

Remarks on a paper by Sargsyan

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(1)

§1.

In this addendum to Sargsyan's paper "Covering with UB operators" [1], we show how to get models of $AD_{\mathbb{R}} + \Theta$ is measurable" from similar hypotheses. The same proof gives significantly stronger large cardinal properties of Θ .

Let κ_0 be measurable, and a limit of Woodins and $< \kappa_0$ -strongs. We have then

$$D(V, \kappa_0) \models AD_{\mathbb{R}} + DC$$

for the derived model of V below κ_0 . We assume

(†) $\text{Hod}^{D(V, \kappa_0)}$ can be analyzed as a hod-premouse.

(†) holds as long as $D(V, \kappa_0)$ has not reached hod pairs of LST-type, and hence we'll pass the target we are aiming at now. Set

$$H_0 = \text{HOD}^{D(V, \kappa_0)}$$

Now fix

$$j: V \rightarrow M$$

with $\text{crit}(j) = \kappa_0$ witnessing κ_0 measurable.

Let

$$\Sigma_{H_0} = \bigoplus_{\alpha < o(H_0)} \Sigma_{H_0(\alpha)}$$

be the join of the strategies for initial segments of H_0 coming out of the hod-limit.

Σ_{H_0} is a $< j(\kappa)$ -strategy for H_0 in M ,

and in fact, letting $\kappa_1 = j(\kappa)$

$$\Sigma_{H_0} \in D(M, \kappa_1).$$

Put

$$H_0^+ = \left(L_P^{\Sigma_{H_0}}(H_0) \right)^{D(M, \kappa_1)}.$$

We assume

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$$(\dagger\dagger) \quad o(H_0^+) < K_0^+.$$

This is true, for example, if $\rightarrow \square_{K_0}$.

Remark We are essentially continuing ~~to~~ from § 11 of Sargisyan's paper. There K_0 is called γ , and H_0^+ is called \mathcal{P} . We have added that K_0 is a limit of Wood's because this simplifies a few things. It implies that the maximal model below K_0 is just $D(V, K_0)$.

Let

$$\Theta^0 = o(H_0).$$

Sargisyan shows

- (1) H_0 is full in $D(M, K_1)$, that is, no level of H_0^+ projects strictly below Θ^0 ,
- (2) $H_0^+ \equiv \Theta^0$ is regular.

Sargsyan also assumes $cf(o(H_0^+)) \neq K_0$,
but ~~we~~ this seems to already follow from
 $o(H_0^+) \neq K_0^+$, as we now show.

Lemma 0. $cf(o(H_0^+)) < K_0$.

Proof. Assume $cf(o(H_0^+)) = K_0$. Let
 $H_1 = j(H_0)$ and $H_1^+ = j(H_0^+)$. Let

$\mathcal{P} \triangleq H_1^+$, $\rho_w(\mathcal{P}) = \Theta^+ = o(H_1)$,
and $j'' o(H_0^+) \subseteq (\Theta^+)^+ \mathcal{P}$. Let

$\delta_1 = j(j)$, and $\delta_1: M \rightarrow M_2$.

\mathcal{P} is a Σ_{H_1} -mouse in

$D(M_2, K_2)$. (Here $K_1 = j(K_0)$

and $K_2 = \delta_1(K_1)$.) Let \mathcal{I} be

the ~~canonical~~ unique iteration strategy \mathcal{I}
for \mathcal{P} with respect to trees

with all critical points $> \theta^\pm$ that moves Σ_{H_1} correctly. We have Λ in $D(M_2, K_2)$.

In $D(M_2, K_2)$ we have a ~~local~~ ~~mod~~ mod pair (W, Φ) such that for some λ

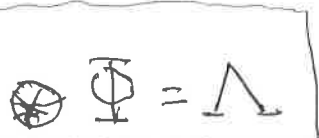
(a) W is a Σ_{H_0} -mod-preimage extending H_0^+

(b) $W \cong \mathbb{S}^2$ is a limit of Woodruff $> 0 (H_0^+)$

(c) For some $d < \lambda$, Φ is projective in $\Psi \uparrow$ trees on $W(d)$.

Thus \mathbb{R} -genericity iterations will put W with α such that Φ^* is reached in the deformed model. A "local HOD-limit" argument shows that we may assume $(W, \Psi) \in M_2$.

Let $E = E_j \cap ([0^\circ]^{rw} \times H_0)$,



and

(3c)

$$U = \text{Ult}(W, E)$$

We have

$$W \xrightarrow{i_E} U \xrightarrow{k} j(W)$$

with $k \circ i_E = j \upharpoonright W$. If j were strong enough (e.g. a rank-to-rank embedding), we could argue at once that U is an iterable hod-pair

over Σ_H , whose derived model reaches Φ . That would imply at once that

$P \in U$. But $P \notin U$, because

$$i_E(o(H_0^+)) = \sup j'' o(H_0^+) = (\Theta^{\sharp+})^P,$$

so P collapses $i_E(o(H_0^+))$.

As in [1], we make do with a weaker j by taking a Skolem hull,

Let $\pi: N \rightarrow V_\delta$ where N is transitive, $V_{k_0} \subseteq N$, $k_0 \in N$, and $|N| = k_0$. Put

$$\bar{W} = \pi^{-1}(W),$$

$$\bar{P} = \pi^{-1}(P),$$

and so on. We have

$$\bar{W} \xrightarrow{\lambda_{\bar{E}}} \bar{U} \xrightarrow{\bar{k}} \overline{j(W)} \xrightarrow{\pi} j(W).$$

Notice that $(j(W), j(\Psi))$ is a hod pair over ~~Σ_{H_1}~~ Σ_{H_1} in $j(M_2)$, and $\pi \in j(M_2)$ because the latter is closed under K -sequences. So we have a pullback strategy

$$\Sigma_{\bar{U}} = \Sigma_{j(W)}^{\pi \circ \bar{k}}$$

for \bar{u} in $j(\mathbb{R}^2)$. Sargsyan has shown that $\Sigma_{\bar{u}}$ extends to $\sum \dot{g}$, where \dot{g} is the canonical term in \bar{u} for the extension of its strategy to generic extensions. He has also shown this remains true for $\Sigma_{\bar{u}}$ - iterates of \bar{u} .

Remark This is theorem 3.76 of the 3/25/10 version of Sargsyan's thesis [2]. See also 3.9, 3.10, and 3.28 of [3]

We also have that $\overline{\Psi} = \sum_{\bar{u}} \dot{g}_{\bar{u}}$, by condensation for $j(\Psi)$.

Let $\mathbb{R}^2 \rightarrow S$ be a genericity iteration above H_0^+ so that for some g on $\text{Coll}(\omega, \delta_0)$ and Φ^* is ~~the~~ wedge

Fix $\alpha < 2^{\bar{w}}$ and a real $z \in D(j(M_2), K_2)$ such that ~~$(\bar{\Psi} \uparrow \bar{W}(\alpha))$~~ (3f)

Code $(\bar{\Phi})$ is Wadge reducible to

Code $(\bar{\Psi} \uparrow \bar{W}(\alpha))$ via z . (Note $\bar{\Phi}$,

$\bar{\Psi}$ are defined on all of $D(j(M_2), K_2)$)

because π exists. Iterate \bar{U}

by $\Sigma_{\bar{U}}$ above $i_E^{\bar{w}}(S_{\alpha}^{\bar{w}})$ to \mathbb{R}

$$\begin{array}{ccc} \bar{U} & \xrightarrow{\Sigma} & \mathbb{R} \\ i_E^{\bar{w}} \uparrow & & \\ \bar{W} & & \end{array}$$

so that z and $i_E^{\bar{w}} \uparrow \bar{W}(\alpha)$ are in Reg , for $g \in \text{Col}(w, S_1^R)$. Then

the derived model of Reg can

compute $\bar{\Psi} \uparrow \bar{W}(\alpha)$ as a pullback

of one of R 's strategies. Since

it has \mathbb{Z} , it has $\overline{\Phi}$.

(39)

this gives that $\overline{\Phi}$ is $OD(\overline{\Sigma}_H)$
in the derived model of R .

that gives $\overline{P} \in R$, so $\overline{P} \in \overline{U}$,
as desired.

□

Remark Sargisyan used this sort of
argument pretty heavily.

$D(V, \kappa_0)$ sees H_0 , but not H_0^+ . (4)

However, we can add H_0^+ to $D(V, \kappa_0)$ without changing much. The following lemma emerged in conversation with Nam Trang.

Lemma 1. Let g be $\text{Col}(\omega, \kappa_0)$ -generic over V ; then

$$(a) \quad P(\mathbb{R}_g^+) \cap L(\text{Hom}_g^+, H_0^+) \stackrel{=}{=} \text{Hom}_g^+,$$

and

$$(b) \quad L(\text{Hom}_g^+, H_0^+) \neq \emptyset \text{ is regular.}$$

Proof. We can obtain $L(\text{Hom}_g^+, H_0^+)$ as a "symmetric Vopenka" extension of H_0^+ . A condition is a pair (n, A) , where for some $\gamma < \theta^0$

$$A \subseteq (\omega_\gamma)^n$$

is OD over $L[\text{Hom}_g^+]$.

Lemma 4

$(n, A) \leq (m, B)$ iff $m \leq n$ and $\forall s \in A (s \upharpoonright m \in B)$.

$V_{op, \omega}$ is the resulting poset, coded as a subset of θ^ω that is OD over $L(Hom^+)$. Thus $V_{op, \omega} \in H_0^+$. Given any G that is $V_{op, \omega}$ -generic over H_0^+ , let

$$S_i^G(n) = \{ \} \text{ iff } \left(\exists (k, A) \in G \left(k > i \wedge \forall t \in A (t(i)(n) = \{ \}) \right) \right)$$

Any $p \in V_{op, \omega}$ can be extended by a G that is H_0^+ -generic and such that

$$\omega \cap L[Hom^+] = \{ S_i^G \mid i \in \omega \}$$

This is proved as usual, looking at G 's induced by H_0^+ -generic maps from ω onto $\bigcup_{\gamma < \theta^\omega} (\gamma)^{L[Hom^+]}$

Claim For any $\sqrt{0p}$ -generic G over H_0^+ , (5)

$$L[\{s_i^G \mid i \in \omega\}, H_0^+] \cap \bigcup_{\gamma < \theta^0} \omega_\gamma \neq \{s_i^G \mid i \in \omega\}.$$

Proof If not, we can reflect inside H_0^+ the failure of this. We get $\alpha < \theta^0$

and $N \triangleleft H_0(\alpha)$ with $H_0 \upharpoonright \delta_\alpha^{H_0} \triangleleft N$

and $p \in \sqrt{0p}$ forcing the contrary over

N . Let

$$\Gamma = \{A \in \text{Hom}_g^* \mid w(A) < \delta_\alpha^{H_0}\}.$$

~~We can arrange~~

~~$$\Gamma \neq P(\mathbb{R}_g^*) \cap L(\Gamma, \mathbb{R}_g^*)$$~~

and we have

$$\Gamma = \text{HOD}_{\Gamma}^{L[\text{Hom}_g^*]} \cap \text{Hom}_g^*.$$

So let G be $\sqrt{0p}$ generic over N and come from a generic map of w

ONTO $\bigcup_{\gamma < \delta_{\alpha}^{H_0}} ({}^{\omega}\gamma)^{L[H_{\alpha}^+]}$ Since $\textcircled{7}$ ~~(8)~~

$N \in \text{HOD}^{L[H_{\alpha}^+]}$, we get

$L(N, \{s_i^{\alpha} | i \in \omega\}) \cap \bigcup_{\gamma < \delta_{\alpha}^{H_0}} ({}^{\omega}\gamma) \subseteq \text{HOD}_{\Gamma}^{L[H_{\alpha}^+]}$

so $= \{s_i^{\alpha} | i \in \omega\}$, contradiction.



