Plus-one premice

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For G an extender, let $\kappa_G = \mathrm{crit}(G)$ and $\lambda_G = i_G(\kappa_G)$. We consider premice in which all extenders on the sequence have space $\leq \kappa_G^+$, that is, the measures of G do not concentrate beyond κ_G^+. We develop some basic fine structure for iterable premice of this kind. The trickiest problems arise from the possibility that the initial segment condition might fail.

Definition 1. A plus-one potential premouse (ppm) is a J-structure N constructed from a sequence $\vec{E}$ of extenders such that if (M, G) is a level of N, and $G \neq \emptyset$, then either

(1) G is a short extender over M, and (M, G) satisfies the Jensen conditions (so that e.g. $M = \mathrm{Ult}(M,G) | (\lambda_G^+)^{\mathrm{Ult}(M,G)}$), or

(2) G is long, and

(a) $M = \mathrm{Ult}(M, G) | (\lambda_G^+)^{\mathrm{Ult}(M,G)}$ (so λ_G is the largest cardinal of M)

(b) letting $\nu(G)$ be the strict sup of the generators of G, we have

$$\nu(G) = \nu + 1$$

for some ν, moreover

$$\forall \xi < \nu \; (G \restriction \xi \in M).$$

(the "weak initial segment condition", weak ISC.)

<u>Remarks</u>

(a) So for long G, there is always a largest generator ν, and $\lambda_G < \nu < (\lambda_G^+)^{\mathrm{Ult}(M,G)}$. G is therefore equivalent to $G \restriction (\lambda_G \cup \{\nu\})$. It would be more in line with Jensen to index G at $(\lambda_G^{++})^{\mathrm{Ult}(M,G)}$, and simply think of G as the amenable predicate $\dot{\lambda}_G^M \restriction (M | \kappa_G^{++M})$. But since there is a largest generator, ~~[illegible]~~ $\lambda_G^{+\,\mathrm{Ult}(M,G)}$ has cofinality $(\kappa_G^{++})^M$ in a natural way, so we can index G at $\mathrm{lh}(G) = (\lambda_G^+)^{\mathrm{Ult}(M,G)}$, and have (M, G) be amenable.

(b) The full ISC for (M, G) would require in (2)(b) above that $G \restriction \nu \in M$. We shall not demand that ~~[illegible]~~ plus-one premice satisfy the full ISC. It is probably false of the levels of the model we ~~[illegible]~~ eventually construct.

The way that the full ISC may fail in the model we construct is isolated in the following definition.

Definition 2 Let (M,G) be a plus-one ppm. We say that (M,G) is type Z_1 iff G is long, and letting $\nu+1 = \nu(G)$, we have that $\nu = \nu(G\restriction\nu)$ is a limit of generators, $E^M_\nu \neq \emptyset$, and setting $F = E^M_\nu$, we have

(i) F is short, $\lambda_F = \lambda_G$, $\nu = \mathrm{lh}\,F$

(ii) $\kappa_G < \kappa_F$,

(iii) $(\kappa_F^+)^M$ is not the space of an extender on the M-sequence,

(iv) for cofinally many $\gamma < (\kappa_F^+)^M$, $i_F(E^M_\gamma) \subseteq G$, and

(v) $(\nu^+)^{\mathrm{Ult}(M,F)} = (\nu^+)^{\mathrm{Ult}(M,G\restriction\nu)}$, and $\mathrm{Ult}(M,F)|\eta = \mathrm{Ult}(M,G\restriction\nu)|\eta$, for $\eta = (\nu^+)^{\mathrm{Ult}(M,F)}$.

Remark. The parallel with type Z short extenders would be better if in the situation above, we said that $G\restriction\nu$ is of type Z_1. If H is type Z short, then $H \restriction (\nu(H)-1)$ goes on, and then its being on the (FSIT-like) M-sequence means H cannot be on it, because $H \restriction (\nu(H)-1)$ is not in the ultrapower of M by H. Similarly, in the type Z_1 case above, F being on the M-sequence prevents $G \restriction \nu$ being put on afterward, because $\mathrm{Ult}(M, G \restriction \nu) \models \nu$ is a cardinal, while F collapses ν.

If (M, G) is of type Z_1, as witnessed by $F = E^M_\nu$, then we can define a long extender $\overline{G}$ over M by: for $\eta < (\kappa_F^+)^M$

$$E^M_\eta \subseteq \overline{G} \text{ iff } i^M_F(E^M_\eta) \subseteq G.$$

So $G \restriction \nu$ is the "stretch by F" of $\overline{G}$.

It is easy to see that $\kappa_{\bar{G}} = \kappa_G$, $\lambda_{\bar{G}} = \kappa_F$, $\nu(\bar{G}) = (\kappa_F^+)^M$, and that $Ult(M, \bar{G})$ agrees with M up to $(\kappa_F^{++})^M = (\nu(\bar{G})^+)^{Ult(M,\bar{G})}$. Thus $\bar{G}$ collapses $(\kappa_F^{++})^M$, and $\bar{G} \notin M$. It follows that $G \restriction \nu \notin M$.

Notice that $\bar{G}$ is as well-backgrounded as G is, because $i_F \restriction \lambda_{\bar{G}} =$ identity. $\bar{G}$ was not put on the M-sequence because it has no largest long generator, and $(M|(\kappa_F^{++})^M, \bar{G})$ may well not project to $\lambda_{\bar{G}}$ at all. Instead, we waited until we could add G, and then F and G put both $G \restriction \nu$ and $\bar{G}$ into the model at the same time G is put in, and the full ISC fails for G.

We continue now towards the definition of "plus-one premouse".

<u>Definition 3a</u> Let (M, G) be a plus-one ppm. Let $\eta < \lambda_G$, and

$$H = G \restriction \eta \quad , \text{ if } G \text{ is short},$$

$$H = G \restriction (\eta \cup \{\nu\}), \text{ if } G \text{ is long with } \nu + 1 = \nu(G).$$

We say that H is <u>whole</u> iff $i_H(\kappa_H) = \eta$.

We say that (M, G) has the <u>Jensen ISC</u> iff whenever H as above is a whole initial segment of G, then $H \in M$.

<u>Remarks</u>

(1) The Jensen ISC is needed to prove comparison. For that, a weaker version suffices though, in that we can require it only for whole H such that H coheres with M, i.e. $\mathrm{Ult}(M, H)$ agrees with M up to $(\lambda_H^+)^{\mathrm{Ult}(M,H)}$. This will be true for arbitrary whole H by condensation, but we don't need to prove condensation in order to see that the Jensen

ISC holds for whole H that cohere with M, if (M,G) has been produced by our background construction.

(2) The Jensen ISC is preserved by Σ_0 ultrapowers of (M,G), if the critical point is $< \lambda_G$. Ultrapowers with critical point λ_G destroy it, as then G itself is a missing whole initial segment of $i(G)$.

We need some restrictions on where projecta can lie.

<u>Definition 3b</u> Let M be a plus-one ppm; then we say <u>M has projectum-free spaces</u> just in case whenever G is a long extender on the M-sequence

(a) if there are η, k such that $\rho_k(M|\eta) \le (\kappa_G^+)^{M|lh\,G}$, then for the $\le_{lex}$ least such $\langle \eta, k\rangle$, $\rho_k(M|\eta) \le \kappa_G$, and

(b) if G is total on M, and M is active with short last extender H such that $\kappa_H = \kappa_G$, then $\rho_1(M) > (\kappa_G^+)^M$.

It is an important that the plus-one ppms we construct

~~Always~~ have projectum-free spaces. This will follow from something called "amenable closure", and we discuss it further on pp. 25 ff. below. (The ideas and results here are due to Woodin [4].) Since we have not yet developed the fine structure needed to make precise sense of ~~Def. 3b~~ $\rho_k(M)$, definition 3b should be regarded as a place-holder. It will be explained fully in [5].

<u>Definition 3c</u> Let (M, G) be a plus-one ppm, with G long, and $\nu + 1 = \nu(G)$. We say that (M, G) (or just G) is <u>Dodd-solid</u> iff $G \restriction \nu \in M$.

<u>Definition 3</u> Let N be a plus-one ppm; then N is a <u>plus-one premouse</u> iff

(1) every proper initial segment of N is fully sound, and has projectum-free spaces,

(2) every initial segment of N satisfies the Jensen ISC,

(3) for any long E on the N-sequence that is total over N,

(a) $\rho_1(N) \neq (\kappa_E^+)^N$, and

(b) if N is active with short last extender H such that $\kappa_H = \kappa_E$, then $\rho_1(N) > (\kappa_E^+)^N$,

(4) if (M, G) is an initial segment of N with G long, then either (M, G) is Dodd solid, or (M, G) is of type Z_1.

