Embedding normalization for mouse pairs

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Abstract

This is a continuation of [2]. We outline a proof that a good background construction done in a strongly uniquely iterable universe yields mouse pairs whose iteration strategies normalize well.

1 Introduction

We assume the reader is familiar with [1], and the corrections to it in [2].

For simplicity, let us assume AD^+. Let (P, Σ) be a level of some good background construction $\mathbb{C}$, done inside some coarse Γ-Woodin pair (N^*, Σ^*). From [2] we have that Σ is defined on stacks of plus trees, has strong hull condensation, and normalizes well with respect to stacks of λ-separated trees. This implies that (P, Σ) can be compared with other such pairs. In this note, we shall outline a proof that Σ normalizes well with respect to arbitrary stacks of plus trees.

The proof has two parts. First, we extend the class of plus trees slightly, by weakening the normality requirement to a property we call *quasi-normality*. We then show directly that Σ quasi-normalizes well, by tracing through how Σ is derived from Σ^*. Basically, we are just recording what the flawed proof of Theorem 4.41 in [1] actually shows. We then use this result and a phalanx comparison argument to show that Σ normalizes well.

2 More plus trees

We wish to enlarge the class of plus trees from [2] ever so slightly. Henceforth, we shall call trees in the larger class plus trees as well.

Notation. If G is a plus extender, let us set $\mathrm{lh}(G) = \mathrm{lh}(G^-)$. We have been calling this ordinal $\gamma(G)$, but the formulae look more familiar if we call it $\mathrm{lh}(G)$.

Definition 2.1 *Let M be a premouse, then a* plus tree *on M is a system $\langle T, \langle E_\alpha \mid \alpha+1 < \mathrm{lh}(\mathcal{T})\rangle, \langle M_\alpha \mid \alpha < \mathrm{lh}(\mathcal{T})\rangle\rangle$ with the usual properties of an iteration tree, except:*

(a) either E_α is on the M_α-sequence, or $E_\alpha = F^+$ for some F on the M_α-sequence,

(b) if $\alpha < \beta$, then

(i) $\nu(E_\alpha) \leq \nu(E_\beta)$, and

(ii) if E_α is of plus type, then $\mathrm{lh}(E_\alpha) < \nu(E_\beta)$.

(c) T-pred$(\alpha+1)$ is the least β such that $\mathrm{crit}(E^{\mathcal{T}}_\alpha) < \nu(E_\beta)$ or $\beta = \alpha+1$, and

(d) $\mathcal{M}^{\mathcal{T}}_{\alpha+1} = \mathrm{Ult}(P, E^{\mathcal{T}}_\alpha)$, where β is as in (c), and $P \trianglelefteq \mathcal{M}^{\mathcal{T}}_\beta$ is as long as possible.

Definition 2.2 *A plus tree $\mathcal{T}$ is* normal *iff whenever $\alpha < \beta < \mathrm{lh}(\mathcal{T}) - 1$, then $\mathrm{lh}(E^{\mathcal{T}}_\alpha) < \mathrm{lh}(E^{\mathcal{T}}_\beta)$.*

The plus trees of [2] are precisely the normal ones. We call the conjunction of (b), (c), and (d) *quasi-normality*.

To see what's going on, suppose $\mathcal{T}$ is a plus tree in which the plus case never occurs. Thus $\nu(E_\alpha) = \lambda(E_\alpha)$. Clause (b) in 2.1 then just weakens the usual length-increasing clause in normality to "λ-nondecreasing". It is easy to see that $\mathcal{T}$ breaks up into disjoint maximal finite intervals $\alpha, \alpha+1, ...\alpha+n$ in which

$$\lambda(E_{\alpha+i}) \leq \lambda(E_{\alpha+i+1}) < \mathrm{lh}(E_{\alpha+i+1}) < \mathrm{lh}(E_{\alpha+i})$$

for all $i < n$. (Of course $n = 0$ is possible.) At the end of such an interval we get $\mathrm{lh}(E_{\alpha+n}) < \mathrm{lh}(E_{\alpha+n+1})$, which implies that $\mathrm{lh}(E_{\alpha+n}) < \lambda(E_{\alpha+n+1})$. We call $[\alpha, \alpha+n]$ a *maximal delay interval*.