The claim easily gives (a) of the lemma, noting that every set in $\text{Hom}_{\Gamma}^{\omega}$ is coded by an \aleph_1 set ${}^{\omega}\gamma$, $\gamma < \Theta^0$.

For (b): if p forces " $\varphi(\cdot, \cdot, s)$ ",

let $f: \alpha \rightarrow \Theta^0$ with $\alpha < \Theta^0$

be definable over $L(H_0^+, \{s_i^{\alpha} | i \in \omega\})$

from ordinals and $s_0^{\alpha} \dots s_{n-1}^{\alpha}$, then

let via the formula φ . Let

$(k, A) \in G$, $n \leq k$, and (k, A) source then φ defines a cofinal map φ^G
 $\cdot \cdot \cdot \xrightarrow{\nu} \Theta^0$ from $S_0^G \cdots S_m^G$. Note that

if $(m, B) \leq (k, A)$ sources " $\varphi^G(\frac{\nu}{\xi}) = \eta$ ",

then in fact $(k, B|k)$ sources

$\varphi^G(\frac{\nu}{\xi}) = \eta$. (Use automorphisms permuting
 coordinates.) But there are $< \Theta^0$

many possible $(k, B|k)$ by $AD_{\mathbb{R}}$

in $L(\text{Hom}_g^+)$. Since $H_0^+ \models \Theta_0$ is regular,

we have a contradiction.

This proves lemma 1.



Truth in $L(\text{Hom}_g^+, H_0^+)$ is independent
 of g , so we write

$$D(V, K_0)^+ = L(\text{Hom}_g^+, H_0^+)$$

in the sloppy analog of the $D(V, K_0)$ notation.

The key to Sargsyan's proof that $H_0^+ \models \emptyset^0$ is regular is Lemma 11.15 of [1], which states that $j \upharpoonright H_0^+$ "has condensation.". We now introduce a slight strengthening of this property, and use it to go on.

Let

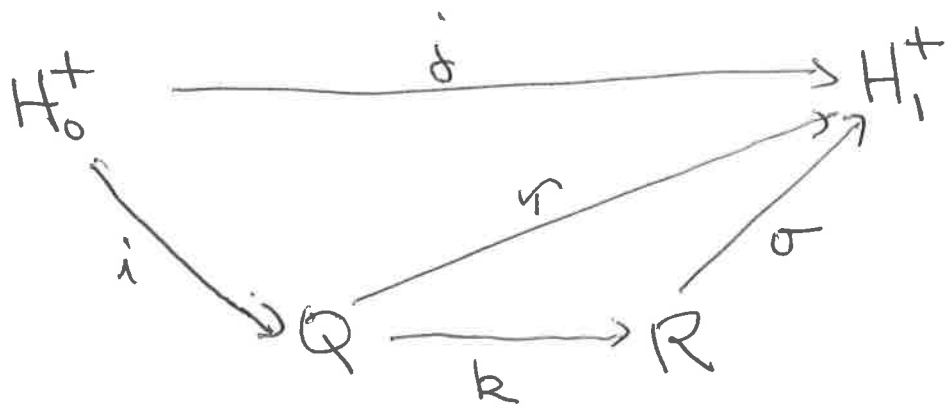
$$H_1 = j(H_0),$$

$$H_1^+ = j(H_0^+).$$

Recall that if $\pi: P \rightarrow Q$ and Σ is an iteration strategy for Q , then Σ^π is the π -pullback of Σ . So Σ^π is a strategy for P .

The proof of the following lemma follows that of 11.15 in [1] pretty much word-for-word.

Lemma 2. Suppose that in $M[h]$, where h is $\text{Col}(\omega, \kappa_1)$ generic over M , we have the commutative diagram



where Q and R are countable in $M[h]$.

Let

$$\Sigma_Q = \left(\Sigma_{H_1^+} \right)^\tau,$$

and

$$\Sigma_R = \left(\Sigma_{H_1^+} \right)^\sigma$$

be the pullback strategies. Then for any $A \in H_0^+$ and any formula φ :

$$\begin{aligned}
 D(M, \kappa_1)^+ \models \forall s \in Q \\
 \left[\varphi(Q, s, \Sigma_Q, H_1^+, j(A)) \iff \varphi(R, k(s), \Sigma_R, H_1^+, j(A)) \right].
 \end{aligned}$$

Remarks

(1) $D(M, k_1)^+ = L(\text{Hom}_h^*, H_1^+)$, in the notation we are using. Note $\Sigma_Q, \Sigma_R, H_1^+$, and $j(A)$ are all in $D(M, k_1)^+$. (E.g.; Σ_Q depends only on $\text{Tr}i(\theta^0)$, which is in $L[\text{Hom}_h^*]$.)

(2) Lemma 11.15 of [1] is the lemma above, but for a certain φ . It ~~is~~ seems that one must first show $H_0^+ \cong \theta^0$ regular, using the concrete φ , ~~expressions~~ before proving Lemma 2 above.

~~(B)~~

~~Corollary 3~~

~~Let $H_0^+ \xrightarrow{i} H_1^+$~~



~~factor j as in the lemma. Let $A \in H_0^+$. Then~~

$$\text{Tr}(i(A)) \cap \theta^1 = j(A) \cap \theta^1.$$

(3) For the sake of completeness, we prove Lemma 2 in an appendix.

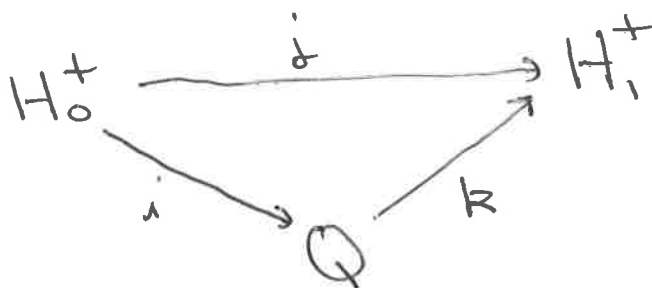
(4) The appendix was revised (correcting an error) in January 2015.

Since $H_0^+ \cong \theta^0$ is regular, we can think of Σ_{H_0} as an iteration strategy for H_0^+ .

We have

Corollary 3 In $D(M, K_1)$, Σ_{H_0} is a fullness preserving strategy for H_0^+ having branch condensation.

Proof Let $i: H_0^+ \rightarrow Q$ be an iteration map by Σ_{H_0} , given by some iteration tree \mathcal{T} on H_0 that we let act on H_0^+ , with \mathcal{T} countable in $D(M, K_1)$. We have



where

$$k(i(f)(a)) = \pi_{Q^-, \infty} \circ j(f)(\pi_{Q^-, \infty}(a)),$$

with $Q^- = Q \upharpoonright i(\theta^0)$, and $\pi_{Q^-, \infty}: Q^- \rightarrow H_1$

being the iteration map according to the tail of Σ_{H_0} . One way to see that k is well-defined, elementary, and the diagram commutes is: let

$$Z = \pi_{Q^-, \infty} \circ i(\theta^0).$$

Then

$$E_i = E_j \uparrow Z,$$

because the measures in E_i concentrate on bounded subsets of θ^0 , and for $a \in [i(\theta^0)]^{\text{rw}}$

- $(a, X) \in E_i$ iff $a \in i(X)$
- iff $\pi_{Q^-, \infty}(a) \in \pi_{Q^-, \infty}(i(X))$
- iff $\pi_{Q^-, \infty}(a) \in j(X)$
- iff $(\pi_{Q^-, \infty}(a), X) \in E_j \uparrow Z$.

We use here that $j \uparrow H_0$ is the iteration map by Σ_{H_0} . But then

$$k: \cup_k \tau(H_0^+, E_i) \rightarrow \cup_k \tau(H_0^+, E_j)$$

is just the natural factor map.

The existence of k implies that Q is wellfounded immediately. But bringing in Lemma 2, we have

$$D(M, \kappa_1) \cong H_0^+ \text{ is full,}$$

so

$$D(M, \kappa_1) \cong Q \text{ is full.}$$

Finally, branch condensation for Σ_{H_0} on H_0 immediately implies branch condensation for Σ_{H_0} on H_0^+ .



We now obtain a measure on $P(\Theta^0) \approx H_0^+$ as follows. Let

$$\gamma_0 = \sup j'' \Theta^0.$$

Note that $\gamma_0 = \pi_{H_0^+, H_1}(\theta^0)$, where $\pi_{H_0^+, H_1}$ is the mod-limit map of $D(M, K, \cdot)$.
 For $A \in \theta^0$ in H_0^+ , put

$$A \in \mathcal{D}_0 \text{ iff } \gamma_0 \in j(A)$$

Lemma 4 (H_0^+, \mathcal{D}_0) is amenable.

Proof Let $\langle A_\alpha \mid \alpha < \theta^0 \rangle \in H_0^+$. Letting

$C = \{ \alpha \mid A_\alpha \in \mathcal{D}_0 \}$, it is enough to see

that C is OD in $D(M, K, \cdot)$ from

H_0^+ and Σ_{H_0} . But $j(\langle A_\alpha \mid \alpha < \theta^0 \rangle) \in H_1^+$,

so letting $\langle B_\alpha \mid \alpha < \theta^0 \rangle$

$$\langle B_\alpha \mid \alpha < \theta^0 \rangle = j(\langle A_\alpha \mid \alpha < \theta^0 \rangle)$$

we have

$$\langle B_\alpha \cap (\gamma_{\theta^0+1}) \mid \alpha < \gamma_0 \rangle \in H_1$$

Moreover

$$\alpha \in C \text{ iff } \gamma_0 \in j(\bar{A})_{j(\alpha)}$$

$$\text{iff } \gamma_0 \in B_{\pi_{H_0^+, H_1}}(\alpha) \cap (\gamma_{\theta^0+1})$$

Since H_1 is OD in $D(M, K_1)$, and π_{H_0, H_1}^M is OD in $D(M, K_1)$ from Σ_{H_0} , we are done.

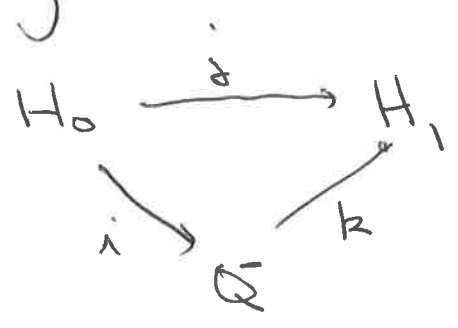


Lemma 5 γ_0 is H_0^+ -normal.

Proof Let $f: \Theta^0 \rightarrow \Theta^0$ be in H_0^+ , and $j(f)(\gamma_0) < \gamma_0$. We must find $\alpha < \Theta^0$ such that

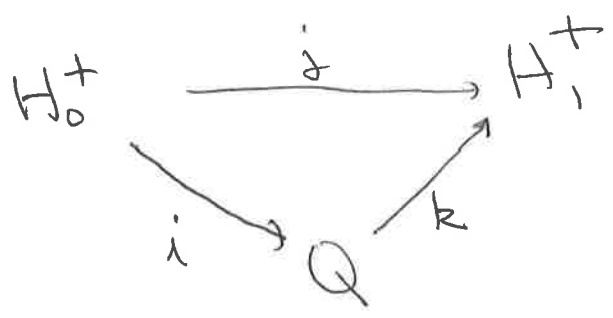
$$j(f)(\gamma_0) = j(\alpha) = \pi_{H_0, H_1}^M(\alpha).$$

Since $j(f)(\gamma_0) < \gamma_0$, we can find a factoring



with i, k being the iteration maps of H_0 by Σ_{H_0} , and $j(f)(\gamma_0) \in \text{ran}(k)$.