The fine structural notions used here will be defined carefully in [5].

We do not demand (3)(a) for the higher projecta, because we want to use the notion of plus-one premouse in showing these projecta behave well.

The following lemma explains why we care about clause ($\cancel{4}$5) of definition 3.

<u>Lemma 4</u> Let (M, G) be a plus-one premouse, and
$$(N, H) = \mathrm{Ult}_0((M, G), E),$$
where E is such that $\mathrm{space}(E) = \mathrm{space}(E')$, for some E' on the M-sequence. Let
$$i : (M, G) \to (N, H)$$
be the canonical embedding. Then

(1) (N, H) is a plus-one premouse, and

(2) if G is long, then $i(\nu(G)) = \nu(H)$.

<u>Proof.</u> We consider only the case that G is long. Let $\partial(G) = \nu + 1$.

We claim that $i(\nu)$ is the largest generator of H. Being a generator is Π_1, so as i is Σ_1 elem., $i(\nu)$ is a generator of H. Also, if $\nu < \xi < lh\, G$, we can write, for $W \in M$ a wellorder of λ_G of type ξ

$$W = i_G(f)(a, \nu) \quad , \quad (a \subseteq \lambda_G),$$

so

$$i(W) = i_H(i(f))(i(a), i(\nu)) \quad , \quad (i(a) \subseteq \lambda_H).$$

Thus $i(\xi)$ is not an H-generator, and in fact there are no H-generators between $i(\nu)$ and $i(\xi)$. Since $\mathrm{ran}(i)$ is cofinal in $lh(H) = o(N)$, then $i(\nu)$ is the largest generator of H.

Without our additional premouse condition (4) in definition 3, it is not clear that (N, H) must satisfy the weak ISC. But using it, we can argue

Claim (N, H) satisfies the weak ISC.

Proof Suppose first that $i(\nu) = \sup i''\nu$. Then for $\xi < \nu$, we have $G \restriction \nu \in M$, and as i is Σ_1 elem.,

$$i(G \restriction \xi) \subseteq H$$

for all $\xi < \nu$. The $i(G \restriction \xi)$ are cofinal in $H \restriction i(\nu)$, so H satisfies the weak ISC.

Suppose then that i is discontinuous at ν, and let $\gamma = \operatorname{cof}(\nu)^M$. Then γ is the space of a measure in E, hence of a measure in E', for some E' on the M-sequence γ.

But (M, G) is a plus-one premouse, so in this case, we must have $G \restriction \nu \in M$.

But then $i(G \restriction \nu) = H \restriction i(\nu)$ witnesses the full ISC for H.

$\boxtimes$

To finish our sketch of the proof of lemma 4 when G is long, notice that we may assume $G \upharpoonright \nu \notin M$, as otherwise $i(G \upharpoonright \nu) = H \upharpoonright i(\nu) \in N$. But then (M, G) is type Z_1, as witnessed by $F = E_\nu^M$. Moreover, $i(\nu) = \sup i''\nu$ by the type Z_1 conditions, and thus $i(\kappa_F^{+M}) = \sup i''(\kappa_F^+)^M$ because $\mathrm{cof}(\nu)^M = (\kappa_F^+)^M$. For cofinally many $\eta < \kappa_F^+$ we have

$$i_F^M(E_\eta^M) \subseteq G$$

and hence

$$i_{i(F)}^N(E_{i(\eta)}^N) \subseteq H.$$

Since $\mathrm{ran}(i)$ is cofinal in $i(\nu)$ and $i(\kappa_F^{+M})$, this shows i_F verifies type Z_1-ness of H.

We leave it to the reader to verify the Jensen ISC for (N, H). It follows easily from the strong inaccessibility of λ_G in M. $\boxtimes$

In order to show that the ultrapowers taken in an iteration tree preserve premouse-hood, we must show the hypothesis of lemma 4 applies.

<u>Definition 4a</u> Let M be a plus-one ppm, and E an extender with $\mathrm{dom}(E) = M|\alpha$, for some cardinal α of M. We say E is <u>close to M</u> iff

(1) E is short, and for all finite $a \subseteq \lambda_E$, E_a is Σ_1^M, and $E_a \cap M|\xi \in M$, for all $\xi < (\kappa_E^+)^M$, or

(2) E is long, and for $\nu + 1 = \nu(E)$,

(a) if $X = \kappa_E^{+M} \cup a$, where $a \subseteq \lambda_E \cup \{\nu\}$ is finite, then $E \restriction X$ is Σ_1^M, and $E \restriction X \cap M|\xi \in M$, for all $\xi < (\kappa_E^{++})^M$,

(b) $(\kappa_E^+)^M$ is the space of a total extender from the M-sequence.

<u>Lemma 4b.</u> Let $\mathcal{T}$ be a k-maximal iteration tree on a plus-one premouse, and $E = E_\alpha^{\mathcal{T}}$ and $M = M_{\alpha+1}^{*\mathcal{T}}$, so that E is applied to M in $\mathcal{T}$; then E is close to M.

The proof goes by an induction that is similar the proof of 6.1.5 of [3]. We shall omit it for now. Lemmas 4 and 4b imply that all the models of a k-maximal iteration tree on a plus-one premouse are themselves plus-one premice.

<u>Remark</u> "k-maximal" means with respect to short extender rules. So $T\text{-pred}(\alpha+1) = \beta$, where β is least such that $\kappa_{E_\alpha} < \lambda_{E_\beta}$.

$$M_{\alpha+1} = \mathrm{Ult}_n(M^*_{\alpha+1}, E_\alpha),$$

where $M^*_{\alpha+1} = M_\beta \upharpoonright \tau$, for τ least such that $\lambda_{E_\beta} \leq \tau$ and either $\tau = o(M_\beta)$ or $\rho_\omega(M_\beta \upharpoonright \tau) \leq \text{space}(E_\alpha)$, and n is least such that $\rho_{n+1}(M^*_{\alpha+1}) \leq \text{space}(E_\alpha)$, or $n = k$ if there was no dropping in model or degree in $[0, \alpha+1]_T$.

It will be important to know that when $\rho_{n+1}(M^*_{\alpha+1}) \leq \text{space}(E_\alpha)$, then in fact $\rho_{n+1}(M^*_{\alpha+1}) \leq \text{crit}(E_\alpha)$. This is needed in order to show that the canonical $i: M^*_{\alpha+1} \to M_{\alpha+1}$ preserves the standard parameter $p_n(M^*_{\alpha+1})$. We get it from the fact that $M^*_{\alpha+1}$ has projectum-free spaces, together with clause 2(b) of closeness.

<u>Remark</u> Clause (b) in the definition of "projectum-free spaces" (Def. 3b), comes up in showing that if $\mathcal{T}$ is a k-maximal iteration tree on a plus-one premouse, then all $\mathcal{M}_\alpha^{\mathcal{T}}$ are plus-one premice. It comes up in the following way. Suppose $E = E_\alpha^{\mathcal{T}}$ is long, and our rules require

$$M_{\alpha+1} = \mathrm{Ult}_n(M^*_{\alpha+1}, E),$$

where $M^*_{\alpha+1}$ happens to be active, with short last extender H. The worry is that ~~we~~ $M_{\alpha+1}$ might be only a "proto-mouse". If $n \geq 1$ this is not a problem, because $i: M^*_{\alpha+1} \longrightarrow M_{\alpha+1}$ is Σ_2 elementary. If $n = 0$ and $\cancel{(\kappa_H^+)^{M^*_{\alpha+1}} \neq (\kappa_E^+)^{M_\alpha}}$ $\kappa_H = \kappa_E$, so that $(\kappa_H^+)^{M^*_{\alpha+1}} = (\kappa_E^+)^{M_\alpha}$, then we have a problem, because then i is discontinuous at $(\kappa_H^+)^{M^*_{\alpha+1}}$, and yet continuous at $o(M^*_{\alpha+1})$, leading to $M_{\alpha+1}$ being only a protomouse.

But notice E is close to $M^*_{\alpha+1}$, so $M^*_{\alpha+1}$ has a total long extender with space $= M^*_{\alpha+1} \restriction (\kappa_E^+)^{M^*_{\alpha+1}}$ on its sequence. Moreover, $M^*_{\alpha+1}$ has projectum-free spaces, so by (3b)(b), $\rho_1(M^*_{\alpha+1}) > (\kappa_E^+)^{M^*_{\alpha+1}}$. So $Ult_1(M^*_{\alpha+1}, E)$ made sense, and this is the ultrapower we should have taken.

