PFA implies $\mathsf{AD}^{L(\mathbb{R})}$.

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In this paper we shall prove

Theorem 0.1 *Suppose there is a singular strong limit cardinal κ such that $\square_\kappa$ fails; then AD holds in L(R).*

See [11] for a discussion of the background to this problem. We suspect that more work will produce a proof of the theorem with its hypothesis that κ is a strong limit weakened to $\forall\alpha < \kappa(\alpha^\omega < \kappa)$, and significantly more work will enable one to drop the hypothesis that κ is a strong limit entirely. At present, we do not see how to carry out even the less ambitious project.

Todorcevic ([23]) has shown that if the Proper Forcing Axiom (PFA) holds, then $\square_\kappa$ fails for all uncountable cardinals κ. Thus we get immediately:

Corollary 0.2 PFA *implies* $\mathsf{AD}^{L(\mathbb{R})}$.

It has been known since the early 90's that PFA implies PD, that PFA plus the existence of a strongly inaccessible cardinal implies $\mathsf{AD}^{L(\mathbb{R})}$, and that PFA plus a measurable yields an inner model of $\mathsf{AD}_\mathbb{R}$ containing all reals and ordinals.[1] As we do here, these arguments made use of Tororcevic's work, so that logical strength is ultimately coming from a failure of covering for some appropriate core models.

In late 2000, A.S. Zoble and the author showed that (certain consequences of) Todorcevic's Strong Reflection Principle (SRP) imply $\mathsf{AD}^{L(\mathbb{R})}$. (See [22].) Since Martin's Maximum implies SRP, this gave the first derivation of $\mathsf{AD}^{L(\mathbb{R})}$ from an "unaugmented" forcing axiom.[2]

It should be possible to adapt the techniques of Ketchersid's thesis [2], and thereby strengthen the conclusions of 0.1 and 0.2 to: there is an inner model of AD^+ plus $\Theta_0 < \Theta$

[1] The first result is due to Woodin, relying heavily on Schimmerling's proof of Δ^1_2 determinacy from PFA. The second result is due to Woodin. For the third, see [1].

[2] In contrast to the arguments referred to in the last paragraph, [22] obtains logical strength from the generic elementary embedding given by a saturated ideal on ω_1, together with simultaneous stationary reflection at ω_2; its argument traces back to Woodin's [24].

which contains all reals and ordinals. Unpublished work of Woodin shows that the existence of such an inner model implies the existence of a nontame mouse.[3] At the moment, the author sees how to adapt the work in chapter 4 and section 5.1 of [2], but the proof of "branch condensation" in section 5.2 of [2] does not adapt to our situation in any straightforward way. This is the point at which Ketchersid brings in some additional properties of his generic embedding (mainly, that its restriction to the ordinals is in V). These properties should not be needed, and possibly the "right" argument in his situation would help in ours as well.

In fact, it should be possible to strengthen the conclusions of 0.1 and 0.2 much further, to the existence of inner models with superstrong and supercompact cardinals respectively. Doing so will require major breakthroughs on the basic open problems of inner model theory.

We shall prove 0.1 in §1. Our proof relies on a mixture of core model theory and descriptive set theory known as the *core model induction* technique. Hugh Woodin was the first to use this technique, in his proof that PFA plus an inaccessible implies $\mathsf{AD}^{L(\mathbb{R})}$. The basic plan in such a proof is to construct mice which are correct for some given level Γ of the Wadge hierarchy (of $L(\mathbb{R})$, or some larger model if one can get that far), via an induction. Descriptive set theory is used to organize the induction, and in particular, the next Γ to consider is the next scaled pointclass. Core model theory is used to construct the mice which are correct for this next Γ.

Our proof of 0.1 builds directly on [8]. There are many tedious details which show up in a core model induction, and so we shall leave some lacunae in our presentation. We believe that everything we have omitted is completely routine. Our intended reader has already read [8], and (ideally) made his way in some detail through at least one core model induction. (The most readily available account of a core model induction is given in [10].)

Section 2 is devoted to a different core model induction. Let $X \prec H_{\omega_3}$ with $|X| = \omega_1$ and $X \cap \omega_2 = \mu \in \omega_2$. We say that X is *amenably closed* iff whenever $A \subset \mu$ and $A \cap \alpha \in X$ for all $\alpha < \mu$, then there is a $B \in X$ such that $A = B \cap \mu$. We shall prove in section 2:

Theorem 0.3 *Assume*

(1) there are stationarily many amenably closed $X \prec H_{\omega_3}$ such that $|X| = \omega_1$,

(2) any function from ω_2 to itself is bounded on a stationary set by a canonical function, and

(3) $2^{\omega_1} = \omega_2$.

Then $\mathsf{AD}^{L(\mathbb{R})}$ holds.

It is easy to show that (1) of 0.3 implies $\neg\mathsf{CH}$, so that together the hypotheses of 0.3 imply that $2^\omega = \omega_2$. Woodin had shown the hypotheses of 0.3 are consistent relative to $\mathsf{AD}_{\mathbb{R}}$ plus "Θ is regular", and asked about the lower bound.

[3] The main part of Woodin's proof is described in [16].

1 Proof of 0.1.

1.1 Framework for the core model induction.

Let κ be our singular strong limit such that $\square_\kappa$ fails.

We shall make use of relativised mice. For A a set of ordinals, an A-premouse is just like an ordinary premouse, except that its hierarchy begins with the rud-closure of $A \cup \{A\}$. All critical points on the sequence of an A-premouse are strictly greater than $\sup(A)$, so that iterations do not move A, and wellfounded iterates are A-premice. The hulls used to define soundness all contain $\sup(A) \cup \{A\}$.

Definition 1.1 *An A-premouse $\mathcal{M}$ is* countably iterable *iff whenever $\mathcal{N}$ is countable and elementarily embeddable in $\mathcal{M}$, then $\mathcal{N}$ is $\omega_1 + 1$-iterable.*

For A a bounded subset of κ^+, let $\mathrm{Lp}(A)$ be the *lower part closure of* A carried out to κ^+. More precisely, set

$$\begin{aligned}
\mathcal{M}_1(A) &= \bigcup\{\mathcal{N} \mid \mathcal{N} \text{ is an } \omega\text{-sound, countably iterable} \\
&\qquad A\text{-premouse and } \rho_\omega(\mathcal{N}) = \sup(A)\}, \\
\mathcal{M}_{\alpha+1}(A) &= \bigcup\{\mathcal{N} \mid \mathcal{N} \text{ is an } \omega\text{-sound, countably iterable} \\
&\qquad A\text{-premouse, } \mathcal{M}_\alpha(A) \trianglelefteq \mathcal{N} \text{ and } \rho_\omega(\mathcal{N}) \leq o(\mathcal{M}_\alpha(A))\}, \text{ and} \\
\mathcal{M}_\lambda(A) &= \bigcup_{\alpha<\lambda} \mathcal{M}_\alpha(A)
\end{aligned}$$

for λ a limit. Here we are using $o(\mathcal{P})$ for the class of ordinals of the premouse $\mathcal{P}$. Of course, the "unions" displayed above have to be taken with a grain of salt. We then set

$$\mathrm{Lp}(A) = \mathcal{M}_{\kappa^+}(A).$$

So $\mathrm{Lp}(A)$ has height κ^+, is countably iterable, has no total measures on its sequence, and is "full" with respect to countably iterable collapsing mice over its initial segments.

Remark 1.2 One problem with working with the Lp operator in the abstract, and not its descriptive-set-theoretically defined approximations, is that we can't show that if $B \in \mathrm{Lp}(A)$, then $\mathrm{Lp}(B) \subseteq \mathrm{Lp}(A)$. The core model induction enables us to work with approximations to the full Lp-closure operation which have this relativisation property.

We need the following fundamental result.

Theorem 1.3 (Schimmerling, Zeman) $\mathrm{Lp}(A) \models \forall \nu \geq \sup(A) \square_\nu$.

It follows then from the failure of $\Box_\kappa$ that there is no largest cardinal of $\mathrm{Lp}(A)$ below κ^+, or put another way

Corollary 1.4 *For any bounded subset A of κ^+, $\mathrm{Lp}(A) \models \mathsf{ZFC}$.*

Let us now fix $A_0 \subseteq \kappa$ such that A_0 codes V_κ in some simple way. Let

$$\lambda = (\kappa^+)^{\mathrm{Lp}(A_0)},$$

so that

$$\lambda < \kappa^+.$$

We show that whenever

$$\mathrm{cof}(\lambda) < \mu < \kappa \text{ and } \mu^\omega = \mu,$$

then

$$V^{\mathrm{Col}(\omega,\mu)} \models \mathsf{AD}^{L(\mathbb{R})}.$$

(In the end, we'll get $\mathsf{AD}^{L(\mathbb{R})}$ in V, and in all $V^{\mathrm{Col}(\omega,\eta)}$ for $\eta \leq \kappa$. But these seem to need getting up to $M_\omega^\sharp$ in $V^{\mathrm{Col}(\omega,\mu)}$, for μ as above.)

So fix such a μ from now on. Let g be V-generic for $\mathrm{Col}(\omega,\mu)$. Let $\mathbb{R}^g$ be the reals in $V[g]$.

We shall show that every Σ_1 formula θ true of some real x in $L(\mathbb{R}^g)$ is *witnessed* to be true by some x-mouse. This is proved by induction on the least α such that $J_\alpha(\mathbb{R}^g) \models \theta[x]$. Our induction hypothesis on the existence of mouse-witnesses easily implies the standard capturing and determinacy hypotheses in a core model induction. Conversely, with a little work, one gets the mouse-witness condition from the standard hypotheses.

There will actually be two hypotheses of the existence of mouse witnesses. In the first, the witnessing mouse is coarse-structural, and in the second, it is an honest fine-structural mouse. We begin with the coarse-structural witness condition.

Definition 1.5 *Let $U \subseteq \mathbb{R}^g$, and $k < \omega$. Let N be countable and transitive, and suppose $\delta_0, ..., \delta_k, S$, and T are such that*

(a) $N \models \mathsf{ZFC} \wedge \delta_0 < ... < \delta_k$ *are Woodin cardinals,*

(b) $N \models S, T$ *are trees which project to complements after the collapse of δ_k to be countable, and*

(c) there is an $\omega_1{+}1$-iteration strategy Σ for N such that whenever $i \colon N \to P$ is an iteration map by Σ and P is countable, then $p[i(S)] \subseteq U$ and $p[i(T)] \subseteq \mathbb{R}^g \setminus U$.

Then we say that N is a coarse (k, U)-Woodin mouse*, as witnessed by $S, T, \Sigma, \delta_0, ..., \delta_k$.*

Our inductive hypothesis is

(W^*_α) Let U be a subset of $\mathbb{R}^g$, and suppose there are scales $\vec{\phi}$ and $\vec{\psi}$ on U and $\mathbb{R}^g \setminus U$ respectively such that $\vec{\phi}^*, \vec{\psi}^* \in J_\alpha(\mathbb{R}^g)$, where $\vec{\phi}^*$ and $\vec{\psi}^*$ are the sequences of prewellorders associated to the scales. Then for all $k < \omega$ and $x \in \mathbb{R}^g$ there are N, Σ such that

(1) $x \in N$, and N is a coarse (k, U)-Woodin mouse, as witnessed by Σ, and

(2) $\Sigma \restriction \mathrm{HC}^{V[g]} \in J_\alpha(\mathbb{R}^g)$.

We emphasize that in W^*_α, it is the *sequences* $\vec{\phi}^*, \vec{\psi}^*$ which are in $J_\alpha(\mathbb{R}^g)$, not just the individual prewellorders in the sequences.

In the end, the mice we construct to verify W^*_α will not be particularly coarse; they will either be ordinary mice constructed from fine extender sequences, or *hybrid* mice, constructed from a fine extender sequence and an iteration strategy. In either case they will have a fine structure, and be suitable for building core models. We will use the core model theory of [13] to construct them.

One can think of W^*_α as asserting, for the given U, that there is a *mouse operator* $x \mapsto \mathcal{M}_x$, defined on $x \in \mathbb{R}^g$ such that $\mathcal{M}_x$ is a (k, U)-Woodin mouse over x.

We now derive some useful consequences of W^*_α.

Lemma 1.6 *If W^*_α holds, then $J_\alpha(\mathbb{R}^g) \models \mathsf{AD}$.*

Proof. By the reflection theorems of Kechris-Solovay or Kechris-Woodin, it is enough to show that U is determined whenever U and $\mathbb{R}^g \setminus U$ admit scales in $J_\alpha(\mathbb{R}^g)$.[4] So fix such a U, and let N be a coarse $(1, U)$-Woodin mouse, as witnessed by S, T, δ_0, δ_1, and Σ. We have that

$$N \models p[S] \text{ is homogeneously Suslin },$$

and hence $p[S]$ is determined in N. (This is a result of Martin, Woodin, and the author; see [19].) Let

$$N \models \tau \text{ is a winning strategy for } p[S].$$

We may assume without loss of generality that N believes τ wins for I. We claim that in $V[g]$, τ wins the game with payoff U for I. For suppose y is a play for II defeating τ; then we can iterate N by Σ, yielding $i\colon N \to P$, with y generic over P at $i(\delta_0)$. Since $\tau(y) \notin U$, and $i(S), i(T)$ are absolute complements over P, $\tau(y) \in p[i(T)]$. By absoluteness of wellfoundedness, $P \models \exists y \;\; \tau(y) \in p[i(T)]$. This contradicts the elementarity of i. $\square$

From Woodin's mouse set argument, we get "capturing on a cone":

[4] Henceforth, this means that the sequence of associated prewellorders is in the model.

Lemma 1.7 *Suppose W^*_α holds; then for a Turing cone of reals x, the following are equivalent, for all reals y:*

(a) y is $OD^{J_\beta(\mathbb{R}^g)}(x)$, for some $\beta < \alpha$,

(b) there is a (fine-structural) x-mouse $\mathcal{M}$ such that $y \in \mathcal{M}$ and $\mathcal{M}$ has an ω_1-iteration strategy in $J_\alpha(\mathbb{R}^g)$.

Proof. See [16]. $\square$

We now want to prove a lightface result on the existence of fine-structural mouse witnesses. We shall call this result W_α. For a technical reason having to do with the real parameters which may enter into the definition of a scale, we are only able to prove W_α in the case that α is a limit ordinal and α begins a (perhaps trivial) gap.

To any Σ_1 formula $\theta(v)$ we associate formulae $\theta^k(v)$ for $k \in \omega$, such that θ^k is Σ_k, and for any γ and any real x,

$$J_{\gamma+1}(\mathbb{R}) \models \theta[x] \Leftrightarrow \exists k < \omega J_\gamma(\mathbb{R}) \models \theta^k[x].$$

Our fine-structural witnesses are as follows.

Definition 1.8 *Suppose $\theta(v)$ is a Σ_1 formula (in the language of set theory expanded by a name for $\mathbb{R}$), and z is a real; then a $\langle \theta, z \rangle$-witnesss is an ω-sound, $(\omega, \omega_1, \omega_1 + 1)$-iterable z-mouse $\mathcal{N}$ in which there are $\delta_0 < ... < \delta_9$, S, and T such that $\mathcal{N}$ satisfies the formulae expressing*

(a) ZFC*,*

(b) $\delta_0, ..., \delta_9$ are Woodin,

(c) S and T are trees on some $\omega \times \eta$ which are absolutely complementing in $V^{\mathrm{Col}(\omega,\delta_9)}$, and

(d) For some $k < \omega$, $p[T]$ is the Σ_{k+3}-theory (in the language with names for each real) of $J_\gamma(\mathbb{R})$, where γ is least such that $J_\gamma(\mathbb{R}) \models \theta^k[z]$.

Remark 1.9 In the phrase $(\omega, \omega_1, \omega_1 + 1)$- *iterable*, the middle ω_1 refers to the fact that our iteration strategy must apply to countable stacks of normal iteration trees. In general, throughout this paper, all iteration trees are linear stacks of normal iteration trees. A *normal* iteration tree is just what was called an *ω-maximal* iteration tree in [5] and [12]. Since people seem to prefer "normal" for this concept, we shall change over. We shall use "maximal" for a different property of iteration trees; see 1.19.

We should note that this is different from the notion of $\langle\theta, z\rangle$-witness defined in [12]. The witnesses in that sense are mice with infinitely many Woodin cardinals, and so they are too crude for our purposes here.

Lemma 1.10 *If there is a $\langle\theta, z\rangle$-witness, then $L(\mathbb{R}) \models \theta[z]$.*

Proof. Suppose $\mathcal{N}$ is a $\langle\theta, z\rangle$-witness, and Σ is its associated $(\omega, \omega_1, \omega_1+1)$-iteration strategy. We may assume that $\mathcal{N}$ is pointwise definable from z, so that Σ is unique, and therefore has the Dodd-Jensen property.[5]

The sentence in part (d) of 1.8 can be expressed in the form

$$\forall y_1 \in \mathbb{R}, \exists y_2 \in \mathbb{R}, \forall y_3 \in \mathbb{R}, \exists y_4 \in \mathbb{R}\psi(y_1, ..., y_4, p[T]),$$

where ψ involves only natural number quantifiers. It follows that this holds not just in $\mathcal{N}$, but after collapsing any $\delta_i, i \leq 5$, over $\mathcal{N}$.

Let A be the set of all reals x such that $x \in p[i(T)]$, for some iteration map $i\colon \mathcal{N} \to \mathcal{Q}$ arising from an iteration tree based on $\mathcal{N}|\delta_0$ played according to Σ. From the Dodd-Jensen property we get that whenever

$$j\colon \mathcal{N} \to \mathcal{P}$$

is an iteration map by Σ, then

$$A \cap \mathcal{P}[h] = p[j(T)] \cap \mathcal{P}[h]$$

for all h which are $\mathcal{P}$-generic over $\mathrm{Col}(\omega, j(\delta_9))$. For suppose $x \in A \cap \mathcal{P}[h]$ and $x \notin p[j(T)]$. Then $x \in p[j(S)]$. Let $i\colon \mathcal{N} \to \mathcal{Q}$ witness $x \in A$, so that $x \in p[i(T)]$. Comparing $\mathcal{Q}$ with $\mathcal{P}$ and using the Dodd-Jensen property, we get iteration maps k and l such that $k \circ i = l \circ j$. But $x \in p[k \circ i(T)] \cap p[l \circ j(S)]$, a contradiction. One argues similarly if $x \in (\mathbb{R} \setminus A) \cap \mathcal{P}[h]$.

We claim that $\forall y_1 \in \mathbb{R}, ..., \exists y_4 \in \mathbb{R}\psi(y_1, ..., A)$, so that A is the first order theory of a level of $L(\mathbb{R})$ satisfying $\theta(z)$, and we are done. To show this, let y_1 be given, and iterate $i\colon \mathcal{N} \to \mathcal{P}$ by Σ so that $y_1 \in \mathcal{P}[h]$ for some h generic over $\mathcal{P}$ for $\mathrm{Col}(\omega, i(\delta_0))$. We get then $y_2 \in \mathcal{P}[h]$ such that $\exists y_3 \forall y_4 \psi(y_1, .., y_4, p[i(T)])$ holds in $\mathcal{P}[h]$. Now let y_3 be given, and iterate $j\colon \mathcal{P} \to \mathcal{Q}$ so that $crit(j) > i(\delta_0)$ and $y_3 \in \mathcal{Q}[h][f]$, where f is $\mathrm{Col}(\omega, j(i(\delta_1)))$-generic. Let $y_4 \in \mathcal{Q}[h][f]$ be such that $\psi(y_1, ..., p[j(i(T))])$ holds. Then by the result of the last paragraph, $\psi(y_1, ..., y_4, A)$ holds, as desired.

$\square$

What we show in our core model induction is just that the converse of 1.10 holds for $L(\mathbb{R}^g)$, at least for α a limit ordinal. More precisely, we show that for α a limit,

[5]See [12]. We actually need the Dodd-Jensen property for compositions of normal trees which are according to Σ, whereas in [6] it is only proved for normal trees. It is possible to show that even when $\mathcal{T}$ is a composition of normal trees by Σ, $\Sigma(\mathcal{T})$ is the unique iterable branch of $\mathcal{T}$, however, and this is what one needs to generalize the argument of [12].

(W_α) If $\theta(v)$ is Σ_1, $z \in R^g$, and $J_\alpha(\mathbb{R}^g) \models \theta[z]$, then there is a $\langle\theta, z\rangle$-witness $\mathcal{N}$ whose associated iteration strategy, when restricted to countable iteration trees, is in $J_\alpha(\mathbb{R}^g)$.

Lemma 1.11 *Let α be a limit ordinal, and suppose that W^*_α holds; then W_α holds.*

Proof. This is very close to the proof of the lightface capturing theorem of [16].

Suppose $\theta(v)$ is Σ_1, $z \in \mathbb{R}^g$, and $J_\alpha(\mathbb{R}^g) \models \theta[z]$. Let $\beta < \alpha$ be least such that

$$J_{\beta+1}(\mathbb{R}^g) \models \theta[z].$$

By 1.7, we can fix $x_0 \geq_T z$ such that if $x \geq_T x_0$, then

$$y \in \mathrm{OD}^{\beta+1}(x) \Leftrightarrow \exists\mathcal{M}(y \in \mathcal{M} \text{ and } J_{\beta+2}(\mathbb{R}^g) \models \mathcal{M} \text{ is an } \omega_1\text{- iterable } x\text{-mouse}).$$

Here $y \in \mathrm{OD}^\gamma(x)$ means that y is definable from x and ordinal parameters over $J_\gamma(\mathbb{R}^g)$. Let

$$U = \text{ universal } \Sigma_3^{J_{\beta+2}(\mathbb{R}^g)} \text{ set of reals,}$$

and let N be a coarse $(10, U)$-Woodin mouse, as witnessed by the trees S_0 and S_1 for U and its complement respectively, the iteration strategy Σ for N, and $\delta_0 < ... < \delta_{10}$. We suppose also that $x_0 \in N$. These objects exist by W^*_α, and by the scale analysis of [17], which implies that every set of reals in $J_{\beta+2}(\mathbb{R}^g)$ has a scale in $J_{\beta+2}(\mathbb{R}^g)$. (If β ends a weak gap this is also true for $\beta+1$, but in any case, β does end a Σ_1-gap.)