As in the proof of Corollary 3, we can extend i, k to



Notice that $\Sigma_{H_1}^k = Q$ -tail of Σ_{H_0} in Σ_{H_0}
 $= (\Sigma_{H_0})_{Q, \Gamma}$, where $i = i^{\square}$. This is because Σ_{H_0} has branch condensation, so it pulls back under its own iteration maps to itself. (I.e. it has pullback condensation.)

Setting $\Sigma_Q = \Sigma_{H_1}^k$, we have

$$D(M, K_1)^+ \cong \left(j(f) \left(\pi_{Q, \infty} (i(\theta^0)) \right) \in \text{ran} \left(\pi_{Q, \infty} \upharpoonright i(\theta^0) \right) \right)$$

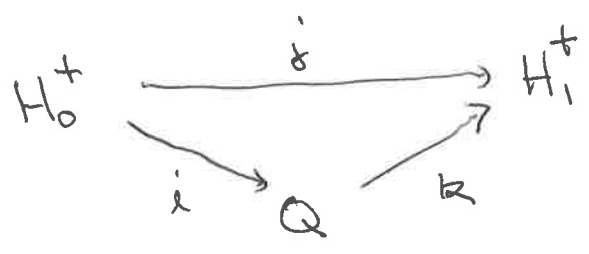
The right hand side has the form $\varphi(Q, i(\theta^0), \Sigma_Q, j(f))$. By Lemma 2,

$\varphi(H_0^+, \theta^0, \Sigma_{H_0}, j(f))$ holds in $D(M, \kappa_1)^+$. But this means that $j(f)(\gamma_0) \in \text{ran}(\pi_{H_0, \theta^0} \upharpoonright \theta^0)$, as desired.



In order to get a model of $AD_{\mathbb{R}}^+$ θ measurable, we'd like to show that $L(H_0^+, \nu_0) \cap P(\theta^0) \subseteq H_0^+$. For this, we need that (H_0^+, ν_0) is iterable. The problem is that ν_0 -ultrapowers are discontinuous at θ^0 .

To illustrate, suppose



where $Q = \cup \pi(H_0^+, \nu_0)$, and

$$k(j(f)(\theta^0)) = j(f)(\gamma_0)$$

is the j -realization map, which exists by our definition of ϑ_0 . Let

$$\Sigma_Q = \Sigma_{H_1^+}^k.$$

Σ_Q extends the Q -tail of Σ_{H_0} , but the latter acts on only $Q \uparrow \text{sup } i'' \theta^0$, whereas Σ_Q can handle iteration trees on the full Q . Since

$$D(M, \kappa_1)^+ \models \Sigma_{H_0} \text{ is a fullness-preserving strategy with branch condensation for } H_0^+,$$

we have

$$D(M, \kappa_1)^+ \models \Sigma_Q \text{ is a fullness-preserving strategy with branch condensation for } Q.$$

Thus Σ_Q really is fullness-preserving and has branch condensation, and (Q, Σ_Q) is in the hod-limit system of $D(M, \kappa_1)$.

Let
$$\pi_{Q, \omega} : Q \longrightarrow H_1,$$

be the map of the system on Q . To go

Further, we need

Claim

(a) $\pi_{Q, \infty} \upharpoonright Q^- = k \upharpoonright Q^-$,

(b) For all $A \in i(\theta^0)$ in Q ,

$A \in i(\theta^0)$ iff $\sup k'' i(\theta^0) \in k(A)$.

Proof. For (a), it's enough to show $\pi_{Q, \infty}$ agrees with k on $i(\theta^0)$. Let $\xi \in i(\theta^0)$.

We have

$$\xi = i(f)(\theta^0),$$

where $f: \theta^0 \rightarrow \theta^0$ is in H_0^+ . But then

$$D(M, \kappa_1)^+ \vDash \pi_{H_0^+, \infty}(f) \equiv j(f) \upharpoonright \pi_{H_0^+, \infty}(\theta^0)$$

so

$$D(M, \kappa_1)^+ \vDash \pi_{Q, \infty}(i(f)) \equiv j(f) \upharpoonright \pi_{Q, \infty}(i(\theta^0)).$$

This gives

$$\begin{aligned}
k(\xi) &= j(f)(\gamma_0) \\
&= \pi_{Q,\infty}(i(f)) \left(\pi_{Q,\infty}(\cancel{\gamma_0}^{\theta^0}) \right) \\
&= \pi_{Q,\infty}(i(f)(\theta^0)) \\
&= \pi_{Q,\infty}(\xi),
\end{aligned}$$

as desired for (a).

For (b), let $\xi < \theta(H_0^+)$ and

$$v_0^\xi = v_0 \cap H_0^+ | \xi.$$

We need to see that $i(v_0^\xi)$ is contained in the measure generated by k . Let

$$P(\theta^0) \cap H_0^+ | \xi = \langle A_\alpha \mid \alpha < \theta^0 \rangle.$$

and $\alpha \in C$ iff $A_\alpha \in v_0^\xi$.

Put $A(\alpha, \beta)$ iff $\beta \in A_\alpha$. Then

$$D(\mathcal{M}, \kappa_1)^+ \equiv \forall \alpha < \theta^0 (A_\alpha \in v_0^\xi \iff \dots)$$

$\pi_{H_0^+}(\theta^0) \in j(\langle A_\alpha \mid \alpha < \theta^0 \rangle)$

Then

$$D(M, k_1)^+ \models \forall \alpha < \theta^0 \left(A_\alpha \in \mathcal{V}_0^\xi \leftrightarrow \pi_{H_0^+, \infty}(\theta^0) \in j(\langle A_\alpha \mid \alpha < \theta^0 \rangle) \right)_{\pi_{H_0^+, \infty}(\alpha)}$$

This is a statement φ about H_0^+ , \mathcal{V}_0^ξ , \vec{A} , Z_{H_0} , and $j(\vec{A})$. By Lemma 2,

$$D(M, k_1)^+ \models \forall \alpha < i(\theta^0) \left(i(\vec{A})_\alpha \in i(\mathcal{V}_0^\xi) \leftrightarrow \pi_{Q, \infty}(i(\theta^0)) \in j(\langle A_\alpha \mid \alpha < \theta^0 \rangle) \right)_{\pi_{Q, \infty}(\alpha)}$$

But $\pi_{Q, \infty}(i(\theta^0)) = \sup k'' i(\theta^0)$ by (a).

So indeed $i(\mathcal{V}_0^\xi)$ is generated by $j(k)$ as desired.

Claim. ~~□~~

By the claim, we can k -realize $Ult(Q, i(\mathcal{V}_0))$, and keep going.

We have now illustrated the main ideas in the proof of

Lemma 6 In $\text{DCM}(K_1)$, there is an iteration strategy Ω for (H_0^+, \mathcal{D}_0) such that if $i: (H_0^+, \mathcal{D}_0) \rightarrow (Q, \mathcal{D}_Q)$ is an iteration map via Ω , and $i = i \circ \vec{i}$, then there is $k: Q \rightarrow H_1^+$ so that

$$\begin{array}{ccc} H_0^+ & \xrightarrow{\vec{i}} & H_1^+ \\ & \searrow i & \nearrow k \\ & Q & \end{array}$$

commutes, and

(a) $\Omega_{Q, \vec{i}} = \left(\sum_{H_1} \right)^k$, for iterations

based on $Q \restriction i(\theta^0)$,

(b) $\pi_{Q \restriction i(\theta^0), \infty} = k \upharpoonright (Q \restriction i(\theta^0))$, where

$\pi_{Q \restriction i(\theta^0), \infty}$ is the iteration map by $\sum_{H_1}^k$,

and

(c) For $A \in i(\theta^0)$ in Q

$$A \in \vec{Q} \text{ iff } \sup k'' i(\theta^0) \in k(A).$$

Proof We proceed by induction on the length of the stack \vec{I} yielding i and Q to show that there are k and $\Omega_{Q, \vec{I}}$ as in (a)-(c). We then show that the resulting Ω is sufficiently absolutely definable that $\Omega \in D(M, K_1)$.

Notice that images of \vec{v}_0 cannot contribute to branching in our normal components of \vec{I} . That is, if

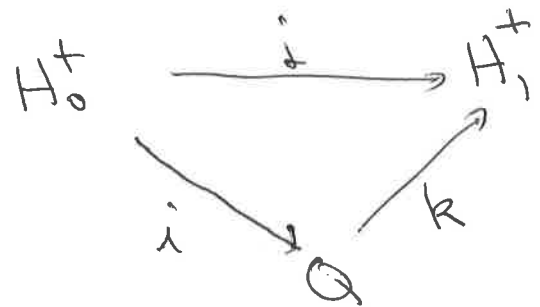
$$E_{\alpha}^{\vec{I}_{\alpha}} = i(\vec{v}_0)$$

where $i: (H_0^+, \vec{v}_0) \rightarrow (M_{\kappa}^{\vec{I}_{\alpha}}, i(\vec{v}_0))$ is an iteration map, then $E_{\alpha}^{\vec{I}_{\alpha}}$ is applied to $M_{\alpha}^{\vec{I}_{\alpha}}$. That is, the next model of \vec{I} is $M_{\alpha+1}^{\vec{I}}$ or M_0 . That is, $M_{\alpha+1}^{\vec{I}} = \text{Ult}(M_{\alpha}^{\vec{I}_{\alpha}}, i(\vec{v}_0))$. Moreover, the rest of \vec{I} can be considered as a normal

stack on $M_{\alpha+1}^{\mathcal{I}_\xi}$, because no model in \mathcal{I}_ξ can have an extender overlapping $i(\delta^\circ)$, which is a limit of Woodruff.

Thus we may as well assume that the trees in our stack $\vec{\mathcal{I}}$ are such that for each ξ , either \mathcal{I}_ξ never uses an image of δ_0 , or else \mathcal{I}_ξ consists of exactly one ultrapower, and that by an image of δ_0 .

Suppose by induction that we have defined enough of \mathcal{I} to determine that $\vec{\mathcal{I}}$ is by \mathcal{I} , and $i = i^{\vec{\mathcal{I}}}$ is such that we have



with (a)-(c) of the lemma being true.

Let

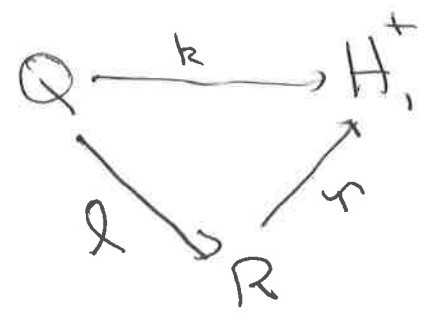
$$l: (Q, \mathcal{D}_Q) \rightarrow (R, \mathcal{D}_R)$$

be such that $l = i^u$, where u is the next normal tree in our stack.

Case 1 u does not involve an image of \mathcal{D}_Q , that is, u is actually a tree on Q .

Prf. Then we do get l (that is, a good branch of u) from $\Omega_{Q, \vec{a}} = \sum_{H_1}^k = \Sigma_Q$.

We have



where $r(l(f)(a)) = k(f)(\pi_{R, \infty}(a))$.