<u>Remark</u> The last remark shows we need to change the definition of "o-maximal" tree slightly, so that it allows us to always take $Ult_1(M^*_{\alpha+1}, E)$ in the situation described here.

So moving from ppm's to premice gives us a condition that is preserved by Σ_0 ultrapowers. We illustrate the usefulness of this by showing (see page 11a)

<u>Theorem 5</u> (Comparison lemma) Let M and N be iterable plus-one premice; then there are iterates R of M and S of N such that $R \trianglelefteq S$ or $S \trianglelefteq R$.

<u>Remark</u> "Iterable" here means via "short extender rules", that is, we ~~[illegible]~~ go back to the earliest model ~~we~~ so that our critical point lies within the <u>short</u> <u>generators</u> of its exit extender. Long generators can be moved along branches of our trees.

This limited iterability can be guaranteed for M constructed using

<u>Remark</u> Because we have not yet introduced the fine structure related to standard parameters necessary to define "premouse", we cannot now give a full proof of Theorem 5. We must ignore the issues related to dropping and preservation of cores. We shall return to them later.

Our goal in the partial proofs of theorems 5 and 6 we present here is to explain the role of the type Z_1 condition, and how our construction leads to ppm which satisfy it.

short extender backgrounds only, as in [1]. That it suffices for comparison of plus-one premice was essentially first proved in [2], and then written up again in [1].

Proof of theorem 5 As a representative special case, let M and N be countable, and let Σ and Γ be ω_1+1 iteration strategies for them. Let $\mathcal{T}$ on M and $\mathcal{U}$ on N be plays of Σ and Γ, where at successor steps player I has chosen least disagreements $E_\alpha^{\mathcal{T}}$ in $M_\alpha^{\mathcal{T}}$ and $E_\alpha^{\mathcal{U}}$ in $M_\alpha^{\mathcal{U}}$, and applied $E_\alpha^{\mathcal{T}}$ to $M_\xi^{\mathcal{T}}$, where ξ is least such that $\mathrm{crit}(E_\alpha^{\mathcal{T}}) < \lambda_{E_\xi^{\mathcal{T}}}$, and similarly for $E_\alpha^{\mathcal{U}}$. Write $M_\alpha = M_\alpha^{\mathcal{T}}$, and $N_\alpha = M_\alpha^{\mathcal{U}}$. Suppose toward contradiction that ~~we know~~ the process does not terminate, so that M_{ω_1} and N_{ω_1} exist. Let

$$\pi : P \longrightarrow V_\theta$$

with P countable transitive, θ large, and everything relevant in $\mathrm{ran}(\pi)$. Let $\pi(\bar{\mathcal{T}}) = \mathcal{T}$, etc. Let $\alpha = \omega_1^P = \mathrm{crit}(\pi)$, with $\pi(\alpha) = \omega_1$.

We have $M_\alpha^{\bar{\mathcal{T}}} = M_\alpha^{\mathcal{T}}$ and $N_\alpha^{\bar{\mathcal{U}}} = N_\alpha^{\mathcal{U}}$.

Thus $M_\alpha, N_\alpha \in P$. Moreover

$$\pi \restriction M_\alpha = i^{\mathcal{T}}_{\alpha\omega_1},$$

and

$$\pi \restriction N_\alpha = i^{\mathcal{U}}_{\alpha,\omega_1}.$$

Let $\gamma+1$ be least in $(\alpha, \omega_1]_T$ and $\eta+1$ least in $(\alpha, \omega_1]_U$, and

$$G = E^{\mathcal{T}}_\gamma,$$

$$H = E^{\mathcal{U}}_\eta.$$

Let

$$P = (M_\gamma \| hG, G),$$
$$Q = (N_\eta \| hH, H).$$

We can lift up P and Q by the short-extender fragments of the branch tail extenders, that is

$$P^* = Ult_0(P, E_{i^{\mathcal{T}}_{\gamma+1,\omega_1}} \restriction \omega_1)$$

and

$$Q^* = Ult_0(Q, E_{i^{\mathcal{U}}_{\eta+1,\omega_1}} \restriction \omega_1).$$

Let $i_0 : P \to P^*$ and $j_0 : Q \to Q^*$ be the canonical embeddings. Thus

$i_0 = i^{\mathcal{T}}_{\gamma+1,\omega_1} \restriction (M_\gamma \| hG)$ (note $M_\gamma \| hG \trianglelefteq M_{\gamma+1}$),

and $j_0 = i^{\mathcal{T}}_{\eta+1,\omega_1} \restriction (N_\eta \| hH)$. ~~Ath~~ Let G^* and H^* be the last extender predicates of P^* and Q^*, i.e.

$$G^* = i_0(G),$$
$$H^* = j_0(H),$$

where we are really applying i_0 and j_0 to fragments of G and H.

Claim G^* and H^* are initial segments of the extender E_π from π. (15)

Proof (See [1] and [2].) Clearly, G and G^* measure the same sets, i.e. $\mathrm{dom}(G) = \mathrm{dom}(G^*)$. For $X \in \mathrm{dom}(G)$

$$\pi(X) = i^{\mathcal{T}}_{\alpha,\omega_1}(X) = i^{\mathcal{T}}_{\gamma+1,\omega_1}(i_G(X)).$$

We'd like to write $i_{\gamma+1,\omega_1}(i_G(X)) =$
$i_{\gamma+1,\omega_1}(i_G)(i_{\gamma+1,\omega_1}(X)) = i_{\gamma+1,\omega_1}(i_G)(X) = i_{G^*}(X)$,

but it is only i_0 that can move i_G as fragment-wise as an amenable predicate of $(M_\gamma \| \mathrm{lh}\, G, G)$, not the full $i_{\gamma+1,\omega_1}$. (If $i_{\gamma+1,\omega_1}$ is a short extender, this is not a problem.) So let

$$\sigma : \mathrm{Ult}_0(M_{\gamma+1}, E_{i^{\mathcal{T}}_{\gamma+1,\omega_1}} \restriction \omega_1) \longrightarrow M_{\omega_1} = \mathrm{Ult}(M_{\gamma+1}, E_{i^{\mathcal{T}}_{\gamma+1,\omega_1}})$$

be the canonical emb. $\sigma \restriction \omega_1 + 1 = \text{identity}$, and so $\mathrm{crit}(\sigma)$ is at least $(\omega_1^{\nu})^+$ of the smaller ultrapower. But $P = (M_\gamma \| \mathrm{lh}\, G, G) = (M_{\gamma+1} \| \mathrm{lh}\, G, G)$, and $lh(G) = \lambda_G^+$ in $M_{\gamma+1}$, so

$$P^* = (M_{\omega_1} | lh G^*, G^*),$$

where $lh(G^*) \le crit(\sigma)$. Taking $x \in dom(G) = dom(G^*)$, we ~~have~~ may assume $x \subseteq (\kappa_G)^{+M_\alpha}$, and we then get

$$\begin{aligned}
\pi(x) &= i^{\mathcal{T}}_{\gamma+1,\omega_1}(i_G(x)) \\
&= \sigma(i_0(i_G(x))) \\
&= \sigma(i_0(i_G)(i_0(x))) \\
&= \sigma(i_{G^*}(x)).
\end{aligned}$$

Since $crit(\sigma) \ge lh(G^*)$, this gives $G^* = E_\pi \cap (M_\alpha^{\mathcal{T}} \times [lh G^*]^{<\omega})$.

Claim $\boxtimes$

<u>Claim</u> At least one of G and H is long.

<u>Proof</u> Assume not. Then $G^* = H^* = E_\pi \cap (M_{\omega_1} \times [\omega_1]^{<\omega})$.

But $G = G^* \restriction \lambda_G$, and $G \notin M_{\omega_1}$, so we see that the Jensen ISC fails for G^*. It held for $(M_\gamma \| lh G, G)$, by our "short-extender rules" for iteration.

It follows that

$$
\begin{aligned}
\lambda_G &= \text{least } \eta \text{ s.t. } G^* \restriction \eta \text{ is whole and } G^* \restriction \eta \notin M_{\omega_1} \\
&= \text{least } \eta \text{ s.t. } H^* \restriction \eta \text{ is whole and } H^* \restriction \eta \notin N_{\omega_1} \\
&= \lambda_H .
\end{aligned}
$$

So $H = H^* \restriction \lambda_H = G^* \restriction \lambda_G = G$, contrary to G being part of a disagreement.

