Remarks on a paper by Sargsyan

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§1.

In this addendum to Sargsyan's paper "Covering with UB operators" [1], we show how to get models of $AD_{\mathbb{R}}$ + "Θ is measurable" from similar hypotheses. The same proof gives significantly stronger large cardinal properties of Θ.

Let κ_0 be measurable, and a limit of Woodins and $<\kappa_0$-strongs. We have then

$$D(V, \kappa_0) \models AD_{\mathbb{R}} + DC$$

for the derived model of V below κ_0. We assume

(†) $\mathrm{Hod}^{D(V,\kappa_0)}$ can be analyzed as a hod-premouse.

(†) holds as long as $D(V, \kappa_0)$ has not reached hod pairs of LST-type, and hence well past the target we are aiming at now. Set

$$H_0 = \mathrm{HOD}^{D(V,\kappa_0)}.$$

Now fix

$$j : V \longrightarrow M$$

with $\mathrm{crit}(j) = \kappa_0$ witnessing κ_0 measurable. Let

$$\Sigma_{H_0} = \bigoplus_{\alpha < o(H_0)} \Sigma_{H_0(\alpha)}$$

be the join of the strategies for initial segments of H_0 coming out of the hod-limit. Σ_{H_0} is a $< j(\kappa)$-strategy for H_0 in M, and in fact, letting $\kappa_1 = j(\kappa)$

$$\Sigma_{H_0} \in D(M, \kappa_1).$$

Put

$$H_0^+ = \left(L_p^{\Sigma_{H_0}}(H_0)\right)^{D(M,\kappa_1)}.$$

We assume

$$(\dagger\dagger) \quad o(H_0^+) < K_0^+ .$$

This is true, for example, if $\neg \Box_{K_0}$.

<u>Remark</u> We are essentially continuing ~~too~~ from §11 of Sargsyan's paper. There K_0 is called η, and H_0^+ is called P. We have added that K_0 is a limit of Woodins because this simplifies a few things. It implies that the maximal model below K_0 is just $D(V, K_0)$.

Let

$$\Theta^0 = o(H_0).$$

Sargsyan shows

(1) H_0 is full in $D(M, K_1)$, that is, no level of H_0^+ projects strictly below Θ^0,

(2) $H_0^+ \models \Theta^0$ is regular.

Sargsyan also assumes $\mathrm{cf}(o(H_0^+)) \neq \kappa_0$, but ~~was~~ this seems to already follow from $o(H_0^+) \neq \kappa_0^+$, as we now show.

<u>Lemma 0</u>. $\mathrm{cf}(o(H_0^+)) < \kappa_0$.

<u>Proof</u>. Assume $\mathrm{cf}(o(H_0^+)) = \kappa_0$. Let $H_1 = j(H_0)$ and $H_1^+ = j(H_0^+)$. Let $\mathcal{P} \trianglelefteq H_1^+$, $\rho_\omega(\mathcal{P}) = \theta' = o(H_1)$, and $j'' o(H_0^+) \subseteq (\theta')^{+\mathcal{P}}$. Let $j_1 = j(j)$, and $j_1 : M \longrightarrow M_2$. $\mathcal{P}$ is a Σ_{H_1}-mouse in $D(M_2, \kappa_2)$. (Here $\kappa_1 = j(\kappa_0)$ and $\kappa_2 = j_1(\kappa_1)$.) Let Λ be the ~~canonical~~ unique iteration strategy Φ for $\mathcal{P}$ with respect to trees

3b

with all critical points $> \theta^1$ that moves Σ_{H_1} correctly. We have Λ in $D(M_2, k_2)$.

In $D(M_2, k_2)$ we have a ~~[illegible]~~ ~~[illegible]~~ hod pair (W, Ψ) such that for some λ

(a) W is a Σ_{H_0}-hod-premouse extending H_0^+

(b) $W \models$ ~~[illegible]~~$_\lambda$ is a limit of Woodins $> o(H_0^+)$

(c) For some $\alpha < \lambda$, Φ is projective in $\Psi \restriction$ trees on $W(\alpha)$.

Thus $\mathbb{R}^{D(M_2, k_2)}$-genericity iterations ~~with~~ ~~part~~ of W ~~with~~ are such that Φ^{*} is reached in the derived model. A "local HOD-limit" argument shows that we may assume $(W, \Psi) \in M_2$. Let

$$E = E_j \cap ([\theta^0]^{<\omega} \times H_0),$$

* $\Phi = \Lambda$

and

$$\mathcal{U} = \mathrm{Ult}(W, E).$$

We have

$$W \xrightarrow[i_E]{} \mathcal{U} \xrightarrow[k]{} j(W)$$

with $k \circ i_E = j \restriction W$. If j were strong enough (e.g. a rank-to-rank embedding), we could argue at once that $\mathcal{U}$ is an iterable hod-pair over Σ_H, whose derived model reaches Φ. That would imply at once that $\mathcal{P} \in \mathcal{U}$. But $\mathcal{P} \notin \mathcal{U}$, because

$$i_E(o(H_0^+)) = \sup j'' o(H_0^+) = (\theta^{*+})^{\mathcal{P}},$$

so $\mathcal{P}$ collapses $i_E(o(H_0^+))$.

As in [1], we make do with a weaker j by taking a Skolem hull.

Let $\pi : N \longrightarrow V_\gamma$ where N is transitive, $V_{\kappa_0} \subseteq N$, $\kappa_0 \in N$, and $|N| = \kappa_0$. Put

$$\bar{W} = \pi^{-1}(W),$$

$$\bar{P} = \pi^{-1}(P),$$

and so on. We have

$$\bar{W} \xrightarrow[i_{\bar{E}}]{} \bar{U} \xrightarrow[\bar{k}]{} \overline{j(W)} \xrightarrow[\pi]{} j(W).$$

Notice that $(j(W), j(\Psi))$ is a hod pair over Σ_{H_1} in $j(M_2)$, and $\pi \in j(M_2)$ because the latter is closed under κ-sequences. So we have a pullback strategy

$$\Sigma_{\bar{U}} = \Sigma_{j(W)}^{\pi \circ \bar{k}}$$

for $\bar{U}$ in $j(M_2)$. Sargsyan has shown that $\Sigma_{\bar{U}}$ extends $\dot{\Sigma}^g$, where $\dot{\Sigma}$ is the canonical term in $\bar{U}$ for the extension of its strategy to generic extensions. He has also shown this remains true for $\Sigma_{\bar{U}}$-iterates of $\bar{U}$.

<u>Remark</u> This is theorem 3.76 of the 3/25/10 version of Sargsyan's thesis [2]. See also 3.9, 3.10, and 3.28 of [3]

We also have that $\bar{\Psi} = \Sigma_{\bar{U}}^{i_E}$, by condensation for $j(\Psi)$.

~~Let $\bar{W} \longrightarrow S$ be a genericity iteration above H_0^+ so that for some g on δ_0^S, Φ^* is Wadge~~

g on $\mathrm{Col}(\omega, \delta_0^S)$ and

Fix $\alpha < \lambda^{\overline{W}}$ and a real $z \in D(j(M_2), \kappa_2)$ (3f)

such that ~~is $\overline{\Psi} \restriction \widehat{W}(\alpha)$ is~~

$\text{Code}(\overline{\Phi})$ is Wadge reducible to $\text{Code}(\overline{\Psi} \restriction \widehat{W}(\alpha))$ via z. (Note $\overline{\Phi}$, $\overline{\Psi}$ are defined on all of $D(j(M_2), \kappa_3)$ because π exists. Iterate $\overline{U}$ by $\Sigma_{\overline{U}}$ above $i_E(\delta_\alpha^{\overline{W}})$ to R

$$\begin{tikzcd} \overline{U} \arrow[r, "\ell"] & R \\ \overline{W} \arrow[u, "i_{\vec{E}}"] & \end{tikzcd}$$

so that z and $i_E \restriction \overline{W}(\alpha)$ are in $R[g]$, for g on $\text{Col}(\omega, \delta_1^R)$. Then the derived model of $R[g]$ can compute $\overline{\Psi} \restriction \overline{W}(\alpha)$ as a pullback of one of R's strategies. Since it has z, it has $\bar{\Phi}$.

This gives that $\bar{\Phi}$ is $OD(\bar{\Sigma}_{H_1})$ in the derived model of R.

that gives $\bar{P} \in R$, so $\bar{P} \in \bar{U}$, as desired.

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<u>Remark</u> Sargsyan used this sort of argument pretty heavily.

$D(V, \kappa_0)$ sees H_0, but not H_0^+. However, we can add H_0^+ to $D(V, \kappa_0)$ without changing much. The following lemma emerged in conversation with Nam Trang.

<u>Lemma 1</u>. Let g be $Col(\omega, <\kappa_0)$-generic over V; then

(a) $P(\mathbb{R}^*_g) \cap L(\mathrm{Hom}^*_g, H_0^+) \subseteq \mathrm{Hom}^*_g$,

and

(b) $L(\mathrm{Hom}^*_g, H_0^+) \models \Theta^0$ is regular.

<u>Proof</u>. We can obtain $L(\mathrm{Hom}^*_g, H_0^+)$ as a "symmetric Vopenka" extension of H_0^+. A condition is a pair (n, A), where for some $\gamma < \Theta^0$

$$A \subseteq (\omega_\gamma)^n$$

is OD over $L[\mathrm{Hom}^*_g]$.

<u>Lemma A</u>

$(n, A) \leq (m, B)$ iff $m \leq n$ and

$\forall s \in A \, (s \upharpoonright m \in B)$.

