce an lpm; moreover, $\lambda(E_\gamma^{\mathcal{S}}) = \lambda(i_{\xi,\gamma+1}^{\mathcal{S}}(F))$, so $\gamma+1$ is now a dead node. One can cope with the fact that $i_{\xi,\gamma+1}^{\mathcal{S}}(F)$ has a missing whole initial segment in the termination arguments; the argument is the same as that of Schindler-Zeman. We shall not give any further details of this anomalous case here.

Now suppose ξ is unstable, and $\langle \beta, n\rangle = \langle \hat{o}(\mathcal{M}_\xi^{\mathcal{S}}), k(\mathcal{M}_\xi^{\mathcal{S}})\rangle$. (Since $\alpha_0 \in M$, this means the anomalous case does not occur.) We look to see whether $\mathcal{M}_{\gamma+1}^{\mathcal{S}}$ is also a model of $\mathcal{U}$. If not, then again we declare $\gamma+1$ to be stable, and go on. Our new last node $\gamma+1$ is stable, so (i)-(vi) are vacuous for $\theta = \gamma+1$.

Finally, if $\mathcal{M}_{\gamma+1}^{\mathcal{S}}$ is also a model of $\mathcal{U}$, then we declare $\gamma+1$ to be unstable, and $\gamma+2$ to be stable. Set

$$\mathcal{M}_{\gamma+2}^{\mathcal{S}} = \text{ transitive collapse of } \text{Hull}^{\mathcal{M}_{\gamma+1}^{\mathcal{S}}}(\alpha_{\gamma+1} \cup i_{0,\gamma+1}^{\mathcal{S}}(q)).$$

Let also $\sigma_{\gamma+1} \colon \mathcal{M}_{\gamma+2}^{\mathcal{S}} \to \mathcal{M}_{\gamma+1}^{\mathcal{S}}$ be the collapse map, and

$$\mathcal{M}_{\gamma+2}^{\mathcal{T}} = \mathcal{M}_{\gamma+1}^{\mathcal{T}}, \text{ and}$$
$$\pi_{\gamma+2} = \pi_{\gamma+1} \circ \sigma_{\gamma+1}.$$

Our new last node is stable. Our induction hypothesis (i) holds for $\theta = \gamma + 1$ because it held for $\theta = \xi$, and because $\lambda_\xi \leq \alpha_\xi$. (iii) is clear. For (ii), we must define $\lambda_{\gamma+1}$. Suppose that there is a least disagreement between $\mathcal{M}^{\mathcal{S}}_{\gamma+2}$ and $M_{\nu,l}$, and lemma 5.64 applies to it, so it involves only some F from the sequence of $\mathcal{M}^{\mathcal{S}}_{\gamma+2}$. If there is no such F, $\mathcal{M}^{\mathcal{S}}_{\gamma+2}$ is the last model of $\mathcal{S}$, and we leave $\lambda^{\mathcal{S}}_{\gamma+1}$ as undefined as $\lambda^{\mathcal{S}}_{\gamma+2}$ is. If F exists, we set

$$\lambda^{\mathcal{S}}_{\gamma+2} = \lambda(F),$$

and

$$\lambda^{\mathcal{S}}_{\gamma+1} = \inf(\lambda^{\mathcal{S}}_{\gamma+2}, \alpha_{\gamma+1}).$$

This insures that (ii) holds at $\theta = \gamma + 1$. It also insures that $\lambda_\gamma < \lambda_{\gamma+1} \leq \lambda_{\gamma+2}$, so that the λ's remain nondecreasing, which is something we want. $\pi_{\gamma+2}$ agrees with $\pi_{\gamma+1}$ on $\lambda^{\mathcal{S}}_{\gamma+1}$, as required. $(\mathcal{M}^{\mathcal{S}}_{\gamma+1}, \mathcal{M}^{\mathcal{S}}_{\gamma+2}, \alpha^{\mathcal{S}}_{\gamma+1})$ is the result of moving up the phalanx.

Remark 5.61 It is possible that $\lambda_{\gamma+1} = \lambda_{\gamma+2}$, and $\mathrm{lh}(F) < \alpha_{\gamma+1}$. Indeed, this will happen a lot. In this case, F will immediately move the phalanx $(\mathcal{M}^{\mathcal{S}}_{\gamma+1}, \mathcal{M}^{\mathcal{S}}_{\gamma+2}, \alpha_{\gamma+1})$ up again. Moreover, since $\lambda_{\gamma+1} = \lambda_{\gamma+2}$, no extender ever gets applied to $\mathcal{M}^{\mathcal{S}}_{\gamma+2}$. It is a "dead node". The phalanx $(\mathcal{M}^{\mathcal{S}}_{\gamma+1}, \mathcal{M}^{\mathcal{S}}_{\gamma+2}, \alpha_{\gamma+1})$ may get moved up repeatedly, along various branches, but that doesn't really involve $\mathcal{M}^{\mathcal{S}}_{\gamma+2}$. After contributing F, it became irrelevant.

Induction hypothesis (iv) is clear. Next we verify (v) and (vi). For this we need

Claim 4. For $a \subset \lambda_E$ finite, $E_a \in \mathcal{M}^{\mathcal{S}}_\xi$.

Proof. Let $\mathcal{M}^{\mathcal{S}}_{\gamma+1} = \mathcal{M}^{\mathcal{U}}_\mu$. By claim 2, $[0, \mu]_U$ does not drop, and $s^{\mathcal{S}}_{\gamma+1} = s^{\mathcal{U}}_\mu$. It follows that E is also used in $\mathcal{U}$. Say $E = E^{\mathcal{U}}_\beta$. Let $\kappa = \mathrm{crit}(E)$. We have

$$\sup_{\tau<\xi} \lambda_\tau \leq \kappa < \lambda_\xi,$$

because we are applying E to $\mathcal{M}^{\mathcal{S}}_\xi$.

Suppose first that E is not the last extender of $\mathcal{M}^{\mathcal{S}}_\gamma$. Then $E_a \in \mathcal{M}^{\mathcal{S}}_\gamma$, and since $\kappa < \lambda^{\mathcal{S}}_\xi \leq \lambda^{\mathcal{S}}_{\xi+1}$, $E_a \subseteq \mathcal{M}^{\mathcal{S}}_{\xi+1} | \lambda^{\mathcal{S}}_{\xi+1}$. Thus by the agreement of models in $\mathcal{S}$, $E_a \in \mathcal{M}^{\mathcal{S}}_{\xi+1}$. If $\alpha_\xi = \mathrm{crit}(\sigma_\xi)$, then α_ξ is a cardinal of $\mathcal{M}^{\mathcal{S}}_{\xi+1}$. If $\mathcal{M}^{\mathcal{S}}_\xi = \mathcal{M}^{\mathcal{S}}_{\xi+1}$, we get $E_a \in \mathcal{M}^{\mathcal{S}}_\xi$, as desired. If not, then $\kappa < \alpha_\xi \leq \mathrm{crit}(\sigma_\xi)$, and $\mathrm{crit}(\sigma_\xi)$ is a cardinal of $\mathcal{M}^{\mathcal{S}}_{\xi+1}$, so $E_a \subseteq \mathcal{M}_{\xi+1} | \mathrm{crit}(\sigma_\xi)$, which yields $E_a \in \mathcal{M}^{\mathcal{S}}_\xi$, as desired.

Suppose next that E is the last extender of $\mathcal{M}^{\mathcal{S}}_\gamma$, and the branch to γ of $\mathcal{S}$ has dropped. Let η be the site of the last drop, i.e. η is least such that $\hat{\imath}^{\mathcal{S}}_{\eta,\gamma}$ maps the full $\mathcal{M}^{\mathcal{S}}_{\eta}$ elementarily to $\mathcal{M}^{\mathcal{S}}_{\gamma}$. Then $\kappa \in \operatorname{ran}(\hat{\imath}^{\mathcal{S}}_{\eta,\gamma})$, and $\gamma \geq (\xi+1)$. This implies $\eta > \xi$.[Proof: $\eta \leq_S \xi$ is impossible since $[0,\xi]_S$ does not drop. So if $\eta < \xi$, and F is the first extender used in $(\eta,\gamma]_S$ such that $\lambda_F > \kappa$, then F is applied to $\mathcal{M}^{\mathcal{S}}_{\tau}$ where $\tau < \xi$. So $\operatorname{crit}(F) < \lambda_\tau \leq \kappa$, and $\kappa \notin \operatorname{ran}(\hat{\imath}^{\mathcal{S}}_{\eta,\gamma}.]$ Thus $\operatorname{crit}(\hat{\imath}^{\mathcal{S}}_{\eta,\gamma}) > \kappa$. Letting $\tau = S\text{-pred}(\eta)$, this easily yields $E_a \in \mathcal{M}^{\mathcal{S}}_{\tau}$. Then we can argue as we did in the preceding paragraph under the hypothesis that $E_a \in \mathcal{M}^{\mathcal{S}}_{\gamma}$, and we get $E_a \in \mathcal{M}^{\mathcal{S}}_{\xi}$ as desired.

Thus we may assume that E is the last extender of $\mathcal{M}^{\mathcal{S}}_{\gamma}$, and the branch of $\mathcal{S}$ to γ (i.e. either $[0,\gamma]_S$ or $[\operatorname{rt}(\gamma),\gamma]_S$) does not drop in model or degree. By a parallel argument, we may assume that E is the last extender of $\mathcal{M}^{\mathcal{U}}_{\beta}$, and the branch $[0,\beta]_U$ does not drop in model or degree. But that means we stop our construction for reason (2), with $\mathcal{M}^{\mathcal{S}}_{\gamma}$ being the last model of $\mathcal{S}$, contrary to our assumption. This proves Claim 4.

□

It is precisely in order to insure Claim 4 that we stop the construction for reason (2).

Claim 5. Items (v) and (vi) of our induction hypotheses hold.

Proof. Let $i = i^{\mathcal{S}}_{\xi,\gamma+1}$, and $k = k_0 = k(M)$. Consider first (vi). For $\beta \leq \alpha_\xi$, let

$$T_\beta = \operatorname{Th}^{\mathcal{M}^{\mathcal{S}}_{\xi}}_{k+1}(\beta \cup i^{\mathcal{S}}_{0,\xi}(q)),$$

and for $\beta \leq \alpha_{\gamma+1}$, let

$$R_\beta = \operatorname{Th}^{\mathcal{M}^{\mathcal{S}}_{\gamma+1}}_{k+1}(\beta \cup i^{\mathcal{S}}_{0,\gamma+1}(q)).$$

If $\beta < \alpha_\xi$, then $T_\beta \in \mathcal{M}^{\mathcal{S}}_{\xi}$, and we can use $i(T_\beta)$ to compute $R_{i(\beta)}$, as usual with solidity witnesses. Since $\alpha_{\gamma+1} = \sup i``\alpha_\xi$, this gives half of (vi). For the other half, assume $R = R_{\alpha_{\gamma+1}}$ is in $\mathcal{M}^{\mathcal{S}}_{\alpha_{\gamma+1}}$, say

$$R = [a,f]^{\mathcal{M}^{\mathcal{S}}_{\xi}}_{E}.$$

Letting $T = T_{\alpha_\xi}$, we then have $\langle\varphi,\mu\rangle \in T$ iff $\langle\varphi, i(\mu)\rangle \in R$ iff for E_a almost every u, $\langle\varphi,\mu\rangle \in f(u)$. Since $E_a \in \mathcal{M}^{\mathcal{S}}_{\xi}$, $T \in \mathcal{M}^{\mathcal{S}}_{\xi}$, a contradiction.

Consider now (v). Let $t = p(\mathcal{M}^{\mathcal{S}}_{\xi})$ and $\sigma = \rho(\mathcal{M}^{\mathcal{S}}_{\xi})$ be the standard parameter and projectum. Let $\tau = \sup i``\sigma$.

Remark 5.62 Our proof shows that $i^{\mathcal{S}}_{0,\xi}(q)$ is an initial segment of t, but it does not show $t = i^{\mathcal{S}}_{0,\xi}(r)$. The standard parameter could move down in its non-solid region.

Let for any $\beta, x \in \mathcal{M}_\xi^{\mathcal{S}}$

$$T_\beta(x) = \mathrm{Th}_{k+1}^{\mathcal{M}_\xi^{\mathcal{S}}}(\beta \cup \{x\}),$$

and for $\beta, x \in \mathcal{M}_{\gamma+1}^{\mathcal{S}}$, let

$$R_\beta(x) = \mathrm{Th}_{k+1}^{\mathcal{M}_{\gamma+1}^{\mathcal{S}}}(\beta \cup \{x\}).$$

If $R_\tau(i(t)) \in \mathcal{M}_{\gamma+1}^{\mathcal{S}}$, say $R_\tau(i(t)) = [a, f]$, then using E_a we can compute $T_\sigma(t)$ inside $\mathcal{M}_\xi^{\mathcal{S}}$, contradiction. Thus $\rho(\mathcal{M}_{\gamma+1}^{\mathcal{S}}) \leq \tau$. On the other hand, let $\kappa \leq \beta < \sigma$ and $x = [a, f]$ in $\mathrm{Ult}(\mathcal{M}_\xi^{\mathcal{S}}, E)$. Then $T_\beta(f) \in \mathcal{M}_\xi^{\mathcal{S}}$, and we can compute $R_{i(\beta)}(x)$ from $i(T_\beta(f))$ in $\mathcal{M}_{\gamma+1}^{\mathcal{S}}$. (First, compute $R_{i(\beta)}(i(f))$. Then note $x = i(f)(a)$, and $a \subset i(\beta)$.) Since $\mathrm{ran}(i)$ is cofinal in τ, we get $\tau \leq \rho(\mathcal{M}_{\gamma+1}^{\mathcal{S}})$.

This proves Claim 5. □

Now let θ be a limit ordinal, and let $b = \Psi(\mathcal{T}\restriction\theta)$ be the branch of $\mathcal{T}$ chosen by Ψ. b may have pairs of the form $\gamma, \gamma+1$ in it where $\mathcal{M}_\gamma^{\mathcal{T}} = \mathcal{M}_{\gamma+1}^{\mathcal{T}}$; this occurs precisely when $\gamma \in b$ is unstable. By construction, the set of such pairs is an initial segment of b that is closed as a set of ordinals.

Suppose first

Case 1. There is a largest $\eta \in b$ such that η is unstable.

Fix this η. There are two subcases.

1(b) for all $\gamma \in b - (\eta+1)$, $\mathrm{rt}(\gamma) = \eta+1$. In this case, $b - (\eta+1)$ is a branch of $\mathcal{S}$. We let $\mathcal{S}$ choose this branch, that is,

$$[\eta+1, \theta)_S = b - (\eta+1),$$

and let $\mathcal{M}_\theta^{\mathcal{S}}$ be the direct limit of the $\mathcal{M}_\gamma^{\mathcal{S}}$ for $\gamma \in b - (\eta+1)$ sufficiently large. The branch embeddings $\hat{\imath}_{\gamma,\theta}^{\mathcal{S}}$, for $\gamma \geq \eta$ in b, are as usual. $\pi_\theta \colon \mathcal{M}_\theta^{\mathcal{S}} \to \mathcal{M}_\theta^{\mathcal{T}}$ is given by the fact that the copy maps commute with the branch embeddings. We declare θ to be stable.

1(b) for all $\gamma \in b - (\eta+1)$, $\mathrm{rt}(\gamma) = \eta$. We let $\mathcal{S}$ choose

$$[0, \theta)_S = (b - \eta) \cup [0, \eta]_S,$$

and let $\mathcal{M}_\theta^{\mathcal{S}}$ be the direct limit of the $\mathcal{M}_\gamma^{\mathcal{S}}$ for $\gamma \in b$ sufficiently large. The branch embeddings $\hat{\imath}_{\gamma,\theta}^{\mathcal{S}}$, for $\gamma \geq \eta$ in b, are as usual. $\pi_\theta \colon \mathcal{M}_\theta^{\mathcal{S}} \to \mathcal{M}_\theta^{\mathcal{T}}$ is given by the fact that the copy maps commute with the branch embeddings. Again, we declare θ to be stable.

In this case, θ is stable, so (i)-(vi) still hold.

Case 2. There are boundedly many unstable ordinals in b, but no largest one. Let η be the sup of the unstable ordinals in b. We let $\mathcal{S}$ choose

$$[0, \theta_S] = (b - \eta) \cup [0, \eta]_S,$$

etc. Again, we declare θ to be stable, and (i)-(vi) still hold.

Case 3. There are arbitrarily large unstable ordinals in b. In this case b is a disjoint union of pairs $\{\gamma, \gamma + 1\}$ such that γ is unstable and $\gamma + 1$ is stable. That is, in $\mathcal{S}$ we have been moving our phalanx up all along b. We set

$$[0, \theta)_S = \{\xi \in b \mid \xi \text{ is unstable }\},$$

and let $\mathcal{M}_\theta^\mathcal{S}$ be the direct limit of the $\mathcal{M}_\xi^\mathcal{S}$ for $\xi \in b$ unstable. There is no dropping of any kind in $[0, \theta)_S$. The branch embeddings $i_{\gamma,\theta}^\mathcal{S}$ and the copy map π_θ are as usual. If $\mathcal{M}_\theta^\mathcal{S}$ is not a model of $\mathcal{U}$, then we declare θ to be stable. Otherwise, we declare θ to be unstable, and set

$$\mathcal{M}_{\theta+1}^\mathcal{S} = \text{ transitive collapse of } \mathrm{Hull}^{\mathcal{M}_\theta^\mathcal{S}}(\lambda_\theta^\mathcal{S} \cup i_{0,\theta}^\mathcal{S}(q)).$$

$\lambda_\theta^\mathcal{S}$ is defined as it was in the unstable successor case: first we define $\lambda_{\theta+1}$, then set

$$\lambda_\theta^\mathcal{S} = \inf(\lambda_{\theta+1}^\mathcal{S}, \alpha_\theta).$$

Let also

$$\sigma_\theta \colon \mathcal{M}_{\theta+1}^\mathcal{S} \to \mathcal{M}_\theta^\mathcal{S}$$

be the collapse map, and

$$\mathcal{M}_{\theta+1}^\mathcal{T} = \mathcal{M}_\theta^\mathcal{T}, \text{ and}$$
$$\pi_{\theta+1} = \pi_\theta \circ \sigma_\theta.$$

$\pi_{\theta+1}$ agrees with π_θ on $\lambda_\theta^\mathcal{S}$, as desired.

(i)-(iv) are clear. Items (v) and (vi) are routine.

We shall use the following proposition in the next section.

Proposition 5.63 *Let θ be a limit ordinal such that θ is stable in $\mathcal{S}_{\nu,l}$, but every $\xi <_{S_{\nu,l}} \theta$ is unstable in $\mathcal{S}_{\nu,l}$; then* $\mathrm{cof}(\theta) = \omega$.

Proof. Let $t = s_\theta^{\mathcal{S}_{\nu,l}}$ be the branch extender of $[0,\theta)_S$, and $\lambda = \mathrm{dom}(t)$. By hypothesis, $t{\restriction}\eta \in \mathcal{U}^{\mathrm{ext}}_{\nu,l}$ for all $\eta < \lambda$, but $t \notin \mathcal{U}^{\mathrm{ext}}_{\nu,l}$. For $\eta < \lambda$, let ξ_η be such that

$$t{\restriction}\eta = s_{\xi_\eta}^{\mathcal{U}_{\nu,l}}.$$

Then $\eta < \gamma$ implies $s^{\mathcal{U}}_{\xi_\eta} \subseteq s^{\mathcal{U}}_{\xi_\gamma}$, and hence $\xi_\eta <_U \xi_\gamma$. Letting $\mu = \sup(\{\xi_\eta \mid \eta < \lambda\})$, and b be the branch of $\mathcal{U}{\restriction}\mu$ determined by the ξ_η's, we have that t is the branch extender of b in $\mathcal{U}$, so $b \neq s^{\mathcal{U}}_\mu$, so $b \neq [0,\mu)_U$. This implies $\mathrm{cof}(\mu) = \omega$, so $\mathrm{cof}(\lambda) = \omega$, so $\mathrm{cof}(\theta) = \omega$, as desired. $\square$

This finishes our construction of the psuedo-tree $\mathcal{S}_{\nu,l}$, and its lift $\mathcal{T}_{\nu,l}$. Notice that every extender used in $\mathcal{S}$ was taken from the sequence of a stable node. Every stable node, except the last model of $\mathcal{S}$, contributes exactly one extender to be used. The last model of $\mathcal{S}$ is stable.

Recall that we assumed that the construction never reached a strategy disagreement between the current model of $\mathcal{S}_{\nu,l}$ and $(M_{\nu,l}, \Omega_{\nu,l})$, and that the extender disagreements involved only empty extenders on the $M_{\nu,l}$ side. Let us record this in a lemma.

Lemma 5.64 *Let $(\mathcal{M}^{\mathcal{S}}_\gamma, \Sigma_\gamma)$ be defined as above, where $\mathcal{S} = \mathcal{S}_{\nu,l}$; then either*

(1) there is a $\langle \tau, n\rangle$ such that $(\mathcal{M}_{\nu,l}|\langle\tau,n\rangle, (\Omega_{\nu,l})_{\langle\tau,n\rangle}) = (\mathcal{M}^{\mathcal{S}}_\gamma, \Sigma_\gamma)$, or

(2) there is a nonempty extender E on the $\mathcal{M}^{\mathcal{S}}_\gamma$ sequence such that, setting $\tau = \mathrm{lh}(E)$,

(i) $\dot{E}^{M_{\nu,l}}_\tau = \emptyset$, and

(ii) $(\Sigma_\gamma)_{\langle\tau,-1\rangle} = (\Omega_{\nu,l})_{\langle\tau,0\rangle}$.

So far as we can see, the lemma can only be proved by going back through the proof of Theorem 4.10, and extending the arguments so that they apply to $\mathcal{S}_{\nu,l}$. That involves generalizing strong hull condensation to psuedo-trees like $\mathcal{S}$, and normalizing well to stacks $\langle\mathcal{S},\mathcal{U}\rangle$, where $\mathcal{U}$ is a normal tree on the last model of $\mathcal{S}$. Then we need to run the construction of 4.10, showing that $W(\mathcal{S},\mathcal{U}^\frown b)$ is a psuedo-hull of $i^*_b(\mathcal{S})$, where b is the branch of $\mathcal{U}$ chosen by $\Omega_{\nu,l}$. There is nothing new in these arguments, but it does not seem possible to get by with quoting our earlier results. We therefore defer the proof of Lemma 5.64 to the next section.

Claim 6. For some $\langle\nu,l\rangle \leq_{\mathrm{lex}} \langle\eta_0,k_0\rangle$, the construction of $\mathcal{S}_{\nu,l}$ stops for either reason 1(b) (that is, $\mathcal{M}^{\mathcal{S}}_\infty \trianglelefteq M_{\nu,l}$), or reason (2).

Proof. If not, then the construction of $\mathcal{S} = \mathcal{S}_{\eta_0,k_0}$ must reach some $\mathcal{M}^{\mathcal{S}}_{\theta}$ such that M_{η_0,k_0} is a proper initial segment of $\mathcal{M}^{\mathcal{S}}_{\theta}$. But M_{η_0,k_0} is a Ψ-iterate of M via a branch of $\mathcal{U}_{\eta_0,k_0}$ that does not drop; let j be the iteration map. We have π_θ from $\mathcal{M}^{\mathcal{S}}_{\theta}$ to the last model of $\mathcal{T}_{\eta_0,k_0}$. Then $\pi_\theta \circ j$ maps M elementarily into a proper initial segment of the last model of $\mathcal{T}_{\eta_0,k_0}$, contrary to the weak Dodd-Jensen property of Ψ. □

The following weaker version of induction hypotheses (v) and (vi) holds more generally.

Claim 7. Let $\mathcal{U} = \mathcal{U}_{\nu,l}$ for some ν, l. Suppose $[0,\eta]_U$ does not drop in model or degree, and let $i = i^{\mathcal{U}}_{0,\eta}$; then

(a) for any $\beta < \alpha_0$, $\mathrm{Th}^{\mathcal{M}^{\mathcal{U}}_{\eta}}_{k_0+1}(i(\beta) \cup i(q)) \in \mathcal{M}^{\mathcal{U}}_{\eta}$,

(b) $\sup i``\rho(M) \leq \rho(\mathcal{M}^{\mathcal{U}}_{\eta}) \leq i(\rho(M))$, and

(c) if $q \neq r$, then $\mathrm{Th}^{\mathcal{M}^{\mathcal{U}}_{\eta}}_{k_0+1}(\rho(\mathcal{M}^{\mathcal{U}}_{\eta}) \cup i(q)) \in \mathcal{M}^{\mathcal{U}}_{\eta}$.

Proof. Part (a) holds because $i(\mathrm{Th}^{M}_{k_0+1}(\beta \cup q))$ can be used to compute $\mathrm{Th}^{\mathcal{M}^{\mathcal{U}}_{\eta}}(i(\beta) \cup i(q))$. Part (b) is proved in Claim 5 of the proof of Theorem 6.2 of [10]. If $q \neq r$, then $\rho < \alpha_0$, and $\rho(\mathcal{M}_\eta) \leq i^{\mathcal{U}}_{0,\eta}(\rho)$, so we get (c) by using (a) with $\beta = \rho$. ‘ □

Let us now fix ν, l as in Claim 6, and let $\mathcal{S} = \mathcal{S}_{\nu,l}$, $\mathcal{U} = \mathcal{U}_{\nu,l}$, and $\mathcal{T} = \mathcal{T}_{\nu,l}$. Let $\mathrm{lh}(\mathcal{S}) = \theta + 1$. We have that $[\mathrm{rt}(\theta), \theta]_S$ does not drop in model or degree. If $0 \leq_S \theta$, this implies that $[0,\theta]_S$ does not drop in model or degree.

Claim 8. For some unstable ξ, $\mathrm{rt}(\theta) = \xi + 1$.

Proof. If not, then $0 \leq_S \theta$, and $[0,\theta]_S$ does not drop. If $\mathcal{S}$ ended for reason 1(b), then $\mathcal{M}^{\mathcal{S}}_{\theta} \trianglelefteq \mathcal{M}^{\mathcal{U}}_{\delta}$ for some δ. But then $\mathcal{M}^{\mathcal{U}}_{\delta} = \mathcal{M}^{\mathcal{S}}_{\theta}$ and $[0,\delta]_U$ does not drop, by weak Dodd-Jensen. If $\mathcal{S}$ ended for reason (2), then again $\mathcal{M}^{\mathcal{U}}_{\delta} = \mathcal{M}^{\mathcal{S}}_{\theta}$ and $[0,\delta]_U$ does not drop.

Standard weak Dodd-Jensen arguments give

$$i^{\mathcal{S}}_{0,\theta} = i^{\mathcal{U}}_{0,\delta}.$$

(This involves copying over to $\mathcal{T}$ in one direction.) But the extenders used in each of these branches can be recovered from the embeddings, using the hull and definability properties. So

$$s^{\mathcal{S}}_{\theta} = s^{\mathcal{U}}_{\delta}.$$

Now let η be least such that η is stable and $\eta \leq_S \theta$. Then $s^{\mathcal{S}}_{\eta} = s^{\mathcal{S}}_{\theta} \restriction \gamma = s^{\mathcal{U}}_{\delta} \restriction \gamma$, for some γ. But there is τ such that $s^{\mathcal{U}}_{\tau} = s^{\mathcal{U}}_{\delta} \restriction \gamma$. Thus $\mathcal{M}^{\mathcal{S}}_{\eta} = \mathcal{M}^{\mathcal{U}}_{\tau}$. If η is a limit ordinal, then by the rules in limit case 3, η was declared unstable, contradiction. If S-pred$(\eta) = \mu$, then μ is unstable, and our rules in the successor case declare η to be unstable. So in any case, we have a contradiction. $\square$

Fix ξ as in Claim 8. Since ξ is unstable, we can fix τ such that $\mathcal{M}_\tau^\mathcal{U} = \mathcal{M}_\xi^\mathcal{S}$. Fix also $\gamma \geq \tau$ such that $M_{\nu,l} \trianglelefteq \mathcal{M}_\gamma^\mathcal{U}$, and hence $\mathcal{M}_\theta^\mathcal{S} \trianglelefteq \mathcal{M}_\gamma^\mathcal{U}$. Set

$$\mu = \rho(\mathcal{M}_{\xi+1}^\mathcal{S}),$$

and

$$t = \sigma_\xi^{-1}(i_{0,\xi}^\mathcal{S}(q)).$$

Claim 9. Either

(i) $\mu = \alpha_\xi$, or

(ii) $\mu < \alpha_\xi \leq \mathrm{crit}(\sigma_\xi)$, and $\mathrm{crit}(\sigma_\xi) = (\mu^+)^{\mathcal{M}_{\xi+1}^\mathcal{S}}$.

Proof. By induction hypothesis (vi), $\mathrm{Th}_{k_0+1}^{\mathcal{M}_{\xi+1}^\mathcal{S}}(\alpha_\xi \cup t) \notin \mathcal{M}_{\xi+1}^\mathcal{S}$, and therefore $\mu \leq \alpha_\xi$.

Suppose $\mu < \alpha_\xi$. We can then find some finite $p \subset \alpha_\xi$ such that $\mathrm{Th}_{k_0+1}^{\mathcal{M}_{\xi+1}^\mathcal{S}}(\mu \cup p \cup t) \notin \mathcal{M}_{\xi+1}^\mathcal{S}$. Since $\max(p) < \alpha_\xi$, we get from (vi) that $R = \mathrm{Th}_{k_0+1}^{\mathcal{M}_\xi^\mathcal{S}}(\mu \cup p \cup i_{0,\xi}^\mathcal{S}(q)) \in \mathcal{M}_\xi^\mathcal{S}$. If $R \in \mathcal{M}_{\xi+1}^\mathcal{S}$, then we have a contradiction, so assume $R \notin \mathcal{M}_{\xi+1}^\mathcal{S}$. Since R is essentially a subset of μ, we get (ii) of Claim 9. $\square$

Claim 10. $\mu = \rho(\mathcal{M}_\theta^\mathcal{S})$.