Here $\pi_{R, \infty}$ is the iteration map by the tail of Σ_Q . r is well-defined and elementary because it is the factor map;

letting

$$Z = \pi_{R, \infty}^{-1} \circ i(\theta^0),$$

(21)

$$E_Q = E_K \upharpoonright Z.$$

And that is true because E_Q ^{component measures of} concentrates on bounded subsets of $i(\theta^0)$, and

$$(a, x) \in E_Q \text{ iff } a \in \mathcal{Q}(x)$$

$$\text{iff } \pi_{R, \infty}(a) \in \pi_{R, \infty}(\mathcal{Q}(x))$$

$$\text{iff } \pi_{R, \infty}(a) \in \pi_{Q, \infty}(x)$$

$$\text{iff } \pi_{R, \infty}(a) \in K(x)$$

$$\text{iff } (\pi_{R, \infty}(a), x) \in E_K.$$

The second-to-last line uses (b) of our induction hypothesis.

This gives us \uparrow . We must see (a)-(c). For (a), we have to see that $\Omega_{R, \vec{\sigma}^n} = \sum H_i$. But this

is true because $\pi \circ (R(\text{loi}(\theta^0)))$ is the iteration map by $\Omega_{R, \pi^* \mathcal{U}}$ by definition, and $\Omega_{R, \pi^* \mathcal{U}}$ has branch, hence pullback, condensation. (By Lemma 2.)

(b) holds because we have defined $\pi \circ \text{loi}(\theta^0)$ to be the iteration map π_{R, θ^0} .

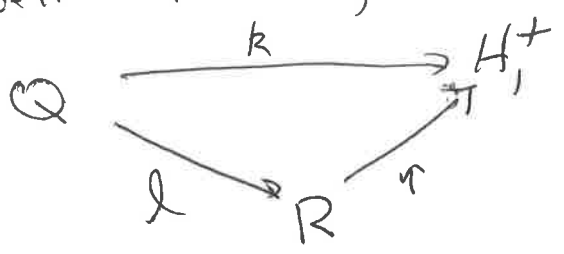
Finally, (c) holds by an application of Lemma 2 that we leave to the reader.

Case 2 \mathcal{U} is the \mathcal{V}_Q -ultrapower, that is $R = \text{Ult}(Q, \mathcal{V}_Q)$ and $\iota: (Q, \mathcal{V}_Q) \rightarrow (R, \mathcal{V}_R)$ is the ultrapower map.

Prf We define $\pi: R \rightarrow H_1^+$ by

$$\pi(\iota(f)(i(\theta^0))) = k(f)(\sup k'' i(\theta^0)).$$

π is well-defined, elementary, and



commutes by induction hypothesis (c).

Now set

$$\Omega_{R, \vec{T}^n U} = \sum_{H_1}^{\uparrow}$$

$\Omega_{R, \vec{T}^n U}$ is fullness-preserving, has branch condensation, and extends $\Omega_{Q, \vec{T}}$ acting on trees based on $R \upharpoonright \text{sup } Q$ "i" (θ^0).

This all follows by applying Lemma 2 to transfer properties of ~~$\Omega_{Q, \vec{T}}$~~ $\sum_{H_1}^R = \sum_Q$ to properties of $\sum_{H_1}^{\uparrow} = \sum_R$. And as in our illustrative example, we can use Lemma 2 to prove that (b) and (c) continue to hold. We omit further detail.

Lemma 6. 

This completes our definition of Ω . We now show that $\Omega \in D(\mathcal{IT}, \kappa, \cdot)$. For this, just notice that we have

defined Ω from the parameters

H_1^+ and $j^* H_0^+$: for g on $\text{Coll}(\omega, \kappa_0)$,

$$\Omega(\mathcal{I}) = b \text{ iff } M \mathcal{E} g \mathcal{I} \models \varphi[\mathcal{I}, b, j^* H_0^+].$$

The reader can easily check that in $M \mathcal{E} g \mathcal{I}$,
there are club many $X \in V_\delta^{M \mathcal{E} g \mathcal{I}}$

(δ large) s.t. if

$$\pi: N \mathcal{E} g \mathcal{I} \rightarrow V_\delta^{M \mathcal{E} g \mathcal{I}}$$

and $\mathcal{I}, b \in N \mathcal{E} g \mathcal{I} \cap \mathcal{K} \mathcal{I}$ (\mathcal{K} on $\text{Coll}(\omega, \aleph)$, $\aleph < \pi^{-1}(j^*(\kappa_0))$),

then

$$\Omega(\mathcal{I}) = b \text{ iff } N \mathcal{E} g \mathcal{I} \cap \mathcal{K} \mathcal{I} \models \varphi[\mathcal{I}, b, \pi^{-1}(j^* H_0^+)].$$

Basically, if you pull back under $\pi^{-1}(j)$,
you'll be pulling back under j .

Since there are club many generically
correct X , we have $\Omega \in D(M, \kappa_1)$.



We can now show that $L(H_0^+, \nu_0) \cap P(\theta^0) \subseteq H_0^+$. In fact,

Lemma 7 $L_p^\Omega(H_0^+, \nu_0) \stackrel{D(M, K_1)}{\cap} P(\theta^0) \subseteq H_0^+$.

Proof If P is a ^{sound} Ω -mouse extending (H_0^+, ν_0) and projecting to θ^0 , then

consider

$$i: P \longrightarrow \text{Ult}(P, \nu_0) =: Q.$$

We can think of Q as a hod-pair over (H_0^+, Σ_{H_0}) . By comparison

(which easily still works ~~on the~~ ~~measurable~~ for Q), P is then

OD in $D(M, K_1)$ from H_0^+ and Σ_{H_0} .

This contradicts the fullness of H_0^+ .



§2. A second measure on Θ^0 .

(32)

It's clear how to go on from here for a while. We may as well index

\mathcal{V}_0 at $(\Theta^0)^{++}$ of $L_p^{\Sigma_{H_0}}[H_0]$, as

usual. Let

$$\mathcal{H}_0 = \left(L_p^{\Sigma_{H_0}}[H_0] / \alpha, \mathcal{V}_0^* \right)$$

where $\alpha = \Theta^{0++}$ and \mathcal{V}_0^* is \mathcal{A}_0

amenable coding of \mathcal{V}_0 . We have that

Ω is an iteration strategy for \mathcal{H}_0

in $D(M, \kappa_1)$, and it is fullness-preserving

and has branch condensation there. Let

$$\mathcal{H}_0^+ = L_p^{\Omega}[\mathcal{H}_0],$$

$$\text{and } \mathcal{H}_1 = j(\mathcal{H}_0), \quad \mathcal{H}_1^+ = j(\mathcal{H}_0^+).$$

Put also

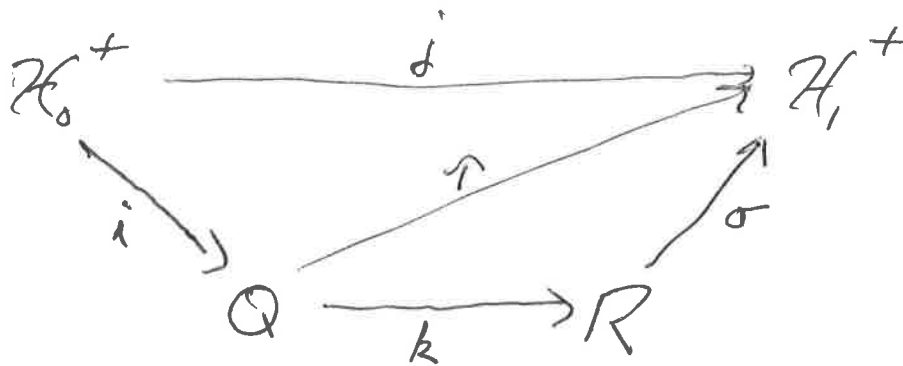
$$\mathcal{D}(M, \kappa_1) = L(\text{Hom}_g^*, \mathcal{H}_1^+),$$

where g is $\text{Col}(w, \langle \kappa_1 \rangle)$ generic / M .

Again, $\text{Hom}_g^* = P(\mathbb{R}_g^*) \cap \mathcal{O}(M, \kappa_1)$,
 and \mathcal{O}^\pm is still regular - show.

The key lemma 2 now reads

Lemma 2' Suppose that in $\text{M}[h]$, where h is $\text{Col}(w, \langle \kappa_1 \rangle)$ generic over M , we have the commutative diagram



where Q, R are countable in $\text{M}[h]$. Let
 $\Sigma_Q = \Sigma_{\mathcal{H}_1^+}^\uparrow$ and $\Sigma_R = \Sigma_{\mathcal{H}_1^+}^\sigma$. Then for any
 $A \in \mathcal{H}_0^+$ and formula φ

$$\begin{aligned}
 \mathcal{O}(M, \kappa_1) \models \forall s \in Q \left(\varphi(Q, s, \Sigma_Q, \mathcal{H}_1^+, j(A)) \right) \\
 \iff \varphi(R, k(s), \Sigma_R, \mathcal{H}_1^+, j(A))
 \end{aligned}$$

That is, it reads the same, with H 's changed to script \mathcal{H} 's. Our pullback strategies Σ_Q and Σ_R are now, like Ω , strategies for hod-promises with measurable limits of Woodins. Indeed, if i is an iteration map by Ω , then Σ_Q will just be a tail of Ω . But i may not be that, e.g., it may come from hitting an ordinal \aleph measure that we are trying now to find.

Now let

$$\pi_{\mathcal{H}_0^+, \aleph} : \mathcal{H}_0^+ \longrightarrow \mathcal{H}_1$$

be the iteration map by Ω , and

$$\delta_1 = \pi_{\mathcal{H}_0^+, \aleph}(\theta^\circ).$$

We put for $A \subseteq \theta^\circ$ in \mathcal{H}_0^+ :

$$A \in \mathcal{V}_1 \text{ iff } \delta_1 \in j(A).$$

We can proceed as we did with \mathcal{V}_0 .

Appendix
Proof of lemma 2

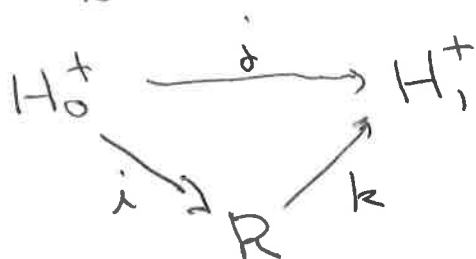
(A1)

The following is just Sargsyan's proof, re-worded.

~~we say that~~
we say that (R, k) factors j iff

$k: R \rightarrow H_1^+$ and $\text{ran}(j) \subseteq \text{ran}(k)$.

So letting $i = k^{-1} \circ j$, we have

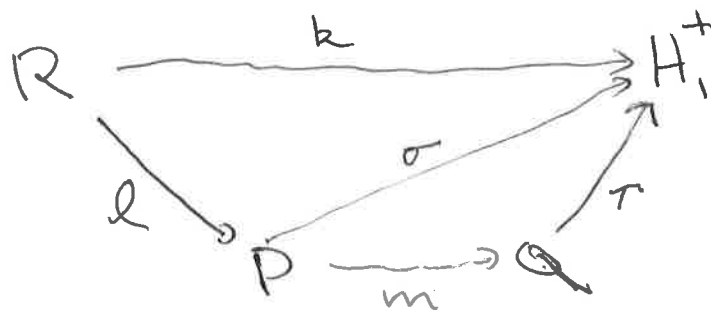


Fix now $A \in H_0^+$. We say that

(R, k) respects $j(A)$ iff (R, k) factors j ,

and R is countable in $D(M, K_1)$,

and whenever



commutes, with P and Q countable in $D(M, K_1)$,

then setting $\Sigma_P = \Sigma_{H_1}^\sigma$ and $\Sigma_Q = \Sigma_{H_1}^\tau$, A2.
 we have

$$(*) \quad D(M, k_1)^+ \models \forall s \in P \left(\varphi(P, s, \Sigma_P, j(A)) \leftrightarrow \varphi(Q, m(s), \Sigma_Q, j(A)) \right).$$

If ~~(*)~~ $(P, Q, \ell, m, \sigma, \tau)$ is factious into k as above, but $(*)$ fails, then we call $(P, Q, \ell, m, \sigma, \tau)$ a $j(A)$ -bad factoring of (R, k) . So (R, k)

respects $j(A)$ just in case it admits no $j(A)$ -bad factorings.