It may seem pointless to consider such trees, because given a maximal delay interval $[\alpha, \alpha+n]$, we could have just skipped using $E_\alpha, ..., E_{\alpha+n-1}$, and taken $E_{\alpha+n}$ out of $\mathcal{M}^{\mathcal{T}}_\alpha$ to continue the iteration. Doing this everywhere would produce a normal iteration tree $\mathcal{S}$ with the same last model as $\mathcal{T}$, differing only in that the nontrivial delay intervals in $\mathcal{T}$ are eliminated. We call $\mathcal{S}$ the *normal companion* of $\mathcal{T}$, and write $\mathcal{S} = \mathcal{T}^{\mathrm{n}}$. So why bother with $\mathcal{T}$, why not just use $\mathcal{T}^{\mathrm{n}}$?[1]

[1] If T-pred$(\beta+1) = \alpha+k$ where $k > 0$, and ¶ $\trianglelefteq \mathcal{M}^{\mathcal{T}}_{\alpha+k}$ is what E_β is applied to in $\mathcal{T}$, then one can check that P is a point in the $\mathrm{lh}(E_{\alpha+k})$-dropdown sequence of $\mathcal{M}^{\mathcal{T}}_\alpha$. So we did not need $\mathcal{M}_{\alpha+1}, ..., \mathcal{M}_{\alpha+n}$ to form future models of $\mathcal{T}^{\mathrm{n}}$.

The answer is that we are considering trees by some iteration strategy Σ. It may happen that $\mathcal{T}$ is by Σ, but its normal companion is not. In the strategy-comparison proof, we have to live with the possibility that this happens when Σ is a background-induced strategy. In general, it happens because the factor map $\sigma\colon \mathrm{Ult}(M,E) \to i_{E^*}(M)$ is not the identity at $\lambda(E)$, so coherence at the M-level is not mirrored by coherence at the background extender level.

For example, suppose $\Sigma = \Omega(\mathbb{C}, M, \Sigma^*)$; that is, Σ is a strategy for M induced by the construction $\mathbb{C}$ and the strategy Σ^* for V. Suppose E is on the M-sequence and E^* is the background for E. If $\lambda(E) \leq \lambda(F) < \mathrm{lh}(F) < \mathrm{lh}(E)$, and $\mathcal{T} = \langle E, F\rangle$, then $\mathcal{T}^{\mathrm{n}} = \langle F\rangle$. Assuming no resurrection, the lift of $\mathcal{T}$ uses E^*, then the background $\sigma(F)^*$ for $\sigma(F)$, where $\sigma\colon \mathrm{Ult}(M,E) \to i_{E^*}(M)$ is the factor map. $\sigma(F)^*$ is provided by the construction $i_{E^*}(\mathbb{C})$. The lift of $\mathcal{T}^{\mathrm{n}}$ uses F^*, provided by $\mathbb{C}$. There is no connection between $\sigma(F)^*$ and F^*. By the coherence of $\mathbb{C}$, F^* is still a background for F in $i_{E^*}(\mathbb{C})$, but the background-induced strategy does not use it when lifting trees that extend $\mathcal{T}$. Because of this, one might have some $\mathcal{U}$ normally extending $\mathcal{T}$ of length ω such that $\Sigma(\mathcal{U}) \neq \Sigma(\mathcal{U}^{\mathrm{n}})$.

In general, any plus tree $\mathcal{T}$ can be reduced to a normal plus tree by eliminating maximal delay intervals. Here a maximal delay interval is a maximal finite interval $\alpha, \alpha+1, ...\alpha+n$ in which no $E_{\alpha+i}$ for $i < n$ is of plus type, and

$$\lambda(E_{\alpha+i}) \leq \nu(E_{\alpha+i+1}) < \mathrm{lh}(E_{\alpha+i+1}) < \lambda(E_{\alpha+i}),$$

for all $i < n$. Eliminating such intervals produces a normal plus tree $\mathcal{T}^{\mathrm{n}}$ that has essentially the same models as $\mathcal{T}$. The important difference is that we may have that $\mathcal{T}$ is by some iteration strategy Σ, while $\mathcal{T}^{\mathrm{n}}$ is not.