Proof. This follows easily from the fact that all extenders used in $[\xi + 1, \theta]_S$ are close to the model to which they are applied, and $\mathrm{crit}(i_{\xi+1,\theta}^\mathcal{S}) \geq \alpha_\xi$. $\square$

Claim 11.

(i) $\mathcal{M}_\theta^\mathcal{S} = \mathcal{M}_\gamma^\mathcal{U}$, and $[0, \gamma]_U$ does not drop in model or degree.

(ii) If $\tau \leq \eta < \gamma$, then $\mathrm{lh}(E_\eta^\mathcal{U}) \geq \alpha_\xi$.

Proof. We have by (vi) that

$$\mathrm{Th}_{k_0+1}^{\mathcal{M}_\xi^\mathcal{S}}(\alpha_\xi \cup i_{0,\xi}^\mathcal{S}(q)) \notin \mathcal{M}_\xi^\mathcal{S}.$$

Suppose $\mathcal{M}_\theta^\mathcal{S} \lhd \mathcal{M}_\gamma^\mathcal{U}$. We have that $[\xi+1,\theta]_S$ does drop in model or degree, and $\mathrm{crit}(i^\mathcal{S}_{\xi+1,\theta}) \geq \alpha_\xi$, so we get

$$\mathrm{Th}_{k_0+1}^{\mathcal{M}_\xi^\mathcal{S}}(\alpha_\xi \cup i^\mathcal{S}_{0,\xi}(q)) = \mathrm{Th}_{k_0+1}^{\mathcal{M}_{\xi+1}}(\alpha_\xi \cup t) \in \mathcal{M}_\gamma^\mathcal{U}.$$

Set

$$R = \mathrm{Th}_{k_0+1}^{\mathcal{M}_{\xi+1}}(\alpha_\xi \cup t).$$

Note that if $E^\mathcal{S}_{\xi+1}$ exists (i.e. $\theta \neq \xi+1$), then $\mathrm{lh}(E^\mathcal{S}_{\xi+1}) \geq \alpha_\xi$. This is because otherwise $\lambda_\xi^\mathcal{S} = \lambda^\mathcal{S}_{\xi+1}$, so $\xi+1$ is a dead node of $\mathcal{S}$, and $\xi+1 <_S \theta$ is impossible. So in any case, $\mathcal{M}_\theta^\mathcal{S}$ agrees with $\mathcal{M}_\xi^\mathcal{S}$ below α_ξ. It follows that $\mathcal{M}^\mathcal{U}_\gamma$ agrees with $\mathcal{M}^\mathcal{S}_\xi$ below α_ξ, and hence with $\mathcal{M}^\mathcal{U}_\tau$ below α_ξ. Thus all $E^\mathcal{U}_\mu$ for $\tau \leq \mu < \gamma$ have length $\geq \alpha_\xi$. But R is essentially a subset of α_ξ, and $R \in \mathcal{M}^\mathcal{U}_\gamma$, so $R \in \mathcal{M}^\mathcal{U}_\tau$, contradiction.

Thus $\mathcal{M}^\mathcal{U}_\gamma = \mathcal{M}^\mathcal{S}_\theta$. The argument also proved (ii).

To see that $[0,\gamma]_U$ does not drop, suppose not, and let the last drop in $[0,\gamma]_U$ occur at $\eta+1$. We must have $\eta+1 \leq \tau$, as otherwise $R \in \mathcal{M}^\mathcal{U}_\tau$. But then $\rho(\mathcal{M}^\mathcal{U}_\gamma) \leq \mathrm{crit}(E^\mathcal{U}_\eta) < \lambda(E^\mathcal{U}_\eta) < \alpha_\xi$, which yields $\rho(\mathcal{M}^\mathcal{S}_\theta) = \rho(\mathcal{M}^\mathcal{U}_\gamma) < \mu$, by Claim 9. This contradicts Claim 10. □

Claim 12. $i^\mathcal{S}_{\xi+1,\theta}(t) = i^\mathcal{U}_{0,\gamma}(q)$.

Proof. Let β be the first (i.e. largest) element of q such that $i^\mathcal{U}_{0,\gamma}(\beta) \neq i^\mathcal{S}_{\xi+1,\theta} \circ \sigma_\xi^{-1} \circ i^\mathcal{S}_{0,\xi}(\beta)$. If

$$i^\mathcal{U}_{0,\gamma}(\beta) < i^\mathcal{S}_{\xi+1,\theta} \circ \sigma_\xi^{-1} \circ i^\mathcal{S}_{0,\xi}(\beta),$$

then

$$\pi_\theta \circ i^\mathcal{U}_{0,\gamma}(\beta) < \pi_\theta \circ i^\mathcal{S}_{\xi+1,\theta} \circ \sigma_\xi^{-1} \circ i^\mathcal{S}_{0,\xi}(\beta) = i^\mathcal{T}_{0,\theta}(\beta).$$

The maps on the two sides above agree at all earlier elements of q, and $\vec{e}$ started out with r, so this contradicts the weak Dodd-Jensen property of Ψ relative to $\vec{e}$. On the other hand, suppose

$$i^\mathcal{U}_{0,\gamma}(\beta) > i^\mathcal{S}_{\xi+1,\theta} \circ \sigma_\xi^{-1} \circ i^\mathcal{S}_{0,\xi}(\beta).$$

Let $\bar\beta = \sigma_\xi^{-1} \circ i^\mathcal{S}_{0,\xi}(\beta)$, and $u = t - (\bar\beta+1)$. Since q is solid at β, and $i^\mathcal{S}_{\xi+1,\theta}(u) = i^\mathcal{U}_{0,\gamma}(q-(\beta+1))$, we get that

$$\mathrm{Th}_{k_0+1}^{\mathcal{M}_\theta^\mathcal{S}}(i^\mathcal{S}_{\xi+1,\theta}((\bar\beta+1) \cup i^\mathcal{S}_{0,\xi}(u)) \in \mathcal{M}^\mathcal{S}_\theta.$$

It follows that $\mathrm{Th}_{k_0+1}^{\mathcal{M}_\theta^\mathcal{S}}(\alpha_\xi \cup i^\mathcal{S}_{\xi+1,\theta}(t)) \in \mathcal{M}^\mathcal{S}_\theta$. But the theory is a subset of $\alpha_x i$, and it is equal to $\mathrm{Th}_{k_0+1}^{\mathcal{M}_{\xi+1}^\mathcal{S}}(\alpha_\xi \cup t)$. So $\mathrm{Th}_{k_0+1}^{\mathcal{M}_\xi^\mathcal{S}}(\alpha_\xi \cup i^\mathcal{S}_{0,\xi}(q)) \in \mathcal{M}_\xi$, contradiction. □

Claim 13. Let η be such that $\eta+1 \leq_U \gamma$ and $\eta \geq \tau$; then $\alpha_\xi \leq \mathrm{crit}(E^\mathcal{U}_\eta)$.

Proof. Let $E = E^{\mathcal{U}}_{\eta}$ and $\beta = U\text{-pred}(\eta+1)$. Let $\kappa = \mathrm{crit}(E)$, and suppose $\kappa < \alpha_\xi$. We have $\mathrm{lh}(E) \geq \alpha_\xi$ by Claim 11.

If $\rho(\mathcal{M}^{\mathcal{U}}_{\beta}) \leq \kappa$, then $\rho(\mathcal{M}^{\mathcal{U}}_{\beta}) = \rho(\mathcal{M}^{\mathcal{U}}_{\gamma}) = \mu$, and so we have $\mu < \alpha_\xi$, and thus (ii) of Claim 9 holds, and $(\mu^+)^{\mathcal{M}^{\mathcal{S}}_{\xi}} > \alpha_\xi$. Now if F is used in $[0,\xi)_S$, then $\lambda(F) < \alpha_\xi$, and so $\lambda(F) \leq \mu \leq \kappa$. Thus if $\beta < \tau$, then $\lambda(E^{\mathcal{U}}_{\beta}) \leq \mu \leq \kappa$, contradiction. So $\beta = \tau$. But then $P(\mu)^{\mathcal{M}^{\mathcal{S}}_{\xi}} = P(\mu)^{\mathcal{M}^{\mathcal{U}}_{\tau}} = P(\mu)^{\mathcal{M}^{\mathcal{U}}_{\gamma}} = P(\mu)^{\mathcal{M}^{\mathcal{S}}_{\theta}} = P(\mu)^{\mathcal{M}^{\mathcal{S}}_{\xi+1}}$, which contradicts (ii) of Claim 9.

Thus $\kappa < \rho(\mathcal{M}^{\mathcal{U}}_{\beta})$. But then

$$\alpha_\xi \leq \sup i_E``(\kappa^+)^{\mathcal{M}^{\mathcal{U}}_{\beta}} \leq \rho(\mathcal{M}^{\mathcal{U}}_{\gamma}) = \mu \leq \alpha_\xi,$$

so $\alpha_\xi = \mu = \mathrm{lh}(E)$. If $q \neq r$, then (c) of Claim 7, applied with $\eta = \gamma$, implies that $\mathrm{Th}^{\mathcal{M}^{\mathcal{U}}_{\gamma}}_{k_0+1}(\alpha_\xi \cup i^{\mathcal{U}}_{0,\gamma}(q)) \in \mathcal{M}^{\mathcal{U}}_{\gamma}$. Hence $\mathrm{Th}^{\mathcal{M}^{\mathcal{S}}_{\xi+1}}_{k_0+1}(\alpha_\xi \cup t) \in \mathcal{M}^{\mathcal{S}}_{\xi+1}$, a contradiction. On the other hand, if $q = r$, then $\alpha_\xi = \rho(\mathcal{M}^{\mathcal{S}}_{\xi})$ is a cardinal of $\mathcal{M}^{\mathcal{S}}_{\xi}$, so $\sup i_E``(\kappa^+)^{\mathcal{M}^{\mathcal{U}}_{\beta}} = \mathrm{lh}(E) > \alpha_\xi$, contrary to the inequality displayed above. □

It follows from Claim 13 that $\tau \leq_U \gamma$, and either $\tau = \gamma$ or $\mathrm{crit}(i^{\mathcal{U}}_{\tau,\gamma}) \geq \alpha_\xi$. In either case

$$(\mu^+)^{\mathcal{M}^{\mathcal{S}}_{\xi}} = (\mu^+)^{\mathcal{M}^{\mathcal{U}}_{\tau}} = (\mu^+)^{\mathcal{M}^{\mathcal{U}}_{\gamma}} = (\mu^+)^{\mathcal{M}^{\mathcal{S}}_{\theta}} = (\mu^+)^{\mathcal{M}^{\mathcal{S}}_{\xi+1}},$$

and all models displayed agree to their common value for μ^+. In particular,

$$\mathcal{M}^{\mathcal{S}}_{\xi}|(\mu^+)^{\mathcal{M}^{\mathcal{S}}_{\xi}} = \mathcal{M}^{\mathcal{S}}_{\xi+1}|(\mu^+)^{\mathcal{M}^{\mathcal{S}}_{\xi+1}}.$$

It follows then from Claim 9 that

$$\mu = \alpha_\xi.$$

Claim 14. r is solid; that is, $q = r$.

Proof. If not, then $\rho(M) < \alpha_0$. It follows by Claim 7 that

$$\rho(\mathcal{M}^{\mathcal{U}}_{\tau}) < \sup i^{\mathcal{U}}_{0,\tau}``\alpha_0 = \sup i^{\mathcal{S}}_{0,\xi}``\alpha_0 = \alpha_\xi = \mu = \rho(\mathcal{M}^{\mathcal{S}}_{\theta}) = \rho(\mathcal{M}^{\mathcal{U}}_{\gamma}).$$

However, $\mathrm{crit}(i^{\mathcal{U}}_{\tau,\gamma}) \geq \alpha_\xi$ or $\gamma = \tau$, so $\rho(\mathcal{M}^{\mathcal{U}}_{\tau}) = \rho(\mathcal{M}^{\mathcal{U}}_{\gamma})$. This is a contradiction. □

By Claim 14, $\alpha_0 = \rho$. It follows from (v) and (vi) that for all unstable η, $\alpha_\eta = \rho(\mathcal{M}^{\mathcal{S}}_{\eta})$. Moreover, by the usual preservation of solid parameters, $i^{\mathcal{S}}_{0,\eta}(r)$ is the standard parameter of $\mathcal{M}^{\mathcal{S}}_{\eta}$. In particular, this is true when $\eta = \xi$. That tells us that the parameter of $\mathcal{M}^{\mathcal{S}}_{\xi}$ is universal:

Claim 15. $i^{\mathcal{S}}_{0,\xi}(r)$ is universal over $\mathcal{M}^{\mathcal{S}}_{\xi}$; that is, $\mathcal{M}^{\mathcal{S}}_{\xi}|\eta = \mathcal{M}^{\mathcal{S}}_{\xi+1}|\eta$, where $\eta = (\alpha^+_\xi)^{\mathcal{M}^{\mathcal{S}}_{\xi}}$.

Proof. This follows from the fact that $\mathcal{M}^{\mathcal{S}}_\theta = \mathcal{M}^{\mathcal{U}}_\gamma$, and $\mathrm{crit}(i^{\mathcal{S}}_{0,\theta}) \geq \alpha_\xi$ and $\mathrm{crit}(i^{\mathcal{U}}_{0,\gamma}) \geq \alpha_\xi$ (and neither branch drops). $\square$

If $\xi = 0$, we are done.

Claim 16. r is universal; that is, $K|(\rho^+)^M = M|(\rho^+)^M$.

Proof. Let us assume $k_0 = 0$ and $\hat{o}(M)$ is a limit ordinal to simplify the fine structure a bit. We may also assume $\xi > 0$.

Suppose first that ρ is regular in M. Let $N \lhd M|(\rho^+)^M$, $\rho(N) = \rho$, and $B \subseteq \rho$ code $\mathrm{Th}^N_n(\rho(N) \cup p(N))$ for $n = k(N)$. We must show $N \lhd K$, and that is equivalent to

(*) For some Σ_1 formula φ, some $b < \rho$, and some $\sigma < \hat{o}(M)$, there is a unique $\langle P, C\rangle$ such that:

  (a) $P \lhd M|\sigma$ and $C \subseteq \rho(P)$ codes $\mathrm{Th}^P_n(\rho(P) \cup p(P))$ for $n = k(P)$, and
  (b) $\mathcal{M}|\sigma \models \varphi[P, C, b, r]$.

  Moreover, for the unique such $\langle P, C\rangle$, we have $C \cap \rho = B$.

We can express (*) as
$$M \models \psi[B, \rho, r],$$
where ψ is Σ_1. Let $i = i^{\mathcal{S}}_{0,\xi}$, and note that $i\colon M \to \mathcal{M}^{\mathcal{S}}_\xi$ is elementary, that is, cofinal and Σ_1-elementary. Moreover, $i(\rho) = \sup i``\rho = \alpha_\xi$, because ρ is regular in M. By Claim 15
$$\mathcal{M}^{\mathcal{S}}_\xi \models \psi[i(B), i(\rho), i(r)].$$
Thus $M \models \psi[B, \rho, r]$, as desired.

Now assume that ρ is singular in M. It will then be enough to show that $P(\rho)^M \subseteq K$. This is because if $\pi\colon K \to M$ is the collapse map, then $\mathrm{crit}(\pi) > \rho$, as otherwise $\mathrm{crit}(\pi) = \rho$ is regular in K, and hence regular in M because $P(\rho)^M \subseteq K$. It follows that $\mathrm{crit}(\pi) \geq (\rho^+)^K = (\rho^+)^M$, which yields Claim 16.

So let $B \subseteq \rho$, $B \in M$, and $B \notin K$. We show by induction on $\eta \leq_S \xi$ that $i^{\mathcal{S}}_{0,\eta}(B) \notin \mathcal{M}^{\mathcal{S}}_{\eta+1}$. The case η is a limit ordinal is easy, so assume $S\text{-pred}(\eta) = \beta$, let $E = E^{\mathcal{S}}_{\eta-1}$, and let $A = i^{\mathcal{S}}_{0,\beta}(B) \cap \alpha_\beta$. So $A \notin \mathcal{M}^{\mathcal{S}}_{\beta+1}$. Let us write i_E for $i^{\mathcal{S}}_{\beta,\eta}$, and let $s = i^{\mathcal{S}}_{0,\beta}(r)$. Suppose toward contradiction that $i_E(A) \cap \alpha_\eta \in \mathcal{M}^{\mathcal{S}}_{\eta+1}$; then we have some $b < \alpha_\eta$, some C, and some Σ_1 formula φ such that
$$\mathcal{M}^{\mathcal{S}}_\eta \models C \text{ is the unique } D \text{ such that } \varphi(D, b, i_E(s)),$$

and $C \cap \alpha_\eta = i_E(A) \cap \alpha_\eta$. Fix b, C, and φ. There are cofinally many ordinals in $\mathcal{M}_\beta^\mathcal{S}$ that are Σ_1 definable from parameters in $\alpha_\beta \cup s$, so we can find such an ordinal σ such that

$$\mathcal{M}_\eta^\mathcal{S}|i_E(\sigma) \models \text{ } C \text{ is the unique } D \text{ such that } \varphi(D, b, i_E(s)),$$

But now let

$$b = [a, f]_E^{\mathcal{M}_\beta^\mathcal{S}}.$$

For E_a almost every u,

$$\mathcal{M}_\beta^\mathcal{S}|\sigma \models \text{ there is a unique } D \text{ such that } \varphi(D, f(u), s).$$

Let C_u be the unique such D, when it exists. The function $u \mapsto C_u$ is definable over $\mathcal{M}_\beta^\mathcal{S}|\sigma$ from f and s. Since $\alpha_\eta = \sup i_E``\alpha_\beta$, we may assume that $f \in \mathcal{M}_\beta^\mathcal{S}|\alpha_\beta$. ($\alpha_\beta$ is a singular cardinal of $\mathcal{M}_\beta^\mathcal{S}$ in the present case.) Moreover, $E_a \in \mathcal{M}_\beta^\mathcal{S}|\alpha_\beta$ by Claim 4. Then for $\delta < \alpha_\beta$,

$$\delta \in A \Leftrightarrow \text{ for } E_a \text{ a.e. } u, \delta \in C_u.$$

This defines A over $\mathcal{M}_\beta^\mathcal{S}|\sigma$ from f, s, and E_a. That implies $A \in \mathcal{M}_{\beta+1}^\mathcal{S}$, a contradiction. □

This completes the proof of Theorem 5.57, modulo Lemma 5.64. □

Corollary 5.65 *Assume $\mathsf{IH}_{\kappa,\delta}$, and there are infinitely many Woodin cardinals below κ. Let w be a wellorder of V_δ, and let $\mathbb{C}$ be a w-construction above κ; then for any $\langle \nu, k\rangle < \mathrm{lh}(\mathbb{C})$, $(\dagger)_{\nu,k}$ holds, that is, the standard parameter of $M_{\nu,k}^\mathbb{C}$ is solid and universal.*

Proof. If not, we have a countable M and $\pi \colon M \to M_{\nu,k}^\mathbb{C}$ elementary such that the standard parameter of M is either non-solid or non-universal. We have that (M, Ω^π) is a least branch hod pair by 5.21. Standard arguments using unique iterability show that Ω^π is $< \kappa$-homogeneously Suslin. Because we have assumed that there are infinitely many Woodin cardinals below κ, $L(\Omega^\pi, \mathbb{R}) \models \mathsf{AD}^+$. Thus the hypotheses of 5.57 are satisfied, and the standard parameter of M is solid and universal, a contradiction. □

Remark 5.66 The argument above really only needs one Ω^π-Woodin cardinal.

We can prove a condensation lemma for lbr hod pairs by the same method. Rather than attempt a general statement, we shall content ourselves with the following simple one, since it is what we need in the next section. The author and Nam Trang have proved a stronger condensation theorem in [37], and used it to generalize the Schimmerling-Zeman characterization of $\{\kappa \mid M \models \Box_\kappa\}$ to the case that M is a least branch hod mouse.

Theorem 5.67 *(***Condensation lemma***) Let M be a countable lpm, and let Ψ be a complete iteration strategy for M defined on all countable M-stacks by Σ. Suppose that whenever s is a countable M-stack by Ψ having last model N, then (N, Ψ_s) is a least branch hod pair. Suppose that Ψ is coded by a set of reals that is Suslin and co-Suslin in some $L(\Gamma, \mathbb{R})$, where $L(\Gamma, \mathbb{R}) \models \mathsf{AD}^+$. Let*

$$\pi \colon H \to M$$

be elementary, with $\mathrm{crit}(\pi) = \rho(H) < \rho(M)$, and H being $k(H)+1$-sound. Suppose also that $\rho(H)$ is a limit cardinal of H; then $H \trianglelefteq M$.

Proof.(Sketch.) We proceed as in the proof of 5.57. Let $\mathbb{C}$ be the construction of some Ψ-Woodin model N^*. We have $\langle \eta_0, k_0 \rangle$ such that (M, Ψ) iterates to $(M^{\mathbb{C}}_{\eta_0,k_0}, \Omega^{\mathbb{C}}_{\eta_0,k_0})$. We may assume that Ψ has the weak Dodd-Jensen property relative to some $\vec{e}$.

For $\langle \nu, l \rangle \leq_{\text{lex}} \langle \eta_0, k_0 \rangle$ we define a psuedo iteration tree $\mathcal{S}_{\nu,l}$ which iterates the phalanx $(M, H, \rho(H))$. $\mathcal{S}_{\nu,l}$ is defined exactly as it was in the proof of 5.57, with one exception with regard to how we move phalanxes up. Note that because $\rho(H) < \rho(M)$, we have $H \in M$.(The theory coding H is a bounded $r\Sigma^M_{k(M)+1}$ subset of $\rho(M)$, hence in M. Since $M|\rho(M) \models \mathsf{KP}$, $H \in M|\rho(M)$.) Now suppose $\gamma+1$ is unstable, and $\xi = S\text{-pred}(\gamma+1)$. We have $\mathcal{M}^{\mathcal{S}}_{\gamma+1} = \mathrm{Ult}(\mathcal{M}^{\mathcal{S}}_\xi, E_\gamma)$ as before. We then set

$$\mathcal{M}^{\mathcal{S}}_{\gamma+2} = i^{\mathcal{S}}_{0,\gamma+1}(H),$$

and

$$\alpha^{\mathcal{S}}_{\gamma+1} = i^{\mathcal{S}}_{0,\gamma+1}(\rho(H)).$$

We have

$$\sigma_{\gamma+1} \colon M^{\mathcal{S}}_{\gamma+2} \to \mathcal{M}^{\mathcal{S}}_{\gamma+1}$$

determined by: $\sigma_{\gamma+1} \restriction \alpha^{\mathcal{S}}_{\gamma+1}$ is the identity, and $\sigma_{\gamma+1}(i^{\mathcal{S}}_{0,\gamma+1}(p(H)) = i^{\mathcal{S}}_{0,\gamma+1}(\pi(p(H))$. If H is not an initial segment of M, then $\mathcal{M}^{\mathcal{S}}_{\gamma+2}$ is not an initial segment of $\mathcal{M}^{\mathcal{S}}_{\gamma+1}$, so we have successfully moved the bad situation up.

There is a similar change at unstable limit ordinals θ. We set $\mathcal{M}^{\mathcal{S}}_{\theta+1} = i^{\mathcal{S}}_{0,\theta}(H)$ and $\alpha^{\mathcal{S}}_\theta = i^{\mathcal{S}}_{0,\theta}(\rho(H))$, etc.

The rest of the construction of $\mathcal{S}_{\nu,l}$, and its conditions for termination, are the same as in the proof of 5.57. Again, the key lemma is the counterpart of Lemma 5.64, according to which no strategy disagreements show up, and least extender disagreements involve only empty extenders on the $M^{\mathbb{C}}_{\nu,l}$ side. We shall prove this lemma in the next section.

We argue as before that for some ν, l, the construction of $\mathcal{S}_{\nu,l}$ terminates at a stable θ such that $\mathcal{M}^{\mathcal{S}}_{\theta} \trianglelefteq \mathcal{M}^{\mathcal{U}}_{\gamma}$, where $\mathcal{U} = \mathcal{U}_{\nu,l}$. (We no longer have $\mathcal{M}^{\mathcal{U}}_{\gamma} \trianglelefteq \mathcal{M}^{\mathcal{S}}_{\theta}$, as the proof of that used that $K \notin M$, whereas $H \in M$.) Using weak Dodd-Jensen, We get that for some unstable ξ, $\mathrm{rt}(\theta) = \xi + 1$.

Let $\mathcal{M}^{\mathcal{U}}_{\tau} = \mathcal{M}^{\mathcal{S}}_{\xi}$. We have that $\mathrm{lh}(E^{\mathcal{U}}_{\tau}) \geq \lambda^{\mathcal{S}}_{\xi+1}$, as otherwise $\xi + 1$ would have been dead. But in the present case, $\lambda^{\mathcal{S}}_{\xi+1}$ is a limit cardinal of $\mathcal{M}^{\mathcal{S}}_{\xi} = \mathcal{M}^{\mathcal{U}}_{\tau}$, so $\mathrm{lh}(E^{\mathcal{U}}_{\tau}) > \lambda^{\mathcal{S}}_{\xi+1}$.

Now we simply follow the proofs of Claims 1-4 in the proof of Theorem 8.2 of [10]. We get from that that $\mathcal{M}^{\mathcal{S}}_{\xi+1}$ is a proper initial segment of $\mathcal{M}^{\mathcal{U}}_{\gamma}$. This implies there are no cardinals of $\mathcal{M}^{\mathcal{U}}_{\gamma}$ strictly between $\lambda^{\mathcal{S}}_{\xi+1}$ and $o(\mathcal{M}^{\mathcal{S}}_{\xi+1})$. It follows that $\mathrm{lh}(E^{\mathcal{U}}_{\tau}) \geq o(\mathcal{M}^{\mathcal{S}}_{\xi+1})$, so that $\mathcal{M}^{\mathcal{S}}_{\xi+1} \trianglelefteq \mathcal{M}^{\mathcal{U}}_{\tau} = \mathcal{M}^{\mathcal{S}}_{\xi}$. But then, as we observed above, $H \trianglelefteq M$, as desired.

$\square$

We get at once

Corollary 5.68 *Assume $\mathsf{IH}_{\kappa,\delta}$, and there are infinitely many Woodin cardinals below κ. Let w be a wellorder of V_δ, let $\mathbb{C}$ be a w-construction above κ, and let $M = M^{\mathbb{C}}_{\nu,k}$. Let*

$$\pi \colon H \to M$$

be elementary, with $\mathrm{crit}(\pi) = \rho(H) < \rho(M)$, and H being $k(H)+1$-sound. Suppose also that $\rho(H)$ is a limit cardinal of H; then $H \trianglelefteq M$.

5.8 Proofs of theorems 0.13 and 0.14

We can easily complete the proofs of these theorems modulo Lemma 5.64. Theorem 0.13, slightly extended, is

Theorem 5.69 *Assume AD^+, let Γ be an inductive-like pointclass with the scale property, and such that all sets in $\check{\Gamma}$ are Suslin. Let (N^*, Ψ) be a coarse Γ-Woodin together with its unique Γ-fullness preserving strategy. (cf. 10.1 of [16]) Let $(M, \Omega^*) = (M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$ be a level of least branch hod pair construction $\mathbb{C}$ done in N^* below δ^{N^*}, and let Ω be the canonical extension of Ω^* to all M-stacks in HC; then*

1. *(M, Ω) is a least branch hod pair, with scope HC,*

2. $(*)(M, \Omega)$, *and*

3. *M has a core; that is, $p(M)$ is solid and universal.*

Proof. Let $\delta = \delta^{N^*}$ be the Woodin of N^*. The iterability hypothesis $\mathsf{IH}_{\omega,\delta}$ holds in N^*. Working in N^*, we get that

$$N^* \models (M, \Omega^*)\text{is an lbr hod-pair with scope } V_\delta.$$

The canonical extension Ω of Ω^* is just the strategy for M induced by lifting to N^* and using Ψ there. Ψ acts on all stacks of trees in HC, not just those in N^*, and we don't need that the stack is in N^* to define its lift to N^*.

Now let η be an inaccessible cardinal of N^* such that $\nu < \eta < \delta$, where $M = M^{\mathbb{C}}_{\nu,k}$. Let Φ be the iteration strategy for $N^*|\eta$ induced by It will be enough to show that Φ normalizes well and has strong hull condensation, not just in N^*, but in V, for then Ω inherits these properties. In V, Φ does not pick unique wellfounded branches, but rather unique branches b of $\mathcal{T}$ such that $C_\Gamma(\mathcal{M}(\mathcal{T})) \subseteq \mathcal{M}_b(\mathcal{T})$, and there is a Q-structure for $\mathcal{M}(\mathcal{T})$ in $C_\Gamma(\mathcal{M}(\mathcal{T}))$. This is still enough to prove that Φ normalizes well and has strong hull condensation, however, essentially because the existence of C_Γ Q-structures passes to hulls that have a tree for a universal Γ set in them. We omit further detail. This proves (1).

Item (2) follows at once from our comparison theorem 5.45. Item (3) follows from Theorem 5.57.

□

So least branch constructions done in a coarse Γ Woodin model do not break down. What is missing is a proof that such constructions go far enough; that is, a proof of HPC. We do get

Theorem 5.70 *Assume* AD^+*; then* LEC *implies* HPC.

Proof. It is enough to show that whenever (P, Σ) is a pure extender mouse pair with scope HC, then there is an lbr hod pair (Q, Ψ) with scope HC such that Σ is definable from parameters over $(\mathrm{HC}, \in, \Psi)$.

