INNER MODELS WITH WELLORDERS IN $L(\mathbb{R})$

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0. Introduction

These notes contain some results which should eventually be part of a proof of the following conjecture.

Conjecture 0.1. Assume $\mathrm{AD}^{L(\mathbb{R})}$. Let $\mathcal{M}$ be an iterable, ω-small, countable premouse. Let $A = \mathbb{R} \cap \mathcal{M}$. Then there are α, $\bar{\alpha}$, n, and j such that

(i) $A = \mathbb{R} \cap J_{\bar{\alpha}}(A)$,
(ii) $j : J_{\bar{\alpha}}(A) \prec_{\Sigma_n} J_\alpha(\mathbb{R})$, and
(iii) there is a wellorder of A that is Δ_{n+1} definable over $J_{\bar{\alpha}}(A)$ from a real parameter in A.

The conjecture breaks naturally into cases depending on the least inner model operator under which A is not closed. We have proved the conjecture in some of the cases. In other cases we have only proved a weaker form of the conjecture in which (ii) and (iii) are replaced by:

(ii) $j : J_{\bar{\alpha}}(A) \prec_{\Sigma_1} J_\alpha(\mathbb{R})$, and
(iii) there is a wellorder of A which is definable over $J_{\bar{\alpha}}(A)$ from a real parameter in A.

Our results on this weaker conjecture are strong enough to show that it is true for all $\alpha \leq \kappa_{\mathbb{R}}$, where $\kappa_{\mathbb{R}}$ is the least $\mathbb{R}$-admissible ordinal.

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1. Ordinal definable reals in $L(\mathbb{R})$.

Throughout the rest of these notes we work in the theory ZF + DC + AD + $V = L(\mathbb{R})$. We assume the reader is familiar with the first three sections of [6], and we will use the notation and terminology of that paper.

Definition 1.1. If $x \in \mathbb{R}$, $1 \le n < \omega$, and $\alpha \ge 1$ then we define $\mathrm{OD}_n^\alpha(x) = \{y \in \mathbb{R} \mid \exists \varphi, \xi$ s.t. φ is a Σ_n formula and ξ is a countable ordinal and for all $w \in \mathrm{WO}$ coding ξ, y is the unique real such that $J_\alpha(\mathbb{R}) \models \varphi[x, y, w, \mathbb{R}]\}$. We also set $\mathrm{OD}^\alpha(x) = \bigcup_n \mathrm{OD}_n^\alpha(x)$. Also, if $\alpha \ge 2$ we set $\mathrm{OD}^{<\alpha}(x) = \bigcup_{\beta<\alpha} \mathrm{OD}^\beta(x)$.

Notice that $\mathrm{OD}^{<\alpha} = \mathrm{OD}_1^\alpha$.

We can also extend the notion of $\mathrm{OD}^{<\alpha}(X)$ to countable transitive sets X. If X is a countable transitive set and $f : \omega \xrightarrow[\text{onto}]{\text{1-1}} X$, we let $x_f(\langle m, n\rangle) = 0$ iff $f(m) \in f(n)$, and call x_f a real coding X. For $B \subseteq X$ and $x = x_f$, we let $B_x(n) = 0$ iff $f(n) \in B$, and say that B_x codes B relative to x.

Definition 1.2. Let X be a countable transitive set, $1 \le n < \omega$, and $\alpha \ge 1$. Then $\mathrm{OD}_n^\alpha(X) = \{B \subseteq X \mid \exists \varphi, \xi$ s.t. φ is a Σ_n formula and ξ is a countable ordinal and for all x coding X and all w coding ξ, B_x is the unique real such that $J_\alpha(\mathbb{R}) \models \varphi[x, B_x, w, \mathbb{R}]\}$. Again, we set $\mathrm{OD}^\alpha(X) = \bigcup_n \mathrm{OD}_n^\alpha(X)$, and if $\alpha \ge 2$ we set $\mathrm{OD}^{<\alpha}(X) = \bigcup_{\beta<\alpha} \mathrm{OD}^\beta(X)$.

If $\alpha \ge \omega_1$ then every hereditarily countable set is in $J_\alpha(\mathbb{R})$ and so $\mathrm{OD}^\alpha(X)$ has a simpler description.

Lemma 1.3. *Suppose X is a countable, transitive set and $\alpha \ge \omega_1$. Then $\mathrm{OD}^\alpha(X) = \{B \subseteq X \mid \exists \varphi, \xi_1, \ldots, \xi_k$ s.t. φ is a formula and $\xi_1, \ldots, \xi_k \in \mathrm{OR} \cap J_\alpha(\mathbb{R})$ and B is the unique set in $J_\alpha(\mathbb{R})$ such that $J_\alpha(\mathbb{R}) \models \varphi[X, B, \xi_1, \ldots, \xi_k, \mathbb{R}]\}$.*

We omit the easy proof of this lemma.

As we mentioned above, to tackle Conjecture 0.1, we break into cases depending on the closure properties of $\mathbb{R} \cap \mathcal{M}$. We know give some definitions which will make this precise.

Definition 1.4. If $\mathcal{M}$ is a premouse we define $\omega_1^{\mathcal{M}}$ to be $\sup\left\{ \xi \in \mathrm{OR}^{\mathcal{M}} \;\middle|\; \rho_\omega(\mathcal{J}_\xi^{\mathcal{M}}) = \omega \right\}$. Now let $\alpha \ge 1$ and $1 \le n < \omega$. We will say that $\mathcal{M}$ is *closed under* OD_n^α iff $\forall x \in \mathbb{R} \cap \mathcal{M}$, $\mathrm{OD}_n^\alpha(x) \subseteq \mathbb{R} \cap \mathcal{M}$. We will say that $\mathcal{M}$ is *strongly closed under* OD_n^α iff $\forall x \in \mathbb{R} \cap \mathcal{M}\ \exists \eta < \omega_1^{\mathcal{M}}\ \mathrm{OD}_n^\alpha(x) \subseteq \mathcal{J}_\eta^{\mathcal{M}}$. Finally we define the *degree of strong closure of* $\mathcal{M}$ to be the lexicographically least pair (α, n) such that $\mathcal{M}$ is not strongly closed under OD_n^α, if there is such a pair.

If $\mathcal{M}$ is as in the hypothesis of Conjecture 0.1, then the degree of strong closure of $\mathcal{M}$ does exist. Let (α, n) be the degree of strong closure of $\mathcal{M}$. We want to consider what type of ordinal α can be. Notice that there is a real $x \in \mathcal{M}$ such that $\mathrm{OD}^\alpha(x) \ne \mathrm{OD}^{<\alpha}(x)$. By the proofs of Theorems 2.7, and 2.9 from [6], there are only two possibilities for α. Either α begins a Σ_1-gap and α is not admissible, or α ends a non-trivial Σ_1-gap.

The proof of Theorem 3.3 from [6] shows that if α ends a *strong* gap, then for a cone of reals x, $\mathrm{OD}^\alpha(x) = \mathrm{OD}^{<\alpha}(x)$. But it is not obvious that there is such an $x \in \mathcal{M}$, and so we are left with the possibility that (α, n) is the degree of strong closure of $\mathcal{M}$ and α ends a strong gap. Luckily, a recent result of Woodin's rules this out.

Lemma 1.5 (Woodin). *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Suppose* α *ends a strong* Σ_1*-gap in* $L(\mathbb{R})$. *Then* $\mathrm{OD}^{\alpha} = \mathrm{OD}^{<\alpha}$.

Definition 1.6. We will say that α is *projective-like* iff α begins a Σ_1 gap and α is not admissible.

Note that if α is projective-like then $[\alpha, \alpha]$ is a Σ_1-gap.

We will break up the projective-like case into 6 different cases, and then with the end-of-a-weak-gap case we will have 7 different cases.

Definition 1.7. Let $\alpha \geq 1$ be an ordinal. We will say that α is

- Type 1 if α is projective-like and $\mathrm{cof}(\alpha) > \omega$.
- Type 2 if α is projective-like and $\mathrm{cof}(\alpha) = \omega$.
- Type 3 if α is projever/e-like and $\alpha = \gamma + 1$ is a successor ordinal and γ is projective-like.
- Type 4 if α is projective-like and $\alpha = \gamma + 1$ is a successor ordinal and $[\gamma, \gamma]$ is a Σ_1-gap and γ is admissible.
- Type 5 if α is projective-like and $\alpha = \gamma + 1$ is a successor ordinal and γ ends a strong gap.
- Type 6 if α is projective-like and $\alpha = \gamma + 1$ is a successor ordinal and γ ends a weak gap.
- Type 7 if α ends a weak gap.

Definition 1.8. Let $\mathcal{M}$ be a premouse. Suppose the degree of strong closure of $\mathcal{M}$ is defined and let (α, n) be the degree of strong closure of $\mathcal{M}$. For $1 \leq k \leq 7$ we will say that $\mathcal{M}$ is Type k iff α is Type k.

Now we can say more precisely how much of Conjecture 0.1 is known. We have proved the conjecture for $\mathcal{M}$ of Type 1. We give that proof in these notes. We have a proof of the weakened form of the conjecture (see the Introduction) for $\mathcal{M}$ of Types 2, 3, and 4, but the details have not yet been written down. Another draft of these notes in the near future will include these additional proofs. Together, types 1,2, and 3 cover all $\alpha \leq \kappa_{\mathbb{R}}$. For $\mathcal{M}$ of types 5, 6, and 7 the conjecture is still open.

Let (α, n) be the degree of strong closure of $\mathcal{M}$. Let us discuss the special case of the conjecture in which $n = 1$. It is not hard to see in this case that $\mathbb{R} \cap \mathcal{M} = \mathrm{OD}^{<\alpha}$ and $\mathrm{OD}^{<\alpha} \neq \mathrm{OD}^{<\beta}$ for any $\beta < \alpha$. It follows that α is a successor ordinal, or a limit ordinal of cofinality ω. Also α cannot be of Types 4, 5, or 7. So α must be of Type 2, 3, or 6. Now there is a $\Sigma_1(J_\alpha(\mathbb{R}))$ wellordering of $\mathrm{OD}^{<\alpha}$. So Conjecture 0.1 in this case boils down to showing that if $A = \mathrm{OD}^{<\alpha}$, then there are $\bar{\alpha}$ and j such that $A = \mathbb{R} \cap J_{\bar{\alpha}}(A)$ and $j : J_{\bar{\alpha}}(A) \prec_{\Sigma_1} J_\alpha(\mathbb{R})$. This is easy to see if α is of Type 2. So suppose that α is of Type 3 or 6. Then $\alpha = \gamma + 1$ and $A = \mathrm{OD}^{\gamma}$ and we must see that there are $\bar{\gamma}$ and j such that $A = \mathbb{R} \cap J_{\bar{\gamma}}(A)$ and $j : J_{\bar{\gamma}}(A) \prec_{\Sigma_\omega} J_\gamma(\mathbb{R})$. This is non-trivial. We believe we have a proof in the case that γ is projective-like, that is, α is of Type 3. This proof will also appear in a future draft of these notes. In the case that γ ends a weak gap, that is α is of Type 6, this is still open.

2. Woodin's mouse set theorem.

Closely connected to Conjecture 0.1, is another conjecture about mouse sets. Below we state a tentative version of the mouse set conjecture in which we only mention the projective-like case. It would probably be reasonable to include also the case that α ends a weak gap.

Conjecture 2.1. Assume $\mathrm{AD}^{L(\mathbb{R})}$. Let α be projective-like, let $n \geq 1$. Then OD^{α}_n is a mouse set.

In these notes we will prove Conjecture 2.1 for α of Type 1.

A related mouse set fact has been proven by Woodin.

Theorem 2.2 (Woodin). *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Let* $\alpha \geq 2$. *Then* $\mathrm{OD}^{<\alpha}$ *is a mouse set.*

Another way to phrase this theorem is that for $\alpha \geq 1$, OD^{α} is a mouse set. Of course this only has content if α is projective-like or α ends a weak gap.

See [8] for a proof of Theorem 2.2. Our proofs in these notes depend heavily on a corollary of the proof in [8]. We turn to this know.

Remark 2.3. Assume $\mathrm{AD}^{L(\mathbb{R})}$. Then every iteration tree in $L(\mathbb{R})$ of length ω_1 has a cofinal branch. This is because ω_1 is measurable in $L(\mathbb{R})$.

Recall the following definitions from [7].

Definition 2.4. Let $\mathcal{M}$ be a premouse and $\delta < \mathrm{OR}^{\mathcal{M}}$.

(a) $k(\mathcal{M}, \delta)$ is the unique $k < \omega$ such that $\mathcal{M}$ is k-sound, $k+1$ solid, and $\rho_{k+1}(\mathcal{M}) \leq \delta < \rho_k(\mathcal{M})$, if such a k exists, and $k(\mathcal{M}, \delta)$ is undefined otherwise.

(b) $\mathcal{M}$ is a *δ-mouse* iff $k(\mathcal{M}, \delta)$ is defined, and letting $k = k(\mathcal{M}, \delta)$, $\mathcal{M} = H^{\mathcal{M}}_{k+1}(\delta \cup \{p_{k+1}(\mathcal{M})\})$.

(c) A putative iteration tree $\mathcal{T}$ on $\mathcal{M}$ is *above δ* iff $\mathrm{crit}(E^{\mathcal{T}}_{\alpha}) \geq \delta$ for all $\alpha + 1 < \mathrm{lh}(\mathcal{T})$.

(d) δ is a cutpoint of $\mathcal{M}$ iff for no E on the $\mathcal{M}$-sequence do we have that $\mathrm{crit}(E) < \delta \leq \mathrm{lh}(E)$.

We will use the notion of "(k, α)-iterable" from [9].

Definition 2.5. If $\mathcal{M}$ is a δ-mouse where δ is a cutpoint of $\mathcal{M}$ then by an "*above-δ α-iteration strategy for* $\mathcal{M}$" we shall mean an above-δ (ω, α)-iteration strategy for $\mathcal{M}$. Similarly for the phrase "$\mathcal{M}$ is ω_1-iterable above δ."

Remark 2.6. Assume $\mathrm{AD}^{L(\mathbb{R})}$. Suppose $\mathcal{M}$ is a countable δ-mouse, where δ is a cutpoint of $\mathcal{M}$, $\mathcal{M}$ is ω-small above δ, and $\mathcal{M}$ is ω_1-iterable above δ. Then $\mathcal{M}$ has a unique above-δ ω_1-iteration strategy (namely the strategy of picking the unique $\Game^{\mathbb{R}} - \Pi^1_1$ iterable branch) and this strategy is in $L(\mathbb{R})$.

Definition 2.7. Let $\mathcal{M}$ be a countable δ-mouse, where δ is a cutpoint of $\mathcal{M}$, and $\mathcal{M}$ is ω-small above δ. Let $\alpha \geq 2$. Then we will say that $\mathcal{M}$ is *α-good above δ* iff for every initial segment $\mathcal{N} \trianglelefteq \mathcal{M}$ such that $\mathcal{N}$ is a δ-mouse, either

(i) there is an above-δ ω_1-iteration strategy for $\mathcal{N}$ in $J_\alpha(\mathbb{R})$; or

(ii) $\mathrm{OD}^{<\alpha}(J^{\mathcal{N}}_{\delta}) \subseteq \mathcal{N}$.

Lemma 2.8. *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Let* $\mathcal{M}$ *be a countable δ-mouse, where δ is a cutpoint of* $\mathcal{M}$, *and* $\mathcal{M}$ *is ω-small above δ. Let $\alpha \geq 2$. Suppose* $\mathcal{M}$ *is α-good above δ and (ii) above holds for* $\mathcal{M}$. *Let* $\mathcal{N}$ *be the $\trianglelefteq$-least initial segment of* $\mathcal{M}$ *such that (ii) holds. Then* $\mathcal{P}(J^{\mathcal{N}}_{\delta}) \cap \mathcal{N} = \mathrm{OD}^{<\alpha}(J^{\mathcal{N}}_{\delta})$.

Proof. Let $B \in \mathcal{P}(J^{\mathcal{N}}_{\delta}) \cap \mathcal{N}$. Then there is a proper initial segment $\mathcal{Q} \triangleleft \mathcal{N}$ such that $\mathcal{Q}$ is a δ-mouse and B is definable over $\mathcal{Q}$. By hypothesis, there is an above-δ ω_1-iteration strategy for $\mathcal{Q}$ in $J_\alpha(\mathbb{R})$. Let ξ be the ordinal height of $\mathcal{Q}$. Then $\mathcal{Q}$ is the unique premouse $\mathcal{R}$ such that: $\mathcal{R}$ is a δ-mouse, δ is a cutpoint of $\mathcal{R}$, $\mathcal{J}^{\mathcal{R}}_{\delta} = \mathcal{J}^{\mathcal{N}}_{\delta}$, $\mathrm{OR} \cap \mathcal{R} = \xi$, and $\mathcal{R}$ has an ω_1 iteration strategy in $J_\alpha(\mathbb{R})$. Since B is definable over $\mathcal{Q}$ from an ordinal parameter, $B \in \mathrm{OD}^{<\alpha}(J^{\mathcal{N}}_{\delta})$. $\square$

The following theorem is a corollary of the proof of Theorem 2.2 above. See [8]. Our proof of Conjecture 0.1 depends heavily on this theorem.

Theorem 2.9 (Woodin). *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Suppose* $\mathcal{M}$ *is a countable δ-mouse, where δ is a cutpoint of* $\mathcal{M}$. *Suppose* $\mathcal{M}$ *is ω-small above δ and ω_1-iterable above δ. Then* $\mathcal{M}$ *is α-good above δ, for all $\alpha \geq 2$.*

3. Projective-like ordinals and good parameters

In this section we briefly review some descriptive set theory material we will need from [4] and [5]. We will also introduce one additional result in this area.

Recall that α is *projective-like* iff

(i) $[\alpha, \alpha]$ is a Σ_1-gap, and
(ii) α is not admissible.

Definition 3.1. If α is projective-like then there is a real x, and there is a total function $f : \mathbb{R} \xrightarrow{\text{cof}} \mathrm{OR} \cap J_\alpha(\mathbb{R})$ such that $f \in \Sigma_1(J_\alpha(\mathbb{R}), \{x\})$. We call such an x a *good parameter for* $J_\alpha(\mathbb{R})$.