In a normal plus tree, M_α agrees with all M_β for $\alpha < \beta$ up to, but not at, $\mathrm{lh}(E_\alpha)$. In an arbitrary plus tree, the agreement is up to $\mathrm{lh}(E_{\alpha+k})$, where $\alpha + k$ is the last point in the delay interval to which α belongs which is $< \beta$.

If $\mathbb{C}$ is a good background construction, then plus trees on $M = M^{\mathbb{C}}_{\nu,k}$ can be lifted to V. The resulting system is

$$\mathrm{lift}(\mathcal{T}, M, \mathbb{C}) = \langle \mathcal{T}^*, \langle \eta_\xi, l_\xi \mid \xi \leq \xi_0\rangle, \langle \psi_\xi \mid \xi \leq \xi_0\rangle\rangle.$$

Here $\langle \eta_0, l_0\rangle = \langle \nu, k\rangle$ and $\psi_0 =$ id. The agreement properties of the lifting maps are given by

Proposition 2.3 *For $\alpha < \beta$,*

(1) $\psi_\beta \restriction \nu(E^{\mathcal{T}}_\alpha) = \mathrm{res}_\alpha \circ \psi_\alpha \restriction \nu(E^{\mathcal{T}}_\alpha)$, and

(2) if $E^{\mathcal{T}}_\alpha$ is of plus type, then

(a) $\psi_\beta \upharpoonright \mathrm{lh}(E_\alpha^{\mathcal{T}}) + 1 = \mathrm{res}_\alpha \circ \psi_\alpha \upharpoonright \mathrm{lh}(E_\alpha^{\mathcal{T}}) + 1$, *and*

(b) *either*

(i) $\psi_\beta \upharpoonright o(E_\alpha^{\mathcal{T}}) + 1 = \mathrm{res}_\alpha \circ \psi_\alpha \upharpoonright o(E_\alpha^{\mathcal{T}}) + 1$, *or*

(ii) $\mathrm{lh}(E_\alpha^{\mathcal{T}}) < \mathrm{crit}(E_{\alpha+1}^{\mathcal{T}}) < \nu(E_{\alpha+1}^{\mathcal{T}}) < o(E_\alpha^{\mathcal{T}})$.

This is the same agreement we stated for the plus trees of [1]. It still holds because although the $\mathrm{lh}(E_\alpha^{\mathcal{T}})$ may decrease strictly in delay intervals, the $\nu(E_\alpha^{\mathcal{T}})$ are always non-decreasing.

In Theorem 4.3, we show that if Σ is background-induced, then $\mathcal{T}$ is by Σ iff $\mathcal{T}^{\mathrm{n}}$ is by Σ. Once we have proved this, we can just work with strategies that have this property, and forget about non-normal plus trees and their delay intervals. But in order to prove 4.3, we need to consider the action of Σ on non-normal plus trees.

3 Quasi-normalizng stacks of plus trees

What the argument of [1] does show is that background-induced strategies quasi-normalize well.

Definition 3.1 *Let $\mathcal{T}$ be a plus tree on M of length $\xi + 1$, and F an extender on the $\mathcal{M}_\xi^{\mathcal{T}}$ sequence.*

(a) $\alpha(\mathcal{T}, F)$ *is the least* γ *such that* F *is on the* $\mathcal{M}_\gamma^{\mathcal{T}}$*-sequence.*

(b) $\alpha_0(\mathcal{T}, F)$ *is the least* γ *such that* $\mathrm{lh}(F) < \nu(E_\gamma^{\mathcal{T}})$, *or* $E_\gamma^{\mathcal{T}}$ *is of plus type and* $\mathrm{lh}(F) < \mathrm{lh}(E_\gamma^{\mathcal{T}})$, *or* $\gamma = \xi$.