So fix (P, Σ), and let Γ be an inductive-like pointclass with the scale property such that Σ is coded in its Δ. Let (N^*, Φ) be a coarse Γ-Woodin together with its unique Γ-fullness preserving strategy, and such that $P \in \mathrm{HC}^{N^*}$. Let $\mathbb{C}$ be the least branch hod pair construction of N^* (relative to its canonical wellorder), and let

$$(Q, \Psi) = (M^{\mathbb{C}}_{\delta,0}, \Omega^{\mathbb{C}}_{\delta,0}),$$

where δ is the Woodin of N^*. Since Φ has scope all of HC, it induces an extension of Ψ with scope HC. We call this extension Ψ as well.

Now let $\mathbb{D}$ be the pure extender $L[E]$ construction of Q, where nice extenders from the Q-sequence are used as backgrounds. The construction never breaks down, and each $(M^{\mathbb{D}}_{\nu,k}, \Omega^{\mathbb{D}}_{\nu,k})$ is a pure extender pair in Q, and hence can be canonically to such a pair in N^*. Working in N^*, we can compare (P, Σ) with each $(M^{\mathbb{D}}_{\nu,k}, \Omega^{\mathbb{D}}_{\nu,k})$. Because the background extenders of $\mathbb{D}$ are assigned background extenders over N^* by $\mathbb{C}$, we can repeat the proof of $(*)(P, \Sigma)$, so (P, Σ) iterates past $(M^{\mathbb{D}}_{\nu,k}, \Omega^{\mathbb{D}}_{\nu,k})$, provided it iterates strictly past all earlier levels of $\mathbb{D}$.

By the Q-filtered backgrounding again, (P, Σ) cannot iterate past $(M^{\mathbb{D}}_{\delta,0}, \Omega^{bbD}_{\delta,0})$. It follows that (P, Σ) iterates to some $(M^{\mathbb{D}}_{\nu,k}, \Omega^{\mathbb{D}}_{\nu,k})$. This is true in N^*, but it is also true in V of (P, Σ) and the canonical extension (M, Ω) of $(M^{\mathbb{D}}_{\nu,k}, \Omega^{\mathbb{D}}_{\nu,k})$, because N^* is sufficiently correct. But then Σ is projective in Ω, and Ω is projective in Ψ, so we are done. $\square$

Remark 5.71 We do not see how to show that under AD^+, HPC implies LEC. That, together with 5.70, suggests that one should try to prove HPC by proving the ostensibly stronger LEC.

Theorem 0.14 is

Theorem 5.72 *Suppose V is normally iterable above μ by the strategy of choosing unique cofinal wellfounded branches. Suppose that there is a $j\colon V \to N$ such that for $\kappa = \operatorname{crit}(j)$, $\kappa > \mu$, $V_{j(\kappa)} \subseteq N$, and $j(\kappa)$ is inaccessible; then there is a canonical inner model M such that $M \models$ "There is a superstrong cardinal", and $M \models$ "I am iterable".*

Proof. Let $\delta = j(\kappa)$, let w be a wellorder of V_δ, and let $\mathbb{C}$ be a w-construction above μ that is maximal. Taking $w = j(w_0)$ where w_0 is a wellorder of V_κ, we may assume that $j(w) \cap V_\delta = w$. By 5.57, the construction never breaks down, so $M^{\mathbb{C}}_{\lambda,0}$ exists. We take $M = M^{\mathbb{C}}_{\lambda,0}$.

We must show that $M \models$"there is a superstrong cardinal". Let

$$E = \{(a, X) \mid a \in [\delta]^{<\omega} \wedge X \in P([\kappa]^{|a|})^M \wedge a \in j(X)\}$$

be the length δ extender of j, restricted to M.

Claim. If $\eta < \delta$ and $E{\restriction}\eta$ is whole, then the trivial completion of $E{\restriction}\eta$ is on the M-sequence.

Proof. We prove this by induction on η. Suppose we know it for $\beta < \eta$, and let F be the trivial completion of $E\restriction\eta$, and $\gamma = i_F^M(\kappa^{+,M})$. We have that $\text{Ult}(M, F) = \text{Ult}(M, E\restriction\eta)$, and there is a natural factor embedding

$$\sigma\colon \text{Ult}(M, F) \to \text{Ult}(M, E)$$

such that $\sigma\restriction\eta = \text{id}$, and $\sigma(\eta) = \delta$. Since η is a limit cardinal of $\text{Ult}(M, F)$, we have that η is a limit cardinal of M. Using the Condensation lemma 5.68 applied to σ, we get that

$$\text{Ult}(M, F)|\langle\gamma, -1\rangle = \text{Ult}(M, E)\restriction\langle\gamma, -1\rangle = M\restriction\langle\gamma, -1\rangle.$$

Since η is a cardinal of M, there must be a stage of $\mathbb{C}$ at which we have $M|\langle\eta, 0\rangle = M_{\nu,0}^{\mathbb{C}}$. After this stage, no projectum drops strictly below η, and stages which project to η are initial segments of M. Thus there is a ν such that

$$(M^{<\nu})^{\mathbb{C}} = M|\langle\gamma, -1\rangle.$$

But then $(M^{<\nu}, F, \emptyset)$ is an lpm. (Coherence we verified above, and the Jensen initial segment condition holds by our induction hypothesis.) Moreover, F has a background certificate that shifts w to itself, namely $E_j\restriction\mu$, for μ the least inaccessible cardinal strictly greater than η. By our bicephalus lemma, proved in the next section,

$$M_{\nu,0}^{\mathbb{C}} = (M^{<\nu}, F, \emptyset).$$

Since η is a cardinal of M and $M_{\nu,0}^{\mathbb{C}}$ projects to η, $M_{\nu,0}^{\mathbb{C}} \lhd M$. Thus $F$is on the M-sequence. $\square$

Since δ is inaccessible in V, there are arbitrarily large $\eta < \delta$ such that $E\restriction\eta$ is whole. Any such η is a cardinal of M, and hence for any such η, $E\restriction\eta$ witnesses that κ is superstrong in M. $\square$

6 Phalanx iteration into a backgrounded construction

In this section we prove that there are no nontrivial iterable bicephali, and we prove Lemma 5.64, thereby completing the proofs of theorems 5.69 and 5.72. Both results involve showing that certain bicephali and phalanxes iterate into background constructions in the same way that ordinary lbr hod pairs do.

We shall also use such a phalanx-comparison argument to show that if (M, Ω) is an lbr hod-pair such that $M \models \mathsf{ZFC}$ + "there are arbitrarily large Woodin cardinals", then whenever g is $\mathbb{P}$-generic over M, $M[g] \models$ " UBH holds for all nice, normal iteration trees that use extenders from $\dot{E}^M$ with critical points strictly above $|\mathbb{P}|^M$". That implies that Ω determines itself on generic extensions of M. We shall use this in the next section to show that if λ is a limit of cutpoint Woodin cardinals in M, and N is a derived model of M below λ, then HOD^N is an Ω-iterate of M.

6.1 The Bicephalus Lemma

Definition 6.1 *An* lpm-bicephalus *is a structure* $\mathcal{B} = (B, \in, \dot{E}^{\mathcal{B}}, \dot{\Sigma}^{\mathcal{B}}, F, G)$ *such that both* $(B, \in, \dot{E}^{\mathcal{B}}, \dot{\Sigma}^{\mathcal{B}}, F, \emptyset)$ *and* $(B, \in, \dot{E}^{\mathcal{B}}, \dot{\Sigma}^{\mathcal{B}}, G, \emptyset)$ *are extender-active least branch premice. We say that* $\mathcal{B}$ *is* nontrivial *iff* $F \neq G$.

We shall usually drop "lpm" from "lpm-bicephalus".

We think of $\mathcal{B}$ as a structure in the language with $\in$ and predicate symbols $\dot{\Sigma}, \dot{E}, \dot{F}$, and $\dot{G}$. We let

$$\mathcal{B}^- = (B, \in, \dot{E}^{\mathcal{B}}, \dot{\Sigma}^{\mathcal{B}}, \emptyset, \emptyset)$$

be the lpm obtained by removing both top extenders. (To be pedantic, $\mathcal{B}$ and $\mathcal{B}^-$ have different languages.) The degree of $\mathcal{B}$ is zero, i.e. $k(\mathcal{B}) = 0$. For $\nu < o(\mathcal{B}) = \hat{o}(\mathcal{B})$, we set $\mathcal{B}|\langle \nu, l\rangle = \mathcal{B}^-|\langle \nu, l\rangle$. The extender sequence of $\mathcal{B}$ is $\dot{E}^{\mathcal{B}}$ together with $\dot{F}^{\mathcal{B}}$ and $\dot{G}^{\mathcal{B}}$; it's not actually a sequence.

A $\mathcal{B}$*-tree* is a tuple $\langle \nu, k, \mathcal{T}\rangle$ such that $\langle \nu, k\rangle \leq_{\text{lex}} \langle \hat{o}(\mathcal{B}), 0\rangle$, and $\mathcal{T}$ is a weakly normal tree on $\mathcal{B}|\langle \nu, k\rangle$. That is, $\mathcal{M}_0^{\mathcal{T}} = \mathcal{B}|\langle \nu, k\rangle$, the extenders used in $\mathcal{T}$ are length-increasing and nonoverlapping along branches, and $E_\alpha^{\mathcal{T}}$ must come from the sequence of $\mathcal{M}_\alpha^{\mathcal{T}}$. If $\mathcal{M}_\alpha^{\mathcal{T}}$ is a bicephalus, this means that the extenders from $\dot{E}^{\mathcal{M}_\alpha}$ together with $\dot{F}^{\mathcal{M}_\alpha}$ and $\dot{G}^{\mathcal{M}_\alpha}$ are eligible. A $\mathcal{B}$*-stack* is a sequence $\langle (\nu_i, k_{i,i}) \mid i \leq n\rangle$ such that $\langle \nu_0, k_{0,0}\rangle$ is a $\mathcal{B}$-tree, and $\langle \nu_{i+1}, k_{i+1}, \mathcal{T}_{i+1}\rangle$ is a $\mathcal{M}_\infty(\mathcal{T}_i)$-tree. A *complete strategy* for $\mathcal{B}$ is a strategy Ω defined on all $\mathcal{B}$-stacks s by Ω such that $s \in N$, for some set N. N is called the scope of Ω.

Definition 6.2 *A* bicephalus pair *is a pair* $(\mathcal{B}, \Omega)$ *such that* $\mathcal{B}$ *is an lpm-bicephalus, and* Ω *is a complete strategy for* $\mathcal{B}$.

Tail strategies are given by $\Omega_s(t) = \Omega(s^\frown t)$. We use $\Omega_{s,N}$ and Ω_N as before. We write Ω^- for $\Omega_{\mathcal{B}^-}$, the complete strategy for $\mathcal{B}^-$ induced by Ω.

We can define the notions of normalizing well, having strong hull condensation, being self-consistent, and being self-aware for bicephalus pairs just as we did before.

The main theorem about bicephali is that there aren't any interesting ones.

Theorem 6.3 *Let* $(\mathcal{B}, \Psi)$ *be a bicephalus pair, where* Ψ *has scope HC. Suppose that* $L(\Psi, \mathbb{R}) \models \mathsf{AD}^+$. *Suppose also that* Ψ *normalizes well and has strong hull condensation, and that* $(\mathcal{B}, \Psi)$ *is self-consistent and self-aware; then* $\dot{F}^{\mathcal{B}} = \dot{G}^{\mathcal{B}}$.

Proof. Let us assume toward contradiction that $\dot{F}^{\mathcal{B}} \neq \dot{G}^{\mathcal{B}}$.

We work in $L(\Psi, \mathbb{R})$. Fix an inductive-like pointclass Γ_0 with the scale property such that Ψ is coded by a set of reals in $\Gamma_0 \cap \check{\Gamma}_0$. We then fix a "coarse Γ_0-Woodin" tuple $(N^*, \Sigma^*, \delta^*, \tau)$, as in theorem 10.1 of [30]. So $N^* \models \delta^*$ is Woodin, and Σ^* is an (ω_1, ω_1) iteration strategy for $N^*|\delta^*$, and fixing a universal Γ_0 set U, $i(\tau)^g = U \cap i(N^*)[g]$ for all g on $\mathrm{Col}(\omega, i(\delta^*))$, whenever i is an iteration map by Σ^*. We also have that the restriction of Σ^* to trees that are definable over $N^*|\delta^*$ is in N^*. We can assume that there is an $\vec{F}$ such that

(a) $N^* \models \vec{F}$ is coarsely coherent,

(b) δ^* is Woodin in N^* via extenders from $\vec{F}$, and

(c) $N^* \models$ "I am strongly uniquely $\vec{F}$-iterable for stacks of trees in V_{δ^*}."

Working now in N^*, let $\mathbb{C}$ be the $\vec{F}$-maximal least branch hod pair construction done in N^*. The construction lasts until we reach some $\langle \nu, k \rangle < \langle \delta^*, 0 \rangle$ such that $(\dagger)_{\nu,k}$ fails, or until we reach $\langle \nu, k \rangle = \langle \delta^*, 0 \rangle$. Let $\langle \eta_0, l_0 \rangle$ be this $\langle \nu, k \rangle$. We write

$$M_{\nu,l} = M^{\mathbb{C}}_{\nu,l} \text{ and } \Omega_{\nu,l} = \Omega^{\mathbb{C}}_{\nu,l},$$

for $\langle \nu, l \rangle \leq \langle \eta_0, l_0 \rangle$.

We now compare $(\mathcal{B}, \Psi)$ with itself, by comparing two versions of it with $(M_{\nu,l}, \Omega_{\nu,l})$. The result will be two trees $\mathcal{S}_{\nu,l}$ and $\mathcal{T}_{\nu,l}$, each on $\mathcal{B}$ and by Ψ. We show that only the two $\mathcal{B}$ sides move in our coiteration, and that no strategy disagreements show up. This is done by induction on $\langle \nu, l \rangle$. It is not possible for our coiterations to terminate because $\mathcal{B}$ is nontrivial, so we end up with $\mathcal{B}$ iterating past $M^{\mathbb{C}}_{\eta_0,l_0}$. This leads to a contradiction.

Let $\mathcal{C}$ be a premouse. For $\eta < \hat{o}(\mathcal{C})$, we let $E^{\mathcal{C}}_\eta = \dot{E}^{\mathcal{C}}_\eta$, and for $\eta = \hat{o}(\mathcal{C})$, we let $E^{\mathcal{C}}_\eta = \dot{F}^{\mathcal{C}}$. If $\mathcal{C}$ is a bicephalus, and $\eta < \hat{o}(\mathcal{C})$, then we set $E^{\mathcal{C}}_\eta = \dot{E}^{\mathcal{C}}_\eta$. If $\eta = \hat{o}(\mathcal{C})$, we leave $E^{\mathcal{C}}_\eta$ undefined.

Fix $\langle \nu, l\rangle$, and suppose we have defined $\mathcal{S}_{\mu,k}$ and $\mathcal{T}_{\mu,k}$ for all $\langle \mu, k\rangle <_{\text{lex}} \langle \nu, l\rangle$. (The trees are empty until $\mathbb{C}$ has gone well past $0^\sharp$.) We define normal trees $\mathcal{S} = \mathcal{S}_{\nu,l}$ and $\mathcal{T} = \mathcal{U}_{\nu,l}$ on $\mathcal{B}$ by induction. At stage α, we have $\mathcal{S}^\alpha$ and $\mathcal{T}^\alpha$ with last models

$$\mathcal{C} = \mathcal{M}^{\mathcal{S}^\alpha}_\infty \text{ and } \mathcal{D} = \mathcal{M}^{\mathcal{T}^\alpha}_\infty.$$

We do not assume $\text{lh}(\mathcal{S}^\alpha) = \text{lh}(\mathcal{T}^\alpha)$.

Case 1. $(M_{\nu,l}, \Omega_{\nu,l}) \trianglelefteq \mathcal{C}$ and $(M_{\nu,l}, \Omega_{\nu,l}) \trianglelefteq \mathcal{D}$.

In this case, we must have that either $(M_{\nu,l}, \Omega_{\nu,l}) \triangleleft \mathcal{C}$, or the branch of $\mathcal{S}_{\nu,l}$ to $\mathcal{C}$ has dropped, because $\mathcal{C}$ is a bicephalus and $M_{\nu,l}$ is not. Similarly on the $\mathcal{D}$ side. (Our claim 0 below implies we never get "half" of a bicephalus lining up with an $M_{\nu,l}$.) We stop the construction of $\mathcal{S}_{\nu,l}$ and $\mathcal{T}_{\nu,l}$, and go on to $\mathcal{S}_{\nu,l+1}$ and $\mathcal{T}_{\nu,l+1}$.

Case 2. Otherwise.

Here the main claim is

Claim 0. There is a γ such that

(a) $M_{\nu,l}|\langle\gamma, 0\rangle$ is extender-passive,

(b) $M_{\nu,l}|\langle\gamma, 0\rangle = \mathcal{C}|\langle\gamma,-1\rangle = \mathcal{D}|\langle\gamma,-1\rangle$, and $(\Omega_{\nu,l})_{\langle\gamma,0\rangle} = \Psi_{\mathcal{S}^\alpha,\langle\gamma,-1\rangle} = \Psi_{\mathcal{T}^\alpha,\langle\gamma,-1\rangle}$, and

(c) at least one of $\mathcal{C}|\langle\gamma, 0\rangle$ and $\mathcal{D}|\langle\gamma, 0\rangle$ is extender-active.

We defer proof of Claim 0 for now.

Let $\gamma = \gamma(\alpha)$ be the unique γ as in Claim 0. We get $\mathcal{S}^{\alpha+1}$ and $\mathcal{T}^{\alpha+1}$ as follows. Let $\eta = o(M_{\nu,l}|\langle\gamma, 0\rangle)$. Let

$$\mathcal{C} = \mathcal{M}^{\mathcal{S}^\alpha}_\xi \text{ and } \mathcal{D} = \mathcal{M}^{\mathcal{T}^\alpha}_\tau.$$

Suppose $\eta < o(\mathcal{C})$, or $\eta = o(\mathcal{C})$ but $\mathcal{C}$ is not a bicephalus, because $[0, \xi]_S$ dropped. We set

$$E^{\mathcal{S}^{\alpha+1}}_\xi = E^{\mathcal{C}}_\eta,$$

if $E_\eta^{\mathcal{C}} \neq \emptyset$, with $\mathcal{S}^{\alpha+1}$ then determined by normality. If $E_\eta^{\mathcal{C}} = \emptyset$, then $\mathcal{S}^{\alpha+1} = \mathcal{S}^\alpha$. Similarly, if $\eta < o(\mathcal{D})$ or $\mathcal{D}$ is not a bicephalus, then we set

$$E_\tau^{\mathcal{T}^{\alpha+1}} = E_\eta^{\mathcal{D}},$$

if $E_\eta^{\mathcal{D}} \neq \emptyset$, with $\mathcal{T}^{\alpha+1}$ then determined by normality. If $E_\eta^{\mathcal{D}} = \emptyset$, then $\mathcal{T}^{\alpha+1} = \mathcal{T}^\alpha$.

If $\eta = o(\mathcal{C})$ and $\mathcal{C}$ is a bicephalus, then if $E_\tau^{\mathcal{T}^{\alpha+1}}$ has already been determined, we let $E_\eta^{\mathcal{S}^{\alpha+1}}$ be the first of $\dot{F}^{\mathcal{C}}$ and $\dot{G}^{\mathcal{C}}$ that is different from $E_\tau^{\mathcal{T}^{\alpha+1}}$. If also $o(\mathcal{D}) = \eta$ and $\mathcal{D}$ is a bicephalus, then we set $E_\xi^{\mathcal{S}^{\alpha+1}} = \dot{F}^{\mathcal{C}}$, and

$$E_\tau^{\mathcal{T}^{\alpha+1}} = \begin{cases} \dot{F}^{\mathcal{D}} & \text{if } \dot{F}^{\mathcal{D}} \neq \dot{F}\mathcal{C} \\ \dot{G}^{\mathcal{D}} & \text{otherwise.} \end{cases}$$

Our definitions guarantee that if one of $E_\xi^{\mathcal{S}}$ and $E_\tau^{\mathcal{T}}$ is a top extender of a bicephalus, then $E_\xi^{\mathcal{S}} \neq E_\tau^{\mathcal{T}}$.

This finishes the definition of $\mathcal{S}^{\alpha+1}$ and $\mathcal{T}^{\alpha+1}$. The limit steps in the construction of $\mathcal{S}_{\nu,l}$ and $\mathcal{T}_{\nu,l}$ are determined by Ψ. Note that $\alpha < \beta \Rightarrow \gamma(\alpha) < \gamma(\beta)$; that is, the common lined up part keeps lengthening.

Eventually, we reach Case 1 above, and the construction of $\mathcal{S}_{\nu,l}$ and $\mathcal{T}_{\nu,l}$ stops. $(\mathcal{B}, \Psi)$ has iterated strictly past $(M_{\nu,l}, \Omega_{\nu,l})$, in two ways. As in the proof of 5.50, this implies $(\dagger)_{\nu,l}$. (When $l = -1$ as well.) It follows then that

$$\eta_0 = \delta^* \text{ and } l_0 = 0.$$

However, $(\mathcal{B}, \Psi)$ cannot iterate past $M_{\delta^*,0}$, by the usual universality argument. Note here that we have $(\dagger)_{\nu,-1}$ for all $\nu < \delta^*$, so the extenders added to the $M_{\nu,-1}$ are unique, and the universality argument applies. This contradiction completes the proof, modulo Claim 0.

Proof of Claim 0. (Sketch) We repeat the proof of Theorem 4.10. Virtually nothing changes, so we shall just mention the main points here.

The main change is the following. We used many times in the proof of 4.10 that for premice Q and R, and Σ an iteration strategy for Q, there is at most one iteration tree $\mathcal{T}$ by Σ such that $R \trianglelefteq M_\alpha(\mathcal{T})$ for $\alpha+1 = \mathrm{lh}(\mathcal{T})$, and $R \ntrianglelefteq \mathcal{M}_\alpha^{\mathcal{T}}$ whenever $\alpha + 1 < \mathrm{lh}(\mathcal{T})$. This uniqueness for normal iterations past a given R clearly fails for bicephali; let $Q = \mathcal{B}$ and $R = \mathrm{Ult}(\mathcal{B}, \dot{F}^{\mathcal{B}})$. What saves us is that in our siuation, with $Q = \mathcal{B}$ and R some initial segment of $M_{\nu,l}$, the trees $\mathcal{S}_{\nu,l}$ and $\mathcal{T}_{\nu,l}$ are being defined together in a way that completely specifies which extender to use at each step on both sides, whether that extender is from the top pair of a bicephalus or not. Moreover, this specification is absolute.

Definition 6.4 *Let R be a premouse, and suppose $\mathcal{S}$ and $\mathcal{T}$ are normal iteration trees on $\mathcal{M}$ of lengths $\alpha + 1$ and $\beta + 1$ respectively such that*

(a) α is the least ξ such that $R \trianglelefteq \mathcal{M}_\xi^{\mathcal{S}}$,

(b) β is the least ξ such that $R \trianglelefteq \mathcal{M}_\xi^{\mathcal{T}}$,

(c) $\mathcal{S}$ and $\mathcal{T}$ are by Ψ, and

(d) the extenders used in $\mathcal{S}$ and $\mathcal{T}$ are chosen according to the rules above, with R playing the role of $M_{\nu,l}$.

Then we call $(\mathcal{S}, \mathcal{T})$ the (R, Ψ)- coiteration.

Subclaim A.

(1) If $R_0 \trianglelefteq R_1$, and $(\mathcal{S}_i, \mathcal{T}_i)$ is the (R_i, Ψ)-coiteration, then $\mathcal{S}_0$ is an initial segment of $\mathcal{S}_1$ and $\mathcal{T}_0$ is an initial segment of $\mathcal{T}_1$.

(2) If S_0 and S_1 are transitive models of ZFC such that $\mathcal{B}, R \in S_i$ and $\Psi \cap S_i \in S_i$ for $i = 0, 1$, and $S_0 \models (\mathcal{S}, \mathcal{T})$ is the $(R, \Psi \cap S_0)$-coiteration, then $S_1 \models (\mathcal{S}, \mathcal{T})$ is the $(R, \Psi \cap S_1)$-coiteration.

Proof. This is obvious. $\square$

Let us assume that Claim 0 is true for $\langle \eta, k \rangle <_{\text{lex}} \langle \nu, l \rangle$. Let $\langle \gamma^*, k^* \rangle$ be least $\langle \gamma, k \rangle$ such that either $(M_{\nu,l}|\langle \gamma, k \rangle, (\Omega_{\nu,l})_{\langle \gamma,k \rangle}) \neq (\mathcal{C}|\langle \gamma, k \rangle, \Psi_{\mathcal{S}^\alpha, \langle \gamma,k \rangle})$, or $(M_{\nu,l}|\langle \gamma, k \rangle, (\Omega_{\nu,l})_{\langle \gamma,k \rangle}) \neq (\mathcal{D}|\langle \gamma, k \rangle, \Psi_{\mathcal{T}^\alpha, \langle \gamma,k \rangle})$. We show first that we are not in the bad case for extender disagreement.

Subclaim B. It is not the case that $k^* = 0$ and $\dot{F}^{M_{\nu,l}|\langle \gamma_0, 0 \rangle} \neq \emptyset$.

Proof. Suppose otherwise, and let $F = \dot{F}^{M_{\nu,l}}|\langle \gamma^*, 0 \rangle$.

We claim first that $l = 0$. For suppose $l = k + 1$. F cannot be on the sequence of $M_{\nu,k}$, since otherwise $\mathcal{S}_{\nu,k}$ would agree with $\mathcal{S}_{\nu,l}$ on all extenders used with length $< \text{lh}(F)$, and similarly for $\mathcal{T}_{\nu,k}$ and $\mathcal{U}_{\nu,l}$. But this would mean Claim 0 failed at $\langle \nu, k \rangle$, contrary to our induction hypothesis. It follows that $M_{\nu,k}$ is not sound. That implies that $M_{\nu,k}$ is the last model of $\mathcal{S}_{\nu,k}$, along a branch that dropped to $M_{\nu,l}$. Similarly, $M_{\nu,k}$ is the last model of $\mathcal{T}_{\nu,k}$, along a branch that dropped to $M_{\nu,l}$. Let α be least such that $M_{\nu,l} \trianglelefteq \mathcal{M}_\alpha^{\mathcal{S}_{\nu,k}}$ and β be least such that $M_{\nu,l} \trianglelefteq \mathcal{M}_\beta^{\mathcal{T}_{\nu,k}}$. From Subclaim A(1), we see that $\mathcal{S}_{\nu,l} = \mathcal{S}_{\nu,k} \upharpoonright (\alpha + 1)$ and $\mathcal{T}_{\nu,l} = \mathcal{T}_{\nu,k} \upharpoonright (\beta + 1)$. Thus $M_{\nu,l}$ is the last model of $\mathcal{S}_{\nu,l}$ and $\mathcal{T}_{\nu,l}$, contradiction.

But then F must be the last extender of $M_{\nu,0}$, for otherwise F is on the sequence of some $M_{\eta,k}$ with $\eta < \nu$, and Claim 0 would fail at $\langle \eta, k\rangle$, contrary to induction hypothesis.

So suppose that $M_{\nu,0}$ is extender-active, with last extender F. Suppose $\mathcal{S} = \mathcal{S}^{\alpha}_{\nu,0}$ and $\mathcal{T} = \mathcal{T}^{\alpha}_{\nu,0}$ have last models $\mathcal{C}$ and $\mathcal{D}$ respectively, and

$$(M_{\nu,-1}, \Omega_{\nu,-1}) = (\mathcal{C}|\langle \nu, -1\rangle, \Psi_{\mathcal{S},\langle \nu,-1\rangle}) = (\mathcal{D}|\langle \nu, -1\rangle, \Psi_{\mathcal{T},\langle \nu,-1\rangle}).$$

So $(\mathcal{S}, \mathcal{T})$ is the $(M_{\nu,-1}, \Psi)$-coiteration. We want to show that F is on the sequences of $\mathcal{C}$ and $\mathcal{D}$, and not as a top extender of a bicephalus in either case. For this, let

$$j \colon V \to \mathrm{Ult}(V, F^{\mathbb{C}}_{\nu})$$

be the canonical embedding, and $\kappa = \mathrm{crit}(j)$. ($V = N^*$ at this moment.) We have that $M_{\nu,-1} \trianglelefteq j(M_{\nu,-1})$ by coherence. (Note $j(M_{\nu,-1})|\nu$ is extender passive.) $j(\mathcal{S}, \mathcal{T})$ is the $(j(M_{\nu,-1}), \Psi)$ coiteration, because $j(\Psi) \subseteq \Psi$. So by Subclaim A, $\mathcal{S}$ is an initial segment of $j(\mathcal{S})$ and $\mathcal{T}$ is an initial segment of $j(\mathcal{T})$.

We have that $\mathcal{M}^{\mathcal{S}}_{\kappa} = \mathcal{M}^{j(\mathcal{S})}_{\kappa}$ and $j{\restriction}\mathcal{M}^{\mathcal{S}}_{\kappa} = i^{j(\mathcal{S})}_{\kappa, j(\kappa)}$, so F is compatible with the first extender G used in $[\kappa, j(\kappa)]_{j(S)}$. $M_{\nu,-1} \triangleleft \mathcal{M}^{j(\mathcal{S})}_{j(\kappa)}$, so G cannot be a proper initial segment of F. But F is not on the sequence of $\mathcal{M}^{j(\mathcal{S})}_{j(\kappa)}$, so F cannot be a proper initial segment of G. Hence $F = G$. Since $\mathcal{S} = j(\mathcal{S}){\restriction}(\xi + 1)$, where $\mathcal{C} = \mathcal{M}^{\mathcal{S}}_{\xi}$, we have that F is on the sequence of $\mathcal{C}$.