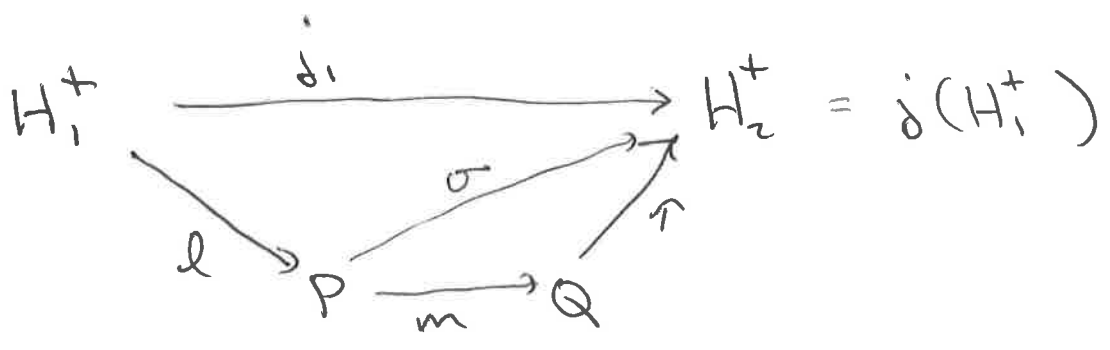
Claim A. There is an $R \in V_{k_1}^M$ and $k \in M$ such that (R, k) factors j , and (R, k) respects $j(A)$.

Before proving the claim, we show that it implies lemma 2. ~~For~~ Lemma 2

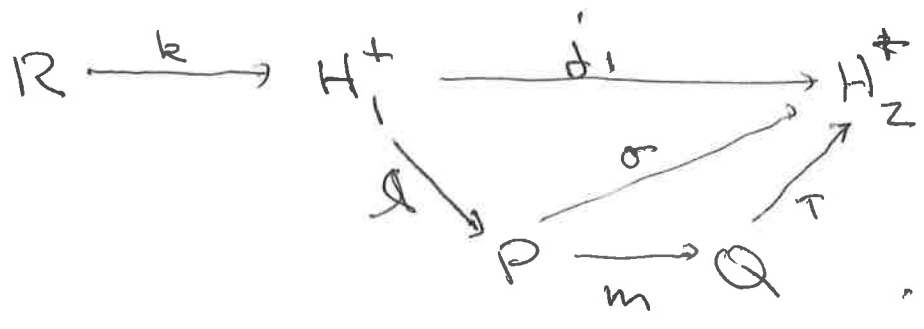
says that (H_0^+, j) respects $j(A)$,
 for all $A \in H_0^+$. Putting $j_1 = j(j)$
 and $M_1 = j(M) = j_1(M)$, it
 is enough to see

$M_1 = (H_1^+, j_1)$ respects $j_1(j(A))$.

Note $j(j(A)) = j_1(j(A))$. So suppose
 that



is a $j_1(j(A))$ - bad factoring of
 (H_1^+, j_1) in M_1 . Now let (R, k)
 be as in the claim. Consider
 the diagram



Note that $j_1(k) = j_1 \circ k$, because $\text{crit}(j_1) = K_1$. It is easy to see then that $(P, Q, \ell \circ k, m, \sigma, \tau)$ is a $j_1(j(A))$ -bad factoring of $(R, j_1(k))$ in M_1 . This is impossible because j_1 is elementary and $j_1(R) = R$.

So it is enough to prove Claim A.

Work in M for a moment. Let $K_0 < \nu < K_1$, with ν an inaccessible limit of Woodruff and $< K_1$ -strongs.

Set $R_\nu^- = \text{HOD}_{\mathcal{I}^0}^{D(M, \nu)}$, and let Σ_ν be the join of the strategies for

$R_2^-(\alpha)$, $\alpha \in \mathcal{S}^{R_2^-}$. Let $\pi_2: R_2^- \rightarrow H_1$
 be the hod-limit map of $D(M, k_1)$,
 which exists because (R_2^-, Σ_2) is a
 "limit hod pair" in $D(M, k_1)$. Let

$$R_2 = L_p^{\Sigma_2}(R_2^-)^{D(M, k_1)}$$

We say $\vec{\nu}$ is good iff

$$R_2 = \text{trans. collapse of } \text{Hull}^{H_1^+}(\pi_2'' R_2^- \cup j'' H_0^+),$$

and if

$$\sigma_2: R_2 \rightarrow H_1^+$$

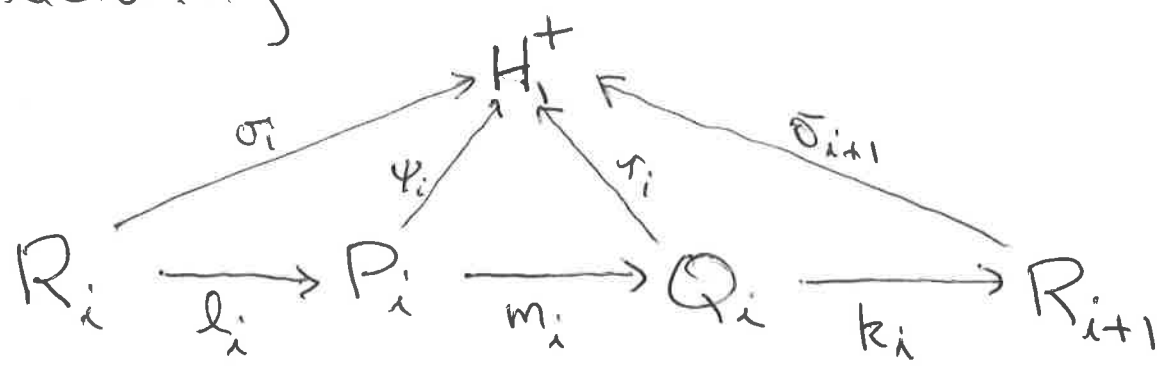
is the transitive collapse, then $\sigma_2 \upharpoonright R_2^- = \pi_2$ is the iteration map by Σ_2 . Goodness can be defined in M , because $j'' H_0^+ \in M$.

Claim

Subclaim A.1 In M , there are measure one many good $\vec{\nu} \prec k_1$.

Proof It is not hard to show that k_1 is good in $j_1(M)$, vis-a-vis $j_0''(j'' H_0^+)$. □

Claim B. Suppose claim A is false. Then there is in \mathcal{M} an increasing infinite sequence $\langle \nu_i \mid i \in \omega \rangle$ of good ν 's such that letting $R_i = R_{\nu_i}$ and $\sigma_i = \sigma_{\nu_i}$, we have $\text{coll}(\omega, \langle \kappa_i \rangle)$ for every i , in $\mathcal{M}^{\text{coll}(\omega, \langle \kappa_i \rangle)}$ a factoring



such that $(P_i, Q_i, l_i, m_i, \psi_i, \tau_i)$ is a $j(A)$ -bad factoring of (R_i, σ_i) .

Proof. This is obvious.



Now fix a sequence $\langle (R_i, \sigma_i) \mid i \in \omega \rangle = \langle \vec{R}, \vec{\sigma} \rangle$

in M , as in claim B. Let

(A7)

$$\pi_i = \sigma_{i+1}^{-1} \circ \sigma_i,$$

so that ~~any~~ $\pi_i: R_i \rightarrow R_{i+1}$. Notice

that if we have

$$\begin{array}{ccc} R_i & \xrightarrow{\pi_i} & R_{i+1} \\ \downarrow \ell & & \uparrow k \\ & P \xrightarrow{m} Q & \end{array}$$

then there are ψ and τ such that

$(P, Q, \ell, m, \psi, \tau)$ is a factoring of (R_i, σ_i) ,

namely $\tau = \sigma_{i+1} \circ k$ and $\psi = \sigma_{i+1} \circ k \circ m$.

So we shall call such a (P, Q, ℓ, m, k) a factoring of (R_i, π_i, R_{i+1}) , and we can speak of its being $j(A)$ -bad.

Now we bring the badness of our factorings down from $D(M, K_i)^+$ into $D(M, K_i)$ itself.

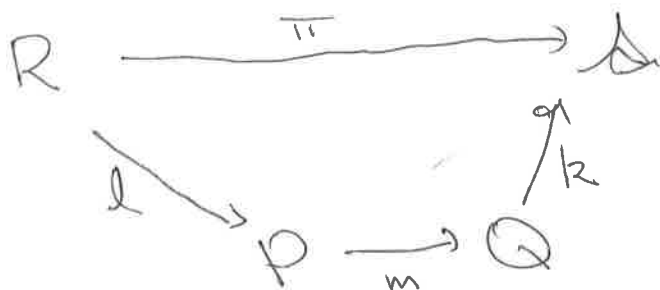
(AD⁺)

(A8)

Def. Let (Δ, Ψ) be a hod-pair,
and let $\pi: R \rightarrow \Delta$. Let $\xi, \gamma, \beta \in \text{ORD}$.

We say that (P, Q, ℓ, m, k) is a
 (ξ, γ, β) -bad factoring of (R, π, Δ, Ψ)

just in case



commutes, and letting $\Sigma_P = \Psi^{k \circ m}$
and $\Sigma_Q = \Psi^k$, we have that for some
 $s \in P$ and formula φ :

$$L_\beta(P_\xi(R)) \models \varphi[P, s, \Sigma_P, \gamma]$$

but $L_\beta(P_\xi(R)) \not\models \varphi[Q, m(s), \Sigma_Q, \gamma]$.

(Here $P_\xi(R) = \{A \subseteq R \mid \omega(A) < \xi\}$, and we assume
 $\Sigma_P, \Sigma_Q \in P_\xi(R)$.)

Def (AD^+) Let us say that

$\langle (\mathcal{D}_n, \pi_n, \Psi_n) \mid n < \omega \rangle$ is a

(ξ, δ, β) -bad sequence iff for all

$n < \omega$

(1) (\mathcal{D}_n, Ψ_n) is a hod pair

(2) $\pi_n: \mathcal{D}_n \rightarrow \mathcal{D}_{n+1}$, and

$$\Psi_n = \Psi_{n+1}^{\pi_n}, \text{ and}$$

(3) there is a (ξ, δ, β) -bad factoring of $(\mathcal{D}_n, \pi_n, \mathcal{D}_{n+1}, \Psi_{n+1})$.

Let us write $\Sigma_i = \Sigma_{\nu_i} = (\Sigma_{H_i})^{\sigma_i}$ for

the strategy ~~in \mathcal{M}_{ν_i}~~ for $R_i = R_{\nu_i}$ that

we have. So $\Sigma_i \in \mathcal{M}$ and acts

on trees in $V_{\kappa_i}^M$, but also extends

canonically to trees in $HC^{M \Sigma_i^h}$, for

h on $Col(\omega, < \kappa_i)$. We write Σ_i^h for

this extension. Note $\Sigma_i^h \in D(M, \kappa_1)^h$.