(c) $\beta(\mathcal{T}, F)$ *is the least* γ *such that* $\mathrm{crit}(F) < \nu(E_\gamma^{\mathcal{T}})$ *or* $\gamma = \xi$.

$\alpha(\mathcal{T}, F)$ is what was called $\alpha(\mathcal{T}, F)$ in [1]. It can also be characterized as the least γ such that $\gamma = \xi$, or $\mathrm{lh}(F) < \mathrm{lh}(E_\gamma^{\mathcal{T}})$. Clearly, $\alpha(\mathcal{T}, F) \leq \alpha_0(\mathcal{T}, F)$. If $\alpha(\mathcal{T}, F) < \alpha_0(\mathcal{T}, F)$, then $\alpha_0(\mathcal{T}, F)$ is in a delay interval that begins with $\alpha(\mathcal{T}, F)$. This interval may end with $\alpha_0(\mathcal{T}, F)$, or continue beyond it.

$\beta(\mathcal{T}, F)$ is just what was called that in [1].

Let $\mathcal{T}$ be a plus tree and F be a plus extender such that F^- is on the sequence of last model of $\mathcal{T}$.[2] Let

$$\alpha = \alpha_0(\mathcal{T}, F^-).$$

[2] We are allowing the possibility that $F = F^-$.

We define the *quasi-normalization* $\mathcal{V} = V(\mathcal{T}, F)$ by

$$\mathcal{V} \restriction \alpha + 1 = \mathcal{T} \restriction \alpha + 1,$$

and

$$\mathcal{M}^{\mathcal{V}}_{\alpha+2} = \mathrm{Ult}(P, F),$$

where P is the appropriate initial segment of $\mathcal{M}^{\mathcal{T}}_{\beta}$, for $\beta = \beta(\mathcal{T}, F)$, and defining $\mathcal{M}^{\mathcal{V}}_{\alpha+1+\xi}$ for $\xi > \beta$ by copying, just as we did in the W-case. Heuristically,

$$V(\mathcal{T}, F) = \mathcal{T} \restriction (\alpha + 1)^\frown \langle F \rangle^\frown i_F \text{``} \mathcal{T}^{>\mathrm{crit}(F)}.$$

This is the same formula that defined $W(\mathcal{T}, F)$ in [1] and [2]. In the case that $\mathcal{T}$ is an ordinary normal tree, all that has changed is the meaning of α. It is easy to see that if $\mathcal{T}$ is normal, then $W(\mathcal{T}, F)$ is the normal companion of $V(\mathcal{T}, F)$. But it could be that $[\alpha - 1, \alpha]$ is a nontrivial delay interval in $V(\mathcal{T}, F)$, so even for normal $\mathcal{T}$, it is possible that $V(\mathcal{T}, F) \neq W(\mathcal{T}, F)$.

More generally, if $\mathcal{S}$ and $\mathcal{T}$ are plus trees on M, and F is a plus extender such that F^- is on the sequence of the last model of $\mathcal{S}$, and $\alpha = \alpha_0(\mathcal{S}, F^-)$ and $\beta = \beta(\mathcal{S}, F)$, and $\mathcal{T} \restriction \beta + 1 = \mathcal{S} \restriction \beta + 1$, and if $\beta < \mathrm{lh}(\mathcal{T})$, then $\mathrm{dom}(F) = \mathcal{M}^{\mathcal{T}}_{\beta} | \eta$ for some $\eta < \lambda(E^{\mathcal{T}}_{\beta})$. We define

$$V(\mathcal{T}, \mathcal{S}, F) = \mathcal{S} \restriction (\alpha + 1)^\frown \langle F \rangle^\frown i_F \text{``} \mathcal{T}^{>\mathrm{crit}(F)}.$$

Again, this is the same formula that defined $W(\mathcal{T}, \mathcal{S}, F)$, but the meaning of α has changed. If $\mathcal{T}$ and $\mathcal{S}$ are normal, then $W(\mathcal{T}, \mathcal{S}, F)$ is the normal companion of $V(\mathcal{T}, \mathcal{S}, F)$. The delay intervals in $V(\mathcal{T}, \mathcal{S}, F)$ are just those in $(\mathcal{S} \restriction \alpha + 1)^\frown \langle F \rangle$, together with the images of delay intervals in $\mathcal{T}$.