$\boxtimes$

Claim Both G and H are long.

Proof Suppose G is short and H is long. Then by the weak ISC, $H \restriction \lambda_H$ is on

18

the sequence of $N_\eta \| \mathrm{lh} H$, and hence $j_0(H \restriction_\gamma \lambda_H) = H^* \restriction \omega_1$ is on the sequence of $N_{\omega_1} \restriction \mathrm{lh} H^*$. But $H^* \restriction \omega_1 = G^*$, and so by $N_{\omega_1} \| \mathrm{lh} H^* \models$ Jensen ISC, $G^* \restriction \lambda_G = G \in N_{\omega_1}$. Since $\mathrm{lh}\, G$ is a cardinal of N_{ω_1}, this is a contradiction.

Claim $\boxtimes$

<u>Claim</u> It is not the case that both G and H are long.

<u>Proof</u> Otherwise, let $\nu+1 = \nu(H)$, and $\xi+1 = \nu(G)$. We have

$$i_0 : (M_\gamma \| \mathrm{lh}\, G, G) \longrightarrow (M_{\omega_1} \| \mathrm{lh}\, G^*, G^*),$$

and

$$j_0 : (N_\eta \| \mathrm{lh}\, H, H) \longrightarrow (N_{\omega_1} \| \mathrm{lh}\, H^*, H^*),$$

moreover G^* and H^* are compatible.

<u>Subclaim</u> $i_0(\xi)$ is the largest generator of G^*, and $\forall \mu < i_0(\xi)$, $G^* \restriction \mu \in M_{\omega_1} | \mathrm{lh}\, G^*$.

<u>Proof</u> ~~then~~ If $G \restriction \xi \in M_\gamma | \mathrm{lh}\, G$, then $i_0(G \restriction \xi) = G^* \restriction i_0(\xi) \in M_{\omega_1} | \mathrm{lh}\, G^*$, so of course $\forall \mu < i_0(\xi)$, $G^* \restriction \mu \in M_{\omega_1} | \mathrm{lh}\, G^*$. But if $G \restriction \xi \notin M_\gamma | \mathrm{lh}\, G$, then since $(M_\gamma | \mathrm{lh}\, G, G)$ is a premouse

$$M_\gamma | \mathrm{lh}\, G \models \mathrm{cof}(\xi) = \tau^+, \text{ for some } \tau < \lambda_G .$$

But i_0 comes from a short extender with critical point λ_G, so we get $i_0(\xi) = \sup i_0{}''\xi$. This again shows $\forall \mu < \xi \; (G^* \restriction \mu \in M_{\omega_1} | \mathrm{lh}\, G^*)$.

We have already computed that $i_0(\xi)$ is the largest generator of G^*.

⊠

<u>Subclaim</u> $j_0(\nu)$ is the largest generator of H^*, and $\forall \mu < j_0(\nu)$, $H^* \restriction \mu \in N_{\omega_1} | \mathrm{lh}\, H^*$.

<u>Proof</u> The same.

⊠

<u>Subclaim</u> $i_0(\xi) = j_0(\eta)$.

<u>Proof</u> Suppose e.g. $i_0(\xi) < j_0(\eta)$. By the last subclaim, $H^* \restriction \omega_1 \cup \{i_0(\xi)\} \in N_{\omega_1} \| lh\, H^*$. That is, $G^* \in N_{\omega_1}$. But $G = G \restriction (\lambda_G \cup \{\xi\}) = G^* \restriction (\lambda_G \cup \{i_0(\xi)\}) \in N_{\omega_1}$ then, contrary to G collapsing λ_G^+ of N_{ω_1}.

$\boxtimes$

<u>Subclaim</u> $G = H$.

<u>Proof</u> We have that

λ_G = least η such that $G^* \restriction (\eta \cup \{i_0(\xi)\})$ is a whole extender that is not in M_{ω_1}

= least η such that $H^* \restriction (\eta \cup \{j_0(\eta)\})$ is a whole extender that is not in N_{ω_1}

$= \lambda_H$.

But then $G = G^* \restriction (\lambda_G \cup \{i_0(\xi)\}) = H^* \restriction (\lambda_H \cup \{j_0(\eta)\}) = H$, as desired.

$\boxtimes$

Clearly $G \neq H$, because they were used in disagreements. So we have proved our last claim.

⊠

The three claims add up to a contradiction, thereby proving Theorem 5.

⊠

The $L[\vec{E}]$ construction of [1] can be modified so as to allow plus-one ppm's as levels. The background certificates remain strictly short extenders (that is, strength$(E) < \lambda_E$.) Assuming SBH for V, we get that the levels of our construction are such that every countable elementary submodel is ω_1+1-iterable for short-extender-rules type trees. See [1] for more about how this goes.

Of course, we have to show by induction on the stages of the construction that each ppm it produces has the fine-structural properties that let us define cores, and go on. Here we focus on our closure-under-initial-segment requirement, clause (4) of definition 3.

Let us call ~~a~~ a construction of the sort we have just indicated a <u>plus-one construction</u>.

<u>Theorem 6</u> Assume SBH. Let ~~[illegible]~~ M be a stage in a plus-one construction; then M is an iterable plus-one premouse.

Our main worry in the proof will be clause (4) of definition 3. Our proof that it holds will resemble the proof of theorem 10.1 of [3], which in turn traces back to earlier work by Mitchell and Jensen in the same vein.

The main part of the proof is:

<u>Lemma 7.</u> Let (M, G) be an iterable plus-one ppm satisfying the Jensen ISC, all of whose proper initial segments are sound premice with projectum-free spaces. Suppose G is long, and $\nu+1 = \nu(G)$, where ν is a limit of generators of G. Suppose $\mathrm{cof}(\nu)^M$ is not the space of a total extender on the M-sequence.

Then (M, G) is a plus-one premouse; that is, (M, G) is either Dodd-solid, or of type Z_1.

<u>Proof of theorem 6 modulo lemma 7.</u>

Suppose that (M, G) is a stage of our plus-one construction, where G is long, and by induction that every proper initial segment of (M, G) is a plus-one premouse. (The case G is short, or $G = \emptyset$, is not problematic.) Let

$$\nu + 1 = \nu(G)$$

and suppose ν is a limit of generators of G. (If there is a largest long generator $\xi < \nu$, then (the completion of) $G \restriction (\xi+1)$ is on the M-sequence by a bicephalus argument. Thus $G \restriction \nu \in M$. If ν is the only long generator of G, then $G \restriction \lambda_G$ is on the M-sequence by a bicephalus argument.)

Let

$$\gamma = \mathrm{cof}(\nu)^M.$$

Lemma 7 finishes things unless $\kappa_H \leq \gamma \leq \mathrm{space}(H)$ for some total H on the M-sequence. So suppose there is such an H, and let H be the first one. So $H \in M$. (Note G is not the first extender with its space.

The rest of the proof uses the amenable closure of M at γ, so we digress to explain amenable closure.

Amenable closure

It was Woodin (cf. [4]) who realized the importance of this phenomenon in long-extender fine structure. The results below are easy adaptations of his arguments.

Definition 8 A ppm M is amenably closed at α iff whenever $A \subseteq M|\alpha$, and for all $\xi < \alpha$, $A \cap M|\xi \in M$, then $A \in M$.

Lemma 9. (Woodin) Let (M, G) be a stage in a plus-one construction that has not yet been cored down. Let γ be a cardinal of M such that $\kappa_G \leq \gamma \leq \text{space}(G)$; then M is amenably closed at γ.

<u>Remark</u> So $\lambda = \kappa_G$ if G is short, and $\lambda \in \{\kappa_G, \kappa_G^{+M}\}$ if G is long. The lemma will hold as stated for "plus-n" constructions,

<u>Proof</u> Let $j : V \to N$ be the background extender for G, as in ΣIJ. So we have a factor embedding $\sigma : \mathrm{Ult}(M,G) \to j(M)$ with

$$\begin{tikzcd}
M \arrow[rr, "j"] \arrow[dr, "i_G^M"'] & & j(M) \\
& \mathrm{Ult}(M,G) \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

commuting, and $\mathrm{crit}(\sigma) = \lambda_G$. Let $A \subseteq \gamma$ be amenable to M. We have that i_G is discontinuous at γ, so j is discontinuous at γ, so the elementarity of j on V gives

$$j(A) \cap \sup j''\gamma \in j(M).$$

Suppose now that $\gamma = \kappa_G$; this means

that $A = j(A) \cap \kappa_G \in j(M)$.
But M agrees with $Ult(M, G)$ to λ_G, and hence with $j(M)$ to λ_G, and λ_G is a cardinal of $j(M)$. By acceptability, $A \in M$.