Let $\lambda > \delta_{10}$ be a limit ordinal such that $S_0, S_1 \in V^N_\lambda$. Working in N, we can find club many $\eta < \delta_9$ such that there is a transitive H and $\pi\colon H \to V_\lambda$ with $S_0, S_1 \in \mathrm{ran}(\pi)$ and $\mathrm{crit}(\pi) = \eta$ and $\pi(\eta) = \delta_9$. For such η and H, we have

$$(b \subseteq V^N_\eta \wedge b \in \mathrm{OD}^{\beta+1}(V^N_\eta)) \Rightarrow b \in H.$$

This is because every $\mathrm{OD}^{\beta+1}(V^N_{\delta_9})$ subset of $V^N_{\delta_9}$ is in N (using the iterability given by Σ and the extender algebra at δ_{10}), and because this fact is recorded in the fact that $(i, y) \in p[S_0]$ for y an N-generic code of $V^N_{\delta_9}$. This fact passes down to H because $S_0 \in \mathrm{ran}(\pi)$. Notice that η is Woodin in H.

In N, we use a full-background-extender construction to build an extender model over z of the form $L[\vec{E}, z]$. We can find an $\eta < \delta_9$ in the club of the last paragraph such that

$$L[\vec{E}, z] \models \eta \text{ is not Woodin.}$$

We can also assume that η is a cardinal of $L[\vec{E}, z]$. Let $\mathcal{Q} \trianglelefteq L[\vec{E}, z]$ be the first level of $L[\vec{E}, z]$ such that

$$L[\vec{E}, z]|\eta \trianglelefteq \mathcal{Q} \text{ and } \mathcal{Q} \notin \mathrm{OD}^{\beta+1}(L[\vec{E}, z]|\eta).$$

Such a level exists because η is Woodin in $L[\vec{E}, z]$ with respect to functions which are $\mathrm{OD}^{\beta+1}(V_\eta^N)$. (We may assume $L[\vec{E}, z]|\eta$ is definable over V_η^N.) Notice that η is a cutpoint of $\mathcal{Q}$, as otherwise we have an iteration strategy for a nontame mouse in $L(\mathbb{R})$.[6] Notice also that η is Woodin in $\mathcal{Q}$.

As in [16], x_0 is generic over $\mathcal{Q}$ for its extender algebra at η, and therefore we can find an $x \geq_T x_0$ and a g generic over $\mathcal{Q}$ for $\mathrm{Col}(\omega, \eta)$ such that $\mathcal{Q}[g] = \mathcal{Q}[x]$, and moreover, there is an x-mouse $\mathcal{R}$ such that $\mathcal{R}|\gamma = \mathcal{Q}|\gamma[g]$ for all γ above the first admissible over $\mathcal{Q}|\eta$. Since $\mathcal{R}$ is the first x-mouse with no iteration strategy in $J_{\beta+2}(\mathbb{R}^g)$, and $x \geq_T x_0$, we get

$$\mathbb{R} \cap \mathcal{R} = \{u \in \mathbb{R}^g \mid u \in \mathrm{OD}^{\beta+1}(x)\}.$$

Letting $\bar{\mathbb{R}} = \mathbb{R} \cap \mathcal{R}$, this implies that there is a unique $\bar{\beta} \in \mathcal{R}$ and Σ_1-elementary

$$\pi\colon J_{\bar{\beta}+2}(\bar{\mathbb{R}}) \to J_{\beta+2}(\mathbb{R}^g).$$

The key here is that every set in $J_{\beta+2}(\mathbb{R}^g)$ has a scale in $J_{\beta+2}(\mathbb{R}^g)$, so that statements about reals in $\bar{\mathbb{R}}$ are witnessed by reals in $\bar{\mathbb{R}}$. This is why we went up to $\beta+2$ in the first place; if β ends a strong gap we cannot use $\mathrm{OD}^\beta(x)$ as our $\bar{\mathbb{R}}$. We have $\bar{\beta} \in \mathcal{R}$ because we can assume $\mathcal{Q}$ has extenders on its sequence with index $> \eta$.

Let $k < \omega$ be such that $J_\beta(\mathbb{R}^g) \models \theta^k[z]$, and let

$$T = \{\langle \varphi, u\rangle \mid \varphi \text{ is } \Sigma_{k+3} \wedge u \in \mathbb{R}^g \wedge J_\beta(\mathbb{R}^g) \models \varphi[u]\}.$$

T and $\mathbb{R}^g \setminus T$ are $\Sigma_1^{J_{\beta+1}(\mathbb{R}^g)}(z)$, and therefore have scales which are $\Sigma_1^{J_{\beta+1}(\mathbb{R}^g)}(z)$. It follows, using π, that the restrictions of these scales are Σ_1-definable over $J_{\bar{\beta}+1}(\bar{\mathbb{R}})$ from z. The trees W_0 and W_1 associated to these restricted scales, being definable over $\mathcal{R}$ from z and ordinals, are in $\mathcal{Q}$ by the homogeneity of $\mathrm{Col}(\omega, \eta)$. It is now easy to see that $\mathcal{Q}$, together with $\delta_0, ..., \delta_8, \eta, W_0$, and W_1 constitutes a $\langle \theta, z\rangle$-witness. □

Note also that we get lightface capturing from W_α.

Lemma 1.12 *Assume W_α holds; then if $x, y \in \mathbb{R}^g$ and x is ordinal definable from y over some $J_\gamma(\mathbb{R}^g)$, where $\gamma < \alpha$, then there is a y-premouse $\mathcal{M}$ such that $x \in \mathcal{M}$, and $J_\alpha(\mathbb{R}^g) \models$ $\mathcal{M}$ is ω_1-iterable.*

A final general remark: although our induction hypothesis W_α^* is to be interpreted in $V[g]$, we are going to have to work for the most part in V, where we can form interesting Skolem hulls of size μ which are closed under ω-sequences. We shall see that W_α gives us size μ mice in V over terms τ such that $\tau^g \in \mathbb{R}^g$. Those are the mice we shall feed into the covering argument. It might have been better to make W_α^* a closure condition on the mice in V, but it does not seem necessary to do so.

[6] This and the corresponding point in the proof of 1.7 are the main points where we need to restrict ourselves to tame mice.

1.2 Scales in $L(\mathbb{R})$

Core model inductions are organized according to the appearance of new definable scales on sets of reals as we go up the Wadge hierarchy. The W^*_α assert that, given $U \subseteq \mathbb{R}^g$, as soon as a scale on U appears, an iteration strategy for a coarse mouse having a forcing term for such a scale appears. Thus it is useful to know how scales appear. Under appropriate determinacy hypotheses, there is in the Wadge hierarchy of $L(\mathbb{R})$ (and beyond) a tight correspondence between the appearance of scales on sets which did not previously admit them, and certain failures of reflection. This correspondence is analyzed in detail in [17] and [18]. Our inductive proof of W^*_α breaks into cases which reflect that analysis.

Definition 1.13 *An ordinal β is* critical *just in case there is some set $U \subseteq \mathbb{R}^g$ such that U and $\mathbb{R}^g \setminus U$ admit scales in $J_{\beta+1}(\mathbb{R}^g)$, but U admits no scale in $J_\beta(\mathbb{R}^g)$.*

(Once again, we are identifying a scale with the *sequence* of its prewellorders here.) Clearly, we need only show that $W^*_{\beta+1}$ holds whenever β is critical, in order to conclude that W^*_α holds for all α.

It follows from [17] that if β is critical, then $\beta+1$ is critical. Moreover, if β is a limit of critical ordinals, then β is critical if and only if $J_\beta(R^g)$ is not an admissible set. Letting β be critical, we then have the following possibilities

(1) $\beta = \eta + 1$, for some critical η;

(2) β is a limit of critical ordinals, and either

  (a) $\mathrm{cof}(\beta) = \omega$, or

  (b) $\mathrm{cof}(\beta) > \omega$, but $J_\beta(\mathbb{R}^g)$ is not admissible;

(3) $\alpha = \sup(\{\eta < \beta \mid \eta \text{ is critical }\})$ is such that $\alpha < \beta$, and either

  (a) $[\alpha, \beta]$ is a Σ_1 gap, or

  (b) $\beta - 1$ exists, and $[\alpha, \beta - 1]$ is a Σ_1 gap.

In each case, the first thing we have to do towards proving $W^*_{\beta+1}$ is to find a way of feeding a description of truth at the bottom of the Levy hierarchy over $J_\beta(\mathbb{R}^g)$ into mice. In cases 1 and 2(a), this is fairly easy: $\Sigma_1^{J_\beta(\mathbb{R}^g)}$ is the class of countable unions of sets belonging to $J_\beta(\mathbb{R}^g)$, so we can just put together countably many mice given by our induction hypothesis. We shall not give any details on these cases in this paper.

We call case 2(b) the *inadmissible case*, and we shall give it a fairly thorough treatment here.

Case 3, the end-of-gap (in scales) case, is the most subtle. In this case, $J_\alpha(\mathbb{R}^g)$ is admissible. In case 3(a), the gap $[\alpha, \beta]$ is weak, and in case 3(b), the gap $[\alpha, \beta - 1]$ is strong unless $\alpha = \beta - 1$. (The case $\alpha = \beta - 1$, which happens for example when α is the least $\mathbb{R}$-admissible, should probably be called an *improper strong gap.*) Here are the basic facts about scales and reflection we shall need in case 3. As to reflection, we have

Theorem 1.14 (Martin [3]) *Assume W^*_β, where β is critical and case 3 holds at β. Then for any $x, y \in \mathbb{R}^g$, if $x \in OD^\gamma(y)$ for some $\gamma < \beta$, then $x \in OD^\gamma(y)$ for some $\gamma < \alpha$.*

As to scale existence, we have

Theorem 1.15 ([17]) *Assume W^*_β, where β is critical and case 3 holds at β; then*

(1) every set of reals $A \in J_\beta(\mathbb{R}^g)$ admits a scale $\vec{\psi}$ such that each prewellorder $\leq_{\psi_i}$ belongs to $J_\beta(\mathbb{R}^g)$, and

(2) letting n be least such that $\rho_n(J_\beta(\mathbb{R}^g)) = \mathbb{R}$, and U be any boldface $\Sigma_n^{J_\beta(\mathbb{R}^g)}$ set of reals, we have $U = \bigcup_{n<\omega} U_n$, where each $U_n \in J_\beta(\mathbb{R}^g)$.

In part (1), the sequence of prewellorders may not belong to $J_\beta(\mathbb{R}^g)$. Part (2) implies that the boldface pointclass $\Sigma_n^{J_\beta(\mathbb{R}^g)}$ is in fact the class of countable unions of sets of reals in $J_\beta(\mathbb{R}^g)$, and has the scale property.

Motivated by 1.15, we make the

Definition 1.16 *A* self-justifying system *is a countable set $\mathcal{A} \subseteq P(\mathbb{R})$ which is closed under complements (in $\mathbb{R}$), and such that every $A \in \mathcal{A}$ admits a scale $\vec{\psi}$ such that $\leq_{\psi_i} \in \mathcal{A}$ for all i.*

Thus if W^*_β holds, β is critical, and case 3 obtains, then for any set of reals $A \in J_\beta(\mathbb{R}^g)$, there is a self-justifying system $\mathcal{A} \subseteq J_\beta(\mathbb{R}^g)$ such that $A \in \mathcal{A}$.

The weak gap/strong gap distinction between 3(a) and 3(b) only comes into the proofs of 1.14 and 1.15, and into the proof of 1.39, a result of Woodin we shall lean on heavily in case 3. Since we shall be using those results rather than proving them, we can ignore the distinction between 3(a) and 3(b) here.

In case 3, the set coding truth at the bottom of the Levy hierarchy over $J_\beta(\mathbb{R}^g)$ which we feed into our mice will be an *iteration strategy* Σ for a mouse $\mathcal{M}$ with a Woodin cardinal which is Lp^α-full, in a sense we shall explain. The structures which witness the truth of $W^*_{\beta+1}$ will be *hybrid* Σ*-mice*, mice over $\mathcal{M}$ constructed from an extender sequence as usual, while simultaneously closing under Σ. The condensation properties of Σ will imply that these hybrid mice behave like ordinary mice, and in particular, we can use core model theory to produce Σ-hybrid mice with any finite number of Woodin cardinals, and thereby capture truth at the higher levels of the Levy hierarchy over $J_\beta(\mathbb{R}^g)$.

1.3 Lifting mouse-closure from μ to κ

The main new idea is just the following: suppose our induction has given us some mouse-closure function $A \mapsto \mathcal{M}(A)$ capturing truth at the bottom of some projective-like hierarchy we now want to climb. We are assuming here that the function $A \mapsto \mathcal{M}(A)$ exists in V, and is defined on all $A \subset \mu$ in V, although its significance as helping verify some W^*_β has to do with $V[g]$. So $\mathcal{M}(A)$ is defined for all subsets A of μ, and $\mathcal{M}$-closed mice with a Woodin cardinal (over arbitrary $A \subseteq \mu$) are what we seek in taking our first step up the projective-like hierarchy ahead. Then in each case of the core model induction, we shall be able to extend the $\mathcal{M}$-operator to act in a natural way on arbitrary subsets of κ. This then puts us in a position to build core models in appropriately closed background universes, and show they reach the desired $\mathcal{M}$-closed mice over subsets of μ, as usual.

In other words, the "cycling" which is typical of a local core model induction takes place between closure-at-μ and closure-at-κ. (There is an intermediate step giving closure at all ν with $\mu < \nu < \kappa$.)

We thanks the referee for pointing out that our lifting arguments in this section do not require the failure of square. What we need is

- κ is a singular strong limit cardinal such that $\kappa^{+\mathrm{Lp}(A)} < \kappa^+$ whenever A is a bounded subset of κ^+, and $\mu = \mathrm{cof}(\kappa^{+\mathrm{Lp}(A_0)})^\omega$, where $A_0 \subseteq \kappa$ and A_0 codes V_κ.

The inadmissible case.

Let β be critical, and suppose case (2) holds at β. That is, β both begins and end a Σ_1-gap, and $J_\beta(\mathbb{R}^g)$ is inadmissible. We have W^*_β by induction, and since β is a limit ordinal, we get W_β.

In this case, we can take $\mathcal{M}(A)$ to be an ordinary, non-hybrid mouse over A. (That is not how we have been doing it up to now, at least in the case $\mathrm{cof}^{V[g]}(\beta) > \omega$, but it is possible, as we shall show.) Let $x = \tau^g$ be a real parameter from which we can define a failure of admissibility. We are assuming that all our A code τ in some simple, uniform way. For $A \subset \nu < \mu^+$, $\mathcal{M}(A)$ is just going to be the first level of $\mathrm{Lp}(A)$ satisfying a certain sentence ψ, where ψ asserts that in $\mathcal{M}[g][h]$, for h generic on $\mathrm{Col}(\omega, \nu)$, there are enough Σ_1-witnesses among initial segments of $\mathcal{M}[g][h]$ to show that the function defined from τ^g witnessing non-admissibility is total on $\mathbb{R} \cap \mathcal{M}[g][h]$.[7] We shall give the details below, but all that really matters now is that

(i) There is a sentence ψ in the language of A-premice such that whenever A codes τ in the specified way, then $\mathcal{M}(A)$ is the first level of $\mathrm{Lp}(A)$ satisfying ψ if there is such a level, and undefined otherwise,

[7] It would be possible to do it this way, but when we get to the details, it is easier to do something approximate.

(ii) The $\mathcal{M}$-operator *relativises well at* μ, in that there is a formula $\theta(u, v, w)$ such that whenever $|A| = \mu$, $\mathcal{M}(A)$ exists, and $A \in L_1(B)$ where $|B| = \mu$, and N is a transitive model of ZFC^- such that $\mathcal{M}(B) \in N$, then $\mathcal{M}(A) \in N$ and $\mathcal{M}(A)$ is the unique $x \in N$ such that $N \models \theta[x, A, \mathcal{M}(B)]$, and

(iii) $\mathcal{M}(A)$ exists for all A bounded in μ^+ and coding τ in the specified way.

In (ii), $L_1(B)$ is the first level of Gödel's L over B; the intent is just to say that A is simply coded into B. It follows from (ii) that for A, B of size μ with A simply coded into B, $\mathcal{M}(A) \in \mathrm{Lp}(B)$. That is all we shall use from (ii) in this section, but we shall need the uniform local definability of $\mathcal{M}(A)$ from $A, \mathcal{M}(B)$ later.

Under these hypotheses, we want to show that $\mathcal{M}(A)$ is defined for arbitrary A bounded in κ^+ coding τ. Let us assume first that A is bounded in κ, and codes τ in the specified way.

Let $\pi \colon H \to V_\eta$, η large, where H is a transitive set of cardinality μ closed under ω-sequences, $\mathrm{crit}(\pi) > \mu$, and $\mathrm{ran}(\pi)$ is cofinal in λ. We assume $\kappa, A, A_0 \in \mathrm{ran}(\pi)$, and write

$$\langle \bar{\kappa}, \bar{A}, \bar{A}_0 \rangle = \pi^{-1}(\langle \kappa, A, A_0 \rangle).$$

It is enough to see that H satisfies "there is a countably iterable $\bar{A}$-mouse satisfying ψ", since then the elementarity of π gives the desired conclusion. However, $\mathcal{M}(\bar{A})$ is such a mouse, and since countable iterability is absolute to H (we can arrange $P(\omega_1) \subseteq H$), it is enough to see that $\mathcal{M}(\bar{A}) \in H$. But note that $\mathrm{Lp}(\bar{A}_0) \cap P(\bar{\kappa}) \subseteq H$ by the usual covering argument, since $\mathrm{ran}(\pi)$ is cofinal in $\lambda = (\kappa^+)^{\mathrm{Lp}(A_0)}$. Since A was bounded in κ, $\bar{A}$ is coded into $\bar{A}_0$, and thus $\mathcal{M}(\bar{A}) \in H$ because the $\mathcal{M}$- operator relativises well at μ.

We next consider arbitrary A bounded in κ^+, and coding τ in the specified way. Let λ^* be the cardinal successor of $\sup(A)$ in $\mathrm{Lp}(A)$. Let $\pi \colon N \to V_\eta$, η large, where N is transitive, closed under ω-sequences, $|N| < \kappa$, and $\mathrm{ran}(\pi)$ is cofinal in λ^*. Let $\pi(\langle \bar{\kappa}, \bar{A} \rangle) = \langle \kappa, A \rangle$. Since $|\bar{A}| < \kappa$, $\mathcal{M}(\bar{A})$ exists. We have $\mathrm{Lp}(\bar{A}) \cap P(\bar{\kappa}) \subseteq N$ by the covering argument, hence $\mathcal{M}(\bar{A}) \in N$. As above, N then satisfies "$\mathcal{M}(\bar{A})$ exists", so V satisfies "$\mathcal{M}(A)$ exists".

The end-of-gap case.

Now let β be critical, and suppose that case 3 holds. Let α be the sup of the critical ordinals $< \beta$, so that either $[\alpha, \beta]$ is a proper weak gap (case 3(a)), or $[\alpha, \beta - 1]$ is a perhaps improper strong gap (case 3(b)). We have W^*_β, and hence W_α, by induction.

The standard move would be to take $\mathcal{M}(A)$ to be a *term-relation hybrid* A-mouse, that is, an ordinary A-mouse expanded by a amenable predicate identifying terms capturing a self-justifying system which knows–and Skolemizes–truth at the end of the gap. Here again we are assuming A codes τ, where $x = \tau^g$ is a real from which each set in our self-justifying system is ordinal definable over some $J_\gamma(\mathbb{R}^g)$, for $\gamma < \beta$. Unfortunately, our lifting of closure at μ to closure at κ has to be done in V, and the term relation hybrid mice are tied to our particular x and g in such a way that their naive pullbacks to V seem useless. So instead of adding the term relations for a self-justifying system, we shall close under a canonical iteration strategy with condensation which we get from the self-justifying system. This gives us something in V which is better behaved than the naive pullbacks of the standard hybrid mice.

Definition 1.17 *For any bounded subset A of μ^+, let $\mathrm{Lp}^\alpha(A)$ be the "union" of all A-mice $\mathcal{N}$ projecting to $\sup(A)$ such that $J_\alpha(\mathbb{R}^g) \models \mathcal{N}$ is ω_1-iterable.*

Note $\mathrm{Lp}^\alpha(A)$ is an initial segment of $\mathrm{Lp}(A)$, since the iteration strategy witnessing $\mathcal{N} \in \mathrm{Lp}^\alpha(A)$ is unique, so that its restriction to V is in V.

Definition 1.18 *Let A be a bounded subset of μ^+. An A-premouse $\mathcal{N}$ is* suitable *iff card $(\mathcal{N}) = \mu$ and*

(a) $\mathcal{N} \models$ there is exactly one Woodin cardinal. We write $\delta^{\mathcal{N}}$ for the unique Woodin cardinal of $\mathcal{N}$.

(b) Letting $\mathcal{M}_0 = \mathcal{N}|\delta^{\mathcal{N}}$, and $\mathcal{M}_{i+1} = \mathrm{Lp}^\alpha(\mathcal{M}_i)$, we have that $\mathcal{N} = \bigcup_{i<\omega} \mathcal{M}_i$. That is, $\mathcal{N}$ is the Lp^α closure of $\mathcal{N}|\delta^{\mathcal{N}}$, up to its ω^{th} cardinal above $\delta^{\mathcal{N}}$.