(A10)
‡

Claim C There is a (ξ, γ, β) such that whenever h on $\text{Col}(w, \kappa_1)$ yields the derived model $D(M, \kappa_1)^h$:

$$D(M, \kappa_1)^h = \langle (R_i, \pi_i, R_{i+1}, \Sigma_{i+1}^h) \mid i < w \rangle$$

is (ξ, γ, β) -bad,

where $\pi_i = \sigma_{i+1}^{-1} \circ \sigma_i$, for each i .

Proof (Sketch.) In $D(M, \kappa_1)^+$, we take a Skolem hull of $L_\eta(P(\mathbb{R}), H_1^+, j(A))$, for η large, throwing in all reals, and $H_1^+, j(A)$ as points. The collapses of H_1^+ and $j(A)$ can be seen to be in H_1 . (That this is true is forced in $\text{Vop}^{<w}$ over H_1^+ , by a reflection argument like before.) Thus the collapses of H_1^+ and $j(A)$ are OD in $D(M, \kappa_1)$, and this is what we need.



Fix $(\xi_0, \gamma_0, \beta_0)$ such that

$$b_0 = \langle (R_i, \pi_i, R_{i+1}, \Sigma_{i+1}) \mid i < \omega \rangle$$

is $(\xi_0, \gamma_0, \beta_0)$ -bad in $D(M, \kappa_1)$. Our rough plan now is to find a ~~Σ_{H_0}~~ Σ_{H_0} -hod-pair (W_0^u, Ψ_0^u) ~~that~~ in $D(M, \kappa_1)$ that has an \mathbb{R}^{MCHT} -genericity iterate (W_0^w, Ψ_0^w) ~~such that~~ whose derived model goes past $P_{\xi_0}(\mathbb{R}^{MCHT})$, and therefore satisfies that b_0 is $(\xi_0, \gamma_0, \beta_0)$ -bad.

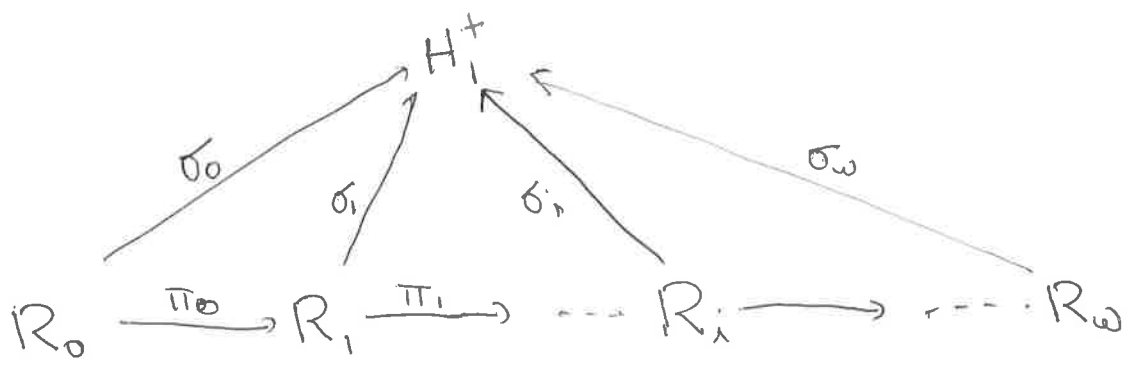
Letting

$$W_{i+1} = \text{Ult}(W_i, E_{\pi_i}),$$

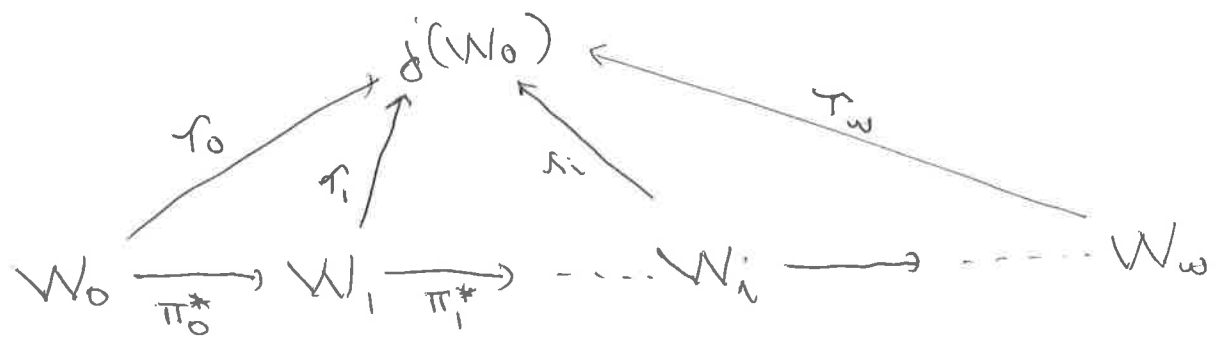
we have that for each i

W_i is a Σ_i -hod-premouse extending R_i ,

and the diagram



lifts to the diagram



It is important here that we can arrange that $(W_0, \Psi_0) \in V$, so we can apply j to it. The maps τ_i are given by

$$\tau_0 = j \uparrow W_0$$

and

$$\tau_{i+1}(\pi_{0i}^*(f)(a)) = j(f)(\sigma_i(a))$$

for all $a \in R_i^-$ and $f \in W_0$.

(Notice here that because σ_0 is cofinal, all of the π_i 's are cofinal, and

thus we can assume $\#$

$$\text{lh } E_{\pi_i} = \pi_{0, i-1}(\theta^0).$$

So our restriction that $a \in R_i^-$ above is ok.)

Let us now simplify things by assuming that j ~~and~~ witnesses that K_0 is huge. This implies that $\tau_w \in j(M)$. But $j(\psi_0)$ is a strategy for $j(W_0)$ in $j(M)$. So we can set

$$\psi_w = j(\psi_0)^{\tau_w},$$

~~and (W_w, ψ_w) is a hod pair~~

and ψ_w acts on all trees in $V_{K_1}^M = V_{K_1}^{j(M)}$. Moreover, ψ_w extends to $\text{HC}M \text{Lh} \uparrow$.

In $\text{MLh} \uparrow$, fix $(P_i, Q_i, l_i, m_i, k_i)$ witnessing that $(R_i, \pi_i, R_{i+1}, \Sigma_{i+1})$ is

$(\xi_0, \gamma_0, \beta_0)$ - bad, for each i .

(A14)

Let

$$U_i = \text{Ult}(W_i, E_{\ell_i})$$

and

$$V_i = \text{Ult}(U_i, E_{m_i}),$$

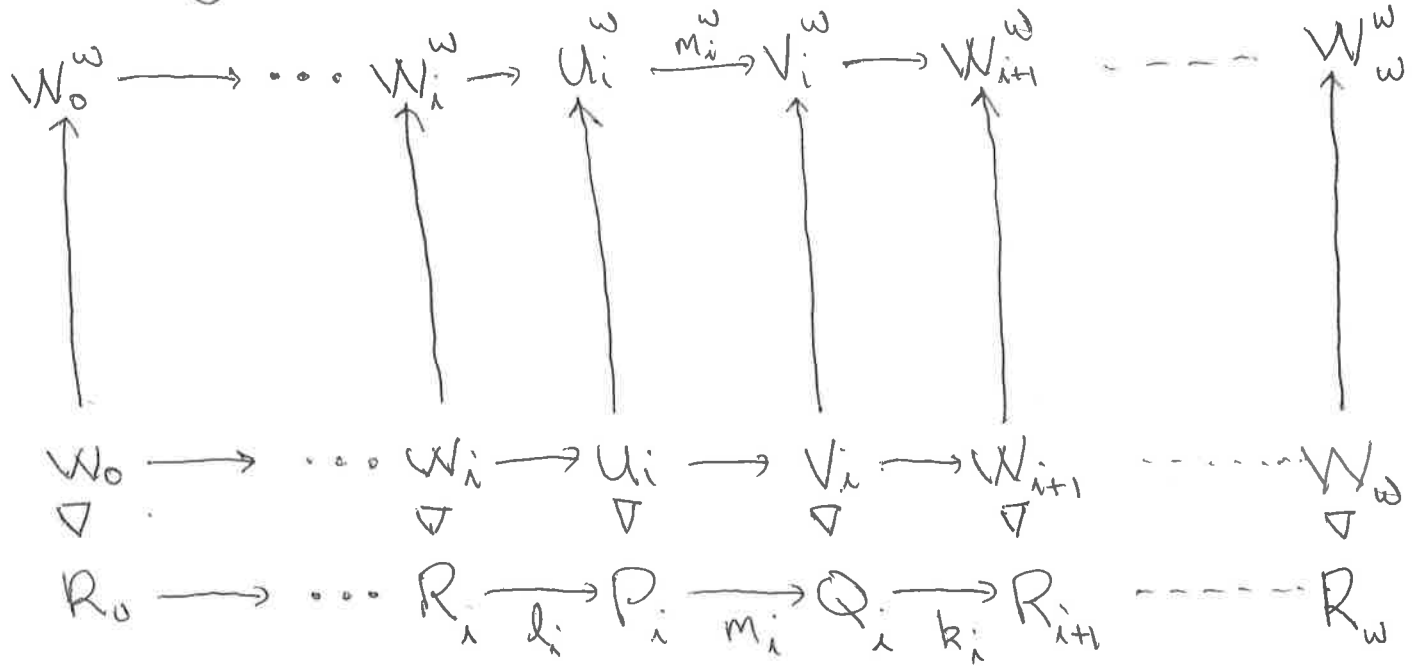
so that we have

$$\begin{array}{ccccccc} W_i & \longrightarrow & U_i & \longrightarrow & V_i & \longrightarrow & W_{i+1} \\ \nabla & & \nabla & & \nabla & & \nabla \\ R_i & \longrightarrow & P_i & \longrightarrow & Q_i & \longrightarrow & R_{i+1} \end{array}$$

All the W_i, U_i , and V_i have iteration strategies obtained by pulling back our strategy Ψ_w for W_w . We can then assume that our genericity iteration $W_0^{\#} \longrightarrow W_0^w$ was dovetailed with \mathbb{R}^{MCH} -genericity iterations of the W_i, U_i, V_i in the standard way. This yields

the diagram

A15



Let \mathcal{Q}

$$C_i = D(W_i^w, \langle \delta_\lambda^{W_i^w} \rangle^h)$$

$\stackrel{\text{M2h3}}{=} \mathbb{R}^h$ -realization of derived model
of W_i^w

$$D_i = D(U_i^w, \langle \delta_\lambda^{U_i^w} \rangle^h)$$

$$E_i = D(V_i^w, \langle \delta_\lambda^{V_i^w} \rangle^h)$$

Claim For all i , $C_i \subseteq D_i \subseteq E_i \subseteq C_{i+1}$.

Proof We are assuming $\lambda^{w'}$ is a limit ordinal, it follows that and of

the form

$$\lambda^w = \alpha + w$$

(A16)

for some α . It follows that the same holds for λ^{w_i} , λ^{u_i} , λ^{v_i} . Moreover, for

$$A \in \mathbb{R}^{M \times H \times J}$$

$$A \in C_i \text{ iff } \exists \beta < \lambda^{w_i}$$

$$A \leq_w \text{code}(\sum^{w_i}(\beta))$$

where $\sum^{w_i}(\beta)$ is the strategy for $w_i(\beta)$, ~~extended~~ which acts on trees in $HC^{M \times H \times J}$,

that we are given by pulling back from Ψ_w . Similar equivalences

characterize D_i and G_i . But

then for $\beta < \lambda^w$, $\sum^{w_i}(\beta)$ is a pullback of $\sum^{u_i}(\lambda_i^*(\beta))$, and hence $\sum^{w_i}(\beta)$ is projective in $\sum^{u_i}(\lambda_i^*(\beta))$.