For each projective-like ordinal $\alpha \geq 2$, we now define an analytical-hierarchy-like sequence of lightface pointclasses.

Definition 3.2. For $\alpha \geq 2$, let

$$\begin{aligned} \Sigma_{(\alpha,0)} &= \mathcal{P}(\mathbb{R}) \cap \Sigma_1(J_\alpha(\mathbb{R})) \\ \text{and for } n \geq 0 \text{ let} \qquad \Pi_{(\alpha,n)} &= \neg\Sigma_{(\alpha,n)} \\ \Delta_{(\alpha,n)} &= \Sigma_{(\alpha,n)} \cap \Pi_{(\alpha,n)} \\ \Sigma_{(\alpha,n+1)} &= \exists^{\mathbb{R}} \Pi_{(\alpha,n)}. \end{aligned}$$

So a set of reals A is in the pointclass $\Sigma_{(\alpha,n)}$, iff A is definable over $J_\alpha(\mathbb{R})$ using a formula φ with a certain restricted syntax. We would like to give a name to this syntax. We will say that a formula φ is $\Sigma_{(\bullet,n)}$, iff φ is the type of formula which is used to define a $\Sigma_{(\alpha,n)}$ set. More specifically, let $\mathcal{L}^{\mathbb{R}}$ be the language with the constant symbol $\dot{\mathbb{R}}$ in addition to the binary relation symbol $\in$. A formula in $\mathcal{L}^{\mathcal{R}}$ is interpreted in the model $(J_\alpha(\mathbb{R}); \in)$ by interpreting $\dot{\mathbb{R}}$ as $\mathbb{R}$. A formula φ is $\Sigma_{(\bullet,0)}$ iff φ is a Σ_1 formula of $\mathcal{L}^{\mathbb{R}}$. A formula φ is $\Pi_{(\bullet,0)}$ iff φ is a Π_1 formula of $\mathcal{L}^{\mathcal{R}}$. If $n \geq 1$, then φ is $\Sigma_{(\bullet,n)}$ iff there is a $\Pi_{(\bullet,n-1)}$ formula ψ such that $\varphi = (\exists v \in \dot{\mathbb{R}})\, \psi$, and φ is $\Pi_{(\bullet,n)}$ iff there is a $\Sigma_{(\bullet,n-1)}$ formula ψ such that $\varphi = (\forall v \in \dot{\mathbb{R}})\, \psi$. (If n is even for example, another appropriate name for a $\Sigma_{(\bullet,n)}$ formula would be a *positive* Σ^1_n/Σ_1 *formula*. We opted for a shorter phrase.)

Since a $\Sigma_{(\bullet,n)}$ formula is, in particular, a Σ_{n+1} formula, $\Sigma_{(\alpha,n)} \subseteq \Sigma_{n+1}(J_\alpha(\mathbb{R}))$. But, in general, $\Sigma_{(\alpha,n)} \neq \Sigma_{n+1}(J_\alpha(\mathbb{R}))$. The reason we are interested in the pointclasses $\Sigma_{(\alpha,n)}$ instead of just $\Sigma_{n+1}(J_\alpha(\mathbb{R}))$, is that the property of being a *scaled* pointclass is propagated by the quantifiers $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$.

Let $\alpha \geq 2$ be projective-like. The collection of pointclasses $\{\, \Sigma_{(\alpha,n)}, \Pi_{(\alpha,n)} \mid n \in \omega \,\}$ has many of the properties of the analytical hierarchy $\{\, \Sigma^1_n, \Pi^1_n \mid n \in \omega \,\}$. For example, $\Sigma_{(\alpha,2n)}$ and $\Pi_{(\alpha,2n+1)}$ are scaled Spector pointclasses. However there is one very important property of the analytical hierarchy which may fail for the collection $\{\, \Sigma_{(\alpha,n)}, \Pi_{(\alpha,n)} \mid n \in \omega \,\}$. This latter collection *may not form a hierarchy.* That is

Warning. For some projective-like α it is not the case that $\Sigma_{(\alpha,n)} \subseteq \Delta_{(\alpha,n+1)}$.

Of course it is true that $\Sigma_{(\alpha,n)} \subseteq \Pi_{(\alpha,n+1)} \subseteq \Sigma_{(\alpha,n+2)}$. Also we do have the following, which explains why we are interested in good parameters: Let $\alpha \geq 2$ be projective-like. Let x be a good parameter for $J_\alpha(\mathbb{R})$. Then $(\forall n) \quad \Sigma_{(\alpha,n)}(x) \subseteq \Delta_{(\alpha,n+1)}(x)$. That is, relative to a good parameter, our pointclasses do form a hierarchy. In fact, we have the following.

Lemma 3.3. *Let $\alpha \geq 2$ be projective-like, and let $n \geq 0$. Let x be a good parameter for $J_\alpha(\mathbb{R})$. Then $\Sigma_{(\alpha,n)}(x) = \mathcal{P}(\mathbb{R}) \cap \Sigma_{n+1}(J_\alpha(\mathbb{R}), \{x\})$.*

Proof. This is Lemma 2.6 from [5]. □

If 0 is a good parameter for $J_\alpha(\mathbb{R})$, then for all n, $\Sigma_{(\alpha,n)} \subseteq \Delta_{(\alpha,n+1)}$, and so we would expect the $\Sigma_{(\alpha,n)}$ pointclasses to behave exactly like the analytical pointclasses. More generally, if x is a good parameter for $J_\alpha(\mathbb{R})$, then we would expect the pointclasses $\Sigma_{(\alpha,n)}(x)$ to behave exactly like the analytical pointclasses. But we want to consider the *lightface* pointclasses $\Sigma_{(\alpha,n)}$, even when 0 is not a good parameter for $J_\alpha(\mathbb{R})$. In this situation, even though $\Sigma_{(\alpha,n)} \subseteq \Delta_{(\alpha,n+1)}$ may fail, many of the proofs of various well-known properties of the analytical hierarchy go through in our setting. In particular there is a *Q-theory* for the pointclasses $\Pi_{(\alpha,2n+1)}$. (See [1].) Below we record some of the facts we shall need.

Lemma 3.4 (Bounded Quantification). *Let $\alpha \geq 2$ be projective-like. Let $n \geq 1$ be odd. Assume* $\mathrm{Det}(J_{\alpha+1}(\mathbb{R}))$. *Let $Q(x,y,z) \in \Pi_{(\alpha,n)}$. Set*

$$P(x,y) \Leftrightarrow \exists z \in \Delta_{(\alpha,n)}(y) Q(x,y,z).$$

Then P is $\Pi_{(\alpha,n)}$.

Proof. This follows from 4D.3 of [3]. □

Lemma 3.5 (Effective Perfect Set Theorem). *Let $\alpha \geq 2$ be projective-like. Let $n \geq 1$ be odd. Assume* $\mathrm{Det}(J_{\alpha+1}(\mathbb{R}))$. *Let A be a countable $\Sigma_{(\alpha,n)}$ set. Then there is a real $z \in \Delta_{(\alpha,n)}$ such that every member of A is recursive in z.*

Proof. The proof of 6E.5 from [3] works here. □

3.1. Q-theory.

Definition 3.6. Let $\alpha \geq 2$ be projective-like. Let $n \geq 1$. Then

$$A_{(\alpha,n)} =_{\mathrm{def}} \left\{ x \in \mathbb{R} \;\middle|\; (\exists \xi < \omega_1)\, x \in \Delta_{(\alpha,n)}(\xi) \right\}.$$

We will write $x \leq_{(\alpha,n)} y$ to mean that $x \in \Delta_{(\alpha,n)}(y)$, and $x \equiv_{(\alpha,n)} y$ to mean $x \leq_{(\alpha,n)} y$ and $y \leq_{(\alpha,n)} x$.

Lemma 3.7 (The Theory of Largest Countable Sets). *Let $\alpha \geq 2$ be projective-like. Assume* $\mathrm{Det}(J_{\alpha+1}(\mathbb{R}))$.

(1) *For $n \geq 1$ odd there is a largest countable $\Pi_{(\alpha,n)}$ set of reals which we will call $C_{(\alpha,n)}$.*
(2) *For $n \geq 2$ even there is a largest countable $\Sigma_{(\alpha,n)}$ set of reals which we will call $C_{(\alpha,n)}$.*
(3) *For all $n \geq 1$, $C_{(\alpha,n)}$ is closed under $\equiv_{(\alpha,n)}$ and prewellordered by $\leq_{(\alpha,n)}$.*
(4) *For $n \geq 2$ even, $C_{(\alpha,n)} = A_{(\alpha,n)}$.*
(5) *For $n \geq 2$ even, $C_{(\alpha,n)} = \left\{ x \in \mathbb{R} \;\middle|\; (\exists y \in C_{(\alpha,n-1)})\, x \leq_T y \right\}$.*
(6) *For $n \geq 2$ even, $C_{(\alpha,n)}$ is closed downward under $\leq_{(\alpha,n)}$.*

Lemma 3.8 (Q-theory). *Let $\alpha \geq 2$ be projective-like. Let $n \geq 1$ be odd. Assume* $\mathrm{Det}(J_{\alpha+1}(\mathbb{R}))$.

(1) *There is a largest $\Pi_{(\alpha,n)}$-bounded set, which we will call $Q_{(\alpha,n)}$.*
(2) *$Q_{(\alpha,n)}$ is the largest $\Sigma_{(\alpha,n)}$-hull.*
(3) *$Q_{(\alpha,n)} = A_{(\alpha,n)}$.*
(4) *$Q_{(\alpha,n)} \subseteq C_{(\alpha,n)}$ and $Q_{(\alpha,n)}$ is closed downward under $\leq_{(\alpha,n)}$.*

When we try to extend to the context of $\Pi_{(\alpha,n)}$ the theory of y^0_3 we find we must be cautious. It is of course possible that $C_{(\alpha,n)} = Q_{(\alpha,n)} = \mathrm{OD}^{<\alpha}$. Since any non-trivial $\Pi_{(\alpha,n)}$ singleton is in $C_{(\alpha,n)} - Q_{(\alpha,n)}$ we find that it is possible that there are no non-trivial $\Pi_{(\alpha,n)}$ singletons. Also, we do not know how to rule out the possibility that $C_{(\alpha,n)} \neq Q_{(\alpha,n)}$ and still there are no non-trivial $\Pi_{(\alpha,n)}$ singletons.

Lemma 3.9. *Let $\alpha \geq 2$ be projective-like. Let $n \geq 1$ be odd. Assume* $\mathrm{Det}(J_{\alpha+1}(\mathbb{R}))$. *Suppose there is a non-trivial $\Pi_{(\alpha,n)}$ singleton and let y be a $\leq_{(\alpha,n)}$-minimal non-trivial $\Pi_{(\alpha,n)}$ singleton. Let $z \in C_{(\alpha,n)} - Q_{(\alpha,n)}$. Then $y \leq_{(\alpha,n)} z$.*

Proof. The proof of Theorem 6.3 from [1] works here. □

Lemma 3.10 (Basis Theorem for $\Sigma_{(\alpha,n)}$)**.** *Let $\alpha \geq 2$ be projective-like. Let $n \geq 1$ be odd. Assume* $\mathrm{Det}(J_{\alpha+1}(\mathbb{R}))$. *Suppose y is a non-trivial $\Pi_{(\alpha,n)}$ singleton. Let $A \in \Sigma_{(\alpha,n)}$, $A \neq \emptyset$. Then $\exists x \in A$ s.t. $x \in \Delta_{(\alpha,n)}(y)$.*

Proof. The proof of Theorem 5.6 from [1] works here. □

Question. Assume $\mathrm{AD}^{L(\mathbb{R})}$. Let $\alpha \geq 2$ be projective-like. Let $n \geq 1$. Is it true that $A_{(\alpha,n)} = \mathrm{OD}^{\alpha}_{n+1}$?

In these notes we will show that the answer is yes if α is of Type 1.

3.2. **Correctness and good parameters.** Recall that if α is projective-like, then a good parameter for $J_\alpha(\mathbb{R})$ is a real x such that there is a total function $f : \mathbb{R} \xrightarrow{\text{cof}} \omega\alpha$ which is $\Sigma_1(J_\alpha(\mathbb{R}), \{x\})$. Here is an alternate characterization.

Lemma 3.11. *Let $\alpha \geq 2$ be projective-like. Let $x \in \mathbb{R}$. The following are equivalent:*

(1) *x is a good parameter for $J_\alpha(\mathbb{R})$.*
(2) *$\Sigma_1(J_\alpha(\mathbb{R})) \subseteq \Delta_{(\alpha,1)}(x)$.*
(3) *x is a 1-name for α.*

Proof. This is Proposition 2.26 from [5]. □

The next definition generalizes condition (2) above.

Definition 3.12. Let $\alpha \geq 2$ be projective-like. Let $n \geq 1$. Let $x \in \mathbb{R}$. We say that x is an *n-good parameter for $J_\alpha(\mathbb{R})$* iff $\Sigma_1(J_\alpha(\mathbb{R})) \subseteq \Delta_{(\alpha,n)}(x)$.

So a 1-good parameter is the same thing as a good parameter. Next we give an alternate characterization of n-good parameters.

Definition 3.13. Let $x \in \mathbb{R}, \alpha \geq 2, n \geq 0$. Then we say that x is an *n-name for α* iff there is a formula θ such that α is the least ordinal so that $J_\alpha(\mathbb{R}) \models \theta[x]$, and θ is a $\Sigma_{(\bullet,n)}$ formula if n is even, or a $\Pi_{(\bullet,n)}$ formula if n is odd.

The following theorem appears in [4]

Theorem 3.14. *Let $\alpha \geq 2$ be projective-like. Assume* $\mathrm{Det}\, J_{\alpha+1}(\mathbb{R})$. *Let $x \in \mathbb{R}$. Let $n \geq 1$. Then the following are equivalent.*

(1) *x is an n-name for α.*
(2) *x is an n-good parameter for $J_\alpha(\mathbb{R})$.*

Proof. (1) $\Rightarrow$ (2). We will assume that n is odd. If n is even the proof is similar. Let x be an n-name for α. Let θ be a $\Pi_{(\bullet,n)}$ formula so that α is least such that $J_\alpha(\mathbb{R}) \models \theta[x]$.

If there is some $\gamma < \omega\alpha$ such that $S_\gamma(\mathbb{R}) \models \theta[x]$, then since $\Pi_{(\bullet,n)}$ formulae are upward absolute for levels of $L(\mathbb{R})$, we must have that α is a successor ordinal, and letting $\alpha = \beta+1$ we have that $\gamma > \omega\beta$. But then there is some formula ψ so that β is least such that $J_\beta(\mathbb{R}) \models \psi[x]$, and so there is some Σ_1 formula φ so that α is least such that $J_\alpha(\mathbb{R}) \models \varphi[x]$. So x is a 0-name for α. In particular x is a 1-name for α, so by Lemma 3.11 x is a good parameter for $J_\alpha(\mathbb{R})$, so in particular x is an n-good parameter. So we may assume that $(\forall\gamma < \omega\alpha)\ S_\gamma(\mathbb{R}) \not\models \theta[x]$.

Let φ be a Σ_1 formula of $\mathcal{L}^{\mathbb{R}}$ so that $\theta =$

$$(\forall v_n \in \dot{\mathbb{R}})(\exists v_{n-1} \in \dot{\mathbb{R}})(\forall v_{n-2} \in \dot{\mathbb{R}}) \ldots (\forall v_1 \in \dot{\mathbb{R}})\ \varphi$$

Since $\Sigma_1(J_\alpha(\mathbb{R})) \subseteq \Pi_{(\alpha,n)}$, must show that $\Sigma_1(J_\alpha(\mathbb{R})) \subseteq \Sigma_{(\alpha,n)}(x)$. Let ψ be any Σ_1 formula of $\mathcal{L}^{\mathbb{R}}$. It suffices to show:

Claim. *Let $z \in \mathbb{R}$. Then the following are equivalent*

(i) $J_\alpha(\mathbb{R}) \models \psi[z]$.

(ii) $(\exists y_n \in \mathbb{R})(\forall y_{n-1} \in \mathbb{R}) \ldots (\exists y_1 \in \mathbb{R})(\forall\beta < \omega\alpha)$
$$\{S_\beta(\mathbb{R}) \models \varphi[x, y_1, \cdots y_n] \rightarrow S_\beta(\mathbb{R}) \models \psi[z]\ \}.$$

Proof of Claim. $(i) \Rightarrow (ii)$. Assume $J_\alpha(\mathbb{R}) \models \psi[z]$ and let β_0 be the least β such that $S_\beta(\mathbb{R}) \models \psi[z]$. To see that (ii) holds it suffices to show that

$$(\exists y_n \in \mathbb{R})(\forall y_{n-1} \in \mathbb{R}) \ldots (\exists y_1 \in \mathbb{R})\{S_{\beta_0}(\mathbb{R}) \not\models \varphi[x, y_1, \cdots y_n]\ \}.$$

But this does hold because otherwise $S_{\beta_0}(\mathbb{R}) \models \theta[x]$ which violates our assumption from the end of the second paragraph.