Similarly, F is on the sequence of $\mathcal{D}$, and used in $j(\mathcal{T})$. But then applying our observation above in $j(V)$, we see that it is not the case that $\mathcal{C}$ is a bicephalus and F is one of its top extenders, or that $\mathcal{D}$ is a bicephalus and F is one of its top extenders. $\square$

By Subclaim B, we may assume that

$$M_{\nu,l}|\langle \gamma^*, k^*\rangle = \mathcal{C}|\langle \gamma^*, k^*\rangle = \mathcal{D}|\langle \gamma^*, k^*\rangle,$$

but there is a strategy disagreement. The situation is symmetric, so we may assume

$$(\Omega_{\nu,l})_{\langle \gamma^*, k^*\rangle} \neq \Psi_{\mathcal{T}^{\alpha},\langle \gamma^*, k^*\rangle}.$$

Let

$$M = M_{\nu,l}|\langle \gamma^*, k^*\rangle.$$

We consider first the case that $M = M_{\nu,l}$, then we reduce to this case using the pullback consistency of Ψ. We derive a contradiction in the case $M = M_{\nu,l}$ by repeating the proof of Theorem 4.10. We shall try to keep the notation close to that in the proof of 4.10.

Let $(\mathcal{S}, \mathcal{T})$ be the (M, Ψ)-coiteration of $\mathcal{B}$. So M is an initial segment of both last models, but $\Omega_{\nu,l} \neq \Psi_{\mathcal{T},M}$. Note that M is an lpm, not a bicephalus. We suppose for simplicity that our strategies diverge on a single weakly normal tree $\mathcal{U}$ on M. That is, letting

$$\Omega = (\Omega_{\nu,l})_{\langle \gamma^*, k^* \rangle},$$

$\mathcal{U}$ is by both Ω and $\Psi_{\mathcal{T},M}$, but

$$\Omega(\mathcal{U}) \neq \Psi(\langle \mathcal{T}, \mathcal{U} \rangle).$$

Let $b = \Omega(\mathcal{U})$. For $\gamma < \mathrm{lh}(\mathcal{U})$ we have the embedding normalizations

$$\mathcal{W}_\gamma = W(\mathcal{T}, \mathcal{U} \upharpoonright (\gamma + 1)) \text{ and } \mathcal{W}_b = W(\mathcal{T}, \mathcal{U}^\frown b).$$

These are defined just as they were for trees on premice of the ordinary or least branch variety. The fact that $\mathcal{U}$ is only weakly normal affects nothing. We adopt all the previous notation; for example, R_γ is the last model of $\mathcal{W}_\gamma$, and $\sigma_\gamma \colon \mathcal{M}^{\mathcal{U}}_\gamma \to R_\gamma$ is the natural map.

Ω is defined by lifting to V. Let

$$\mathrm{lift}(\mathcal{U}, M_{\nu,l} | \langle \gamma^*, k^* \rangle, \mathbb{C}) = \langle \mathcal{U}^*, \langle \eta_\tau, l_\tau \mid \tau < \mathrm{lh}\,\mathcal{U} \rangle, \langle \psi^{\mathcal{U}}_\tau \mid \tau < \mathrm{lh}\,\mathcal{U} \rangle \rangle.$$

Here $\langle \eta_0, l_0 \rangle = \langle \nu, l \rangle$ and $\psi^{\mathcal{U}}_0 = \mathrm{id}$. Let

$$S_\gamma = \mathcal{M}^{\mathcal{U}^*}_\gamma,$$

and for $\langle \mu, k \rangle \leq_{\mathrm{lex}} \langle \nu, l \rangle$ let

$$(\mathcal{V}^*_{\mu,k}, \mathcal{W}^*_{\mu,k}) = \text{ the } (M_{\mu,k}, \Psi)\text{-coiteration of } \mathcal{B},$$

For $\gamma < \mathrm{lh}(\mathcal{U})$ or $\gamma = b$, let

$$(\mathcal{V}^*_\gamma, \mathcal{W}^*_\gamma) = (\mathcal{V}^*_{\eta_\gamma, l_\gamma}, \mathcal{W}^*_{\eta_\gamma, l_\gamma})^{S_\gamma}.$$

So if $[0, \gamma]_U$ does not drop in model or degree, $(\mathcal{V}^*_\gamma, \mathcal{W}^*_\gamma) = i^{\mathcal{U}^*}_{0,\gamma}((\mathcal{S}, \mathcal{T}))$.

We define by induction psuedo-hull embeddings Φ_γ from $\mathcal{W}_\gamma$ into $\mathcal{W}^*_\gamma$, for $\gamma < \mathrm{lh}(\mathcal{U})$ or $\gamma = b$, just as before. Let

$$\Phi_\gamma = \langle u^\gamma, \langle t^{0,\gamma}_\beta \mid \beta \leq z(\gamma) \rangle, \langle t^{1,\gamma}_\beta \mid \beta < z(\gamma) \rangle, p^\gamma \rangle.$$

Let us just say a few words about how to obtain $\Phi_{\gamma+1}$, because this is where the main point lies.

We have $t^\gamma\colon R_\gamma \to N_\gamma$, where N_γ is the last model of $\mathcal{W}^*_\gamma$. Let $F = \sigma_\gamma(E^{\mathcal{U}}_\gamma)$, and let $\mu = U\text{-pred}(\gamma+1)$. (Sadly, we can't use "$\nu$" for this ordinal.) So $\mathcal{W}_{\gamma+1} = W(\mathcal{W}_\mu, F)$. Let us assume for simplicity that $(\mu, \gamma+1]_U$ is not a drop in model or degree. Let

$$\mathrm{res}_\gamma = (\sigma_{\eta_\gamma, l_\gamma}[M_{\eta_\gamma, l_\gamma}|\langle \mathrm{lh}\, \psi^{\mathcal{U}}_\gamma(E^{\mathcal{U}}_\gamma), 0\rangle])^{S_\gamma},$$

and let

$$G = \mathrm{res}_\gamma(t^\gamma(F)).$$

We have $t^\gamma \circ \sigma_\gamma = \psi^{\mathcal{U}}_\gamma$, so $G = \mathrm{res}_\gamma(\psi^{\mathcal{U}}_\gamma(E^{\mathcal{U}}_\gamma)$. Let G^* be the background extender for G provided by $i^{\mathcal{U}^*}_{0,\gamma}(\mathbb{C})$, so that

$$S_{\gamma+1} = \mathrm{Ult}(S_\mu, G^*).$$

Since we are not dropping,

$$\mathcal{W}^*_{\gamma+1} = i_{G^*}(\mathcal{W}^*_\mu),$$

where $i_{G^*} = i^{\mathcal{U}^*}_{\mu,\gamma+1}$. The main thing we need to see is that G is used in $\mathcal{W}^*_{\gamma+1}$.

Let $P = N_\gamma|\langle \mathrm{lh}(t^\gamma(F), 0\rangle$, θ be least such that $P \trianglelefteq \mathcal{M}^{\mathcal{V}^*_\gamma}_\theta$, and τ least such that $P \trianglelefteq \mathcal{M}^{\mathcal{W}^*_\gamma}_\tau$. Let $(\mathcal{V}^{**}_\gamma, \mathcal{W}^{**}_\gamma)$ be the $(\mathrm{res}_\gamma(P), \Psi)$-coiteration of $\mathcal{B}$. By the counterpart of Lemma 4.5,

(i) $\mathcal{W}^{**}_\gamma$ extends $\mathcal{W}^*_\gamma \restriction (\tau+1)$,

(ii) letting $\xi = \mathrm{lh}\, \mathcal{W}^{**}_\gamma - 1$, G is on the $\mathcal{M}^{\mathcal{W}^{**}_\gamma}_\xi$ sequence, and not on the $\mathcal{M}^{\mathcal{W}^{**}_\gamma}_\alpha$ sequence for any $\alpha < \xi$,

(iii) $\tau \leq_{W^{**}_\gamma} \xi$, and $\hat{\imath}^{\mathcal{W}^{**}_\gamma}_{\tau,\xi} \restriction (\mathrm{lh}\, t^\gamma(F)+1) = \mathrm{res}_\gamma \restriction (\mathrm{lh}\, t^\gamma(F)+1)$, and

(iv) similarly for $\mathcal{V}^{**}_\gamma$ vis-a-vis $\mathcal{V}^*_\gamma$.

$P, \mathrm{res}_\gamma(P)$, and N_μ all agree up to $\mathrm{dom}(G)$, so

$$\mathrm{res}_\gamma(P)|\langle \mathrm{lh}(G), -1\rangle \trianglelefteq i_{G^*}(N_\mu),$$

and $i_{G^*}(N_\mu)|\langle \mathrm{lh}(G), 0\rangle$ is extender-passive, by coherence. We then get that $\mathcal{V}^{**}_\gamma$ is an initial segment of $\mathcal{V}^*_{\gamma+1}$, $\mathcal{W}^{**}_\gamma$ is an initial segment of $\mathcal{W}^*_{\gamma+1}$ and G is used in both $\mathcal{V}^*_{\gamma+1}$ and $\mathcal{W}^*_{\gamma+1}$. It matters here that $\mathrm{res}_\gamma(P)$ is a premouse, not a bicephalus, so both trees are forced to use G by our rules.

Now let $M = M_{\nu,l}|\langle\gamma^*, l^*\rangle$, where $\langle\gamma^*, l^*\rangle <_{\text{lex}} \langle\hat{o}(M_{\nu,l}), l\rangle$. Let

$$\langle\nu_0, l_0\rangle = \text{Res}_{\nu,l}[M] \text{ and } \pi = \sigma_{\nu,l}[M].$$

$(\Omega_{\nu,l})_M$ is defined by $(\Omega_{\nu,l})_M = \Omega^{\pi}_{\nu_0,l_0}$. By induction, the (M_{ν_0,l_0}, Ψ) coiteration is a pair $(\mathcal{V}^*, \mathcal{W}^*)$ such that M_{ν_0,l_0} is the last model of $\mathcal{W}^*$, and $\Omega_{\nu_0,l_0} = \Psi_{\mathcal{W}^*, M_{\nu_0,l_0}}$. By the counterpart of Lemma 4.5, the last drop along the main branch of $\mathcal{W}^*$ was to M, and the branch embedding is the resurrection map π, that is,

$$\pi = \hat{\imath}^{\mathcal{W}^*}_{\xi,\theta} : M \to M_{\nu_0,l_0}.$$

Here ξ is least such that $M \trianglelefteq \mathcal{M}^{\mathcal{W}^*}_{\xi}$, so the (M, Ψ) coiteration $(\mathcal{S}, \mathcal{T})$ of $\mathcal{B}$ is such that

$$\mathcal{W}^* \restriction (\xi + 1) = \mathcal{T}.$$

But then

$$\begin{aligned}\Psi_{\mathcal{T},M} &= (\Psi_{\mathcal{W}^*, M_{\nu_0,l_0}})^{\hat{\imath}^{\mathcal{W}^*}_{\xi,\theta}} \\ &= (\Omega_{\nu_0,l_0})^{\pi} \\ &= (\Omega_{\nu,l})_M.\end{aligned}$$

The first equality holds because Ψ normalizes well and has strong hull condensation, and is therefore pullback consistent.

This finishes our proof of 6.3.

$\square$

Corollary 6.5 *Assume $\mathsf{IH}_{\kappa,\delta}$, and there are infinitely many Woodin cardinals below κ. Let w be a wellorder of V_δ, let $\mathbb{C}$ be a w-construction above κ; then $\mathbb{C}$ gives rise to no nontrivial bicephali. That is, if $\langle\nu, -1\rangle < \text{lh}(\mathbb{C})$, then $\mathbb{C}$ satisfies $(\dagger)_{\nu,-1}$.*

6.2 Proof of Lemma 5.64

Recall that we have (M, Ψ) an lbr hod-pair in $L(\Psi, \mathbb{R})$, and Ψ is Suslin-co-Suslin captured by (N^*, Σ^*). We are working in N^*, where $\mathbb{C}$ is a backgrounded construction such that (M, Ψ) iterates to $(M^{\mathbb{C}}_{\eta_0,k_0}, \Omega^{\mathbb{C}}_{\eta_0,k_0})$. For $\langle\nu, l\rangle \leq \langle\eta_0, k_0\rangle$, we have the tree $\mathcal{U}_{\nu,l}$ of minimal length whereby (M, Ψ) iterates past $(M_{\nu,l}, \Omega_{\nu,l})$.

We have also the psuedo-tree $\mathcal{S}_{\nu,l}$ on the phalanx (M, K, α). We had $\pi \colon K \to M$ with $\text{crit}(\pi) \geq \alpha$. Implicit in the construction of $\mathcal{S}$ is a pullback iteration strategy

$$\Phi = \Psi^{(\text{id},\pi)}$$

for (M, K, α). We used $\mathrm{id}\colon M \to M$ and $\pi\colon K \to M$ to lift $\mathcal{S}$ to a tree

$$\mathcal{T} = (\mathrm{id}, \pi)\mathcal{S}$$

on M, then chose the branch chosen as a branch of $\mathcal{T}$ by Ψ. That is

$$\Phi(\mathcal{S}) = \Psi((\mathrm{id}, \pi)\mathcal{S}).$$

Φ is actually a strategy for a slightly stronger iteration game than the usual game producing a normal tree on a phalanx. Namely, Φ wins $\mathcal{G}_0$, where in $\mathcal{G}_0$ the opponent, player I, plays not just the extenders $E^{\mathcal{S}}_\gamma$, but also decides whether nodes are unstable. We demand that if I declares θ unstable, then he must have declared all $\tau <_S \theta$ unstable, and $0 \leq_S \theta$, and $[0, \theta]_S$ does not drop in model or degree. We then set $\alpha_\theta = \sup i^{\mathcal{S}}_{0,\theta}``\alpha$ and $\mathcal{M}^{\mathcal{S}}_{\theta+1} = \mathrm{Hull}^{\mathcal{M}^{\mathcal{S}}_\theta}(\alpha_\theta \cup i^{\mathcal{S}}_{0,\theta}(q))$. I must then declare $\theta + 1$ to be stable, and take his next extender from $\mathcal{M}^{\mathcal{S}}_{\theta+1}$. If I declares θ to be stable, he must take his next extender from $\mathcal{M}^{\mathcal{S}}_\theta$. The rest of $\mathcal{G}_0$ is as in the normal iteration game. Let us call a play $\mathcal{V}$ of $\mathcal{G}_0$ in which no one has yet lost a *psuedo iteration tree* on (M, K, α).

Remark 6.6 We can generalize $\mathcal{G}_0$ much further, to a game in which I is allowed to gratuitously drop to Skolem hulls whenever he pleases. With some minimal conditions, Ψ will pull back to a strategy for this game. We don't need that generality, so we won't go into it.

The psuedo-tree $\mathcal{S}_{\nu,l}$ from the proof of 5.57 was a play by Φ in which I followed certain rules for picking his extenders and declaring nodes unstable.

Let $\mathcal{G}$ be the game in which I and II play $\mathcal{G}_0$ until someone loses, or I decides that they should play the game $G^+(N, \omega, \delta^*)$ for producing finite stacks of weakly normal trees on the last model N of their play of $\mathcal{G}_0$. Clearly, we can pull back Ψ via (id, π) to a winning strategy for II in this game. We again call this strategy Φ, and write

$$\Phi = \Psi^{(\mathrm{id}, \pi)}$$

for it.

Let $\mathcal{V}$ be a psuedo-tree on (M, K, α_0) with last model N, and $s = \langle(\nu_i, k_i, \mathcal{U}_i) | i \leq n\rangle$ an N-stack. We can define the embedding normalization $\mathcal{W} = W(\mathcal{V}, s)$ in essentially the same way that we did when no psuedo-trees were involved. For example, suppose that s consists of just one weakly normal tree $\mathcal{U}$ on N. Being the last model, N has been declared stable in $\mathcal{V}$. We define

$$\mathcal{W}_\gamma = W(\mathcal{V}, \mathcal{U}\restriction(\gamma + 1))$$

by induction on γ. Each $\mathcal{W}_\gamma$ is a psuedo-tree with last model R_γ, and we have $\sigma_\gamma\colon \mathcal{M}^{\mathcal{U}}_\gamma \to R_\gamma$. We set $\mathcal{W}_0 = \mathcal{W}$. The successor step is given by

$$\mathcal{W}_{\gamma+1} = \mathcal{W}_\gamma{\restriction}(\theta+1)^\frown\langle F\rangle^\frown i_F``(\mathcal{W}_\nu^{\geq\beta}),$$

where $F = \sigma_\gamma(E^{\mathcal{U}}_\gamma)$, $\theta = \alpha_F$ is the least *stable* node of $\mathcal{W}_\gamma$ such that F is on the $\mathcal{M}^{\mathcal{W}_\gamma}_\theta$-sequence, $\nu = U\text{-pred}(\gamma+1)$, and β is least such that $\mathrm{crit}(F) < \lambda^{\mathcal{W}_\nu}_\beta$. (This is the case that $(\nu,\gamma+1]_U$ does not drop.) We have $\varphi\colon\ \mathrm{lh}(\mathcal{W}_\nu) \to \mathrm{lh}(\mathcal{W}_{\gamma+1})$ given by

$$\varphi(\xi) = \begin{cases} \xi & \text{if } \xi < \beta \\ (\theta+1)+(\xi-\beta) & \text{otherwise.} \end{cases}$$

A node η of $\mathcal{W}_{\gamma+1}$ is stable just in case $\eta \leq \theta$ and η is stable as a node of $\mathcal{W}_\gamma$, or $\eta = \varphi(\xi)$, where ξ is stable as a node of $\mathcal{W}_\nu$. We define by induction on $\xi \geq \beta$ the models $\mathcal{M}^{\mathcal{W}_{\gamma+1}}_{\varphi(\xi)}$ and maps $\pi_\xi\colon \mathcal{M}^{\mathcal{W}_\nu}_\xi \to \mathcal{M}^{\mathcal{W}_{\gamma+1}}_{\varphi(\xi)}$ as before.

For example, suppose $\xi = \beta$. We let

$$\mathcal{M}^{\mathcal{W}_{\gamma+1}}_{\theta+1} = \mathrm{Ult}(\mathcal{M}^{\mathcal{W}_\nu}_\beta, F),$$

and π_β be the canonical embedding. If β is stable in $\mathcal{W}_\nu$, then $E^{\mathcal{W}_{\gamma+1}}_{\theta+1} = \pi_\beta(E^{\mathcal{W}_\nu}_\beta)$, and

$$\mathcal{M}^{\mathcal{W}_{\gamma+1}}_{\theta+2} = \mathrm{Ult}(P, E^{\mathcal{W}_{\gamma+1}}_{\theta+1}),$$

where P is the appropriate initial segment of some $\mathcal{M}^{\mathcal{W}_\nu}_\tau$ We determine $\pi_{\beta+1}$ using the Shift Lemma as before. (I.e., $\pi_{\beta+1}([a,f]) = [\pi_{\theta+1}(a), \pi_\tau(f)]$ if $\tau \neq \beta$, or if $\tau = \beta$ and $\mathrm{crit}(F) \leq \mathrm{crit}(E^{\mathcal{W}_{\gamma+1}}_{\theta+1})$. Otherwise, $\pi_{\beta+1}([a,f]) = [\pi_{\theta+1}(a), f]$.) So nothing changes.

On the other hand, if β is unstable in $\mathcal{W}_\nu$, then $\theta+1$ is unstable in $\mathcal{W}_{\gamma+1}$. We set

$$(\alpha_{\theta+1})^{\mathcal{W}_{\gamma+1}} = \sup i^{\mathcal{W}_{\gamma+1}}_{0,\theta+1}``(\alpha_0),$$

and as we must,

$$\mathcal{M}^{\mathcal{W}_{\gamma+1}}_{\theta+2} = \text{ collapse of } \mathrm{Hull}^{\mathcal{M}^{\mathcal{W}_{\gamma+1}}_{\theta+1}}(\alpha_{\theta+1} \cup i^{\mathcal{W}_{\gamma+1}}_{0,\theta+1}(q)).$$

Let σ be the uncollapse map. Let $\tau\colon \mathcal{M}^{\mathcal{W}_\nu}_{\beta+1} \to \mathcal{M}^{\mathcal{W}_\nu}_\beta$ be the uncollapse map. Note that $\mathcal{W}_\nu{\restriction}(\beta+2) = \mathcal{W}_\gamma{\restriction}(\beta+2) = \mathcal{W}_{\gamma+1}{\restriction}(\beta+2)$ in the present case. We set

$$\pi_{\beta+1} = \sigma^{-1} \circ i^{\mathcal{W}_{\gamma+1}}_{\beta,\theta+1} \circ \tau.$$

We set $E^{\mathcal{W}_{\gamma+1}}_{\theta+2} = \pi_{\beta+1}(E^{\mathcal{W}_\nu}_{\beta+1})$. (Let's ignore the case $\beta + 2 = \mathrm{lh}(\mathcal{W}_\nu)$.) We have $\lambda^{\mathcal{W}_\nu}_\beta = \inf(\alpha^{\mathcal{W}_\nu}_\beta, \lambda(E^{\mathcal{W}_\nu}_{\beta+1}))$, and we set

$$\lambda^{\mathcal{W}_{\gamma+1}}_{\theta+1} = \inf(\alpha^{\mathcal{W}_{\gamma+1}}_{\theta+1}, \lambda(E^{\mathcal{W}_{\gamma+1}}_{\theta+2})).$$

It is easy to see that $\mathcal{M}^{\mathcal{W}_{\gamma+1}}_{\theta+2}|\lambda_{\theta+1} = \mathcal{M}^{\mathcal{W}_{\gamma+1}}_{\theta+1}|\lambda_{\theta+1}$. (We are ingoring the anomalous case here.) We also have

$$\pi_\beta \restriction \lambda^{\mathcal{W}_\nu}_\beta = \pi_{\beta+1} \restriction \lambda^{\mathcal{W}_\nu}_\beta,$$

which is the agreement we need to continue defining $W(\mathcal{W}_\nu, F)$.

This finishes our discussion of the normalization $W(\mathcal{V}, s)$, for $\mathcal{V}$ a psuedo-tree on (M, K, α), and s a stack on the last model of $\mathcal{V}$. We say that strategy Λ for the game $\mathcal{G}$ *normalizes well* iff whenever $\langle \mathcal{V}, s\rangle$ is according to Λ, then $W(\mathcal{V}, s)$ is according to Λ.

Lemma 6.7 *Let $\Phi = \Psi^{(id,\pi)}$ be the iteration strategy for (M, K, α) obtained by pulling back the strategy Ψ for M; then Φ normalizes well.*

Proof.(Sketch.) Ψ itself normalizes well. But normalizing commutes with copying in this context, as it did in the case of ordinary iteration trees. That is

$$(\mathrm{id}, \pi)W(\mathcal{T}, \mathcal{U}) = W((\mathrm{id}, \pi)\langle \mathcal{T}, \mathcal{U}\rangle),$$

where on the right the stack $\langle \mathcal{T}, \mathcal{U}\rangle$ is lifted by (id, π) in the natural way. So

$$\begin{aligned} W(\mathcal{T}, \mathcal{U}) \text{ is by } \Phi &\Leftrightarrow (\mathrm{id}, \pi)W(\mathcal{T}, \mathcal{U}) \text{ is by } \Psi \\ &\Leftrightarrow W((\mathrm{id}, \pi)\langle \mathcal{T}, \mathcal{U}\rangle) \text{ is by } \Psi \\ &\Leftrightarrow (\mathrm{id}, \pi)\langle \mathcal{T}, \mathcal{U}\rangle \text{ is by } \Psi \\ &\Leftrightarrow \langle \mathcal{T}, \mathcal{U}\rangle \text{ is by } \Psi\ , \end{aligned}$$

as desired. See the proof of Theorem 3.3. □

We turn to strong hull condensation. The changes we need to make in order to accomodate psuedo-trees are straightforward, but we may as well spell them out.

If $\mathcal{T}$ is a psuedo-tree on (M, K, α), then we set $\mathrm{stab}(\mathcal{T}) = \{\beta < \mathrm{lh}(\mathcal{T}) \mid \beta \text{ is } \mathcal{T}\text{-stable }\}$. We let $\mathrm{Ext}(\mathcal{T})$ be the set of extenders used, and $\mathcal{T}^{\mathrm{ext}}$ the extender tree of $\mathcal{T}$. $\mathcal{T}$ is determined by $\mathrm{stab}(\mathcal{T})$ and $\mathrm{Ext}(\mathcal{T})$. (Psuedo-trees are normal, by definition.)

Definition 6.8 *For $\mathcal{T}$ a psuedo-tree, we put $\xi \leq^*_T \eta$ iff*

(a) $\xi \leq_T \eta$, or

(b) there is a $\gamma \leq_T \eta$ such that ξ and γ are stable roots of $\mathcal{T}$, and $\xi - 1 \leq_T \gamma - 1$.

In case (b), we let $\hat{\imath}^{\mathcal{T}}_{\xi,\gamma} \colon \mathcal{M}^{\mathcal{T}}_{\xi} \to \mathcal{M}^{\mathcal{T}}_{\gamma}$ be given by

$$\hat{\imath}^{\mathcal{T}}_{\xi,\eta} = \hat{\imath}^{\mathcal{T}}_{\gamma,\eta} \circ (\tau^{-1} \circ i^{\mathcal{T}}_{\xi-1,\gamma-1} \circ \sigma),$$

where $\sigma \colon \mathcal{M}_{\xi} \to \mathcal{M}_{\xi-1}$ and $\tau \colon \mathcal{M}_{\gamma} \to \mathcal{M}_{\gamma-1}$ are the maps from the Skolem hulls.

Here is a diagram:

$$\begin{tikzcd}
 & \mathcal{M}^{\mathcal{T}}_{\eta} \\
\mathcal{M}^{\mathcal{T}}_{\gamma-1} & \mathcal{M}^{\mathcal{T}}_{\gamma} \arrow[l, "\tau"'] \arrow[u, "\hat{\imath}_{\gamma,\eta}"'] \\
\mathcal{M}^{\mathcal{T}}_{\xi-1} \arrow[u, "\hat{\imath}_{\xi,\gamma-1}"] & \mathcal{M}^{\mathcal{T}}_{\xi} \arrow[l, "\sigma"'] \arrow[u, "\hat{\imath}_{\xi,\gamma}"'] \\
\mathcal{M}^{\mathcal{T}}_{0} \arrow[u, "\hat{\imath}_{0,\xi-1}"] & \mathcal{M}^{\mathcal{T}}_{1} \arrow[l] \arrow[u, "\hat{\imath}_{1,\xi}"']
\end{tikzcd}$$

Thus the stable roots of $\mathcal{T}$ have a branch structure themselves, with 1 at its root.

Definition 6.9 *Let $\mathcal{T}$ and $\mathcal{U}$ be normal psuedo-iteration trees on (M, K, α_0). A* pseudo-hull embedding *of $\mathcal{T}$ into $\mathcal{U}$ is a system*

$$\langle u, \langle t^0_\beta \mid \beta < \operatorname{lh} \mathcal{T} \rangle, \langle t^1_\beta \mid \beta + 1 < \operatorname{lh} \mathcal{T} \wedge \beta \in \operatorname{stab}(\mathcal{T}) \rangle, p \rangle$$

such that

1. *$u : \{\alpha \mid \alpha + 1 < \operatorname{lh} \mathcal{T} \ \wedge \alpha \in \operatorname{stab}(\mathcal{T})\} \to \{\alpha \mid \alpha + 1 < \operatorname{lh} \mathcal{U} \ \wedge \alpha \in st(\mathcal{U})\}$, $\alpha < \beta \Rightarrow u(\alpha) < u(\beta)$, and λ is limit iff $u(\lambda)$ is limit.*

2. *$p : \operatorname{Ext}(\mathcal{T}) \to \operatorname{Ext}(\mathcal{U})$ is such that E is used before F on the same branch of $\mathcal{T}$ iff $p(E)$ is used before $p(F)$ on the same branch of $\mathcal{U}$. Thus p induces $\hat{p} : \mathcal{T}^{\text{ext}} \to \mathcal{U}^{\text{ext}}$.*

3. *Let* $v : \mathrm{lh}\,\mathcal{T} \to \mathrm{lh}\,\mathcal{U}$ *be given by*

$$v(\beta) = \begin{cases} 0 & \text{if } \beta = 0 \\ v(\alpha) + 1 & \text{if } \beta = \alpha + 1 \wedge s^{\mathcal{T}}_{\beta} = \emptyset \\ \text{unique } \xi \text{ such that } s^{\mathcal{U}}_{\xi} = \hat{p}(s^{\mathcal{T}}_{\beta}) & \text{otherwise.} \end{cases}$$

   Then $v(\alpha) \leq^{*}_{U} u(\alpha)$*, and*

   (i) $\alpha \in \mathrm{stab}(\mathcal{T}) \Leftrightarrow v(\alpha) \in \mathrm{stab}(\mathcal{U})$,

   (ii) if $\alpha \in \mathrm{stab}(\mathcal{T})$*,* α *is a limit ordinal, and* $[0, \alpha)_T \cap \mathrm{stab}(\mathcal{T}) = \emptyset$*, then* $v(\alpha) = u(\alpha)$.

4. *For any* β,

$$t^0_{\beta} : M^{\mathcal{T}}_{\beta} \to M^{\mathcal{U}}_{v(\beta)}$$

   is total and elementary. Moreover, for $\alpha <^{*}_{T} \beta$,

$$t^0_{\beta} \circ \hat{\imath}^{\mathcal{T}}_{\alpha,\beta} = \hat{\imath}^{\mathcal{U}}_{v(\alpha),v(\beta)} \circ t^0_{\alpha}.$$

   In particular, the two sides have the same domain.