$Vop_{<\omega}$ is the resulting poset, coded as a subset of Θ^0 that is OD over $L(Hom^*_g)$. Thus $Vop_{<\omega} \in H_0^+$. Given any G that is $Vop_{<\omega}$-generic over H_0^+, let

$$s_i^G(n) = \xi \text{ iff } \exists (k, A) \in G \, (k > n \wedge \forall t \in A \, (t(i)(n) = \xi)).$$

Any $p \in Vop_{<\omega}$ can be extended by a G that is H_0^+-generic and such that

$$^{\omega}\Theta^0 \cap L[Hom^*_g] = \{ s_i^G \mid i \in \omega \}.$$

This is proved as usual, looking at G's induced by H_0^+-generic maps from ω onto $\bigcup_{\gamma < \Theta^0} ({}^{\omega}\gamma)^{L[Hom^*_g]}$,

<u>Claim</u> For any $\mathbb{V}_{op<\omega}$-generic G over H_0^+,

$$L[\{s_i^G \mid i\in\omega\}, H_0^+] \cap \bigcup_{\gamma<\Theta^0} {}^{\omega}\gamma = \{s_i^G \mid i \in \omega\}$$

<u>Proof</u> If not, we can reflect inside H_0^+ the failure of this. We get $\alpha < \Theta^0$ and $N \trianglelefteq H_0(\alpha)$ with $H_0 | \delta_\alpha^{H_0} \triangleleft N$ and $p \in \overline{\mathbb{V}_{op}}$ forcing the contrary over N. Let

$$\Gamma = \{A \in \mathrm{Hom}^*_g \mid w(A) < \delta_\alpha^{H_0}\}.$$

~~We can arrange~~

$$\cancel{\Gamma = P(\mathbb{R}^*_g) \cap L(\Gamma, \mathbb{R}^*_g)}$$

and We have

$$\Gamma = \mathrm{HOD}_\Gamma^{L[\mathrm{Hom}^*_g]} \cap \mathrm{Hom}^*_g .$$

So let G be $\overline{\mathbb{V}_{op}}$ generic over N and come from a generic map of ω

onto $\bigcup_{\gamma<\delta_\alpha^{H_0}} ({}^\omega\gamma)^{L[\mathrm{Hom}^*_g]}$. Since (7)

$N \in \mathrm{HOD}^{L[\mathrm{Hom}^*_g]}$, we get

$L(N, \{s_i^G \mid i\in\omega\}) \cap \bigcup_{\gamma<\delta_\alpha^{H_0}} ({}^\omega\gamma) \subseteq \mathrm{HOD}_\Gamma^{L[\mathrm{Hom}^*_g]}$,

so $= \{s_i^G \mid i\in\omega\}$, contradiction.

$\boxtimes$

The claim easily gives (a) of the lemma, noting that every set in Hom^*_g is coded by an ~~a~~ se ${}^\omega\gamma$, $\gamma<\Theta^0$.

For (b): ~~if p forces " $\varphi(\cdot,\cdot,s$~~

let $f : \alpha \longrightarrow \Theta^0$ with $\alpha<\Theta^0$ be definable over $L(H_0^+, \{s_i^G \mid i\in\omega\})$ from ordinals and $s_0^G \ldots s_{n-1}^G$, ~~(k)~~ via the formula φ. Let

$(k, A) \in G$, $n \leq k$, and (k, A) force that φ defines a cofinal map $\varphi^G : \check{\alpha} \longrightarrow \check{\Theta}^0$ from $S_0^G \ldots S_{n-1}^G$. Note that if $(m, B) \leq (k, A)$ forces "$\varphi^G(\check{\xi}) = \check{\eta}$", then in fact $(k, B \restriction k)$ forces $\varphi^G(\check{\xi}) = \check{\eta}$. (Use automorphisms permuting coordinates.) But there are $< \Theta^0$ many possible $(k, B \restriction k)$ by $AD_{\mathbb{R}}$ in $L(\mathrm{Hom}_g^*)$. Since $H_0^+ \models \Theta_0$ is regular, we have a contradiction.

This proves lemma 1.

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Truth in $L(\mathrm{Hom}_g^*, H_0^+)$ is independent of g, so we write

$$D(V, \kappa_0)^+ = L(\mathrm{Hom}_g^*, H_0^+)$$

in the sloppy analog of the $D(V, \kappa_0)$ notation.

The key to Sargsyan's proof that $H_0^+ \models \theta^0$ is regular is Lemma 11.15 of [1], which states that $j \restriction H_0^+$ "has condensation.". We now introduce a slight strengthening of this property, and use it to go on.

Let

$$H_1 = j(H_0),$$
$$H_1^+ = j(H_0^+).$$

Recall that if $\pi : P \to Q$ and Σ is an iteration strategy for Q, then Σ^π is the π-pullback of Σ. So Σ^π is a strategy for P.

The proof of the following lemma follows that of 11.15 in [1] pretty much word-for-word.

<u>Lemma 2</u>. Suppose that in $M[h]$, where h is $\mathrm{Col}(\omega, <\kappa_1)$ generic over M, we have the commutative diagram

$$\begin{tikzcd}
H_0^+ \arrow[rr, "j"] \arrow[dr, "i"'] & & H_1^+ \\
& Q \arrow[ur, "\tau"] \arrow[r, "k"'] & R \arrow[u, "\sigma"']
\end{tikzcd}$$

where Q and R are countable in $M[h]$.

Let
$$\Sigma_Q = \left(\Sigma_{H_1^+}\right)^{\tau},$$
and
$$\Sigma_R = \left(\Sigma_{H_1^+}\right)^{\sigma}$$

be the pullback strategies. Then for any $A \in H_0^+$ and any formula φ:

$$D(M, \kappa_1)^+ \models \forall s \in Q$$
$$\left[\varphi(Q, s, \Sigma_Q, H_1^+, j(A)) \Longleftrightarrow \varphi(R, k(s), \Sigma_R, H_1^+, j(A)) \right].$$

Remarks

(1) $D(M, k_1)^+ = L(\mathrm{Hom}^*_h, H_1^+)$, in the notation we are using. Note Σ_Q, Σ_R, H_1^+, and $j(A)$ are all in $D(M, k_1)^+$. (E.g.; Σ_Q depends only on $\tau r i(\Theta^0)$, which is in $L[\mathrm{Hom}^*_h]$.)

(2) Lemma 11.15 of [21] is the lemma above, but for a certain φ. It ~~is~~ ~~not~~ seems that one must first show $H_0^+ \models \Theta^0$ regular, using the concrete φ, ~~[illegible]~~ before proving Lemma 2 above.

~~(3)~~

~~**Corollary 3** Let~~

$$\begin{tikzcd} H_0^+ \arrow[rr, "j"] \arrow[dr, "i"'] & & H_1^+ \\ & Q \arrow[ur, "r"'] & \end{tikzcd}$$

~~factor j as in the lemma. Let $A \in H_0^+$. Then~~

$$\cancel{r(i(A)) \cap \Theta' = j(A) \cap \Theta'.}$$

(3) For the sake of completeness, we prove Lemma 2 in an appendix.

(4) The appendix was revised (correcting an error) in January 2015.

Since $H_0^+ \models \theta^0$ is regular, we can think of Σ_{H_0} as an iteration strategy for H_0^+. We have

<u>Corollary 3</u> In $D(M, \kappa_1)$, Σ_{H_0} is a fullness preserving strategy for H_0^+ having branch condensation.

<u>Proof</u> Let $i : H_0^+ \longrightarrow Q$ be an iteration map by Σ_{H_0}, given by some iteration tree $\mathcal{T}$ on H_0 that we let act on H_0^+, with $\mathcal{T}$ countable in $D(M, \kappa_1)$. We have

$$\begin{tikzcd}
H_0^+ \arrow[rr, "j"] \arrow[dr, "i"'] & & H_1^+ \\
& Q \arrow[ur, "k"'] &
\end{tikzcd}$$

where

$$k(i(f)(a)) = j(f)(\pi_{Q^-,\infty}(a)),$$

with $Q^- = Q | i(\theta^0)$, and $\pi_{Q^-,\infty} : Q^- \longrightarrow H_1$

being the iteration map according to the tail of Σ_{H_0}. One way to see that k is well-defined, elementary, and the diagram commutes is: let

$$Z = \pi_{\vec{Q},\infty}{}''\, i(\theta^0).$$

Then

$$E_i = E_j \restriction Z,$$

because the measures in E_i concentrate on bounded subsets of θ^0, and for $a \in [i(\theta^0)]^{<\omega}$

$$\begin{aligned}(a, X) \in E_i \text{ iff } & a \in i(X) \\ \text{iff } & \pi_{\vec{Q},\infty}(a) \in \pi_{\vec{Q},\infty}(i(X)) \\ \text{iff } & \pi_{\vec{Q},\infty}(a) \in j(X) \\ \text{iff } & (\pi_{\vec{Q},\infty}(a), X) \in E_j \restriction Z.\end{aligned}$$

We use here that $j \restriction H_0$ is the iteration map by Σ_{H_0}. But then

$$k : \mathrm{Ult}(H_0^+, E_i) \longrightarrow \mathrm{Ult}(H_0^+, E_j)$$

is just the natural factor map.

The existence of k implies that Q is wellfounded immediately. But bringing in Lemma 2, we have

$$D(M, \kappa_1) \models H_0^+ \text{ is full},$$

so

$$D(M, \kappa_1) \models Q \text{ is full}.$$

Finally, branch condensation for Σ_{H_0} on H_0 immediately implies branch condensation for Σ_{H_0} on H_0^+.

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We now obtain a measure on $P(\theta^0) \cap H_0^+$ as follows. Let

$$\gamma_0 = \sup j'' \theta^0.$$

Note that $\gamma_0 = \pi_{H_0^+, H_1}(\theta^0)$, where $\pi_{H_0^+, H_1}$ is the hod-limit map of $D(M, \kappa_1)$.