$(ii) \Rightarrow (i)$. Assume that (ii) holds. Then to show that $J_\alpha(\mathbb{R}) \models \psi[z]$ it suffices to show that it is *not* the case that

$$(\exists y_n \in \mathbb{R})(\forall y_{n-1} \in \mathbb{R}) \ldots (\exists y_1 \in \mathbb{R})(\forall\beta < \omega\alpha)\{S_\beta(\mathbb{R}) \not\models \varphi[x, y_1, \cdots y_n]\ \}.$$

But the above condition is equivalent to

$$(\exists y_n \in \mathbb{R})(\forall y_{n-1} \in \mathbb{R}) \ldots (\exists y_1 \in \mathbb{R})\{J_\alpha(\mathbb{R}) \not\models \varphi[x, y_1, \cdots y_n]\ \}.$$

And this contradicts our assumption that $J_\alpha(\mathbb{R}) \models \theta[x]$. $\diamondsuit$

(2) $\Rightarrow$ (1). Now let x be an n-good parameter for $J_\alpha(\mathbb{R})$. For this direction we give the proof in both the odd case and the even case. First assume that n is odd. Then we have $\mathrm{OD}^{<\alpha}(x) \in \Sigma_1(J_\alpha(\mathbb{R}), \{x\}) \subseteq \Sigma_{(\alpha,n)}(x)$. As $\mathrm{OD}^\alpha(x)$ is countable and n is odd, by the Effective Perfect Set Theorem, Lemma 3.5, there is a real $y \in \Delta_{(\alpha,n)}(x)$ such that $(\forall z \in \mathrm{OD}^\alpha(x))$ z is recursive in $\langle x, y\rangle$. Then $y \notin \mathrm{OD}^{<\alpha}(x)$. So α is least such that $y \in \Delta_{(\alpha,n)}(x)$. So $\langle x, y\rangle$ is an n-name for α. As $\{y\} \in \Sigma_{(\alpha,n)}(x)$, x is an n-name for α.

Now assume that $n \geq 2$ is even. Let $A = \mathbb{R} - \mathrm{OD}^{<\alpha}(x)$. Then $A \in \Pi_1(J_\alpha(\mathbb{R}), \{x\}) \subseteq \Sigma_{(\alpha,n)}(x)$. Thus there is a $y \in A$ s.t. $y \in \Delta_{(\alpha,n)}(x)$. Since α is least such that $y \in \Delta_{(\alpha,n)}(x)$, $\langle x, y\rangle$ is an n-name for α. Since $y \in \Delta_{(\alpha,n)}(x)$, x is an n-name for α. $\square$

Now we turn to the topic of correctness.

Definition 3.15. If $A \subset \mathbb{R}$, $\alpha \geq 1$ and $1 \leq n \leq \omega$ then we say that A is $\Sigma_n(J_\alpha(\mathbb{R}))$-*correct* iff there is an ordinal $\bar{\alpha}$ and a function j such that

(i) $A = \mathbb{R} \cap J_{\bar{\alpha}}(A)$,and

(ii) $j : J_{\bar{\alpha}}(A) \prec_{\Sigma_n} J_\alpha(\mathbb{R})$.

If Γ is a pointclass and A a pointset then we say that A is a *basis* for Γ iff $B \cap A \neq \emptyset$ for all $B \in \Gamma$.

Lemma 3.16. *Suppose α begins a Σ_1-gap. Let A be a nonempty set of reals. Then the following are equivalent:*

(a) *A is $\Sigma_1(J_\alpha(\mathbb{R}))$-correct.*
(b) *Letting*
$$H = \{\, a \in J_\alpha(\mathbb{R}) \mid (\exists x \in A)\ \{a\} \in \Sigma_1(J_\alpha(\mathbb{R}), \{x\}) \,\}$$
we have that $H \prec_1 J_\alpha(\mathbb{R})$ and $\mathbb{R} \cap H = A$.
(c) *A is closed under join, and for all $x \in A$, A is a basis for $\Sigma_1(J_\alpha(\mathbb{R}), \{x\})$.*

Proof. This is Lemma 2.28 from [5]. □

Remark 3.17. If α begins a Σ_1-gap and A is $\Sigma_1(J_\alpha(\mathbb{R}))$-correct then the $\bar{\alpha}$ and the j of definition 3.15 are unique. We will use the notation j_A to refer to this unique j.

Corollary 3.18. *Suppose α begins a Σ_1-gap. Let $n \geq 2$. Let A be a non-empty set of reals. Then the following are equivalent:*

(a) *A is $\Sigma_n(J_\alpha(\mathbb{R}))$-correct.*
(b) *A is $\Sigma_1(J_\alpha(\mathbb{R}))$-correct and j_A is Σ_n elementary.*
(c) *A is closed under join and $(\forall x \in A)$ A is a basis for $\Sigma_n(J_\alpha(\mathbb{R}), \{x\})$.*

Proof. This is Corollary 2.30 from [5]. □

Recall that for $n \geq 1$ we write $x \leq_{(\alpha,n)} y$ to mean that $x \in \Delta_{(\alpha,n)}(y)$. For $n = 0$ we will write $x \leq_{(\alpha,0)} y$ iff $\{x\} \in \Sigma_1(J_\alpha(\mathbb{R}), \{y\})$.

In the projective hierarchy, if $n \geq 2$ is even and a non-empty set of reals A is closed under join and downward closed under $\leq_n$, then A is Σ^1_n-correct. The next two lemmas show that in our setting a good parameter is also required for correctness.

Lemma 3.19. *Suppose α begins a Σ_1-gap. Assume* $\mathrm{Det}(J_\alpha(\mathbb{R}))$. *Let A be a non-empty set of reals. Suppose A is closed under join and downward closed under $\leq_{(\alpha,0)}$.*

(1) *If α is a limit ordinal then A is $\Sigma_1(J_\alpha(\mathbb{R}))$-correct.*
(2) *If $\alpha = \beta + 1$ and there is no $x \in A$ such that x is a 0-name for α, then A is $\Sigma_1(J_\alpha(\mathbb{R}))$-correct.*
(3) *If $\alpha = \beta + 1$ and A is closed downward under $\leq_{(\alpha,2)}$ then A is $\Sigma_1(J_\alpha(\mathbb{R}))$-correct.*
(4) *If $\alpha = \beta + 1$ and there is an $x \in A$ such that x is a 0-name for α, and β is projective-like, then the following are equivalent:*
    (a) *A is $\Sigma_1(J_\alpha(\mathbb{R}))$-correct.*
    (b) *A is $\Sigma_\omega(J_\beta(\mathbb{R}))$-correct.*
    (c) *$(\exists z \in A)$ z is a good parameter for $J_\beta(\mathbb{R})$.*

Proof. This is Lemma 2.31 from [5]. □

The following result appeared in [5] without condition (c). We will need condition (c) for our proof of the wellordering theorem in the uncountable cofinality case.

Lemma 3.20. *Let $\alpha \geq 2$ be projective-like. Assume* $\mathrm{Det}(J_{\alpha+1}(\mathbb{R}))$. *Let $n \geq 2$ be even. Let A be a non-empty set of reals, closed under join and closed down under $\leq_{(\alpha,n)}$. Then the following are equivalent.*

(a) *A is $\Sigma_{n+1}(J_\alpha(\mathbb{R}))$-correct.*
(b) *$(\exists x \in A)$ x is a good parameter for $J_\alpha(\mathbb{R})$.*
(c) *$(\exists x \in A)$ x is an n-good parameter for $J_\alpha(\mathbb{R})$.*

Proof. (a) $\Rightarrow$ (b). This follows from the fact that the set of good parameters for $J_\alpha(\mathbb{R})$ is $\Pi_2(J_\alpha(\mathbb{R}))$.

(b) $\Rightarrow$ (c). Trivial.

(c) $\Rightarrow$ (b). Let $x \in A$ be an n-good parameter for $J_\alpha(\mathbb{R})$. Let S be the set of good parameters for $J_\alpha(\mathbb{R})$. Notice that S is a $[\Pi_1(J_\alpha(\mathbb{R})) \wedge \Pi_{(\alpha,1)}]$ set. Since $\Pi_{(\alpha,1)} \subseteq \Sigma_{(\alpha,n)}$ and $\Pi_1(J_\alpha(\mathbb{R})) \subseteq \Sigma_{(\alpha,n)}(x)$, S is a $\Sigma_{(\alpha,n)}(x)$ set. So there is an element of S which is $\Delta_{(\alpha,n)}(x)$. So there is an element of S in A.

(b) $\Rightarrow$ (a). Let x be as in (b). Now let $z \in A$ and let B be a non-empty set of reals with $B \in \Sigma_{n+1}(J_\alpha(\mathbb{R}), \{z\})$. Then $B \in \Sigma_{(\alpha,n)}(\langle x, z\rangle)$. So $(\exists y \in B)$ $y \leq_{(\alpha,n)} \langle x, z\rangle$. So $B \cap A \neq \emptyset$. □

In the theory of the projective sets, C_{2n} is closed downward under $\leq_{2n}$, and so it is Σ^1_{2n} correct. In our context, $C_{(\alpha,2n)}$ is closed downward under $\leq_{(\alpha,2n)}$. By the previous lemma, $C_{(\alpha,2n)}$ is correct iff it contains a good parameter. The next theorem says that unless $C_{(\alpha,2n)}$ is trivial, it does contain a good parameter. The theorem also says this about $C_{(\alpha,2n+1)}$, but it is less clear what significance this has. It is an open question whether the following theorem is also true for $Q_{(\alpha,2n+1)}$. This result appeared in [4].

Theorem 3.21. *Let $\alpha \geq 2$ be projective-like. Assume* $\mathrm{Det}(J_{\alpha+1}(\mathbb{R}))$. *Let $n \geq 1$. Suppose $C_{(\alpha,n)} - \mathrm{OD}^{<\alpha} \neq \emptyset$. Then $(\exists z \in C_{(\alpha,n)})$ such that z is a good parameter for $J_\alpha(\mathbb{R})$.*

Proof. Since $C_{(\alpha,2n)}$ is the Turing hull of $C_{(\alpha,2n-1)}$, it suffices to prove the theorem for odd n. So let $n \geq 1$ be odd. Let $x \in C_{(\alpha,n)} - \mathrm{OD}^{<\alpha}$. Then α is least so that $x \in C_{(\alpha,n)}$. So x is an n-name for α, so x is an n-good parameter for $J_\alpha(\mathbb{R})$.

Let S be the set of good parameters for $J_\alpha(\mathbb{R})$. Then $S \in \Pi_2(J_\alpha(\mathbb{R})) \subseteq \Pi_{(\alpha,n)}(x)$. So $(\exists z \in S)$ $\{z\} \in \Pi_{(\alpha,n)}(x)$. So $(\exists z \in S)$ $\{z\} \subseteq C_{(\alpha,n)}(x) \subseteq C_{(\alpha,n)}$. □

The following somewhat technical Lemma considers Σ_{n+1} correctness when n is odd. This is a new result which we will need in our proof of the wellordering theorem.

Lemma 3.22. *Let $\alpha \geq 2$ be projective-like. Assume* $\mathrm{Det}(J_{\alpha+1}(\mathbb{R}))$. *Let $n \geq 1$ be odd. Suppose $A \subseteq \mathbb{R}$ is closed under join, closed downward under $\leq_{(\alpha,n)}$, and $(\forall x \in A)(\exists y \in A)$ s.t. $y \in C_{(\alpha,n)}(x) - Q_{(\alpha,n)}(x)$. Then A is $\Sigma_2(J_\alpha(\mathbb{R}))$-correct. Moreover, if there is a good parameter for $J_\alpha(\mathbb{R})$ in A then A is $\Sigma_{n+1}(J_\alpha(\mathbb{R}))$-correct.*

Proof. First we prove the "moreover" part. Suppose $z \in A$ is a good parameter for $J_\alpha(\mathbb{R})$. Then Q-theory for $\Pi_{(\alpha,n)}(z)$ is isomorphic to Q-theory for Π^1_3, and so for all $x \in \mathbb{R}$ there exists a non-trivial $\Pi_{(\alpha,n)}(\langle x, z\rangle)$ singleton. Let $x \in A$ and let $B \in \Sigma_{n+1}(J_\alpha(\mathbb{R}), \{x\})$. Then $B \in \Sigma_{(\alpha,n)}(x, z)$. Let $y' \in A$ with $y' \in C_{(\alpha,n)}(\langle x, z\rangle) - Q_{(\alpha,n)}(\langle x, z\rangle)$.

Let y be a $\leq_{(\alpha,n)}$-minimal non-trivial $\Pi_{(\alpha,n)}(\langle x,z\rangle)$ singleton. By Lemma 3.9, $y \leq_{(\alpha,n)} y'$ and so $y \in A$. By Lemma 3.10, there is a $w \in \Delta_{(\alpha,n)}(\langle x,z,y\rangle)$ s.t. $w \in B$. So $w \in A$ and we have shown that $\forall x \in A$, A is a basis for $\Sigma_{n+1}(J_\alpha(\mathbb{R}), \{x\})$. So A is $\Sigma_{n+1}(J_\alpha(\mathbb{R}))$-correct.

Now let us remove the assumption that there is a good parameter in A. Let $x \in A$ and let $B \in \Sigma_2(J_\alpha(\mathbb{R}), \{x\})$. Let $z \in C_{(\alpha,n)} - Q_{(\alpha,n)}$. Then α is least such that $z \in C_{(\alpha,n)}$ so z is an n-name for α so z is an n-good parameter for $J_\alpha(\mathbb{R})$. It follows that $B \in \Sigma_{(\alpha,n)}(\langle x,z\rangle)$.

Let S be the set of good parameters for $J_\alpha(\mathbb{R})$. Then $S \in \Pi_2(J_\alpha(\mathbb{R})) \subseteq \Pi_{(\alpha,n)}(z)$. So there is a good parameter v for $J_\alpha(\mathbb{R})$ such that $\{v\} \in \Pi_{(\alpha,n)}(z)$. If $v \in \Delta_{(\alpha,n)}(\langle x,z\rangle)$ then $v \in A$ and so we are done by the previous paragraph. So suppose that $v \notin \Delta_{(\alpha,n)}(\langle x,z\rangle)$. Then v is a non-trivial $\Pi_{(\alpha,n)}(\langle x,z\rangle)$ singleton. The important point is that there *is some* non-trivial $\Pi_{(\alpha,n)}(\langle x,z\rangle)$ singleton.

Let y be a $\leq_{(\alpha,n)}$-minimal non-trivial $\Pi_{(\alpha,n)}(\langle x,z\rangle)$ singleton. As above, by Lemma 3.9, $y \in A$. By Lemma 3.10, there is a $w \in \Delta_{(\alpha,n)}(\langle x,z,y\rangle)$ s.t. $w \in B$. So $w \in A$ and we have shown that $\forall x \in A$, A is a basis for $\Sigma_2(J_\alpha(\mathbb{R}), \{x\})$. So A is $\Sigma_2(J_\alpha(\mathbb{R}))$-correct. $\square$

4. The mouse set theorem in the uncountable cofinality case. Part I.

For the rest of this section fix an ordinal λ which is of Type 1. That is, λ is projective-like and $\mathrm{cof}(\lambda) > \omega$. In this section we will give half of the proof that for all $n \geq 1$, $A_{(\lambda,n)} = \mathrm{OD}^\lambda_{n+1}$, and $A_{(\lambda,n)}$ is a mouse set.

First we give some definitions which will help us to characterize exactly the mouse $\mathcal{M}$ such that $\mathbb{R} \cap \mathcal{M} = A_{(\lambda,n)}$.

Definition 4.1. Let $\mathcal{M}$ be a countable ω-small δ-mouse. Let $\mathcal{T}$ be an ω-maximal iteration tree on $\mathcal{M}$ above δ. Suppose that for all limit $\lambda < \mathrm{lh}(\mathcal{T})$, letting $b = [0,\lambda]_T$, we have that $Q(b, \mathcal{T}\restriction\lambda)$ is ω_1-iterable above $\delta(b, \mathcal{T}\restriction\lambda)$. Then we will say that $\mathcal{T}$ is *correct*.

Definition 4.2. Let $\mathcal{M}$ be a premouse and $\delta \in \mathrm{OR}^{\mathcal{M}}$. Then $\mathcal{M}$ is *explicitly* $(\lambda,0)$*-big above* δ iff

(1) $\mathrm{OD}^{<\lambda}(J^{\mathcal{M}}_\delta) \subseteq \mathcal{M}$, and
(2) $\mathcal{M}$ is a δ-mouse.

We also say that $\mathcal{M}$ is $(\lambda,0)$*-big above* δ iff there is an initial segment $\mathcal{N} \trianglelefteq \mathcal{M}$ such that $\mathcal{N}$ is explicitly $(\lambda,0)$-big above δ.

Now let $1 \leq n < \omega$. Then $\mathcal{M}$ is *explicitly* (λ,n)*-big above* δ iff there are ordinals $\delta_1 \dots \delta_n$ such that $\delta < \delta_1 < \cdots < \delta_n$ and $\delta_1, \dots, \delta_n$ are Woodin in $\mathcal{M}$ and $\mathcal{M}$ is explicitly $(\lambda,0)$-big above δ_n.

Also, for $n \geq 1$ we will say that $\mathcal{M}$ is (λ,n)*-big above* δ iff either:

(a) there is an initial segment $\mathcal{N} \trianglelefteq \mathcal{M}$ such that $\mathcal{N}$ is explicitly (λ,n)-big above δ, or
(b) there is an initial segment $\mathcal{M}' \trianglelefteq \mathcal{M}$ such that $\mathcal{M}'$ is a δ-mouse and there is an above-δ, ω-maximal, correct iteration tree $\mathcal{T}$ on $\mathcal{M}'$ of some countable length $\theta+1$, and there is an initial segment $\mathcal{N} \trianglelefteq \mathcal{M}^{\mathcal{T}}_\theta$ such that $\mathcal{N}$ is explicitly (λ,n)-big above δ.

Finally, for all $n \in \omega$ we say that $\mathcal{M}$ is (λ,n)*-small above* δ iff $\mathcal{M}$ is not (λ,n)-big above δ.

The following simple fact is the key to our being able to prove a comparison lemma for (λ,n)-small mice by induction on n.

Lemma 4.3. *Let $n \in \omega$. Suppose $\mathcal{M}$ is a countable δ-mouse and $\mathcal{M}$ is $(\lambda,n+1)$-small above δ. Let $\mathcal{T}$ be an above-δ, ω-maximal, correct iteration tree on $\mathcal{M}$ of countable length $\theta+1$. Let $\delta' > \delta$ be an ordinal such that $\mathrm{lh}(E^{\mathcal{T}}_\xi) < \delta'$ for all $\xi < \theta$. Let $\mathcal{Q} \trianglelefteq \mathcal{M}^{\mathcal{T}}_\theta$ be such that δ' is Woodin in $\mathcal{Q}$. Then $\mathcal{Q}$ is (λ,n)-small above δ'.*

Proof. Assume towards a contradiction that $\mathcal{Q}$ is (λ, n)-big above δ'. If there is an initial segment $\mathcal{N} \trianglelefteq \mathcal{Q}$ such that $\mathcal{N}$ is explicitly (λ, n)-big above δ', then $\mathcal{N}$ is explicitly $(\lambda, n+1)$-big above δ, and so $\mathcal{T}$ witnesses that $\mathcal{M}$ is $(\lambda, n+1)$-big above δ, contrary to our hypothesis.