This tells us that $C_i \subseteq D_i$.

The inclusions $D_i \subseteq G_i \subseteq C_{i+1}$ follow by the same argument.



The claim implies that $L_{\beta_0}(P_{\xi_0}(\mathbb{R}^{\text{MATH}}))$ is a Wadge initial segment of all $C_i, D_i,$ and G_i .

Since W_w^w is an iterate of W_w by φ_w , W_w^w is wellfounded. Thus every ordinal is eventually fixed by the $l_i^w, m_i^w,$ and k_i^w . So we can find an i such that

$$m_i^w((\xi_0, \beta_0, \gamma_0)) = (\xi_0, \beta_0, \gamma_0).$$

But m_i^w is elementary, $m_i^w \upharpoonright P_i = m_i$,
 and m_i^w moves the UB code of Σ_{P_i}
 to the UB code of Σ_{Q_i} . Thus for
 any $s \in P_i$ and any φ

$$L_{\beta_0}(P_{\xi_0}(R))^{D_i} \neq \varphi[P_i, s, \Sigma_{P_i}, \gamma_0]$$

iff

$$L_{\beta_0}(P_{\xi_0}(R))^{G_i} \neq \varphi[Q_i, m_i(s), \Sigma_{Q_i}, \gamma_0].$$

Since $L_{\beta_0}(P_{\xi_0}(R))^{D_i} = L_{\beta_0}(P_{\xi_0}(R))^{G_i} =$

$L_{\beta_0}(P_{\xi_0}(R))^{MZH}$, this contradicts

$(P_i, Q_i, \delta_i, m_i, k_i)$ being a $(\xi_0, \beta_0, \gamma_0)$ -
 bad factoring of $(R_i, \pi_i, R_{i+1}, \Sigma_{i+1})$.



We have now pretty much proved lemma 2 in the case that j is a hugeness embedding. We shall adapt the argument above to the general case by replacing the R_i 's and W_i 's by Skolem hulls of themselves having size κ_0 .

We have $(\xi_0, \delta_0, \beta_0)$ that such that $b_0 = \langle (R_i, \pi_i, R_{i+1}, \Sigma_{i+1}) \mid i < \omega \rangle$ is a $(\xi_0, \delta_0, \beta_0)$ -bad sequence in $D(M, \kappa_1)$. We fix a ~~hod pair~~ Σ_{H_0} -hod-pair (W_0, ψ_0) extending H_0^+ in $D(M, \kappa_1)$ such that

- (a) $\lambda^{W_0} = \alpha + \omega$ for some α
- (b) $L_{\beta_0}(P_{\xi_0}(M))^{M \text{ th } J}$ is coded by a set of reals that is \leq_W code $(\langle \langle W_0 \rangle \rangle^{\psi_0}, \psi_0|_{W_0(\eta)})$, for some $\eta \in \lambda^{W_0}$, for any h on $\text{Coll}(\omega, < \kappa_1)$,
- (c) $(W_0, \psi_0) \in V$.

Parts (a) and (b) we can satisfy because we are assuming that the hod analysis (relativised, analyzing $HOD_{\Sigma_{Ho}}$) works in $D(M, K_1)$. Part (c) we can add by a "local HOD-limit" argument, analogous to the way we got the R_α 's. It follows from (a)-(c) that if W_0^w is an \mathbb{R}^{MLHJ} -genericity iteration by Φ_0 , then $L_{\Phi_0}(P_{\Phi_0}(\mathbb{R}))^{MLHJ}$ is an proper initial segment of the derived model of W_0^w .

We need a structure N that will certify the relationship of (W_0, Φ_0) to the (R_i, Σ_i) 's, so that this relationship will be preserved when we replace ~~them~~ them by Skolem hulls of themselves (as a relationship ^{still} holding in $MLHJ$), not

just in some Skolem hull of $M \Sigma H J$.)

For this, let (N, Φ) be a hod-pair
 over $\overline{W_0} \text{TC} (\{W_\alpha\}_{\alpha < \omega} \cup \{R_\alpha \mid \alpha < \omega\})$ relative
 to $\Psi_0, \langle \Sigma_\alpha \mid \alpha < \omega \rangle$. That is, we put X
 at the bottom, and feed in the strategies
 Ψ_0, Σ_α ($\alpha < \omega$) throughout. We take
 N to have ω Woodin cardinals, i.e.
 $\aleph^N = \omega$, and satisfy ZFC. Let Ψ_0
 and Σ_α be the canonical terms in N
 for $\Psi_0 \upharpoonright (HC)^{\mathbb{D}(N, \langle \Sigma_\alpha^N \rangle)}$ and $\Sigma_\alpha \upharpoonright (HC)^{\mathbb{D}(N, \langle \Sigma_\alpha^N \rangle)}$.

Working inside N , we can do a
 " $V_{\delta_\omega}^N$ -genericity iteration" of W_0 by Ψ_0 ,
 successively making the $V_{\delta_\alpha}^N$'s generic
 for the extender algebras at the images of
 the $W_0(\alpha+n)$'s (where $\aleph^{W_0} = \alpha + \omega$).

(A22)

This yields $i: W_0 \rightarrow W_0^\omega$ inside N ,
 and W_0^ω has the property that whenever
 \mathbb{R}^* is the reals of a symmetric collapse
 over N below δ_w^N , then \mathbb{R}^* is also the
 reals of a symmetric collapse over W_0^ω
 below $\delta_{i(\kappa)}^{W_0^\omega}$.

By a local hod-limit argument, we
 can arrange that $\kappa \in V_{\kappa_1}^M$, $\Phi \upharpoonright M \in M$,
 and $\Phi \upharpoonright M$ determines Φ^h with domain
 $HC^{M \Sigma h^T}$, for any h on $Col(\omega, < \kappa_1)$.

Claim D. There is a $(\xi_1, \beta_1, \gamma_1) \in W_0^\omega$ such
 that whenever g is N -generic over
 $Col(\omega, < \delta_w^N)$, then letting D be the
 resulting derived model of W_0^ω ,

$D \neq \langle (R_i, \pi_i, R_{i+1}, \Sigma_{i+1}^g \upharpoonright HC) \mid i < \omega \rangle$ is
 $(\xi_1, \beta_1, \gamma_1)$ -bad.

Proof we are claiming

$$N = \varphi [W_0, \dot{\Psi}_0, \langle (R_i, \dot{\Sigma}_i) \mid i \leq \omega \rangle]$$

for a certain φ . But let

$$t: N \longrightarrow N/\omega$$

be an $\mathbb{R}^{M \times H \times J}$ -genericity iteration by $\overline{\Phi}$ done in/over $M \times H \times J$. Then for l on $\text{col}(\omega, \langle t(\delta_\omega^N) \rangle$,

$$\Psi_0 = t(\dot{\Psi})_l \cap \text{HC}^{M \times H \times J}$$

and

$$\Sigma_i = t(\dot{\Sigma}_i)_l \cap \text{HC}^{M \times H \times J}$$

That is, $\overline{\Phi}$ moves the terms for Ψ_0 and Σ_i correctly (because ~~the~~ it moves $\overline{\Phi} \cap N$ and $\Sigma_i \cap N$ correctly, and these yield $\dot{\Psi}$ and $\dot{\Sigma}_i$) ~~various~~ But then

by our choice of W_0 and Ψ_0 ,

$$N_{\infty} = \varphi [\tau(W_0), \tau(\Psi_0), \tau(\langle (R_i, \dot{\Sigma}_i) \mid i < \omega \rangle)]$$

Since $\tau: N \rightarrow N_{\infty}$ is elementary, we are done.



Now let $W_{i+1} = \text{Ult}(W_i, E_{\pi_i})$,

$\pi_i^*: W_i \rightarrow W_{i+1}$ lift π_i , and $W_{\omega} = \text{dir lim } W_i$ as before. We have

$$\tau_i: W_i \rightarrow j(W_0), \quad \tau_{\omega}: W_{\omega} \rightarrow j(W_0)$$

again as before. Our problem is that $\tau_{\omega} \notin j(M)$ is possible, so that we cannot make sense of $j(\Psi_0)^{\tau_{\omega}}$. So

we let

$$\rho: P \longrightarrow V_{\xi}^M$$

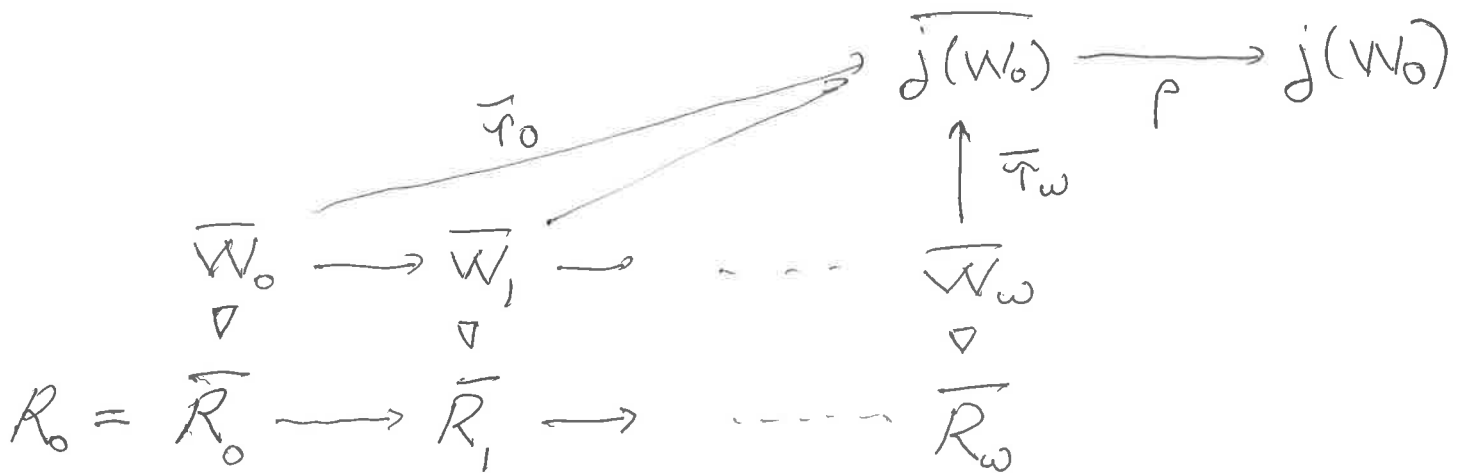
where P is transitive, $|P| = \kappa_0$,

ξ is large, and everything relevant is in $\text{ran}(\rho)$. We put

$$\overline{W}_i = \rho^{-1}(W_i),$$

$$\overline{\pi}_i = \rho^{-1}(\pi_i),$$

and so on. Inside P , we have the collapse of one of our previous diagrams:



That is, all of this diagram is in P , except for ρ and $j(W_0)$. Now notice that $\rho \circ \tau_w$ is essentially a K_0 -sequence of elements of $j(W_0) \in j(M) = j_1(M)$, and $\rho \circ \tau_w \in M$, so $\rho \circ \tau_w \in j(M)$.