Definition 3.2 *Let $\mathcal{T}$ and $\mathcal{U}$ be plus trees on M; then a* tree embedding *of $\mathcal{T}$ into $\mathcal{U}$ is a system $\Phi = \langle u, v, \langle s_\alpha \mid \alpha < \mathrm{lh}(\mathcal{T}) \rangle, \langle t_\alpha \mid \alpha + 1 < \mathrm{lh}(\mathcal{T}) \rangle \rangle$ satisfying the same properties required by [2] in the case $\mathcal{T}$ and $\mathcal{U}$ are normal.*

Remark 3.3 It can happen that a non-normal plus tree $\mathcal{T}$ is tree embedded into a normal plus tree $\mathcal{U}$. In this case, if $[\alpha, \alpha + 1]$ is a delay in $\mathcal{T}$, then $u(\alpha) + 1 = v(\alpha + 1) <_U u(\alpha + 1)$. It can also happen that $\mathcal{T}$ is normal and $\mathcal{U}$ is not.

The agreement properties of the component maps in a tree embedding are like those listed in 5.3 of [2]. That is, letting

$$\nu^{\mathcal{T}}_{\alpha} = \sup\{\tau_\xi \mid \xi < \alpha\},$$

where

$$\tau_\xi = \begin{cases} \mathrm{lh}(E_\xi^{\mathcal{T}}) + 1 & \text{if the plus case occurs in } \mathcal{T} \text{ at } \xi \\ \lambda(E_\xi^{\mathcal{T}}) & \text{otherwise,} \end{cases}$$

we have

Proposition 3.4 *Let $\mathcal{T}$ and $\mathcal{U}$ be plus trees on M, and let $\Phi = \langle u, v, \langle s_\alpha \mid \alpha < \mathrm{lh}(\mathcal{T})\rangle, \langle t_\alpha \mid \alpha + 1 < \mathrm{lh}(\mathcal{T})\rangle\rangle$ be a tree embedding from $\mathcal{T}$ into $\mathcal{U}$; then*

(1) If $\alpha + 1 < \mathrm{lh}(\mathcal{T})$, then s_α agrees with t_α on $\nu_\alpha^{\mathcal{T}}$.

(2) If $\alpha < \beta < \mathrm{lh}(\mathcal{T})$, then s_β agrees with t_α on $\nu(E_\alpha^{\mathcal{T}}) + 1$, and on $\mathrm{lh}(E_\alpha^{\mathcal{T}})$ if α ends a maximal delay interval.

(3) If $\alpha < \beta$, then s_α agrees with s_β on $\nu_\alpha^{\mathcal{T}}$.

(4) If $\alpha < \beta$ and $\beta + 1 < \mathrm{lh}(\mathcal{T})$, then t_β agrees with t_α on $\nu(E_\alpha^{\mathcal{T}})$, and on $\mathrm{lh}(E_\alpha^{\mathcal{T}}) + 1$ if α ends a maximal delay interval.

Notice that if $E_\alpha^{\mathcal{T}}$ is of plus type, then α ends a maximal delay interval, so we have the greater agreement between t_α and future s and t maps given by (2) and (4). There is a natural tree embedding Φ of $\mathcal{T}$ into $V(\mathcal{T}, \mathcal{S}, F)$. If $\mathcal{T}$ is normal and $\Phi^*\colon \mathcal{T} \to W(\mathcal{T}, \mathcal{S}, F)$ is the natural map, then Φ is essentially the same as Φ^*, modulo some re-indexing.