So it must be the case that $n > 0$ and there is an initial segment $\mathcal{M}' \trianglelefteq \mathcal{Q}$ such that $\mathcal{M}'$ is a δ'-mouse and there is an above-δ', ω-maximal, correct, iteration tree $\mathcal{U}$ on $\mathcal{M}'$ of some countable length $\Omega + 1$ and there is an initial segment $\mathcal{N} \trianglelefteq \mathcal{M}^{\mathcal{T}}_{\Omega}$ such that $\mathcal{N}$ is explicitly (λ, n)-big above δ'. Since δ' is Woodin in $\mathcal{N}$, $\mathcal{N}$ is explicitly $(\lambda, n+1)$-big above δ. As the critical point of every extender from $\mathcal{U}$ is greater than the length of every extender from $\mathcal{T}$, the concatenation of $\mathcal{T}$ and $\mathcal{U}$ is an ω-maximal iteration tree on $\mathcal{M}$ which witnesses that $\mathcal{M}$ is $(\lambda, n+1)$-big above δ, contrary to hypothesis. □

We will prove that $A_{(\lambda,n)} = \mathbb{R} \cap \mathcal{M}$ where $\mathcal{M}$ is the minimal, iterable ω-mouse that is (λ, n)-big. As usual, our mouse set theorem consists of two directions. First we prove a series of comparison lemmas which will enable us to show that $\mathbb{R} \cap \mathcal{M} \subseteq A_{(\lambda,n)}$. Our first and simplest comparison lemma doesn't require any iterating.

Lemma 4.4. *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Let* $\alpha \geq 2$. *Suppose that* $\mathcal{M}$ *and* $\mathcal{N}$ *are countable* ω*-small* δ*-mice, where* δ *is a cutpoint of* $\mathcal{M}$ *and* $\mathcal{N}$. *Suppose* $\mathcal{J}^{\mathcal{M}}_{\delta} = \mathcal{J}^{\mathcal{N}}_{\delta}$, *and let* $\mathcal{P} = \mathcal{J}^{\mathcal{M}}_{\delta}$. *Suppose* $\mathcal{M}$ *is* ω_1*-iterable above* δ *and* $\mathrm{OD}^{<\alpha}(\mathcal{P}) \not\subseteq \mathcal{M}$. *Suppose* $\mathcal{N}$ *is* α*-good above* δ. *Then* $\mathcal{M} \trianglelefteq \mathcal{N}$ *or* $\mathcal{N} \trianglelefteq \mathcal{M}$.

Proof. Either

(i) there is an above-δ ω_1-iteration strategy for $\mathcal{N}$ in $J_\alpha(\mathbb{R})$; or
(ii) $\mathrm{OD}^{<\alpha}(\mathcal{P}) \subset \mathcal{N}$.

If (i) holds then the conclusion of the lemma is clear, so suppose that (ii) holds. Let $\mathcal{N}'$ be the $\trianglelefteq$-least initial segment of $\mathcal{N}$ such that (ii) holds. We will show that $\mathcal{M} \triangleleft \mathcal{N}'$. Now $\mathcal{M}$ is comparable with every proper initial segment of $\mathcal{N}'$ which projects to δ. Suppose towards a contradiction that every such initial segment is a proper initial segment of $\mathcal{M}$. Since $\mathcal{M}$ is a δ-mouse, there is an $A \subset \delta$ which is definable over $\mathcal{M}$ but not in $\mathcal{M}$. By theorem 2.9, $\mathcal{M}$ is also α-good above δ, so our hypothesis on $\mathcal{M}$ implies that there is an above-δ ω_1-iteration strategy for $\mathcal{M}$ in $J_\alpha(\mathbb{R})$. This implies that $A \in \mathrm{OD}^{<\alpha}(\mathcal{P})$. Thus $A \in \mathcal{N}'$, and so A is definable over some proper initial segment of $\mathcal{N}'$ which projects to δ, and so $A \in \mathcal{M}$. Contradiction. □

If $\mathcal{M}$ is a bigger mouse than the one in the previous lemma, then we need for $\mathcal{N}$ to satisfy a stronger iterability condition than the one in that lemma.

Definition 4.5. Let $\mathcal{M}$ be a countable δ-mouse, where δ is a cutpoint of $\mathcal{M}$, and $\mathcal{M}$ is ω-small above δ. Then $\mathcal{M}$ is $\Pi_1(J_\lambda(\mathbb{R}))$*-iterable above* δ iff whenever $\mathcal{T}$ is a countable, above-δ, ω-maximal, putative iteration tree on $\mathcal{M}$ then either:
(i) $\mathcal{T}$ has a last wellfounded model; or
(ii) for all $\alpha < \lambda$ there is a maximal wellfounded branch b of $\mathcal{T}$ such that $Q(b, \mathcal{T})$ is α-good above $\delta(b, \mathcal{T})$.

Remarks 4.6.

(a) If $\mathcal{M}$ is ω_1-iterable above δ then $\mathcal{M}$ is $\Pi_1(J_\lambda(\mathbb{R}))$-iterable above δ. This follows from Theorem 2.9.
(b) $\Pi_1(J_\lambda(\mathbb{R}))$-iterability is a $\Pi_1(J_\lambda(\mathbb{R}))$-definable condition.

Lemma 4.7. *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Suppose that* $\mathcal{M}$ *and* $\mathcal{N}$ *are countable,* ω*-small* δ*-mice, where* δ *is a cutpoint of* $\mathcal{M}$ *and* $\mathcal{N}$. *Suppose* $\mathcal{J}^{\mathcal{M}}_{\delta} = \mathcal{J}^{\mathcal{N}}_{\delta}$. *Suppose* $\mathcal{M}$ *is* ω_1*-iterable above* δ *and* $(\lambda, 1)$*-small above* δ, *and* $\mathcal{N}$ *is* $\Pi_1(J_\lambda(\mathbb{R}))$*-iterable above* δ. *Then* $\mathcal{M} \trianglelefteq \mathcal{N}$ *or* $\mathcal{N} \trianglelefteq \mathcal{M}$.

Proof. Let Σ be an ω_1-iteration strategy for $\mathcal{M}$. Compare $\mathcal{M}$ with $\mathcal{N}$. On the $\mathcal{M}$-side produce an ω-maximal iteration tree on $\mathcal{M}$ above δ using Σ at limit steps. On the $\mathcal{N}$-side, produce an ω-maximal iteration tree $\mathcal{T}$ on $\mathcal{N}$ above δ by choosing at limit steps the unique cofinal branch b such that for all $\alpha < \lambda$, $Q(b, \mathcal{T})$ is α-good above $\delta(b, \mathcal{T})$, if there is a unique such branch.

If the comparison lasts ω_1 stages then, since ω_1 is measurable in $L(\mathbb{R})$, we get cofinal ω_1 branches on both sides and thus a contradiction.

So assume the comparison terminates at some countable stage. If the comparison terminates successfully, then by Lemma 1.11 from [7], $\mathcal{M} \trianglelefteq \mathcal{N}$ or $\mathcal{N} \trianglelefteq \mathcal{M}$ and we are done.

So now assume the comparison terminates unsuccessfully with a putative iteration tree $\mathcal{T}$ on $\mathcal{N}$, and an iteration tree $\mathcal{U}$ on $\mathcal{M}$. There are three possibilities: (i) $\mathcal{T}$ has two cofinal wellfounded branches b and c such that for all $\alpha < \lambda$, $Q(b, \mathcal{T})$ is α-good above $\delta(b, \mathcal{T})$, and similarly for c. (ii) $\mathcal{T}$ has limit length but has no cofinal branch b as above. (iii) $\mathcal{T}$ has successor length and has a last model which is illfounded. We will derive a contradiction in all three cases.

Suppose first that possibility (i) occurs. Let $\delta' = \delta(b, \mathcal{T}) = \delta(c, \mathcal{T})$ and let $\mathcal{P}$ be the common lined-up part of height δ'. Let $\mathcal{Q}$ be the corresponding Q-structure on the $\mathcal{M}$ side. If $\mathcal{Q} = \mathcal{P}$, then $\mathcal{Q} \trianglelefteq Q(b, \mathcal{T})$ and $\mathcal{Q} \trianglelefteq Q(C, \mathcal{T})$. As in [7] this is a contradiction. (See also Lemma 7.15 from [5].) So assume that $\delta' \in \mathcal{Q}$. Since $\mathcal{U}$ is consistent with Σ, $\mathcal{U}$ is correct. Then since $\mathcal{M}$ is $(\lambda, 1)$-small above δ and δ' is Woodin in $\mathcal{Q}$ we have by Lemma 4.3 that $\mathcal{Q}$ is $(\lambda, 0)$-small above δ'. Now $\mathcal{Q}$ is a δ'-mouse so by definition of $(\lambda, 0)$-big we must have that $\mathrm{OD}^{<\lambda}(\mathcal{P}) \not\subseteq \mathcal{Q}$. Thus there is an $\alpha < \lambda$ such that $\mathrm{OD}^{<\alpha}(\mathcal{P}) \not\subseteq \mathcal{Q}$. Since $Q(b, \mathcal{T})$ is α-good above δ' and $\mathcal{Q}$ is ω_1-iterable above δ' we have by Lemma 4.4 that $Q(b, \mathcal{T}) \trianglelefteq \mathcal{Q}$ or $\mathcal{Q} \trianglelefteq Q(b, \mathcal{T})$. Similarly $Q(c, \mathcal{T}) \trianglelefteq \mathcal{Q}$ or $\mathcal{Q} \trianglelefteq Q(c, \mathcal{T})$. As above, this is a contradiction.

Next we consider cases (ii) and (iii). In either case it follows that for each $\alpha < \lambda$ we have distinct maximal branches b_α such that $Q(b_\alpha, \mathcal{T})$ is α-good above $\delta(b_\alpha, \mathcal{T})$. Since $\mathcal{T}$ is countable and $\mathrm{cof}(\lambda) > \omega$, there is some limit $\gamma \leq \mathrm{lh}(\mathcal{T})$ such that for cofinally many $\alpha < \lambda$, b_α is cofinal in $\mathcal{T}\restriction\gamma$. But then we get the same contradiction as in the previous paragraph, using $\mathcal{T}\restriction\gamma$ instead of $\mathcal{T}$. □

If the $\mathcal{M}$ of the previous lemma is not $(\lambda, 1)$-small, but rather (λ, n)-small for some $n > 1$, then we will need for $\mathcal{N}$ to satisfy a stronger iterability condition. To prove our mouse set theorem we will need for this stronger iterability condition on $\mathcal{N}$ to have an optimally simple definition. The method for getting this optimally simple definition is due to Steel and first occurs in [7]. The idea is to use the Bounded Quantification Theorem, Lemma 3.4.

Fix some way to code hereditarily countable sets by reals, and for $x \in \mathbb{R}$ let x^* be the hereditarily countable set coded by x.

Definition 4.8. If $x \in \mathbb{R}$ then we say that x is a *mouse code* iff $x^* = (\mathcal{M}, \delta)$ where $\mathcal{M}$ is a countable premouse and $\delta \in \mathrm{OR}$ and either

(1) $\delta = \mathrm{OR}^{\mathcal{M}}$, or
(2) $\delta \in \mathrm{OR}^{\mathcal{M}}$ and $\mathcal{M}$ is a δ-mouse.

In [5] we defined, for $n \geq 1$, x a mouse code, and $\alpha < \kappa^{\mathbb{R}}$, the iteration game $G_{(\alpha,n)}(x)$. We now assume the reader is familiar with this material. Since we are dealing with ordinals $\lambda > \kappa^{\mathbb{R}}$ in these notes, we must slightly modify the definition of $G_{(\alpha,n)}$ from [5]. Since we have fixed a Type 1 ordinal λ for this whole section, we shall refer to our revised definition as $G_{(\lambda,n)}$.

For even $n \geq 2$, the definition of $G_{(\alpha,n)}$ in [5] employed a $\Pi_1(J_\alpha(\mathbb{R}))$ notion of iterability called "rank-$\omega\alpha$-iterability." Here we modify this so that the definition of $G_{(\lambda,n)}$ employs instead our notion from above of $\Pi_1(J_\lambda(\mathbb{R}))$-iterability.

For odd $n \geq 1$ the definition of $G_{(\alpha,n)}$ in [5] employed the following $\Sigma_1(J_\alpha(\mathbb{R}))$ notion of iterability for a premouse $\mathcal{M}$:

There exists an iteration strategy in $J_\alpha(\mathbb{R})$ for the $\Game^{\mathbb{R}}$-closed iteration game on $\mathcal{M}$.

Here we modify this so that the definition of $G_{(\lambda,n)}$ employs instead the following $\Sigma_1(J_\lambda(\mathbb{R}))$ notion of iterability for $\mathcal{M}$:

There exists an ω_1-iteration strategy for $\mathcal{M}$ in $J_\lambda(\mathbb{R})$.

We shall say no more about the definitions of $G_{(\lambda,n)}$. Exactly as in [5] we get:

Lemma 4.9. *$\{x \mid x$ is a mouse code and II wins $G_{(\lambda,n)}(x)\ \}$ is $\Pi_{(\lambda,n)}$.*

Definition 4.10. Let $\mathcal{M}$ be a countable δ-mouse. Let $n \geq 1$. Then $\mathcal{M}$ is $\Pi_{(\lambda,n)}$ *iterable above* δ iff for some (equivalently, for all) $x \in \mathbb{R}$ such that $x^* = (\mathcal{M}, \delta)$, there is a winning quasi-strategy for player *II* in $G_{(\lambda,n)}(x)$.

For completeness let us also say that $\mathcal{M}$ is $\Pi_{(\lambda,0)}$ *iterable* above δ iff $\mathcal{M}$ is $\Pi_1(J_\lambda(\mathbb{R}))$ iterable above δ.

Theorem 4.11. *Assume $\mathrm{AD}^{L(\mathbb{R})}$. Let $n \in \omega$.*

(1) *Let $\mathcal{M}$ be a countable δ-mouse which is $(\lambda, n+1)$-small above δ, where δ is a cutpoint of $\mathcal{M}$. Suppose $\mathcal{M}$ is ω_1-iterable above δ. Let $\mathcal{T}$ be an ω-maximal iteration tree on $\mathcal{M}$ of countable limit length, with $\mathcal{T}$ above δ, such that $\mathcal{T}$ is correct. Let b be the unique cofinal branch of $\mathcal{T}$ such that $Q(b,\mathcal{T})$ is ω_1-iterable above $\delta(b,\mathcal{T})$. If $n \geq 1$ then b is the unique cofinal branch c of $\mathcal{T}$ such that $\mathcal{M}_c^{\mathcal{T}}$ is wellfounded and $Q(c,\mathcal{T})$ is $\Pi_{(\lambda,n-1)}$-iterable above $\delta(\mathcal{T})$.*

(2) *Let $\mathcal{M}$ be a countable δ-mouse which is $(\lambda, n+1)$-small above δ, where δ is a cutpoint of $\mathcal{M}$. Suppose $\mathcal{M}$ is ω_1-iterable above δ. Then $\mathcal{M}$ is $\Pi_{(\lambda,n)}$ iterable above δ.*

(3) *Suppose $\mathcal{M}$ and $\mathcal{N}$ are countable ω-small, δ-mice, where δ is a cutpoint of $\mathcal{M}$ and $\mathcal{N}$. Suppose $\mathcal{J}_\delta^{\mathcal{M}} = \mathcal{J}_\delta^{\mathcal{N}}$. Suppose $\mathcal{M}$ is ω_1-iterable above δ, and $(\lambda, n+1)$-small above δ, and that $\mathcal{N}$ is $\Pi_{(\lambda,n)}$ iterable above δ. Then $\mathcal{M} \trianglelefteq \mathcal{N}$ or $\mathcal{N} \trianglelefteq \mathcal{M}$.*

Proof. By induction on n. Let $n = 0$. For (1) the theorem makes no claim in this case; (2) is true by remark 4.6a; and (3) is true by Lemma 4.7.

For $n > 0$ the inductive steps are exactly the same as in the proof of Theorem 7.16 from [5], which itself is based heavily on the proof of Lemma 2.2 from [7]. $\square$

Corollary 4.12. *Assume $\mathrm{AD}^{L(\mathbb{R})}$. Let $n \in \omega$. Let $\mathcal{M}$ be an ω_1-iterable premouse. Suppose that every proper initial segment of $\mathcal{M}$ is $(\lambda, n+1)$-small. Then $\mathbb{R} \cap \mathcal{M} \subseteq A_{(\lambda,n+1)}$.*

Proof. Let $x \in \mathbb{R} \cap \mathcal{M}$. Let ξ be the rank of x in the order of construction of $\mathcal{M}$. Let $\mathcal{N} \trianglelefteq \mathcal{M}$ be the $\trianglelefteq$-least initial segment of $\mathcal{M}$ containing x. Then $\mathcal{N}$ is an ω-mouse. By part (2) of Theorem 4.11, $\mathcal{N}$ is $\Pi_{(\lambda,n)}$-iterable. By part (3) of Theorem 4.11, if $\mathcal{N}'$ is any ω-mouse, and $\mathcal{N}'$ is $\Pi_{(\lambda,n)}$ iterable, then $\mathcal{N} \trianglelefteq \mathcal{N}'$ or $\mathcal{N}' \trianglelefteq \mathcal{N}$. Thus, x is the unique real such that there is some $\Pi_{(\lambda,n)}$ iterable ω-mouse $\mathcal{N}'$ such that x is the ξ^{th} real in the order of construction of $\mathcal{N}'$. $\square$

5. The mouse set theorem in the uncountable cofinality case. Part II.

Again fix an ordinal λ which is of Type 1. That is, λ is projective-like and $\mathrm{cof}(\lambda) > \omega$. Now we turn to the other direction of the mouse set theorem. We will show that if $\mathcal{M}$ is (λ, n)-big then $A_{(\lambda,n)} \subseteq \mathbb{R} \cap \mathcal{M}$. In fact we will show the ostensibly stronger fact that $\mathrm{OD}^{\lambda}_{n+1} \subseteq \mathbb{R} \cap \mathcal{M}$.