(c) If $\xi < \delta^{\mathcal{N}}$ is a cardinal of $\mathcal{N}$, then $\mathrm{Lp}^\alpha(\mathcal{N}|\xi) \models \xi$ is not Woodin.

We say an iteration tree $\mathcal{U}$ on a premouse $\mathcal{N}$ *lives below* η if $\mathcal{U}$ can be regarded as an iteration tree on $\mathcal{N}|\eta$. If $\mathcal{U}$ is normal, then as usual we write $\delta(\mathcal{U})$ for $\sup\{\mathrm{lh}(E^{\mathcal{U}}_\alpha \mid \alpha < \mathrm{lh}(\mathcal{U})\}$, and $\mathcal{M}(\mathcal{U})$ for $\bigcup\{\mathcal{M}^{\mathcal{U}}_\alpha| \mathrm{lh}(E^{\mathcal{U}}_\alpha \mid \alpha < \mathrm{lh}(\mathcal{U})\}$.

Definition 1.19 *Let $\mathcal{U}$ be a normal iteration tree of length $< \mu^+$ on a suitable $\mathcal{N}$, and suppose $\mathcal{U}$ lives below $\delta^{\mathcal{N}}$; then $\mathcal{U}$ is* short *iff for all limit $\xi \leq \mathrm{lh}(\mathcal{U})$, $\mathrm{Lp}^\alpha(\mathcal{M}(\mathcal{U} \restriction \xi)) \models \delta(\mathcal{U} \restriction \xi)$ is not Woodin. Otherwise, we say $\mathcal{U}$ is* maximal.

Just to emphasize, a non-normal iteration tree is neither short nor maximal. Similarly, a tree on $\mathcal{N}$ which cannot be regarded as a tree on $\mathcal{N}|\delta^{\mathcal{N}}$ is neither short nor maximal.

Definition 1.20 *Let Σ be a (μ^+, μ^+)-iteration strategy on a suitable $\mathcal{N}$; then Σ is* fullness-preserving *iff whenever $\mathcal{P}$ is an iterate of $\mathcal{N}$ by Σ, via a tree which lives below $\delta^{\mathcal{N}}$, then*

(1) if $\mathcal{N}$-to-$\mathcal{P}$ does not drop, then $\mathcal{P}$ is suitable, and

(2) if $\mathcal{N}$-to-$\mathcal{P}$ drops, then $J_\alpha(\mathbb{R}) \models \mathcal{P}$ is ω_1-iterable.

It is not hard to see that in case (2) of 1.20, we have that for all ξ, $\mathrm{Lp}^\alpha(\mathcal{P}|\xi) \models \xi$ is not Woodin, and thus no initial segment of $\mathcal{P}$ is suitable.

Of course, we should really speak of α-suitability, etc., but α has been fixed for this section.

Lemma 1.21 *Suppose Σ is a fullness-preserving iteration strategy for $\mathcal{N}$, and $\mathcal{T}$ is an iteration tree living below $\delta^{\mathcal{N}}$, played by Σ, which has a last normal component tree $\mathcal{U}$ having base model $\mathcal{P}$ and of limit length. Let b be the branch of $\mathcal{U}$ chosen by Σ; then*

(1) if $\mathcal{N}$-to-$\mathcal{P}$ drops, then $\mathcal{U}$ is short, and $\mathcal{Q}(b,\mathcal{U})$ is a proper initial segment of $\mathrm{Lp}^\alpha(\mathcal{M}(\mathcal{U}))$, and

(2) if $\mathcal{N}$-to-$\mathcal{P}$ does not drop, so that $\mathcal{P}$ is suitable, then

(a) for all $\xi < \mathrm{lh}(\mathcal{U})$, $\mathcal{U} \upharpoonright \xi$ is short,

(b) if $\mathcal{U}$ is short, then $\mathcal{Q}(b,\mathcal{U})$ exists and is a proper initial segment of $\mathrm{Lp}^\alpha(\mathcal{M}(\mathcal{U}))$, and

(c) if $\mathcal{U}$ is maximal, then b does not drop, and $i_b^{\mathcal{U}}(\delta^{\mathcal{P}})) = \delta(\mathcal{U})$.

We shall omit the straightforward proof of this lemma.

According to this lemma, a fullness-preserving strategy is guided by $\mathcal{Q}$-structures in Lp^α, unless, for the current normal component $\mathcal{U}$, there is no such $\mathcal{Q}$-structure. That is case (2)(c) above, and then from (2)(c) we see that $\mathcal{U}$ has no normal continuation. Moreover, although Lp^α cannot tell us what b is, it can identify $\mathcal{M}_b(\mathcal{U})$, since

$$\mathcal{M}_b(\mathcal{U}) = (\mathrm{Lp}^\alpha) - \text{closure of } (\mathcal{M}(\mathcal{U}))$$

up to its ω^{th} cardinal. This important insight is due to Woodin. It means that Lp^α can "track" a fullness-preserving iteration strategy, in that it can find the models of an evolving iteration tree, although it cannot always find the branches and embeddings.[8]

We wish to describe a condensation property for iteration strategies. For the notion of a finite support in an iteration tree, see [14]. Let $\mathcal{T}$ be an iteration tree on $\mathcal{N}$, and

$$\sigma\colon \beta \to \mathrm{lh}(\mathcal{T})$$

an order preserving map such that $\mathrm{ran}(\sigma)$ is support-closed. Then there is a unique iteration tree $\mathcal{S}$ on $\mathcal{N}$ of length β such that there are maps

$$\pi_\gamma\colon \mathcal{M}_\gamma^{\mathcal{S}} \to \mathcal{M}_{\sigma(\gamma)}^{\mathcal{T}},$$

[8]For infinite stacks of normal trees, more work is needed even to find the models using only Lp^α as a guide. Using "quasi-iterations", Woodin has solved this problem. We shall not need quasi-iterations for our proof of $\mathsf{AD}^{L(\mathbb{R})}$, but they are needed in adapting Ketchersid's work.

for $\gamma < \beta$, which commute with the tree embeddings, with

$$\pi_\gamma(E^{\mathcal{S}}_\gamma) = E^{\mathcal{T}}_{\sigma(\gamma)}$$

for all $\gamma < \beta$, and $\pi_{\gamma+1}$ determined by the shift lemma. (Support-closure is just what we need to keep this process going.)

Definition 1.22 *Let $\mathcal{S}$ and $\mathcal{T}$ be iteration trees related as above; then we say that $\mathcal{S}$ is a* hull *of $\mathcal{T}$, as* witnessed by *σ and the π_γ, for $\gamma < \mathrm{lh}(\mathcal{T})$.*

Definition 1.23 *An iteration strategy Σ* condenses well *iff whenever $\mathcal{T}$ is an iteration tree played according to Σ, and $\mathcal{S}$ is a hull of $\mathcal{T}$, then $\mathcal{S}$ is according to Σ.*

It is clear that if Σ is the unique iteration strategy[9] on $\mathcal{N}$, then Σ condenses well. More generally, if $\mathcal{N}$ is an initial segment of $\mathcal{M}$, and Γ is the unique iteration strategy for $\mathcal{M}$, and Σ is the strategy for $\mathcal{N}$ which is determined by Γ, then Σ condenses well. One can think of an iteration strategy which condenses well as the "trace" of a unique iteration strategy on a stronger mouse.

Remark 1.24 The notions we have just introduced are in essence due to Woodin, and are used to great effect in Ketchersid's thesis [2].

We shall see that the self-justifying system definable from τ^g in $L(\mathbb{R}^g)$ gives us, back in V, a suitable τ-mouse $\mathcal{N}$, together with a fullness-preserving (μ^+, μ^+)-iteration strategy Σ for $\mathcal{N}$ which condenses well. The following lemma then lifts the closure at μ given by Σ to closure at κ.

Lemma 1.25 *In V: let $\mathcal{N}$ be a suitable premouse (over some A bounded in μ^+), and let Σ be a fullness-preserving (μ^+, μ^+)-iteration strategy for $\mathcal{N}$ which condenses well. Then Σ has a unique extension Γ to a (κ^+, κ^+)- iteration strategy which condenses well.*

Proof. Uniqueness is easy: suppose Γ_0 and Γ_1 are two extensions of Σ which condense well, and that $\mathcal{T}$ is according to both, but $\Gamma_0(\mathcal{T}) = b$ and $\Gamma_1(\mathcal{T}) = c$, where $b \neq c$. Taking a size μ Skolem hull of the universe and collapsing, we get that both $\bar{\mathcal{T}}^\frown \bar{b}$ and $\bar{\mathcal{T}}^\frown \bar{c}$ are according to Σ, a contradiction since $\bar{b} \neq \bar{c}$.

For existence, we define the restriction of the desired Γ to iteration trees of length $< \xi$[10] by induction on ξ. Here $\mu^+ \leq \xi < \kappa^+$, as for $\xi < \mu^+$ we just use Σ. Let us call this restriction Γ_ξ. Clearly, if ξ is a limit, then we must set

$$\Gamma_\xi = \bigcup_{\eta < \xi} \Gamma_\eta,$$

[9] For some reasonable sort of iteration game.

[10] Our iteration tree is a composition of normal trees, and its length is the sum of the lengths of its normal components.

and Γ_ξ condenses well if all Γ_η for $\eta < \xi$ condense well. Now suppose Γ_ξ is given, extending Σ and condensing well. If ξ is not a limit ordinal, there is nothing for our strategy to decide, so we have $\Gamma_\xi = \Gamma_{\xi+1}$. So let ξ be a limit ordinal.

As before, we consider first the case $\xi < \kappa$. Let $\mathcal{T}$ be an iteration tree on $\mathcal{N}$ which is according to Γ_ξ; we have to choose a cofinal wellfounded branch of $\mathcal{T}$.

Fix $\theta > \kappa^+$ a limit ordinal. Let us call X *nice* iff $X \prec V_\theta$, $|X| = \mu$, $\mu + 1 \subseteq X$, $\kappa, \mathcal{T}, A_0 \in X$, X is closed under ω-sequences, and $X \cap \lambda$ is cofinal in λ. (Recall that λ is the cardinal successor of κ in $\mathrm{Lp}(A_0)$.) If X is nice, then we let

$$\pi_X \colon H_X \to V_\theta$$

be the anticollapse map. Let

$$\langle \mathcal{T}_X, \xi_X, \kappa_X, \lambda_X, A_X \rangle = \pi_X^{-1}(\langle \mathcal{T}, \xi, \kappa, \lambda, A_0 \rangle),$$

and let

$$b_X = \Sigma(\mathcal{T}_X).$$

(Note here that $\mathcal{T}_X$ has size μ and is according to Σ. It may well be that $b_X \notin H_X$, however.) If $X \prec Y$ and X, Y are nice, let

$$\pi_{X,Y} \colon H_X \to H_Y$$

be the collapse of the inclusion map, and set

$$c_{X,Y} = \text{ downward closure in } \mathcal{T}_Y \text{ of } \pi_{X,Y}``b_X.$$

Definition 1.26 *X is* $\mathcal{T}$-stable *iff X is nice, and*

$$\forall Y (Y \textit{ is nice and } X \prec Y \Rightarrow c_{X,Y} \subseteq b_Y).$$

We show

Claim There is a nice Z such that whenever $Z \prec X$ and X is nice, then X is $\mathcal{T}$-stable.

Proof. There are three cases, based on the cofinality of ξ.

Suppose first that $\omega_1 \leq \mathrm{cof}(\xi) \leq \mu$. We show that all nice X are $\mathcal{T}$-stable. For let X, Y be nice and $X \prec Y$; then as $\mathrm{crit}(\pi_{X,Y}) > \mu$, $\pi''_{X,Y}\xi_X$ is cofinal in ξ_Y. It follows that $c_{X,Y}$ is a cofinal branch of $\mathcal{T}_Y$. But branches in iteration trees are closed as sets of ordinals, and $\mathrm{cof}(\xi_Y) > \omega$, so $c_{X,Y} = b_Y$, as desired.

Now suppose $\mathrm{cof}(\xi) = \omega$, and suppose toward contradiction that no X is $\mathcal{T}$-stable. We can then form an elementary chain $\langle X_\nu | \nu < \mu^+ \rangle$ which is continuous at limit ordinals, and such that X_ν is nice whenever ν is a successor ordinal or a limit ordinal of uncountable cofinality[11], and such that

$$c_{X_\nu, X_{\nu+1}} \neq b_{X_{\nu+1}}$$

[11]Recall that a nice X must be closed under ω-sequences.

whenever ν is a successor or has uncountable cofinality. To save ink, let us drop the "X" in all our subscripts, so that $b_{X_\nu} = b_\nu$, $c_{X_\nu, X_{\nu+1}} = c_{\nu,\nu+1}$, and so forth. Now notice that

$$\mathrm{cof}(\nu) > \omega \Rightarrow b_\nu \in H_\nu,$$

since b_ν is determined by any ω-sequence cofinal in it, and H_ν is closed under ω-sequences. The usual Fodor argument then gives us a stationary $S \subseteq \mu^+$ such that $\nu \in S \Rightarrow \mathrm{cof}(\nu) > \omega$ and

$$\nu, \gamma \in S \wedge \nu < \gamma \Rightarrow \pi_{\nu,\gamma}(b_\nu) = b_\gamma.$$

Fix $\nu, \gamma \in S$ such that $\nu < \gamma$. Clearly, $\pi_{\nu+1,\gamma}$ witnesses that $(\mathcal{T}_{\nu+1})^\frown \pi_{\nu,\nu+1}(b_\nu)$ is a hull of $\mathcal{T}_\gamma{}^\frown b_\gamma$, and is therefore according to Σ. This means that $c_{\nu,\nu+1} = b_{\nu+1}$, a contradiction.

So if $\mathrm{cof}(\xi) = \omega$, then there is a $\mathcal{T}$-stable Z. But then Z witnesses the claim. For let $Z \prec X \prec Y$, where X, Y are nice; then since $\pi_{Z,X}$ and $\pi_{X,Y}$ are continuous at ξ_Z and ξ_X respectively,

$$\begin{aligned} c_{X,Y} &= \text{downward closure in } \mathcal{T} \text{ of } \pi_{X,Y}\text{"}b_X \\ &= \text{downward closure in } \mathcal{T} \text{ of } \pi_{X,Y} \circ \pi_{Z,X}\text{"}b_Z \\ &= c_{Z,Y} = b_Y, \end{aligned}$$

as desired.

Finally, suppose $\mathrm{cof}(\xi) \geq \mu$. As in the first case, we show that all nice X are $\mathcal{T}$-stable. Note that if $\mathcal{T}$ has no last normal component, then it has a unique cofinal branch which is easily definable from $\mathcal{T}$. It is then easy to see that all nice X are $\mathcal{T}$-stable. So assume $\mathcal{T}$ has a last normal component $\mathcal{U}$, and let $\mathcal{U}_X = \pi_X^{-1}(\mathcal{U})$ whenever X is nice. Let b_X^* be the part of b_X which lies in $\mathcal{U}_X$. Fix a nice X, and let $X \prec Y$ where Y is nice.

Suppose first that $\mathcal{U}_X$ is short. Since X is cofinal in λ, $\mathrm{Lp}(A_X) \cap P(\kappa_X) \subseteq H_X$. But $\mathcal{T}_X$ is coded into A_X since $\xi < \kappa$, and thus $\mathrm{Lp}^\alpha(\mathcal{M}(\mathcal{U}_X)) \subseteq H_X$. Thus $\mathcal{Q}(b_X^*, \mathcal{U}_X) \in H_X$, and as usual, we can recover b_X^* from its $\mathcal{Q}$-structure, so that $b_X^* \in H_X$, and hence $b_X \in H_X$. But then $\pi_{X,Y}(b_X)$ is a cofinal branch of $\mathcal{T}_Y$, and since $\mathrm{lh}(\mathcal{T}_Y)$ has uncountable cofinality in V, we have $\pi_{X,Y}(b_X) = b_Y$. Clearly, $c_{X,Y}$ is an initial segment of $\pi_{X,Y}(b_X)$, so $c_{X,Y} \subseteq b_Y$, as desired.

Suppose next that $\mathcal{U}_X$ is maximal. We have then by 1.21 that

$$i_{0,b_X}^{\mathcal{T}_X}(\delta^{\mathcal{N}}) = \delta(\mathcal{U}_X).$$

Letting $\sup(c_{X,Y}) = \eta$, we get, since η has uncountable cofinality, that $c_{X,Y} = b_Y$ if $\eta = \mathrm{lh}(\mathcal{T}_Y)$. So assume $\eta < \mathrm{lh}(\mathcal{T}_Y)$, in which case we get

$$c_{X,Y} = [0, \eta]_{T_Y}.$$

We claim that

$$i_{0,\eta}^{\mathcal{T}_Y}(\delta^{\mathcal{N}}) = \delta(\mathcal{T}_Y \restriction \eta).$$

For let $\rho < \delta^{\mathcal{N}}$; we can then find $\beta \in b_X^*$ such that $\mathrm{crit}(i_{\beta,b_X}^{\mathcal{T}_X}) > i_{0,\beta}^{\mathcal{T}_X}(\rho)$, because $\delta^{\mathcal{N}}$ is sent to $\delta(\mathcal{T}_X)$ by $i_{0,b_X}^{\mathcal{T}_X}$. Although b_X may not be in the domain of $\pi_{X,Y}$, nevertheless $\mathcal{T}_X$ is, and this is enough to conclude that $\mathrm{crit}(i_{\gamma,\eta}^{\mathcal{T}_Y}) > i_{0,\gamma}^{\mathcal{T}_Y}(\rho)$, where $\gamma = \pi_{X,Y}(\beta)$. Thus $i_{0,\eta}^{\mathcal{T}_Y}(\rho) < \delta(\mathcal{T}_Y \restriction \eta)$. Since $i_{0,\eta}^{\mathcal{T}_Y}$ is continuous at $\delta^{\mathcal{N}}$, we have $i_{0,\eta}^{\mathcal{T}_Y}(\delta^{\mathcal{N}}) = \delta(\mathcal{T}_Y \restriction \eta)$.

But now recall that $\mathcal{T}$ is a tree on $\mathcal{N}|\delta^{\mathcal{N}}$, and that it is a composition of normal trees. Since $\mathrm{lh}(E_\eta^{\mathcal{T}_Y}) < i_{0,\eta}^{\mathcal{T}_Y}(\delta^{\mathcal{N}}) = \delta(\mathcal{T}_Y \restriction \eta)$, $\mathcal{T}_Y$ is not a "normal continuation" of $\mathcal{T}_Y \restriction \eta$, which implies that

$$\mathcal{T}_Y = (\mathcal{T}_Y \restriction \eta)^\frown \mathcal{V},$$

where $\mathcal{V}$ is a tree on $\mathcal{M}_\eta^{\mathcal{T}_Y}$. Thus

$$c_{X,Y} = [0, \eta]_{T_Y} \subseteq b_Y,$$

as desired. This proves the claim. □

We can now define $\Gamma_{\xi+1}(\mathcal{T})$. For any nice X, let

$$c_X = \text{ downward closure in } \mathcal{T} \text{ of } \pi_X``b_X,$$

and let

$$\Gamma_{\xi+1}(\mathcal{T}) = \bigcup\{c_X \mid X \text{ is } \mathcal{T}\text{-stable }\}.$$

It is not hard to see that $\Gamma_{\xi+1}(\mathcal{T})$ is a cofinal wellfounded branch of $\mathcal{T}$. It is also not hard to see that $\Gamma_{\xi+1}$ condenses well. This completes the definition of Γ_ξ for $\xi < \kappa$.

The definition of $\Gamma_{\xi+1}$ from Γ_ξ when $\xi \geq \kappa$ is quite similar. Given $\mathcal{T}$ of length ξ by Γ_ξ, let

$$\lambda^* = (\xi^+)^{\mathrm{Lp}(\mathcal{T})}.$$

We now modify the notion of *nice* X by replacing $|X| = \mu$ with the requirement that $|X| < \kappa$, and demanding that X be cofinal in λ^*. Since $\xi \geq \kappa$ and Γ_ξ condenses well, all $\mathcal{T}_X$ for X nice are according to Γ_ξ. This enables us to duplicate the argument above, completing the proof of 1.25. □

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The reader may wonder how 1.25 fits with our plan of constructing a mouse-closure function $A \mapsto \mathcal{M}(A)$ defined on A of size κ as the basis for climbing the next projective-like hierarchy. In the weak gap case, we shall construct $\mathcal{N}$ and Σ satisfying the hypotheses of 1.25, and by 1.25, we can then assume Σ is a (κ^+, κ^+)-iteration strategy. For A which code $\mathcal{N}$ is some specified way, we let

$$L^\Sigma(A) = \text{ minimal } \Sigma\text{-closed model of height } \kappa^+ \text{ containing } A.$$

Since Σ condenses well, $L^{\Sigma}(A)$ has a fine structure above $\sup(A)$, and in particular satisfies $\Box$ above $\sup(A)$. Covering arguments will then give

$$\mathcal{M}(A) = L^{\Sigma}(A)^{\sharp}$$

exists for all bounded $A \subseteq \kappa^{+}$. This operator will provide our starting point in climbing the next projective-like hierarchy.

1.4 Details in the inadmissible case

We now fill in a more-or-less complete proof of $W^*_{\alpha+1}$ from W^*_{α} in the case α begins a Σ_1 gap in $L(\mathbb{R}^g)$, $J_\alpha(\mathbb{R}^g)$ is inadmissible, and α has uncountable cofinality in $V[g]$. The countable cofinality and successor cases are similar, but somewhat simpler. Since α is a limit ordinal, we have W_α by Lemma 1.11.