Finally, if $\mathcal{T}$ is a plus tree, and $\mathcal{U}$ is a plus tree on the last model of $\mathcal{T}$, then we define $V(\mathcal{T}, \mathcal{U})$ by induction on $\mathrm{lh}(\mathcal{U})$, just as in the W-case. Setting $\mathcal{V}_\xi = V(\mathcal{T}, \mathcal{U} \restriction \xi + 1)$, we have

$$\mathcal{V}_{\gamma+1} = V(\mathcal{V}_\nu, \mathcal{V}_\gamma, F_\gamma),$$

where $\nu = U\text{-pred}(\gamma + 1)$ and $F_\gamma = \sigma_\gamma(E_\gamma^{\mathcal{U}})$, for $\sigma_\gamma\colon \mathcal{M}_\gamma^{\mathcal{U}} \to \mathcal{M}_{z(\gamma)}^{\mathcal{V}_\gamma}$ the natural map.

Definition 3.5 *Let $\langle \mathcal{T}, \mathcal{U}\rangle$ be a stack of plus trees; then $V(\mathcal{T}, \mathcal{U})$ is the* quasi-normalization *of $\langle \mathcal{T}, \mathcal{U}\rangle$. For longer stacks s, the quasi-normalization $V(s)$ is defined "bottom up": $V(s^\frown\langle \mathcal{U}\rangle) = V(V(s), \pi\mathcal{U})$, for π the t-map on last models, with direct limits under the associated tree embeddings for s of limit length.*

Remark 3.6 It is easy to see that if $\mathcal{T}$ is λ-separated, then for any plus tree $\mathcal{U}$ on its last model, $V(\mathcal{T}, \mathcal{U}) = W(\mathcal{T}, \mathcal{U})$.

Definition 3.7 *A* complete (η, θ) iteration strategy *for M with scope H_δ is an iteration strategy for M defined on stacks of length $< \eta$ consisting of plus trees in H_δ.*

Definition 3.8 *Let Σ be a complete (η, θ)-iteration strategy for M; then*

(1) Σ quasi-normalizes well for 2-stacks *iff whenever $\langle \mathcal{T}, \mathcal{U} \rangle$ is a 2-stack of plus trees by Σ such that $\mathcal{U}$ has last model Q, then*

   (a) $V(\mathcal{T}, \mathcal{U})$ is by Σ, and

   (b) letting $\mathcal{V} = V(\mathcal{T}, \mathcal{U})$ have last model R, and $\pi \colon Q \to R$ be the last σ-map of the quasi-normalization, we have that $\Sigma_{\langle \mathcal{T}, \mathcal{U} \rangle, Q} = (\Sigma_{\mathcal{V}, R})^{\pi}$.

(2) Σ quasi-normalizes well *iff all its tails Σ_s quasi-normalize well for 2-stacks.*

Clearly, if Σ quasi-normalizes well, then so do all its tail strategies. In fact, the tails quasi-normalize well for finite stacks. (See [1], section 4.1.)

Remark 3.9 If $\mathcal{T}$ is λ-separated, then $V(\mathcal{T}, \mathcal{U}) = W(\mathcal{T}, \mathcal{U})$. Thus a strategy that quasi-normalizes well must normalize well in the usual W-sense for stacks of λ-separated trees.

Using the proof of Theorem 6.2 from [2] (which corrects the the proof of Theorem 4.41 in [1], we get

Theorem 3.10 *Let $\mathbb{C}$ be a good background construction done in the universe of some coarse Γ-Woodin pair (N^*, Σ^*). Let M be a model of $\mathbb{C}$, and $\Sigma = \Omega(\mathbb{C}, M, \Sigma^*)$ be the induced complete (η, θ) strategy; then Σ quasi-normalizes well.*

Notice here that Σ^* is defined only on stacks of normal trees, while Σ is defined on stacks of merely quasi-normal trees. This is not a problem, because whenever $\mathcal{T}$ is quasi-normal, then $\mathrm{lift}(\mathcal{T}, M, \mathbb{C})_0$ is normal.