The next lemma tells us that if $\mathcal{M}$ is explicitly $(\lambda, 0)$-big then we will be able to iterate $\mathcal{M}$ and preserve its $(\lambda, 0)$-bigness.

Lemma 5.1. *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Let* $\mathcal{M}$ *be a countable,* ω*-small* δ*-mouse where* δ *is a cutpoint of* $\mathcal{M}$. *Suppose* $\mathrm{OD}^{<\lambda}(J_\delta^{\mathcal{M}}) \subseteq \mathcal{M}$. *Let* $k = k(\mathcal{M}, \delta)$. *Suppose* $\mathcal{M}$ *is* (k, ω_1)*-iterable* (above 0) *and let* Σ *be a* (k, ω_1)*-iteration strategy for* $\mathcal{M}$. *Let* $\mathcal{T}$ *be a* k*-maximal iteration tree on* $\mathcal{M}$ *of countable length* $\theta + 1$ *such that*

(a) $\mathcal{T}$ *is according to* Σ, *and*
(b) $D^{\mathcal{T}} = \emptyset$, *and*
(c) $\mathrm{lh}\, E_\xi^{\mathcal{T}} < i_{0,\xi}^{\mathcal{T}}(\delta)$ *for all* $\xi < \theta$.

Let $\mathcal{M}' = \mathcal{M}_\theta^{\mathcal{T}}$, *let* $\pi = i_{0,\theta}^{\mathcal{T}}$, *and let* $\delta' = \pi(\delta)$. *Then* $\mathcal{M}'$ *is a* δ'*-mouse with* $k(\mathcal{M}', \delta') = k$, *and* $\mathrm{OD}^{<\lambda}(J_{\delta'}^{\mathcal{M}'}) \subseteq \mathcal{M}'$.

Proof. Let $p = p_{k+1}(\mathcal{M}) = \langle r, u \rangle$. Let $p' = \pi(p) = \langle r', u' \rangle$. To see that the first statement of the conclusion holds it suffices to see that

(i) $\rho_k(\mathcal{M}') > \delta'$.
(ii) $\mathcal{M}'$ is k-sound.
(iii) $\mathcal{M}' = H_{k+1}^{\mathcal{M}'}(\delta' \cup \{p'\})$.
(iv) r' is $(k+1)$-solid over $(\mathcal{M}', u')$.

(i) and (ii) follow from the fact that π is a k-embedding.

For (iii), let $x \in \mathcal{M}'$. Since $\mathcal{T}$ only uses extenders below δ and its images there is a term $\tau \in \mathrm{Sk}_k$ and an ordinal $\gamma < \delta'$ and a $z \in \mathcal{M}$ such that $x = \tau^{\mathcal{M}'}[\gamma, \pi(z)]$. Since $\mathcal{M}$ is a δ-mouse, there is a term $\sigma \in \mathrm{Sk}_{k+1}$ and an $\eta < \delta$ such that $z = \sigma^{\mathcal{M}}[\eta, p]$. Since π is $r\Sigma_{k+1}$ elementary, $\pi(z) = \sigma^{\mathcal{M}'}[\pi(\eta), p']$. So $x \in H_{k+1}^{\mathcal{M}'}(\delta' \cup \{p'\})$.

For (iv), the proof of Lemmas 4.6 and 4.7 from [2] show that r' is $(k+1)$-solid over $(\mathcal{M}', u')$. The hypotheses in those lemmas that $\mathcal{M}$ is $(k+1)$-sound and that $\mathcal{T}$ is above $\rho_{k+1}(\mathcal{M})$ are not needed for this fact.

Now we turn to the second statement in the conclusion. Since $\mathcal{T}$ is according to Σ, $\mathcal{M}'$ is ω_1-iterable above δ'. Suppose towards a contradiction that $\mathrm{OD}^{<\lambda}(J_{\delta'}^{\mathcal{M}'}) \not\subseteq \mathcal{M}'$. By Theorem 2.9, $\mathcal{M}'$ is λ-good above δ'. By our assumption then, there is an above-δ' ω_1-iteration strategy for $\mathcal{M}'$ in $J_\lambda(\mathbb{R})$. Let us now call this strategy Σ. (We will have no more use for the original Σ.) Since π is a k-embedding and $\rho_{k+1}^{\mathcal{M}} \leq \delta$, we can use π to copy any ω-maximal iteration tree on $\mathcal{M}$ above δ to an ω-maximal iteration tree on $\mathcal{M}'$ above δ'. Thus Σ and π induce an above-δ, ω_1-iteration strategy Σ^π on $\mathcal{M}$.

Clearly $\Sigma^\pi \in J_\lambda(\mathbb{R})$. Since $\mathcal{M}$ is a δ-mouse, there is an $A \subseteq \delta$ such that A is definable over $\mathcal{M}$ but not in $\mathcal{M}$. Let φ and ξ be such that $A = \{\, \eta < \delta \mid \mathcal{M} \models \varphi[\eta, \xi] \,\}$. Let $\zeta = \mathrm{OR} \cap \mathcal{M}$. Let $\mathcal{P} = J_\delta^{\mathcal{M}}$. Then A is the unique subset of δ such that

> there exists a δ-mouse $\mathcal{N}$ such that $\mathcal{P} \lhd \mathcal{N}$ and δ is a cutpoint of $\mathcal{N}$ and $\zeta = \mathrm{OR} \cap \mathcal{N}$ and there is an above-δ ω_1-iteration strategy for $\mathcal{N}$ in $J_\lambda(\mathbb{R})$ and $A = \{\, \eta < \delta \mid \mathcal{N} \models \varphi[\eta, \xi] \,\}$.

Thus $A \in \mathrm{OD}^{<\lambda}(\mathcal{P})$. So $A \in \mathcal{M}$. Contradiction. $\square$

In what follows we seem to need a notion of iterability stronger than (ω, ω_1)-iterability. Below we use the notion of (k, α, θ)-iterability from [9].

Definition 5.2. Let $\mathcal{M}$ be a countable, ω-small premouse, $\beta \in \mathrm{OR}^{\mathcal{M}}$, and $n \geq 1$. The we will say that $\mathcal{M}$ is *properly* (λ, n)*-big above* β iff $\mathcal{M}$ is explicitly (λ, n)-big above β as witnessed by $\delta_1, \ldots, \delta_n$, and letting $k = k(\mathcal{M}, \delta_n)$ we have that $\mathcal{M}$ is (k, ω_1, ω)-iterable (above 0).

Definition 5.3. Let $\mathcal{M}$ be a countable, ω-small premouse, $\beta \in \mathrm{OR}^{\mathcal{M}}$, and $n \geq 1$. Then we will say that $\mathcal{M}$ is *minimally* (λ, n)*-big above* β iff $\mathcal{M}$ is explicitly (λ, n)-big above β as witnessed by $\delta_1, \ldots, \delta_n$, and letting $k = k(\mathcal{M}, \delta_n)$, there is a (k, ω_1, ω)-iteration strategy Σ for $\mathcal{M}$ such that, letting Γ be the (k, ω_1)-iteration strategy for $\mathcal{M}$ derived from Σ, we have that whenever $\mathcal{T}$ is a k-maximal iteration tree on $\mathcal{M}$ of some countable length $\theta + 1$ such that

(a) $\mathcal{T}$ is according to Γ, and
(b) $D^{\mathcal{T}} = \emptyset$, and
(c) $\mathrm{lh}\, E_\xi^{\mathcal{T}} < i_{0,\xi}^{\mathcal{T}}(\delta_1)$ for all $\xi < \theta$, and
(d) $\mathcal{T}$ is above β,

and whenever $\mathcal{N}$ is a proper initial segment of $\mathcal{M}_\theta^{\mathcal{T}}$ such that $i_{0,\theta}^{\mathcal{T}}(\delta_1) \in \mathcal{N}$ and $\rho_\omega(\mathcal{N}) = i_{0,\theta}^{\mathcal{T}}(\delta_1)$, then $\mathcal{N}$ is $(\lambda, n-1)$-small above $i_{0,\theta}^{\mathcal{T}}(\delta_1)$.

Lemma 5.4. *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Let* $n \geq 1$. *Let* $\mathcal{M}$ *be a countable,* ω*-small premouse and* β *a cardinal of* $\mathcal{M}$. *Suppose that* $\mathcal{M}$ *is properly* (λ, n)*-big above* β. *Then there is another premouse* $\mathcal{M}'$ *such that* $\mathcal{M}$ *and* $\mathcal{M}'$ *agree through* β, β *is a cardinal of* $\mathcal{M}'$, *and* $\mathcal{M}'$ *is minimally* (λ, n)*-big above* β.

Proof. Let $\delta_1, \ldots, \delta_n$ witness that $\mathcal{M}$ is explicitly (λ, n)-big above β. Let $k = k(\mathcal{M}, \delta_n)$. Let Σ be a (k, ω_1, ω)-iteration strategy for $\mathcal{M}$. We will define a sequence of mice $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \ldots$ and a sequence of maximal iteration trees $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots$ such that $\mathcal{T}_i$ is a tree on $\mathcal{M}_i$, $\mathcal{M}_{i+1}$ is a proper initial segment of the last model of $\mathcal{T}_i$, and the concatenation of the $\mathcal{T}_i$ is a play of the (k, ω_1, ω)-iteration game on $\mathcal{M}$ which is according to Σ. Such a sequence cannot be infinite, and the final $\mathcal{M}_i$ will be our $\mathcal{M}'$.

Let $\mathcal{M}_0 = \mathcal{M}$. Let $\kappa_0 = \delta_1$. If Σ witnesses that $\mathcal{M}$ is minimally (λ, n)-big above β then we are done, so suppose otherwise. Let Γ_0 be the (k, ω_1)-iteration strategy for $\mathcal{M}_0$ derived from Σ. Let $\mathcal{U}_0$ be a k-maximal iteration tree on $\mathcal{M}_0$ of some countable length $\theta + 1$ such that

(a) $\mathcal{U}_0$ is according to Γ_0, and
(b) $D^{\mathcal{U}_0} = \emptyset$, and
(c) $\mathrm{lh}\, E_\xi^{\mathcal{U}_0} < i_{0,\xi}^{\mathcal{U}_0}(\kappa_0)$ for all $\xi < \theta$, and
(d) $\mathcal{U}_0$ is above β,

and such that $\mathcal{U}_0$ witnesses that Σ doesn't witness that $\mathcal{M}$ is minimally (λ, n)-big above β. Let $\mathcal{N}_1 = \mathcal{M}_\theta^{\mathcal{U}_0}$ and let $\kappa_1 = i_{0,\theta}^{\mathcal{U}_0}(\kappa_1)$. By hypothesis there is a proper initial segment $\mathcal{P} \lhd \mathcal{N}_1$ such that $\mathcal{P}$ projects to κ_1 and $\mathcal{P}$ is $(\lambda, n-1)$-big above κ_1. Let $\mathcal{P}_1$ be the $\trianglelefteq$-least such $\mathcal{P}$. If there is an initial segment of $\mathcal{P}_1$ which is *explicitly* $(\lambda, n-1)$-big above κ_1, then let $\mathcal{M}_1$ be the $\trianglelefteq$-least such initial segment and set $\mathcal{T}_1 = \mathcal{U}_1$. Otherwise, $n \geq 2$ and there is an ω-maximal, countable, correct, above-κ_1 iteration tree $\mathcal{V}_1$ on $\mathcal{P}_1$ with last model $\mathcal{P}_1'$ such that $\mathcal{P}_1'$ has an initial segment which is explicitly $(\lambda, n-1)$-big above κ_1. In this case let $\mathcal{M}_1$ be the $\trianglelefteq$-least initial segment of $\mathcal{P}_1'$ that is explicitly $(\lambda, n-1)$-big above κ_1, and let $\mathcal{T}_1$ be the concatenation of $\mathcal{U}_1$ and $\mathcal{V}_1$. Notice that since $\mathcal{V}_1$ is correct, $\mathcal{T}_1$ is consistent with Γ_0. Also notice that in all cases, $\mathcal{M}_1$ agrees with $\mathcal{M}_0$ through β, and $\mathcal{M}_1$ is properly (λ, n)-big above β as witnessed by a sequence of Woodin cardinals in $\mathcal{M}_1$, the first of which is κ_1.

If Σ witnesses that $\mathcal{M}_1$ is minimally (λ, n)-big above β then we can set $\mathcal{M}' = \mathcal{M}_1$ and we are done, so suppose otherwise. Let Γ_1 be the (ω, ω_1)-iteration strategy for $\mathcal{M}_1$ derived from Σ.

Then let $\mathcal{T}_2$, $\mathcal{M}_2$, κ_2, and Γ_2 be defined from $\mathcal{M}_1$, κ_1, and Γ_1 in the same way that $\mathcal{T}_1$, $\mathcal{M}_1$, κ_1, and Γ_1 were defined from $\mathcal{M}_0$, κ_0, and Γ_0.

Continuing in this way we construct $\mathcal{T}_k$, $\mathcal{M}_k$, κ_k, and Γ_k for $k = 0, 1, 2, \ldots$. Eventually we must construct an $\mathcal{M}_k$ which is minimally (λ, n)-big above β, because otherwise the concatenation of the $\mathcal{T}_k$ is an infinite play consistent with Σ with cofinally many drops. Let $\mathcal{M}' = \mathcal{M}_k$ where $\mathcal{M}_k$ is minimally (λ, n)-big above β. $\square$

Lemma 5.5. *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Let* $\mathcal{M}$ *be a countable,* ω*-small* δ*-mouse where* δ *is a cutpoint of* $\mathcal{M}$. *Suppose that* $\mathcal{M}$ *is* $(\lambda, 0)$*-big above* δ*, but no proper initial segment of* $\mathcal{M}$ *is. Suppose also that* $\mathcal{M}$ *is* ω_1*-iterable above* δ. *Let* G *be* $\mathcal{M}$*-generic for* $\mathrm{Coll}(\omega, \delta)$. *Let* $A = \mathbb{R} \cap \mathcal{M}[G]$. *Then there is an ordinal* $\bar{\lambda} \in \mathcal{M}$ *and an embedding* j *such that*

(i) $A = \mathbb{R} \cap J_{\bar{\lambda}}(A)$, *and*

(ii) $j : J_{\bar{\lambda}}(A) \prec_{\Sigma_1} J_{\lambda}(\mathbb{R})$, *and*

(iii) $\bar{\lambda}$ *is the least ordinal* γ *such that there is a wellordering of* A *definable over* $J_{\gamma}(A)$.

Proof. Let $X = J_{\delta}^{\mathcal{M}}$. Let x be a real coding X such that $\mathcal{M}[G] = \mathcal{M}[x]$. By Lemma 2.8, $\mathcal{P}(X) \cap \mathcal{M} = \mathrm{OD}^{<\lambda}(X)$. Let $\Gamma = \mathcal{P}(\mathbb{R}) \cap \Sigma_1(J_{\lambda}(\mathbb{R}))$. Since $\mathrm{cof}(\lambda) > \omega$, Γ is a *good* pointclass in the sense of [8], and $\mathrm{OD}^{<\lambda}(X) = C_{\Gamma}(X)$. Let T be the tree of a Γ-scale on a universal Γ set. By Theorem 3.4 of [8] (which is due to Harrington and Kechris) $C_{\Gamma}(X) = \mathcal{P}(X) \cap L(X \cup \{T, X\})$. Thus $\mathbb{R} \cap \mathcal{M}[G] = \mathbb{R} \cap L[T, x] = C_{\Gamma}(x) = \mathrm{OD}^{<\lambda}(x)$. By part (1) of Lemma 3.19, $A = \mathbb{R} \cap \mathcal{M}[G]$ is $\Sigma_1(J_{\lambda}(\mathbb{R}))$-correct. So there is an ordinal $\bar{\lambda}$ and a function j such that

(i) $A = \mathbb{R} \cap J_{\bar{\lambda}}(A)$, and

(ii) $j : J_{\bar{\lambda}}(A) \prec_{\Sigma_1} J_{\lambda}(\mathbb{R})$.

Since there is a wellorder of $C_{\Gamma}(x)$ which is $\Gamma(x)$ definable, there is a wellorder of A which is $\Sigma_1(J_{\bar{\lambda}}(A), \{x\})$. Since there is no wellorder of $\mathbb{R}$ in $J_{\lambda}(\mathbb{R})$, there is no wellorder of A in $J_{\bar{\lambda}}(A)$. This proves (iii).

We only need to show that $\bar{\lambda} \in \mathcal{M}$. Since $\lambda < \theta^{L(\mathbb{R})}$, and $L[T, x] \models \mathrm{ZFC}$, we know that $\bar{\lambda} \leq (\omega_2)^{L[T,x]}$. So it suffices to see that $(\omega_2)^{L[T,x]} \in \mathcal{M}$.