So we can set

$$\overline{\Psi}_w = j(\Psi_0)^{\rho \circ \tau_w}$$

$$\overline{\Psi}_n = \left(\overline{\Psi}_w \right)^{\overline{\tau}_{nw}^*} = j(\Psi_0)^{\rho \circ \tau_n},$$

and we have that $\overline{\Psi}_w, \overline{\Psi}_n$ can be extended to act on all trees in HC^{MESH}

(as this is contained in $HC^{dim \leq 1}$ for ℓ on $col(w, < j(K_i))$). Similarly, we have

the pullbacks

$$\overline{\Sigma}_i = (\Sigma_i)^{\rho}$$

acting on all trees in HC^{MESH} . The reader may object that we are mis-using our bar notation, but in fact

$$\rho^{-1}(\Psi_0) \equiv \overline{\Psi}_0 \cap P$$

and

$$\rho^{-1}(\Sigma_i) = \overline{\Sigma}_i \cap P$$

Claim E.

$$(1) \rho^{-1}(\Sigma_i) = \Sigma_i^{\rho} \cap P$$

$$(2) \rho^{-1}(\Psi_0) = \Psi_0^{\rho} \cap P \\ = j(\Psi_0)^{\rho \circ \bar{\tau}_0} \cap P$$

Proof

(1) If $\mathcal{I} \in P$ is by $\rho^{-1}(\Sigma_i)$, then $\rho(\mathcal{I})$ is by Σ_i , so $\rho\mathcal{I}$ is by Σ_i because it is a hull of $\rho(\mathcal{I})$, so \mathcal{I} is by Σ_i^{ρ} .

(2) Similarly, $\rho^{-1}(\Psi_0) = \Psi_0^{\rho} \cap P$. For the second equality, we have

$$\begin{array}{ccc} W_0 & \xrightarrow{j} & j(W_0) \\ \rho \uparrow & & \uparrow \rho \\ \overline{W_0} & \xrightarrow{\bar{\tau}_0} & \overline{j(W_0)} \end{array}$$

If \mathcal{I} is by $\rho^{-1}(\Psi_0)$, then $j(\rho(\mathcal{I}))$ is by $j(\Psi_0)$. But $(\rho \circ \bar{\tau}_0)\mathcal{I}$ is a hull of $j(\rho(\mathcal{I})) = \rho(\bar{\tau}_0(\mathcal{I}))$, and $(\rho \circ \bar{\tau}_0)\mathcal{I} \in j(M)$

because $p \circ \bar{\tau}_0 \in j(M)$. (And in fact, its image as a subset of $j(p(\mathcal{T}))$ is also in $j(M)$.) So $(p \circ \bar{\tau}_0)\mathcal{T}$ is by ~~$j(p(\mathcal{T}))$~~ $j(\Psi)$.



We also let

$$\bar{\Phi} = \Phi^p,$$

and we have that $p^{-1}(\bar{\Phi}) \subseteq \bar{\Phi}$, as above.

$\bar{\Phi}$ is a strategy for \bar{N} acting on all trees in HC^{MLKJ} . In $MLKJ^{col(w, k)}$

we do on \mathbb{R}^{MLKJ} genericity iteration

$$\nu : \bar{N} \longrightarrow \bar{N}_\infty$$

of \bar{N} . Let

$$W_0^* = \nu(\overline{W_0^w}),$$

so that we have in \bar{N}_∞ an $\mathbb{R}^{V_{sw}}$ -genericity iteration from

$\overline{W}_0 \Rightarrow v(\overline{W}_0)$ to W_0^* . Let

$$(\beta_2, \gamma_2, \delta_2) = v((\overline{\beta}_1, \overline{\gamma}_1, \overline{\delta}_1)),$$

where $(\beta_1, \gamma_1, \delta_1) \in W_0^w$ and witness the truth of claim D. \mathbb{R}^{MCHT} is the reals of a derived model of \overline{N}_0 , and hence of W_0^* . Let D_0^* be this derived model of W_0^* . Let also

$$\Psi_0^* = v(\overline{\Psi}_0)^{\mathbb{R}^{MCHT}}$$

and

$$\Sigma_i^* = v(\overline{\Sigma}_i)^{\mathbb{R}^{MCHT}}$$

be the interpretations of $v(p^{-1}(\overline{\Psi}_0))$ and $v(p^{-1}(\overline{\Sigma}_i))$ in the \mathbb{R}^{MCHT} -realization of $D(\overline{N}_0, \overline{S}_w^*)$. We have

Claim F.

$W_0^* = \langle (\bar{R}_i, \bar{\pi}_i, \bar{R}_{i+1}, \Sigma_{i+1}^*) \mid i < \omega \rangle$ is
 $(\beta_2, \xi_2, \gamma_2)$ - bad.

Proof This follows from the fact that
 $(\beta_1, \xi_1, \gamma_1)$ witnesses the truth of Claim D
 in N , and that v and ρ are elementary,
 and $v(\bar{R}_i, \bar{\pi}_i) = (R_i, \pi_i)$.

□

The next claim makes a key
 connection.

Claim G.

$$(1) \quad \Psi_0^* = \overline{\Psi_0},$$

$$(2) \quad \Sigma_i^* = \overline{\Sigma_i}.$$

Proof This is proved in Saugsoyan's

thesis. (Claim E is a first step,
but claim G is a good deal harder to
prove.) See 3.76 of [2],
and 3.9, 3.10, and 3.28 of [3].



Now we can reach a contradiction
as before. Fix for each i a tuple
 $(P_i, Q_i, l_i, m_i, k_i)$ such that

~~the~~ $D_0^* \models (P_i, Q_i, l_i, m_i, k_i)$ is
a $(\beta_2, \beta_2, \delta_2)$ -bad factoring
of $(\bar{R}_i, \bar{\pi}_i, \bar{R}_{i+1}, \Sigma_{i+1}^*)$.

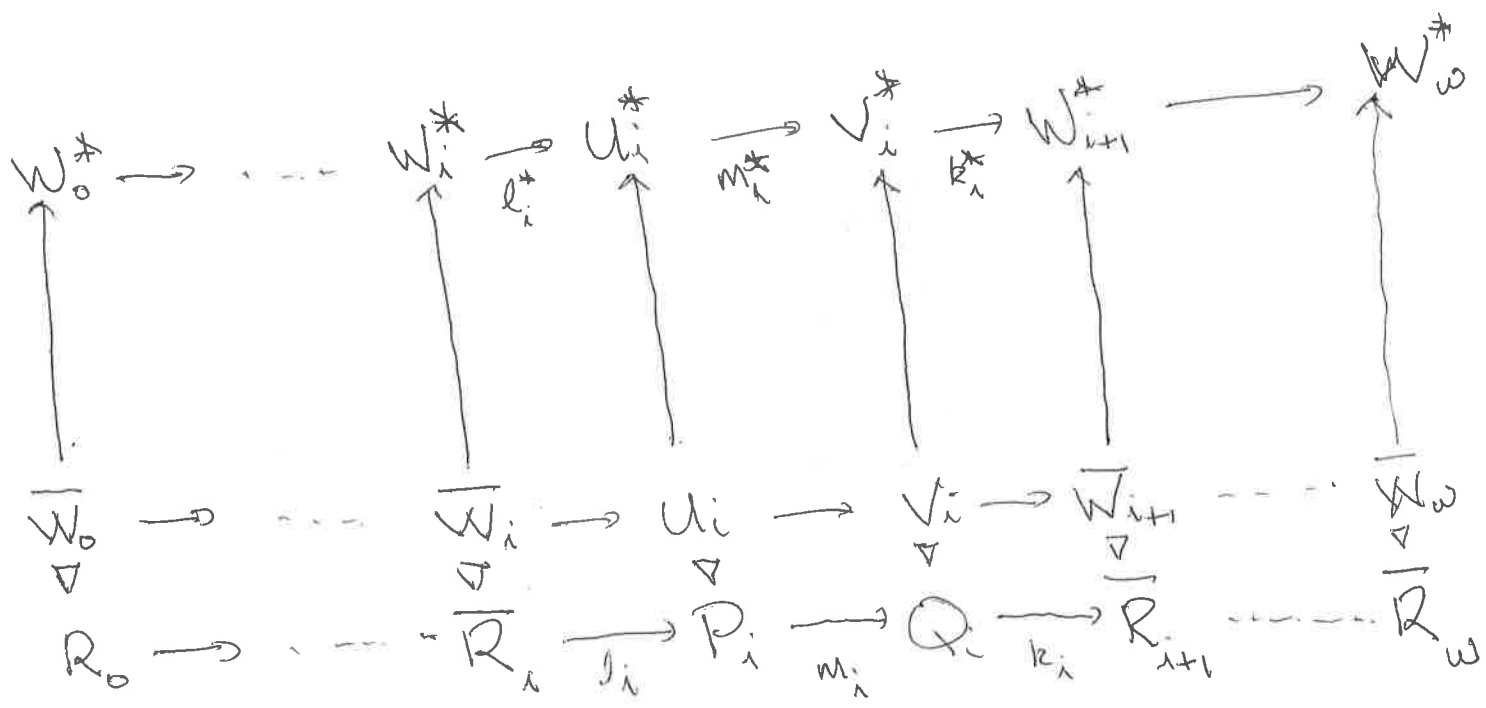
Let

$$U_i = \text{Ult}(\bar{W}_i, E_{P_i}),$$

$$V_i = \text{Ult}(U_i, E_{m_i}),$$

with strategies ~~$\Psi_{i+1}^{m_i l_i}$~~ $\bar{\Psi}_{i+1}^{m_i l_i}$ and $\bar{\Psi}_{i+1}^{m_i}$

respectively. We can lift the iteration $\overline{W}_0 \rightarrow W_0^*$ to an iteration $U_0 \rightarrow U_0^{**}$, and then generically iterate $U_0^{**} \rightarrow U_0^*$ to make \mathbb{R}^{METH} the reals of a derived model of U_0^* . Then we lift $U_0 \rightarrow U_0^*$ to $V_0 \rightarrow V_0^{**}$, and generically iterate $V_0^{**} \rightarrow V_0^*$ to make \mathbb{R}^{METH} the reals of a derived model of V_0^* . ETC. We get



Again, let D_i^* , E_i^* , F_i^* be the derived models of W_i^* , U_i^* , V_i^* . We have

$$D_A^* \subseteq E_i^* \subseteq F_i^* \subseteq D_{i+1}^* .$$

Since W_w^* is from a stack of trees via $\bar{\Psi}_w$, it is wellfounded, this gives us an i such that

$$m_i^* ((\beta_2, \gamma_2, \delta_2)) = (\beta_2, \gamma_2, \delta_2) .$$

Now let $\ddot{\Lambda} \in W_0$ be the canonical name for Σ_0 on its derived model,

and $\dot{\Lambda} = \beta^{-1}(\ddot{\Lambda})$, and let

$\dot{\Lambda}_i$ and $\dot{\Lambda}_{i+1}$ be the images of $\dot{\Lambda}$ in U_i^* and V_i^* (by whatever route).

Λ_i^* = interpretation of Λ_i in E_i^* ,

Ω_i^* = interpretation of Ω_i in F_i^* .

As with claim G, we get

Claim H $\Lambda_i^* = (\Sigma_{i+1}^*)^{k_i \circ m_i}$, end

$$\Omega_i^* = (\Sigma_{i+1}^*)^{k_i}$$

That is, m_i^* moves the name for Σ_{P_i} on ~~the~~ derived model of U_i^* to the name for Σ_{Q_i} on the derived model of V_i^* .

This is a contradiction, as before.



References

[1] G. Sargsyan, Covering with universally Baire operators

[2] G. Sargsyan, A tale of hybrid mice,
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