Suppose then that $\gamma = (\kappa_G^+)^M$, so that G is long. Then

$$H =_{df} G \restriction \lambda_G$$

is on the sequence of $Ult(M, G)$, so $\sigma(H)$ is on the $j(M)$-sequence, say

$$\sigma(H) = E_\nu^{j(M)}.$$

But the embedding of $\sigma(H)$ agrees with j up to κ_G^{+M}, i.e.

$$i_{\sigma(H)}^{j(M)} \restriction \kappa_G^{+M} = j \restriction \kappa_G^{+M}.$$

To see this, let $B \subseteq \kappa_G$ be in M, then

$$i_G^M(B) = i_H^M(B),$$

so

$$j(B) = \sigma(i_G^M(B))$$
$$= \sigma(i_H^M(B)) = i_{\sigma(H)}^{j(M)}(\sigma(B))$$
$$= i_{\sigma(H)}^{j(M)}(B).$$

But then, working in $j(M)$ we can recover A from $j(A) \cap \nu$ and $j \restriction (\kappa^+)^M$. So $A \in j(M)$, and then $A \in M$ as before.

$\boxtimes$

<u>Corollary 10</u> Let M be a stage in a plus-one construction, and suppose M has not been cored down yet. Let G be an extender on the M-sequence, and suppose that $P(\kappa_G)^M \subseteq M \| \text{lh}\, G$; then

(a) G is total on M

(b) for all M-cardinals γ with $\kappa_G \leq \gamma \leq \text{space}(G)$,

(i) M is amenably closed at γ

(ii) for all n, $\rho_n(M) \neq \gamma$

(c) if G is long and M is active with short last extender H such $\kappa_H = \kappa_G$, then $\rho_1(M) > (\kappa_H^+)^M$.

Proof Let (N, L) be the "ancestor" of $(M \| lh\, G, G)$ in our construction. Since $\rho(\kappa_G)^M \subseteq M \| lh\, G$, we never projected $\le \kappa_L = \kappa_G$ between stage (N, L) and stage M. Moreover, we never projected to any γ such that $\kappa_G \le \gamma \le \text{space}(L) = \text{space}(G)$, since a new subset of the projectum would yield a failure of amenable closure. So G and L have the same domain (measure the same sets). (But $G \ne L$ is possible, as we may core down above $\text{dom}(G)$.) This yields (a) and (b).

For (c), let for $\alpha < (\kappa_H^+)^M$

$$\alpha \in A \text{ iff } (M, H) \models \varphi[\alpha, p]$$

where φ is Σ_1, and we are thinking of H as $i_H : M | (\kappa_H^+)^M \longrightarrow M$, which is amenable. Let $H_\xi = i_H \restriction (M|\xi)$ and $\gamma_\xi = \sup i_H``\xi$, for $\xi < \kappa_H^+$. Put

$$(\alpha, \xi) \in B \text{ iff } (M|\gamma_\xi, H_\xi) \models \varphi[\alpha, \beta].$$

Then $B \subseteq M \restriction \text{space}(G)$, and B is amenable to M, so $B \in M$. Thus $A \in M$, as desired.

$\boxtimes$

<u>Corollary 11</u> Let M be a stage in a plus-one construction; then M has projectum-free spaces.

Corollary 11 follows easily from corollary 10, so we omit proof. We note that ~~in~~ one can have a stage M of a plus-one construction, a long G on the M-sequence, and a least $\langle \eta, k \rangle$ such that $\rho_k(M|\eta) \leq (\kappa_G^+)^M$, with $\rho_k(M|\eta) = \kappa_G$. In this case, κ_G was the critical point of the uncoring map at some stage.

This ends our digression on amenable closure.

Proof of theorem 6 modulo lemma 7:

Recall that H was the first total extender on the M-sequence with space $= \gamma = \mathrm{cof}(\nu)^M$. Let

$$i : (M, G) \longrightarrow Ult_0((M,G), H)$$

be the canonical embedding. Let

$$\eta = \sup i''\nu,$$

so that $\eta < i(\nu)$.

Claim $i(G) \upharpoonright \eta \notin Ult_0((M,G), H)$

Proof Let $i(G)\upharpoonright\eta = [b, f]^M_H$. Then for $a \in [\nu]^{<\omega}$, $X \subseteq [\kappa_G^+]^{|a|}$ in M,

$(a, X) \in G$ iff for H_b-a.e. u,
$(a, X) \in f(u)$.

Since $H \in M$, $G \upharpoonright \nu \in M$, contradiction.

$\boxtimes$

<u>Claim</u> η is a generator of $i(G)$.

<u>Proof</u> Since ν is a limit of generators of G, η is a limit of generators of $i(G)$. Let

$$(N, i(G)) = Ult_0((M,G), H).$$

Then $\eta = \lambda^{+}_{i(G)}$ of $Ult(N, i(G) \upharpoonright \eta)$.

So η is the critical point of the factor map from $Ult(N, i(G)\upharpoonright\eta)$ to $Ult(N, i(G))$.

(Note here that $i(\nu) < \lambda^{+}_{i(G)}$ of $Ult(N, i(G))$.)

$\boxtimes$

Now let K be the trivial completion of $i(G)\upharpoonright \lambda_{i(G)} \cup \{\eta\}$, so that $(N \| lh K, K)$ is a plus-one ppm, and iterable, and $\nu(K) = \eta + 1$ with η a limit of generators of K, and

$\kappa \restriction \eta \in N \| \mathrm{lh} K$. Note also that

$$\mathrm{cof}(\eta)^N = \gamma .$$

(This follows easily from $i "\gamma \in N$.)

Thus $\mathrm{cof}(\eta)^N$ is not the space of a total extender on the N-sequence. Applying lemma 7 to (N, K), we get

$$F = E_\eta^N$$

is a short extender, $\kappa_{i(G)} \le \kappa_F < \lambda_{i(G)} = \lambda_F$, and for cofinally many $\xi < \kappa_F^{+N}$,

$i_F(E_\xi^N) \subseteq i(G)$. Thus

$$\gamma = (\kappa_F^+)^N .$$

But $\gamma = \mathrm{space}(H)$, so $\kappa_H = \kappa_F = \mathrm{crit}(i)$.

So $\kappa_{i(G)} < \mathrm{crit}(i)$, so $\kappa_G = \kappa_{i(G)}$.

<u>Claim</u> N is amenably closed at γ.

<u>Proof</u> Let $A \subseteq \gamma$ be amenable to N. Since $M|\gamma = N|\gamma$, A is amenable to M. Since H exists and is total on M, corollary 10 tells us that $A \in M$. Thus $i(A) \in N$. However, $i \restriction \gamma \in N$ because H was long. Thus $A \in N$, as desired. $\boxtimes$

But now let

$$A = \{\xi < \gamma \mid i_F(E_\xi^N) \subseteq i(G)\}.$$

Since for all $\rho < \eta$, $i(G) \restriction \rho \in N$, A is amenable to N. Thus $A \in N$.

But that gives $i(G) \restriction \eta \in N$, a contradiction. This proves theorem 6 modulo lemma 7. $\boxtimes$

Proof of lemma 7 Let

$$M_0 = (M, G).$$

We may assume that M_0 is countable, and has an ω_1+1 - iteration strategy with the weak Dodd-Jensen property. Let Σ_0 be ~~the~~ such a strategy. Let

$$M_1 = Ult_0(M_0, G\restriction \nu)$$

and

$$N_0 = (M, G).$$

We compare (M_0, M_1, λ_G) with N_0, iterating least disagreements. On the N_0-side, we use short extender rules to decide which model to go back to, and use Σ_0 at limit steps to pick branches. On the (M_0, M_1, λ_G)-side, we proceed as follows. Let ~~[illegible]~~ $\mathcal{T}$ be the tree on (M_0, M_1, λ_G) we are producing. Let

$$P = M_0 | \eta, \text{ where } \eta \geq \nu \text{ is least such that } \rho_k(M_0|\eta) \leq \lambda_G, \text{ some } k.$$

So $P \trianglelefteq M_0$ is the collapsing structure for ν.