5. *For* $\alpha + 1 < \mathrm{lh}\,\mathcal{T}$ *and* $\alpha \in \mathrm{stab}(\mathcal{T})$,

$$t^1_{\alpha} = \hat{\imath}^{\mathcal{U}}_{v(\alpha),u(\alpha)} \circ t^0_{\alpha},$$

   and

$$\begin{aligned} p(E^{\mathcal{T}}_{\alpha}) &= t^1_{\alpha}(E^{\mathcal{T}}_{\alpha}) \\ &= E^{\mathcal{U}}_{u(\alpha)}. \end{aligned}$$

   Moreover, for $\alpha < \beta < \mathrm{lh}\,\mathcal{T}$ *and* $\alpha \in \mathrm{stab}(\mathcal{T})$,

$$t^0_{\beta} \restriction \mathrm{lh}(E^{\mathcal{T}}_{\alpha}) + 1 = t^1_{\alpha} \restriction \mathrm{lh}(E^{\mathcal{T}}_{\alpha}).$$

6. *If* $\alpha \notin \mathrm{stab}(\mathcal{T})$*, then*

$$t^0_{\alpha+1} = \sigma^{-1} \circ t^0_{\alpha} \circ \tau,$$

   where $\tau \colon \mathcal{M}^{\mathcal{T}}_{\alpha+1} \to \mathcal{M}^{\mathcal{T}}_{\alpha}$ *and* $\sigma \colon \mathcal{M}^{\mathcal{U}}_{v(\alpha)+1} \to \mathcal{M}^{\mathcal{U}}_{v(\alpha)}$ *are the Skolem hull maps.*

7. *If* $\beta = T\text{-pred}(\alpha{+}1)$ *(and hence* $\alpha \in \mathrm{stab}(\mathcal{T})$*) , then letting* $\beta^{*} = U\text{-pred}(u(\alpha){+}1)$,

(ii) *if β is $\mathcal{T}$-stable, then $v(\beta) \leq^*_U \beta^* \leq^*_U u(\beta)$,*

(iii) *if β is $\mathcal{T}$-unstable, then $v(\beta) \leq_U \beta^* \leq_U u(\beta+1) - 1$.*

In any case,

$$t^0_{\alpha+1}([a,f]^P_{E^{\mathcal{T}}_\alpha}) = [t^1_\alpha(a), \hat{\imath}^{\mathcal{U}}_{v(\beta),\beta^*} \circ t^0_\beta(f)]^{P^*}_{E^{\mathcal{U}}_{u(\alpha)}},$$

where $P \trianglelefteq M^{\mathcal{T}}_\beta$ is what $E^{\mathcal{T}}_\alpha$ is applied to, and $P^* \trianglelefteq M^{\mathcal{U}}_{\beta^*}$ is what $E^{\mathcal{U}}_{u(\alpha)}$ is applied to.

Here is a diagram that goes with the last clause of the definition, in the case that $\alpha+1$ and β are both $\mathcal{T}$-unstable.

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{T}}_{\alpha+2} \arrow[rrrr, "t^0_{\alpha+2}"] \arrow[dr] & & & & \mathcal{M}^{\mathcal{U}}_{v(\alpha+2)} \arrow[dll] & \\
& \mathcal{M}^{\mathcal{T}}_{\alpha+1} \arrow[r, "t^0_{\alpha+1}"] & \mathcal{M}^{\mathcal{U}}_{v(\alpha+1)} & & & \\
& & & & & \mathcal{M}^{\mathcal{U}}_{u(\beta+1)} \\
& & \mathcal{M}^{\mathcal{U}}_{\beta^*} \arrow[uu, "E^{\mathcal{U}}_{u(\alpha)}"'] & & \mathcal{M}^{\mathcal{U}}_{\beta^*+1} \arrow[ll] \arrow[ur] \arrow[uuu, "\hat{\imath}^{\mathcal{U}}_{\beta^*+1,v(\alpha+2)}"'] & \\
& \mathcal{M}^{\mathcal{T}}_\beta \arrow[r, "t^0_\beta"] \arrow[ur] \arrow[uuu, "E^{\mathcal{T}}_\alpha"] & \mathcal{M}^{\mathcal{U}}_{v(\beta)} \arrow[u, "i^{\mathcal{U}}_{v(\beta),\beta^*}"'] & & & \\
\mathcal{M}^{\mathcal{T}}_{\beta+1} \arrow[rrrr, "t^0_{\beta+1}"] \arrow[ur] \arrow[uuuuu, "\hat{\imath}^{\mathcal{T}}_{\beta+1,\alpha+2}"] & & & & \mathcal{M}^{\mathcal{U}}_{v(\beta+1)} \arrow[ull] \arrow[uu, "i^{\mathcal{U}}_{v(\beta+1),\beta^*+1}"'] &
\end{tikzcd}$$

Definition 6.10 *Let Λ be a winning strategy for* II *in $\mathcal{G}_0$; then Λ has* strong hull condensation *iff whenever $\mathcal{U}$ is a psuedo-tree according to Λ, and there is a psuedo-hull embedding from $\mathcal{T}$ into $\mathcal{U}$, then $\mathcal{T}$ is according to Λ.*

Lemma 6.11 *Let (N, Σ) be an lbr hod-pair, let $\pi \colon K \to M$ with* $\text{crit}(\pi) \geq \alpha$, *and let $\Lambda = \Sigma^{(id,\pi)}$ be the pullback strategy for* II *in $\mathcal{G}_0$; then Λ has strong hull condensation.*

Proof.(Sketch.) This is like the proof of 3.6. If $\mathcal{U}$ is a play by Λ, and $\mathcal{T}$ is a psuedo-hull of $\mathcal{U}$, then $(\mathrm{id}, \pi)\mathcal{T}$ is a psuedo-hull of $(\mathrm{id}, \pi)\mathcal{U}$. $\square$

Thus our strategy $\Phi = \Psi^{(\mathrm{id}, \pi)}$ for (M, K, α) normalizes well and has strong hull condensation. Returning to the proof of 5.64, we have $\mathcal{S} = \mathcal{S}_{\nu,l}$, and $\gamma < \mathrm{lh}(\mathcal{S})$. We want to show that either

(1) there is a $\langle \tau, n \rangle$ such that $(\mathcal{M}_{\nu,l}|\langle \tau, n\rangle, (\Omega_{\nu,l})_{\langle \tau, n\rangle}) = (\mathcal{M}_\gamma^{\mathcal{S}}, \Sigma_\gamma)$, or

(2) there is a nonempty extender E on the $\mathcal{M}_\gamma^{\mathcal{S}}$ sequence such that, setting $\tau = \mathrm{lh}(E)$,

   (i) $\dot{E}_\tau^{M_{\nu,l}} = \emptyset$, and

   (ii) $(\Sigma_\gamma)_{\langle \tau, -1\rangle} = (\Omega_{\nu,l})_{\langle \tau, 0\rangle}$.

Here $\Sigma_\gamma = \Phi_{\mathcal{S}\restriction\gamma+1}$, in the current notation. It is a complete strategy for the lpm $\mathcal{M}_\gamma^{\mathcal{S}}$. Assume not. Since (1) fails, there is a least disagreement between $(\mathcal{M}_\gamma^{\mathcal{S}}, \Sigma_\gamma)$ and $(M_{\nu,l}, \Omega_{\nu,l})$. Since (2) fails, the least disagreement either involves a nonempty extender from $M_{\nu,l}$, or is a strategy disagreement.

As in the proof of the bicephalus lemma, the main thing not present in earlier arguments is that the way $\mathcal{S}$ is formed, and in particular the way stability declarations are made by I, is sufficiently absolute. To formalize this,

Definition 6.12 *For an lpm R, we say that $(\mathcal{V}, \mathcal{W})$ is the* (Φ, Ψ, R)-coiteration *(of (M, K, α) with M) iff*

(a) $\mathcal{V}$ is a psuedo-tree by Φ on (M, K, α) with last model P,

(b) $\mathcal{W}$ is a normal tree by Ψ on M with last model Q,

(c) $R \trianglelefteq P$ and $R \trianglelefteq Q$, and $\mathcal{V}$ and $\mathcal{W}$ are of minimal length such that this is true, and

(d) stability (and hence the next model) in $\mathcal{V}$ is determined by the rules we have given: θ is unstable iff $[0, \theta]_V$ does not drop, and $s_\theta^{\mathcal{V}} = s_\tau^{\mathcal{U}}$ for some τ.

We remark that the internal strategy $\dot{\Sigma}^R$ is relevant in (c), but no external strategy is relevant. (c) tells us that $\mathcal{V}$ and $\mathcal{W}$ proceed by hitting the least extender disagreement with R, and that the corresponding R-extenders are all empty.

Suppose now that (2) fails because there is a nonempty extender on the $M_{\nu,l}$ side at the least disagreement between $(\mathcal{M}_\gamma^{\mathcal{S}}, \Sigma_\gamma)$ with $(M_{\nu,l}, \Omega_{\nu,l})$. As in the proof of the bicephalus lemma, we can reduce to the case that $l = 0$, and the least disagreement involves $F = \dot{F}^{M_{\nu,0}}$, with $F \neq \emptyset$. Letting $\mathcal{U} = \mathcal{U}_{\nu_0,l}$, we then have that $(\mathcal{S},\mathcal{U})$ is the $(\Phi, \Psi, M_{\nu,-1})$- coiteration. Let P and Q be the last models of $\mathcal{S}$ and $\mathcal{U}$. So

$$(M_{\nu,-1}, \Omega_{\nu,-1}) = (P|\langle \nu, -1\rangle, \Phi_{\mathcal{S},\langle \nu,-1\rangle}) = (Q, \Psi_{\mathcal{U},\langle \nu,-1\rangle}).$$

Let

$$j \colon V \to \mathrm{Ult}(V, F^{\mathbb{C}}_{\nu})$$

be the canonical embedding, and $\kappa = \mathrm{crit}(j)$. ($V = N^*$ at this moment.) We have that $M_{\nu,-1} \trianglelefteq j(M_{\nu,-1})$ by coherence. (Note $j(M_{\nu,-1})|\nu$ is extender passive.) $j(\mathcal{S},\mathcal{U})$ is the $(\Phi, \Psi, j(M_{\nu,-1}))$ coiteration, because $j(\Psi) \subseteq \Psi$, and hence $j(\Phi) \subseteq \Phi$. So $\mathcal{U}$ is an initial segment of $j(\mathcal{U})$. It follows that $\mathcal{S}$ is an initial segment of $j(\mathcal{S})$. (In particular, $\mathrm{stab}(\mathcal{S}) = \mathrm{stab}(j(\mathcal{S})) \cap \mathrm{lh}(\mathcal{S})$.)

We have that $\mathcal{M}^{\mathcal{S}}_{\kappa} = \mathcal{M}^{j(\mathcal{S})}_{\kappa}$ and $j{\restriction}\mathcal{M}^{\mathcal{S}}_{\kappa} = i^{j(\mathcal{S})}_{\kappa,j(\kappa)}$, so F is compatible with the first extender G used in $[\kappa, j(\kappa)]_{j(S)}$. $M_{\nu,-1} \lhd \mathcal{M}^{j(\mathcal{S})}_{j(\kappa)}$, so G cannot be a proper initial segment of F. But F is not on the sequence of $\mathcal{M}^{j(\mathcal{S})}_{j(\kappa)}$, so F cannot be a proper initial segment of G. Hence $F = G$, and F is used in $j(\mathcal{S})$. Since $\mathcal{S} = j(\mathcal{S}){\restriction}(\xi + 1)$, where $P = \mathcal{M}^{\mathcal{S}}_{\xi}$, we have that F is on the sequence of P, contradiction.

So we may assume that we have $\langle \gamma^*, k^*\rangle$ such that

$$M_{\nu,l}|\langle \gamma^*, k^*\rangle = \mathcal{M}^{\mathcal{S}}_{\gamma}|\langle \gamma^*, k^*\rangle$$

but there is a strategy disagreement, that is

$$(\Omega_{\nu,l})_{\langle \gamma^*,k^*\rangle} \neq \Phi_{\mathcal{S},\langle \gamma^*,k^*\rangle}.$$

Let

$$Q = M_{\nu,l}|\langle \gamma^*, k^*\rangle.$$

Again we consider first the case that $Q = M_{\nu,l}$, then we reduce to this case using the pullback consistency of Φ. We derive a contradiction in the case $Q = M_{\nu,l}$ by repeating the proof of Theorem 4.10.

Letting $\mathcal{U} = \mathcal{U}_{\nu,l}$, we have that $(\mathcal{S},\mathcal{U})$ is the (Φ, Ψ, Q))-coiteration of (M, K, α) with M. We suppose for simplicity that our strategies diverge on a single weakly normal tree $\mathcal{V}$ on Q. That is, letting

$$\Omega = \Omega_{\nu,l},$$

$\mathcal{V}$ is by both Ω and $\Phi_{\mathcal{S},Q}$, but

$$\Omega(\mathcal{V}) \neq \Phi(\langle \mathcal{S}, \mathcal{V}\rangle).$$

Let $b = \Omega(\mathcal{V})$. For $\gamma < \mathrm{lh}(\mathcal{U})$ we have the embedding normalizations

$$\mathcal{W}_\gamma = W(\mathcal{S}, \mathcal{V}{\restriction}(\gamma+1)) \text{ and } \mathcal{W}_b = W(\mathcal{S}, \mathcal{V}^\frown b).$$

Ω is defined by lifting to V. Let

$$\mathrm{lift}(\mathcal{V}, M_{\nu,l}, \mathbb{C}) = \langle \mathcal{V}^*, \langle \eta_\tau, l_\tau \mid \tau < \mathrm{lh}\,\mathcal{V}\rangle, \langle \psi_\tau^{\mathcal{V}} \mid \tau < \mathrm{lh}\,\mathcal{V}\rangle\rangle.$$

Here $\langle \eta_0, l_0\rangle = \langle \nu, l\rangle$ and $\psi_0^{\mathcal{V}} = \text{ id }$. Let

$$S_\gamma = \mathcal{M}_\gamma^{\mathcal{V}^*},$$

and for $\langle \mu, k\rangle \leq_{\mathrm{lex}} \langle \nu, l\rangle$ let $\mathcal{W}^*_{\mu,k}$ be such that

$$(\mathcal{W}^*_{\mu,k}, \mathcal{U}_{\mu,k}) = \text{ the } (\Phi, \Psi, M_{\mu,k})\text{-coiteration of } (M, K, \alpha) \text{ with } M.$$

For $\gamma < \mathrm{lh}(\mathcal{V})$ or $\gamma = b$, let

$$(\mathcal{W}^*_\gamma, \mathcal{U}_\gamma) = (\mathcal{W}^*_{\eta_\gamma, l_\gamma}, \mathcal{U}_{\eta_\gamma, l_\gamma})^{S_\gamma}.$$

So if $[0,\gamma]_V$ does not drop in model or degree, $(\mathcal{W}^*_\gamma, \mathcal{U}_\gamma) = i^{\mathcal{V}^*}_{0,\gamma}((\mathcal{S}, \mathcal{U}))$.

We define by induction psuedo-hull embeddings Δ_γ from $\mathcal{W}_\gamma$ into $\mathcal{W}^*_\gamma$, for $\gamma < \mathrm{lh}(\mathcal{V})$ or $\gamma = b$, by induction on γ. Let

$$\Delta_\gamma = \langle u^\gamma, \langle t_\beta^{0,\gamma} \mid \beta \leq z(\gamma)\rangle, \langle t_\beta^{1,\gamma} \mid \beta < z(\gamma)\rangle, p^\gamma\rangle.$$

Again, we shall just describe briefly how to obtain $\Delta_{\gamma+1}$ from Δ_γ.

We have $t^\gamma \colon R_\gamma \to N_\gamma$, where N_γ is the last model of W^*_γ. Let $F = \sigma_\gamma(E_\gamma^{\mathcal{V}})$, and let $\mu = V\text{-pred}(\gamma+1)$. So $\mathcal{W}_{\gamma+1} = W(\mathcal{W}_\mu, F)$. Let us assume for simplicity that $(\mu, \gamma+1]_V$ is not a drop in model or degree. Let

$$\mathrm{res}_\gamma = (\sigma_{\eta_\gamma, l_\gamma}[M_{\eta_\gamma, l_\gamma} | \langle \mathrm{lh}\,\psi_\gamma^{\mathcal{V}}(E_\gamma^{\mathcal{V}}), 0\rangle])^{S_\gamma},$$

and let

$$G = \mathrm{res}_\gamma(t^\gamma(F)).$$

We have $t^\gamma \circ \sigma_\gamma = \psi_\gamma^{\mathcal{V}}$, so $G = \mathrm{res}_\gamma(\psi_\gamma^{\mathcal{V}}(E_\gamma^{\mathcal{V}}))$. Let G^* be the background extender for G provided by $i^{\mathcal{U}^*}_{0,\gamma}(\mathbb{C})$, so that

$$S_{\gamma+1} = \mathrm{Ult}(S_\mu, G^*).$$

Since we are not dropping,

$$W^*_{\gamma+1} = i_{G^*}(W^*_\mu),$$

where $i_{G^*} = i^{\mathcal{V}^*}_{\mu,\gamma+1}$. The main thing we need to see is that G is used in $W^*_{\gamma+1}$.

Let $Q = N_\gamma | \langle \mathrm{lh}(t^\gamma(F), 0\rangle$, τ be least in $\mathrm{stab}(\mathcal{W}^*_\gamma)$ such that $Q \trianglelefteq \mathcal{M}_\tau^{\mathcal{W}^*_\gamma}$, and θ least such that $Q \trianglelefteq \mathcal{M}_\theta^{U_\gamma}$. Let $(\mathcal{W}^{**}_\gamma, \mathcal{U}^{**}_\gamma)$ be the $(\Phi, \Psi, \mathrm{res}_\gamma(Q))$-coiteration of (M, K, α) with M. By the counterpart of Lemma 4.5,

(i) $\mathcal{W}_\gamma^{**}$ extends $\mathcal{W}_\gamma^*{\restriction}(\tau+1)$,

(ii) letting $\xi = \mathrm{lh}(\mathcal{W}_\gamma^{**}) - 1$, G is on the $\mathcal{M}_\xi^{\mathcal{W}_\gamma^{**}}$ sequence, and not on the $\mathcal{M}_\alpha^{\mathcal{W}_\gamma^{**}}$ sequence for any $\alpha < \xi$,

(iii) $\tau \leq_{W_\gamma^{**}} \xi$, and $\hat{\imath}_{\tau,\xi}^{W_\gamma^{**}}{\restriction}(\mathrm{lh}\, t^\gamma(F)+1) = \mathrm{res}_\gamma{\restriction}(\mathrm{lh}\, t^\gamma(F)+1)$, and

(iv) similarly for $\mathcal{U}_\gamma^{**}$ vis-a-vis $\mathcal{U}_\gamma$.

Proof. (Sketch.) Item (i) includes the agreement on stability declarations and next models. The point is that the $(\Phi, \Psi, \mathrm{res}_\gamma(Q))$-coiteration reaches models extending Q on both sides by the proof of Lemma 4.5. Let η be least such that $\eta \leq_{W_\gamma^{**}} \xi$ and $Q \trianglelefteq \mathcal{M}_\eta^{\mathcal{W}_\gamma^{**}}$. We have that from the proof of 4.5 that

$$\hat{\imath}_{\eta,\xi}^{W_\gamma^{**}}{\restriction}(\mathrm{lh}\, t^\gamma(F)+1) = \mathrm{res}_\gamma{\restriction}(\mathrm{lh}\, t^\gamma(F)+1).$$

The proof also shows that either $\eta = \xi$, or the first ultrapower taken in $(\eta, \xi]_{W_\gamma^{**}}$ involves a drop in model or degree. In either case, η is stable in $\mathcal{W}_\gamma^{**}$. Let also δ be least such that $P \trianglelefteq \mathcal{M}_\delta^{\mathcal{U}_\gamma^{**}}$. We then have that $(\mathcal{W}_\gamma^{**}{\restriction}(\eta+1), \mathcal{U}_\gamma^{**}{\restriction}(\delta+1))$ is the (Φ, Ψ, Q) coiteration. But $Q \trianglelefteq N_\gamma$, so this is an initial segment of the (Φ, Ψ, N_γ) coiteration, that is, of $(\mathcal{W}_\gamma^*, \mathcal{U}_\gamma)$. This implies $\eta = \tau$ and $\delta = \theta$. $\square$

$Q, \mathrm{res}_\gamma(Q)$, and N_μ all agree up to $\mathrm{dom}(G)$, so

$$\mathrm{res}_\gamma(Q)|\langle \mathrm{lh}(G), -1\rangle \trianglelefteq i_{G^*}(N_\mu),$$

and $i_{G^*}(N_\mu)|\langle \mathrm{lh}(G), 0\rangle$ is extender-passive, by coherence. We then get that $\mathcal{U}_\gamma^{**}$ is an initial segment of $\mathcal{U}_{\gamma+1}$, $\mathcal{W}_\gamma^{**}$ is an initial segment of $\mathcal{W}_{\gamma+1}^*$ and G is used in both $\mathcal{U}_{\gamma+1}$ and $\mathcal{W}_{\gamma+1}^*$.

Let β be least such that $\mathrm{crit}(F) < \lambda_\beta^{\mathcal{W}_\mu}$, and let δ be least such that F is on the $\mathcal{M}_\delta^{\mathcal{W}_\gamma}$ sequence. $\Delta_{\gamma+1}{\restriction}(\delta+1) = \Delta_\gamma{\restriction}(\delta+1)$, and this is ok because $\mathcal{W}_\gamma{\restriction}(\delta+1) = \mathcal{W}_{\gamma+1}{\restriction}(\delta+1)$ and $\mathcal{W}_\gamma^*{\restriction}v^\gamma(\delta) = \mathcal{W}_{\gamma+1}^*{\restriction}v^{\gamma+1}(\delta)$. We set

$$u^{\gamma+1}(\delta) = \xi = \mathrm{lh}(\mathcal{W}_\gamma^{**}) - 1,$$

so that

$$p^{\gamma+1}(F) = G.$$

Our proof above showed that some $\eta \leq_{\mathcal{W}_\gamma^{**}} \xi$ was stable, so that ξ is stable in $\mathcal{W}_\gamma^{**}$, and hence in $\mathcal{W}_{\gamma+1}^*$.

Let $\beta^* = W^*_{\gamma+1}$-pred$(\xi+1)$. Let us verify that β^* is located where it should be in $\mathcal{W}^*_{\gamma+1}$ according to definition 6.9. Basically, we just run through the proof of Sublemma 4.12.1, taking into account the stability structure now present. So let

$$\kappa = \mathrm{crit}(F),$$

and

$$P = \mathcal{M}_\beta^{\mathcal{W}_\mu}|(\kappa^+)^{\mathcal{M}_\beta^{\mathcal{W}_\mu}} = \mathcal{M}_\beta^{\mathcal{W}_\gamma}|(\kappa^+)^{\mathcal{M}_\beta^{\mathcal{W}_\gamma}} = \mathcal{M}_\beta^{\mathcal{W}_{\gamma+1}}|(\kappa^+)^{\mathcal{M}_\beta^{\mathcal{W}_{\gamma+1}}} = \mathrm{dom}(F).$$

Recall here we are assuming $(\mu, \gamma+1]_V$ does not drop. Let

$$\kappa^* = t^\mu(\kappa) = t^\gamma(\kappa) = \mathrm{crit}(G),$$

and

$$P^* = t^\mu(P) = t^\gamma(P) = \mathrm{dom}(G).$$

We can characterize β and β^* by

Claim 1. Let τ be least such that $P \trianglelefteq \mathcal{M}_\tau^{\mathcal{W}_\mu}$. Say that τ is *special* iff τ is unstable in $\mathcal{W}_\mu$, $P \trianglelefteq \mathcal{M}_{\tau+1}^{\mathcal{W}_\mu}$, and $\alpha_\tau^{\mathcal{W}_\mu} \le \kappa$. Then either

(i) τ is special, and $\beta = \tau + 1$, or

(ii) τ is not special, and $\beta = \tau$.

Proof. $P \trianglelefteq \mathcal{M}_\beta^{\mathcal{W}_\mu}$ because $\kappa < \lambda_\beta^{\mathcal{W}_\mu} = \lambda_\beta^{\mathcal{W}_\gamma}$, and $P \trianglelefteq \mathcal{M}_\delta^{\mathcal{W}_\gamma}$. If it is not the case that $P \trianglelefteq \mathcal{M}_\eta^{\mathcal{W}_\mu}$ for some $\eta < \beta$, then $\beta = \tau$. Moreover, τ is not special, since if τ is special, then $\lambda_\tau^{\mathcal{W}_\gamma} \le \alpha_\tau^{\mathcal{W}_\gamma}$. (Note $\mathcal{W}_\mu \restriction (\tau+1) = \mathcal{W}_\gamma \restriction (\tau+1)$ as psuedo-trees, and the two agree to $\tau + 2$ if τ is unstable in one, or equivalently, both.) So $\lambda_\beta \le \kappa$, contradiction. So we have alternative (ii).

Suppose $P \trianglelefteq \mathcal{M}_\eta^{\mathcal{W}_\mu}$, where $\eta < \beta$. Note first that η cannot be stable in $\mathcal{W}_\mu$. For otherwise, $E_\eta = E_\eta^{\mathcal{W}_\mu}$ exists, and $\lambda(E_\eta) \le \kappa$. But if $\lambda(E_\eta) = \kappa$, then $P \ntrianglelefteq \mathcal{M}_\eta^{\mathcal{W}_\mu}$, because P is passive, and E_η is indexed at $o(P)$. Thus $\mathrm{lh}(E_\eta) < \kappa$. But E_η is not on the P-sequence, because it is not on the $\mathcal{M}_\beta^{\mathcal{W}_\mu}$-sequence, so again $P \ntrianglelefteq \mathcal{M}_\eta^{\mathcal{W}_\mu}$, contradiction.

So η is unstable in $\mathcal{W}_\mu$. Arguing as above, we get that $\kappa < \lambda(E_{\eta+1}^{\mathcal{W}_\mu})$, so that $\beta \le \eta + 1$. But then, in $\mathcal{W}_\mu$,

$$\lambda_\eta = \inf(\alpha_\eta, \lambda_{\eta+1}) \le \kappa < \lambda_{\eta+1}.$$

It follows that $\alpha_\eta \leq \kappa$. Thus $\tau = \eta$, τ is special, and $\beta = \tau + 1$. So we have alternative (i).

$\square$

Similarly,

Claim 2. Let τ^* be least such that $P^* \trianglelefteq \mathcal{M}_{\tau^*}^{\mathcal{W}_\mu^*}$. Say that τ^* is *special* iff τ^* is unstable in $\mathcal{W}_\mu^*$, $P^* \trianglelefteq \mathcal{M}_{\tau^*+1}^{\mathcal{W}_\mu^*}$, and $\alpha_{\tau^*}^{\mathcal{W}_\mu^*} \leq \kappa^*$. Then either

(i) τ^* is special, and $\beta^* = \tau^* + 1$, or

(ii) τ^* is not special, and $\beta^* = \tau^*$.

Claim 3. τ is special iff τ^* is special.

Claim 4. If β is unstable in $\mathcal{W}_\mu$, then $u^\mu\restriction(\beta+2) = u^{\gamma+1}\restriction(\beta+2)$, $\mathcal{W}_\mu^*\restriction(\beta^*+1) = \mathcal{W}_{\gamma+1}^*\restriction(\beta^*+1)$, and

(a) $v^{\gamma+1}(\beta) \leq_{W_{\gamma+1}^*} \beta^* \leq_{W_{\gamma+1}^*} u^{\gamma+1}(\beta+1) - 1$,

(b) β^* is unstable in $\mathcal{W}_{\gamma+1}^*$, and

(c) $\xi + 1$ is unstable in $\mathcal{W}_{\gamma+1}^*$.

Proof. The agreement between u^μ and $u^{\gamma+1}$, and between $\mathcal{W}_\mu^*$ and $\mathcal{W}_{\gamma+1}^*$, is clear from the absoluteness of being the $(\Phi, \Psi, P^*))$-coiteration.

We have $v^\mu(\beta) \leq_{W_\mu^*} \eta$, where $\eta = u^\mu(\beta+1) - 1$ is unstable, in this case. Since F is being applied to $\mathcal{M}_\beta^{\mathcal{W}_\mu}$, $o(P) \leq \lambda_\beta^{\mathcal{W}_\mu} \leq \alpha_\beta^{\mathcal{W}_\mu}$, so $P \trianglelefteq \mathcal{M}_{\beta+1}^{\mathcal{W}_\mu}$; moreover

$$P^* = t_{\beta+1}^{1,\mu}(P) = \hat{\imath}_{v(\beta),\eta}^{\mathcal{W}_\mu^*} \circ t_\beta^{0,\mu}(P).$$

So $P^* \in \operatorname{ran}(\hat{\imath}_{v(\beta),\eta}^{\mathcal{W}_\mu^*})$. We can then argue as before that $\tau^* = \beta^*$, and $v^\mu(\beta) \leq_{W_\mu^*} \beta^* \leq_{W_\mu^*} \eta$, giving (a). Since $\beta^* \leq_{\mathcal{W}_\mu^*} \eta$, β^* is unstable in $\mathcal{W}_\mu^*$, and by absoluteness, it is unstable in $\mathcal{W}_{\gamma+1}^*$.