Proof. (Sketch.) The proof is the same as the proof given in [2] for the case that $\mathcal{T}$ is λ-separated. Let us adopt all the notation there, and jump to the place where the repair of [1] occurred, namely, the proof of Claim 6.4 in [2]. We have now

$$\alpha = \alpha_0(\mathcal{V}_\gamma, F),$$

and we are assuming $\alpha + 1 < \mathrm{lh}(\mathcal{V}_\gamma)$ for definiteness. We let

$$E = E_\alpha^{\mathcal{V}_\gamma},$$

and what we need to see is that

$$\psi^\gamma_{z(\gamma)} \restriction \mathrm{lh}(E) = \mathrm{res}_E^{\mathbb{C}^\gamma_\alpha} \circ \psi^\gamma_\alpha \restriction \mathrm{lh}(E).$$

We no longer know that the plus case occurs at α in $\mathcal{V}_\gamma$, as we did in the case that $\mathcal{T}$ is λ-separated. But we have set $\alpha = \alpha_0(\mathcal{V}_\gamma, F)$, instead of $\alpha = \alpha(\mathcal{V}_\gamma, F)$, so either $\mathrm{lh}(F) < \nu(E)$, or E is of plus type and $\mathrm{lh}(F) < \mathrm{lh}(E)$. In both cases, Proposition 2.3 implies that $\psi^\gamma_{z(\gamma)} \restriction \mathrm{lh}(E) = \mathrm{res}_E^{\mathbb{C}^\gamma_\alpha} \circ \psi^\gamma_\alpha \restriction \mathrm{lh}(E)$. □

4 Mouse pairs

We have extended the scope of our iteration strategies so that they act on stacks of plus trees, in the new sense. So strong hull condensation is a stronger property. Nevertheless, the old proof show that background-induced strategies have the property.

Theorem 4.1 *Let $N^* \models \mathsf{ZFC}$ + "$\mathbb{C}$ is a background construction". Let Σ^* be a (η, θ)-iteration strategy for $(N^*, \vec{F}^{\mathbb{C}})$. Suppose that $\langle \nu, k \rangle < \mathrm{lh}(\mathbb{C})$, and $\Sigma = \Omega(\mathbb{C}, M^{\mathbb{C}}_{\nu,k}, \Sigma*)$ is the complete iteration strategy for $M^{\mathbb{C}}_{\nu,k}$ induced by Σ^*. Suppose finally that Σ^* has strong hull condensation; then Σ has strong hull condensation.*

Definition 4.2 (M, Ω) *is a* pure extender pair *with scope H_δ iff*

(1) M is a pure extender premouse, and $M \in H_\delta$,

(2) Ω is a (δ, δ)- iteration strategy for M, defined on stacks of plus trees,

(3) Ω has strong hull condensation, and

(4) Ω quasi-normalizes well.

Good background constructions yield such pairs, by the arguments in [1],[2], and this note.

If (M, Ω) is a pure extender pair, then the restriction of Ω to stacks of λ-separated trees is determined by its action on single λ-separated trees. The action of Ω on single plus trees is determined by its action on λ-separated trees, as in [1] for the case of normal plus trees.[3] This then determines the action of Ω on arbitrary stacks of plus trees, since Ω quasi-normalizes well.

We have not yet shown that the strategy in a pure extender pair must normalize well in the standard W-sense. This follows from

Theorem 4.3 *Assume AD^+, and let (P, Σ) be a pure extender pair with scope HC. Let $\mathcal{T}$ be a plus tree on P; then*

(a) $\mathcal{T}$ is by Σ iff $\mathcal{T}^n$ is by Σ, and

(b) if $\mathcal{T}$ is by Σ, then $\Sigma_{\mathcal{T}} = \Sigma_{\mathcal{T}^n}$.

[3]We can define $\mathcal{T}^s$ for arbitrary plus trees as in [1, 5.7], and we have that $\mathcal{T}$ is a psuedo-hull of $\mathcal{T}^s$, so $\mathcal{T}$ is by Ω iff $\mathcal{T}^s$ is by Ω. In fact, for Ω induced by a good background construction, $\mathcal{T}$ and $\mathcal{T}^s$ lift to the same tree on the background universe.

Proof.(Sketch.) Let $\mathcal{T}$ be of minimal length so that the theorem fails. Let $b = \Sigma(\mathcal{T})$ and $c = \Sigma(\mathcal{T}^n)$. We compare the two phalanxes $\Phi(\mathcal{T}^\frown b)$ and $\Phi((\mathcal{T}^{\mathrm{n}})^\frown c)$, thinking of them as phalanxes of mouse pairs, and the comparison as involving comparison of iteration strategies. It suffices to line up the strategies acting on λ-separated trees, and a λ-separated iteration into a level of the construction in a background universe that captures the two strategies will do that, by [2].

