§2. Construction of $\mathcal{H}_0$: adding an extender with length below κ_0^+

We have $j : V \to M$, $\mathrm{crit}(j) = \kappa_0$, as in section 0. Let g be V-generic over $\mathrm{Col}(\omega, <\kappa_0)$. ~~that AD(V, κ₀) = L(Hom*g, R*g) is a model of AD_R + DC~~ Let $\Gamma \subseteq \mathrm{Hom}^*_g$ be such that

$$L(\Gamma, \mathbb{R}^*_g) \models AD_{\mathbb{R}} + DC + \text{“} HOD \restriction \Theta$$

is the direct limit of all hod-pairs (P, Σ) s.t. Σ is fullness-pres. and has branch condensation, under the comparison maps”.

Put

$$\Theta^0 = \Theta^{L(\Gamma, \mathbb{R})},$$

$$H_0 = HOD^{L(\Gamma, \mathbb{R})} \restriction \Theta^0.$$

Let $\hat{j} : V[g] \to M[h]$ extend j, and

$$H_0^+ = Lp^{\Sigma_{H_0}}(H_0)^{\hat{j}(\Gamma)},$$

where $\Sigma_{H_0} = \bigoplus_{\alpha < \theta^0} \Sigma_{H_0(\alpha)}$, and $\Sigma_{H_0(\alpha)}$ is the common tail of all $\hat{j}(\Sigma_P)$, where $H_0(\alpha)$ is a Σ_P-iterate of P. If $\Gamma = \mathrm{Hom}_g^*$, then $\hat{j}(\Gamma) = \mathrm{Hom}_h^*$. Otherwise, any complete Γ set A has a Hom_g^* code, and $\hat{j}$ moves this code to the code of the complete $\hat{j}(\Gamma)$ set $\hat{j}(A)$.

Working in $M[h]$, we shall build a $\hat{j}(\Gamma)$-hod-pair $(\mathcal{H}, \Sigma_{\mathcal{H}})$ that extends (H_0^+, Σ_{H_0}). $\mathcal{H}$ will have a top block, and ~~that~~ θ^0 will begin that block. We shall show that if $\Gamma \neq \mathrm{Hom}_g^*$, then $(\mathcal{H}, \Sigma_{\mathcal{H}})$ either reaches $\Theta_h^{\mathbb{P}}$, or is a pointclass generator for $\hat{j}(\Gamma)$ in $M[h]$. We shall show that if $\Gamma = \mathrm{Hom}_g^*$, then $(\mathcal{H}, \Sigma_{\mathcal{H}})$ reaches $\Theta_h^{\mathbb{P}}$.

In fact, $\mathcal{H}$ and $\Sigma_{\mathcal{H}}$ are constructed in M, and $M[h]$ is just used to record properties of them. So we'll have $\mathcal{H} \in M$, and an iteration strategy Ψ for $\mathcal{H}$ with $\Psi \in M$, and defined on all $\vec{\mathcal{T}} \in M$ of size $< j(\kappa)$. Ψ will "determine itself on generic extensions" in such a way that for all l on $col(\omega, < j(\kappa))$ there is $\dot{\Psi}^l \supseteq \Psi$ defined on $HC^{M[l]}$. What we are calling $\Sigma_{\mathcal{H}}$ is then $\dot{\Psi}^h$. (So $M \models \Vdash^{col(\omega,<\kappa,)} \check{\Psi} \subseteq \dot{\Psi}$.) $\dot{\Psi}$ is a "symmetric name", in that $\dot{\Psi}^l = \dot{\Psi}^m$ whenever $HC^{M[l]} = HC^{M[m]}$.

So in M, we are constructing by induction on ξ pairs $(\mathcal{N}_\xi, \dot{\Psi}_\xi)$ such that the following induction hypotheses hold. We call them $(\dagger)$.