For $A \subseteq \theta^0$ in H_0^+, put

$$A \in \nu_0 \text{ iff } \gamma_0 \in j(A).$$

<u>Lemma 4</u> (H_0^+, ν_0) is amenable.

<u>Proof</u> Let $\langle A_\alpha \mid \alpha < \theta^0 \rangle \in H_0^+$. Letting $C = \{\alpha \mid A_\alpha \in \nu_0\}$, it is enough to see that C is OD in $D(M, \kappa_1)$ from H_0^+ and Σ_{H_0}. But $j(\langle A_\alpha \mid \alpha < \theta^0 \rangle) \in H_1^+$, so letting $\cancel{\langle B_\alpha \rangle}$

$$\langle B_\alpha \mid \alpha < \theta^1 \rangle = j(\langle A_\alpha \mid \alpha < \theta^0 \rangle)$$

we have

$$\langle B_\alpha \cap (\gamma_0 + 1) \mid \alpha < \gamma_0 \rangle \in H_1.$$

Moreover

$$\alpha \in C \text{ iff } \gamma_0 \in j(\vec{A})_{j(\alpha)}$$

$$\text{iff } \gamma_0 \in B_{\pi^M_{H_0, H_1}(\alpha)} \cap (\gamma_0 + 1).$$

Since H_1 is OD in $D(M, \kappa_1)$, and $\pi^M_{H_0, H_1}$ is OD in $D(M, \kappa_1)$ from Σ_{H_0}, we are done.

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<u>Lemma 5</u> ν_0 is H_0^+-normal.

<u>Proof</u> Let $f : \theta^0 \to \theta^0$ be in H_0^+, and $j(f)(\gamma_0) < \gamma_0$. We must find $\alpha < \theta^0$ such that

$$j(f)(\gamma_0) = j(\alpha) = \pi_{H_0, H_1}(\alpha).$$

Since $j(f)(\gamma_0) < \gamma_0$, we can find a factoring

$$\begin{tikzcd} H_0 \arrow[rr, "j"] \arrow[dr, "i"'] & & H_1 \\ & Q \arrow[ur, "k"'] & \end{tikzcd}$$

with i, k being the iteration maps ~~[illegible]~~ by Σ_{H_0}, and $j(f)(\gamma_0) \in \operatorname{ran}(k)$.

As in the proof of Corollary 3, we can extend i, k to

$$\begin{tikzcd} H_0^+ \arrow[rr, "j"] \arrow[dr, "i"'] & & H_1^+ \\ & Q \arrow[ur, "k"'] & \end{tikzcd}$$

Notice that $\Sigma^k_{H_1}$ = Q-tail of Σ_{H_0} = $(\Sigma_{H_0})_{Q,\mathcal{T}}$, where $i = i^{\mathcal{T}}$. This is because Σ_{H_0} has branch condensation, so it pulls back under its own iteration maps to itself. (I.e. it has pullback condensation.)

Setting $\Sigma_Q = \Sigma^k_{H_1}$, we have

$$D(M, K_1)^+ \models \text{"} j(f)(\pi_{Q,\infty}(i(\Theta^0))) \in \operatorname{ran}(\pi_{Q,\infty} \restriction i(\Theta^0)) \text{"}.$$

The right hand side has the form $\varphi(Q, i(\Theta^0), \Sigma_Q, j(f))$. By Lemma 2,

$\varphi(H_0^+, \theta^0, \Sigma_{H_0}, j(f))$ holds in $D(M, \kappa_1)^+$. But this means that $j(f)(\gamma_0) \in \mathrm{ran}(\pi_{H_0,\infty} \restriction \theta^0)$, as desired.

⊠

In order to get a model of $AD_{\mathbb{R}}$ + Θ measurable, we'd like to show that $L(H_0^+, \nu_0) \cap P(\theta^0) \subseteq H_0^+$. For this, we need that (H_0^+, ν_0) is iterable. The problem is that ν_0-ultrapowers are discontinuous at θ^0.

To illustrate, suppose

$$\begin{tikzcd} H_0^+ \arrow[rr, "j"] \arrow[dr, "i"'] & & H_1^+ \\ & Q \arrow[ur, "k"'] & \end{tikzcd}$$

where $Q = \mathrm{Ult}(H_0^+, \nu_0)$, and

$$k(i(f)(\theta^0)) = j(f)(\gamma_0)$$

is the j-realization map, which exists by our definition of $\mathcal{D}_0$. Let

$$\Sigma_Q = \Sigma^k_{H_1^+}.$$

Σ_Q extends the Q-tail of Σ_{H_0}, but the latter acts on only $Q \restriction \sup i''\theta^0$, whereas Σ_Q can handle iteration trees on the full Q. Since

$D(M,\kappa_1)^+ \models \Sigma_{H_0}$ is a fullness-preserving strategy with branch condensation for H_0^+,

we have

$D(M,\kappa_1)^+ \models \Sigma_Q$ is a fullness-preserving strategy with branch condensation for Q.

Thus Σ_Q really is fullness-preserving and has branch condensation, and (Q, Σ_Q) is in the hod-limit system of $D(M,\kappa_1)$.

Let

$$\pi_{Q,\infty} : Q \longrightarrow H_1$$

be the map of the system on Q. To go

further, we need

<u>Claim</u>

(a) $\pi_{Q,\infty} \restriction Q^- = k \restriction Q^-$,

(b) For all $A \subseteq i(\theta^0)$ in Q,

$$A \in i(\mathcal{U}_0) \text{ iff } \sup k'' i(\theta^0) \in k(A).$$

<u>Proof</u>. For (a), it's enough to show $\pi_{Q,\infty}$ agrees with k on $i(\theta^0)$. Let $\xi < i(\theta^0)$. We have

$$\xi = i(f)(\theta^0),$$

where $f : \theta^0 \to \theta^0$ is in H_0^+. But then

$$D(M,\kappa_1)^+ \models \pi_{H_0^+,\infty}(f) = j(f) \restriction \pi_{H_0^+,\infty}(\theta^0)$$

so

$$D(M,\kappa_1)^+ \models \pi_{Q,\infty}(i(f)) = j(f) \restriction \pi_{Q,\infty}(i(\theta^0)).$$

This gives

$$k(\xi) = j(f)(\gamma_0)$$

$$= \pi_{Q,\infty}(i(f))\left(\pi_{Q,\infty}(\theta^0)\right)$$

$$= \pi_{Q,\infty}\left(i(f)(\theta^0)\right)$$

$$= \pi_{Q,\infty}(\xi),$$

as desired for (a).

For (b), let $\xi < o(H_0^+)$ and

$$\nu_0^\xi = \nu_0 \cap H_0^+|\xi.$$

We need to see that $i(\nu_0^\xi)$ is contained in the measure generated by k. Let

$$P(\theta^0) \cap H_0^+|\xi = \langle A_\alpha \mid \alpha < \theta^0 \rangle.$$

~~and~~

$$\cancel{\alpha \in C \text{ iff } A_\alpha \in \nu_0^\xi.}$$

~~Put $A(\alpha,\beta)$ iff $\beta \in A_\alpha$. Then~~

$$\cancel{D(\ldots,\kappa_1)^+ \models \forall \alpha < \theta^0 \left(A_\alpha \in \nu_0^\xi \leftrightarrow \pi_{H_0^+,\infty}(\theta^0) \in j(\langle A_\alpha \mid \alpha < \theta^0 \rangle)\right)}$$

Then

$$D(M,\kappa_1)^+ \models \forall \alpha < \theta^0 \; \Big(A_\alpha \in \mathcal{V}_0^\xi \longleftrightarrow \pi_{H_0^+,\infty}(\theta^0) \in j(\langle A_\alpha \mid \alpha < \theta^0 \rangle)_{\pi_{H_0^+,\infty}(\alpha)} \Big)$$

This is a statement φ about H_0^+, $\mathcal{V}_0^\xi$, $\vec{A}$, Z_{H_0}, and $j(\vec{A})$. By Lemma 2,

$$D(M,\kappa_1)^+ \models \forall \alpha < i(\theta^0) \; \Big(i(\vec{A})_\alpha \in i(\mathcal{V}_0^\xi) \longleftrightarrow \pi_{Q,\infty}(i(\theta^0)) \in j(\langle A_\alpha \mid \alpha < \theta^0 \rangle)_{\pi_{Q,\infty}(\alpha)} \Big).$$

But $\pi_{Q,\infty}(i(\theta^0)) = \sup k'' i(\theta^0)$ by (a).

So indeed $i(\mathcal{V}_0^\xi)$ is generated by k, as desired.

Claim. $\boxtimes$

By the claim, we can k-realize $Ult(Q, i(\mathcal{V}_0))$, and keep going.

We have now illustrated the main ideas in the proof of

<u>Lemma 6</u> In $D(M, K_1)$, there is an iteration strategy Ω for (H_0^+, ϑ_0) such that if $i : (H_0^+, \vartheta_0) \longrightarrow (Q, \vartheta_Q)$ is an iteration map via Ω, and $i = i^{\vec{T}}$, then there is $k : Q \longrightarrow H_1^+$ so that

$$\begin{tikzcd}
H_0^+ \arrow[rr, "i"] \arrow[dr, "i"'] & & H_1^+ \\
& Q \arrow[ur, "k"'] &
\end{tikzcd}$$

commutes, and

(a) $\Omega_{Q,\vec{T}} = (\Sigma_{H_1})^k$, for iterations based on $Q|i(\theta^0)$,

(b) $\pi_{Q|i(\theta^0),\infty} = k \upharpoonright (Q|i(\theta^0))$, where $\pi_{Q|i(\theta^0),\infty}$ is the iteration map by $\Sigma_{H_1}^k$,

and (c) For $A \subseteq i(\theta^0)$ in Q

$A \in \vartheta_Q$ iff $\sup k''i(\theta^0) \in k(A)$.