Note that the models in $\mathcal{T}$ that do not appear in $\mathcal{T}^n$ are not models that we ever apply an extender to in the comparison process. So in effect, for $\mathcal{S} = \mathcal{T}^{\mathrm{n}}$, we are comparing $\Phi(\mathcal{S}^\frown b)$ with $\Phi(\mathcal{S}^\frown c)$, using $\Sigma_{\mathcal{T}^\frown b}$ and $\Sigma_{\mathcal{S}^\frown c}$ to iterate the two phalanxes.[4]

We reach a contradiction as in the proof from [1] that UBH holds in strategy mice. Note here that Σ has the Dodd-Jensen property, because strong hull condensation implies pullback consistency. The Dodd-Jensen part is actually a little simpler here, because we don't need to lift the tree on either phalanx. $\square$

As a corollary, we get that pure extender pairs normalize well in the sense of [1].

Corollary 4.4 *Assume AD^+, and let (P, Σ) be a pure extender pair with scope HC. Let s be a stack of normal plus trees on P with last model N, and let $\langle \mathcal{T}, \mathcal{U} \rangle$ be a stack of normal plus trees on N by Σ_s with last model Q; then*

(a) $W(\mathcal{T}, \mathcal{U})$ is by Σ_s, and

(b) letting $\mathcal{W} = W(\mathcal{T}, \mathcal{U})$ have last model R, and $\pi \colon Q \to R$ be the last σ-map of the embedding normalization, we have that $\Sigma_{s^\frown \langle \mathcal{T}, \mathcal{U} \rangle, Q} = (\Sigma_{s^\frown \langle \mathcal{W} \rangle, R})^\pi$.

Proof. Since Σ quasi-normalizes well, $V(\mathcal{T}, \mathcal{U})$ is by Σ_s. So by Theorem 4.3, $V(\mathcal{T}, \mathcal{U})^{\mathrm{n}}$ is by Σ. But $W(\mathcal{T}, \mathcal{U}) = V(\mathcal{T}, \mathcal{U})^{\mathrm{n}}$, so we have (a). We get part (b) from clause (1)(b) in Definition 3.8 and clause (b) in 4.3. $\square$

Remark 4.5 We haven't thought much about how one would normalize a stack of non-normal plus trees. Probably the natural thing is just to normalize the stack of their normal companions.

Least branch premice must be defined so that the strategy on λ-separated trees is inserted in the strategy predicate. One could go further and add the strategy on plus trees, or stacks of plus trees, but this is determined by the strategy on single λ-separated trees in a way the model can unravel. This leads to

[4] We can move up phalanxes in the comparison process by the method of the UBH proof in [1], or we can form meta-trees on $\mathcal{S}^\frown b$ and $\mathcal{S}^\frown c$. The former amounts to using a step of full normalization when you move up, and the latter amounts to using a step of embedding normalization when you move up.

Definition 4.6 (M, Ω) *is a* least branch hod pair *with scope* H_δ *iff*

(1) M *is a least branch premouse, and* $M \in H_\delta$*,*

(2) Ω *is a* (δ, δ)*- iteration strategy for* M*, defined on stacks of plus trees,*

(3) Ω *has strong hull condensation,*

(4) Ω *quasi-normalizes well, and*

(5) if s *is a stack of plus trees by* Ω *with last model* N*, then* $\dot{\Sigma}^N \subseteq \Omega_{s,N}$*.*

Again, good background constructions yield such pairs, by [1] and [2]. Theorem 4.3 and Corollary 4.4 hold for such pairs.

References

[1] J. Steel, *Normalizing iteration trees and comparing iteration strategies*, available at www.math.berkeley.edu/∼steel

[2] J. Steel, *Some λ-errors in NITCIS*, available at www.math.berkeley.edu/∼steel