Let $\phi(v_0, v_1)$ and $x \in \mathbb{R}^g$ determine the failure of admissibility, so that ϕ is Σ_1,

$$\forall y \in \mathbb{R}^g \exists \beta < \alpha J_\beta(\mathbb{R}^g) \models \phi[x, y],$$

and letting $\beta(x, y)$ be the least such β,

$$\alpha = \sup\{\beta(x, y) \mid y \in \mathbb{R}^g\}.$$

(Since α begins a gap, $J_\alpha(\mathbb{R}^g)$ is the Σ_1 hull of its reals, so the parameter from which a failure of admissibility is defined can be taken to be a real.) Let $x = \tau^g$.

Let $p \in g$ force over V all the properties of τ which we have listed as properties of x in $V[g]$ so far. Let $\mu < \nu < \mu^+$, and $A \subseteq \nu$, and suppose A codes up τ in some simple way.

Notice that if $\mathcal{M}$ is an A-premouse, and $G \times H$ is $\mathcal{M}$-generic over $\mathrm{Col}(\omega, \mu) \times \mathrm{Col}(\omega, \nu)$, then $\mathcal{M}[G][H]$ can be regarded as a z-premouse, where $z = z(G, H)$ is a real obtained in some simple fashion from G, H, and A, and which in turn codes G, H, and A in some simple fashion. (See [18].) Also, there is a term $\sigma = \sigma_A$ defined uniformly from A in $\mathcal{M}$ such that whenever $G \times H$ is generic as above, then $\sigma^{G \times H} \in \mathbb{R}$ and

$$(\sigma^{G \times H})_0 = \tau^G,$$

and

$$\{(\sigma^{G\times H})_i \mid i > 0\} = \{\rho^{G \times H} \mid \rho \in L_1(A) \text{ and } \rho^{G\times H} \in \mathbb{R}\}.$$

Here $(w)_i$ is the i^{th} real coded into the real w, in some fixed simple way, and $L_1(A)$ is the first level of Gödel's L over A. For $n < \omega$, let ϕ^*_n be the Σ_1 formula

$$\phi^*_n(v) = \exists\alpha(J_\alpha(\mathbb{R}) \models \forall i \in \omega(i > 0 \Rightarrow \phi((v)_0, (v)_i)) \wedge (\alpha + \omega n) \text{ exists }).$$

Now let ψ be the natural sentence in the language of A-premice (having therefore a name for A) such that for any A-premouse $\mathcal{M}$:

$$\mathcal{M} \models \psi$$

iff whenever $G \times H$ is $\mathcal{M}$-generic over $\mathrm{Col}(\omega, \mu) \times \mathrm{Col}(\omega, \nu)$ and $p \in G$, then for any n there is a $\gamma < o(M)$ such that

$$\mathcal{M}[z(G,H)]|\gamma \text{ is a } \langle \phi_n^*, \sigma_A^{G\times H}\rangle\text{- witness.}$$

Definition 1.27 *For any bounded subset A of κ^+, $\mathcal{M}(A)$ is the shortest initial segment of* $\mathrm{Lp}(A)$ *which satisfies ψ, if it exists, and is undefined otherwise.*

Lemma 1.28 *For any bounded subset A of μ^+ coding τ in the specified way, $\mathcal{M}(A)$ exists, and moreover, $\mathcal{M}(A)$ is ordinal definable from A over $J_\gamma(\mathbb{R}^g)$, for some $\gamma < \alpha$.*

Sketch of Proof. Since we stated W_α as a closure condition on the mice over reals in $V[g]$, there is some work to be done in going back to V, as is done in this lemma.

Working in $V[g]$ we have an iterable mouse $\mathcal{N}$ over (A, g) such that whenever H is $\mathcal{N}$-generic for $\mathrm{Col}(\omega, \sup(A))$, then $\mathcal{N}[H]$ (regarded as a mouse over $z(g \times H)$) is a $\langle \phi^*, \sigma_A^{G\times H}\rangle$-witness. Moreover, $\mathcal{N}[H]$ has an iteration strategy in $J_\alpha(\mathbb{R}^g)$. (We use $\mathrm{cof}^{V[g]}(\alpha) > \omega$ at this point, as we have to sup over the stages $\gamma < \alpha$ at which Σ_1 witnesses for $\phi(\tau^g, \sigma^h)$, for $\sigma \in L_1(A)$, have iteration strategies.) Now let $\mathcal{M}(A)$ be the structure constructed from A and the extender sequence of $\mathcal{N}$. One can show that in $V[g]$, $\mathcal{M}(A)$ is an iterable mouse over A such that $\mathcal{M}(A)[g] = \mathcal{N}$. But then $\mathcal{M}(A)$ is in V by the homogeniety of $\mathrm{Col}(\omega, \mu)$, as is the restriction of the canonical iteration strategy for $\mathcal{M}(A)$. $\square$

The reader should see the proof of Theorem 3.9 of [18] for a much more detailed version of this argument.

Lemma 1.29 *The $\mathcal{M}$-operator relativises well at μ.*

Proof. Let A, B have size μ, with A coded into B and τ coded into A. We give an informal description of how to compute $\mathcal{M}(A)$ from A and $\mathcal{M}(B)$, and leave it to the reader to convert this into a formula which defines $\mathcal{M}(A)$ from A and $\mathcal{M}(B)$ over any transitive model P of ZFC^- containing $\mathcal{M}(B)$. Our descriptions will make use of generic extensions of $\mathcal{M}(A)$ and $\mathcal{M}(B)$, but in the end we are only using the forcing relations of these models, so the description works in P.

Let $\sup(A) = \nu$, and $G \times H$ be $\mathcal{M}(A)$-generic over $\mathrm{Col}(\omega, \mu) \times \mathrm{Col}(\omega, \nu)$ and $p \in G$. For any n, let γ_n be the least γ such that $\mathcal{M}[z(G,H)]|\gamma$ is a $\langle \phi_n^*, \sigma_A^{G\times H}\rangle$- witness. Since $o(\mathcal{M}(A)) = \sup_n \gamma_n$, it is enough to fix n, and recover $\mathcal{M}(A)|\gamma_n$ from A and $\mathcal{M}(B)$. For this, let β_n be the least β such that $J_\beta(\mathbb{R}^g) \models \varphi_n^*[\sigma_A^{G\times H}]$. Our capturing hypothesis W_α, together with the proof of 1.28, guarantees that $\mathcal{M}(A)|\gamma_n$ has an iteration strategy in $J_{\beta_n+1}(\mathbb{R}^g)$, and thus $\mathcal{M}(A)|\gamma_n$ is definable from A over $J_{\beta_n+1}(\mathbb{R}^g)$. Now let $\sup(B)=\eta$, and let I be such that $G\times I$ is $\mathcal{M}(B)$-generic over $\mathrm{Col}(\omega,\mu)\times\mathrm{Col}(\omega,\eta)$, and H is coded into I. Let θ be least such that $J_\theta(\mathbb{R}^g)\models\varphi^*_{n+1}[\sigma_B^{G\times I}]$.i It is easy to see that $\beta_n<\theta$. Since $\mathcal{M}(B)[z(G\times I]$ has an initial segment which is a $\langle\varphi^*_{n+1},\sigma_B^{G\times I}\rangle$-witness, it "knows" the theory of $J_\theta(\mathbb{R}^g)$. Using the homogeneity of the relevant forcings, we get that $\mathcal{M}(A)|\gamma_n\in\mathcal{M}(B)$, and a way of defining it over $\mathcal{M}(B)$. □

From our work in the section on lifting closure at μ to closure at κ, we get immediately

Corollary 1.30 *For all A bounded in κ^+ and coding τ as specified, $\mathcal{M}(A)$ exists.*

We now climb the finitely-many-$\mathcal{M}$-closed-Woodins hierachy in completely standard fashion, using the basic argument getting one Woodin cardinal from failure of square at a singular strong limit from [8]. For A a bounded subset of κ^+ coding τ in the specified way, and $\mathcal{P}$ an A-premouse, let us say that $\mathcal{P}$ is *$\mathcal{M}$-closed* iff for all $\xi<\mathrm{OR}\cap\mathcal{P}$ such that $\xi\geq\sup(A)$ and $\mathcal{P}\models\xi$ is a cardinal, $\mathcal{M}(\mathcal{P}|\xi)\trianglelefteq\mathcal{P}$.

Definition 1.31 *For any $n\geq 0$, let $P_n^\sharp(A)$ be the minimal active countably iterable $\mathcal{M}$-closed mouse over A having n Woodins, if there is such a mouse, and undefined otherwise.*

Let us say that an A-premouse $\mathcal{Q}$ is *$P_n^\sharp$-closed* iff for all $\xi<\mathrm{OR}\cap\mathcal{Q}$ such that $\xi\geq\sup(A)$ and $\mathcal{Q}\models\xi$ is a cardinal, $P_n^\sharp(\mathcal{Q}|\xi)\trianglelefteq\mathcal{Q}$.

Definition 1.32 *For any A bounded in κ^+ and coding τ as specified, let $P_n^{\sharp\sharp}(A)$ be the minimal active countably iterable $P_n^\sharp$-closed A-premouse, if there is one, and undefined otherwise.*

Lemma 1.33 *For all $n<\omega$ and all A coding τ and bounded in κ^+, $P_n^\sharp(A)$ exists.*

Proof. The proof is by induction on n, considering first the case $\sup(A)<\mu^+$, then lifting to arbitrary A as done above. (At the same time, we get that all $P_n^\sharp(A)$ are (κ^+,κ^+)-iterable in V.)

For $n=0$, we get that $P_0^\sharp(A)$ exists for all A bounded in μ^+, by a covering argument applied to the minimal $\mathcal{M}$-closed model over A of height κ^+, noting that this model does not compute κ^+ correctly. It is also important here that the $\mathcal{M}$-operation condenses to itself. The basic arguments here familiar, but there are some subsidiary points at which care is needed. First, our notion of $\mathcal{M}$-closure requires only that the *levels* of the model be closed under $\mathcal{M}$. One obtains the minimal $\mathcal{M}$-closed model $\mathcal{N}$ over A as follows: set $\mathcal{N}_0=L_1(A),\mathcal{N}_{\alpha+1}=\mathcal{M}(\mathcal{N}_\alpha)$, and $\mathcal{N}_\lambda=\bigcup_{\eta<\lambda}\mathcal{N}_\eta$ for $\lambda\leq\kappa^+$ a limit ordinal; and then take $\mathcal{N}=\mathcal{N}_{\kappa^+}$. One can show that $\mathcal{N}$ is a mouse, and in particular *all* its levels are sound, not just those of the form $\mathcal{N}_\eta$. The proof is the same as that for L, the key being that the $\mathcal{M}$ operation condenses to itself. The model $\mathcal{N}$ is closed under the $\mathcal{M}$-operation on arbitrary sets simply coding A, and indeed can define this operation from its extender sequence, because the $\mathcal{M}$-operation relativises well. (This is where we use the full strength of our notion of relativising well.) It would have been awkward to have built $\mathcal{N}$ by closing explicitly under the $\mathcal{M}$-operation on arbitrary sets coding A, because we need a model which has a fine structure theory at all its levels.

It is easy to see that the $P_0^\sharp$ operator relativises well, using 1.29. Also, $P_0^\sharp(A)$ is the first initial segment of $\mathrm{Lp}(A)$ satisfying a certain sentence. Thus our results on lifing mouse-closure apply, and we have that $P_0^\sharp(A)$ exists for all A bounded in κ^+.

We now consider the case $n = 1$. First, we can run the arguments of the last paragraphs one more time and get that for all A bounded in κ^+ and coding τ, $P_0^{\sharp\sharp}(A)$ exists. Let

$$R(A) = \text{ minimal } P_0^\sharp\text{-closed model of height } \kappa^+ \text{ over } A,$$

and let

$$\Omega_A = \text{ first indiscernible for } R(A).$$

Notice that $R(A)$ is closed under, and can define, the $P_0^\sharp$ operator on all its sets which simply code A.

Now given B a bounded subset of μ^+ coding τ, and $A \subset \kappa$ coding A_0 (and perhaps more), we build K^c over B in $R(A)$ below Ω_A via the construction of [13]. Call the resulting model $K^c(B)^A$. Here $K^c(B)^A$ is to be built inside $R(A)$, starting with B, closing under first order definability, adding extenders to the sequence subject to the background condition of [13], and taking cores at each step as in [13].

It is enough to see some such $K^c(B)^A$ reaches a level $\mathcal{Q}$ such that for some δ, $\mathcal{Q} \models \delta$ is Woodin and $P_0^\sharp(\mathcal{Q}|\delta) = \mathcal{Q}$, for then $\mathcal{Q}$ is the desired $P_1^\sharp(B)$. (A codes V_κ, so countable iterabilty is absolute between V and $R(A)$.) We shall say $K^c(B)^A$ *reaches* $P_1^\sharp(B)$ in this case.

Claim. If $K^c(B)^A$ does not reach $P_1^\sharp(B)$, then $R(A) \models K^c(B)^A$ is Ω_A-iterable.

Proof. We work in $R(A)$, which is closed under the $P_{n-1}^\sharp$ operator, so by the usual reflection argument, it is enough to see that the $\mathcal{Q}$ structure for a size μ tree $\mathcal{T}$ on a size μ elementary submodel of $K^c(B)$ exists, and is an initial segment of $P_0^\sharp(\mathcal{M}(\mathcal{T}))$. (All background extenders for the K^c construction were κ-complete in $R(A)$.) For this, granted our smallness assumption on $K^c(B)$, it is enough to see that $K^c(B) \models$ there are no Woodin cardinals. If not, let δ be the largest Woodin of $K^c(B)$. (There must be a largest Woodin of $K^c(B)$, as otherwise $K^c(B)$ reaches $M_\omega^\sharp(B)$, and therefore it reaches $P_1^\sharp(B)$. See remark 1.36 below.) Working in $R(A)$, we compare $P_0^\sharp(K^c(B)|\delta)$ with $K^c(B)$.

The key point is that we have Ω_A-iterability on both sides (in the $K^c(B)$-case, only above δ). In the case of $K^c(B)$, this follows by countable iterability and the standard reflection argument. In the case of $P_0^\sharp(K^c(B)|\delta)$, this is because any $\mathcal{M}$-closed universe is sufficiently correct that it knows how to iterate its mice of the form $\mathcal{M}(C)$. This correctness is implicit in the proof of 1.29 that the $\mathcal{M}$-operator relativises well.

But then, $P_0^\sharp(K^c(B)|\delta)$ must iterate past $K^c(B)$, contradicting the universality of the latter model. $\square$

Assume now toward contradiction that no model $K^c(B)^A$ reaches $P_1^\sharp(B)$. It follows from the claim, and the arguments of [13], that for any A, there is a "true core model" $K(B)^A$ derived from $K^c(B)^A$ in $R(A)$. As in [8], these local $K(B)$'s stack up below their versions of κ^+, and since their square sequences can't stack up to a full square sequence, we can find an A such that $R(A)$ thinks κ is a singular strong limit where $K(B)$ fails to have weak covering. This is a contradiction.

The case $n > 1$ is essentially the same. The additional ingredient we need is

Lemma 1.34 *If $P_n^\sharp(A)$ exists for all A bounded in μ^+ and coding τ, then the $P_n^\sharp$-operator relativises well at μ.*

The proof of 1.34 uses a lemma on the universality of extender models which has some independent interest.

Lemma 1.35 *Let $\mathcal{E} \subseteq V_\delta$ be a collection of extenders such that δ is Woodin via the extenders in $\mathcal{E}$, and let Let W be an extender model built inside V_δ via a maximal construction using background extenders from $\mathcal{E}$. Then W is universal, in the following sense: let $\eta < \delta$ be a cardinal of W, and a cutpoint of W. Let M be a premouse over $W|eta$ which is η-sound and projects to η, with a $\delta + 1$-iteration strategy Σ (for trees above η) such that for all $E \in \mathcal{E}$,*

$$i_E(\Sigma) = \Sigma \cap \mathit{Ult}(V, E).$$

Then $M \trianglelefteq W$.

Proof sketch. We compare M with W. The proof of Theorems 2.5 and 3.2 of [8] shows that W does not move in this comparison. (Thus we need not know how to iterate W in order to do the comparison.) It is enough to see that $\mathcal{M}$ does not iterate past W.

If it does, we have an iteration tree $\mathcal{T}$ on $\mathcal{M}$ such that $W = \mathcal{M}_\delta^{\mathcal{T}}|\delta$. For notational simplicity, let us assume $[0, \delta]_T$ does not drop; otherwise, we can just work beyond its last drop. Let

$$A = \{(\alpha, x, y) \in V_\delta | y \in i^{\mathcal{T}}_{\alpha,\delta)}(x)\},$$

and for $\alpha \in [0, \delta)_T$, let

$$E_\alpha = E_\gamma^{\mathcal{T}}, \text{ where } (\gamma + 1)T\delta \wedge \mathrm{pd}_T(\gamma + 1) = \alpha,$$

and

$$f(\alpha) = \text{ least inaccessible cardinal } > \nu(E_\alpha).$$

Set $f(\alpha) = 0$ if $\alpha \notin [0, \delta]_T$. Let E^W be the extender sequence of W. Since δ is Woodin, we can find an embedding $j: V \to N$ such that for $\kappa = \mathrm{crit}(j)$, we have $\kappa \in [0, \delta]_T$ and $j(f)(\kappa) = f(\kappa), V_{(f)(\kappa)} \in N, j(A) \cap V_{f(\kappa)} = A \cap V_{f(\kappa)}$, and $j(E^W) \cap V_{f(\kappa)} = E^W \cap V_{f(\kappa)}$. By Lemma 11.4 of [5], we have that letting $F = (E_j \cap W)|f(\kappa)$, all proper initial segments of F are on E^W, or an ultrapower away. Since j shifts A to itself, we see that E_κ is compatible with F, and hence an initial segment of F. By the initial segment condition, E_κ is on the W-sequence. This contradicts the fact that E_κ was used in $\mathcal{T}$. □

Proof of 1.34. Let B be simply coded into A, where A and B are bounded subsets of μ^+. We show how to obtain $P_n^\sharp(B)$ from $P_n^\sharp(A)$ in a way which is uniformly definable over models of ZFC^- containing $P_n^\sharp(A)$. For this, let δ be the top Woodin cardinal of $P = P_n^\sharp(A)$, and let W be the result of the K^c-construction inside V_δ^P, using full background extenders from the P-sequence. By lemma 1.35, W is $P_0^\sharp$-closed below its height. (If ξ is a cardinal of W, let E be the first extender overlapping ξ on the W-sequence. It is a first order property of W that $P_0^\sharp(W|\eta) \trianglelefteq W|\eta$ whenever η is a cardinal cutpoint of W. The first order property holds in $\mathrm{Ult}(W, E)$, where ξ is a cardinal cutpoint. Thus $P_0^\sharp(W|\xi) \trianglelefteq \mathrm{Ult}(W, E)$, and hence $P_0^\sharp(W|\xi) \trianglelefteq W$.)

We claim that if $\mathcal{Q}$ is a proper initial segment of $P_0^\sharp(W)$ such that $\delta < o(\mathcal{Q})$, then $\rho_\omega(\mathcal{Q}) \geq \delta$. For if not, working in P, we can take a Skolem hull and get

$$\pi \bar{\mathcal{Q}} \to \mathcal{Q}, \mathrm{crit}(\pi) = \bar{\delta}, \pi(\bar{\delta}) = \delta,$$

with $\bar{\delta}$ a cardinal of W, and $\rho_\omega(\bar{\mathcal{Q}}) < \bar{\delta}$. We can also arrange, by condensation, that $\bar{\mathcal{Q}}$) is a proper initial segment of $P_0^\sharp(W|\bar{\delta})$. But W is $P_0^\sharp$-closed, so $\bar{\mathcal{Q}} \trianglelefteq W$, contrary to the fact that $\bar{\delta}$ is a cardinal of W. This proves our claim.

Since δ is Woodin in P, which is the background universe for a maximal K^c construction giving rise to W, the claim of the last paragraph implies δ is Woodin in $P_n^\sharp(W)$. But then $P_n^\sharp(B)$ is just the core of $P_n^\sharp(W)$, and hence $P_n^\sharp(B)$ can be recovered from P. This proves 1.34. □

We now set, for $A \subseteq V_\kappa$ coding V_κ,

$$R(A) = \text{ minimal } P_{n-1}^\sharp\text{-closed model of height } \kappa^+ \text{ over } A,$$

$$\Omega_A = \text{ first indiscernible for } R(A),$$

and proceed as in the case $n = 1$. This completes the proof of Lemma 1.33. □

Remark 1.36 Some of the complexity in our proof of 1.33 is due to the fact that we chose to *prove* that $K^c(B)^A$ is $\mathcal{M}$-closed, rather than change the K^c-construction so as to explicitly close the levels of the variant K^c under the $\mathcal{M}$-operator. Changing the construction leads to some difficulties.(For example, if $\mathcal{N}$ is a level of the variant construction, then we cannot simply let $\mathcal{M}(\mathcal{N})$ be the next level, as some proper initial segment of $\mathcal{M}(\mathcal{N})$ may project across $o(\mathcal{N})$.) However, these difficulties can be overcome, and the result is a more general argument, in that one has no need to assume one is "below $M_\omega^\sharp$", as we did in the proof given here.

We can now prove $W^*_{\alpha+1}$. Note first

Lemma 1.37 *For any A bounded in μ^+ and coding τ, and any $n < \omega$, $J_{\alpha+1}(\mathbb{R}^g) \models P_n^\sharp(A)$ is ω_1-iterable.*

Sketch of proof. As part of our construction, we have shown that $P_n^\sharp(A)$ has a unique κ^+-iteration strategy Σ in V. One can show that Σ extends uniquely to trees in $V[g]$, basically because the $\mathcal{Q}$-structures used to define Σ can be extended to $V[g]$. Finally, the $\mathcal{M}$-operator extends to $V[g]$, and on sets in $\mathrm{HC}^{V[g]}$ is Σ_1-definable over $J_\alpha(\mathbb{R}^g)$ from τ. As in [21], we then get that the extensions of the $P_n^\sharp$- operators to $\mathrm{HC}^{V[g]}$ are Σ_{3n}-definable over $J_\alpha(\mathbb{R}^g)$, and that the canonical ω_1-iteration strategy for $P_n^\sharp(A)$ is Σ_{3n+1} definable over $J_\alpha(\mathbb{R}^g)$. □

We shall actually prove something slightly stronger than $W^*_{\alpha+1}$, namely that $W_{\alpha+1}$ holds, not for all z, but for a cone of z.