Proof We proceed by induction on the length of the stack $\vec{\mathcal{I}}$ yielding i and Q to show that there are k and $\Omega_{Q,\vec{\mathcal{I}}}$ as in (a)–(c). We then show that the resulting Ω is sufficiently absolutely definable that $\Omega \in D(M, \kappa_1)$.

Notice that images of ν_0 cannot contribute to branching in our normal components of $\vec{\mathcal{I}}$. That is, if

$$E_\alpha^{\mathcal{I}_\xi} = i(\nu_0)$$

where $i : (H_0^+, \nu_0) \to (M_\alpha^{\mathcal{I}_\xi}, i(\nu_0))$ is an iteration map, then $E_\alpha^{\mathcal{I}_\xi}$ is applied to $M_\alpha^{\mathcal{I}_\xi}$. ~~That is, the next model of $\vec{\mathcal{I}}$ (either $M_{\alpha+1}^{\mathcal{I}_\xi}$ or $M_0^{\mathcal{I}_{\xi+1}}$~~ That is, $M_{\alpha+1}^{\mathcal{I}_\xi} = \mathrm{Ult}(M_\alpha^{\mathcal{I}_\xi}, i(\nu_0))$. Moreover, the rest of $\vec{\mathcal{I}}$ can be considered as a normal stack on $M_{\alpha+1}^{\mathcal{T}_\xi}$, because no model in $\mathcal{T}_\xi$ can have an extender overlapping $i(\delta^0)$, which is a limit of Woodins.

Thus we may as well assume that the trees in our stack $\vec{\mathcal{T}}$ are such that for each ξ, either $\mathcal{T}_\xi$ never uses an image of ϑ_0, or else $\mathcal{T}_\xi$ consists of exactly one ultrapower, and that by an image of ϑ_0.

Suppose by induction that we have defined enough of Ω to determine that $\vec{\mathcal{T}}$ is by Ω, and $i = i^{\vec{\mathcal{T}}}$ is such that we have

$$\begin{tikzcd} H_0^+ \arrow[rr, "i"] \arrow[dr, "i"'] & & H_1^+ \\ & Q \arrow[ur, "k"'] & \end{tikzcd}$$

with (a)-(c) of the lemma being true.

Let

$$l : (Q, \vartheta_Q) \longrightarrow (R, \vartheta_R)$$

be such that $l = i^{\mathcal{U}}$, where $\mathcal{U}$ is the next normal tree in our stack.

<u>Case 1</u> $\mathcal{U}$ does not involve an image of ϑ_Q, that is, $\mathcal{U}$ is actually a tree on Q.

<u>Prf</u>. Then we do get l (that is, a good branch of $\mathcal{U}$) from $\Omega_{Q,\vec{\mathcal{I}}} = \Sigma^k_{H_1} = \Sigma_Q$.

We have

$$\begin{tikzcd} Q \arrow[rr, "k"] \arrow[dr, "l"'] & & H_1^+ \\ & R \arrow[ur, "\sigma"'] & \end{tikzcd}$$

where $\sigma(l(f)(a)) = k(f)(\pi_{R,\infty}(a))$.

Here $\pi_{R,\infty}$ is the iteration map by the tail of Σ_Q. σ is well-defined and elementary because it is the factor map;

letting
$$Z = \pi_{R,\infty}{}'' \, l \circ i(\theta^0),$$

$$E_l = E_k \upharpoonright Z .$$

And that is true because component measures of E_l concentrate on bounded subsets of $i(\theta^0)$, and

$$\begin{aligned}
(a,x) \in E_l \quad &\text{iff } a \in l(x) \\
&\text{iff } \pi_{R,\infty}(a) \in \pi_{R,\infty}(l(x)) \\
&\text{iff } \pi_{R,\infty}(a) \in \pi_{Q,\infty}(x) \\
&\text{iff } \pi_{R,\infty}(a) \in k(x) \\
&\text{iff } (\pi_{R,\infty}(a), x) \in E_k .
\end{aligned}$$

The second-to-last line uses (b) of our induction hypothesis.

This gives us ψ. We must see (a)–(c). For (a), we have to see that $\Omega_{R,\vec{a}^\frown u} = \Sigma^{\tau}_{H_1}$. But this is true because $\sigma \restriction (R|(\ell \circ i(\theta^0)))$ is the iteration map by $\Omega_{R,\mathcal{T}^\frown\mathcal{U}}$ by definition, and $\Omega_{R,\mathcal{T}^\frown\mathcal{U}}$ has branch, hence pullback, condensation. (By Lemma 2.)

(b) holds because we have defined $\sigma \restriction \ell \circ i(\theta^0)$ to be the iteration map $\pi_{R,\infty}$.

Finally, (c) holds by an application of Lemma 2 that we leave to the reader.

<u>Case 2</u> $\mathcal{U}$ is the ν_Q-ultrapower, that is $R = \mathrm{Ult}(Q, \nu_Q)$ and $\ell : (Q, \nu_Q) \to (R, \nu_R)$ is the ultrapower map.

<u>Prf</u> We define $\sigma : R \to H_1^+$ by

$$\sigma(\ell(f)(i(\theta^0))) = k(f)(\sup k'' i(\theta^0)).$$

σ is well-defined, elementary, and

$$\begin{tikzcd} Q \arrow[rr, "k"] \arrow[dr, "\ell"'] & & H_1^+ \\ & R \arrow[ur, "\sigma"'] & \end{tikzcd}$$

commutes by induction hypothesis (e).

Now set

$$\Omega_{R,\vec{\jmath}^\frown u} = \Sigma^{r}_{H_1}.$$

$\Omega_{R,\vec{\jmath}^\frown u}$ is fullness-preserving, has branch condensation, and extends $\Omega_{Q,\vec{\jmath}}$ acting on trees based on $R \mid \sup \ell''1(\Theta^{\circ})$.
This all follows by applying Lemma 2 to transfer properties of ~~$\Omega_{Q,\vec{\jmath}}$~~ $\Sigma^{k}_{H_1} = \Sigma_Q$ to properties of $\Sigma^{r}_{H_1} = \Sigma_R$. And as in our illustrative example, we can use Lemma 2 to prove that (b) and (c) continue to hold. We omit further detail.

Lemma 6. $\boxtimes$

This completes our definition of Ω. We now show that $\Omega \in D(\mathcal{M}, \kappa_1)$. For this, just notice that we have defined Ω from the parameters H_1^+ and $j \restriction H_0^+$: for g on $Coll(\omega, \kappa_0)$,

$$\Omega(\mathcal{T}) = b \text{ iff } M[g] \models \varphi[\mathcal{T}, b, j \restriction H_0^+].$$

The reader can easily check that in $M[g]$, there are club many $X \prec V_\gamma^{M[g]}$ (γ large) s.t. if

$$\pi : N[g] \longrightarrow V_\gamma^{M[g]}$$

and $\mathcal{T}, b \in N[g][h]$ (h on $Coll(\omega, \nu)$, $\nu < \pi^{-1}(j(\kappa_0))$), then

$$\Omega(\mathcal{T}) = b \text{ iff } N[g][h] \models \varphi[\mathcal{T}, b, \pi^{-1}(j \restriction H_0^+)].$$

Basically, if you pull back under $\pi^{-1}(j)$, you'll be pulling back under j.

Since there are club many generically correct X, we have $\Omega \in D(M, \kappa_1)$.

$\boxtimes$

We can now show that $L(H_0^+, \nu_0) \cap P(\Theta^0) \subseteq H_0^+$. In fact,

<u>Lemma 7</u> $Lp^{\Omega}(H_0^+, \nu_0)^{D(M,\kappa_1)} \cap P(\Theta^0) \subseteq H_0^+$.

<u>Proof</u> If $\mathcal{P}$ is any sound Ω-mouse extending (H_0^+, ν_0) and projecting to Θ^0, then consider

$$i : \mathcal{P} \longrightarrow \mathrm{Ult}(\mathcal{P}, \nu_0) = Q.$$

We can think of Q as a hod-pair over (H_0^+, Σ_{H_0}). By comparison (which easily still works ~~on the measurable~~ for Q), $\mathcal{P}$ is then OD in $D(M, \kappa_1)$ from H_0^+ and Σ_{H_0}. This contradicts the fullness of H_0^+.

$\boxtimes$

§2. A second measure on Θ^0.

It's clear how to go on from here for a while. We may as well index ϑ_0 at $(\Theta^0)^{++}$ of $Lp^{\Sigma_{H_0}}[H_0]$, as usual. Let

$$\mathcal{H}_0 = (Lp^{\Sigma_{H_0}}[H_0] \mid \alpha, \vartheta_0^*)$$

where $\alpha = \Theta^{0++}$ and ϑ_0^* is the amenable coding of ϑ_0. We have that Ω is an iteration strategy for $\mathcal{H}_0$ in $D(M, \kappa_1)$, and it is fullness-preserving and has branch condensation there. Let

$$\mathcal{H}_0^+ = Lp^{\Omega}[\mathcal{H}_0],$$

and $\mathcal{H}_1 = j(\mathcal{H}_0)$, $\mathcal{H}_1^+ = j(\mathcal{H}_0^+)$.

Put also

$$\mathcal{D}(M, \kappa_1) = L(Hom_g^*, \mathcal{H}_1^+),$$

where g is $Coll(\omega, <\kappa_1)$ generic $/M$.
Again, $Hom_g^* = P(\mathbb{R}_g^*) \cap L\Theta(M, \kappa_1)$,
and Θ^I is still regular there.