First let us show that $\mathcal{M} \models$ "δ^+ exists." Let $\eta = \sup\left\{ \xi \in \mathcal{M} \;\middle|\; \rho_{\omega}(\mathcal{J}_{\xi}^{\mathcal{M}}) \leq \delta \right\}$. We want to see that $\eta \in \mathcal{M}$, for then $\eta = (\delta^+)^{\mathcal{M}}$. Let $\mathcal{R} = \mathcal{J}_{\eta}^{\mathcal{M}}$. We want to see that $\mathcal{R} \neq \mathcal{M}$. For this it suffices to see that $\rho_{\omega}(\mathcal{R}) > \delta$. Now $\mathcal{P}(\delta) \cap \mathcal{R} = \mathcal{P}(\delta) \cap L(X \cup \{T, X\})$. So to see that $\rho_{\omega}(\mathcal{R}) > \delta$ it suffices to see that $\mathcal{R} \in L(X \cup \{T, X\})$. Let $S = \left\{ \mathcal{J}_{\xi}^{\mathcal{M}} \;\middle|\; \delta < \xi < \mathrm{OR}^{\mathcal{M}} \wedge \rho_{\omega}(\mathcal{J}_{\xi}^{\mathcal{M}}) \leq \delta \right\}$. Since δ is a cutpoint of $\mathcal{M}$, $\mathcal{R}$ is passive. Thus $\mathcal{R} = \bigcup S$. So it suffices to see that $S \in L(X \cup \{T, X\})$. Now $S \subset L(X \cup \{T, X\})$ because if $\mathcal{N} \in S$ then $\mathcal{N}$ is coded by a subset of X in $\mathcal{M}$. Furthermore, if $\mathcal{N} \in S$ then by Theorem 2.9, $\mathcal{N}$ has an above-δ ω_1-iteration strategy in $J_{\lambda}(\mathbb{R})$. So $J_{\bar{\lambda}}(A) \models$ "$\exists$ an above-δ, ω_1-iteration strategy for $\mathcal{N}$." So S is definable in $J_{\bar{\lambda}}(A)$ from X. As G is $L(X \cup \{T, X\})$-generic for $\mathrm{Coll}(\omega, \delta)$, which is homogeneous, $S \in L(X \cup \{T, X\})$.

In summary $\eta = \mathrm{OR} \cap \mathcal{R} = (\delta^+)^{\mathcal{M}} = (\delta^+)^{L(X \cup \{T, X\})} = (\omega_1)^{L[T,x]}$, and $\eta \in \mathcal{M}$. Notice that the argument above also shows that $\mathcal{M} \notin L(X \cup \{T, X\})$.

Now η is a cutpoint of $\mathcal{M}$ and $\mathcal{M}$ is an *eta*-mouse and $\mathcal{M}$ is ω_1-iterable above η. By Theorem 2.9, $\mathcal{M}$ is λ-good above η. It can't be true that $\mathcal{M}$ has an above-η ω_1-iteration strategy in $J_{\lambda}(\mathbb{R})$, because then $\mathcal{M} \in L(\mathcal{R} \cup \{T, \mathcal{R}\}) = L(X \cup \{T, X\})$, but this is not true, as we mentioned above. Thus it must be true that $\mathrm{OD}^{<\lambda}(\mathcal{R}) \subseteq \mathcal{M}$. So $\mathcal{M}$ is also $(\lambda, 0)$-big above η. Let $\mathcal{N}$ be the $\trianglelefteq$-least initial segment of $\mathcal{M}$ which is $(\lambda, 0)$-big above η. Then $\mathcal{N}$ bears the same relationship to η as $\mathcal{M}$ does to δ. So by the same argument as above, $(\eta^+)^{\mathcal{N}} \in \mathcal{N}$. But $(\eta^+)^{\mathcal{N}} = (\eta^+)^{L(\mathcal{R} \cup \{T, \mathcal{R}\})} = (\eta^+)^{L(X \cup \{T, X\})} = (\delta^{++})^{L(X \cup \{T, X\}))} = (\omega_2)^{L[T,x]}$. $\square$

Theorem 5.6. *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Let* $n \geq 1$. *Let* $\mathcal{M}$ *be a countable,* ω*-small premouse, let* $\mathbb{P}$ *be a poset in* $\mathcal{M}$ *and let* $\kappa \geq \omega$ *be such that* $\mathbb{P} \subseteq J_\kappa^{\mathcal{M}}$. *Suppose* $\mathcal{M} \models \kappa^+$ *exists and let* $\beta = (\kappa^+)^{\mathcal{M}}$. *Assume that* $\mathcal{M}$ *is properly* (λ, n)*-big above* β. *Let* G *be* $\mathcal{M}$*-generic over* $\mathbb{P}$. *Then* $\mathbb{R} \cap \mathcal{M}[G]$ *is closed under* $\mathrm{OD}_{n+1}^\lambda$.

Proof. (By induction on $n \geq 1$.) By Lemma 5.4, we may assume that $\mathcal{M}$ is *minimally* (λ, n)-big above β. Fix ordinals $\delta_1, \ldots, \delta_n$ witnessing that $\mathcal{M}$ is explicitly (λ, n)-big above β. Let $k = k(\mathcal{M}, \delta_n)$. Fix a (k, ω_1, ω)-iteration strategy Σ for $\mathcal{M}$ witnessing that $\mathcal{M}$ is minimally (λ, n)-big above β. Let Γ be the (k, ω_1)-iteration strategy for $\mathcal{M}$ induced by Σ.

Let $x_0 \in \mathbb{R} \cap \mathcal{M}[G]$, and let $y_0 \in \mathrm{OD}_{n+1}^{<\lambda}(x_0)$. We will show that $y_0 \in \mathcal{M}[G]$. Recall that there is a $\Sigma_1(J_\lambda(\mathbb{R}))$ partial function from $\mathbb{R}$ onto $J_\lambda(\mathbb{R})$. This implies that $\Sigma_{n+1}(J_\lambda(\mathbb{R})) = \exists^{\mathbb{R}}[\Pi_n(J_\lambda(\mathbb{R})) \wedge \Sigma_1(J_\lambda(\mathbb{R}))]$. Fix a formula φ and a $\xi < \omega_1$ such that for all $w \in \mathrm{WO}$ with $|w| = \xi$,

$$\{y_0\} = \{\, y \mid (\exists z \in \mathbb{R}) J_\lambda(\mathbb{R}) \models \varphi[x_0, y, z, w] \,\}.$$

If $n > 1$ take φ to be a Π_n formula. If $n = 1$ take φ to be the conjunction of a Π_1 formula and a Σ_1 formula. Fix $z_0, w_0 \in \mathbb{R}$ such that $|w_0| = \xi$ and $J_\lambda(\mathbb{R}) \models \varphi[x_0, y_0, z_0, w_0]$. Also fix a good parameter r_0 for $J_\lambda(\mathbb{R})$.

We now use Woodins technique of "iterating to make a real generic." This gives us a partial order $\mathbb{Q} \subseteq J_{\delta_1}^{\mathcal{M}}$, with $\mathbb{Q} \in \mathcal{M}$, and a k-maximal iteration tree $\mathcal{T}$ on $\mathcal{M}$ of countable length $\theta + 1$ such that

(a) $\mathcal{T}$ is according to Γ, and
(b) $D^{\mathcal{T}} = \emptyset$ so that $i_{0,\theta}^{\mathcal{T}}$ is defined, and
(c) $\operatorname{lh} E_\xi^{\mathcal{T}} < i_{0,\xi}^{\mathcal{T}}(\delta_1)$ for all $\xi < \theta$, and
(d) $\operatorname{crit} E_\xi^{\mathcal{T}} > \beta$ for all $\xi < \theta$ (so G is $\mathcal{M}_\theta^{\mathcal{T}}$-generic over $\mathbb{P}$), and
(e) $\langle y_0, z_0, w_0, r_0 \rangle$ is $\mathcal{M}_\theta^{\mathcal{T}}[G]$-generic over $i_{0,\theta}^{\mathcal{T}}(\mathbb{Q})$.

Let $\mathcal{M}' = \mathcal{M}_\theta^{\mathcal{T}}$, and let $\delta_i' = i_{0,\theta}^{\mathcal{T}}(\delta_i)$. Since $\mathbb{R} \cap \mathcal{M}'[G] = \mathbb{R} \cap \mathcal{M}[G]$, it suffices to show that $y_0 \in \mathcal{M}'[G]$.

By Lemma 5.1, $\mathcal{M}'$ is a δ_n'-mouse with $k(\mathcal{M}', \delta_n') = k$, and $\mathrm{OD}^{<\lambda}(J_{\delta_n'}^{\mathcal{M}'}) \subseteq \mathcal{M}'$. Also, since $\mathcal{T}$ is according to Γ, $\mathcal{M}'$ is (k, ω_1, ω)-iterable. Thus $\mathcal{M}'$ is properly (λ, n)-big above β, as witnessed by $\delta_1', \ldots, \delta_n'$. Also since Σ witnesses that $\mathcal{M}$ is minimally (λ, n)-big above β, every proper initial segment of $\mathcal{M}'$ that projects to δ_1' is $(\lambda, n-1)$-small above δ_1'. (We will only use this last fact in the case $n = 1$. We will appeal to Lemma 5.4 again later in the proof in a more important way.)

Let H be $\bar{\mathcal{M}}[G]$ generic over $\mathrm{Coll}(\omega, \bar{\delta}_1)$ and such that $\langle y_0, z_0, w_0, r_0 \rangle \in \bar{\mathcal{M}}[G][H]$. Let $A = \mathbb{R} \cap \mathcal{M}'[G][H]$.

Now we break into two cases depending on whether n is even or odd.

Case 1. $n \geq 1$ *is odd.*

Claim 1. *There is an ordinal* $\bar{\lambda} \in \mathcal{M}'$ *and a function* j *such that*

(i) $A = \mathbb{R} \cap J_{\bar{\lambda}}(A)$*, and*
(ii) $j : J_{\bar{\lambda}}(A) \prec_{\Sigma_n} J_\lambda(\mathbb{R})$.

Proof of Claim 1. If $n = 1$ this is true by Lemma 5.5. (We do not need property (iii) of j from Lemma 5.5 here. We will need it at a different place later in the proof.)

So now assume that $n \geq 3$, n odd. Let $\beta' = ((\delta_1')^+)^{\mathcal{M}'}$. Then $\mathcal{M}'$ is properly $(\lambda, n-1)$-big above δ_1'. So by induction, A is closed under OD_n^λ. In particular A is closed under $A_{(\lambda, n-1)}$. Since $n-1$ is even, $A_{(\lambda, n-1)} = C_{(\lambda, n-1)}$ and $C_{(\lambda, n-1)}$ is closed downward under $\leq_{(\lambda, n-1)}$. Thus

A is closed downward under $\leq_{(\lambda,n-1)}$. Also the good parameter $r_0 \in A$. Since $n-1 \geq 2$ is even, we can apply Lemma 3.20 to conclude that A is $\Sigma_n(J_\lambda(\mathbb{R}))$-correct. This means that there is an ordinal $\bar{\lambda}$ and a function j such that

(i) $A = \mathbb{R} \cap J_{\bar{\lambda}}(A)$,and

(ii) $j : J_{\bar{\lambda}}(A) \prec_{\Sigma_n} J_\lambda(\mathbb{R})$.

We just need to see that $\bar{\lambda} \in \mathcal{M}'$. Since $\lambda < \theta^{L(\mathbb{R})}$ and $J_{\delta_2'}^{\mathcal{M}'} \models \mathrm{ZFC}$ + "$(\delta_1')^{++}$ exists," this is true. $\diamondsuit$

Since φ is a Π_n formula (or the conjunction of a Π_1 and a Σ_1 formula if $n = 1$) and $y_0, z_0, w_0 \in A$, it follows that in $\mathcal{M}'[G][H]$ the following is true:

$$\{y_0\} = \left\{ y \mid (\exists z, w \in \mathbb{R})\left[w \in \mathrm{WO} \wedge |w| = \xi \wedge J_{\bar{\lambda}}(\mathbb{R}) \models \varphi[x_0, y, z]\right] \right\}.$$

As the collapse forcing is homogeneous, this implies that $y_0 \in \mathcal{M}'[G]$.

Case 2. *$n \geq 2$ is even.*

Let $\beta' = ((\delta_1')^+)^{\mathcal{M}'}$. Then $\mathcal{M}'$ is properly $(\lambda, n-1)$-big above β'. By Lemma 5.4, there is a premouse $\mathcal{M}''$ which agrees with $\mathcal{M}'$ through β', such that $\beta' = ((\delta_1')^+)^{\mathcal{M}''}$, and such that $\mathcal{M}''$ is *minimally* $(\lambda, n-1)$-big above β'. Thus G is $\mathcal{M}''$ generic over $\mathbb{P}$ and $\mathbb{R} \cap \mathcal{M}'[G] = \mathbb{R} \cap \mathcal{M}''[G]$; and H is $\mathcal{M}''[G]$ generic over $\mathrm{Coll}(\omega, \delta_1')$ and $\mathbb{R} \cap \mathcal{M}'[G][H] = \mathbb{R} \cap \mathcal{M}''[G][H]$. Thus without loss of generality we may assume that $\mathcal{M}'$ is minimally $(\lambda, n-1)$-big above β'.

We will show that y_0 is ordinal definable in $\mathcal{M}'[G][H]$ from x_0. Since collapse forcing is homogeneous, this will give us that $y_0 \in \mathbb{R} \cap \mathcal{M}'[G] = \mathbb{R} \cap \mathcal{M}[G]$.

Let $\delta_2', \ldots, \delta_n'$ witness that $\mathcal{M}'$ is explicitly $(\lambda, n-1)$ big above β', let $k = k(\mathcal{M}', \delta_n')$, and let Σ be a (k, ω_1, ω) iteration strategy for $\mathcal{M}'$ which witnesses that $\mathcal{M}'$ is minimally $(\lambda, n-1)$-big above β'. (We are recycling some variable names here.) Let Γ' be the (k, ω_1)-iteration strategy for $\mathcal{M}'$ derived from Σ.

Claim 2. *Let $\mathcal{T}'$ be a k-maximal iteration tree on $\mathcal{M}'$ of some countable length $\theta' + 1$ such that*

(a) *$\mathcal{T}'$ is according to Γ', and*

(b) *$D^{\mathcal{T}'} = \emptyset$, and*

(c) *$\mathrm{lh}\, E_\xi^{\mathcal{T}'} < i_{0,\xi}^{\mathcal{T}'}(\delta_2')$ for all $\xi < \theta$, and*

(d) *$\mathcal{T}'$ is above β'.*

Let $\mathcal{M}^* = \mathcal{M}_{\theta'}^{\mathcal{T}'}$ and let $\delta^* = i_{0,\theta'}^{\mathcal{T}'}(\delta_i')$, for $2 \leq i \leq n$. (Since $\mathcal{T}'$ is above β', G is $\mathcal{M}^*$-generic over $\mathbb{P}$, and H is $\mathcal{M}^*[G]$-generic over $\mathrm{Coll}(\omega, \delta_1')$.) *Let K be $\mathcal{M}^*[G][H]$-generic for* $\mathrm{Coll}(\omega, \delta_2^*)$. *Let $A = \mathbb{R} \cap \mathcal{M}^*[G][H][K]$. Then there is an ordinal $\bar{\lambda} \in \mathcal{M}^*$, and an embedding j such that:*

(i) *$A = \mathbb{R} \cap J_{\bar{\lambda}}(A)$, and*

(ii) *$j : J_{\bar{\lambda}}(A) \prec_{\Sigma_{n-1}} J_\lambda(\mathbb{R})$, and*

(iii) *$\bar{\lambda}$ is the least ordinal γ such that there is a wellorder of A definable over $J_\gamma(A)$.*

Proof of Claim 2. By Lemma 5.1, $\mathcal{M}^*$ is a δ_n^*-mouse with $k(\mathcal{M}^*, \delta_n^*) = k$, and $\mathrm{OD}^{<\lambda}(J_{\delta_n^*}^{\mathcal{M}^*}) \subseteq \mathcal{M}^*$. Also, since $\mathcal{T}'$ is according to Γ', $\mathcal{M}^*$ is (k, ω_1, ω)-iterable. Also since Σ witnesses that $\mathcal{M}'$ is minimally $(\lambda, n-1)$-big above β', every proper initial segment of $\mathcal{M}^*$ that projects to δ_2^* is $(\lambda, n-2)$-small above δ_2^*.

If $n = 2$ then we have that $\mathcal{M}^*$ is $(\lambda, 0)$-big above δ_2^* and no proper initial segment of $\mathcal{M}^*$ is $(\lambda, 0)$-big above δ_2^* and $\mathcal{M}^*$ is ω_1-iterable above δ_2^*. So our claim is true by Lemma 5.5.

Now assume that $n \geq 4$ is even. The proof of Claim 1 above (in the $n-1$ case of this theorem) gives us $\bar{\lambda}$ and j so that (i) and (ii) hold. We must see that (iii) holds. Since there is no wellorder of $\mathbb{R}$ in $J_\lambda(\mathbb{R})$, there is no wellorder of A in $J_{\bar\lambda}(A)$. So we must see that there is a wellorder of A definable over $J_{\bar\lambda}(A)$.

Let s be a real in A which codes $\left\langle \mathcal{J}^{\mathcal{M}^*}_{\delta_2^*}, K \right\rangle$. It suffices to see that there is a wellorder of A which is $\Sigma_{n-1}(J_\lambda(\mathbb{R}), \{s\})$. For this it suffices to see that there is a wellorder of $\mathcal{P}(\mathcal{J}^{\mathcal{M}^*}_{\delta_2^*}) \cap \mathcal{M}^*$ which is $\Sigma_{n-1}(J_\lambda(\mathbb{R}), \left\{ \mathcal{J}^{\mathcal{M}^*}_{\delta_2^*} \right\})$. Let $<^*$ be the order of construction in $\mathcal{M}^*$ restricted to $\mathcal{P}(\mathcal{J}^{\mathcal{M}^*}_{\delta_2^*})$. If $X, Y \subset \mathcal{J}^{\mathcal{M}^*}_{\delta_2^*}$, then by Theorem 4.11, $X <^* Y$ iff

> There exists a δ_2^*-mouse $\mathcal{N}$ such that δ_2^* is a cutpoint of $\mathcal{N}$ and $\mathcal{J}^{\mathcal{N}}_{\delta_2^*} = \mathcal{J}^{\mathcal{M}}_{\delta_2^*}$ and $\mathcal{N}$ is $\Pi_{(\lambda,n-3)}$ iterable above δ_2^* and $X, Y \in \mathcal{N}$ and $X <^{\mathcal{N}} Y$.

Since $\Pi_{(\lambda,n-3)}$ iterability is $\Pi_{n-2}(J_\lambda(\mathbb{R}))$ definable, this shows that $<^*$ is $\Sigma_{n-1}(J_\lambda(\mathbb{R}), \{\mathcal{J}^{\mathcal{M}^*}_{\delta^*}\})$. $\diamondsuit$

Recall that we have fixed a Π_n formula φ. Since there is a $\Sigma_1(J_\lambda(\mathbb{R}))$ partial function from $\mathbb{R}$ onto $J_\lambda(\mathbb{R})$, $\Pi_n(J_\lambda(\mathbb{R})) = \forall^{\mathbb{R}}[\Sigma_{n-1}(J_\lambda(\mathbb{R})) \vee \Pi_1(J_\lambda(\mathbb{R}))]$. Fix a formula ψ such that φ is equivalent to $(\forall v \in \mathbb{R})\psi$ over $J_\lambda(\mathbb{R})$, and ψ is the disjunction of a Σ_{n-1} formula and a Π_1 formula.