Lemma 1.38 *$W^*_{\alpha+1}$ holds.*

Sketch of proof. Let U be a set of reals in $J_{\alpha+1}(\mathbb{R}^g)$, and $k < \omega$; we seek a coarse (k, U)-Woodin mouse. Suppose that U is Σ_n-definable over $J_\alpha(\mathbb{R}^g)$ from the real parameter z.[12] Let $z = \rho^g$, and

$$P = P^\sharp_{k+n+2}(\langle \tau, \rho \rangle).$$

We show that P is the desired witness.

Let δ_0 be the largest Woodin cardinal of P, and δ_1 the next-to-largest. Let W be the universal $\Sigma_1^{J_\alpha(\mathbb{R}^g)}$ set of reals, and θ a Σ_1 formula which defines it over $J_\alpha(\mathbb{R}^g)$. Let Σ be the canonical iteration strategy for P, and hence for $P[g]$. There is a term $\dot{W} \in P[g]$ such that whenever

$$i \colon P[g] \to Q[g]$$

is an iteration map by Σ, and h is $\mathrm{Col}(\omega, i(\delta_1))$- generic over $Q[g]$, and $y \in \mathbb{R} \cap Q[g][h]$, then

$$y \in W \Leftrightarrow y \in i(\dot{W})^h.$$

Roughly speaking, the term $\dot{W}$ asks: if we Levy collapse δ_0 via l, and then using $\mathcal{M}(P[g][h][l])|\delta_0$ as our oracle for the theory of the first level of $L(\mathbb{R})$ at which $\phi(x, \sigma^l)$ is seen to be true for

[12]There is a lightface Σ_1 partial map of $\mathbb{R}^g$ onto $J_\alpha(\mathbb{R}^g)$.

all terms $\sigma \in L_1(P[g][h]|\delta_0$, do we see that $\theta(y)$ has been verified before that level? Since any real t can be obtained as such a σ^l after an iteration of $Q[g][h]$ above $i(\delta_1)$ and below $i(\delta_0)$, and since the ordinals we called $\beta(x,t)$ were cofinal in α, $\dot{W}$ behaves as advertised.

Since α is inadmissible and begins a gap, the Σ_n theory of $J_\alpha(\mathbb{R}^g)$ can be computed from the Σ^1_n theory of $(\mathbb{R}, W, x)$. Let δ be the k^{th} Woodin cardinal (from the bottom) of $P[g]$. Using the Woodins above δ to answer one-real-quantifier questions as above, we get a term $\dot{U}$ in $P[g]$ such that if h is P-generic over $\mathrm{Col}(\omega, \delta)$ and y is a real in $P[g][h]$, then

$$y \in U \Leftrightarrow y \in \dot{U}^h.$$

Moreover, letting $\gamma = (\delta_0^+)^P$, and $\pi\colon \mathcal{Q}[g] \to P[g]|\gamma$ and $\pi(\dot{Z}) = \dot{U}$, and h is $\mathcal{Q}[g]$ generic over $\mathrm{Col}(\omega, \pi^{-1}(\delta))$, then again, $\dot{Z}^h = U \cap \mathcal{Q}[g][h]$.[13] In $P[g]$ we can then construct the absolutely complementing trees S and T required by 1.5: T_y tries to build $\pi, \mathcal{Q}, h$ as above with $y \in \dot{Z}^h$, and S_y tries to build $\pi, \mathcal{Q}, h$ as above with $y \notin \dot{Z}^h$. □

1.5 Details in the end-of-gap gap case

Now let β be critical, and suppose that case 3 holds. Let α be the sup of the critical ordinals $< \beta$, so that either $[\alpha, \beta]$ is a proper weak gap (case 3(a)), or $[\alpha, \beta - 1]$ is a perhaps improper strong gap (case 3(b)). We have W^*_β, and hence W_α, by induction. We shall prove $W^*_{\beta+1}$.

The main thing we need here is a mouse closure operation which will serve as the basis for a more-or-less standard induction on the projective-like hierarchy on the sets of reals definable over $J_\beta(\mathbb{R}^g)$. As explained above, the mice under which we close will be hybrid mice, like ordinary $L[\vec{E}]$-mice, but having an iteration strategy Σ for some suitable $\mathcal{N}$ fed in in addition to the extenders (all of which have critical point $> \mathrm{OR}^{\mathcal{N}}$). The appropriate $\mathcal{N}$ and Σ are given by

Lemma 1.39 *There is, in V, a suitable $\mathcal{N}$ and a fullness-preserving μ^+-iteration strategy Σ for $\mathcal{N}$ such that Σ condenses well.*

Remark 1.40 The reader may notice that we have backtracked a bit, in that Σ is only a μ^+-iteration strategy, rather that a (μ^+, μ^+)-strategy. That is, it only operates on normal iteration trees. This will be enough for our purpose. It is possible to get a full (μ^+, μ^+)-iteration strategy, but this involves Woodin's theory of quasi-iterability, and we prefer to avoid the extra complexity that introduces.

Proof. We work in $V[g]$ for a while. Recall that $\mathrm{OD}^\gamma(z)$ is the collection of sets which are ordinal definable from z over $J_\gamma(\mathbb{R}^g)$; we write $\mathrm{OD}^{<\xi}(z)$ for $\bigcup_{\gamma<\xi} \mathrm{OD}^\gamma(z)$.

[13]This fact about Skolem hulls follows from the construction. It comes down to the fact that an elementary submodel of an iterable structure is still iterable.

Let $\langle A_i | i \in \omega\rangle$ be a self-justifying system, with each $A_i \in J_\beta(\mathbb{R}^g)$, and A_0 the universal $\Sigma_1^{J_\alpha(\mathbb{R}^g)}$ set. Let

$$x^* = \tau^g$$

be a real such that for all i, A_i is $\mathrm{OD}^{<\beta}(x^*)$. Here τ is (essentially) a subset of μ, and of course $\tau \in V$. The suitable $\mathcal{N}$ we seek will be a τ-mouse.

We need some concepts and results, due to Woodin, which are explained in more detail in [16] and [15]. First

Lemma 1.41 (Woodin) *Let $\mathcal{N}$ be a suitable premouse over some $z \in HC$ which simply codes x^*, and let $\nu \geq \delta^{\mathcal{N}}$ be a cardinal of $\mathcal{N}$, let $A \subseteq \mathbb{R}^g$ be $OD^{<\beta}(z)$; then there is a term $\sigma \in \mathcal{N}$ such that whenever h is $\mathcal{N}$-generic for* $\mathrm{Col}(\omega, \nu)$*, then*

$$\sigma^h = A \cap \mathcal{N}[h].$$

Definition 1.42 *For $\mathcal{N}$, z,ν, and A as in 1.41, $\tau^{\mathcal{N}}_{A,\nu}$ is the unique standard term σ such that $\sigma^h = A \cap \mathcal{N}[h]$ for all* $\mathrm{Col}(\omega, \nu)$*-generics h over $\mathcal{N}$. We write $\tau^{\mathcal{N}}_A$ for $\tau^{\mathcal{N}}_{A,\delta}$, where $\delta = \delta^{\mathcal{N}}$.*

See [16] for further explanation. Woodin proved the following key condensation result:

Theorem 1.43 (Woodin) *Let $\mathcal{N}$ be a suitable premouse over $z \in HC$, and let $\mathcal{B}$ be a self-justifying family of subsets of $\mathbb{R}^g$ containing the universal $\Sigma_1^{J_\alpha(\mathbb{R}^g)}$ set, and such that each $B \in \mathcal{B}$ is $OD^{<\beta}(z)$. Suppose*

$$\pi : \mathcal{M} \to \mathcal{N}$$

is Σ_0-elementary and such that

$$\forall B \in \mathcal{B} \forall \nu \geq \delta^{\mathcal{N}} \tau^{\mathcal{N}}_{B,\nu} \in \mathrm{ran}(\pi).$$

Then $\mathcal{M}$ is suitable, and for all $B \in \mathcal{B}$,

$$\pi(\tau^{\mathcal{M}}_{B,\bar{\nu}}) = \tau^{\mathcal{N}}_{B,\nu},$$

where $\pi(\bar{\nu}) = \nu$.

See [15].

Definition 1.44 *If $\mathcal{N}$ is suitable, and $\mathcal{T}$ is a maximal normal iteration tree on $\mathcal{N}$, then $\mathcal{M}(\mathcal{T})^+$ is the unique suitable $\mathcal{P}$ such that $\mathcal{M}(\mathcal{T}) = \mathcal{P}|\delta^{\mathcal{P}}$.*

Definition 1.45 *Let $\mathcal{N}$ be suitable z-premouse, where $z \in HC$ and codes x^*, and $A \subseteq \mathbb{R}^g$ be $OD^{<\beta}(z)$. We say $\mathcal{N}$ is* weakly A-iterable *just in case for all $n < \omega$, there is a fullness-preserving winning strategy Σ for* II *in the iteration game $\mathcal{G}(\omega, n, \omega_1)$*[14] *such that whenever*

$$i: \mathcal{N} \to \mathcal{P}$$

is an iteration map produced by an iteration according to Σ, then

$$i(\tau^{\mathcal{N}}_{A,\nu}) = \tau^{\mathcal{P}}_{A,i(\nu)}$$

for all cardinals $\nu \geq \delta^{\mathcal{N}}$ of $\mathcal{N}$.

We should remark that if $\mathcal{N}$ is weakly A-iterable, and Σ, Γ are iteration strategies for $\mathcal{G}(\omega, n, \omega_1)$ and $\mathcal{G}(\omega, k, \omega_1)$ witnessing this with $n \leq k$, then Σ and Γ can only disagree at some maximal normal component $\mathcal{U}$, and then their disagreement has no effect on the remainder of either game, since they agree that $\mathcal{M}(\mathcal{U})^+$ will be the base model for the next normal component. In particular, any model reached using Σ is itself weakly A-iterable.

We rely heavily on the following basic result of Woodin.

Theorem 1.46 (Woodin) *Let $z \in HC^{V[g]}$, and let $A \subseteq \mathbb{R}^g$ be $OD^{<\beta}(z)$; then there is a suitable, weakly A-iterable z-premouse.*

The reader can find a proof of 1.46, in the weak gap case, outlined in [20]. (See lemma 1.12.1 there.) We also need the result in the strong gap case, where we do not yet know the proof.

Theorem 1.46, together with our self-justifying system, yields a fullness-preserving strategy that condenses well, as we now show.

Definition 1.47 *Let $\mathcal{N}$ be a suitable z-premouse, and $\mathcal{A}$ a collection of $OD^{<\beta}(z)$ sets of reals; then we say $\mathcal{N}$ is* weakly $\mathcal{A}$-iterable *iff for all finite $F \subseteq \mathcal{A}$, $\mathcal{N}$ is weakly $\oplus F$-iterable, where $\oplus F$ is the join of the sets of reals in F.*

Corollary 1.48 (Woodin) *Let $\mathcal{A}$ be a countable collection of $OD^{<\beta}(z)$ sets of reals, where $z \in HC^{V[g]}$ and codes x^*; then there is a suitable, weakly $\mathcal{A}$-iterable z-premouse.*

Proof. For each $F \subseteq \mathcal{A}$ finite, we have by theorem 1.46 a suitable, weakly $\oplus F$-iterable $\mathcal{N}_F$. Let Σ_F be a fullness-preserving strategy for II in $\mathcal{G}(\omega, 1, \omega_1)$ for $\mathcal{N}_F$. We now simultaneously coiterate all the $\mathcal{N}_F$, using Σ_F to iterate $\mathcal{N}_F$.

Claim. The coiteration ends successfully at some countable ordinal.

[14]The output of this game is a linear stack of n normal iteration trees, the first one being on $\mathcal{N}$.

Proof. Let M be the Lp^α-closure of $\langle \mathcal{N}_F | F \in [\mathcal{A}]^{<\omega}\rangle$, so that $\omega_1^M < \omega_1$. (This is true because any nice hull of size μ is Lp^α-closed, as Lp^α relativises well.) M can track the coiteration generated by the Σ_F until some maximal tree $\mathcal{U}_F$ on $\mathcal{N}_F$ is produced. (Note that coiterations always generate normal trees.) But as soon as that happens, the coiteration is over. For let $\mathcal{P}_G$ be the next model selected by Σ_G to continue $\mathcal{U}_G$, for all $G \in [\mathcal{A}]^{<\omega}$. As $\mathcal{U}_F$ is maximal, $\mathcal{P}_F = \mathcal{M}(\mathcal{U}_F)^+$ is suitable. If $\mathcal{N}_G$-to-$\mathcal{P}_G$ drops, then because Σ_G is fullness-preserving, $\mathcal{M}(\mathcal{U}_F)$ has a $\mathcal{Q}$-structure in $\mathrm{Lp}^\alpha(\mathcal{M}(\mathcal{U}_F))$[15], a contradiction. But then $\mathcal{P}_G$ is suitable, and the minimality condition in suitability easily implies $\mathcal{P}_G = \mathcal{P}_F$, for all G.

The usual regressive function argument shows the coiteration cannot be tracked in M for $\omega_1^M + 1$ steps. Thus it must terminate successfully at some stage $\leq \omega_1^M$. This proves the claim. □

The proof of the claim also shows that if $\mathcal{P}_F$ is the last model on the tree $\mathcal{U}_F$ produced in the successful coiteration by Σ_F, then no branch $\mathcal{N}_F$-to-$\mathcal{P}_F$ drops, and $\mathcal{P}_F = \mathcal{P}_G$ for all F, G. (*Some* branch doesn't drop by general coiteration theory, and then the proof of the claim gives the rest.) It is clear that the common last model $\mathcal{P}$ is suitable, and weakly $\mathcal{A}$-iterable. □

Definition 1.49 *Let $\mathcal{N}$ be a suitable z-premouse, where $z \in HC^{V[g]}$, let $\mathcal{A}$ be a collection of $OD^{<\beta}(z)$ subsets of $\mathbb{R}^g$, and let Σ be an ω_1-iteration strategy for $\mathcal{N}$. We say Σ is* guided by *$\mathcal{A}$ just in case Σ is fullness preserving, and whenever $\mathcal{T}$ is a countable (necessarily normal) iteration tree by Σ of limit length, and $b = \Sigma(\mathcal{T})$, then*

(a) if $\mathcal{T}$ is short, then $\mathcal{Q}(b, \mathcal{T})$ exists and $\mathcal{Q}(b, \mathcal{T}) \in \mathrm{Lp}^\alpha(\mathcal{M}(\mathcal{T}))$, and

(b) if $\mathcal{T}$ is maximal, then
$$i_b(\tau^{\mathcal{N}}_{A,\nu}) = \tau^{\mathcal{M}^{\mathcal{T}}_b}_{A,i(\nu)}$$
for all $A \in \mathcal{A}$ and cardinals $\nu \geq \delta^{\mathcal{N}}$ of $\mathcal{N}$.

Notice that in case (b) above, b does not drop and $\mathcal{M}_b = \mathcal{M}(\mathcal{T})^+$, as Σ is fullness-preserving.

Theorem 1.50 (Woodin) *Let $z \in HC^{V[g]}$, and let $\mathcal{A}$ be a countable, self-justifying system of $OD^{<\beta}(z)$ sets which contains the universal $\Sigma_1^{J_\alpha(\mathbb{R}^g)}$ set. Then there is a suitable z-premouse $\mathcal{N}$, and a unique fullness-preserving ω_1-iteration strategy for $\mathcal{N}$ which is guided by $\mathcal{A}$; moreover, this strategy condenses well.*

[15]Some initial segment of $\mathcal{P}_G$ is a $\mathcal{Q}$-structure for $\mathcal{M}(\mathcal{U}_F)$ because of the drop. This $\mathcal{Q}$-structure cannot lie beyond $\mathrm{Lp}^\alpha(\mathcal{M}(\mathcal{U}_F))$, as otherwise $\mathcal{P}_G$ would have a suitable initial segment.

Proof. By 1.48, we have a suitable z-premouse $\mathcal{N}$ which is weakly $\mathcal{A}$-iterable. Let $\mathcal{A} = \{A_k \mid k < \omega\}$, and for each $k < \omega$, let Γ_k be a fullness-preserving ω_1-iteration strategy witnessing that $\mathcal{N}$ is weakly $A_0 \oplus ... \oplus A_k$-iterable. The desired strategy Σ will be a sort of limit of the Γ_k.

So long as all Γ_k agree, Σ simply plays according to their common prescription. So suppose $\mathcal{T}$ is a normal tree of limit length which has been played according to all Γ_k, but there are k and l such that $\Gamma_k(\mathcal{T}) \neq \Gamma_l(\mathcal{T})$. Since the Γ_k are fullness-preserving and guided by Lp^α $\mathcal{Q}$-structures when these exist, $\mathcal{T}$ must be maximal, and letting

$$b_k = \Gamma_k(\mathcal{T})$$

for all k, and

$$i_k : \mathcal{N} \to \mathcal{M}(\mathcal{T})^+ = \mathcal{M}^{\mathcal{T}}_{b_k}$$

be the canonical embedding, we have that i_k moves the term relations for all A_i with $i \leq k$ correctly.

For $k < \omega$, let ν_k be the k^{th} cardinal of $\mathcal{M}(\mathcal{T})^+$ which is $\geq \delta(\mathcal{T})$, and set

$$\mathcal{M}_k = \mathcal{M}(\mathcal{T})^+ | \nu_k,$$

$$\tau_{j,k} = \tau^{\mathcal{M}(\mathcal{T})^+}_{A_j,\nu_k},$$

and

$$\gamma_k = \sup\{\xi \mid \xi \text{ is definable over } \mathcal{M}_k \text{ from points of the form } \tau_{i,j}, \text{ where } i, j \leq k.\}.$$

Let $\mathcal{M} = \mathcal{M}(\mathcal{T})$.

Claim 1. The γ_k are cofinal in $\delta(\mathcal{T})$.

Proof. Let γ be the sup of the γ_k. Let $\pi : \bar{\mathcal{M}} \to \mathcal{M}$ be the transitive collapse of the set of points definable over some $\mathcal{M}_k$ from the $\tau_{j,l}$ for $j, l < \omega$ and ordinals $< \gamma$. Using the regularity of $\delta(\mathcal{T})$ in $\mathcal{M}$, we get that $\pi \restriction \gamma =$identity, and $\pi(\gamma) = \delta(\mathcal{T})$. From 1.43, we then have that $\bar{\mathcal{M}}$ is suitable. [16] The minimality condition in the suitability of $\mathcal{M}$ then implies $\gamma = \delta(\mathcal{T})$, as desired. □

The usual uniqueness proof for good branches in iteration trees [17] yields

Claim 2. Let $k \leq l$, and let E be an extender of length $\leq \gamma_k$; then E is used in b_k if and only if E is used in b_l.

Proof. This is a simple consequence of the fact that $\mathrm{ran}(i_k) \cap \mathrm{ran}(i_l)$ is cofinal in γ_k. □

[16] The reason is essentially that Lp^α-fullness is a $\Pi_1^{J_\alpha(\mathbb{R}^g)}$ statement, true of reals coding $\mathcal{M}_k$ added by collapsing ν_k, and Skolemized by the τ's.

[17] The "zipper argument".

Define now
$$\xi \in b \Leftrightarrow \exists k \forall l \geq k(\xi \in b_l),$$
so that E is used in b iff E is used in b_k, for all sufficiently large k.

Claim 3. b is cofinal in $\mathrm{lh}(\mathcal{T})$.

Proof. Suppose $\eta = \bigcup b < \delta(\mathcal{T})$. Fix k such that
$$\mathrm{lh}(E_\eta^{\mathcal{T}}) < \gamma_k.$$
By claim 2,
$$b \subseteq b_l, \text{ for all } l \geq k.$$
This implies that $\eta \in b$. (If not, then b is cofinal in η, but then all b_l for $l \geq k$ are cofinal in η, so $\eta \in b_l$ for all $l \geq k$ since branches are closed.) By the definition of γ_k, we can find ξ such that
$$\mathrm{lh}(E_\eta) \leq \xi < \gamma_k \wedge \forall l \geq k(\xi \in \mathrm{ran}(i_l)).$$
Now let F_l be the extender applied to $\mathcal{M}_\eta^{\mathcal{T}}$ along the branch b_l, for $l \geq k$. We have $\mathrm{crit}(F_l) < lh(E_\eta) \leq \xi$ for all $l \geq k$. Pick $l > k$ such that $F_l \neq F_k$; such an l exists as η was largest in b. Then by the standard argument, $\mathrm{ran}(i_k) \cap \mathrm{ran}(i_l) \subseteq \mathrm{crit}(F_k)$, contrary to $\xi \in \mathrm{ran}(i_k) \cap \mathrm{ran}(i_l)$. □

Now set
$$T_k^{\mathcal{M}} = \text{ Th }^{\mathcal{M}_{k+1}}(\delta^{\mathcal{M}} \cup \{\tau_{i,j} \mid i, j < k\}),$$
and let $T_k^{\mathcal{N}}$ be defined from $\mathcal{N}$ and its capturing terms in parallel fashion. Thus we have
$$i_k(T_k^{\mathcal{N}}) = T_k^{\mathcal{M}}$$
because i_k moves the relevant term relations correctly.

Claim 4. For all k, $i_b(T_k^{\mathcal{N}}) = T_k^{\mathcal{M}}$.