Finally, a key point. Recall that $(\mathcal{W}_\gamma^{**}, \mathcal{U}_\gamma^{**})$ is the $(\Phi, \Psi, \operatorname{res}_\gamma(Q))$ coiteration. Letting $\rho + 1 = \operatorname{lh}(\mathcal{U}^* *_\gamma)$, we have that G is on the sequence of $\mathcal{M}_\rho^{\mathcal{U}_\gamma^{**}}$, but not on the sequence of any earlier model. It follows that

$$\mathcal{U}_{\gamma+1}\restriction(\rho+1) = \mathcal{U}_\gamma^{**},$$

and

$$E_\rho^{\mathcal{U}_{\gamma+1}} = G.$$

Since β^* is stable in $\mathcal{W}^*_{\gamma+1}$, we have τ such that

$$\mathcal{M}_\tau^{\mathcal{U}_{\gamma+1}} = \mathcal{M}_{\beta^*}^{\mathcal{W}^*_{\gamma+1}}.$$

But then G must be applied to $\mathcal{M}_\tau^{\mathcal{U}_\gamma+1}$ in $\mathcal{U}_{\gamma+1}$, leading to

$$\mathcal{M}_{\tau+1}^{\mathcal{U}_{\gamma+1}} = \mathcal{M}_{\xi+1}^{\mathcal{W}^*_{\gamma+1}},$$

so that $\xi+1$ is unstable in $\mathcal{W}^*_{\gamma+1}$, as desired for (c). □

Claim 5. If β is stable in $\mathcal{W}_\mu$, $u^\mu\restriction(\beta+1) = u^{\gamma+1}\restriction(\beta+1)$, $\mathcal{W}^*_\mu\restriction(\beta^*+1) = \mathcal{W}^*_{\gamma+1}\restriction(\beta^*+1)$, and

(a) $v^{\gamma+1}(\beta) \leq_{W^*_{\gamma+1}} \beta^* \leq_{W^*_{\gamma+1}} u^{\gamma+1}(\beta)$,

(b) β^* is stable in $\mathcal{W}^*_{\gamma+1}$, and

(c) $\xi+1$ is stable in $\mathcal{W}^*_{\gamma+1}$.

Proof. Deferred for now. □

These claims show that $\Delta_{\gamma+1}\restriction(\delta+1)$ is a psuedo-hull embedding. The rest of $\Delta_{\gamma+1}$ is determined by

$$u^{\gamma+1}(\varphi(\eta)) = i_{G^*}(u^\mu(\eta)),$$

where $\varphi\colon \mathrm{lh}(\mathcal{W}_\mu) \to \mathrm{lh}(\mathcal{W}_{\gamma+1})$ is the map from embedding normalization. One must check that the associated $v^{\gamma+1}$ preserves stability. Here we use proposition 5.63. In general, $v^{\gamma+1}(\varphi(\eta)) = \sup i_{G^*}``v^\mu(\eta)$. However, if $\varphi(\eta)$ is a stable limit ordinal in $\mathcal{W}_{\gamma+1}$, then η is stable in $\mathcal{W}_\mu$, so $\mathrm{cof}(\eta) = \mathrm{cof}(\varphi(\eta)) = \omega$. But then $\mathrm{cof}(v^\mu(\eta)) = \omega$, so i_{G^*} is continuous at $v^\mu(\eta)$. Thus $v^{\gamma+1}(\varphi(\eta)) = i_{G^*}(v^\mu(\eta))$, hence $v^{\gamma+1}(\varphi(\eta))$ is stable in $\mathcal{W}^*_{\gamma+1}$ by the elementarity of i_{G^*}.

This ends our sketch of the proof of Lemma 5.64.

6.3 UBH holds in hod mice

In this section, we adapt the proof in [31] that a suitable form of UBH is true in pure extender models. We show thereby that whenever (M, Ω) is an lbr hod pair with scope HC, and Ω is Suslin-co-Suslin in some model of AD^+, then the corresponding form of UBH holds in M. As in the pure extender case, the proof involves a comparison of phalanxes of the form $\Phi(\mathcal{T}^\frown b)$ and $\Phi(\mathcal{T}^\frown c)$.

We shall use this theorem to show that if (M, Ω) is as above, and λ is a limit of Woodin cardinals in M, then for each $\xi < \lambda$ there is a term $\tau \in M$ such that for all g generic over M for a poset belonging to $M|\lambda$,

$$\tau^g = \Omega_{M|\xi} \cap (M|\lambda)[g].$$

This generic interpretability result is important in showing that the HOD of the derived model of M below λ is an iterate of $M|\lambda$. It has other uses as well.

Definition 6.13 *Let $\mathcal{T}$ be a normal iteration tree on an lpm M. We say that $\mathcal{T}$ is a* plus-2 tree *on M iff whenever $\alpha + 1 < \mathrm{lh}(\mathcal{T})$, there is an cardinal ρ of $\mathcal{M}_\alpha^\mathcal{T}$ such that*

$$\mathcal{M}_\alpha^\mathcal{T} \models \rho^{++} < \lambda(E_\alpha^\mathcal{T}) < \rho^{+++}.$$

We write $\rho_\alpha^\mathcal{T}$ for the unique such ρ.

We are only interested in plus-2 trees that do not drop anywhere. In such a tree $\mathcal{T}$, if T-pred$(\beta + 1) = \alpha$, then $\mathrm{crit}(E_\beta^\mathcal{T}) \leq \rho_\alpha^\mathcal{T}$, because $\mathrm{crit}(E_\beta^\mathcal{T})$ is a limit cardinal of $\mathcal{M}_\alpha^\mathcal{T}$ below $\lambda(E_\alpha^\mathcal{T})$.

Theorem 6.14 *Assume AD^+, and let (M, Ω) be a least branch hod pair with scope HC. Suppose $M \models \mathsf{ZFC}^-$, and Ω is coded by a Suslin-co-Suslin set of reals. Let δ be a cutpoint of M, $\mu > \delta$ a regular cardinal of M, and let $\langle \mu, 0, \mathcal{T} \rangle$ be an M-tree such that*

(a) $\mathcal{T}$ is a plus-2 tree of limit length that does not drop anywhere,

(b) $\mathcal{T}$ has all critical points $> \delta$, and

(c) $\mathcal{T} \in (M|\mu)[g]$, for some g that is M-generic over $\mathrm{Col}(\omega, \delta)$.

Then

$$M[g] \models \mathcal{T} \textit{ has at most one cofinal, wellfounded branch.}$$

Proof. (Sketch.) Suppose not. Let $\dot{\mathcal{T}} \in M|\mu$ be the M-least name such that 1 forces $\dot{\mathcal{T}}$ to be a counterexample. Let g be M-generic over $\mathrm{Col}(\omega, \delta)$, and $\mathcal{T} = \dot{\mathcal{T}}^g$. Let

$$\pi \colon N \to M|\mu$$

be elementary, and such that $\text{crit}(\pi) > \delta$, and N is pointwise definable from ordinals $\leq \delta$. Thus $\dot{\mathcal{T}} \in \text{ran}(\pi)$. Let

$$\hat{\pi} \colon N[g] \to (M|\mu)[g]$$

be the canonical extension of π, and let

$$\hat{\pi}(\mathcal{S}) = \mathcal{T}.$$

By assumption, $\mathcal{T}$ has distinct, cofinal, wellfounded branches in $(M|\mu)[g]$, so we have b, c such that

$$N[g] \models \text{ } b \text{ and } c \text{ are distinct cofinal, wellfounded branches of } \mathcal{S}.$$

Let $\Phi(\mathcal{S}^\frown b)$ be the weak phalanx $(\langle \mathcal{M}_\alpha^{\mathcal{S}} \mid \alpha < \text{lh}(\mathcal{S})\rangle, \langle \rho_\alpha^{\mathcal{S}} \mid \alpha + 1 < \text{lh}(\mathcal{S})\rangle)$. We get an iteration strategy for $\Phi(\mathcal{S}^\frown b)$ by finding maps $\pi_\alpha \colon \mathcal{M}_\alpha^{\mathcal{S}} \to M|\gamma_\alpha$ for $\alpha < \text{lh}(\mathcal{S})$, along with $\pi_b \colon \mathcal{M}_b^{\mathcal{S}} \to M|\mu$ so that

$$\pi_b = \pi,$$

and

$$\pi_\alpha \restriction \rho_\alpha^{+,\mathcal{M}_\alpha^{\mathcal{S}}} = \pi \restriction \rho_\alpha^{+,\mathcal{M}_\alpha^{\mathcal{S}}}.$$

This is done by working in the wellfounded model $M_b^{\mathcal{T}}[g]$, where we have $i_b^{\mathcal{T}}(\pi)$ to play the role of π, and can use condensation and an absoluteness argument to find the other maps. (It is important here that we dropped the requisite agreement of the π_α by one cardinal.) See [31] for more details on this argument. Our iteration strategy for $\Phi(\mathcal{S}^\frown b)$ is then just the pullback of Ω under the π_α, for $\alpha < \text{lh}(\mathcal{S})$ or $\alpha = b$. Call this strategy Ψ.

Similarly, we get an iteration strategy Σ for the weak phalanx $\Phi(\mathcal{S}^\frown c)$ by pulling back Ω under maps σ_α, for $\alpha < \text{lh}(\mathcal{S})$ or $\alpha = c$.

Let $(N^*, \Sigma^*, \delta^*)$ be a coarse Γ Woodin model, where Ω is coded by a $\Gamma \cap \check{\Gamma}$ set of reals. Let $\mathbb{C}$ be a maximal w-construction below δ^* in N^*. We compare $\Phi(\mathcal{S}^\frown b)$ with $\Phi(\mathcal{S}^\frown c)$ by defining, for each ν, l, the $(\Psi, \Sigma, M_{\nu,l}^{\mathbb{C}})$-*coiteration* (of $\Phi(\mathcal{S}^\frown b)$ with $\Phi(\mathcal{S}^\frown c)$). This is a pair of psuedo trees $(\mathcal{W}_{\nu,l}, \mathcal{V}_{\nu,l})$ according to Ψ and Σ respectively, obtained by iterating away least disagreements with $M_{\nu,l}^{\mathbb{C}}$, as in the proof of Theorem 5.57. The process of moving a phalanx up is a little different, so let us look at it briefly.

Let $\theta + 1 = \text{lh}(\mathcal{S}^\frown b)$. We have

$$\mathcal{M}_\alpha^{\mathcal{W}} = \mathcal{M}_\alpha^{\mathcal{S}}$$

for $\alpha < \theta$, and

$$\mathcal{M}_\theta^{\mathcal{W}} = \mathcal{M}_b^{\mathcal{S}}.$$

The exchange ordinals of $\mathcal{W}$ at the outset are

$$\lambda_\alpha^{\mathcal{W}} = \rho_\alpha^{+,\mathcal{M}_\alpha^{\mathcal{S}}},$$

for $\alpha < \mathrm{lh}(\mathcal{S})$. We say that θ is stable in $\mathcal{W}$, and all $\alpha < \theta$ are unstable. At any stage, the current last model $\mathcal{M}_\gamma^{\mathcal{W}}$ of $\mathcal{W}$ is stable, and we let $E_\gamma^{\mathcal{W}}$ be the first extender on its sequence that is part of a disagreement with $M_{\nu,l}^{\mathbb{C}}$. We show that the corresponding extender on $M_{\nu,l}^{\mathbb{C}}$ is empty, and no strategy disagreements ever show up.

Let

$$E = E_\gamma^{\mathcal{W}},$$

$\kappa = \mathrm{crit}(E)$, and α be least such that $\kappa < \lambda_\alpha^{\mathcal{W}}$. We set $\lambda_\gamma^{\mathcal{W}} = \lambda(E)$. We shall have $\alpha = W\text{-pred}(\gamma+1)$. If α is stable, we just proceed as usual, creating one new model $\mathcal{M}_{\gamma+1}^{\mathcal{W}}$, which is stable. Similarly, if α is unstable but $\mathrm{Ult}(\mathcal{M}_\alpha, E)$ does not occur in $\mathcal{V}$, we create only one new model, and it is stable. So suppose α is unstable, and $\mathrm{Ult}(\mathcal{M}_\alpha^{\mathcal{W}}, E)$ does occur in $\mathcal{V}$.

Let β be least such that $\alpha < \beta$ and β is stable. (E.g. if $\alpha < \theta$, then $\beta = \theta$.) For $0 \leq \xi \leq (\beta - \alpha)$, we set

$$\mathcal{M}_{(\gamma+1)+\xi}^{\mathcal{W}} = \mathrm{Ult}(\mathcal{M}_{\alpha+\xi}^{\mathcal{W}}, E).$$

If $\xi < (\beta - \alpha)$, we declare that $\gamma + 1 + \xi$ is unstable, and set

$$\lambda_{\gamma+1+\xi}^{\mathcal{W}} = i_{\alpha+\xi,\gamma+1+\xi}^{\mathcal{W}}(\lambda_{\alpha+\xi}^{\mathcal{W}}).$$

We declare $\gamma + 1 + (\beta - \alpha)$ to be stable. It is the new last node of $\mathcal{W}$, from which we shall take the next extender.

By induction, we have that for every node ξ of $\mathcal{W}$, there is a unique root $\tau \leq \theta$ such that $\tau \leq_W \xi$. If ξ is unstable, then so is τ; that is, $\tau < \theta$. Moreover, if ξ is unstable, then $[\tau, \xi)_W$ does not drop in model or degree, and $\lambda_\xi^{\mathcal{W}} = i_{0,\xi}^{\mathcal{W}}(\tau)$.

As before, the maps π_α, for $\alpha < \mathrm{lh}(\mathcal{S})$ or $\alpha = b$, yield a pullback strategy for a more general iteration game on $\Phi(\mathcal{S}^\frown b)$. We also call this strategy Ψ. In the more general game, I makes stability declarations and creates new models according to the rules above. Of course, there are no $M_{\nu,l}$ and $\mathcal{V}$ in the setting of the general game. I picks the next extender E freely (subject to normality), and if E is to be applied to an unstable $\mathcal{M}_\alpha$, I may decide whether $\mathrm{Ult}(\mathcal{M}_\alpha, E)$ is stable as he pleases. If he decides against stability, he must create new models as above. At limit γ such that the branch to γ II has chosen consists of unstable nodes, I is again free to decide whether γ is stable. If he decides for unstability, he must create new models in the way we are about to describe.

At limit steps in the construction of $\mathcal{W}$, we use Ψ to pick a branch a to be $[0,\gamma)_W$. We take γ to be stable unless every $\xi \in a$ is unstable (so a does not drop), and $\mathcal{M}_\gamma^\mathcal{W}$ is a model of $\mathcal{V}$. (Equivalently, $s_\gamma^\mathcal{W} = s_\tau^\mathcal{V}$, for some τ.) In this case, we declare γ to be unstable. Let τ be the unique root such that $\tau <_W \gamma$. For $0 \le \xi \le (\theta - \tau)$, we set

$$\mathcal{M}_{\gamma+\xi}^\mathcal{W} = \mathrm{Ult}(\mathcal{M}_{\tau+\xi}^\mathcal{W}, E),$$

where E is the branch extender of a. If $\xi < (\theta - \tau)$, then $\gamma + \xi$ is unstable, and

$$\lambda_{\gamma+\xi}^\mathcal{W} = i_E(\lambda_\xi^\mathcal{W}).$$

$\gamma + (\theta - \tau)$ is stable, and we take the next extender from $\mathcal{M}_{\gamma+(\theta-\tau)}^\mathcal{W}$.

Similarly, the σ_α for $\alpha < \mathrm{lh}(\mathcal{S})$ or $\alpha = c$ yield a pullback strategy Σ for the more general game on $\Phi(\mathcal{S}^\frown c)$. Using Σ, choosing extenders according to least disagreement with $M_{\nu,l}$, and making stability declarations by looking at $\mathcal{W}$, we get a tree $\mathcal{V}$ on $\Phi(\mathcal{S}^\frown c)$. Although the constructions of $\mathcal{W}$ and $\mathcal{V}$ determine stability by looking at each other, the reader can check that there is no circularity: when it comes time to determine whether γ is stable in $\mathcal{W}$, the relevant part of $\mathcal{V}$ is already determined.

Remark 6.15 Our process of moving phalanxes up amounts to a step of full normalization. We could have used a step of embedding normalization instead, and thereby arranged that our $\mathcal{W}$ and $\mathcal{V}$ are actually normal iteration trees on N. The cost would be dealing with more embeddings. It may be that as we have defined them, $\mathcal{W}$ and $\mathcal{V}$ are normal trees on N, but we have not shown that, and we do not need it.

Let us consider how the coiteration can terminate. Note first that $M|\delta = N|\delta$ is a cutpoint initial segment of $N = \mathcal{M}_0^\mathcal{S}$, and $\delta < \lambda_0^\mathcal{S}$. So both $\mathcal{W}$ and $\mathcal{V}$ begin with an iteration tree $\mathcal{U}$ on $N|\delta$ that is by $\Omega_{\langle\delta,0\rangle}$ and has last model $P = M_{\nu,l}|\langle\delta_0, 0\rangle$, with the strategy agreement $\Omega_{\mathcal{U},P} = (\Omega_{\nu,l}^\mathbb{C})_{\langle\delta_0,0\rangle}$. This follows from Theorem 5.45. We assume here that $\langle\nu, l\rangle$ is large enough that $(N|\delta, \Omega_{\langle\delta,0\rangle})$ does not iterate past $(M_{\nu,l}, \Omega_{\nu,l})$. Thinking of $\mathcal{U}$ as a tree on N, its last model is $\mathcal{M}_{\tau_0}^\mathcal{U} = Q$, where P is a cutpoint initial segment of Q. Q is pointwise definable from the ordinals $< \delta_0$. (In most cases, $\tau_0 = \delta_0$.) Let E be the branch extender of $i_{0,\tau_0}^\mathcal{U}$; then E is also the branch extender of some branch $[0,\tau)$ of both $\mathcal{W}$ and $\mathcal{V}$, with

$$Q = \mathcal{M}_\tau^\mathcal{W} = \mathcal{M}_\tau^\mathcal{V},$$

and for $0 \le \xi \le \theta$,

$$\mathcal{M}^{\mathcal{W}}_{\tau+\xi} = \mathrm{Ult}(\mathcal{M}^{\mathcal{S}^\frown b}_{\xi}, E),$$

and

$$\mathcal{M}^{\mathcal{V}}_{\tau+\xi} = \mathrm{Ult}(\mathcal{M}^{\mathcal{S}^\frown c}_{\xi}, E).$$

(As we have set it up, it is not quite true that $\tau = \tau_0$, or $\mathcal{W}\restriction(\tau+1) = \mathcal{U}$, or $\mathcal{W}\restriction(\tau+1) = \mathcal{V}\restriction(\tau+1)$. There are various lifts of $\Phi(\mathcal{S}^\frown b)$ and $\Phi(\mathcal{S}^\frown c)$ inside $\mathcal{W}$ and $\mathcal{V}$ ocurring before the lifts displayed above. Those earlier lifts play no real role anywhere. The extenders chosen from their last models could just as well have been chosen from their first models.) Let

$$Z = \mathrm{Th}^N(\delta),$$

and

$$Z_0 = \mathrm{Th}^Q(\delta_0) = i^{\mathcal{W}}_{0,\tau}(Z) = i^{\mathcal{V}}_{0,\tau}(Z).$$

Q is pointwise definable from ordinals $< \delta_0$, so it is completely determined by Z_0. All critical points in $\mathcal{S}$ are above δ, so $Z = \mathrm{Th}^{\mathcal{M}^{\mathcal{S}}_\alpha}(\delta)$ for all $\alpha < \mathrm{lh}(\mathcal{S})$, and also for $\alpha = b$ or $\alpha = c$. Thus for all $\xi \le \theta$,

$$Z_0 = \mathrm{Th}^{\mathcal{M}^{\mathcal{W}}_{\tau+\xi}}(\delta_0) = \mathrm{Th}^{\mathcal{M}^{\mathcal{V}}_{\tau+\xi}}(\delta_0).$$

Moreover, for $\eta \ge \tau$, the critical points of $E^{\mathcal{W}}_\eta$ or $E^{\mathcal{V}}_\eta$ (if they exist) are $> \delta_0$. So the rest of $\mathcal{W}$ can be considered as a psuedo-tree on the phalanx $(\langle \mathcal{M}^{\mathcal{W}}_{\tau+\xi} \mid \xi \le \theta\rangle, \langle \lambda^{\mathcal{W}}_{\tau+\xi} \mid \xi < \theta\rangle)$, and similarly for $\mathcal{V}$. Let us call the $\tau+\xi$ for $\xi \le \theta$ the *new roots* of $\mathcal{W}$ and $\mathcal{V}$.

If γ is a new root of $\mathcal{W}$, and $\gamma \le_W \eta$, then for no proper initial segment P of $\mathcal{M}^{\mathcal{W}}_\eta$ do we have $Z_0 = \mathrm{Th}^P(\delta_0)$. Moreover, $Z_0 = \mathrm{Th}^{\mathcal{M}^{\mathcal{W}}_\eta}(\delta_0)$ iff $[\gamma, \eta)_W$ does not drop. Similarly for $\mathcal{V}$. Motivated by this, let us call $\langle \nu, l\rangle$ *relevant* iff

(a) $(\mathcal{M}^{\mathcal{W}}_\tau | \langle \delta_0, 0\rangle, \Omega_{\mathcal{U},\langle \delta_0,0\rangle}) = (M^{\mathbb{C}}_{\nu,l}|\langle \delta_0, 0\rangle, (\Omega^{\mathbb{C}}_{\nu,l})_{\langle \delta_0,0\rangle})$,

(b) δ_0 is a cardinal cutpoint of $M^{\mathbb{C}}_{\nu,l}$, and

(c) for no proper initial segment P of $M^{\mathbb{C}}_{\nu,l}$ do we have $Z_0 = \mathrm{Th}^P(\delta_0)$.

Let us call $\langle \nu, l\rangle$ *exact* iff it is relevant, and $Z_0 = \mathrm{Th}^{M^{\mathbb{C}}_{\nu,l}}(\delta_0)$.

If $\langle \nu, l\rangle$ is relevant, then neither $\mathcal{W}_{\nu,l}$ nor $\mathcal{V}_{\nu,l}$ can reach a last model that is a proper initial segment of $M_{\nu,l}$.

Lemma 6.16 *If $\langle \nu, l\rangle$ is relevant, then in the $(\Psi, \Sigma, M^{\mathbb{C}}_{\nu,l})$ coiteration, no strategy disagreements show up, and no nonempty extender on the $M^{\mathbb{C}}_{\nu,l}$ side is part of a least disagreement.*

We omit the proof. It is like the proofs of the earlier results along the same lines.

Claim 1. There is an exact $\langle \nu, l\rangle <_{\text{lex}} \langle \delta^*, 0\rangle$.

Proof. Otherwise $\langle \delta^*, 0\rangle$ is relevant, so the $(\Psi, \Sigma, \mathcal{M}^{\mathbb{C}}_{\delta^*,0})$ coiteration produces $(\mathcal{W}, \mathcal{V})$ with last models extending $\mathcal{M}^{\mathbb{C}}_{\delta^*,0}$. This contradicts the universality of $\mathcal{M}^{\mathbb{C}}_{\delta^*,0}$. $\square$

Now let $\langle \nu, l\rangle$ be the unique exact pair.

Claim 2. $l = 0$ and $\mathrm{Th}^{M_{\nu,0}}(\delta_0) = Z_0$.

Proof. If not, then $\rho(M_{\nu,l}) < \delta_0$. But letting P be the last model of $\mathcal{W}$, we have that δ_0 is a cardinal of P, $P|\delta_0 = M_{\nu,l}|\delta_0$, and $\rho(P) \geq \delta_0$. It follows that P is a proper initial segment of $M_{\nu,l}$, and the branch of $\mathcal{W}$ to P does not drop. But then $\langle \nu, l\rangle$ is not relevant, contradiction. $\square$

It is easy to see that $M_{\nu,0} \models \mathsf{ZFC}^-$, so $\rho(M_{\nu,k}) = o(M_{\nu,k})$ for all $k < \omega$, but $\rho(M_{\nu+1,0}) = \delta_0$.

Let $\mathcal{W} = \mathcal{W}_{\nu,0}$ and $\mathcal{V} = \mathcal{V}_{\nu,0}$ have lengths γ_0 and γ_1.

Claim 3. $\mathcal{M}^{\mathcal{W}}_{\gamma_0} = \mathcal{M}^{\mathcal{V}}_{\gamma_1} = M_{\nu,0}$; moreover, the branches of $\mathcal{W}$ and $\mathcal{V}$ to γ_0 and γ_1 do not drop.

Proof. Neither side can iterate to a proper initial segment of $M_{\nu,0}$ because $\langle \nu, 0\rangle$ is relevant. Neither side can iterate strictly past $M_{\nu,0}$ because $\langle \nu, 0\rangle$ is exact. $\square$

Let $\eta_0 \leq_W \gamma_0$ and $\eta_1 \leq_V \gamma_1$ be the new roots of the two trees below γ_0 and γ_1. Let

$$i_0 \colon Q \to \mathcal{M}^{\mathcal{W}}_{\eta_0} \text{ and } i_1 \colon Q \to \mathcal{M}^{\mathcal{V}}_{\eta_1}$$

be the embeddings given by the fact that $Z_0 = \mathrm{Th}^{\mathcal{M}^{\mathcal{W}}_{\eta_0}}(\delta_0) = \mathrm{Th}^{\mathcal{M}^{\mathcal{V}}_{\eta_1}}(\delta_0)$. These are just the lifts under $i^{\mathcal{U}}_{0,\tau_0}$ of the branch embeddings $i^{\mathcal{S}}_{0,\eta_0-\tau}$ and $i^{\mathcal{S}}_{0,\eta_1-\tau}$. We have that

$$i^{\mathcal{W}}_{\eta_0,\gamma_0} \circ i_0 = i^{\mathcal{V}}_{\eta_1,\gamma_1} \circ i_1,$$

since both embeddings are the embedding given by Q being the transitive collapse of $\mathrm{Hull}^{M_{\nu,0}}(\delta_0)$.

We now get a contradiction using the hull and definability properties in $M_{\nu,0}$ as usual.

Definition 6.17 *For M an lpm, we say that M has the* definability property at α *iff α is first order definable over M from some ordinals $b \in [\alpha]^{<\omega}$, and write* $\mathrm{Def}(M,\alpha)$ *in this case. We say that M has the* hull property at α *iff whenever $A \subset \alpha$ and $A \in M$, there is a $B \in M$ such that B is definable over M from some $b \in [\alpha]^{<\omega}$, and $B \cap \alpha = A$. We write* $\mathrm{Hp}(M,\alpha)$ *in this case.*

Claim 4. $\eta_0 = \eta_1$.

Proof. Suppose toward contradiction that $\eta_0 = \tau + \xi_0$ and $\eta_1 = \tau + \xi_1$, where $\xi_0 \neq \xi_1$. Let

$$j_0 \colon \mathcal{M}^{\mathcal{S}}_{\xi_0} \to \mathrm{Ult}(\mathcal{M}^{\mathcal{S}}_{\xi_0}, E) = \mathcal{M}^{\mathcal{W}}_{\eta_0},$$

and

$$j_1 \colon \mathcal{M}^{\mathcal{S}}_{\xi_1} \to \mathrm{Ult}(\mathcal{M}^{\mathcal{S}}_{\xi_1}, E) = \mathcal{M}^{\mathcal{V}}_{\eta_1}$$

be the canonical embeddings. Suppose first that ξ_0 and ξ_1 are incomparable in $\mathcal{S}$, and let $F = E^{\mathcal{S}}_{\alpha}$ and $G = E^{\mathcal{S}}_{\beta}$, where $\alpha + 1 \leq_S \xi_0$, $\beta + 1 \leq_S \xi_1$, $\alpha \neq \beta$, and S-pred$(\alpha+1) = S$-pred$(\beta+1) = \rho$. We may assume $\mathrm{lh}(F) < \mathrm{lh}(G)$. Let $\lambda = \sup\{\lambda(E^{\mathcal{S}}_{\alpha}) \mid \alpha + 1 \leq_S \rho\}$. Letting $\kappa_0 = \mathrm{crit}(F)$, we have

$$\kappa_0 = \text{ least } \mu \geq \lambda \text{ such that } \neg\mathrm{Def}(\mathcal{M}^{\mathcal{S}}_{\xi_0}, \mu).$$

Because the generators of j_0 (i.e. the generators of E) are contained in δ_0, we get

$$\begin{aligned} j_0(\kappa_0) &= \text{ least } \mu \geq j_0(\lambda) \text{ such that } \neg\mathrm{Def}(\mathcal{M}^{\mathcal{W}}_{\eta_0}, \mu) \\ &= \text{ least } \mu \geq j_0(\lambda) \text{ such that } \neg\mathrm{Def}(M_{\nu,0}, \mu). \end{aligned}$$

The second line comes from using $i^{\mathcal{W}}_{\eta_0,\gamma_0}$ to move up to $\mathcal{M}^{\mathcal{W}}_{\gamma_0} = M_{\nu,0}$. Note here that $j_0(\kappa_0) < j_0(\lambda^{\mathcal{S}}_{\xi_0}) = \lambda^{\mathcal{W}}_{\eta_0} \leq \mathrm{crit}(i^{\mathcal{W}}_{\eta_0,\gamma_0})$. Similarly, letting $\kappa_1 = \mathrm{crit}(G)$, we get

$$\begin{aligned} j_1(\kappa_1) &= \text{ least } \mu \geq j_1(\lambda) \text{ such that } \neg\mathrm{Def}(\mathcal{M}^{\mathcal{V}}_{\eta_1}, \mu) \\ &= \text{ least } \mu \geq j_1(\lambda) \text{ such that } \neg\mathrm{Def}(M_{\nu,0}, \mu). \end{aligned}$$

So $j_0(\kappa_0) = j_1(\kappa_1)$. But $\kappa_0, \kappa_1 < \mathrm{lh}(F)$, and $j_0 \restriction \mathrm{lh}(F) = j_1 \restriction \mathrm{lh}(F)$, so $\kappa_0 = \kappa_1$.