The key lemma 2 now reads

__Lemma 2'__ Suppose that in $M[h]$, where h is $Coll(\omega, <\kappa_1)$ generic over M, we have the commutative diagram

$$\begin{tikzcd} \mathcal{H}_0^+ \arrow[rr, "j"] & & \mathcal{H}_1^+ \\ & Q \arrow[ul, "i"] \arrow[ur, "\tau"] \arrow[r, "k"'] & R \arrow[u, "\sigma"'] \end{tikzcd}$$

where Q, R are countable in $M[h]$. Let $\Sigma_Q = \Sigma_{\mathcal{H}_1^+}^{\tau}$ and $\Sigma_R = \Sigma_{\mathcal{H}_1^+}^{\sigma}$. Then for any $A \in \mathcal{H}_0^+$ and formula φ

$$L\Theta(M, \kappa_1) \models \forall s \in Q \, \big(\varphi[Q, s, \Sigma_Q, \mathcal{H}_1^+, j(A)]$$
$$\longleftrightarrow \varphi[R, k(s), \Sigma_R, \mathcal{H}_1^+, j(A)] \big).$$

That is, it reads the same, with H's changed to script $\mathcal{H}$'s. Our pullback strategies Σ_Q and Σ_R are now, like Ω, strategies for hod-premice with measurable limits of Woodins. Indeed, if i is an iteration map by Ω, then Σ_Q will just be a tail of Ω. But i may not be that, e.g., it may come from hitting an order I measure that we are trying now to find.

Now let

$$\pi_{\mathcal{H}_0^+,\infty} : \mathcal{H}_0^+ \longrightarrow \mathcal{H}_1$$

be the iteration map by Ω, and

$$\gamma_1 = \pi_{\mathcal{H}_0^+,\infty}(\theta^0).$$

We put for $A \subseteq \theta^0$ in $\mathcal{H}_0^+$:

$$A \in \mathcal{U}_1 \text{ iff } \gamma_1 \in j(A).$$

We can proceed as we did with $\mathcal{U}_0$.

Appendix

Proof of lemma 2

The following is just Sargsyan's proof, re-worded.

We say that (R,k) factors j iff

$k : R \to H_1^+$ and $\operatorname{ran}(j) \subseteq \operatorname{ran}(k)$.

So letting $i = k^{-1} \circ j$, we have

$$\begin{tikzcd}
H_0^+ \arrow[rr, "j"] \arrow[dr, "i"'] & & H_1^+ \\
& R \arrow[ur, "k"'] &
\end{tikzcd}$$

Fix now $A \in H_0^+$. We say that (R,k) respects $j(A)$ iff (R,k) factors j, and R is countable in $D(M,K_1)$, and whenever

$$\begin{tikzcd}
R \arrow[rrr, "k"] \arrow[dr, "l"'] & & & H_1^+ \\
& P \arrow[urr, "\sigma"] \arrow[r, "m"'] & Q \arrow[ur, "\tau"'] &
\end{tikzcd}$$

commutes, with P and Q countable in $D(M,K_1)$, then setting $\Sigma_P = \Sigma^{\sigma}_{H_1}$ and $\Sigma_Q = \Sigma^{\tau}_{H_1}$, we have

$$(*)\quad D(M,\kappa_1^*)^+ \models \forall s \in P\,\big(\varphi(P,s,\Sigma_P,j(A)) \longleftrightarrow \varphi(Q,m(s),\Sigma_Q,j(A))\big).$$

If (P,Q,ℓ,m,σ,τ) factors into k as above, but $(*)$ fails, then we call (P,Q,ℓ,m,σ,τ) a $\underline{j(A)\text{-bad}}$ $\underline{\text{factoring of } (R,k)}$. So (R,k) respects $j(A)$ just in case it admits no $j(A)$-bad factorings.

<u>Claim A.</u> There is an $R \in V^M_{\kappa_1}$ and $k \in M$ such that (R,k) factors j, and (R,k) respects $j(A)$.

Before proving the claim, we show that it implies lemma 2. ~~For~~ Lemma 2 says that (H_0^+, j) respects $j(A)$, for all $A \in H_0^+$. Putting $j_1 = j(j)$ and $M_1 = j(M) = j_1(M)$, it is enough to see

$$M_1 \models (H_1^+, j_1) \text{ respects } j_1(j(A)).$$

Note $j(j(A)) = j_1(j(A))$. So suppose that

$$\begin{tikzcd} H_1^+ \arrow[rrr, "j_1"] \arrow[dr, "l"'] & & & H_2^+ = j(H_1^+) \\ & P \arrow[urr, "\sigma"] \arrow[r, "m"'] & Q \arrow[ur, "\tau"'] & \end{tikzcd}$$

is a $j_1(j(A))$ – bad factoring of (H_1^+, j_1) in M_1. Now let (R, k) be as in the claim. Consider the diagram

$$\begin{tikzcd}
R \arrow[r, "k"] & H_1^+ \arrow[rr, "j_1"] \arrow[dr, "l"'] & & H_2^* \\
& & P \arrow[ur, "\sigma"] \arrow[r, "m"'] & Q \arrow[u, "\tau"']
\end{tikzcd}.$$

Note that $j_1(k) = j_1 \circ k$, because $\mathrm{crit}(j_1) = \kappa_1$. It is easy to see then that $(P, Q, l \circ k, m, \sigma, \tau)$ is a $j_1(j(A))$-bad factoring of $(R, j_1(k))$ in M_1. This is impossible because j_1 is elementary and $j_1(R) = R$.

So it is enough to prove Claim A.

Work in M for a moment. Let $\kappa_0 < \nu < \kappa_1$, with ν an inaccessible limit of Woodins and $<\kappa_1$-strongs. Set $R_\nu^- = \mathrm{HOD}_{|\Theta}^{D(M,\nu)}$, and let Σ_ν be the join of the strategies for

$R_\nu^-(\alpha)$, $\alpha < \delta^{R_\nu^-}$. Let $\pi_\nu : R_\nu^- \to H_1$ be the hod-limit map of $D(M, \kappa_1)$, which exists because (R_ν^-, Σ_ν) is a "limit hod pair" in $D(M, \kappa_1)$. Let

$$R_\nu = Lp^{\Sigma_\nu}(R_\nu^-)^{D(M,\kappa_1)}.$$

We say ν is <u>good</u> iff

$$R_\nu = \text{trans. collapse of } \mathrm{Hull}^{H_1^+}(\pi_\nu " R_\nu^- \cup j" H_0^+),$$

and if

$$\sigma_\nu : R_\nu \to H_1^+$$

is the transitive collapse, then $\sigma_\nu \restriction R_\nu^- = \pi_\nu$ is the iteration map by Σ_ν. Goodness can be defined in M, because $j" H_0^+ \in M$.

~~Claim A~~
<u>Subclaim A.1</u> In M, there are measure one many good $\nu < \kappa_1$.

<u>Proof</u> It is not hard to show that κ_1 is good in $j_1(M)$, vis-a-vis $j_{01}(j" H_0^+)$. $\boxtimes$

<u>Claim B</u>. Suppose claim A is false. Then there is in M an increasing ~~infinite~~ sequence $\langle \nu_i \mid i \in \omega \rangle$ of good ν's such that letting $R_i = R_{\nu_i}$ and $\sigma_i = \sigma_{\nu_i}$, we have ~~[illegible]~~ for every i, in $M^{\mathrm{Coll}(\omega, <\kappa_1)}$, a factoring

$$\begin{tikzcd}
 & & & H_i^+ & & & \\
R_i \arrow[urrr, "\sigma_i"] \arrow[rr, "l_i"'] & & P_i \arrow[ur, "\psi_i"] \arrow[rr, "m_i"'] & & Q_i \arrow[ul, "r_i"'] \arrow[rr, "k_i"'] & & R_{i+1} \arrow[ulll, "\sigma_{i+1}"']
\end{tikzcd}$$

such that $(P_i, Q_i, l_i, m_i, \psi_i, r_i)$ is a $j(A)$-bad factoring of (R_i, σ_i).

<u>Proof</u>. This is obvious. $\boxtimes$

Now fix a sequence $\langle (R_i, \sigma_i) \mid i \in \omega \rangle = (\vec{R}, \vec{\sigma})$

in M, as in claim B. Let

$$\pi_i = \sigma_{i+1}^{-1} \circ \sigma_i ,$$

so that $\pi_i : R_i \longrightarrow R_{i+1}$. Notice that if we have

$$\begin{tikzcd}
R_i \arrow[rrr, "\pi_i"] \arrow[dr, "l"'] & & & R_{i+1} \\
& P \arrow[r, "m"'] & Q \arrow[ur, "k"'] &
\end{tikzcd}$$

then there are φ and ψ such that $(P, Q, l, m, \varphi, \psi)$ is a factoring of (R_i, σ_i), namely $\psi = \sigma_{i+1} \circ k$ and $\varphi = \sigma_{i+1} \circ k \circ m$. So we shall call such a (P, Q, l, m, k) a factoring of (R_i, π_i, R_{i+1}), and we can speak of its being $j(A)$-bad.

Now we bring the badness of our factorings down from $D(M, \kappa_1)^+$ into $D(M, \kappa_1)$ itself.

<u>Def</u> (AD^+). Let (Δ, Ψ) be a hod-pair, and let $\pi : R \to \Delta$. Let $\xi, \gamma, \beta \in ORD$.