Proof. Fix k. We regard $T_k^{\mathcal{N}}$ as a subset of $\delta^{\mathcal{N}}$. Since b is cofinal, it is enough to see that $i_b(T_k^{\mathcal{N}}) \cap \mathrm{lh}(E) = T_k^{\mathcal{M}} \cap \mathrm{lh}(E)$ whenever E is used in b. But fixing such an E, we can find $l \geq k$ such that E is used in b_l. It follows that $i_b(X) \cap \mathrm{lh}(E) = i_l(X) \cap \mathrm{lh}(E)$ for all $X \in \mathcal{N}$, and applyinng this to $X = T_k^{\mathcal{N}}$, we have the desired conclusion. □

It is easy to see using 1.43 that $\mathcal{N}$ is pointwise Σ_0-definable from ordinals $< \delta^{\mathcal{N}}$ and the $\tau_{i,j}^{\mathcal{N}}$. Thus $\mathcal{N}$ is coded by the join of the $T_k^{\mathcal{N}}$, so that $\mathcal{M}_b^{\mathcal{T}}$ is coded by the join of the $i_b(T_k^{\mathcal{N}})$. It follows from claim 4 that $\mathcal{M}_b^{\mathcal{T}} = \mathcal{M}$ and i_b moves all the term relations correctly. Thus b satisfies all the requirements for the choice of a fullness-preserving, $\mathcal{A}$-guided iteration strategy, and we can set $\Sigma(\mathcal{T}) = b$. Since $\mathcal{T}$ was maximal, the iteration game we were playing is now over, and Σ has won.

We leave it to the reader to show that the strategy Σ we have just defined condenses well. The term-condensation lemma 1.43 is of course the key. This finishes the proof of 1.50. $\square$

We are finally ready to complete the proof of Lemma 1.39. Roughly speaking, 1.50 gives us what we want, except that it exists in $V[g]$, and depends on g. By considering all possible finite variants of g, and comparing the mice associated to each of them, we shall produce a mouse which does not depend on g. We shall then show that this mouse has the form $\mathcal{N}[g]$, where $\mathcal{N}$ is a mouse over τ in V.[18]

Let $p_0 \in g$ be a condition which forces everything about τ and $V[g]$ which we have used so far. For each $p \leq p_0$, let g_p be given by

$$g_p = p \cup g \upharpoonright (\omega \setminus \mathrm{dom}(p)).$$

Here we are identifying g with $\bigcup g \colon \omega \to \mu$. So g_p is V-generic, and $V[g_p] = V[g]$, for all $p \leq p_0$. For $p \leq p_0$, let $\mathcal{A}_p$ be the self-justifying system of sets which are $\mathrm{OD}^{<\beta}(\tau^{g_p})$ associated to τ^{g_p}. Let

$$z_p = \langle \tau, g_p \rangle,$$

so that the sets in $\mathcal{A}_p$ are all $\mathrm{OD}^{<\beta}(z_p)$. Let

$$\mathcal{A} = \bigcup_{p \leq p_0} \mathcal{A}_p,$$

and notice that since z_p easily computes z_q, all sets in $\mathcal{A}$ are $\mathrm{OD}^{<\beta}(z_p)$, for all p. Let $\dot{\mathcal{A}}$ be a symmetric term for $\mathcal{A}$, that is

$$\forall p \leq p_0 \dot{\mathcal{A}}^{g_p} = \mathcal{A}.$$

From now on, let's assume $p_0 = \emptyset$ to save ink. For each p, we have by 1.50 a term $\dot{N}_p, \dot{\Sigma}_p$ such that

$$\begin{aligned} p \Vdash \ & \dot{\Sigma}_p \text{ is an } \dot{\mathcal{A}}\text{-guided, fullness-preserving} \\ & \text{strategy for the } \langle \tau, \dot{g} \rangle \text{ mouse } \dot{\mathcal{N}} \\ & \text{such that } \dot{\Sigma} \text{ condenses well.} \end{aligned}$$

Let $\mathcal{N}_p = \dot{\mathcal{N}}^{g_p}$ and $\Sigma_p = \dot{\Sigma}^{g_p}$. Now $\mathcal{N}_p$ is a z_p-mouse, but it can also be regarded as a z_q mouse for any q, since z_p and z_q compute each other easily. It therefore makes sense to simultaneously compare all the $\mathcal{N}_p$ in $V[g]$, using the Σ_p to iterate them. Let

$$\mathcal{N}_\infty = \text{ common iterate of all } \mathcal{N}_p.$$

[18]The *Boolean-valued comparison* method is due to Woodin.

Because the Σ_p are fullness-preserving, $\mathcal{N}_p$-to-$\mathcal{N}_\infty$ does not drop for all p, and $\mathcal{N}_\infty$ can be regarded as a suitable z_p-mouse, for each p. These are different presentations so perhaps we should write $\mathcal{N}_\infty^p$, but there is a fixed extender sequence

$$\vec{E}_\infty = \dot{E}^{\mathcal{N}_\infty^p}, \text{ for all } p.$$

Moreover, $\mathcal{N}_\infty$ is weakly $\mathcal{A}$-iterable, and thus by 1.50 has a unique $\mathcal{A}$-guided strategy Σ which is fullness-preserving and condenses well.

Since the comparison which produced $\mathcal{N}_\infty$ depends only on the *set* of all $\dot{\mathcal{N}}^{g_p}$, and not any enumeration of this set, we have symmetric terms for $\vec{E}_\infty$ and Σ; that is $\dot{\vec{E}}_\infty$ and $\dot{\Sigma}$ such that

$$\dot{\vec{E}}_\infty^{\,g_p} = \vec{E}_\infty \wedge \dot{\Sigma}^{g_p} = \Sigma$$

for all p. It follows from the homogeneity of $\mathrm{Col}(\omega, \mu)$ that any subset of V which is definable in $V[g]$ from $\{g_p \mid p \leq p_0\}$, $\vec{E}_\infty$, Σ, and elements of V is itself in V.

In V, we can now inductively build a τ-mouse $\mathcal{N}$. We maintain

$$\mathcal{N}|\eta[g] = \mathcal{N}_\infty^\emptyset[g]|\eta,$$

by induction on η. The first few levels of $\mathcal{N}$ are just initial segments of $L(\tau)$. Given $\mathcal{N}|\eta$, we get $\mathcal{N}|\eta+1$ by letting the next extender be

$$\dot{E}_\eta^{\mathcal{N}} = (\vec{E}_\infty)_\eta \cap \mathcal{N}|\eta.$$

Note that $\dot{E}_\eta^{\mathcal{N}}$ is in V, and can be defined from η over V uniformly in η. One can show that $\mathcal{N}$ is a τ-mouse, and $\mathcal{N}[g] = \mathcal{N}_\infty$. The proof is given in [18]. It relies on the fact that fine-structure is preserved, level-by-level, by small forcing. That also implies that any iteration tree $\mathcal{T}$ on $\mathcal{N}$ can be regarded as a tree $\mathcal{T}^*$ on $\mathcal{N}[g] = \mathcal{N}_\infty$, with the same drop and degree structure, and $\mathcal{M}_\xi^{\mathcal{T}^*} = \mathcal{M}_\xi^{\mathcal{T}}[g]$ for all ξ. Thus Σ induces a μ^+-iteration strategy, which we also call Σ, on $\mathcal{N}$. Moreover, $\Sigma \in V$. We leave it to the reader to check that Σ condenses well in V. This proves 1.39. $\square$

By Lemma 1.25, we can assume that Σ is a κ^+-iteration strategy for $\mathcal{N}$ in V which condenses well. (The hypothesis of 1.25 was that Σ is a (μ^+, μ^+)-strategy. but we can lift μ^+ strategies by the same proof.)

We need to use hybrid mice obtained by constructing from some A coding $\mathcal{N}$, A bounded in κ^+, adding extenders to a coherent sequence we are building, and at the same time closing the model we are building under Σ. This is parallel to the method of building K^c's in the inadmissible case which we alluded to in remark 1.36, but did not actually use. In the present situation, we have no way to argue that a pure extender model over $\mathcal{N}$ must be closed under Σ. Iterability for these hybrid mice includes the provision that Σ is moved correctly. (All critical points on the coherent sequence must be $> \sup(A)$, and hence $> \mu$.) If this is done in a natural way, the resulting model has a fine structure.[19] The key to the fine structure is that Σ condenses well. Condensation for Σ is also used in the realizability proof that size μ elementary submodels of levels of $K^c_\Sigma(A)$ are countably iterable in $V[g]$.[20] Let us call such mice Σ-*hybrid mice.*

Definition 1.51 *Let A be bounded in κ^+ and code $\mathcal{N}$ in the specified way; then $P^\Sigma_n(A)^\sharp$ is the minimal iterable Σ-hybrid mouse over A which is active, and satisfies "there are n Woodin cardinals".*

Lemma 1.52 *For all $n < \omega$ and all A bounded in κ^+, $P^\Sigma_n(A)^\sharp$ exists, and is (κ^+, κ^+)-iterable.*

The proof of 1.52 is an induction on n which is very close to the proof of the corresponding lemma in the inadmissible case, lemma 1.33. We therefore give no further detail.

We are ready to prove $W^*_{\beta+1}$. Let $U \subseteq \mathbb{R}^g$ be in $J_{\beta+1}(\mathbb{R}^g)$ and $k < \omega$; we seek a coarse (k, U)-Woodin mouse. Let U be $\Sigma_n^{J_\beta(\mathbb{R}^g)}$ in the real parameter z, and $z = \sigma^g$. Our desired witness will be

$$P = P^\Sigma_{k+n}(\langle \mathcal{N}, \sigma \rangle)^\sharp[g].$$

Note first

Claim 1. $P[g]$ has a unique ω_1-iteration strategy in $V[g]$.

Proof. As in the inadmissible case–we have the $\mathcal{Q}$-structures we need to compute it. $\square$

Let Γ be the strategy for $P[g]$ given by claim 1. Let $\langle A_i \mid i < \omega \rangle$ be our self-justifying system of sets which are $\mathrm{OD}^{<\beta}(\langle \tau, g \rangle)$. If j is least such that $\rho_j^{J_\beta(\mathbb{R}^g)} = \mathbb{R}^g$, then Σ_j-truth at β is coded in a simple way into

$$W = \oplus_{i<\omega} A_i.$$

Claim 2. For any $\nu \in P$, there is a term $\dot{W} \in P$ relative to $\mathrm{Col}(\omega, \mu) \times \mathrm{Col}(\omega, \nu)$ such that whenever $i \colon P[g] \to \mathcal{Q}[g]$ is an iteration map by Γ (constructed in $V[g]$), and h is $\mathcal{Q}[g]$-generic over $\mathrm{Col}(\omega, i(\nu))$, then

$$\dot{W}^{g \times h} = W \cap \mathcal{Q}[g][h].$$

[19]Woodin found the following trick for closing under Σ in such a way that the levels of the model we build are all *amenable* structures, which is important for fine structure: if we are at a level $\mathcal{P}$ appropriate for closing further under Σ, and $\mathcal{T}$ is the $\mathcal{P}$-least iteration tree of limit length α which is by Σ, but such that we have not yet told our model what $\Sigma(\mathcal{T})$ is, we let $\mathcal{Q}$ be the structure of height $o(\mathcal{P}) + \alpha$ obtained by doing α steps of the usual constructible closure starting with $\mathcal{P}$, and then take $(\mathcal{Q}, B)$ to be the next level of our model, where $B = \{o(\mathcal{P}) + \beta \mid \beta \in \Sigma(\mathcal{T})$.

[20]It would be possible to talk only about countable iterability in V. Given $\pi \colon \mathcal{M} \to \mathcal{Q}$, where $\mathcal{M}$ is countable and $\mathcal{Q}$ is a level of $K^c_\Sigma(A)$, iterability for $\mathcal{M}$ means that the collapse of Σ is moved to its pullback Σ^π. By condensation for Σ, this is what happens along realizable branches of trees on $\mathcal{M}$.

Proof. Basically, $\dot{W}$ asks what the $\tau_{A_i}^{\mathcal{N}}$ are moved to in the iteration of $\mathcal{N}$ which makes $P|\nu^+$ generic over the extender algebra of the iterate. This iteration is done inside $\mathcal{P}$, using what it knows of Σ. $\square$

Claim 3. Let δ be the k^{th} Woodin cardinal of P; then for any $\Sigma_n^{J_\beta(\mathbb{R}^g)}(z)$ set Y, there is a term $\dot{Y} \in P$ relative to $\mathrm{Col}(\omega, \mu) \times \mathrm{Col}(\omega, \delta)$ such that whenever $i\colon P[g] \to \mathcal{Q}[g]$ is an iteration map by Γ (constructed in $V[g]$), and h is $\mathcal{Q}[g]$-generic over $\mathrm{Col}(\omega, i(\delta))$, then

$$\dot{Y}^{g\times h} = Y \cap \mathcal{Q}[g][h].$$

Proof. $\dot{Y}$ is constructed from the term $\dot{W}$ given by claim 2, applied at the $k+n^{th}$ Woodin of $\mathcal{P}$. The n-Woodins above δ are used to answer the relevant n-real-quantifier statements. $\square$

We can now see that P is the desired coarse witness. The trees in $\mathcal{P}$ which are moved appropriately by Γ are obtained just as in the inadmissible case. $\square$

2 Amenably closed hulls

In this section we prove theorem 0.3. Let us assume the hypotheses of that theorem.

2.1 Framework for the core model induction

As in section 1, we shall prove that V is closed under the appropriate inner model operators. In this case, we shall cycle between closure at ω_1 and closure at ω_2, so that ω_1 and ω_2 will play the roles that μ and κ played in section 1.

For any $A \subseteq \omega_2$, let $\mathrm{Lp}(A)$ be the lower part closure over A carried out to ω_3; that is,

$$\mathrm{Lp}(A) = \mathcal{M}_\xi(A),$$

where ξ is least such that $o(\mathcal{M}_\xi(A)) = \omega_3$ [21].

One can show that our hypotheses imply that for all $A \subseteq \omega_2$, $\mathrm{Lp}(A) \models \mathsf{ZFC}$. (This is the analog of 1.4.) However, the following related result is what we really need.

Lemma 2.1 *Let A be a bounded subset of ω_2; then*

(1) ω_2 is inaccessible in $\mathrm{Lp}(A)$*, and*

(2) $\mathrm{Lp}(A) \models (\omega_2^V)^+$ *exists.*

[21] One can show that our hypotheses imply that ω_3 is a limit of cardinals of $\mathrm{Lp}(A)$, so that $\xi = \omega_3$, but we do not need this, so we omit the proof.

Proof. We first prove (1). Suppose toward contradiction that ω_2 is the cardinal successor in $\mathrm{Lp}(A)$ of ν. We may assume $A \subseteq \nu$. Let

$$\pi \colon N \cong X \prec H_{\omega_3}$$

be the transitive collapse map, where X is amenably closed, with $\nu + 1 \cup \{A\} \subseteq X$, and $|X| = \omega_1$. Let

$$\kappa = X \cap \omega_2 = \mathrm{crit}(\pi).$$

Thus $\pi(\kappa) = \omega_2$, so that $\kappa = (\nu^+)^N$.

It is easy to get a contradiction using $\square$ in $\mathrm{Lp}(A)$. [22] Here is a more elementary argument. Let $\mathcal{P}$ be the first level of $\mathrm{Lp}(A)$ not in N and such that $\rho_\omega(\mathcal{P}) < \kappa$. For simplicity, assume that $\rho_1(\mathcal{P}) < \kappa$, $\mathcal{P}$ is passive, and $o(\mathcal{P})$ is a limit ordinal. Let f be a partial $\Sigma_1^{\mathcal{P}}(q)$ function from ν onto κ, and for $\xi < o(\mathcal{P})$ such that $q \in \mathcal{P}|\xi$, let f_ξ be the result of interpreting the Σ_1 definition of f in $\mathcal{P}|\xi$. Thus $f = \bigcup_\xi f_\xi$, and each f_ξ is in $\mathcal{P}$. For $\alpha < \kappa$, let

$$\xi_\alpha = \text{ least } \xi \text{ such that } \mathrm{ran}(f_\xi) \cap (\kappa \setminus \alpha) \neq \emptyset,$$

so that the ξ_α are increasing, and cofinal in $o(\mathcal{P})$. Let

$$A = \{\langle \alpha, \beta, \gamma \rangle \mid \alpha < \kappa \wedge f_{\xi_\alpha}(\beta) = \gamma\}.$$

A is essentially a subset of κ, and it is amenable to N. Thus $A \in N$. But f is easily computed from A, so then κ is not a cardinal of N, a contradiction.

We now prove (2). Let $f \colon \omega_2 \to \omega_2$ be given by

$$f(\alpha) = (\alpha^+)^{\mathrm{Lp}(A)},$$

and let W be a wellorder of ω_2 such that for stationarily many $\alpha < \omega_2$,

$$f(\alpha) < \text{ order type of } (W \cap (\alpha \times \alpha)).$$

Let γ be the order type of W, and suppose toward contradiction that $|\gamma| = \omega_2^V$ in $\mathrm{Lp}(A)$. Let $Y \prec V_\eta$ for some large η, with $|Y| = \omega_1$ and $W, A \in Y$, and $\alpha = Y \cap \omega_2 \in \omega_2$, and $f(\alpha) < \text{o.t.}(W \cap (\alpha \times \alpha))$. Let M be the transitive collapse of Y, and π the collapse map. Then $\pi^{-1}(W) = W \cap (\alpha \times \alpha)$, so

$$\text{order type of } (W \cap (\alpha \times \alpha)) < (\alpha^+)^{\mathrm{Lp}(A)^M}.$$

However, by condensation, $\mathrm{Lp}(A)^M|\tau \trianglelefteq \mathrm{Lp}(A)$, for $\tau = (\alpha^+)^{\mathrm{Lp}(A)^M}$. Thus $W \cap (\alpha \times \alpha)$ has order type $< f(\alpha)$, a contradiction. $\square$

[22]The proof using $\square$ is due to Woodin. Let $\langle C_\alpha \mid \alpha < \omega_2 \rangle$ be a sequence witnessing that $\square_\nu$ holds in $\mathrm{Lp}(A)$. We have that $C_\alpha = \pi^{-1}(C_\alpha) \in N$ for all $\alpha < \kappa$, and therefore C_κ is amenable to N. Thus $C_\kappa \in N$, so that κ is singular in N, a contradiction.

Remark 2.2 Our hypothesis concerning bounding by canonical functions, (2) of 0.3, is used only in the proof of part (2) of 2.1. (Woodin pointed out that it could be used for this purpose.) We do not know whether (1) and (3) of 0.3 have any strength; for example, we do not know whether their conjunction implies that $0^\sharp$ exists. Here are some comments due to the referee which suggest that (1) and (3) together might be weaker than $0^\sharp$, and in fact approximately that of a remarkable cardinal (see [9] for this concept). First, arguments like those below show that (1) implies that for $\kappa = \omega_2^V$, $L \models \kappa$ is κ^+-remarkable. On the other hand, starting from a remarkable cardinal, Räsch and Schindler have constructed a model of the form $L[A], A \subseteq \omega_2$, in which there are stationarily many $X \prec H_{\omega_3}$ such that $\mu = X \cap \omega_2 \in \omega_2$ and if $H \cong X$ is transitive then $H \cap \mathrm{OR} = (\mu^+)^{L[A\cap\mu]}$. (See [7].) This is the consequence of (1) we used, along with (2) of 0.3, in our proof of part (2) of 2.1. However, the full (1) of 0.3 does not hold in the Räsch-Schindler model.

Fix for the remainder of this section a V-generic object g over $\mathrm{Col}(\omega, \omega_1)$. Let W_α^* be obtained from the corresponding assertion in section 1 by replacing the g on $\mathrm{Col}(\omega, \mu)$ involved there with our current g. Let W_α be obtained similarly from the corresponding assertion in section 1. The basic results of section 1.1 had nothing to do with g, so we have:

- W_α^* implies $J_\alpha(\mathbb{R}^g) \models \mathsf{AD}$ (1.6),
- W_α^* implies capturing on a cone (1.7),
- for α a limit ordinal, W_α^* implies W_α (1.11),
- Σ_1 assertions witnessed as in W_α are true in $L(\mathbb{R}^g)$ (1.10), and
- W_α implies lightface capturing (1.12).

It will suffice to show that W_α^* holds for all α. Again, we shall prove this by induction on α, following the pattern of scales in $L(\mathbb{R}^g)$.

2.2 Lifting mouse-closure from ω_1 to ω_2

Again, we shall have to extend mouse-closure functions $A \mapsto \mathcal{M}(A)$ defined on bounded subsets of ω_2 so that they act also on bounded subsets of ω_3. In the case that $\mathcal{M}(A)$ is an ordinary, non-hybrid mouse, the key is the following lemma.

Lemma 2.3 *Let $\pi\colon N \to H_{\omega_3}$ be elementary, where N is transitive, $|N| = \omega_1$, and $\omega_1 \in N$. Suppose $\mathrm{ran}(\pi)$ is amenably closed, and let $A \subseteq \kappa = \mathrm{crit}(\pi)$; then $\mathrm{Lp}(A) \cap P(\kappa) \subseteq N$.*

Proof. Assume not, and let $\mathcal{P}$ be the first level of $\mathrm{Lp}(A)$ which projects to κ and is not in N. For simplicity, we assume $\rho_1(\mathcal{P}) \leq \kappa$, $\mathcal{P}$ is passive, and $o(\mathcal{P})$ is a limit ordinal. Let φ be a Σ_1 formula, and $q \in \mathcal{P}$ a parameter, such that putting

$$\alpha \in T \Leftrightarrow \alpha < \kappa \wedge \mathcal{P} \models \varphi[\alpha, q],$$

we have that $T \notin N$. By the amenable closure of $\mathrm{ran}(\pi)$, we can fix $\eta < \kappa$ such that $T \cap \eta \notin N$.