Now let ν_0 be the sup of the generators of F; that is, the least γ such that every $\alpha < \lambda(F)$ is of the form $i_F(f)(a)$, for some $a \in [\gamma]^{<\omega}$. For $\beta < \nu_0$, $F \restriction \beta \in \mathrm{Ult}(\mathcal{M}^{\mathcal{S}}_{\rho}, F)$, by closure under initial segment. (This not one of the axioms on premice in the Jensen theory, but it can be proved to hold of sound, iterable premice.) On the other hand, $F \restriction \beta \notin \mathrm{Ult}(\mathcal{M}^{\mathcal{S}}_{\rho}, F \restriction \beta)$. From this we get

$$|\nu_0|^{\mathcal{M}^{\mathcal{S}}_{\xi_0}} = \text{ least } \mu \geq \kappa_0 \text{ such that } \neg\mathrm{Hp}(\mathcal{M}^{\mathcal{S}}_{\xi_0}, \mu).$$

Similarly, letting ν_1 be the sup of the generators of G, we get

$$|\nu_1|^{\mathcal{M}^{\mathcal{S}}_{\xi_1}} = \text{ least } \mu \geq \kappa_0 \text{ such that } \neg\mathrm{Hp}(\mathcal{M}^{\mathcal{S}}_{\xi_1}, \mu).$$

Using $i^{\mathcal{W}}_{\eta_0,\gamma_0} \circ j_0$ and $i^{\mathcal{V}}_{\eta_1,\gamma_1} \circ j_1$ to move up to $M_{\nu,0}$, and considering the hull property there, we get as above that $j_0(|\nu_0|^{\mathcal{M}^{\mathcal{S}}_{\xi_0}}) = j_1(|\nu_1|^{\mathcal{M}^{\mathcal{S}}_{\xi_1}})$. Thus $|\nu_0|^{\mathcal{M}^{\mathcal{S}}_{\xi_0}} = |\nu_0|^{\mathcal{M}^{\mathcal{S}}_{\xi_0}}$. However, G was used strictly after F in $\mathcal{S}$, so $\mathrm{lh}(F)$ is a cardinal of $\mathcal{M}^{\mathcal{S}}_{\beta}|\mathrm{lh}(G)$, and thus

$$\nu_0 < \mathrm{lh}(F) \leq |\nu_1|^{\mathcal{M}^{\mathcal{S}}_{\xi_1}}.$$

This is a contradiction.

We are left to consider the case $\xi_0 <_S \xi_1$. Let G be the extender used in $[0, \xi_1)_S$ and applied to $\mathcal{M}^{\mathcal{S}}_{\xi_0}$. Let $\kappa_1 = \mathrm{crit}(G)$ and ν_1 be the sup of the generators of G. Let $\lambda = \sup\{\lambda(E^{\mathcal{S}}_\alpha) \mid \alpha + 1 \leq_S \xi_0\}$. Then again,

$$\begin{aligned} j_1(\kappa_1) = &\text{ least } \mu \geq j_1(\lambda) \text{ such that } \neg\mathrm{Def}(\mathcal{M}^{\mathcal{V}}_{\eta_1}, \mu) \\ = &\text{ least } \mu \geq j_1(\lambda) \text{ such that } \neg\mathrm{Def}(\mathcal{M}^{\mathcal{V}}_{\gamma_1}, \mu). \end{aligned}$$

Note that γ_0 is stable, and η_0 is unstable, so $\eta_0 <_W \gamma_0$. Let F be the extender used in $[\eta_0, \gamma_0)_W$ and applied to $\mathcal{M}^{\mathcal{W}}_{\eta_0}$. Let $\kappa_0 = \mathrm{crit}(F)$. Then since $\mathcal{M}^{\mathcal{S}}_{\eta_0}$ has the definability property everywhere above $j_1(\lambda)$, using $i^{\mathcal{W}}_{\eta_0,\gamma_0}$ we see that κ_0 is the least $\mu \geq j_1(\lambda)$ such that $\neg\mathrm{Def}(M_{\nu,0}, \mu)$. Thus $\kappa_0 = j_1(\kappa_1)$. But $F = E^{\mathcal{W}}_\eta$ for some $\eta \geq \tau + \theta$, so $j_1(\lambda(G)) < \lambda(F)$, so $j_1(\lambda(G)) < \nu(F)$, and the hull property fails in $\mathrm{Ult}(\mathcal{M}^{\mathcal{W}}_{\eta_0}, F)$ at all η such that $\kappa_0 < \eta \leq j_1(\lambda(G))$. Moving up to $M_{\nu,0}$,

$$\forall \eta(\kappa_0 < \eta \leq j_1(\lambda(G)) \Rightarrow \neg\mathrm{Hp}(M_{\nu,0}, \eta).$$

However, $\mathcal{M}^{\mathcal{S}}_{\xi_1}$ does have the hull property at $\nu_1 = \nu(G) \leq \lambda(G)$. This gives $\mathrm{Hp}(M^{\mathcal{V}}_{\eta_1}, j_1(\nu_1))$, and thus $\mathrm{Hp}(M_{\nu,0}, j_1(\nu_1))$, noting here that $\mathrm{crit}(i^{\mathcal{V}}_{\eta_1,\gamma_1}) \geq j_1(\nu_1)$. This is a contradiction. □

Claim 5. $\eta_0 < \tau + \theta$.

Proof. Otherwise $\xi_0 = b$ and $\xi_1 = c$. Let F be the first extender used in $b - c$ and G the first extender used in $c - b$. We get a contradiction just as we did in the proof of Claim 4, in the case ξ_0 and ξ_1 were S-incomparable. □

Now let s be the increasing enumeration of the extenders used in $(\eta_0, \gamma_0)_W$ and t the increasing enumeration of the extenders used in $(\eta_1, \gamma_1)_V$. Using the hull and definability properties in $M_{\nu,0}$, we get that $s = t$. But this implies that γ_0 and γ_1 are unstable, a contradiction. That completes the proof of Theorem 6.14. □

7 HOD in the derived model of hod mouse

In this section, we prove Theorem 0.16. For the reader's convenience, we re-state it here.

Theorem 7.1 *Suppose V is normally iterable above κ by the strategy of choosing unique cofinal wellfounded branches. Suppose there is a superstrong cardinal $\lambda > \kappa$, and suppose there are arbitrarily large Woodin cardinals; then there is a Wadge cut Γ in Hom_∞ such that $L(\Gamma, \mathbb{R}) \models \mathsf{AD}_\mathbb{R}$, and*

$$HOD^{L(\Gamma,\mathbb{R})} \models \mathsf{GCH} + \text{ there is a superstrong cardinal.}$$

The theorem follows easily from

Theorem 7.2 *Assume AD^+, and let (M, Ψ) be an lbr hod pair with scope HC, and such that Ψ is coded by a Suslin-co-Suslin set of reals. Suppose*

$$M \models \mathsf{ZFC} + \lambda \text{ is a limit of Woodin cardinals.}$$

Let g be $\mathrm{Col}(\omega, <\lambda)$-generic over M, $\mathbb{R}^*_g = \bigcup\{\mathbb{R} \cap M[g\restriction\omega \times \alpha] \mid \alpha < \lambda\}$, and $Hom^*_g = \{p[T] \cap \mathbb{R}^*_g \mid \exists\alpha < \lambda(M[g\restriction\omega \times \alpha] \models T \text{ is } < \lambda\text{-absolutely complemented }\}$. Then

$$L(Hom^*_g, \mathbb{R}^*_g) \models \mathsf{AD}_\mathbb{R}.$$

and

- *(a) if λ is a limit of cutpoints in M, then $HOD^{L(Hom^*_g,\mathbb{R}^*_g))}|\theta^{L(Hom^*_g,\mathbb{R}^*_g)}$ is a non-dropping iterate of $M|\lambda$ by $\Psi_{\langle\lambda,0\rangle}$, and*
- *(b) if $\kappa < \lambda$ is least so that $o(\kappa) \geq \lambda$ in M, then there is an iteration map $i \colon M \to N$ by Ψ coming from a stack s on $M|\lambda$ such that $HOD^{L(Hom^*_g,\mathbb{R}^*_g)}|\theta^{L(Hom^*_g,\mathbb{R}_g)} = N|i(\kappa)$.*

The model $L(\mathrm{Hom}^*_g, \mathbb{R}^*_g)$ above is the "old" derived model of M below λ. Because $\mathsf{AD}_\mathbb{R}$ holds in it, $P(\mathbb{R}^*_g) \cap L(\mathrm{Hom}^*_g, \mathbb{R}^*_g) = \mathrm{Hom}^*_g$. It is clear that in case (b) above, $i(\kappa)$ is regular in $\mathrm{HOD}^{L(\mathrm{Hom}^*_g,\mathbb{R}^*_g)}$, and hence $L(\mathrm{Hom}^*_g, \mathbb{R}^*_g) \models \mathsf{AD}_\mathbb{R} +$ "θ is regular". [35] produces a model of the "largest Suslin axiom", or LSA, from a hypothesis on the existence of lbr hod pairs. (Sargsyan [18] had already produced a model of LSA from a somewhat weaker assumption on the existence of hybrid mice.)

We shall need the following *generic interpretability* lemma. Its proof follows the same basic outline as Sargsyan's proof of the corresponding fact for rigidly layered hod pairs below LSA.(See [16] and [18].)

Lemma 7.3 *(Generic interpretability) Assume* AD^+*, and let* (P, Σ) *be an lbr hod pair with scope HC, and such that* Σ *is coded by a Suslin-co-Suslin set of reals. Let*

$$P \models \mathsf{ZFC}^- + \delta \text{ is Woodin;}$$

then there is a term $\tau \in P$ *such that whenever* $i \colon P \to Q$ *is the iteration map associated to a non-dropping* P*-stack* s *by* Σ*, and* g *is* $\mathrm{Col}(\omega, < i(\delta))$*-generic over* Q*, then*

$$i(\tau)^g = \Sigma_{s,<i(\delta)} \restriction HC^{Q[g]}.$$

Proof. For $\xi < \eta < \delta$ and $k < \omega$, we shall define a term $\tau_{\xi,k,\eta}$ such that whenever g is P-generic over $\mathrm{Col}(\omega, \eta)$, then $\tau^g_{\xi,k,\eta} = \Sigma_{\langle \xi,k \rangle} \restriction \mathrm{HC}^{P[g]}$. We then take τ to be the join of the $\tau_{\xi,k,\eta}$. Clearly then $\tau^g = \Sigma_{<\delta} \restriction \mathrm{HC}^{P[g]}$ whenever g is $\mathrm{Col}(\omega, < \delta)$ generic over P. It will be clear that this property of τ is preserved by Σ-iteration.

So fix $\xi < \eta < \delta$ and $k < \omega$. Let g be P-generic over $\mathrm{Col}(\omega, \eta)$. We shall define $\Sigma_{\langle \xi,k \rangle} \restriction \mathrm{HC}^{P[g]}$ from $\xi, k, P|\delta$ and g. The definition will be uniform in g, giving us the desired term.

Let $\mu = (\eta^+)^P$. We may assume that μ is a cutpoint of P. For if not, let E be the first extender on the P-sequence such that $\mathrm{crit}(E) < \mu < \mathrm{lh}(E)$, and set $Q = \mathrm{Ult}(P, E)$. Then μ is a cutpoint of Q, $\mathrm{HC}^{P[g]} = \mathrm{HC}^{Q[g]}$, and by strategy coherence, $\Sigma_{\langle E \rangle, \langle \xi,k \rangle} = \Sigma_{\langle \xi,k \rangle}$. A definition of $\Sigma_{\langle E \rangle, \langle \xi,k \rangle} \restriction \mathrm{HC}^{Q[g]}$ from $Q|i_E(\delta), \xi, k$, and g will then give the desired definition of $\Sigma_{\langle \xi,k \rangle} \restriction \mathrm{HC}^{P[g]}$. So we assume μ is a cutpoint of P.

Let w be the canonical wellorder of $P|\delta$, and working in P, let $\mathbb{C}$ be a w-construction of length δ that is above μ, and such that

(i) Each $F^{\mathbb{C}}_\nu$ is a plus-2 extender on the P-sequence, and

(ii) $\mathbb{C}$ adds extenders whenever possible, subject to (i).

Item (i) involves a slight inconsistency with our previous definition of w-construction. There we required that the strength of $F^{\mathbb{C}}_\nu$ be at least an inaccessible cardinal $\eta > \mathrm{lh}(\dot{E}^{M_{\nu,0}})$, and because we minimized in Mitchell order, the strength could not be more than that. Here we mean to require that the strength be $(\rho^{++})^P$, for $\rho = \mathrm{lh}(\dot{E}^{M_{\nu,0}})$, and not more, because we minimize in the order of extenders on the P-sequence. That implies that $\lambda(F^{\mathbb{C}}_\nu) < (\rho^{+++})^P$. This change does not affect anything we proved about w-constructions earlier. It has the consequence that for any $\mathcal{T}$ on $M^{\mathbb{C}}_{\nu,k}$, the iteration tree $\mathcal{T}^*$ on P that is part of $\mathrm{lift}(\mathcal{T}, M_{\nu,k}, \mathbb{C})$ is a plus-2 tree using extenders from the P-sequence. So by 6.14, if $\mathcal{T} \in P[g]$, then UBH holds for $\mathcal{T}^*$.

We also have CBH for plus-2 trees $\mathcal{T}^*$ on P such that $\mathcal{T}^* \in P$. This is because $\dot{\Sigma}^P(\mathcal{T}^*)$ is defined, in P, and wellfounded. Thus in P, the $\Omega^{\mathbb{C}}_{\nu,l}$ are total. In P, they are induced by $\dot{\Sigma}^P$, but $\dot{\Sigma}^P \subseteq \Sigma$, and Σ is total on V. So Σ induces a total-on-V strategy $\Omega^*_{\nu,l}$ for $M_{\nu,l}$ such that $\Omega^{\mathbb{C}}_{\nu,l} \subseteq \Omega^*_{\nu,l}$. The $\Omega^*_{\nu,l}$ are Suslin-co-Suslin in V because Σ is. Since they are induced by Σ, they have strong hull condensation and normalize well. In fact, each $(M^{\mathbb{C}}_{\nu,l}, \Omega^*_{\nu,l})$ is an lbr hod pair in V. Moreover, $V \models \mathsf{AD}^+$, so in V we can carry out the comparisons needed to see each $(M_{\nu,l}, \Omega^*_{\nu,l})$ has a core. Thus $(M_{\nu,l}, \Omega_{\nu,l})$ has a core in P, and $\mathbb{C}$ does not break down in P.

Claim 1. In P, there is a $\nu < \delta$ such that $(P|\langle \xi, k\rangle, \dot{\Sigma}^P_{\langle \xi,k\rangle})$ iterates to $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$.

Proof. Suppose not. Working in P, we claim that for all $\langle \nu, l\rangle$ such that $\nu < \delta$, $(P|\langle \xi, k\rangle, \dot{\Sigma}^P_{\langle \xi,k\rangle})$ iterates strictly past $(M_{\nu,l}, \Omega_{\nu,l})$. This almost follows from the comparison theorem 5.45. However, to simply quote 5.45, we would need to know that $\dot{\Sigma}^P_{\langle \xi,k\rangle}$ is $< \delta$ universally Baire in P. That is part of the theorem we are proving now. Nevertheless, the proof of 5.45 works here. The consequence of universal Baireness we need is just that if $\mathcal{T}$ is a normal tree by $\dot{\Sigma}^P_{\langle \xi,k\rangle}$, and $i\colon P \to Q$ is an iteration map by Σ with $\mathrm{crit}(i) > \xi$, then $i(\mathcal{T})$ is by $\dot{\Sigma}^P_{\langle \xi,k\rangle}$. This much is true by the strategy coherence of Σ.

But then $(P|\langle \xi, k\rangle, \dot{\Sigma}^P_{\langle \xi,k\rangle})$ iterates past $M_{\delta,0}$ in P. This contradicts the Woodinness of δ in P. $\square$

Let $\mathcal{T}$ be the normal tree by $\dot{\Sigma}^P_{\langle \xi,k\rangle}$ whose last model is $M^{\mathbb{C}}_{\nu,k}$ given by claim 1, and let $i\colon P|\langle \xi, k\rangle \to M_{\nu,k}$ be its canonical embedding.

Claim 2. $\Sigma_{\mathcal{T},M_{\nu,k}} = \Omega^*_{\nu,k}$.

Proof. The proof that the two strategies agree on all trees in P actually shows that they agree on all trees in V. [Let $\mathcal{U}$ be by both strategies, and $b = \Omega^*_{\nu,k}(\mathcal{T})$. Let $\mathcal{U}^*$ be the tree according to Σ that is part of $\mathrm{lift}(\mathcal{U}, M_{\nu,k}, \mathbb{C})$; again, we do not need $\mathcal{U} \in P$ to make sense of lifting. Then $W(\mathcal{T}, \mathcal{U}^\frown b)$ is a psuedo-hull of $i^{\mathcal{U}^*}_b(\mathcal{T})$ by our previous calculations. However, $i^{\mathcal{U}^*}_b(\mathcal{T})$ is by $\Sigma_{\langle \xi,k\rangle}$ by strategy coherence, so $W(\mathcal{T}, \mathcal{U}^\frown b)$ is by $\Sigma_{\langle \xi,k\rangle}$ because $\Sigma_{\langle \xi,k\rangle}$ normalizes well, so $b = \Sigma_{\mathcal{T},M_{\nu,k}}(\mathcal{U})$.] $\square$

Now let $\mathcal{U}$ be a normal tree on $P|\langle \xi, k\rangle$ of limit length that is according to $\Sigma_{\langle \xi,k\rangle}$, and such that $\mathcal{U}$ is countable in $P[g]$. We wish to find $\Sigma_{\langle \xi,k\rangle}(\mathcal{U})$ in $P[g]$, and define it from the relevant parameters. But $\Sigma_{\langle \xi,k\rangle}$ is pullback consistent, so

$$\begin{aligned} \Sigma_{\langle \xi,k\rangle}(\mathcal{U}) = b &\text{ iff } \Sigma_{\mathcal{T},M_{\nu,k}}(i\mathcal{U}) = b \\ &\text{ iff } \Omega^*_{\nu,k}(i\mathcal{U}) = b. \end{aligned}$$

So it will be enough to show

Claim 3. If $\mathcal{S}$ is countable in $P[g]$, of limit length, and by $\Omega^*_{\nu,k}$, and $b = \Omega_{\nu,k}(\mathcal{S})$, then $b \in P[g]$. Moreover, b is uniformly definable over $P[g]$ from $\mathcal{S}$ and $\mathbb{C}$.

Proof. Let $\mathcal{S}^*$ be the plus-2 tree on P that it part of $\text{lift}(\mathcal{S}, M_{\nu,k}, \mathbb{C})$. enough to show $b \in P[g]$, and to define there from the relevant parameters, uniformly.

We know from 6.14 that in $P[g]$, $\mathcal{S}^*$ has at most one cofinal, wellfounded branch. Since all critical points in $\mathcal{S}^*$ are strictly above μ, we can think of $\mathcal{S}^*$ as a plus-2 tree on $P[g]$. Then by [8], since $\mathcal{S}^*$ is countable in $P[g]$, it has exactly one cofinal wellfounded branch b in $P[g]$. Moreover, again by [8], $\mathcal{S}^*$ is continuously illfounded off b. It follows that $b = \Sigma(\mathcal{S}^*)$, and therefore $b = \Omega^*_{\nu,k}(\mathcal{S})$, as desired. □

This completes the proof of Lemma 7.3. □

Assume AD^+, and let (M, Ω) be an lbr hod pair with scope HC. Suppose s and t are stacks by Ω on M with last models P and Q such that M-to-P and M-to-Q do not drop. By 5.54 and Dodd-Jensen, we can then find stacks u and v by Ω_s and Ω_t with a common last model such that neither stack drops getting to N, and such that $\Omega_{s^\frown u} = \Omega_{t^\frown v}$. By Dodd-Jensen, for any such s, t, u, and v, $i_u \circ i_s = i_v \circ i_t$, where these are the the iteration maps in question. Thus we have a well-defined direct limit system.

Definition 7.4 *Let (P, Σ) be an lbr hod pair; then*

(1) $\mathcal{F}(P, \Sigma)$ is the collection of all (Q, Ψ) such that there is an P-stack s by Σ with last model Q, such that P-to-Q does not drop, and $\Psi = \Sigma_s$.

(2) For $(Q, \Psi) \in \mathcal{F}(P, \Sigma)$, $\pi_{(P,\Sigma),(Q,\Psi)} \colon P \to Q$ is the unique iteration map given by any and all stacks by Σ.

(3) $M_\infty(P, \Sigma)$ is the direct limit of $\mathcal{F}(P, \Sigma)$ under the $\pi_{(Q,\Psi),(R,\Phi)}$.

(4) $\pi_{(P,\Sigma),\infty} \colon P \to M_\infty(P, \Sigma)$ is the direct limit map.

Of course, $M_\infty(P, \Sigma) = M_\infty(Q, \Psi)$ for all $(Q, \Psi) \in \mathcal{F}(P, \Sigma)$. Clearly, $(P, \Sigma) \equiv^* (Q, \Psi)$ iff $M_\infty(P, \Sigma) = M_\infty(Q, \Psi)$. Thus $M_\infty(P, \Sigma) \in \text{HOD}$, being definable from the rank of (P, Σ) in the mouse order.

Not all of Σ is actually used in forming $M_\infty(P, \Sigma)$. Let us call a normal tree $\mathcal{T}$ *relevant* iff $\mathcal{T}$ is by Σ, and there is a normal $\mathcal{S}$ by Σ such that $\mathcal{T} \subseteq \mathcal{S}$, and $\mathcal{S}$ has a last model Q, and the branch P-to-Q does not drop. Call a P-stack s relevant if for $i + 1 < \text{dom}(s)$, the branch of $\mathcal{T}_i(s)$ to $M_\infty(\mathcal{T}_i(s))$ does not drop, and for $i + 1 = \text{dom}(s)$, $\mathcal{T}_i(s)$ is relevant. Let Σ^{rel} be the restriction of Σ to relevant trees. The Σ-iterations that go into forming $M_\infty(P, \Sigma)$ are all relevant, so Σ^{rel} is what we need to construct $M_\infty(P, \Sigma)$. The author has shown in [33]:

Theorem 7.5 *Assume AD^+, and let (P, Σ) be an lbr hod pair with scope HC. Let κ be the cardinality of $o(M_\infty(P, \Sigma))$, and let Code(Σ^{rel}) be the set of reals coding stacks by Σ^{rel}; then*

(a) Code(Σ^{rel}) and its complement are κ-Suslin, and

(b) Code(Σ) is not α-Suslin, for any $\alpha < \kappa$.

In particular, κ is a Suslin cardinal.

The one can show the irrelevant part of Σ is also Suslin, but perhaps not $o(M_\infty(P, \Sigma))$-Suslin. (It is possible that $M_\infty(P, \Sigma) = P$, because there are no non-dropping iterations of P!) So one gets

Theorem 7.6 *Assume AD^+, and let (P, Σ) be an lbr hod pair with scope HC. and let Code(Σ) be the set of reals coding stacks by Σ; then Code(Σ) and its complement are Suslin.*

Note here that since Σ is total on stacks by Σ, if Code(Σ) is β-Suslin, then so is its complement.

Part (b) of the Theorem 7.5 follows at once from the Kunen-Martin theorem, and the fact that there is a wellfounded relation W on $\mathbb{R}$ of rank at least $o(M_\infty(P, \Sigma))$ such that W is arithmetic in Code(Σ). [Let $(t, b)W(s, a)$ iff s and t are stacks by Σ with last models M and N, $s \subseteq t$, P-to-N does not drop, and $i^t_{M,N}(a) > b$.] Part (a) is easy if Σ has branch condensation. We get it in the general case from some tail Σ_s fully normalizing well. See [33] for proofs of 7.5 and 7.6.

We mention Theorem 7.6 because we have many theorems under AD^+ in which the phrase "let (P, Σ) be an lbr hod pair such that Σ is Suslin and co-Suslin" occurs. Here the "co-Suslin" part is trivially redundant, and the "Suslin" part is nontrivially redundant.

Proof of Theorem 7.2. The techniques here are pretty well known. Let (M, Ψ) and g be as in the hypotheses. For $\nu < \lambda$, let

$$\Psi^g_{\langle \nu, k\rangle} = \Psi_{\langle \nu, k\rangle} \restriction \mathrm{HC}^{M(\mathbb{R}^*_g)}.$$

Fixing a coding of elements of HC by reals, we can identify $\Psi^g_{\langle \nu, k\rangle}$ with a subset of $\mathbb{R}^*_g$.

Claim 1. If $\nu < \lambda$, then $\Psi^g_{\langle \nu, k\rangle} \in \mathrm{Hom}^*_g$.

Proof. Let $h = g \cap \mathrm{Col}(\omega, <\nu^+)$. In $M[h]$ we have, for each $\mu < \lambda$, a term τ such that for all l that are $\mathrm{Col}(\omega, \mu)$-generic over $M[h]$,

$$\tau^l = \Psi_{\langle \nu, k\rangle} \restriction \mathrm{HC}^{M[h][l]}.$$

For the specific such term τ given to us by Lemma 7.3, it is not hard to see that for all sufficiently large γ,

$$M[h] \models \text{ there are club many generically } \tau\text{-correct hulls of } V_\gamma.$$

That is, in $M[h]$, whenever N is countable and transitive, and

$$\pi \colon N[h] \to (M|\gamma)[h]$$

is elementary, and everything relevant is in $\mathrm{ran}(\pi)$, and

$$\pi(\langle \bar\tau, \bar\mu\rangle) = \langle \tau, \mu\rangle,$$

then for any l that is $\mathrm{Col}(\omega, \bar\mu)$-generic over N,

$$\bar\tau^l = \Psi_{\langle \nu, k\rangle} \ \cap \mathrm{HC}^{N[l]}.$$

The proof of this is similar to the proof of Theorem 5.1 of [30]. Working in M, let $\mathbb{C}$ be the background construction and

$$i \colon M|\langle \nu, k\rangle \to M^{\mathbb{C}}_{\eta,k}$$

be the iteration map by $\Psi_{\langle \nu, k\rangle}$ that is described in τ. Let $\bar{\mathbb{C}} = \pi^{-1}(\mathbb{C})$ and $\bar{i} = \pi^{-1}(i)$, etc. So these are described in $\bar\tau$. Suppose $\mathcal{U}$ is according to $\bar\tau^l$. Let

$$\mathcal{W} = \mathrm{lift}^N(\bar{i}\mathcal{U})$$

be the plus-2 tree on N that is given to us by $\bar\tau^l$. $\mathcal{W}$ is countable and plus-2 in $N[h, l]$, so by 6.14, it picks unique cofinal wellfounded branches there. This implies that $\mathcal{W}$ is continuously illfounded off the branches it chooses. But then $\pi\mathcal{W}$ is continuously illfounded off the branches it chooses, so $\pi\mathcal{W}$ is by Ψ. But lifting commutes with copying, so

$$\begin{aligned}\pi\mathcal{W} &= \pi\,\mathrm{lift}^N(\bar{i}\mathcal{U})\\ &= \mathrm{lift}^M((\pi\circ\bar{i})\mathcal{U})\\ &= \mathrm{lift}^M(i\mathcal{U}).\end{aligned}$$

Note here that π is the identity on the base model of $\mathcal{U}$, so $\pi \circ \bar{i}$ agrees with $\pi(\bar{i}) = i$ on the base model of $\mathcal{U}$. This gives the last equality.

So $\text{lift}^M(i\mathcal{U})$ is by Ψ, and hence $i\mathcal{U}$ is by $(\Omega^{\mathbb{C}}_{\eta,k})^{h,l}$. But we saw in the proof of 7.3 that this means $i\mathcal{U}$ is by the tail strategy$(\Psi_{\langle \nu,k\rangle})_{\mathcal{T},M^{\mathbb{C}}_{\eta,k}}$, where $\mathcal{T}$ is the tree giving rise to i. Since $\Psi_{\langle \nu,k\rangle}$ is pullback consistent, $\mathcal{U}$ is by $\Psi_{\langle \nu,k\rangle}$, as desired. $\square$

Remark 7.7 We could also prove Claim 1 by quoting Theorem 7.6, and using the fact that every Suslin-co-Suslin set of $L(\text{Hom}^*_g, \mathbb{R}^*_g)$ is in Hom^*_g.

Claim 2. The $\Psi^g_{\langle \nu,k\rangle}$, for $\nu < \lambda$ are Wadge-cofinal in Hom^*_g.

Proof. Let $\eta < \lambda$ and

$$M[g\restriction\omega \times \eta] \models T \text{ and } T^* \text{ are } < \lambda\text{-absolute complements.}$$

Let $\eta < \delta < \lambda$, and $M \models \delta$ is Woodin. Let $\mu = (\delta^{++})^M$. Put $\pi \in \mathcal{I}$ iff there is a non-dropping, normal iteration tree $\mathcal{U}$ on $M|\mu$ such that

(i) $\mathcal{U}$ is by $\Psi^g_{\langle \mu,0\rangle}$, with last model N,

(ii) all critical points in $\mathcal{U}$ are strictly above η, and

(iii) $\pi\colon M[g\restriction\omega \times \eta] \to N[g\restriction\omega \times \eta]$ is the lift of the iteration map.