We follow short extender rules for $\mathcal{T}$, i.e. $\mathcal{T}\text{-pred}(\alpha+1) = 0$ if $\mathrm{crit}(E_\alpha^{\mathcal{T}}) < \lambda_G$, and $\mathcal{T}\text{-pred}(\alpha+1) = $ least β s.t. $\mathrm{crit}(E_\alpha^{\mathcal{T}}) < \lambda_{E_\beta^{\mathcal{U}}}$ if the least such β is ≥ 1, except in the following case: if $\mathrm{crit}(E_\alpha^{\mathcal{T}}) = \lambda_G$, and $E_\alpha^{\mathcal{T}}$ is short, then

$$M_{\alpha+1}^{\mathcal{T}} = \mathrm{Ult}_k(P, E_\alpha^{\mathcal{T}})$$

where k is least s.t. $\rho_{k+1}(P) = \lambda_G$.

Remark Short-extender rules would have us apply $E_\alpha^{\mathcal{T}}$ to M_1 in this case. Note

$\nu = (\lambda_G^+)^{M_1}$, and $P \notin M_1$, but $P \restriction \nu = M_1 \restriction \nu$. So we can apply a short $E_\alpha^\mathcal{T}$ with critical point λ_G to P. We may not be able to apply a long $E_\alpha^\mathcal{T}$ with critical point λ_G to P, and we do not — we apply it to M_1.

Claim 4 Σ_0 induces an iteration strategy for (M_0, M_1, λ_G), with respect to trees formed as above.

Proof We lift such trees to trees on M_0 played by Σ_0, as follows. We start with

$$\begin{array}{ccc} M_0 = M_0^* & \longrightarrow & \mathrm{Ult}(M_0, G) = M_1^* \\ \uparrow {\scriptstyle \pi_0 = id} & & \uparrow {\scriptstyle \pi_1} \\ M_0 & \longrightarrow & \mathrm{Ult}(M_0, G|\nu) = M_1 \end{array},$$

where π_1 is the factor embedding, so that $\pi_1 \restriction \nu$ = identity. We then just lift the evolving $\mathcal{T}$ on (M_0, M_1, λ_G) to a tree $\mathcal{T}^*$ on (M_0, M_1^*, λ_G) in the obvious way, and use Σ_0 to choose branches of $\mathcal{T}^*$. The main observation is just that $P \trianglelefteq M_1^*$, because G coheres with M_0 past the collapsing structure of ν. So if we have $\mathrm{crit}(E_\alpha^{\mathcal{T}}) = \lambda_G$, and $E_\alpha^{\mathcal{T}}$ is short, ~~and~~ then we can set

$$M_{\alpha+1}^* = \mathrm{Ult}_k(P, \pi_\alpha(E_\alpha^{\mathcal{T}})),$$

and

$$\pi_{\alpha+1}([a,f]^P_{E_\alpha}) = [\pi_\alpha(a), f]^P_{\pi_\alpha(E_\alpha)}.$$

We have by induction that $\pi_\alpha \restriction \nu =$ identity, so this makes sense. We get that $\pi_{\alpha+1} \restriction E_\alpha = \pi_\alpha \restriction E_\alpha$, and so we can continue. At the $\mathcal{T}^*$ level, we think of P as having been reached by dropping inside M_1^*, so Σ_0 has to allow such a move.

We omit further detail in the proof of claim 1.

$\boxtimes$

<u>Claim 2</u>. The comparison of (M_0, M_1, λ_G) vs. N_0 terminates.

<u>Proof</u> We can use the proof of theorem 5. The key is that our iterations preserve the weak ISC for (M, G) and its initial segments. For G itself,

this is because, for example, if $\mathcal{M}_\alpha^{\mathcal{T}} = (Q, H)$, and $[0, \alpha]_T$ does not drop, and $i : \mathcal{M}_\alpha^{\mathcal{T}} \longrightarrow Ult_0(\mathcal{M}_\alpha^{\mathcal{T}}, E_\beta^{\mathcal{T}}) = \mathcal{M}_{\beta+1}^{\mathcal{T}}$, then i is continuous at $i_{0\alpha}^{\mathcal{T}}(\vartheta) = \vartheta(H) - 1$. (That follows as in the proof of lemma 4b.) The other cases follow as in the proof of lemma 4.

$\boxtimes$

We are writing $\mathcal{T}$ for the tree on the (M_0, M_1, λ_G) side. Let us stipulate that $0 \leq_T 1$, and that

$$i_{01}^{\mathcal{T}} : M_0 \longrightarrow M_1 = Ult(M_0, G \restriction \vartheta)$$

is the canonical embedding. Let also $\mathcal{U}$ be the tree on the N_0 side. Put $M_\alpha = M_\alpha^{\mathcal{T}}$ and $N_\alpha = M_\alpha^{\mathcal{U}}$. Let α and β be such that

$$M_\alpha \trianglelefteq N_\beta \quad \text{or} \quad N_\beta \trianglelefteq M_\alpha .$$

<u>Claim 3</u> $\mathcal{I} \leq_T \alpha$ ~~and if $\gamma+1 \in [1, \alpha]$ with $T\text{-pred}(\gamma+1) =$ and $\mathrm{crit}(E_\gamma^{\mathcal{T}}) = \lambda_\alpha$, then $E_\gamma^{\mathcal{T}}$ is not short~~

<u>Proof</u> Suppose not. Let $\mathcal{J}^*$, with $M_\xi^* = M_\xi^{\mathcal{J}^*}$, be the lift of $\mathcal{J}$ to M_0^*. Suppose $0 \leq_T \alpha$. We have the diagram

$$\begin{tikzcd}
M_\alpha^* \arrow[d, dash, dashed] & M_\alpha \arrow[l, "\pi_\alpha"'] \arrow[d, dash, dashed] & N_\beta \arrow[d, dash, dashed] \\
M_0^* & M_0 \arrow[l, "id"'] \arrow[r, phantom, "="] & N_0
\end{tikzcd}$$

We argue as usual. If $[0, \beta]_{\mathcal{U}}$ drops, then $M_\alpha \triangleleft N_\beta$ and $[0, \alpha]_T$ does not drop, so $i_{0\alpha}^{\mathcal{J}}$ maps M_0 to a proper initial segment of a Σ_0-iterate of M_0, contrary to weak Dodd-Jensen for Σ_0.

If $[0, \alpha]_T$ drops, then $N_\beta \triangleleft M_\alpha$ and $i_{0\beta}^{\mathcal{U}}$ exists, so $\pi_\alpha \circ i_{0\beta}^{\mathcal{U}}$ maps M_0 to a proper initial segment of a $\mathcal{E}_0$-iterate. So neither branch drops. Similarly, we get

$$M_\alpha = N_\beta,$$

and

$$i^{\mathcal{T}}_{0\alpha} = i^{\mathcal{U}}_{0\beta}.$$

But then let H and K be the extenders applied to M_0 and N_0 in $[0,\alpha]_T$ and $[0,\beta]_U$. We argue just as in the proof that comparison terminates that $H = K$. (Note here that $H \neq G \restriction \nu$, because we have assumed $1 \not\leq_T \alpha$!). This contradicts the fact that H and K were part of disagreements when they were used.

$\boxtimes$

<u>Claim 4</u> If $\gamma+1 \in (1, \alpha]_T$ and $\mathrm{crit}(E_\gamma^{\mathcal{T}}) = \lambda_G$, then $E_\gamma^{\mathcal{T}}$ is not short.

<u>Proof</u> If $E_\gamma^{\mathcal{T}}$ is short, then $M_{\gamma+1} = \mathrm{Ult}_k(P, E_\gamma^{\mathcal{T}})$ still projects to λ_G, and is unsound. This implies that $M_\alpha^{\mathcal{T}}$ is unsound, perhaps because of further dropping. That gives $N_\beta \trianglelefteq M_\alpha$, and $i_{0\beta}^{\mathcal{U}}$ exists. Letting $\pi_\alpha : M_\alpha \longrightarrow M_\alpha^*$, we have that $\pi_0 \circ i_{0\beta}^{\mathcal{U}}$ maps M_0 to a Σ_0 iterate reached along a dropping branch. (The drop occurred when we dropped from M_1^* to P.) This contradicts weak Dodd-Jonsen for Σ_0.

$\boxtimes$

Claim 5 $M_\alpha = N_\beta$, $[0,\alpha]_T$ and $[0,\beta]_U$ do not drop, and $i^{\mathcal{T}}_{0\alpha} = i^{\mathcal{U}}_{0\beta}$.

Proof This follows as above from the weak Dodd-Jensen property of Σ_0.

$\boxtimes$

Claim 6 $\alpha \neq \mathcal{I}$, and moreover if $\gamma+1$ is least in $(1,\alpha]_T$, then $\mathrm{crit}(E^{\mathcal{T}}_\gamma) = \lambda_G$.