Claim. $\mathrm{cof}(o(\mathcal{P})) = \omega_1$.

Proof. If not, let $\langle \xi_n \mid n < \omega \rangle$ be an increasing sequence cofinal in $o(\mathcal{P})$, with $q \in \mathcal{P}|\xi_0$. Let

$$\mathcal{H}_n = \text{ transitive collapse of } \mathrm{Hull}_1^{\mathcal{P}|\xi_n}(\kappa \cup \{q, \xi_0, ..., \xi_{n-1}\}),$$

and $\sigma^n \colon \mathcal{H}_n \to \mathcal{P}|\xi_n$ the collapse map. Each $\mathcal{H}_n$ is coded by a subset of κ in $\mathcal{P}$, and hence each $\mathcal{H}_n$ is in N. Working in N, where $\kappa = \omega_2$, we can form in N a continuous chain $\langle X^n_\gamma \mid \gamma < \kappa \rangle$ of elementary submodels of $\mathcal{H}_n$, each of size ω_1 in N, with $\omega_1 \cup \eta \subseteq X^n_0$, and $\bigcup_{\gamma<\kappa} X^n_\gamma = \mathcal{H}_n$. Let

$$\sigma^n_\gamma \colon \mathcal{H}^n_\gamma \cong X^n_\gamma \prec \mathcal{H}_n$$

be the collapse map, and $\kappa^n_\gamma = \mathrm{crit}(\pi^n_\gamma)$.

Now $\{\kappa^n_\gamma \mid \gamma < \kappa\}$ is club in κ, and $\mathrm{cof}(\kappa) = \omega_1$ by the amenable closure of $\mathrm{ran}(\pi)$. Thus we can find $\bar{\kappa}, \gamma < \kappa$ such that

$$\bar{\kappa} = \kappa^n_\gamma, \text{ for all } n \ .$$

It is easy then to see that if we set

$$\mathcal{R} = \bigcup_n \mathcal{H}^n_\gamma \text{ and } \sigma = \bigcup_n \pi_n \circ \pi^n_\gamma,$$

then $\sigma \colon \mathcal{R} \to \mathcal{P}$ is Σ_0 elementary and cofinal, with

$$\eta < \bar{\kappa} = \mathrm{crit}(\sigma) \wedge \sigma(\bar{\kappa}) = \kappa.$$

We also have that $q \in \mathrm{ran}(\sigma)$, so that $T \cap \eta$ is $\mathbf{\Sigma}_1^{\mathcal{R}}$. Note that $\mathcal{R}$ is an $(A \cap \bar{\kappa})$- mouse, countable iterability being guaranteed by σ. We have $\rho_1(\mathcal{R}) = \bar{\kappa}$. Let

$$\mathcal{Q} = \mathfrak{C}_1(\mathcal{R})$$

be the first core of $\mathcal{R}$; then $\mathcal{Q}$ is ω-sound, and $T \cap \eta$ is $\mathbf{\Sigma}_1^{\mathcal{Q}}$. [23] Thus $\mathcal{Q} \trianglelefteq \mathrm{Lp}(A \cap \bar{\kappa})$, so $T \cap \eta \in \mathrm{Lp}(A \cap \bar{\kappa})$. But ω_2 is inaccessible in $\mathrm{Lp}(A \cap \bar{\kappa})$, and using π, we see then that κ is

[23] Let $\mathcal{Q}^*, \mathcal{R}^*, \kappa^*, T^*$ and η^* be the collapses of $\mathcal{Q}, \mathcal{R}, \bar{\kappa}, T$, and η in some countable elementary submodel of V. $\mathcal{Q}^*$ and $\mathcal{R}^*$ have a common iterate $\mathcal{N}$. Since the iteration is above κ^* in both cases, we have that $T^* \cap \eta^*$ is $\mathbf{\Sigma}_1^{\mathcal{Q}^*}$, using the closeness of extenders to the model to which they are applied in an iteration tree.

a limit cardinal in $\mathrm{Lp}(A \cap \bar{\kappa})$. It follows that $T \cap \eta \in N$, a contradiction. This proves the claim. $\square$

Let $f\colon \omega_1 \to \eta$ be surjection, with $f \in N$. For any $\gamma < \omega_1$, let

$$\begin{aligned} g(\gamma) &= \text{ least } \xi \text{ such that } \mathcal{P}|\xi \models \varphi[f(\gamma), q]), \text{ if } f(\gamma) \in T, \\ &= \kappa, \text{ if } f(\gamma) \notin T, \end{aligned}$$

and

$$\begin{aligned} h(\gamma) &= \text{ least } \mu > \eta \text{ s.t. } \mathrm{Hull}_\omega^{\mathcal{P}|g(\gamma)}(\mu \cup \{q\}) \cap \kappa = \mu, \text{ if } f(\gamma) \in T, \\ &= 0, \text{ if} f(\gamma) \notin T. \end{aligned}$$

Notice that $h(\gamma) < \kappa$ for all γ, as $g(\gamma) < o(\mathcal{P})$ and κ is regular in $\mathcal{P}$.

We claim that for any $\gamma < \omega_1$, $h \restriction \gamma \in N$. For let $\xi < o(\mathcal{P})$ be such that $q \in \mathcal{P}|\xi$ and $g``\gamma \subseteq \xi$; there is such a ξ because $\mathrm{cof}(o(\mathcal{P})) > \omega$. Let S be the first order theory of $\mathcal{P}|\xi$ in parameters from $\kappa \cup \{q\}$. Since $S \in \mathcal{P}$, and S is essentially a subset of κ, we have $S \in N$. But now it is easy to see that S records enough information about $\mathcal{P}$ that from it, together with f, N can compute $h \restriction \gamma$.

We claim that $\mathrm{ran}(h)$ is cofinal in κ. For if not, let $\mu = \sup(\mathrm{ran}(h))$. It follows from our definitions that $\mathcal{H} = \mathcal{H}_1^{\mathcal{P}}(\mu \cup \{q\})$ is a countably iterable $A \cap \mu$-mouse, and that $T \cap \eta$ is $\mathbf{\Sigma_1^{\mathcal{H}}}$. As in the proof of the claim, this implies that $T \cap \eta \in \mathrm{Lp}(A \cap \mu)$, and thus $T \cap \eta \in N$, a contradiction.

Let $h^*(\gamma) = \sup(\mathrm{ran}(h \restriction \gamma))$. It is easy to see that h^* (i.e. its graph) is an amenable-to-N subset of $\kappa \times \kappa$. Since $\mathrm{ran}(\pi)$ is amenably closed, we have $h^* \in N$. But then κ is singular in N, a contradiction. $\square$

If $\kappa \leq \nu$, and $B \subset \nu$, then we say that a set $A \subseteq \kappa^3$ *codes* B iff $W = \{\langle \alpha, \beta \rangle \mid \langle 0, \alpha, \beta \rangle \in A\}$ is a wellorder, and $B = \{\gamma \mid \exists \xi (\langle 1, 1, \xi \rangle \in A \wedge \gamma = |\xi|_W\}$.

Corollary 2.4 *Let ψ be a sentence of the language of relativised premice, and suppose that for all bounded $A \subset \omega_2$ there is a countably iterable A-mouse $\mathcal{M}(A)$ such that $\mathcal{M}(A) \models \psi$. Suppose also that the $\mathcal{M}$-operator relativises well, in that whenever A and B are bounded subsets of ω_2 such that A codes B, then $\mathcal{M}(B) \in \mathrm{Lp}(A)$. Then for all bounded $B \subset \omega_3$, there is a countably iterable B-mouse which satisfies ψ.*

Proof. Let B be given, and let $A \subseteq \omega_3^3$ code B. Let $\pi\colon N \to H_{\omega_3}$ with $\mathrm{ran}(\pi)$ amenably closed, N transitive, and $\pi(\bar{A}) = A$ and $\pi(\bar{B}) = B$. Clearly, $\bar{A}$ codes $\bar{B}$, and thus $\mathcal{M}(\bar{B}) \in \mathrm{Lp}(\bar{A})$. So $\mathcal{M}(\bar{B}) \in N$ by lemma 2.3. Clearly $N \models (\mathcal{M}(\bar{B}) \models \psi)$, so by the elementarity of π, it is enough to see that $N \models \mathcal{M}(\bar{B})$ is countably iterable. But let $N \models \mathcal{P}$ is a countable premouse embeddable in $\mathcal{M}(\bar{B})$. Then $\mathcal{P}$ really is such a premouse, so $\mathcal{P}$ is $\omega_1 + 1$-iterable, and thus $H_{\omega_3} \models \mathcal{P}$ is $\omega_1 + 1$-iterable. Since $\pi(\mathcal{P}) = \mathcal{P}$, we have $N \models \mathcal{P}$ is $\omega_1 + 1$-iterable, as desired. $\square$

We shall also have to lift mouse-closure operations given by strategy-hybrid mice. We can do this just as it is done in lemma 2.4 for ordinary mice, once we show how to lift the iteration strategies themselves.

As in section 1, fix some α which begins a proper weak gap in $L(\mathbb{R}^g)$, and assume W_α^*. For A bounded in ω_2, we define $\mathrm{Lp}^\alpha(A)$, suitability for A-mice, and the notion of a fullness-preserving (ω_2, ω_2)-iteration strategy on a suitable A-mouse $\mathcal{N}$, just as in section 1.2. Lemma 1.25 now goes over verbatim.

Lemma 2.5 *In V: let $\mathcal{N}$ be a suitable premouse (over some A bounded in ω_2), and let Σ be a fullness-preserving (ω_2, ω_2)-iteration strategy for $\mathcal{N}$ which condenses well. Then Σ has a unique extension Γ to an (ω_3, ω_3)- iteration strategy which condenses well.*

Proof. The proof is quite similar to that of 1.25. In particular, uniqueness follows from the argument of that proof.

For existence, again we define the restriction of the desired Γ to iteration trees of length $< \xi$, by induction on ξ. Here $\omega_2 \leq \xi < \omega_3$, as for $\xi < \omega_2$ we just use Σ. Let us call this restriction Γ_ξ. Clearly, if ξ is a limit, then we must set

$$\Gamma_\xi = \bigcup_{\eta<\xi} \Gamma_\eta,$$

and Γ_ξ condenses well if all Γ_η for $\eta < \xi$ condense well. Now suppose Γ_ξ is given, extending Σ and condensing well. If ξ is not a limit ordinal, there is nothing for Γ to decide, so let ξ be a limit ordinal. Let $\mathcal{T}$ be an iteration tree on $\mathcal{N}$ which is according to Γ_ξ; we have to choose a cofinal wellfounded branch of $\mathcal{T}$.

Let

$$S = \{X \prec H_{\omega_3} \mid X \text{ is amenably closed and } \mathcal{T} \in X\},$$

and for $X \in S$, let

$$\pi_X \colon H_X \to H_{\omega_3}$$

be the anticollapse map, and $\kappa_X = \mathrm{crit}(\pi_X)$. Let

$$\langle \mathcal{T}_X, \xi_X \rangle = \pi_X^{-1}(\langle \mathcal{T}, \xi \rangle),$$

and let

$$b_X = \Sigma(\mathcal{T}_X).$$

Again, $b_X \notin H_X$ is possible. If $X \prec Y$ and $X, Y \in S$, let

$$\pi_{X,Y} \colon H_X \to H_Y$$

be the collapse of the inclusion map, and set

$$c_{X,Y} = \text{ downward closure in } \mathcal{T}_Y \text{ of } \pi_{X,Y}\text{``}b_X.$$

We write $\forall^* X \in S\ \ \varphi(X)$ to mean that $\{X \in S \mid \neg\varphi(X)\}$ is not stationary in $P_{\omega_2}(H_{\omega_3})$.

Claim. For any $\gamma < \xi$, either

$$\forall^* X \in S\ \ \pi_X^{-1}(\gamma) \in b_X,$$

or

$$\forall^* X \in S\ \ \pi_X^{-1}(\gamma) \notin b_X.$$

Proof. There are three cases, based on the cofinality of ξ.

Suppose first that $\mathrm{cof}(\xi) = \omega_1$. If the claim is false, we can find a $X, Y \in S$ such that $\pi_X^{-1}(\gamma) \in b_X$ and $\pi_Y^{-1}(\gamma) \notin b_Y$. However, since π_X and π_Y have critical point $> \omega_1$, the downward closures of $\pi_X\text{''}b_X$ and $\pi_Y\text{''}b_Y$ are cofinal in ξ, and since branches in an iteration tree contain all their limit points, they are the same. Thus $\gamma \in \pi_X\text{''}b_X$ iff $\gamma \in \pi_Y\text{''}b_Y$, a contradiction.

Next, suppose $\mathrm{cof}(\xi) = \omega$. Let f map ω_2 onto ξ, and let $f_X = \pi_X^{-1}(f)$ whenever $X \in S$ and $f \in X$. For any $X \in S$, let

$$\alpha_X = \text{ least } \alpha < \kappa_X \text{ such that } b_X \cap f_X\text{''}\alpha \text{ is cofinal in } \xi_X.$$

By Fodor's lemma, we can fix a stationary $U \subseteq S$ and α such that

$$X \in U \Rightarrow \alpha_X = \alpha.$$

By thinning U we can stabilize the truth value of $\pi_X^{-1}(\gamma) \in b_X$, and by symmetry, we may as well assume that

$$X \in U \Rightarrow \pi_X^{-1}(\gamma) \in b_X.$$

We claim that then $\forall^* X \in S\ \ \pi_X^{-1}(\gamma) \in b_X$. For if not, we can fix an $X \in S$ such that $\alpha < \kappa_X$ and $\pi_X^{-1}(\gamma) \notin b_X$. Since U is stationary, we can then find $Y \in U$ such that $X \prec Y$. Now

$$\pi_{X,Y}\text{''}f_X\text{''}\alpha = f_Y\text{''}\alpha$$

is cofinal in b_Y, and thus $\mathcal{T}_X^\frown \pi_{X,Y}^{-1}\text{''}b_Y$ is a hull of $\mathcal{T}_Y$. Since Σ condenses well, we have that $\pi_{X,Y}^{-1}\text{''}b_Y = b_X$. This is a contradiction, as $\pi_Y^{-1}(\gamma) \in b_Y$, but $\pi_X^{-1}(\gamma) \notin b_X$.

Suppose next that $\omega_1 \leq \mathrm{cof}(\xi) \leq \mu$. We show that all nice X are $\mathcal{T}$-stable. For let X, Y be nice and $X \prec Y$; then as $\mathrm{crit}(\pi_{X,Y}) > \mu$, $\pi''_{X,Y}\xi_X$ is cofinal in ξ_Y. It follows that $c_{X,Y}$ is a cofinal branch of $\mathcal{T}_Y$. But branches in iteration trees are closed as sets of ordinals, and $\mathrm{cof}(\xi_Y) > \omega$, so $c_{X,Y} = b_Y$, as desired.

Suppose finally that $\mathrm{cof}(\xi) = \omega_2$. If the claim fails, then we can find $X, Y \in S$ such that $\gamma \in X \prec Y$, and $\pi_x^{-1}(\gamma) \in b_X \Leftrightarrow \pi_Y^{-1}(\gamma) \notin b_Y$. We now proceed to a contradiction just as in the case $\mathrm{cof}(\xi) > \mu$ of 1.25. The main point is the following. Let A code $\mathcal{T}_X$, with $A \subseteq \kappa_X$ and $A \in H_X$. By lemma 2.3, $\mathrm{Lp}(A) \cap P(\kappa_X) \subseteq H_X$. Letting $\mathcal{U}_X$ be the last normal component of $\mathcal{T}_X$, we then get that $\mathrm{Lp}^\alpha(\mathcal{M}(\mathcal{U}_X)) \subseteq H_X$ (using W^*_α at this point to see that $\mathrm{Lp}^\alpha(\mathcal{M}(\mathcal{U}_X)) \subseteq \mathrm{Lp}^\alpha(A)$). With these observations, we can proceed exactly as in 1.25. □

We can now put

$$\gamma \in \Gamma_{\xi+1}(\mathcal{T}) \Leftrightarrow \forall^* X \in S(\pi_X^{-1}(\gamma) \in b_X.$$

This completes the proof that the desired (ω_3, ω_3)-iteration strategy Γ exists. It is easy to show that it condenses well. □

2.3 Finitely many Woodins

We now prove that $V[g]$ satisfies PD, or equivalently, that in $V[g]$, for all reals x and all $n < \omega$, $M_n^\sharp(x)$ exists and is countably iterable. This is equivalent in turn to W_1^*.[24]

We want to work in V, where we can use our hypothesis on the existence of amenably closed hulls. So working in V, we show

$(*)_n$ For every bounded $A \subseteq \omega_3$, there is an active $\mathcal{M} \trianglelefteq \mathrm{Lp}(A)$ such that $\mathcal{M} \models$ there are n Woodin cardinals.

(The least such active $\mathcal{M}$ is what we mean by $M_n^\sharp(A)$.) The proof is by induction on n. The reader who has some familiarity with core model theory will see at once how the proof must go in outline. Given $\pi\colon N \cong X \prec H_{\omega_3}$, where X is amenably closed, we use the Lp-closure of N to see that the extender E_π measures enough sets that it can be added to some relevant K. To see that it is actually on this K, we need to see that the phalanx $(K, Ult(K, E_\pi), \alpha)$ is iterable, for the appropriate α. We cannot insure iterability by arranging that N is closed under ω-sequences, because $2^\omega = \omega_2 > |N|$. However, there are stationarily many candidates for X, and this lets us use the Mitchell-Schimmerling proof of weak covering in the non-countably-closed case. See [4]. Many of the complexities of [4] are irrelevant here; see [8, theorem 3.4] for an argument very close to the one we give here. We shall give more detail here than the reader familiar with that argument will need, as a service to those who are less familiar with it.

Lemma 2.6 $(*)_0$ *holds; that is, for every bounded $A \subseteq \omega_3$, $A^\sharp$ exists.*

[24]We set $J_0(\mathbb{R}) = \mathrm{HC}$, and then ordinal definability over $J_0(\mathbb{R})$ can be understood literally, rather than via some coding of the countable ordinals by reals.

Proof. By lemma 2.4, we may assume A is bounded in ω_2. Let

$$\pi\colon N \to H_{\omega_3}$$

be elementary, where N is transitive, $|N| = \omega_1$, $\pi(A) = A$, and $\mathrm{ran}(\pi)$ is amenably closed. Let $\kappa = \mathrm{crit}(\pi)$, and

$$E = \{B \subseteq \kappa \mid B \in L[A] \wedge \kappa \in \pi(B)\}.$$

Since $P(\kappa) \cap L[A] \subseteq N$ by 2.3, E is an $L[A]$-ultrafilter. It will be enough to show $\mathrm{Ult}(L[A], E)$ is wellfounded, and for this, it suffices to show that E is countably complete.

By 2.1 we have $(\omega_2^V)^{+L[A]} < \omega_3$, and hence by Jensen's covering theorem, we may assume that $\mathrm{cof}((\omega_2^V)^{+L[A]}) = \omega_2^V$, as otherwise $A^\sharp$ exists, and we are done. It follows that

$$\mathrm{cof}(\kappa^{+L[A]}) = \mathrm{cof}(\kappa) = \omega_1.$$

(Note that $\mathrm{cof}(\kappa) \neq \omega$ by amenable closure.) So if $\mathcal{B} \subseteq P(\kappa) \cap L[A]$ is countable, then we can find $\mu < \kappa^{+L[A]}$ such that $\mathcal{B} \subseteq J_\mu[A]$. Letting f map κ onto $P(\kappa) \cap J_\mu[A]$ be in $L[A]$, we can then find $\alpha < \kappa$ such that $\mathcal{B} \subseteq f``\alpha$. Let $E^* = E \cap f``\alpha$; then for $\gamma < \alpha$ we have

$$f(\gamma) \in E^* \Leftrightarrow \kappa \in \pi(f)(\gamma),$$

and since $\pi(f) \in L[A]$, this implies $E^* \in L[A]$, and hence $E^* \in N$.

$$\kappa \in \bigcap \pi``E^* = \bigcap \pi(E^*),$$

so $\pi(E^*)$ has nonempty intersection, so by elementarity E^* has nonempty intersection. Since $\mathcal{B} \subseteq E^*$, we are done. $\square$

Now assume $(*)_n$. In order to prove $(*)_{n+1}$, it suffices to show that $M^\sharp_{n+1}(A)$ exists for all A bounded in ω_2. This we shall do by showing that $K^c(A)$, as computed in some appropriate background universe R, reaches a putative $M^\sharp_{n+1}(A)$, that is, a first active level $\mathcal{P}$ satisfying "there are $n+1$ Woodin cardinals". The K^c-construction guarantees that $\mathcal{P}$ is countably iterable in R, and since we shall have $H_{\omega_2} \subseteq R$, this implies $\mathcal{P}$ is countably iterable.

R will be a model of height ω_3, closed under the function $A \mapsto M^\sharp_n(A)$. This closure is necessary in order to prove the iterability with respect to uncountable iteration trees which one needs to move from $K^c(A)$ to $K(A)$ in the case $K^c(A)$ does not reach the desired $\mathcal{P}$. Of course, we can construct such an R using $(*)_n$, but we also need some $\Omega < \omega_3$ carrying an external measure to play the role of the measurable cardinal in the basic theory of [13]. This forces us to essentially repeat the proof of lemma 2.6.