Standard arguments show that for $x \in \mathbb{R}^*_g$,

$$x \in p[T] \Leftrightarrow \exists \pi \in \mathcal{I}(x \in p[\pi(T \cap \omega \times \delta^{+,M})]).$$

This shows that $p[T]$ is projective in $\Psi^g_{\langle \mu,0\rangle}$. This easily implies the claim. $\square$

Working in $L(\text{Hom}^*_g, \mathbb{R}^*_g)$, we define a directed system of lbr hod pairs whose direct limit is HOD. Let us say that an lbr hod pair (P, Σ) is *full* iff Σ is Suslin-co-Suslin, and

(a) $P \models \mathsf{ZFC}^-$, P has a largest cardinal δ, and $k(P) = 0$, and

(b) whenever s is a P-stack by Σ with last model Q, and the branch P-to-Q of s does not drop, and $i_s\colon P \to Q$ is the iteration map, then there is no lbr hod pair (R, Φ) such that Φ is Suslin-co-Suslin, $Q \trianglelefteq^{\text{ct}} R$, $\rho(R) \le i_s(\delta)$, and $\Phi_{\langle o(Q),0\rangle} = \Sigma_s$.

We write δ^P for the largest cardinal of P.

Claim 3. Let η be a successor cardinal of M, and $\eta < \lambda$; then $(M|\eta, \Psi^g_{\langle \eta, 0\rangle})$ is a full lbr hod pair in $L(\mathrm{Hom}^*_g, \mathbb{R}^*_g)$.

Proof. $(M|\eta, \Psi_{\langle \eta,0\rangle})$ is an lbr hod pair in V, so $(M|\eta, \Psi^g_{\langle \eta,0\rangle})$ is an lbr hod pair in $L(\mathrm{Hom}^*_g, \mathbb{R}^*_g)$. We must see that $(M|\eta, \Psi_{\langle \eta,0\rangle})$ is full.

Suppose toward contradiction that in $L(\mathrm{Hom}^*_g, \mathbb{R}^*_g)$ we have

(i) an $M|\eta$-stack s by $\Psi_{\langle \eta,0\rangle}$ with last model Q, such that the branch $M|\eta$-to-Q of s does not drop, and

(ii) an lbr hod pair (R, Φ) such that Φ is Suslin-co-Suslin, $Q \trianglelefteq^{\mathrm{ct}} R$, $\rho(R) \leq \delta^Q$, and $\Phi_{\langle o(Q),0\rangle} = \Psi_{s,\langle o(Q),0\rangle}$.

Since η is a cardinal of M, s is in fact an M-stack, and regarding it this way, it has a last model S such that $Q \trianglelefteq S$, and the branch M-to-S of s does not drop. Since $o(Q)$ is a cardinal of S, $R \notin S$.

However, working in V now, we can find an $\mathbb{R}^*_g$-genericity iteration of $S|\lambda$ by Ψ_s so that all its critical points are strictly above $o(Q)$. Let W be the final model of this genericity iteration; then we have h being $\mathrm{Col}(\omega, <\lambda)$ generic over W so that

$$\mathbb{R}^*_h = \mathbb{R}^*_g,$$

where $\mathbb{R}^*_h$ are the reals in some $W[h \cap (\omega \times \nu)]$, for $\nu < \lambda$. Moreover, as in Claim 2, the strategies $(\Psi_s)^h_{\langle \nu,k\rangle}$ for $\nu < \lambda$ are Wadge cofinal in Hom^*_h, and clearly $(\Psi_s)^h_{\langle \nu,k\rangle} = (\Psi_s)^h_{\langle \nu,k\rangle}$. It follows that

$$\mathrm{Hom}^*_h = \mathrm{Hom}^*_g.$$

Note that R is ordinal definable in $L(\mathrm{Hom}^*_g, \mathbb{R}^*_g)$ from Q and $\Psi_{s,\langle o(Q),0\rangle}$. The following little lemma isolates the familiar reason.

Lemma 7.8 *Assume* AD^+. *Let* (P, Σ) *and* (N, Ω) *be lbr hod pairs with scope HC such that for some* (Q, Φ),

(a) $(Q, \Phi) \trianglelefteq^{\mathrm{ct}} (P, \Sigma)$ *and* $(Q, \Phi) \trianglelefteq^{\mathrm{ct}} (N, \Omega)$,

(b) P *and* N *are each* $o(Q)$*-sound, and project to* $o(Q)$*, and*

(c) $P(o(Q)) \cap N = P(o(Q)) \cap P$;

then $(P, \Sigma) = (N, \Omega)$.

Proof. If there are any counterexamples, then there is a Suslin-co-Suslin counterexample by Woodin's basis theorem. Let N^* be a coarse Γ Woodin, for a sufficiently large Γ, and let $\mathbb{C}$ be the maximal hod pair construction of N^*. Let (P, Σ) iterate to $(M^{\mathbb{C}}_{\nu,k}, \Omega^{\mathbb{C}}_{\nu,k})$ and (N, Ω) iterate to $(M^{\mathbb{C}}_{\eta,l}, \Omega^{\mathbb{C}}_{\eta,l})$, and assume without loss of generality that $\langle \nu, k\rangle \leq_{\text{lex}} \langle \eta, l\rangle$. Thus (N, Ω) iterates past $(M_{\nu,k}, \Omega_{\nu,k})$. Let $\mathcal{T}$ and $\mathcal{U}$ be the normal trees by which the two sides iterate to and past $(M_{\nu,k}, \Omega_{\nu,k})$. The branch P-to-$M_{\nu,k}$ of $\mathcal{T}$ does not drop, so $\mathcal{T}$ is a tree on Q by Φ. Let $R \trianglelefteq^{\text{ct}} M_{\nu,k}$ be the image of Q along P-to-$M_{\nu,k}$. Then $\mathcal{T}$ is the initial segment $\mathcal{U}\restriction\gamma + 1$ of $\mathcal{U}$ where extenders disagreeing with R are used, and $R = i^{\mathcal{U}}_{0,\gamma}(Q)$. Let $\text{lh}(\mathcal{U}) = \tau + 1$. Since R is a cutpoint in $\mathcal{M}^{\mathcal{U}}_{\gamma}$, $\text{crit}(\hat{\imath}^{\mathcal{U}}_{\gamma,\tau}) > o(R)$. Since $M_{\nu,k}$ is $o(R)$ sound and projects to $o(R)$, and $M_{\nu,k} \trianglelefteq \mathcal{M}^{\mathcal{U}}_{\tau}$, we must have $\gamma = \tau$. That is, $\mathcal{T} = \mathcal{U}$. Moreover, $M_{\nu,k}$ and $\mathcal{M}^{\mathcal{U}}_{\gamma}$ have the same subsets of $o(R)$ in them, because P and N had the same subsets of $o(Q)$. It follows that $M_{\nu,k} = \mathcal{M}^{\mathcal{U}}_{\gamma}$. So letting E be the branch extender of the main branch of $\mathcal{T} = \mathcal{U}$, we have that

$$\text{Ult}(P, E) = \text{Ult}(N, E).$$

It is easy then to see that $P = N$.

Since Σ and Ω are pullback consistent, we also get

$$\Sigma = (\Omega^{\mathbb{C}}_{\nu,k})^{\pi} = \Omega,$$

where π is the main branch embedding of $\mathcal{T} = \mathcal{U}$.

$\square$

Applying the lemma, we have that R is ordinal definable in $L(\text{Hom}^*_h, \mathbb{R}^*_h)$ from Q and $\Psi^h_{s,\langle o(Q),0\rangle}$. But $\Psi^h_{s,\langle o(Q),0\rangle}$ is definable over $W(\mathbb{R}^*_h)$ from parameters in W, by 7.3. By homogeneity, $R \in W$, and thus $R \in S$, a contradiction.

$\square$

We define in $L(\text{Hom}^*_g, \mathbb{R}^*_g)$: for $(P, \Sigma), (Q, \Psi) \in \mathcal{F}$,

$$(P, \Sigma) \prec^* (Q, \Psi) \text{ iff } \exists (R, \Phi)[(R, \Phi) \trianglelefteq^{\text{ct}} (Q, \Psi) \wedge (P, \Sigma) \text{ iterates to } (R, \Phi)].$$

If $(P, \Sigma) \prec^* (Q, \Psi)$, then

$$\pi_{(P,\Sigma),(Q,\Psi)} \colon P \to R \trianglelefteq^{\text{ct}} Q$$

is the iteration map. By Dodd-Jensen, it is well-defined, that is, independent of the choice of stack witnessing that (P, Σ) iterates to some $(R, \Phi) \trianglelefteq^{\text{ct}} (Q, \Psi)$. The π's commute, and $\prec^*$ is directed, so we have a direct limit system. Set

$$M_\infty = \text{ direct limit of } (\mathcal{F}, \prec^*) \text{ under the } \pi_{(P,\Sigma),(Q,\Psi)},$$

and let

$$\pi_{(P,\Sigma),\infty}\colon P \to M_\infty$$

be the direct limit map. Another way to characterize M_∞ is that it is the lpm N of minimal height such that for all $(P,\Sigma) \in \mathcal{F}$, $M_\infty(P,\Sigma) \trianglelefteq^{\mathrm{ct}} M_\infty$. Our two definitions of $\pi_{(P,\Sigma),\infty}$ are consistent with one another.

Let us write

$$\Theta = \theta^{L(\mathrm{Hom}^*_g,\mathbb{R}^*_g)},$$

and

$$\mathrm{HOD} = \mathrm{HOD}^{L(\mathrm{Hom}^*_g,\mathbb{R}^*_g)}.$$

Claim 4. $M_\infty \subseteq \mathrm{HOD}|\Theta$.

Proof. This is easy. $\square$

For the reverse inclusion, we show first

Claim 5. $\Theta = o(M_\infty)$.

Proof. We must show that $\Theta \le o(M_\infty)$. This is easy to do if we appeal to 7.6. For let $\alpha < \Theta$, and let $B \in \mathrm{Hom}^*_g$ be a prewellorder of length $(\alpha^+)^{L(\mathrm{Hom}^*_g,\mathbb{R}^*_g)}$. By the proofs of claims 2 and 3, there is a $(P,\Sigma) \in \mathcal{F}$ such that B is Wadge reducible to $\mathrm{Code}(\Sigma^{\mathrm{rel}})$. By 7.5, $\mathrm{Code}(\Sigma^{\mathrm{rel}})$ is κ-Suslin in $L(\mathrm{Hom}^*_g,\mathbb{R}^*_g)$, where $\kappa = |o(M_\infty(P,\Sigma))|$. So B is κ-Suslin, so $\alpha^+ \le \kappa$ by Kunen-Martin. So $\alpha < o(M_\infty)$.

We now give a proof that avoids 7.5. Let η be a cardinal of M, and let

$$B = \Psi^g_{\langle \eta, 0\rangle}.$$

Let $M_0 = \mathrm{Ult}(M,E)$, for E the first extender on M overlapping η, if there is one. Let $M_0 = M$ otherwise. Let

$$\delta_0 = \text{ least } \delta > \eta \text{ such that } M_0 \models \delta \text{ is Woodin.}$$

So η and δ_0 are cutpoints of M_0. Letting $N = M_0|(\delta_0^+)^{M_0}$ and $\Phi = \Psi_{\langle E\rangle,N}$ or $\Phi = \Psi_N$ as appropriate, we have that $(N,\Phi) \in \mathcal{F}$. We shall show that

$$\pi_{(N,\Phi),\infty}(\delta_0) \ge \Theta(B).$$

Here $\Theta(B)$ is the sup of the lengths of $\mathrm{OD}(B)$ prewellorders of $\mathbb{R}$, in $L(\mathrm{Hom}^*_g,\mathbb{R}^*_g)$ of course.

Remark 7.9 We believe that a little more work shows that $\pi_{(N,\Phi),\infty}(\delta_0) = \Theta(B)$.

To see this, let $f\colon \mathbb{R}^*_g \to \tau$ be a surjection, and

$$f(x) = \xi \text{ iff } L_\alpha(\mathrm{Hom}^*_g, \mathbb{R}^*_g) \models \varphi[x, \xi, B].$$

We must show that $\tau \leq \pi_{(N,\Phi),\infty}(\delta_0)$. It is more convenient here to consider the relativised direct limit system $\mathcal{F}^\eta(N,\Phi)$, in which all iterations must be strictly above η. It is not hard to see that $\mathcal{F}^\eta(N,\Phi)$ is directed. Let $M^\eta_\infty(N,\Phi)$ be its direct limit, and $\pi^\eta_{(N,\Phi),\infty}$ be the direct limit map. We shall show

$$\tau \leq \pi^\eta_{(N,\Phi),\infty}(\delta_0).$$

Since $\mathcal{F}^\eta(N,\Phi)$ is a subsystem of the full $\mathcal{F}(N,\Phi)$, this is enough.

Working in V, let

$$\mathbb{R}^*_g = \{x_i \mid i < \omega\},$$

and let s be a run of $G^+(N,\omega,\omega_1)$ by Φ that is cofinal in $\mathcal{F}^\eta(N,\Phi)$, so that

$$N_\omega = M^\eta_\infty(N,\Phi),$$

where N_ω is the direct limit along s, and $i^s_{0,\omega} = \pi^\eta_{(N,\Phi),\infty}$. Let $N_0 = N$, and N_k be the last model of $s{\restriction}k$, for $k > 0$. Let $\delta_k = i^s_{0,k}(\delta_0)$. We can arrange that whenever $i < k$, then $x_i \in N_k[H]$, for some H that is generic over N_k for the extender algebra at δ_k.

We have $N_0 \trianglelefteq^{\mathrm{ct}} M_0$. The stack s is according to Ψ_{M_0}, so thinking of s as a stack on M_0, and letting M_k be the last model of $s{\restriction}k$ in this context, we have

$$N_k \trianglelefteq^{\mathrm{ct}} M_k,$$

and

$$i_{k,l}\colon M_k \to M_l$$

the iteration map given by s, for $k, l \leq \omega$.

Now we do the usual dovetailed $\mathbb{R}^*_g$- genericity iterations, iterating each $(M_k, \Psi_{s{\restriction}k, M_k})$, strictly above δ_k to (Q_k, Ω_k), and arranging that $L(\mathrm{Hom}^*_g, \mathbb{R}^*_g)$ is also a derived model of Q_k. Let

$$j_k\colon M_k \to Q_k$$

be the map of the $\mathbb{R}^*_g$ genericity iteration, and let

$$\sigma_{k,l}\colon Q_k \to Q_l$$

be the copy map, which exists because we dovetailed the genericity iterations together. (See for example the proof of Theorem 6.29 of [38] for the details of this well-known construction.) Here is a diagram.

$$\begin{tikzcd}
Q_0 \arrow[r, "\sigma_{0,k}"] & Q_k \arrow[r, "\sigma_{k,\omega}"] & Q_\omega \\
M_0 \arrow[u, "j_0"] \arrow[r, "i_{0,k}"] & M_k \arrow[u, "j_k"] \arrow[r, "i_{k,\omega}"] & M_\omega \arrow[u, "j_\omega"] \\
N_0 \arrow[u, "\mathrm{id}"] \arrow[r, "i_{0,k}"] & N_k \arrow[u, "\mathrm{id}"] \arrow[r, "i_{k,\omega}"] & N_\omega \arrow[u, "\mathrm{id}"]
\end{tikzcd}$$

We have for each $k < \omega$ a Q_k-generic h_k such that $\mathbb{R}^*_{h_k} = \mathbb{R}^*_g$ and $\mathrm{Hom}^*_{h_k} = \mathrm{Hom}^*_g$. The latter holds because for each $\xi < \lambda$, the critical points in j_k are eventually above $j_k(\xi)$, and the initial segment of the iteration that gets us to this point acts only on some $M|\gamma$ for $\gamma < \lambda$. This tells us that $(\Omega_k)^{h_k}_{\langle j_k(\xi),0\rangle}$ is projective in $\Psi^g_{\langle\gamma,0\rangle}$. That implies $\mathrm{Hom}^*_{h_k} \subseteq \mathrm{Hom}^*_g$. The reverse inclusion comes from the fact that each $\Psi_{\langle\gamma,0\rangle}$ is a pullback of some $\Omega_{\langle\xi,0\rangle}$.

Note that we have for each $k < \omega$ a term $\dot{B}_k \in Q_k$ such that

$$\dot{B}_k^{Q_k[l]} = B$$

for all l that are $\mathrm{Col}(\omega, <\lambda)$ generic over Q_k and such that $\mathbb{R}^*_l = \mathbb{R}^*_g$. Moreover,

$$\sigma_{k,n}(\dot{B}_k) = \dot{B}_n$$

for $k < n < \omega$. This follows from 7.3, the fact that all embeddings in the diagram above have critical point $> \eta$, and strategy coherence. Let $\mathbb{W}_k$ be the extender algebra of Q_k at δ_k, and put

$$\begin{aligned}\xi \in Y_k \text{ iff } Q_k \models \exists b \in \mathbb{W}_k[b \Vdash (\mathrm{Col}(\omega, <\lambda) \Vdash \\ \check{\xi} \text{ is the least } \gamma \text{ such that } L_{\check{\alpha}}(\mathrm{Hom}^*_{\dot{G}}, \mathbb{R}^*_{\dot{G}}) \models \varphi[\dot{x}, \gamma, \dot{B}_k])]\end{aligned}$$

Because $\mathbb{W}_k$ has the δ_k-chain condition in Q_k,

$$Q_k \models |Y_k| < \delta_k.$$

Now we define an order preserving map

$$p\colon \tau \to \pi^\eta_{(N,\Phi),\infty}(\delta_0) = i_{0,\omega}(\delta_0).$$

Let $\xi < \tau$, and pick any x such that $f(x) = \xi$. Let $k < \omega$ be sufficiently large that

(i) $x = x_i$ for some $i < k$, and

(ii) for $k \leq m \leq n < \omega$, $\sigma_{m,n}(\alpha) = \alpha$ and $\sigma_{m,n}(\xi) = \xi$.

Since Q_ω is wellfounded, we can find such a k. By (i), x is $\mathbb{W}_k$-generic over Q_k. It follows that $\xi \in Y_k$; say that

$$\xi = \text{ the } \gamma\text{-th element of } Y_k$$

in its increasing enumeration. We then set

$$p(\xi) = i_{k,\omega}(\gamma) = \sigma_{k,\omega}(\gamma).$$

We must check that $p(\xi)$ is independent of the choice of x, and that p is order preserving. For this, let $f(y) = \tau$. Let $k_{x,\xi}$ and $k_{y,\tau}$ be as in (i) and (ii) above, for (x, ξ) and (y, τ) respectively. Let $\gamma_{x,\xi}$ and $\gamma_{y,\tau}$ be the corresponding γ's. Taking $n \geq \max(k_{x,\xi}, k_{y,\tau})$, we have $\xi, \tau \in Y_n$, and

$$\xi = \text{ the } \sigma_{k_{x,\xi},n}(\gamma_{x,\xi})\text{-th element of } Y_n.$$

This is because $\sigma_{k_{x,\xi},n}(\xi) = \xi$. Similarly,

$$\tau = \text{ the } \sigma_{k_{y,\tau},n}(\gamma_{y,\tau})\text{-th element of } Y_n.$$

So

$$\begin{aligned} \xi \leq \tau \text{ iff } & i_{k_{x,\xi},n}(\gamma_{x,\xi}) \leq i_{k_{y,\tau},n}(\gamma_{y,\tau}) \\ \text{iff } & i_{k_{x,\xi},\omega}(\gamma_{x,\xi}) \leq i_{k_{y,\tau},\omega}(\gamma_{y,\tau}), \end{aligned}$$

as desired. This proves Claim 5. □

Claim 6. $\mathrm{HOD}|\Theta \subseteq M_\infty$.

Proof. The proof uses ideas from the proof of claim 5. Let A be a bounded subset of Θ, and

$$\xi \in A \text{ iff } L_\alpha(\mathrm{Hom}^*_g, \mathbb{R}^*_g) \models \varphi[\xi].$$

Let (N, Ψ^g_N) be an initial segment of M such that $(N, \Psi^g_N) \in \mathcal{F}$, and setting $\Phi = \Psi^g_N$, we have

$$A \subseteq \pi_{(N,\Phi),\infty}(\delta_0),$$

for some δ_0 that we now fix. Let $M_0 = \mathrm{Ult}(M, E)$, where E is the first extender on the M-sequence overlapping $o(N)$, if there is one. Otherwise, let $M_0 = M$. Let s be a generic run of $G^+(N, \omega, \omega_1)$ by Φ, so that

$$N_\omega = M^\eta_\infty(N, \Phi),$$

where N_ω is the direct limit along s, and $i^s_{0,\omega} = \pi^\eta_{(N,\Phi),\infty}$. Let $N_0 = N$, and N_k be the last model of $s \restriction k$, for $k > 0$. Let $\delta_k = i^s_{0,k}(\delta_0)$. Thinking of s as a stack on M_0, and letting M_k be the last model of $s \restriction k$ in this context, we have

$$N_k \trianglelefteq^{\mathrm{ct}} M_k,$$

and

$$i_{k,l} \colon M_k \to M_l$$

the iteration map given by s, for $k, l \leq \omega$.

Again, we do the usual dovetailed $\mathbb{R}^*_g$- genericity iterations, iterating each $(M_k, \Psi_{s \restriction k, M_k})$, strictly above δ_k to (Q_k, Ω_k), and arranging that $L(\mathrm{Hom}^*_g, \mathbb{R}^*_g)$ is also a derived model of Q_k. Let

$$j_k \colon M_k \to Q_k$$

be the map of the $\mathbb{R}^*_g$ genericity iteration, and let

$$\sigma_{k,l} \colon Q_k \to Q_l$$

be the copy map. The diagram in the proof of claim 5 applies to our current situation.

Now let

$$\begin{aligned} \xi \in A_k \text{ iff } & i_{k,\omega}(\xi) \in A\} \\ \text{iff } & L_\alpha(\mathrm{Hom}^*_g, \mathbb{R}^*_g) \models \varphi[\pi_{(N_k,(\Omega_k)_{N_k}),\infty}(\xi)]. \end{aligned}$$

A_k can be defined over Q_k from α and N_k, because the forcing leading from Q_k to its derived model $L(\mathrm{Hom}^*_g, \mathbb{R}^*_g)$ is homogeneous. So $A_k \in Q_k$, so $A_k \in N_k$. Let l be large enough that $\sigma_{m,n}(\alpha = \alpha$ whenever $l \leq m \leq n < \Omega$. Then for $l \leq m \leq n < \omega$,

$$i_{m,n}(A_m) = A_n.$$

It follows that

$$A = i_{l,\omega}(A_l) = \pi_{(N_l,(\Omega_l)_{N_l}),\infty}(A_l).$$

So $A \in M_\infty$, as desired. $\square$

Claim 6 nearly finishes the proof of Theorem 7.2. What is left is to identify M_∞ as the iterate of M described in clause (a) if λ is a limit of cutpoints of M, and in clause (b) otherwise. This is easy, and we leave it to the reader. $\square$

Proof of Theorem 7.1

Under the hypotheses of 7.1, we have shown that there is an lbr hod pair (M, Ψ) with scope HC such that for some λ, $M \models$ "λ is a limit of cutpoint Woodins, and there is a superstrong $< \lambda$." Moreover, we have that $\mathrm{Code}(\Psi)$ is Hom_∞. So we can apply 7.2, and we get that the HOD of the derived model $D(M, < \lambda)$ is an iterate of M, and satisfies "there is a superstrong cardinal". But then via an $\mathbb{R}$-genericity iteration M-to-M^*, we can realize $D(M^*, < \lambda)$ as $L(\Gamma, \mathbb{R})$, for some $\Gamma \subsetneq \mathrm{Hom}_\infty$. This proves the theorem. $\square$

With more work, this HOD-computation can be localized. That is, assuming the hypotheses of Theorem 7.1, and letting $\Gamma \subseteq \mathrm{Hom}_\infty$ be the pointclass witnessing its conclusion, then whenever $\Gamma_0 \subsetneq \Gamma$, then

$$L(\Gamma_0, \mathbb{R}) \models V_\Theta^{\mathrm{HOD}} \text{ is the universe of an lpm.}$$

This is proved in [33]. What we have done in the present section is analogous, in the pure extender model case, to Theorem 5.1 of [30], according to which the Mouse Set Conjecture holds in the derived model of a mouse (provided the mouse has some natural closure properties). The work of [33] is parallel to Theorem 16.1 of [30], according to which the Mouse Set Conjecture implies its local versions.

It is also interesting to see what strong determinacy theories are true in the derived models of lbr hod pairs (P, Σ) such that P reaches reasonably large cardinals. There are some results in this direction in [35].

References

[1] A. Andretta, I. Neeman, and J. Steel. The domestic levels of K^c are iterable. *Israel J. Math.*, 125:157–201, 2001.

[2] S. Baldwin. Between strong and superstrong *J. Symbolic Logic*, 51(3):547–559 (1986).

[3] G. Fuchs. λ-structures and s-structures: translating the models. *Annals of Pure and Applied Logic*, 162(4):257–317 (2011).

[4] G. Fuchs, λ-structures and s-structures: translating the iteration strategies. *Annals of Pure and Applied Logic*, 162(9):710–751 (2011).

[5] G. Fuchs, I. Neeman, and R. Schindler. A criterion for coarse iterability. *Arch. Math. Logic*, 49(4):447–467, 2010.

[6] P. Koellner and W. H. Woodin Large cardinals from determinacy. in Handbook of Set Theory, M. Foreman and A. Kanamori eds., Springer-Verlag, v. 3, pp. 1951–2120 (2010).

[7] K. Kunen. Some applications of iterated ultrapowers in set theory. *Annals of Math. Logic*, 1:179–227 (1970).

[8] D. A. Martin and J. R. Steel. Iteration trees. *J. Amer. Math. Soc.*, 7(1):1–73, 1994.

[9] W. J. Mitchell. Embeddings of iteration trees. Unpublished.

[10] W. J. Mitchell and John R. Steel. Fine structure and iteration trees. *Lecture notes in Logic*, vol. 3, Springer-Verlag, Berlin, 1994, 130pp.

[11] I. Neeman. Inner models in the region of a Woodin limit of Woodin cardinals. *Annals of Pure and Applied Logic*, 116(1-3):67–155, 2002.

[12] I. Neeman. Determinacy in $L(\mathbb{R})$. In *Handbook of set theory. Vols. 1, 2, 3*, pages 1877–1950. Springer, Dordrecht, 2010.

[13] I. Neeman and J. R. Steel A weak Dodd-Jensen lemma *J. Symbolic Logic* 64 (3), 1285-1294, 1999.

[14] I. Neeman and J. R. Steel. Counterexamples to the unique and cofinal branches hypotheses. *J. Symbolic Logic* 71 (3) 977–988, 2006.

[15] I. Neeman and J. R. Steel. Equiconsistencies at subcompacts. Archive for Math. Logic, v. 55 (2016), pp. 207-238.

[16] G. Sargsyan. A tale of hybrid mice. ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–University of California, Berkeley.

[17] G. Sargsyan Hod mice and the mouse set conjecture Memoirs of the Amer. Math. Soc., vol. 236, no. 1111 (2015)

[18] G. Sargsyan and N. Trang. The largest Suslin axiom. Preprint.

[19] E. Schimmerling and M. Zeman. Characterization of $\square_\kappa$ in core models, J. of Mathematical Logic, 4(01): pp. 1–72 (2004).

[20] E. Schimmerling and J. R. Steel. The maximality of the core model. *Trans. Amer. Math. Soc.*, 351(8):3119–3141, 1999.

[21] E. Schimmerling and J. R. Steel. Fine structure for tame inner models. *J. Symbolic Logic* 61 (2), 1996.

[22] R. Schindler and J. Steel. The self-iterability of $L[\vec{E}]$. *J. Symbolic Logic* 74 (2009) pp. 751-779.

[23] R. Schindler and M. Zeman. Fine structure. *Handbook of Set Theory*, M. Foreman. A. Kanamori eds., volume 1.

[24] R. Schindler, J. Steel, and M. Zeman. Deconstructing inner model theory *J. Symbolic Logic*, 67, June 2002, pp. 721-736.

[25] F. Schlutzenberg Reconstructing copying and condensation Preprint, 2015, available at https://sites.google.com/site/schlutzenberg/home

[26] F. Schlutzenberg. Iterability for stacks. Preprint, August 2015.

[27] F. Schlutzenberg and J. R. Steel Normalizing iteration trees In preparation.

[28] F. Schlutzenberg and N. Trang Scales in hybrid mice preprint 2016, available at math.uci.edu/∼ntrang

[29] J. R. Steel An outline of inner model theory In *Handbook of Set Theory*, vol. 3, M. Foreman and A. Kanamori eds., North Holland, pp. 580-665.

[30] J. R. Steel. Derived models associated to mice. In *Computational prospects of infinity. Part I. Tutorials*, volume 14 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 105–193. World Sci. Publ., Hackensack, NJ, 2008.

[31] J. R. Steel Local K^c constructions. *J. Symbolic Logic*, 72, Sept. 2007, 721-737.

[32] J. R. Steel Hod mice below LST^- Handwritten notes October 2013, available at www.math.berkeley.edu/∼steel

[33] J. R. Steel Local HOD computation. Handwritten notes July 2016, available at www.math.berkeley.edu/∼steel.

[34] J. R. Steel. Hod pair capturing and short tree strategies. Handwritten notes October 2016, available at www.math.berkeley.edu/∼steel.

[35] J. R. Steel LSA from least branch hod pairs. Handwritten notes July 2016, available at www.math.berkeley.edu/∼steel.

[36] J. R. Steel A theorem of Woodin on mouse sets In *Ordinal definability and recursion theory: the Cabal Seminar, volume III*, A.S Kechris et. al. editors, volume 43 of *Lecture Notes in Logic*, ASL (2016) pp. 243-256.

[37] J. R. Steel and N. Trang, Square in least branch hod mice In preparation.

[38] J. R. Steel and W. H. Woodin HOD as a core model In *Ordinal definability and recursion theory: the Cabal Seminar, volume III*, A.S. Kechris et. al. editors, volume 43 of *Lecture Notes in Logic*, ASL (2016) pp. 257-348.

[39] W. H. Woodin. Suitable extender models I. *Journal of Mathematical Logic*, 10(1-2):101–339, 2010.

[40] M. Zeman Inner models with large cardinals De Gruyter, 2001.

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