We say that (P, Q, l, m, k) is a

<u>(ξ, γ, β)-bad factoring of (R, π, Δ, Ψ)</u>

just in case

$$\begin{tikzcd} R \arrow[rrr, "\pi"] \arrow[dr, "l"'] & & & \Delta \\ & P \arrow[r, "m"'] & Q \arrow[ur, "k"'] & \end{tikzcd}$$

commutes, and letting $\Sigma_P = \Psi^{k \circ m}$ and $\Sigma_Q = \Psi^k$, we have that for some $s \in P$ and formula φ:

$$L_\beta(P_\xi(\mathbb{R})) \models \varphi[P, s, \Sigma_P, \gamma]$$

but

$$L_\beta(P_\xi(\mathbb{R})) \not\models \varphi[Q, m(s), \Sigma_Q, \gamma].$$

(Here $P_\xi(\mathbb{R}) = \{A \subseteq \mathbb{R} \mid w(A) < \xi\}$, and we assume $\Sigma_P, \Sigma_Q \in P_\xi(\mathbb{R})$.)

(Aa)

Def (AD^+) ~~Let~~ We say that $\langle (\mathcal{S}_n, \pi_n, \Psi_n) \mid n<\omega \rangle$ is a (ξ, γ, β)-bad sequence iff for all $n<\omega$

(1) $(\mathcal{S}_n, \Psi_n)$ is a hod pair

(2) $\pi_n : \mathcal{S}_n \longrightarrow \mathcal{S}_{n+1}$, and

$\Psi_n = \Psi_{n+1}^{\pi_n}$, and

(3) there is a (ξ, γ, β)-bad factoring of $(\mathcal{S}_n, \pi_n, \mathcal{S}_{n+1}, \Psi_{n+1})$.

Let us write $\Sigma_i = \Sigma_{\nu_i} = (\Sigma_{H_1})^{\sigma_i}$ for the strategy ~~in~~ for $R_i = R_{\nu_i}$ that we have. So $\Sigma_i \in$ ~~M~~ M and acts on trees in $V_{\kappa_1}^M$, but also extends canonically to trees in $HC^{M[h]}$, for h on $Col(\omega, <\kappa_1)$. We write Σ_i^h for this extension. Note $\Sigma_i^h \in D(M,\kappa_1)^h$.

Claim C There is a (ξ, γ, β) such that whenever h on $\mathrm{Col}(\omega, <\kappa_1)$ yields the derived model $D(M,\kappa_1)^h$:

$$D(M,\kappa_1)^h \models \langle (R_i, \pi_i, R_{i+1}, \Sigma_{i+1}^h) \mid i<\omega \rangle$$
$$\text{is } (\xi,\gamma,\beta)\text{-bad},$$

where $\pi_i = \sigma_{i+1}^{-1} \circ \Phi_i$, for each i.

Proof (Sketch.). In $D(M,\kappa_1)^+$, we take a Skolem hull of $L_\eta(P(\mathbb{R}), H_1^+, j(A))$, for η large, throwing in all reals, and H_1^+, $j(A)$ as points. The collapses of H_1^+ and $j(A)$ can be seen to be in H_1. (That this is true is forced in $V_{op}^{<\omega}$ over H_1^+, by a reflection argument like before.) Thus the collapses of H_1^+ and $j(A)$ are OD in $D(M,\kappa_1)$, and this is what we need.

$\boxtimes$

Fix $(\xi_0, \gamma_0, \beta_0)$ such that

$$b_0 = \langle (R_i, \pi_i, R_{i+1}, \Sigma_{i+1}) \mid i < \omega \rangle$$

is $(\xi_0, \gamma_0, \beta_0)$-bad in $D(M, \kappa_1)$. Our rough plan now is to find a Σ_{H_0}-hod-pair (W_0^{*}, Ψ_0^{*}) in $D(M, \kappa_1)$ that has an $\mathbb{R}^{M[h]}$-genericity iterate $(W_0^{\omega}, \Psi_0^{\omega})$ whose derived model goes past $P_{\xi_0}(\mathbb{R}^{M[h]})$, and therefore satisfies that b_0 is $(\xi_0, \gamma_0, \beta_0)$-bad.

Letting

$$W_{i+1} = \mathrm{Ult}(W_i, E_{\pi_i}),$$

we have that for each i

$$W_i \text{ is a } \Sigma_i\text{-hod-premouse extending } R_i,$$

and the diagram

$$\begin{tikzcd}
 & & & H_1^+ & & & \\
R_0 \arrow[urrr, "\sigma_0"] \arrow[r, "\pi_0"] & R_1 \arrow[urr, "\sigma_1"] \arrow[r, "\pi_1"] & \cdots & R_i \arrow[u, "\sigma_i"] \arrow[r] & \cdots & R_\omega \arrow[ull, "\sigma_\omega"']
\end{tikzcd}$$

lifts to the diagram

$$\begin{tikzcd}
 & & & j(W_0) & & & \\
W_0 \arrow[urrr, "\tau_0"] \arrow[r, "\pi_0^*"'] & W_1 \arrow[urr, "\tau_1"] \arrow[r, "\pi_1^*"'] & \cdots & W_i \arrow[u, "\tau_i"] \arrow[r] & \cdots & W_\omega \arrow[ull, "\tau_\omega"']
\end{tikzcd}$$

It is important here that we can arrange that $(W_0, \Psi_0^-) \in V$, so we can apply j to it. The maps τ_i are given by

$$\tau_0 = j \restriction W_0$$

and

$$\tau_{i+1}(\pi_{0i}^*(f)(a)) = j(f)(\sigma_i(a))$$

for all $a \in R_i^-$ and $f \in W_0$. ~~with~~

(Notice here that because σ_0 is cofinal, all of the π_i's are cofinal, and

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Thus we can assume ~~ε~~

$$\mathrm{lh}\, E_{\pi_i} = \pi_{0,i-1}(\theta^0).$$

So our restriction that $a \in R_i^-$ above is ok.)

Let us now simplify things by assuming that j ~~with~~ witnesses that κ_0 is huge. This implies that $\tau_\omega \in j(M)$. But $j(\Psi_0)$ is a strategy for $j(W_0)$ in $j(M)$. So we can set

$$\Psi_\omega = j(\Psi_0)^{\tau_\omega},$$

~~and (W_ω, Ψ_ω) is a hod pair~~

and Ψ_ω acts on all trees in $V_{\kappa_1}^M = V_{\kappa_1}^{j_1(M)} = V_{\kappa_1}^{j(M)}$. Moreover, Ψ_ω extends to $HC^{M[h]}$.

In $M[h]$, fix $(P_i, Q_i, l_i, m_i, k_i)$ witnessing that $(R_i, \pi_i, R_{i+1}, \Sigma_{i+1})$ is

$(\xi_0, \gamma_0, \beta_0)$ – bad, for each i.

Let

$$U_i = \text{Ult}(W_i, E_{l_i})$$

and

$$V_i = \text{Ult}(U_i, E_{m_i}),$$

so that we have

$$\begin{array}{ccccccc} W_i & \longrightarrow & U_i & \longrightarrow & V_i & \longrightarrow & W_{i+1} \\ \triangledown & & \triangledown & & \triangledown & & \triangledown \\ R_i & \longrightarrow & P_i & \longrightarrow & Q_i & \longrightarrow & R_{i+1} \end{array}$$

All the W_i, U_i, and V_i have iteration strategies obtained by pulling back our strategy Ψ_ω for W_ω. We can then assume that our genericity iteration $W_0 \longrightarrow W_0^\omega$ was dovetailed with $\mathbb{R}^{\text{MChs}}$ – genericity iterations of the W_i, U_i, V_i in the standard way. This yields the diagram

$$\begin{tikzcd}
W_0^\omega \arrow[r] & \cdots W_i^\omega \arrow[r] & U_i^\omega \arrow[r, "m_i^\omega"] & V_i^\omega \arrow[r] & W_{i+1}^\omega \arrow[r, dashed] & W_\omega^\omega \\
W_0 \arrow[u] \arrow[r] & \cdots W_i \arrow[u] \arrow[r] & U_i \arrow[u] \arrow[r] & V_i \arrow[u] \arrow[r] & W_{i+1} \arrow[u] \arrow[r, dashed] & W_\omega \arrow[u] \\
\triangledown & \triangledown & \triangledown & \triangledown & \triangledown & \triangledown \\
R_0 \arrow[r] & \cdots R_i \arrow[r, "l_i"'] & P_i \arrow[r, "m_i"'] & Q_i \arrow[r, "k_i"'] & R_{i+1} \arrow[r, dashed] & R_\omega
\end{tikzcd}$$

Let $\cancel{\mathcal{B}}$

$$C_i = \cancel{D(W_i^\omega, \delta^{W_i^\omega})^h} \; D(W_i^\omega, < \delta_\lambda^{W_i^\omega})^h$$

$= \cancel{R}$ $\mathbb{R}^{MThS}$- realization of derived model of W_i^ω

$$D_i = D(U_i^\omega, < \delta_\lambda^{U_i^\omega})^h$$

$$G_i = D(V_i^\omega, < \delta_\lambda^{V_i^\omega})^h .$$

<u>Claim</u> For all i, $C_i \subseteq D_i \subseteq E_i \subseteq C_{i+1}$.

<u>Proof</u> We are assuming $\lambda^{K'}$ is a limit ordinal, ~~IA follows that~~ and of

(A16)

the form

$$\lambda^W = \alpha + \omega$$

for some α. It follows that the same holds for λ^{W_i}, λ^{U_i}, λ^{V_i}. Moreover, for $A \subseteq \mathbb{R}^{M[h]}$

$$A \in C_i \text{ iff } \exists \beta < \lambda^{W_i}$$
$$A \leq_w \text{code}(\Sigma^{W_i(\beta)})$$

where $\Sigma^{W_i(\beta)}$ is the strategy for $W_i(\beta)$, ~~extended to~~ which acts on trees in $HC^{M[h]}$, that we are given by pulling back from Ψ_ω. Similar equivalences characterize D_i and G_i. But then for $\beta < \lambda^W$, $\Sigma^{W_i(\beta)}$ is a pullback of $\Sigma^{U_i(l_i^*(\beta))}$, and hence $\Sigma^{W_i(\beta)}$ is projective in $\Sigma^{U_i(l_i^*(\beta))}$.