Definition 2.7 *For any A, $M_{n+\frac{1}{2}}(A)$ is the minimal countably iterable A-mouse $\mathcal{P}$ of height ω_3 such that whenever η is a cutpoint of $\mathcal{P}$, then $M^\sharp_n(\mathcal{P}|\eta) \lhd \mathcal{P}$. $M^\sharp_{n+\frac{1}{2}}(A)$ is the minimal active countably iterable A-mouse $\mathcal{P}$ such that whenever η is a cutpoint of $\mathcal{P}$, then $M^\sharp_n(\mathcal{P}|\eta) \lhd \mathcal{P}$.*

There is a sentence $\psi =$ "I am $M_{n+\frac{1}{2}}(A)$" such that for $\mathcal{P}$ a countably iterable A-mouse of height ω_3, $\mathcal{P} \models \psi \Leftrightarrow \mathcal{P} = M_{n+\frac{1}{2}}(A)$. From this we get

Lemma 2.8 *Let $j\colon M_{n+\frac{1}{2}}(A) \to \mathcal{P}$ be elementary, and $j(A) = A$. Suppose $\mathcal{P}$ is countably iterable, and $o(\mathcal{P}) = \omega_3$; then*

(1) $\mathcal{P} = M_{n+\frac{1}{2}}(A)$, and

(2) letting $\kappa = \mathrm{crit}(j)$, and $\mathcal{B} \subseteq P(\kappa)$, $\mathcal{B} \in M_{n+\frac{1}{2}}(A)$, be such that $M_{n+\frac{1}{2}}(A) \models |\mathcal{B}| = \kappa$, we have that $\{B \in \mathcal{B} \mid \kappa \in j(B)\} \in M_{n+\frac{1}{2}}(A)$.

One of the standard proofs that the existence of a nontrivial elementary embedding from L to L implies $0^\sharp$ exists generalizes, and gives

Lemma 2.9 *If there is a nontrivial elementary $j\colon M_{n+\frac{1}{2}}(A) \to M_{n+\frac{1}{2}}(A)$ such that $j \restriction (\sup(A)+1) =$ identity, then $M^\sharp_{n+\frac{1}{2}}(A)$ exists.*

As there is nothing new here, we omit the proof.

Similarly, there is a sentence expressing "I am $M^\sharp_{n+\frac{1}{2}}(A)$". If $M^\sharp_{n+\frac{1}{2}}(A)$ exists, then $M_{n+\frac{1}{2}}(A)$ is the model of height ω_3 left behind when the last extender of $M^\sharp_{n+\frac{1}{2}}(A)$ is iterated ω_3 times.

Lemma 2.10 *If $(*)_n$ holds, then for all A bounded in ω_3, $M^\sharp_{n+\frac{1}{2}}(A)$ exists.*

Proof. It is enough to prove the lemma for A which are bounded in ω_2. Fix such an A. By $(*)_n$, $M_{n+\frac{1}{2}}(A)$ exists. We have that ω_2 is inaccessible in $M_{n+\frac{1}{2}}(A)$, and that $(\omega_2)^{+M_{n+\frac{1}{2}}(A)} < \omega_3$ by 2.1. (It is easy to see that $M_{n+\frac{1}{2}}(A)$ is a definable submodel of $\mathrm{Lp}(A)$. Alternatively, one can just repeat the proof of 2.1 at this point.)

Claim. If $\mathrm{cof}((\omega_2^V)^{+M_{n+\frac{1}{2}}(A)}) < \omega_2$, then $M^\sharp_{n+\frac{1}{2}}(A)$ exists.

Proof. This is a straightforward adaptation of Jensen's proof of covering for L. $\square$

We can now simply repeat the proof of 2.6. Let $\pi\colon N \to H_{\omega_3}$, where N is transitive, $|N| = \omega_1$, with $\omega_1, \sup(A) < \kappa = \mathrm{crit}(\pi)$, and $\mathrm{ran}(\pi)$ amenably closed. We have

$$P(\kappa) \cap M_{n+\frac{1}{2}}(A) \subseteq N,$$

and hence π generates an ultrafilter E_π over $M_{n+\frac{1}{2}}(A)$. By the argument of 2.6, E_π is countably complete. It follows at once that $\mathrm{Ult}(M_{n+\frac{1}{2}}(A), E_\pi)$ is countably iterable, and so 2.8 and 2.9 imply that $M^\sharp_{n+\frac{1}{2}}(A)$ exists. $\square$

Lemma 2.11 $(*)_n \Rightarrow (*)_{n+1}$.

Proof. It is enough to show $M^\sharp_{n+1}(A)$ exists for all A bounded in ω_2, so fix such an A. For $B \subseteq \omega_2$ such that $H_{\omega_2} \subseteq L[B]$, let

$$\Omega_B = \text{ critical point of the last extender of } M^\sharp_{n+\frac{1}{2}}(B),$$

and let $K^c_B(A)$ be the output of the K^c-construction over A, as done in $M_{n+\frac{1}{2}}(B)$ up to Ω_B. All tame levels of $K^c_B(A)$ are countably iterable in $M_{n+\frac{1}{2}}(B)$, and hence really countably iterable. We may therefore assume that no active level of $K^c_B(A)$ satisfies "there are n Woodin cardinals". The closure of $M_{n+\frac{1}{2}}(B)$ under the function $x \mapsto M^\sharp_n(x)$ then implies

$$M_{n+\frac{1}{2}}(B) \models K^c_B(A) \text{ is } (\omega, \Omega_B + 1)\text{-iterable .}$$

Thus, working in $M_{n+\frac{1}{2}}(B)$, we can derive a true $K(A)$ from $K^c_B(A)$, and the basic results of [13] go through for it. We set

$$K^B(A) = K(A)^{M_{n+\frac{1}{2}}(B)}.$$

Because we consider only B which code H_{ω_2}, we see from the inductive definition of K in section 6 of [13] that $K^B(A)|\omega_2 = K^C(A)|\omega_2$ for all B, C. In fact, a slightly closer inspection of the inductive definition shows that

$$B \in L[C] \Rightarrow K^B(A)|\mu_B \trianglelefteq K^C(A),$$

where μ_B is the cardinal successor of ω_2 in $K^B(A)$. (See lemma 3.1.1 of [8] for a more general result along these lines.) Now by hypothesis (2) of 0.3, we get that

$$\sup(\{\mu_B \mid B \subseteq \omega_2 \wedge H_{\omega_2} \subseteq L[B]\}) < \omega_3,$$

and so we can fix B such that

$$\forall C \subseteq \omega_2 (H_{\omega_2} \subseteq L[C] \Rightarrow \mu_C \leq \mu_B.$$

We shall derive a contradiction by showing that in $M_{n+\frac{1}{2}}(B)$ (in fact, in H_{ω_2}) there are extenders which ought to go on the sequence of $K^B(A)$, but are not there.

Let $\mathcal{S}$ be the set of all $X \prec H_{\omega_3}$ such that X is amenably closed, and $A, B \in X$, and $\sup(A) + 1 \subseteq X$, and $|X| = \omega_1$. So $\mathcal{S}$ is stationary. For $X \in \mathcal{S}$, let

$$\pi_X \colon N_X \cong X \prec H_{\omega_3}$$

be the anticollapse map,

$$\kappa_X = \mathrm{crit}(\pi_X),$$

$$E_X = \ (\kappa_X, \omega_2)\text{-extender over } N_X \text{ derived from } \pi_X.$$

and

$$\mu_X = \pi_X^{-1}(\mu_B) \wedge K_X = \pi_X^{-1}(K^B(A)).$$

Claim 1. For any $X \in \mathcal{S}$, κ_X is inaccessible in $K^B(A)$, μ_X is the cardinal successor of κ_X in $K^B(A)$, and $K_X|\mu_X = K^B(A)|\mu_X$.

Proof. ω_2 is inaccessible in $K^B(A)$ by the proof of 2.1, so κ_X is inaccessible in K_X, and a limit cardinal in $K^B(A)$. Thus it will be enough to show that μ_X is the cardinal successor of κ_X in $K^B(A)$, and $K_X|\mu_X = K^B(A)|\mu_X$. Note that $K_X|\kappa_X = K^B(A)|\kappa_X$.

Let $\mathcal{P} \trianglelefteq K_X$ with $\rho_\omega(\mathcal{P}) = \kappa_X$. Note that $\mathcal{P} \in M_{n+\frac{1}{2}}(B)$. In order to show that $\mathcal{P} \trianglelefteq K^B(A)$, it will suffice to show that the phalanx $(K^B(A), \mathcal{P}, \kappa_X)$ is $\Omega_B + 1$-iterable in $M_{n+\frac{1}{2}}(B)$. For this, it suffices to show that the phalanx is countably iterable (where this has the obvious meaning) in $M_{n+\frac{1}{2}}(B)$, or equivalently, in V. But we can lift iteration trees on $(K^B(A), \mathcal{P}, \kappa_X)$ to trees on $(K^B(A), K^B(A), \kappa_X)$ using π_X, so the countable iterability of $K^B(A)$ yields the desired conclusion.

Similarly, let $\mathcal{P} \trianglelefteq K^B(A)$ with $\rho_\omega(\mathcal{P}) = \kappa_X$. Let $U \subseteq \kappa_X$ code the theory in $\mathcal{P}$ of parameters in $\kappa_X \cup p_\omega(\mathcal{P})$, so that $\mathcal{P}$ is coded by U. Then $U \cap \alpha \in K^B(A)|\kappa_X$ for all $\alpha < \kappa_X$, so $U \in N_X$, so $\mathcal{P} \in N_X$. Note that the phalanx $(K_x, \mathcal{P}, \kappa_X)$ is countably iterable in V, since trees on it can be lifted using π_X to trees on $(K^B(A), \mathcal{P}, \kappa_X)$, which is countably iterable in V. Let $C \subseteq \kappa_X$ code $B, \mathcal{P}$, with $C \in N_X$. It will be enough to show that $(K^C(A)^{N_X}, \mathcal{P}, \kappa_X)$ is countably iterable in N_X, for then $\mathcal{P} \trianglelefteq K^C(A)^{N_X}$ (as countable iterability implies full iterability in a universe having the necessary $\mathcal{Q}$-structures), so by our choice of B, $\mathcal{P} \trianglelefteq K_X$. Working in N_X, it suffices then to show that whenever $\psi: M \to H$ where M, H are transitive, $|M| = \omega_1$, and everything relevant is in $\mathrm{ran}(\psi)$, then $\psi^{-1}((K^C(A)^{N_X}, \mathcal{P}, \kappa_X))$ is countably iterable. However, $\psi^{-1}(K^C(A)^{N_X})$ is $\psi^{-1}(\kappa_X)$-strong in $M_{n+\frac{1}{2}}(C)^{N_X}$[25], and since $\pi_X^{-1}(B)$ and C both code $H_{\kappa_X}^{N_X}$, we have $\psi^{-1}(K^C(A)^{N_X})$ is $\psi^{-1}(\kappa_X)$-strong in $\pi_X^{-1}(M_{n+\frac{1}{2}}(B))$. This implies that $\psi^{-1}((K^C(A)^{N_X}, \mathcal{P}, \kappa_X))$ is countably iterable in $\pi_X^{-1}(M_{n+\frac{1}{2}}(B))$, or equivalently, in N_X, as desired. □

It follows that for $X \in \mathcal{S}$, E_X measures all sets in $K^B(A)$. Now for $\alpha < \omega_2$, $E_X \restriction \alpha \in H_{\omega_2} \subseteq M_{n+\frac{1}{2}}(B)$, and as κ_X is not Shelah in $K^B(A)$, some $E_X \restriction \alpha$ is not in $K^B(A)$. It follows from the arguments of [13] and the equivalence between countable iterability and iterability in our present situation that for each $X \in \mathcal{S}$, the phalanx

$$(K^B(A), \mathrm{Ult}(K^B(A), E_X), \omega_2))$$

[25]Although $\psi \notin M_{n+\frac{1}{2}}(C)^{N_x}$ is possible, we can use the fact that ψ exists to verify the inductive, joint-iterability definition is satisfied in N_X, or equivalently, in $M_{n+\frac{1}{2}}(C)^{N_X}$.

is not countably iterable.

As in [4] and [8], we find countably many functions representing in $\mathrm{Ult}(K^B(A), E_X)$ enough to guarantee the failure of countable iterability of $(K^B(A), \mathrm{Ult}(K^B(A), E_X), \omega_2)$. Taking appropriate Skolem hulls of finitely many of these functions, we find premice

$$\mathcal{H}_n^X \lhd K^B(A)|\mu_X$$

for $n < \omega$, with embeddings $\tau_n^X : \mathcal{H}_n^X \to \mathcal{H}_{n+1}^X$ such that $\tau_n^X \in K^B(A)|\mu_X$ as well, such that setting

$$\mathcal{H}^X = \lim_n \mathcal{H}_n^X$$

we have that

$$(K^B(A), \mathrm{Ult}(\mathcal{H}^X, E_X), \omega_2) \text{ is not countably iterable .}$$

If $(K^B(A), \mathcal{Q}, \kappa_X)$ is a countably iterable phalanx, then we say $\mathcal{Q}$ is *good at* X. If in addition, $(K^B(A), \mathrm{Ult}(\mathcal{Q}, E_X), \omega_2)$ is not countably iterable, then we say $\mathcal{Q}$ *lifts badly to* H_{ω_3}. Thus $\mathcal{H}^X$ is good at X, but lifts badly to H_{ω_3}. For any $X \in \mathcal{S}$, let us pick a premouse $\mathcal{Q}_X$ which is good at X, but lifts badly to H_{ω_3}, and is *minimal at* X, in the sense that whenever $\mathcal{U}$ is an iteration tree on the phalanx $(K^B(A), \mathcal{Q}_X, \kappa_X)$, then

- no proper initial segment of $\mathcal{M}_\infty^{\mathcal{U}}$ lifts badly to H_{ω_3}, and
- if $\mathcal{M}_\infty^{\mathcal{U}}$ lifts badly to H_{ω_3}, then $\mathcal{M}_\infty^{\mathcal{U}}$ lies above $\mathcal{Q}_X$ in the tree $\mathcal{U}$, and there is no dropping on the branch from $\mathcal{Q}_X$ to $\mathcal{M}_\infty^{\mathcal{U}}$.

(This is just the definition of [8, p. 3134].) As in [4], the iterability of $K^B(A)$ implies that there is such a minimal $\mathcal{Q}_X$.

Let

$$f : \omega_2 \overset{\text{onto}}{\to} H_{\mu_B}^{K^B(A)}.$$

We may assume that $f \in X$ for all $X \in \mathcal{S}$, and set $f_X = \pi_X^{-1}(f)$. Thus $\mathrm{dom}(f_X) = \kappa_X$, and for all $n < \omega$, $\mathcal{H}_n^X, \tau_n^X \in \mathrm{ran}(f_X)$. Now κ_X has cofinality ω_1, so we can find $\alpha_X < \kappa_X$ such that for all n, $\mathcal{H}_n^X, \tau_n^X \in f_X``\alpha_X$. By Fodor's lemma, we may assume that α_X is constant on $\mathcal{S}$; that is, we may fix $\alpha < \omega_2$ and assume that

$$\forall n < \omega \ \ (\mathcal{H}_n^X, \tau_n^X \in \mathrm{ran}(f_X \restriction \alpha).$$

Now fix $X \in \mathcal{S}$. Since $\mathcal{S}$ is stationary, we can fix $Y \in \mathcal{S}$ such that $X \in Y$. We have that $N_X \cup \{N_X, \pi_X, E_X\} \subseteq Y$, and letting $\pi_{X,Y} = \pi_Y^{-1}(\pi_X)$ and $E_{X,Y} = \pi_Y^{-1}(E_X)$,

$$\pi_{X,Y} : N_X \to N_Y,$$

and $E_{X,Y}$ is the (κ_X, κ_Y)-extender over N_X generated by $\pi_{X,Y}$. Clearly,

$$E_{X,Y} = E_X \restriction \kappa_Y.$$

Note that we may assume $\mathcal{Q}_X = \pi_Y^{-1}(\mathcal{Q}_X) \in N_Y$, and by the elementarity of π_Y

$$N_Y \models (K_Y, \mathrm{Ult}(\mathcal{Q}_X, E_{X,Y}), \kappa_Y) \text{ is not countably iterable.}$$

Since N_Y has the relevant $\mathcal{Q}$-structures, this implies that $(K_Y, \mathrm{Ult}(\mathcal{Q}_X, E_{X,Y}), \kappa_Y)$ is truly not countably iterable. In general, let us say that a premouse $\mathcal{R}$ which is good at X *lifts badly to* Y iff $(K_Y, \mathrm{Ult}(\mathcal{R}, E_{X,Y}), \kappa_Y)$ is not countably iterable. Thus $\mathcal{Q}_X$ lifts badly to Y.

For any n, $\mathcal{H}_n^Y \in K_Y|\mu_Y$, so we can find $\beta < \alpha$ such that $f_Y(\beta) = \mathcal{H}_n^Y$. But $\pi_{X,Y}(f_X(\beta)) = f_Y(\beta)$, so

$$\mathcal{H}_n^Y \in \mathrm{ran}(\pi_{X,Y})$$

for all n. Similarly, the τ_n^Y are in the range of $\pi_{X,Y}$. Let

$$\mathcal{M}^* = \text{ direct limit of } \pi_{X,Y}^{-1}(\mathcal{H}_n^Y) \text{ under the } \pi_{X,Y}^{-1}(\tau_n^Y).$$

It is easy to see that $\pi_{X,Y}$ induces an embedding $\sigma\colon \mathcal{M}^* \to \mathcal{Q}_Y$, and the extender of σ restricted to κ_Y is just $E_{X,Y}$. Since we can lift trees on $(K^B(A), \mathcal{M}^*, \kappa_X)$ to trees on $(K^B(A), \mathcal{Q}_Y, \kappa_Y)$ using $(\mathrm{id}\,, \sigma)$, $\mathcal{M}^*$ is good at X. Since $\mathcal{Q}_Y$ is good at Y, and there is a natural $\psi\colon \mathrm{Ult}(\mathcal{M}^*, E_{X,Y}) \to \mathcal{Q}_Y$ with $\psi \restriction \kappa_Y + 1 =$identity, $\mathcal{M}^*$ does not lift badly to Y. In fact, $\mathcal{M}^*$ *hereditarily* does not lift badly from X to Y, in the sense that $\mathcal{M}_\infty^\mathcal{V}$ does not lift badly to Y whenever $\mathcal{V}$ is an iteration tree on $(K^B(A), \mathcal{M}^*, \kappa_X)$. The reason is that $\mathcal{V}$ lifts via (id $, \sigma)$ to a tree $\mathcal{U}$ on $(K^B(A), \mathcal{Q}_Y, \kappa_Y)$. But then $(K^B(A), \mathcal{M}_\infty^\mathcal{U}, \kappa_Y)$ is countably iterable, and the copy map $\tau\colon \mathcal{M}_\infty^\mathcal{V} \to \mathcal{M}_\infty^\mathcal{U}$ generates the extender $E_{X,Y}$ because it agrees with σ on $P(\kappa_X)$. This gives an embedding from $\mathrm{Ult}(\mathcal{M}_\infty^\mathcal{V}, E_{X,Y})$ to $\mathcal{M}_\infty^\mathcal{U}$ which is the identity on $\kappa_Y + 1$, and thus the iterability of $(K^B(A), \mathrm{Ult}(\mathcal{M}_\infty^\mathcal{V}, E_{X,Y}, \kappa_Y)$.

Let $(\mathcal{U}, \mathcal{V})$ be the coiteration of the phalanxes $(K^B(A), \mathcal{Q}_X, \kappa_X)$ and $(K^B(A), \mathcal{M}^*, \kappa_X)$. This coiteration can be done in $M_{n+\frac{1}{2}}(B)$, and so we can use the hull and definability properties of $K^B(A)$ in $M_{n+\frac{1}{2}}(B)$ to analyze it. This yields that either

(a) $\mathcal{M}_\infty^\mathcal{U}$ lies above $\mathcal{Q}_X$, with no dropping in $\mathcal{U}$ along the branch $\mathcal{Q}_X$-to-$\mathcal{M}_\infty^\mathcal{U}$, or

(b) $\mathcal{M}_\infty^\mathcal{V}$ lies above $\mathcal{M}^*$, with no dropping in $\mathcal{V}$ along the branch $\mathcal{M}^*$-to-$\mathcal{M}_\infty^\mathcal{V}$.

We claim (a) holds. Otherwise, by (b) we have an embedding $\psi\colon \mathcal{M}^* \to \mathcal{M}_\infty^\mathcal{U}$, with $\psi \restriction \kappa_X$ the identity. Moreover, $\mathcal{M}_\infty^\mathcal{U}$ is strictly below $\mathcal{Q}_X$ in the relation with respect to which $\mathcal{Q}_X$ is minimal. Thus $\mathcal{M}_\infty^\mathcal{U}$, and hence $\mathcal{M}^*$, do not lift badly to H_{ω_3}. On the other hand,

$$\mathrm{Ult}(\mathcal{M}^*, E_X) = \mathrm{Ult}(\mathrm{Ult}(\mathcal{M}^*, E_{X,Y}), E_Y) = \mathrm{Ult}(\lim_n \mathcal{H}_n^Y, E_Y),$$

and $\lim_n \mathcal{H}_n^Y$ lifts badly to H_{ω_3}. This is a contradiction.

Thus (a) holds, and we have $\psi\colon \mathcal{Q}_X \to \mathcal{M}_\infty^\mathcal{V}$ which is the identity on κ_X. But $\mathcal{M}_\infty^\mathcal{V}$ does not lift badly to Y, since $\mathcal{M}^*$ hereditarily does not lift badly to Y. This implies that $\mathcal{Q}_X$ does not lift badly to Y. This contradiction completes the proof of lemma 2.11. $\square$

We get at once

Theorem 2.12 *Assume the hypotheses of theorem 0.3; then*

(1) $(*)_n$ *holds for all* $n < \omega$*, and*

(2) if g *is* V*-generic over* $\mathrm{Col}(\omega, \omega_2)$*, then* $V[g] \models \mathsf{PD}$.

2.4 The transfinite levels of $L(\mathbb{R})$.

One can join the arguments and results of subsections 2.1 and 2.2 exactly as we did in our proof of theorem 0.1 in section 1. This yields a proof of theorem 0.3. We omit further detail.

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