Proof Suppose otherwise; then $i^{\mathcal{T}}_{0\alpha}(K_G) = \lambda_G$. It follows that $i^{\mathcal{U}}_{0\beta}(K_G) = \lambda_G$. All extenders E used in $\mathcal{U}$ satisfy $\lambda_E \geq \lambda_G$, because N_0

already agrees with M_1 to ν. Letting $E = E_\eta^{\mathcal{U}}$ be the first extender used in $i_{0\beta}^{\mathcal{U}}$, we must then have that $\kappa_E = \kappa_G$ and $\lambda_E = \lambda_G$. Moreover, $\operatorname{crit} i_{\eta+1,\beta}^{\mathcal{U}} > \lambda_E$, because $i_{0\beta}^{\mathcal{U}}(\kappa_E) = \lambda_E$.

Also

$$\ln E = (\lambda_E^+)^{N_{\eta+1}} = (\lambda_E^+)^{N_\beta}$$

$$= (\lambda_G^+)^{M_\alpha} = (\lambda_G^+)^{M_1}$$

$$= \nu.$$

Since E was on the N_η-sequence, it is generated by $\lambda_E \cup \{\xi\}$ for some $\xi < \nu$. (We may not need any ξ.) Thus

$$\nu \subseteq \{ i_{0\beta}^{\mathcal{U}}(f)(a) \mid a \in [\lambda_E \cup \{\xi\}]^{<\omega} \wedge f \in N_0 \}$$

so

$$\nu \subseteq \{ i_{0\alpha}^{\mathcal{T}}(f)(a) \mid a \in [\lambda_G \cup \{\xi\}]^{<\omega} \wedge f \in M_0 \}.$$

But this contradicts that ν is a limit of generators of G.

$\boxtimes$

So let $\gamma+1$ be least in $[1, \alpha]_T$, and notice that by claims 4 and 6, $\mathrm{crit}(E^{\mathcal{I}}_{\gamma}) = \lambda_G$ and $E^{\mathcal{I}}_{\gamma}$ is long. Let

$$E = E^{\mathcal{I}}_{\gamma} \restriction \lambda_{E^{\mathcal{I}}_{\gamma}}$$

be its "superstrong part", and notice that E is on the $M_{\gamma+1}$-sequence, by the weak ISC and coherence. Let

$$E^* = i^{\mathcal{I}}_{\gamma+1, \alpha}(E).$$

A calculation just like that in the proof of the comparison theorem (p. 15) shows that

<u>Claim 7</u> $E^* = F \restriction i^{\mathcal{T}}_{0\alpha}(\kappa_G)$, where F is the extender of $i^{\mathcal{T}}_{1,\alpha}$.

Moreover, E^* is on the sequence of M_α. In fact, $\nu = (\kappa_E^+)^{M_1}$, so

$$E = E^{M_{\gamma+1}}_{\sup(i_{1,\gamma+1}"\nu)},$$

so

$$E^* = E^{M_\alpha}_{\sup \nu^*},$$

where

$$\nu^* = \sup i^{\mathcal{T}}_{1,\alpha}"\nu.$$

Now we look at the first extenders used on the two branches $[0,\alpha]_T$ and $[0,\beta]_U$. For $[0,\alpha]_T$, we have already shown that it is $G \restriction \nu$. For $[0,\beta]_U$, let it be

$$H = E^{\mathcal{U}}_\eta.$$

Let

$$Q = (N_\eta \| h H, H)$$

$$Q^* = Ult_0\left(Q, E_{i^{\mathcal{U}}_{\eta+1,\beta}} \upharpoonright i^{\mathcal{U}}_{0\beta}(\kappa_G)\right)$$

$$= (N_\beta \| h H^*, H^*)$$

be the result of stretching H by the short part of the branch-tail embedding $i^{\mathcal{U}}_{\eta+1,\beta}$) as we did in the comparison proof.

Let also

$$G^* = \bigcup_{\xi<\nu} i^{\mathcal{T}}_{1,\alpha}(G|\xi)$$

$$= \bigcup_{\xi<\nu} i_{E^*}(G\upharpoonright\xi)$$

be the stretch of $G\upharpoonright\nu$ by the short part of $i^{\mathcal{T}}_{1,\alpha}$. We have ~~that~~ as in the comparison proof that

Claim 8 G^* and H^* are initial segments of the extender of $i^{\mathcal{T}}_{0\alpha} = i^{\mathcal{U}}_{0\beta}$.

This gives

Claim 9 ν^* is the largest generator of H^*, and $G^* = H^* \upharpoonright \nu^*$.

Proof Clearly $\nu^* = lh\, G^*$, and G^* has no largest generator. H^* does have a largest generator, because either ~~H~~ H was the image of G along the branch $[0, \eta]_{\mathcal{U}}$, in which case our assumptions about $\operatorname{cof}(\nu(G)-1)$ not being a space plus weak solidity carry the day, or else $(M_\eta \| lh H, H)$ is a plus-one premouse itself, and lemma 4 works. In fact, letting $\nu(H) = \xi+1$, we have

$$\nu(H^*) = j_0(\xi) + 1$$

where $j_0 : Q \to Q^*$ is the ultrapower embedding.

$H^* \notin N_\beta$, as otherwise $H = H^* \upharpoonright \lambda_H \cup \{j_0(\xi)\} \in N_\beta$, contrary to $\text{lh}\, H$ being a cardinal of N_β. It follows that H^* is not a proper initial segment of G^*, as otherwise $H^* = i_{\omega_1}(G|\gamma) \upharpoonright \text{lh}\, H^* \in N_\beta$. But $H^* \neq G^*$, because one has a largest generator and the other doesn't. Thus $G^* = H^* \upharpoonright \nu^*$.

~~However~~ But also, $G^* \notin N_\beta$, since if $G^* \in N_\beta = M_\alpha$, then since $E^* \in M_\alpha$, we get ~~[illegible]~~ ν

$G \restriction \nu \in M_\alpha$, contrary to $G \restriction \nu$ collapsing $(\nu^+)^{M_\alpha}$.

It follows from the weak ISC for H^* that ν^* is its largest generator.

$\boxtimes$

Notice that $(N_\beta \| lh\, H^*, H^*)$ is a plus-one premouse of type Z_1. Here E^* plays the role of the stretching extender F in definition 2, stretching the initial segments of $G \restriction \nu$ into those of $H^* \restriction \nu^*$. So $G \restriction \nu$ is the $\overline{H^*}$ of the discussion following definition 2. Clause (v) of def. 2 holds because $Ult(N_\beta, G \restriction \nu)$ agrees with N_β up to $(\lambda_G^{++})^{N_\beta} = (\nu^+)^{N_\beta}$. That in turn is true because the extender $E_\gamma^{\mathcal{I}}$ applied to M_1 was long, and total on M_1, and had critical point λ_G.

If Q is the image of (M, G) along $[0, \eta]_U$, we can pull this back to (M, G).

<u>Claim 10</u> Suppose $[0,\eta]_{\mathcal{U}}$ does not drop, and H is the last extender of N_η (so that $Q = N_\eta$); then (M, G) satisfies alternative (b) in the conclusion of lemma 7.

<u>Proof</u> ν was the largest generator of G, so $i^{\mathcal{U}}_{0\eta}(\nu)$ is the largest generator of H, so

$$\nu^* = j_0(i^{\mathcal{U}}_{0\eta}(\nu)).$$

Since $E^* = E^{M_\alpha}_{\nu^*} = E^{N_\beta}_{\nu^*} = E^{Q^*}_{\nu^*}$, we can pull back by $j_0 \circ i^{\mathcal{U}}_{0\eta}$ to get

$$E^M_\nu \neq \emptyset.$$

If $\kappa_G \ge \mathrm{crit}(E^M_\nu)$, then $\mathrm{crit}(E^*) = i^{\mathcal{U}}_{0\eta}(\mathrm{crit}(E^M_\nu)) \le \kappa_G$. This is because $i^{\mathcal{U}}_{0\eta} \restriction \kappa_G + 1 =$ identity (as $\kappa_H = \kappa_G$), and

$i_{0\eta}(G) = H$), and because

$\mathrm{crit}(E^*) < \mathrm{crit}(j_0) = \lambda_H$. (The latter because $\mathrm{crit}(E^*) < j_0(\lambda_H) = i^{\mathcal{U}}_{0\beta}(\kappa_G)$.) Thus

$$\kappa_G < \mathrm{crit}(E^M_\nu).$$

Also $\lambda_{E^*} = j_0(\lambda_H)$ ~~$= i^{u}_{0\beta}(\kappa_G)$, giving~~

$= j_0(i^{u}_{0\eta}(\lambda_G))$, giving

$$\lambda_{E^M_\nu} = \lambda_G.$$

~~Finally, we're done if we show~~

~~Subclaim For cofinally many $\xi < \nu$, $G \restriction \xi \in \mathrm{ran}\, i_{E^M_\nu}$.~~

~~Proof Suppose not; let ξ_0 be such that $\xi_0 \le \xi < \nu \Rightarrow G \restriction \xi \notin \mathrm{ran}\, i_{E^M_\nu}$.~~

Now let $\gamma = \text{crit}(E_\nu^M)$. We are done if we show

<u>Subclaim</u> For cofinally many $\eta < \gamma^{+M}$, $i_{E_\nu^M}(E_\eta^M) \subseteq G$.