This tells us that $C_i \subseteq D_i$.

The inclusions $D_i \subseteq G_i \subseteq C_{i+1}$ follow by the same argument.

$\boxtimes$

The claim implies that $L_{\beta_0}(P_{\xi_0}(\mathbb{R}^{M[h]}))$ is a Wadge initial segment of all C_i, D_i, and G_i.

Since W_ω^ω is an iterate of W_ω by φ_ω, W_ω^ω is wellfounded. Thus every ordinal is eventually fixed by the l_i^ω, m_i^ω, and k_i^ω. So we can find an i such that

$$m_i^\omega((\xi_0, \beta_0, \gamma_0)) = (\xi_0, \beta_0, \gamma_0).$$

A18

But m_i^ω is elementary, $m_i^\omega \restriction P_i = m_i$, and m_i^ω moves the UB code of Σ_{P_i} to the UB code of Σ_{Q_i}. Thus for any $s \in P_i$ and any φ

$$L_{\beta_0}(P_{\xi_0}(\mathbb{R}))^{D_i} \models \varphi[P_i, s, \Sigma_{P_i}, \gamma_0]$$

iff

$$L_{\beta_0}(P_{\xi_0}(\mathbb{R}))^{G_i} \models \varphi[Q_i, m_i(s), \Sigma_{Q_i}, \gamma_0].$$

Since $L_{\beta_0}(P_{\xi_0}(\mathbb{R}))^{D_i} = L_{\beta_0}(P_{\xi_0}(\mathbb{R}))^{G_i} = L_{\beta_0}(P_{\xi_0}(\mathbb{R}))^{M[h]}$, this contradicts $(P_i, Q_i, \mathcal{l}_i, m_i, k_i)$ being a $(\xi_0, \beta_0, \gamma_0)$-bad factoring of $(R_i, \pi_i, R_{i+1}, \Sigma_{i+1})$.

$\boxtimes$

A19

We have now pretty much proved lemma 2 in the case that j is a hugeness embedding. We shall adapt the argument above to the general case by replacing the R_i's and W_i's by Skolem hulls of themselves having size κ_0.

We have $(\xi_0, \gamma_0, \beta_0)$ ~~that~~ such that $b_0 = \langle (R_i, \pi_i, R_{i+1}, \Sigma_{i+1}) \mid i < \omega \rangle$ is a $(\xi_0, \gamma_0, \beta_0)$-bad sequence in $D(M, \kappa_1)$. We fix a ~~hod pair~~ Σ_{H_0}-hod-pair (W_0, Ψ_0) extending H_0^+ in $D(M, \kappa_1)$ such that

(c) $\lambda^{W_0} = \alpha + \omega$ for some α

(b) $L_{\beta_0}(P_{\xi_0}(\mathbb{R}))^{M[h]}$ is coded by a set of reals that is $\leq_W$ code$(\Psi_{0_{W_0(\eta)}})$, for some $\eta < \lambda^{W_0}$, for any h on $Col(\omega, <\kappa_1)$,

(a) $(W_0, \Psi_0) \in V$.

A20

Parts (a) and (b) we can satisfy because we are assuming that the hod analysis (relativised, analyzing $HOD_{\Sigma_{H_0}}$) works in $D(M, \kappa_1)$. Part (c) we can add by a "local HOD-limit" argument, analogous to the way we got the R_y's. It follows from (a)–(c) that if W_0^ω is an $\mathbb{R}^{M[h]}$-genericity iteration by Ψ_0, then $L_{p_0}(P_{\Psi_0}(\mathbb{R}))^{M[h]}$ is an proper initial segment of derived model of W_0^ω.

We need a structure N that will certify the relationship of (W_0, Ψ_0) to the (R_i, Σ_i)'s, so that this relationship will be preserved when we replace ~~them~~ them by Skolem hulls of themselves (as a relationship still holding in $M[h]$, not

just in some Skolem hull of $M[h]$.)
For this, let (N, Φ) be a hod-pair over $X = TC(\{W_0\} \cup \{R_i \mid i<\omega\})$ relative to Ψ_0, $\langle \Sigma_{R_i} \mid i<\omega \rangle$. That is, we put X at the bottom, and feed in the strategies Ψ_0, Σ_i $(i<\omega)$ throughout. We take N to have ω Woodin cardinals, i.e. $\lambda^N = \omega$, and satisfy ZFC. Let $\dot{\Psi}_0$ and $\dot{\Sigma}_i$ be the canonical terms in N for $\Psi_0 \restriction (HC)^{D(N, <\delta_\lambda^N)}$ and $\Sigma_i \restriction (HC)^{D(N, <\delta_\lambda^N)}$.

Working inside N, we can do a "$V^N_{\delta_\omega}$-genericity iteration" of W_0 by Ψ_0, successively making the $V^N_{\delta_n}$'s generic for the extender algebras at the images of the $W_0(\alpha+n)$'s (where $\lambda^{W_0} = \alpha+\omega$).

This yields $i : W_0 \longrightarrow W_0^\omega$ inside N, and W_0^ω has the property that whenever $\mathbb{R}^*$ is the reals of a symmetric collapse over N below δ_ω^N, then $\mathbb{R}^*$ is also the reals of a symmetric collapse over W_0^ω below $\delta_{i(\lambda)}^{W_0^\omega}$.

By a local hod-limit argument, we can arrange that $\mathcal{N} \in V_{\kappa_1}^M$, $\Phi \restriction M \in M$, and $\Phi \restriction M$ determines Φ^h with domain $HC^{M[h]}$, for any h on $Col(\omega, <\kappa_1)$.

<u>Claim D</u>. There is a $(\xi_1, \beta_1, \gamma_1) \in W_0^\omega$ such that whenever g is N-generic over $Col(\omega, <\delta_\omega^N)$, then letting D be the resulting derived model of W_0^ω,

$$D \models \langle (R_i, \pi_i, R_{i+1}, \Sigma_{i+1}^g \restriction HC) \mid i < \omega \rangle \text{ is } (\xi_1, \beta_1, \gamma_1)\text{-bad}.$$

A23

Proof We are claiming

$$N \models \varphi[W_0^{*}, \mathring{\Psi}_0^{*}, \langle (R_i, \mathring{\Sigma}_i) \mid i<\omega \rangle]$$

for a certain φ. But let

$$t : N \longrightarrow N_\infty$$

be an $\mathbb{R}^{M[h]}$-genericity iteration by $\bar{\Phi}$ done in/over $M[h]$. Then for l on $\mathrm{col}(\omega, < t(\delta_\omega^N))$,

$$\Psi_0 = t(\mathring{\Psi})_l \cap HC^{M[h]}$$

and

$$\Sigma_i = t(\mathring{\Sigma}_i)_l \cap HC^{M[h]}.$$

That is, Φ moves the terms for Ψ_0 and Σ_i correctly (because ~~the~~ it moves $\Phi \cap N$ and $\Sigma_i \cap N$ correctly, and these yield $\mathring{\Psi}$ and $\mathring{\Sigma}_i$ ~~...~~). But then by our choice of W_0 and Ψ_0,

$N_\infty \models \varphi[t(W_0), t(\mathring{\Psi}_0), t(\langle(R_i, \mathring{\Sigma}_i) \mid i<\omega\rangle)]$.

Since $t: N \to N_\infty$ is elementary, we are done.

$\boxtimes$

Now let $W_{i+1} = Ult(W_i, E_{\pi_i})$, $\pi_i^*: W_i \to W_{i+1}$ lift π_i, and $W_\omega =$ dir lim W_i as before. We have $\tau_i: W_i \to j(W_0)$, $\tau_\omega: W_\omega \to j(W_0)$, again as before. Our problem is that $\tau_\omega \notin j(M)$ is possible, so that we cannot make sense of $j(\Psi_0)^{\tau_\omega}$. So we let

$$\rho: P \longrightarrow V_\xi^M$$

where P is transitive, $|P| = \aleph_0$,

ξ is large, and everything relevant is in $\operatorname{ran}(\rho)$. We put

$$\overline{W}_i = \rho^{-1}(W_i),$$

$$\overline{\pi}_i = \rho^{-1}(\pi_i),$$

and so on. Inside P, we have the collapse of one of our previous diagrams:

$$\begin{tikzcd}
 & & & \overline{j(W_0)} \arrow[r, "\rho"'] & j(W_0) \\
\overline{W}_0 \arrow[r] \arrow[urrr, bend left=10, "\overline{\tau}_0"] \arrow[d, phantom, "\triangledown"] & \overline{W}_1 \arrow[r] \arrow[urr] \arrow[d, phantom, "\triangledown"] & \cdots & \overline{W}_\omega \arrow[u, "\overline{\tau}_\omega"'] \arrow[d, phantom, "\triangledown"] & \\
R_0 = \overline{R}_0 \arrow[r] & \overline{R}_1 \arrow[r] & \cdots & \overline{R}_\omega &
\end{tikzcd}$$

That is, all of this diagram is in P, except for ρ and $j(W_0)$. Now notice that $\rho \circ \overline{\tau}_\omega$ is essentially a κ_0-sequence of elements of $j(W_0) \in j(M) = j_1(M)$, and $\rho \circ \overline{\tau}_\omega \in M$, so $\rho \circ \overline{\tau}_\omega \in j(M)$.