To finish the proof we will show that y_0 is the unique real y such that $\mathcal{M}'[G][H]$ satisfies the following:

> $\exists z, w \in \mathbb{R}$ with $w \in \mathrm{WO}$ and $|w| = \xi$ s.t. it is it is forced over $\mathrm{Coll}(\omega, \delta_2')$ that $\exists\bar\lambda$ s.t. $\bar\lambda$ is the least ordinal γ s.t. there is a wellorder of $\mathbb{R}$ definable over $J_\gamma(\mathbb{R})$, and $J_{\bar\lambda}(\mathbb{R}) \models (\forall v \in \mathbb{R})\psi[x_0, y, z, w, v]$.

This will be enough because then y_0 is ordinal definable from x_0 in $\mathcal{M}'[G][H]$, and so $y_0 \in \mathcal{M}'[G]$.

First notice that y_0 does satisfy the condition above. $J_\lambda(\mathbb{R}) \models (\forall v \in \mathbb{R})\psi[x_0, y_0, z_0, w_0, v]$. Then by Claim 2 (with $\mathcal{T} = \emptyset$) if K is $\mathcal{M}[G][H]$ generic over $\mathrm{Coll}(\omega, \delta_2')$, then $J_{\bar\lambda}(A) \models (\forall v \in \mathbb{R})\psi[x_0, y_0, z_0, w_0, v]$, where $\bar\lambda$ and A are as in Claim 2.

Now let y be any real other than y_0, suppose that y satisfies the condition above, and we will derive a contradiction. Fix $z, w \in \mathbb{R} \cap \mathcal{M}'[G][H]$ as above. Since $y \neq y_0$, there exists a $v_0 \in \mathbb{R}$ such that $J_\lambda(\mathbb{R}) \models \neg\psi[x_0, y, z, w, v_0]$.

We now use Woodins technique of iterating to make a real generic again. This gives us a partial order $\mathbb{Q} \subseteq J^{\mathcal{M}'}_{\delta_2'}$, with $\mathbb{Q} \in \mathcal{M}'$, and a k-maximal iteration tree $\mathcal{T}'$ on $\mathcal{M}'$ of countable length $\theta' + 1$ such that

(a) $\mathcal{T}'$ is according to Γ', and
(b) $D^{\mathcal{T}'} = \emptyset$ so that $i^{\mathcal{T}'}_{0,\theta'}$ is defined, and
(c) $\mathrm{lh}\, E^{\mathcal{T}'}_\xi < i^{\mathcal{T}'}_{0,\xi}(\delta_2')$ for all $\xi < \theta'$, and
(d) $\mathcal{T}'$ is above β', and
(e) $\langle v_0 \rangle$ is $\mathcal{M}^{\mathcal{T}'}_{\theta'}[G][H]$-generic over $i^{\mathcal{T}'}_{0,\theta'}(\mathbb{Q})$.

Let $\pi = i^{\mathcal{T}'}_{0,\theta'}$. Let $\mathcal{M}^* = \mathcal{M}^{\mathcal{T}'}_{\theta'}$ and let $\delta^* = i^{\mathcal{T}'}_{0,\theta'}(\delta_i')$, for $2 \le i \le n$. Since $\mathrm{crit}(\pi) > \beta'$, π can be extended to an embedding $\hat\pi : \mathcal{M}'[G][H] \to \mathcal{M}^*[G][H]$. Let K be $\mathcal{M}^*[G][H]$-generic for $\mathrm{Coll}(\omega, \delta_2^*)$ with $v_0 \in \mathcal{M}^*[G][H][K]$. Let $A = \mathbb{R} \cap \mathcal{M}^*[G][H][K]$. By Claim 2 there is an ordinal $\bar\lambda \in \mathcal{M}^*$, and an embedding j such that:

(i) $A = \mathbb{R} \cap J_{\bar\lambda}(A)$, and
(ii) $j : J_{\bar\lambda}(A) \prec_{\Sigma_{n-1}} J_\lambda(\mathbb{R})$, and
(iii) $\bar\lambda$ is the least ordinal γ such that there is a wellorder of A definable over $J_\gamma(A)$.

We have that $J_{\bar\lambda}(A) \models \neg\psi[x_0, y, z, w, v_0]$. But the existence of the embedding $\hat\pi$ tells us that in $\mathcal{M}^*[G][H]$ it is forced over $\mathrm{Coll}(\omega, \delta_2^*)$ that $J_{\bar\lambda}(\mathbb{R}) \models (\forall v \in \mathbb{R})\psi[x_0, y, z, w, v]$. This is a contradiction. $\square$

Corollary 5.7. *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Let* λ *be a projective-like ordinal with* $\mathrm{cof}(\lambda) > \omega$. *Let* $\mathcal{M}$ *be an* ω*-small, sound,* $(\omega, \omega_1, \omega)$*-iterable premouse. Suppose that* $\mathcal{M}$ *is* (λ, n)*-big. Then* $\mathrm{OD}^{\lambda}_{n+1} \subseteq \mathbb{R} \cap \mathcal{M}$.

Proof. Without loss of generality assume that no proper initial segment of $\mathcal{M}$ is (λ, n)-big. It follows that either $\mathcal{M}$ is explicitly (λ, n)-big, or there is an ω-maximal, correct iteration tree $\mathcal{T}$ on $\mathcal{M}$ of some countable length $\theta + 1$, and there is an initial segment $\mathcal{N} \trianglelefteq \mathcal{M}^{\mathcal{T}}_{\theta}$ such that $\mathcal{N}$ is explicitly (λ, n)-big. Assume it is the latter case that occurs. If $\mathcal{M}$ itself is explicitly (λ, n)-big the argument is similar and simpler.

It suffices to see that $\mathrm{OD}^{\lambda}_{n+1} \subseteq \mathbb{R} \cap \mathcal{N}$. Let Σ be an $(\omega, \omega_1, \omega)$-iteration strategy for $\mathcal{M}$ and let Γ be the (ω, ω_1)-iteration strategy derived from Σ. Since $\mathcal{T}$ is correct, $\mathcal{T}$ is according to Γ. Thus $\mathcal{N}$ is (s, ω_1, ω)-iterable, where $s = \omega$ if $\mathcal{N} \triangleleft \mathcal{M}^{\mathcal{T}}_{\theta}$, and $s = \deg^{\mathcal{T}}(\mathcal{M}^{\mathcal{T}}_{\theta})$ if $\mathcal{N} = \mathcal{M}^{\mathcal{T}}_{\theta}$. Assume that $\mathcal{N} = \mathcal{M}^{\mathcal{T}}_{\theta}$. If $\mathcal{N} \triangleleft \mathcal{M}^{\mathcal{T}}_{\theta}$ then the argument is similar and simpler.

Let $\delta_1 < \cdots < \delta_n$ witness that $\mathcal{N}$ is explicitly (λ, n)-big. By definition, $\mathcal{N}$ is a δ_n-mouse. Let $k = k(\mathcal{N}, \delta_n)$. Then $\mathcal{N}$ is k-sound and so we must have that $k \geq \deg^{\mathcal{T}}(\mathcal{M}^{\mathcal{T}}_{\theta}) = s$. So $\mathcal{N}$ is *properly* (λ, n)-big. By Theorem 5.6 (with $\mathbb{P} = \emptyset$) $\mathrm{OD}^{\lambda}_{n+1} \subseteq \mathbb{R} \cap \mathcal{N}$. $\square$

Theorem 5.8. *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Let* λ *be a projective-like ordinal with* $\mathrm{cof}(\lambda) > \omega$. *Let* $n \geq 1$. *Then* $A_{(\lambda,n)} = \mathrm{OD}^{\lambda}_{n+1}$ *and* $A_{(\lambda,n)}$ *is a mouse set.*

Proof. By Theorem 2.2, OD^{λ} is a mouse set. So if $A_{(\lambda,n)} = \mathrm{OD}^{\lambda}$ we are done. So assume that $A_{(\lambda,n)} \subsetneq \mathrm{OD}^{\lambda}$. Then the proof of Theorem 2.2 gives us an ω-small ω-mouse $\mathcal{M}_1$ such that $A_{(\lambda,n)} \subseteq \mathbb{R} \cap \mathcal{M}_1$ and $\mathcal{M}_1$ is $(\omega, \omega_1, \omega)$-iterable. By Corollary 4.12, $\mathcal{M}_1$ is (λ, n)-big.

Let $\mathcal{M} \trianglelefteq \mathcal{M}_1$ be $\trianglelefteq$-least such that $\mathcal{M}$ is (λ, n)-big. By Corollary 4.12, $\mathbb{R} \cap \mathcal{M} \subseteq A_{(\lambda,n)}$. By Corollary 5.7, $\mathrm{OD}^{\lambda}_{n+1} \subseteq \mathbb{R} \cap \mathcal{M}$. Thus $A_{(\lambda,n)} = \mathbb{R} \cap \mathcal{M} = \mathrm{OD}^{\lambda}_{n+1}$.

$\square$

6. The Wellordering theorem in the uncountable cofinality case.

Theorem 6.1. *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Let* $\mathcal{M}$ *be an* ω*-small, countable, sound,* $(\omega, \omega_1, \omega)$*-iterable premouse. Let* $A = \mathbb{R} \cap \mathcal{M}$. *Let* (α, n) *be the degree of strong closure of* $\mathcal{M}$*, and suppose that* α *is projective-like and* $\mathrm{cof}(\alpha) > \omega$. *Then there exists an ordinal* $\bar\alpha$*, an integer* $m \geq 1$*, and an embedding* j *such that*

(i) $A = \mathbb{R} \cap J_{\bar\alpha}(A)$,
(ii) $j : J_{\bar\alpha}(A) \prec_{\Sigma_m} J_\alpha(\mathbb{R})$, *and*
(iii) *there is a wellordering of* A *that is* $\Delta_{m+1}(J_{\bar\alpha}(A), A)$.

Proof. Let $\beta_0 < \omega_1^{\mathcal{M}}$ be such that $\rho_\omega(\mathcal{J}^{\mathcal{M}}_{\beta_0}) = \omega$ and for all γ with $\beta_0 < \gamma < \omega_1^{\mathcal{M}}$, $\mathrm{OD}^{\alpha}_{n}(\mathcal{J}^{\mathcal{M}}_{\beta_0}) \not\subseteq J^{\mathcal{M}}_{\gamma}$. Let z_0 be a real in $\mathcal{M}$ coding $\mathcal{J}^{\mathcal{M}}_{\beta_0}$. For any mouse $\mathcal{N}$, let $<^{\mathcal{N}}$ be the order of construction in $\mathcal{N}$, restricted to $\mathbb{R} \cap \mathcal{N}$.

For any $\alpha' < \alpha$, $\mathbb{R} \cap \mathcal{M}$ is closed under $\mathrm{OD}^{\alpha'}$. In particular $\mathbb{R} \cap \mathcal{M}$ is closed downward under $\leq_{(\alpha,0)}$. By part (1) of Lemma 3.19, $A = \mathbb{R} \cap \mathcal{M}$ is $\Sigma_1(J_\alpha(\mathbb{R}))$-correct. So there is an ordinal $\bar\alpha$ and an embedding j such that

(i) $A = \mathbb{R} \cap J_{\bar\alpha}(A)$, and
(ii) $j : J_{\bar\alpha}(A) \prec_{\Sigma_1} J_\alpha(\mathbb{R})$.

We will show that for some $m \in \omega$, j is Σ_m elementary, and $<^{\mathcal{M}}$ is $\Delta_{m+1}(J_{\bar{\alpha}}(A), \{z_0\})$.

Remark 6.2. Recall that (α, n) is the degree of strong closure of $\mathcal{M}$. The reader may expect that if m is as in the previous paragraph, then $m = n - 1$. We will not necessarily get that $m = n - 1$. There is one case of our proof that leaves open the possibility that $m = 1$ and $n \geq 4$. We conjecture that this case actually cannot happen.

Let $x, y \in \mathbb{R} \cap \mathcal{M}$. Let $\mathcal{N} \trianglelefteq \mathcal{M}$ be $\trianglelefteq$-least such that $x, y, \beta_0 \in \mathcal{N}$. By definition of β_0, $\mathrm{OD}^{\alpha}_{n}(\mathcal{J}^{\mathcal{M}}_{\beta_0}) \not\subseteq \mathcal{N}$. So by Corollary 5.7, relativized to z_0, $\mathcal{N}$ is $(\alpha, n-1)$-small above β_0.

Consider for a moment the case $n = 1$. Then $\mathcal{N}$ is $(\alpha, 0)$-small above β_0. That means that $\mathrm{OD}^{<\alpha}(\mathcal{J}^{\mathcal{N}}_{\beta_0}) \not\subseteq \mathcal{N}$. By Theorem 2.9, there is an above-β_0 ω_1-iteration strategy for $\mathcal{N}$ in $J_\alpha(\mathbb{R})$. Then we have that $x <^{\mathcal{M}} y$ iff

> $\exists \mathcal{N}' \in \mathcal{M}$ s.t. $\mathcal{N}'$ is an ω-mouse, $\mathcal{J}^{\mathcal{N}'}_{\beta_0} = \mathcal{J}^{\mathcal{M}}_{\beta_0}$, $x <^{\mathcal{N}'} y$, and there is an above-β_0 ω_1-iteration strategy for $\mathcal{N}'$ in $J_\alpha(\mathbb{R})$.

Thus $<^{\mathcal{M}}$ is $\Delta_1(J_{\bar{\alpha}}(A), \{z_0\})$. This proves the theorem with $m = 0 = n - 1$. In this case we got that j is Σ_{m+1} elementary, not just Σ_m-elementary.

So from now on suppose $n \geq 2$. Since $\mathcal{N}$ is $(\lambda, n-1)$-small above β_0 we have by parts (2) and (3) of Theorem 4.11 that $x <^{\mathcal{M}} y$ iff

> $\exists \mathcal{N}' \in \mathcal{M}$ s.t. $\mathcal{N}$ prime is an ω-mouse, $\mathcal{J}^{\mathcal{N}'}_{\beta_0} = \mathcal{J}^{\mathcal{M}}_{\beta_0}$, $x <^{\mathcal{N}'} y$, and $\mathcal{N}'$ is $\Pi_{(\alpha,n-2)}$-iterable above β_0.

Recall that $\Pi_{(\alpha,n-2)}$-iterability is a $\Pi_{n-1}(J_\alpha(\mathbb{R}))$-condition. Thus it suffices to prove that $\mathbb{R} \cap \mathcal{M}$ is at least $\Sigma_{n-1}(J_\alpha(\mathbb{R}))$-correct. This would prove the theorem with $m = n-1$. Below we break into several cases. In all but one case we will prove that $\mathbb{R} \cap \mathcal{M}$ is $\Sigma_{n-1}(J_\alpha(\mathbb{R}))$-correct. There is one case in which $n \geq 4$ and yet we can only prove that $\mathbb{R} \cap \mathcal{M}$ is $\Sigma_1(J_\alpha(\mathbb{R}))$-correct. In this case we will still manage to prove the theorem by showing that $<^{\mathcal{M}}$ is $\Delta_2(J_{\bar{\alpha}}(A), \{z_0\})$ even though $n \geq 4$.

Case 1. $n = 2$.

Since we already know that $\mathbb{R} \cap \mathcal{M}$ is $\Sigma_1(J_\alpha(\mathbb{R}))$-correct, we are done in this case.

Case 2. *$n \geq 3$ is odd.*

In this case we will prove that $\mathbb{R} \cap \mathcal{M}$ is $\Sigma_{n-1}(J_\alpha(\mathbb{R}))$-correct.

We proved in the previous section that for $k \geq 1$, $A_{(\alpha,k)} = \mathrm{OD}^{J_\alpha(\mathbb{R})}_{k-1}$. Since $n-2$ is odd, $A_{(\alpha,n-2)} = Q_{(\alpha,n-2)}$. Thus the fact that $\mathcal{M}$ is strongly closed under $\mathrm{OD}^{\alpha}_{n-1}$ means that $\forall x \in \mathbb{R}^{\mathcal{M}} \exists \eta < \omega_1^{\mathcal{M}}\, Q_{(\alpha,n-2)}(x) \subseteq \mathcal{J}^{\mathcal{M}}_{\eta}$.

Claim 1. $\forall x \in \mathbb{R}^{\mathcal{M}} \exists y \in \mathbb{R}^{\mathcal{M}}\ y \in C_{(\alpha,n-2)}(\langle x, z_0\rangle) - Q_{(\alpha,n-2)}(\langle x, z_0\rangle)$.