<u>Proof</u> Suppose not; let η_0 be such that whenever $\eta_0 < \eta < \gamma^{+M}$, $i_{E_\nu^M}(E_\eta^M) \nsubseteq G$. For each $\eta < \gamma^{+M}$, let

$$A_\eta = \text{least } (a, X) \text{ in } i_{E_\nu^M}(E_\eta^M) \,\Delta\, G.$$

So $A_\eta \in M|\nu$. Let

$$A = \{(\eta, A_\eta) \mid \eta_0 < \eta < \gamma^{+M}\}.$$

Then A is amenable to $M|\nu$. Set

$$A^* = \bigcup_{\xi < \nu} j_0 \circ i_{0\eta}(A \cap \xi).$$

The reader can easily check that $\mathcal{A}^*$ witnesses that for ~~cofinally many~~ all sufficiently large $\eta < (\kappa_{E^*}^+)^{(N_\beta | lh H^*, H^*)}$, $i_{E^*}(E_\eta^{N_\beta}) \notin H^*$.

This is a contradiction.

Subclaim $\boxtimes$

Claim 10 $\boxtimes$

Finally

<u>Claim 11</u> If $[0,\eta]_U$ drops, or H is not the last extender of N_η, then $G \upharpoonright \nu \in M$.

<u>Proof</u> Let γ be the largest generator of H. We have

$$j_0(\gamma) = \nu,$$

and then

$$j_0(E_\gamma^{N_\eta}) = E^*,$$

so that for all $\xi < \nu$,

$$i_{E_\gamma^{N_\eta}}(G\restriction\xi) \subseteq H\restriction\gamma$$

by the elementarity of j_0. Thus if H is not the last extender of N_η, then $G\restriction\nu \in N_\eta$. But this implies $G\restriction\nu \in M = N_0$ by

~~Subclaim For all $\xi \geq 1$, $G\restriction\nu \notin N_\xi$. (if $G\restriction\nu \notin$~~

~~Proof~~

Subclaim. If $G\restriction\nu \notin M$, then $G\restriction\nu \notin N_\xi$ for all ξ.

Proof. $M = N_0$ agrees with $M_1 = \mathrm{Ult}(M, G\restriction\nu)$ up to ν, so $\mathrm{lh}\, E_0^{\mathcal{U}} \geq \nu$. Since $E_0^{\mathcal{U}} \in N_0$, $G\restriction\nu \in \mathrm{Ult}(N_0, E_0^{\mathcal{U}}) \Rightarrow G\restriction\nu \in N_0$. Thus $G\restriction\nu \notin N_1$.

M_1 agrees with M_α up to $(\nu^+)^{M_1}$, as otherwise the long extender applied to M_1 in $[0, \alpha)_T$ would force the branch to drop. Thus $(\nu^+)^{M_1} = (\nu^+)^{M_\alpha} = (\nu^+)^{N_\beta}$, so $\mathrm{lh}\, E_1^{\mathcal{U}} \geq (\nu^+)^{M_1}$. It follows that $(\nu^+)^{M_1}$ is a cardinal of N_ξ, for all $\xi \geq 2$. Thus $G\restriction\nu \notin N_\xi$, for all $\xi \geq 2$.

⊠

So we may suppose that H is the last extender of N_η. So by the hypotheses of claim 11, $[0,\eta]_U$ drops. Let $\gamma+1 \in [0,\eta]_U$ be largest in D^U. Let $\theta = U\text{-pred}(\gamma+1)$, and $N^*_{\gamma+1} \trianglelefteq N_\theta$ be the initial segment of N_θ to which $i_{E_\gamma^N} = i^*$ is applied. So we have

$$N^*_{\gamma+1} = (Q', H')$$

and

$$i_{\gamma+1,\eta} \circ i^* : (Q', H') \xrightarrow{\Sigma_1} (Q, H) \underset{\xi}{=} N_\eta .$$

It is easy to see that $\mathrm{crit}(i^*) \geq \lambda_G$, as ~~and~~ λ_G is an inaccessible cardinal in all N_ξ's, and they all agree that far, so $\mathrm{crit}(i^*) < \lambda_G$ would not require a drop. But

$$\lambda_G = \mathrm{crit}(E_\gamma^{N_\eta}), \text{ where } \gamma+1 = \nu(H),$$

so $\lambda_G \in \mathrm{ran}(i_{\gamma+1,\eta} \circ i^*)$. It follows that

$\mathrm{crit}(i^*) > \lambda_G$, so $\mathrm{crit}(i^*) > \nu$.

Letting $\gamma' + 1 = \nu(H')$, we then have by the elementarity of $i_{\eta} \circ i^*$ that

$$F = E_{\gamma'}^{N_\theta} \text{ is short and}$$

$$\kappa_{H'} < \kappa_F < \lambda_{H'} = \lambda_F,$$

and for cofinally many $\xi < \nu = (\kappa_F^+)^{N_\theta}$

$$i_F(G \restriction \xi) \subseteq H' \restriction \gamma'.$$

Since $H' \in N_\theta$, this gives $G \restriction \nu \in N_\theta$.

By the subclaim then, $\theta = 0$, and $G \restriction \nu \in M$, as desired.

Claim 4. $\boxtimes$

This finishes the proof of Lemma 7.

$\boxtimes$

There is a description of how $G \restriction \nu$

gets into M, when it does, that is implicit in the proof of lemma 7. We now make it explicit.

Definition 9. We say that (M, G) adds $\bar{G}$ iff (M, G) is a plus-one pfpm of type Z_1, and for $\nu+1 = \nu(G)$ and $F = E_\nu^M$

$$\bar{G} = \bigcup_{\substack{\xi < \nu \\ G\restriction\xi \in \operatorname{ran} i_F}} i_F^{-1}(G\restriction\xi).$$

and

$$M \mid \nu(\bar{G})^{+M} = \operatorname{Ult}(M, \bar{G}) \restriction \nu(\bar{G})^{+\operatorname{Ult}(M, \bar{G})}.$$

Remark The second condition implies $\nu(\bar{G})^+$ of $\operatorname{Ult}(M, \bar{G})$ is a cardinal of M, so that $\bar{G} \notin M$. Thus (M, G) projects $\leq \nu(\bar{G})$, and collapses $\nu(\bar{G})^{+M}$.

We don't know whether the first condition plus iterability implies that $\operatorname{Ult}(M, \bar{G})$ agrees

with M to ~~...~~ $\delta(\overline{G})^{+\,Ult(M,\overline{G})}$.
Clearly, they agree to $\delta(\overline{G})$. We don't know whether the first condition plus $\overline{G} \notin M$ implies the second, modulo iterability.

<u>Definition 10</u> We say that <u>M certifies $\overline{G}$</u> just in case some initial segment of M adds $\overline{G}$.

The initial segment of M ~~...~~ adding $\overline{G}$ is uniquely determined, as the first level of M collapsing $\delta(\overline{G})^{+\,Ult(M,\overline{G})}$. $\overline{G}$ is uniquely determined by this level, as the unique extender it adds.

What our proof of theorem 6 showed

is

<u>Theorem 11</u> Assume SBH, and let (M, G) be a stage in a plus-one construction, with G long. Let $\nu(G) = \nu + 1$; then exactly one of the following holds

(1) G has a largest generator $< \nu$, and $G \restriction \nu = E_\nu^M$ is on the M-sequence,

(2) ν is a limit of generators of G, and some proper initial segment of M adds $G \restriction \nu$,

(3) $G \restriction \nu \notin M$, (M, G) is of type Z_1, and (M, G) adds some $\bar{G}$.

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