So we can set

$$\overline{\Psi}_\omega = j(\Psi_0)^{\rho \circ \overline{\tau}_\omega}$$

$$\overline{\Psi}_n = \left(\overline{\Psi}_\omega\right)^{\overline{\pi}^*_{n\omega}} = j(\Psi_0)^{\rho \circ \overline{\tau}_n},$$

and we have that $\overline{\Psi}_\omega$, $\overline{\Psi}_n$ can be extended to act on all trees in $HC^{M[h]}$ (as this is contained in $HC^{j(M)[l]}$ for l on $col(\omega, < j(\kappa_1))$). Similarly, we have the pullbacks

$$\overline{\Sigma}_i = (\Sigma_i)^\rho$$

acting on all trees in $HC^{M[h]}$. The reader may object that we are mis-using our bar notation, but in fact

$$\rho^{-1}(\Psi_0) = \overline{\Psi}_0 \cap P$$

and

$$\rho^{-1}(\Sigma_i) = \overline{\Sigma}_i \cap P$$

Claim E.

(1) $\rho^{-1}(\Sigma_i) = \Sigma_i^{\rho} \cap P$

(2) $\rho^{-1}(\Psi_0) = \Psi_0^{\rho} \cap P$

$= j(\Psi_0)^{\rho \circ \bar{\tau}_0} \cap P$

Proof

(1) If $\mathfrak{I} \in P$ is by $\rho^{-1}(\Sigma_i)$, then $\rho(\mathfrak{I})$ is by Σ_i, so $\rho\mathfrak{I}$ is by Σ_i because it is a hull of $\rho(\mathfrak{I})$, so $\mathfrak{I}$ is by Σ_i^{ρ}.

(2) Similarly, $\rho^{-1}(\Psi_0) = \Psi_0^{\rho} \cap P$. For the second equality, we have

$$\begin{tikzcd}
W_0 \arrow[r, "j"] & j(W_0) \\
\overline{W_0} \arrow[u, "\rho"] \arrow[r, "\bar{\tau}_0"] & \overline{j(W_0)} \arrow[u, "\rho"']
\end{tikzcd}$$

If $\mathfrak{I}$ is by $\rho^{-1}(\Psi_0)$, then $j(\rho(\mathfrak{I}))$ is by $j(\Psi_0)$. But $(\rho \circ \bar{\tau}_0)\mathfrak{I}$ is a hull of $j(\rho(\mathfrak{I})) = \rho(\bar{\tau}_0(\mathfrak{I}))$, and $(\rho \circ \bar{\tau}_0)\mathfrak{I} \in j(M)$

because $p \circ \bar{\pi}_0 \in j(M)$. (And in fact, its image as a subtree of $j(p(\mathcal{T}))$ is also in $j(M)$.) So $(p \circ \bar{\pi}_0)\mathcal{T}$ is by $j(\Psi_0)$.

$\boxtimes$

We also let

$$\bar{\Phi} = \Phi^p,$$

and we have that $p^{-1}(\Phi) \subseteq \bar{\Phi}$, as above. $\bar{\Phi}$ is a strategy for $\bar{N}$ acting on all trees in $HC^{M[h]}$. In $M[h]^{Col(\omega,\kappa_1)}$, we do an $\mathbb{R}^{M[h]}$ genericity iteration

$$\nu : \bar{N} \longrightarrow \bar{N}_\infty$$

of $\bar{N}$. Let

$$W_0^* = \nu(\overline{W_0^\omega}),$$

so that we have in $\bar{N}_\infty$ an $\mathbb{B}V_{\delta\omega}$-genericity iteration from

$\overline{W}_0 = \upsilon(\overline{W}_0)$ to W_0^*. Let

$$(\beta_2, \xi_2, \gamma_2) = \upsilon((\overline{\beta}_1, \overline{\gamma}_1, \overline{\xi}_1)),$$

where $(\beta_1, \gamma_1, \xi_1) \in W_0^\omega$ and witness the truth of claim D. $\mathbb{R}^{MCMJ}$ is the reals of a derived model of ~~W~~ $\overline{N}_\infty$, and hence of W_0^*. Let D_0^* be this derived model of W_0^*. Let also

$$\psi_0^* = \upsilon(\overline{\dot{\psi}_0})^{\mathbb{R}^{MCMJ}}$$

and

$$\Sigma_i^* = \upsilon(\overline{\dot{\Sigma}_i})^{\mathbb{R}^{MCMJ}}$$

be the interpretations of $\upsilon(p^{-1}(\dot{\psi}_0))$ and $\upsilon(p^{-1}(\dot{\Sigma}_i))$ on the $\mathbb{R}^{MCMJ}$-realization of $D(\overline{N}_\infty, \delta_\omega^{\overline{N}_\infty})$. We have

Claim F.

$$W_0^* \models \langle (\bar{R}_i, \bar{\pi}_i, \bar{R}_{i+1}, \Sigma^*_{i+1}) \mid i<\omega \rangle \text{ is}$$

$(\beta_2, \xi_2, \gamma_2)$ - bad.

Proof This follows from the fact that $(\beta_1, \xi_1, \gamma_1)$ witnesses the truth of Claim D in N, and that ν and ρ are elementary, and $\nu((\bar{R}_i, \bar{\pi}_i)) = (\bar{R}_i, \bar{\pi}_i)$.

$\boxtimes$

The next claim makes a key connection.

Claim G.

(1) $\psi_0^* = \bar{\psi}_0$,

(2) $\Sigma_i^* = \bar{\Sigma}_i$.

Proof This is proved in Sargsyan's

thesis. (Claim E is a first step, but claim G is a good deal harder to prove.) See 3.76 of [2], and 3.9, 3.10, and 3.28 of [3].

$\boxtimes$

Now we can reach a contradiction as before. Fix for each i a tuple $(P_i, Q_i, l_i, m_i, k_i)$ such that

$$D_0^* \models (P_i, Q_i, l_i, m_i, k_i) \text{ is a } (\beta_2, \xi_2, \gamma_2)\text{-bad factoring of } (\bar{R}_i, \bar{\pi}_i, \bar{R}_{i+1}, \Sigma^*_{i+1}).$$

Let

$$U_i = \mathrm{Ult}(\bar{W}_i, E_{l_i}),$$

$$V_i = \mathrm{Ult}(U_i, E_{m_i}),$$

with strategies $\bar{\Psi}^{m_i \circ l_i}_{i+1}$ and $\bar{\Psi}^{m_i}_{i+1}$

respectively. We can lift the iteration $\overline{W}_0 \longrightarrow W_0^*$ to an iteration $U_0 \longrightarrow U_0^{**}$, and then genericity iterate $U_0^{**} \longrightarrow U_0^*$ to make $\mathbb{R}^{M_2 h_y}$ the reals of a derived model of U_0^*. Then we lift $U_0 \longrightarrow U_0^*$ to $V_0 \longrightarrow V_0^{**}$, and genericity iterate $V_0^{**} \longrightarrow V_0^*$ to make $\mathbb{R}^{M_2 h_y}$ the reals of a derived model of V_0^*. Etc. We get

$$
\begin{tikzcd}
W_0^* \arrow[r] & \cdots & W_i^* \arrow[r, "l_i^*"'] & U_i^* \arrow[r, "m_i^*"'] & V_i^* \arrow[r, "k_i^*"'] & W_{i+1}^* \arrow[r] & W_\omega^* \\
\overline{W}_0 \arrow[u] \arrow[r] \arrow[d, phantom, "\triangledown"] & \cdots & \overline{W}_i \arrow[u] \arrow[r] \arrow[d, phantom, "\triangledown"] & U_i \arrow[u] \arrow[r] \arrow[d, phantom, "\triangledown"] & V_i \arrow[u] \arrow[r] \arrow[d, phantom, "\triangledown"] & \overline{W}_{i+1} \arrow[u] \arrow[d, phantom, "\triangledown"] & \cdots \quad \overline{W}_\omega \arrow[u] \arrow[d, phantom, "\triangledown"] \\
R_0 \arrow[r] & \cdots & \overline{R}_i \arrow[r, "j_i"'] & P_i \arrow[r, "m_i"'] & Q_i \arrow[r, "k_i"'] & \overline{R}_{i+1} & \cdots \quad \overline{R}_\omega
\end{tikzcd}
$$

Again, let D_i^*, E_i^*, F_i^* be the derived models of W_i^*, U_i^*, V_i^*. We have $D_i^* \subseteq E_i^* \subseteq F_i^* \subseteq D_{i+1}^*$. Since W_ω^* is from a stack of trees via $\bar{\Psi}_\omega$, it is wellfounded. This gives us an i such that

$$m_i^*((\beta_2, \xi_2, \gamma_2)) = (\beta_2, \xi_2, \gamma_2).$$

Now let $\ddot{\Lambda} \in W_0$ be the canonical name for Σ_0 on its derived model, and $\dot{\Lambda} = \rho^{-1}(\ddot{\Lambda})$, and let $\dot{\Lambda}_i$ and $\dot{\Lambda}_{i+1}$ be the images of $\dot{\Lambda}$ in U_i^* and V_i^* (by whatever route).

Λ^*_i = interpretation of $\mathring{\Lambda}_i$ in E^*_i,

Ω^*_i = interpretation of $\mathring{\Lambda}_i$ in F^*_i.

As with claim G, we get

<u>Claim H</u> $\Lambda^*_i = \left(\Sigma^*_{i+1}\right)^{k_i \circ m_i}$, and

$\Omega^*_i = \left(\Sigma^*_{i+1}\right)^{k_i}$.

That is, m^*_i moves the name for Σ_{P_i} on the derived model of U^*_i to the name for Σ_{Q_i} on the derived model of V^*_i.

This is a contradiction, as before.

$\boxtimes$

References

[1] G. Sargsyan, Covering with universally Baire operators

[2] G. Sargsyan, A tale of hybrid mice, Ph.D. thesis, March 2010 version

[3] G. Sargsyan, HOD mice and the Mouse Set Conjecture, Memoirs of the AMS