Proof of Claim 1. Let $x \in \mathbb{R}^{\mathcal{M}}$. Let $\mathcal{N} \trianglelefteq \mathcal{M}$ be $\trianglelefteq$-least such that $x, \beta_0 \in \mathcal{N}$, $\mathcal{N}$ is an ω-mouse, and $Q_{(\alpha,n-2)}(\langle x, z_0\rangle) \subseteq \mathcal{N}$. By the proof of Theorem 5.8, $\mathbb{R} \cap \mathcal{N} = Q_{(\alpha,n-2)}(\langle x, z_0\rangle)$. As we argued above, $\mathcal{N}$ is $(\alpha, n-1)$-small above β_0. Let y be the canonical real definable over $\mathcal{N}$ such that y codes $\mathcal{N}$. We will show that $y \in C_{(\alpha,n-2)}(\langle x, z_0\rangle)$. Let $S = \{\mathcal{P} \mid \mathcal{P}$ is an ω-mouse, $\mathcal{J}^{\mathcal{P}}_{\beta_0} = \mathcal{J}^{\mathcal{M}}_{\beta_0}$, $\mathcal{P}$ is $\Pi_{(\alpha,n-2)}$-iterable above β_0, and $(\forall v \in \mathbb{R} \cap \mathcal{P})\ v \in Q_{(\alpha,n-2)}(\langle x, z_0\rangle)\}$. Let $T = \{ w \mid w$ codes some $\mathcal{P} \in S$ and w is the canonical real definable over $\mathcal{P}\}$. Then T is a $\Pi_{(\alpha,n-2)}(\langle x, z_0\rangle)$ set and $y \in T$. So it suffices to show that T is countable. For this it suffices to show that $S \subseteq \mathcal{M}$. Let $\mathcal{P} \in \mathcal{S}$. By part (3) of Theorem 4.11, either $\mathcal{P} \trianglelefteq \mathcal{N}$ or $\mathcal{N} \trianglelefteq \mathcal{P}$. Since $\mathbb{R} \cap \mathcal{P} \subseteq Q_{(\alpha,n-2)}(\langle x, z_0\rangle)$, $y \notin \mathcal{P}$. So it can't be true that $\mathcal{N} \triangleleft \mathcal{P}$. Thus $\mathcal{P} \trianglelefteq \mathcal{N}$ and so $\mathcal{P} \in \mathcal{M}$. $\diamondsuit$

By Lemma 3.22, $\mathbb{R} \cap \mathcal{M}$ is $\Sigma_2(J_\alpha(\mathbb{R}))$-correct. If $n = 3$ we are done, so assume now that $n \geq 5$. By Lemma 3.22 again, in order to see that $\mathbb{R} \cap \mathcal{M}$ is $\Sigma_{n-1}(J_\alpha(\mathbb{R}))$-correct it suffices to prove:

Claim 2. *There is a good parameter for $J_\alpha(\mathbb{R})$ in $\mathcal{M}$.*

Proof of Claim 2. Suppose $\mathcal{Q}$ is some δ-mouse where δ is a cutpoint of $\mathcal{Q}$. Suppose $\mathcal{Q}$ is $(\alpha, 0)-$*big* above δ, $(\alpha, 2)$-small above δ, and ω_1-iterable above δ. Notice that $\mathcal{Q}$ is $\Pi_{(\alpha,1)}$-iterable above δ, but if $\alpha' < \alpha$ then $\mathcal{Q}$ is not $\Pi_{(\alpha',1)}$ iterable above δ. Thus if x is a code for $\mathcal{Q}$ and w is a code for δ then $\langle x, w\rangle$ is a 1-name for α, and so a good parameter for $J_\alpha(\mathbb{R})$. Thus it suffices to show that there is such a $\mathcal{Q}$ and a δ in $\mathcal{M}$ and countable in $\mathcal{M}$.

Now there is an $\mathcal{N} \lhd \mathcal{M}$ such that $\mathcal{N}$ is countable in $\mathcal{M}$, $\beta_0 \in \mathcal{N}$, $\mathcal{N}$ is $(\alpha, n-2)$-big above β_0 and $(\alpha, n-1)$-small above β_0. Notice that if $\mathcal{N}$ were *explicitly* $(\alpha, n-2)$-big above β_0, or even explicitly $(\alpha, n-3)$-big above β_0, then we would be done. For suppose $\delta_1, \ldots, \delta_{n-3}$ were ordinals such that $\beta_0 < \delta_1 < \cdots < \delta_{n-3}$, and $\delta_1, \ldots, \delta_{n-3}$ are Woodin cardinals in $\mathcal{N}$, and $\mathcal{N}$ is $(\alpha, 0)$-big above δ_{n-3}. Since $\mathcal{N}$ is $(\alpha, n-1)$-small above β_0, we have that $\mathcal{N}$ is $(\alpha, 2)$-small above δ_{n-3}. So as we argued above, there is a good parameter in $\mathcal{M}$. The problem is that we cannot prove that $\mathcal{N}$ is *explicitly* $(\alpha, n-3)$-big above β_0. But by definition, there is an iterate $\mathcal{Q}$ of $\mathcal{N}$ which is explicitly $(\alpha, n-3)$-big above β_0. Using the $\Sigma_2(J_\alpha(\mathbb{R}))$-correctness of $\mathcal{M}$, we will show, essentially, that there is such a $\mathcal{Q}$ in $\mathcal{M}$.

Let $\mathcal{N}_0 \lhd \mathcal{M}$ be $\trianglelefteq$-least such that $\mathcal{N}_0$ is $(\alpha, 0)$-big above β_0. It follows that $\mathcal{N}_0$ is a β_0-mouse. If $\mathcal{N}_0$ is $(\alpha, 1)$-small above β_0 we are done, so suppose that $\mathcal{N}_0$ is $(\alpha, 1)$-big above β_0. Recall that this means that either:

(a) $\mathcal{N}_0$ is explicitly $(\alpha, 1)$-big above β_0, or

(b) there is an above-β_0, ω-maximal, correct iteration tree $\mathcal{T}$ on $\mathcal{N}_0$ of some countable length $\theta + 1$, and there is an initial segment $\mathcal{N} \trianglelefteq \mathcal{M}^{\mathcal{T}}_\theta$ such that $\mathcal{N}$ is explicitly $(\alpha, 1)$-big above β_0.

If (a) holds, then let $\mathcal{N}_1 = \mathcal{N}_0$ and let δ_1 witness that $\mathcal{N}_1$ is explicitly $(\alpha, 1)$-big above β_0. We now describe how to define $\mathcal{N}_1$ and δ_1 in case (b) holds. So suppose now that (b) holds. We wish to show that the $\mathcal{N}$ mentioned in (b) above can be taken to be in $\mathcal{M}$. Notice that (b) implies that:

> There exists an above-β_0, ω-maximal iteration tree $\mathcal{T}$ on $\mathcal{N}_0$ of some countable length $\theta + 1$, and an $\mathcal{N}' \trianglelefteq \mathcal{M}^{\mathcal{T}}_\theta$ and a $\delta \in \mathrm{OR}^{\mathcal{N}'}$ such that $\mathcal{N} \models$"δ is Woodin" and such that:
> (1) there exists a $\gamma < \alpha$ s.t. for all limit ordinals $\eta < \theta$, if we let $b = [0, \eta]_T$, then there is an above-$\delta(\mathcal{T}{\restriction}\eta)$ ω_1-iteration strategy for $Q(b, \mathcal{T}{\restriction}\eta)$ in $J_\gamma(\mathbb{R})$, and
> (2) for all ω-small δ-mice $\mathcal{P}$ such that δ is a cutpoint of $\mathcal{P}$ and $\mathcal{J}^{\mathcal{N}}_\delta = \mathcal{J}^{\mathcal{P}}_\delta$, if there is an above-$\delta$ ω_1-iteration strategy for $\mathcal{P}$ in $J_\alpha(\mathbb{R})$, then $\mathcal{P} \trianglelefteq \mathcal{N}$, and
> (3) if θ is a limit ordinal then either $\delta = \delta(\mathcal{T}{\restriction}\theta)$, or letting $b = [0, \theta]_T$ we have that there is an above-$\delta(\mathcal{T}{\restriction}\theta)$ ω_1-iteration strategy for $Q(b, \mathcal{T}{\restriction}\theta)$ in $J_\alpha(\mathbb{R})$.

The reason we can have (1) is the following. If for some $\eta < \theta$ $Q(b, \mathcal{T}{\restriction}\eta)$ were $(\alpha, 0)$-big above $\delta(\mathcal{T}{\restriction}\eta)$, then we could replace $\mathcal{T}$ by $\mathcal{T}{\restriction}\eta$. Thus if $\mathcal{T}$ is as in (b) and no proper initial segment of $\mathcal{T}$ is as in (b) then for all limit ordinals $\eta < \theta$, if we let $b = [0, \eta]_T$, then there is an above-$\delta(\mathcal{T}{\restriction}\eta)$ ω_1-iteration strategy for $Q(b, \mathcal{T}{\restriction}\eta)$ in $J_\alpha(\mathbb{R})$. Then the γ from (1) comes from the fact that $\mathcal{T}$ is countable and $\mathrm{cof}(\alpha) > \omega$.

The reason we can have (3) is that either $Q(b, \mathcal{T}{\restriction}\theta)$ is $(\alpha, 0)$-big above $\delta(\mathcal{T}{\restriction}\theta)$, in which case we can take $\mathcal{N} \trianglelefteq Q(b, \mathcal{T}{\restriction}\theta)$ and $\delta = \delta(\mathcal{T}{\restriction}\theta)$, or $Q(b, \mathcal{T}{\restriction}\theta)$ is $(\alpha, 0)$-small above $\delta(\mathcal{T}{\restriction}\theta)$, in which case there is an above-$\delta(\mathcal{T}{\restriction}\theta)$ ω_1-iteration strategy for $Q(b, \mathcal{T}{\restriction}\theta)$ in $J_\alpha(\mathbb{R})$.

The above is a $\Sigma_2(J_\alpha(\mathbb{R}))$ fact about $\mathcal{N}_0$ and β_0. Since $\mathcal{M}$ is $\Sigma_2(J_\alpha(\mathbb{R}))$-correct, there exists a $\mathcal{T} \in \mathcal{M}$ as above. Let $\theta + 1$ be the length of $\mathcal{T}$. Let $\mathcal{N}$ and δ be as given above.

For all limit $\eta < \theta$, $[0, \eta]_T$ is the unique ω_1-iterable branch, so $\mathcal{T} \restriction \theta$ is correct. If θ is a successor ordinal, then $\mathcal{T}$ is correct and we set $\mathcal{N}_1 = \mathcal{N}$ and $\delta_1 = \delta$.

Assume then that θ is a limit ordinal. Let $b = [0, \theta]_T$. Let $\delta' = \delta(\mathcal{T} \restriction \theta)$. If there is an above-$\delta'$ ω_1-iteration strategy for $Q(b, \mathcal{T} \restriction \theta)$ in $J_\alpha(\mathbb{R})$, then again $\mathcal{T}$ is correct and we set $\mathcal{N}_1 = \mathcal{N}$ and $\delta_1 = \delta$.

Assume then that there is not an above-δ' ω_1-iteration strategy for $Q(b, \mathcal{T} \restriction \theta)$ in $J_\alpha(\mathbb{R})$. Then by (3) $\delta = \delta'$. It follows that $\mathcal{N} \trianglelefteq \mathcal{Q}$. In this case we have said nothing to ensure that b is iterable. Let c be the unique cofinal branch of $\mathcal{T} \restriction \theta$ such that $Q(c, \mathcal{T} \restriction \theta)$ is ω_1-iterable above δ. If $c = b$ then again $\mathcal{T}$ is correct and we set $\mathcal{N}_1 = \mathcal{N}$ and $\delta_1 = \delta$.

So now assume that $c \neq b$. Let $\mathcal{Q}' = Q(c, \mathcal{T} \restriction \theta)$. Since $\mathcal{N}_0$ is $(\alpha, n-1)$-small above β_0, $\mathcal{Q}'$ is $(\alpha, n-2)$-small above δ. Thus $\mathcal{Q}' \in \mathrm{OD}^\alpha_{n-1}(\mathcal{J}^{\mathcal{Q}'}_\delta)$. Since $\mathcal{M}$ is $(\alpha, n-1)$-closed, $\mathcal{Q}' \in \mathcal{M}$. Notice that $\mathcal{Q}'$ is $(\alpha, 0)$-big above δ. This is because otherwise $\mathcal{Q}' \in \mathrm{OD}^{<\alpha}(\mathcal{J}^{\mathcal{Q}'}_\delta)$ and so $\mathcal{Q}' \trianglelefteq \mathcal{N} \trianglelefteq \mathcal{Q}$, which is impossible. In this case we set $\mathcal{N}_1$ to be the $\trianglelefteq$-least initial segment of $\mathcal{Q}'$ which is $(\alpha, 0)$-big above δ and we set $\delta_1 = \delta$.

So now in any case we have the following:

- $\mathcal{J}^{\mathcal{M}}_{\beta_0} \trianglelefteq \mathcal{N}_1$.
- $\mathcal{N}_1 \in \mathcal{M}$ and $\mathcal{N}_1$ is countable in $\mathcal{M}$.
- $\mathcal{N}_1 \models$"$\beta_0 < \delta_1$ and δ_1 is Woodin."
- $\mathcal{N}_1$ is $(\alpha, 0)$-big above δ_1.
- $\mathcal{N}_1$ is a δ_1-mouse and letting $k = k(\mathcal{N}_1, \delta_1)$ we have that $\mathcal{N}_1$ is (k, ω_1, ω)-iterable.

If $\mathcal{N}_1$ is $(\alpha, 1)$-small above δ_1, then as we argued above there is a good parameter for $J_\alpha(\mathbb{R})$ in $\mathcal{M}$ and we are done. So suppose that $\mathcal{N}_1$ is $(\alpha, 1)$-big above δ_1. Then taking the argument which gave us $\mathcal{N}_1$ from $\mathcal{N}_0$ and repeating it $n-4$ more times, we find that unless there is a good parameter for $J_\alpha(\mathbb{R})$ in $\mathcal{M}$, there is a mouse $\mathcal{N}_{n-3}$ such that

- $\mathcal{J}^{\mathcal{M}}_{\beta_0} \trianglelefteq \mathcal{N}_{n-3}$.
- $\mathcal{N}_{n-3} \in \mathcal{M}$ and $\mathcal{N}_{n-3}$ is countable in $\mathcal{M}$.
- $\mathcal{N}_{n-3} \models$"$\beta_0 < \delta_1 < \cdots < \delta_{n-3}$ and the δ_i are Woodin."
- $\mathcal{N}_{n-3}$ is $(\alpha, 0)$-big above δ_{n-3}.
- $\mathcal{N}_{n-3}$ is a δ_{n-3}-mouse and letting $k = k(\mathcal{N}_1, \delta_{n-3})$ we have that $\mathcal{N}_{n-3}$ is (k, ω_1, ω)-iterable.

Since $\mathcal{N}_{n-3}$ is countable in $\mathcal{M}$, $\mathrm{OD}^\alpha_n(\mathcal{J}^{\mathcal{M}}_{\beta_0}) \not\subseteq \mathcal{N}_{n-3}$. It follows from the proof of Lemma 5.7 that $\mathcal{N}_{n-3}$ is not $(\lambda, n-1)$-big above β_0. It then follows that $\mathcal{N}_{n-3}$ is $(\lambda, 2)$-small above δ_{n-3}. As we argued above, this gives us a good parameter in $\mathcal{M}$. $\diamondsuit$

Case 3. *$n \geq 4$ is even and there is a good parameter for $J_\alpha(\mathbb{R})$ in $\mathcal{M}$.*

In this case we will see that $\mathbb{R} \cap \mathcal{M}$ is $\Sigma_{n-1}(J_\alpha(\mathbb{R}))$-correct. Since $n-2$ is even, $\mathrm{OD}^{J_\alpha(\mathbb{R})}_{n-1} = C_{(\alpha, n-2)}$. Thus the fact that $\mathcal{M}$ is strongly closed under OD^α_{n-1} means that $\forall x \in \mathbb{R}^{\mathcal{M}} \exists \eta < \omega_1^{\mathcal{M}}\ C_{(\alpha, n-2)}(x) \subseteq \mathcal{J}^{\mathcal{M}}_\eta$. In particular, $\mathbb{R} \cap \mathcal{M}$ is closed downward under $\Delta_{(\alpha, n-2)}$. Since there is a good parameter for $J_\alpha(\mathbb{R})$ in $\mathcal{M}$, $\mathbb{R} \cap \mathcal{M}$ is $\Sigma_{n-1}(J_\alpha(\mathbb{R}))$-correct.

Case 4. *$n \geq 4$ is even and there is* not *a good parameter for $J_\alpha(\mathbb{R})$ in $\mathcal{M}$.*

In this case we cannot prove that $\mathbb{R} \cap \mathcal{M}$ is any more than $\Sigma_1(J_\alpha(\mathbb{R}))$-correct. We conjecture that this case cannot happen but we don't see how to prove this. We can prove that in this case, $<^{\mathcal{M}}$ is $\Delta_2(J_{\bar{\alpha}}(A), \{z_0\})$.

Towards this end, consider the following condition P on mice $\mathcal{N}$. Say $P(\mathcal{N})$ iff

$\mathcal{N}$ is an ω-mouse, $\mathcal{J}^{\mathcal{M}}_{\beta_0} \triangleleft \mathcal{N}$, and for all $\alpha' < \alpha$, $\mathcal{N}$ is $\Pi_{(\alpha', n-2)}$-iterable above β_0.

Notice that P is a $\Pi_1(J_\alpha(\mathbb{R}), \{z_0\})$ condition and $P(\mathcal{N})$ holds for all ω-mice $\mathcal{N} \trianglelefteq \mathcal{M}$ with $\beta_0 \in \mathcal{N}$. To see that $<^{\mathcal{M}}$ is $\Delta_2(J_{\bar{\alpha}}(A), \{z_0\})$ it suffices to see that there does *not* exist an $\mathcal{N} \in \mathcal{M}$ such that $P(\mathcal{N})$ and $\mathcal{N} \ntrianglelefteq \mathcal{M}$.

Suppose towards a contradiction that there were such an $\mathcal{N}$. Let $\mathcal{Q} \triangleleft \mathcal{M}$ be an ω-mouse such that $\beta_0 \in \mathcal{Q}$ and $\mathcal{Q} \ntrianglelefteq \mathcal{N}$ and $\mathcal{N} \ntrianglelefteq \mathcal{Q}$. As we argued previously, $\mathcal{Q}$ is $(\alpha, n-1)$-small above β_0. By part (3) of Theorem 4.11 then, $\mathcal{N}$ is not $\Pi_{(\alpha,n-2)}$ iterable above β_0. Let $w \in \mathcal{M}$ be a real coding $\langle \mathcal{N}, \beta_0 \rangle$. Since $P(\mathcal{N})$ holds, w is an $(n-2)$-good parameter for $J_\alpha(\mathbb{R})$. Since there is such a w in $A = \mathbb{R} \cap \mathcal{M}$, and $n-2$ is even, we have by Lemma 3.20 that there is a good parameter for $J_\alpha(\mathbb{R})$ in A. This is a contradiction. $\square$

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