Induction hypotheses $(\dagger)_\xi$: In M, the following hold, for $(\mathcal{N}, \dot{\varphi}) = (\mathcal{N}_\xi, \dot{\varphi}_\xi)$:

(a) $\mathcal{N}$ is a hod premouse extending H_0^+,

(b) $\dot{\varphi}$ is a $col(\omega, <\kappa_1)$-name such that $\dot{\varphi}^g = \dot{\varphi}^h$ whenever $HC^{M[g]} = HC^{M[h]}$,

(c) $\Vdash^{col(\omega,<\kappa_1)} (\check{\mathcal{N}}, \dot{\varphi})$ is a $j(\dot{\Gamma})$-hod-pair such that $\dot{\varphi}$ is $j(\dot{\Gamma})$-fullness-preserving and has branch condensation and is positional.

It is easiest to describe the construction of $\mathcal{N}_{\xi+1}$ and $\dot{\varphi}_{\xi+1}$ if we have $j \upharpoonright \mathcal{N}_\xi \in M$. This is of course true if $o(\mathcal{N}_\xi) < \kappa_0^+$, and may be true beyond that if j witnesses more than measurability of κ_0.

Remarks

(1) If $\neg\Box_{\kappa_0}$, and we reach ξ such that $o(\mathcal{N}_\xi) = \kappa_0^+$, then

$M_\xi \models ZFC$ + "θ_0 is a strong limit of Woodins".

So although we haven't reached $O_h^{\mathbb{P}}$, we're close.

(2) If j witnesses κ_0 is huge, then we can go up to M_{κ_1}, and we will definitely reach $O_h^{\mathbb{P}}$ before that.

(3) In clause (†)(c), $\dot{\Gamma}$ is a symmetric name for Γ. We assume $\dot{\Gamma} \in V$ for simplicity; in general it will be in some size $< \kappa_0$ extension of V.

We have one further induction hypothesis: for $(M, \dot{\varphi}) = (M_\xi, \dot{\varphi}_\xi)$,

$(*)_\xi$ if $j \restriction M \in M$, then

$$M \models \Vdash^{Col(\omega, <\kappa_1)} \dot{\varphi} = j(\dot{\varphi})^+.$$

(Here on the right side, "j" should be replaced by "$j \restriction M$", to be precise.) (In the superscripts only.)

Some explanation is in order. Let $j_1 = j(j)$, with $j_1 : M \to N$, and $\kappa_2 = j_1(\kappa_1)$. We have that $j(\dot{\varphi})$ is a $col(\omega, <\kappa_2)$ name in N for a strategy for $j(\mathcal{N})$. But letting h be $col(\omega, <\kappa_1)$-generic $/M$, we can make sense of $j(\dot{\varphi})_h$ by the symmetry of $j(\dot{\varphi})$: it is the common value of all $j(\dot{\varphi})_\ell \cap V_{\kappa_2}^{N[h]}$ for $h \subseteq \ell$, ℓ on $col(\omega, <\kappa_2)$. So $j(\dot{\varphi})_h \in M[h]$.

Thus its $j \upharpoonright \mathcal{N}$ - pullback $j(\dot{\varphi})_h^j$ makes sense in $M[h]$. It is defined on all $HC^{M[h]}$. (Note $HC^{M[h]} \cup \{j \upharpoonright \mathcal{N}\} \subseteq N[h]$.)

$(*)_\xi$ then says that for any such h,

$$\dot{\varphi}_h = \left(j(\dot{\varphi})_h\right)^j .$$

This almost follows from ~~branch~~ hull condensation. Namely, let $\mathring{\varphi}_g$ be the common value of all $\mathring{\varphi}_h \cap V_{\kappa_1}^{M[g]}$, for $g \leq h$, h on $Col(\omega, <\kappa_1)$ and g on $Col(\omega, <\kappa_0)$. $\mathring{\varphi}_g \in V[g]$, and has branch condensation. Moreover, with $\hat{j}: V[g] \to M[h]$, we have $\hat{j}(\mathring{\varphi}_g) = j(\mathring{\varphi})_h$. But letting $\mathcal{T} \in V[g]$ be by $\mathring{\varphi}_g$, we have $\hat{j}(\mathcal{T})$ is by $j(\mathring{\varphi})_h$, so $\hat{j}\,\mathcal{T} \cong \hat{j}\,"\mathcal{T}$ is by $j(\mathring{\varphi})_h$. [The fact that $\hat{j}\,"\mathcal{T} \notin M[h]$ can be overcome with an absoluteness argument.] Thus $\mathcal{T}$ is by $(j(\mathring{\varphi})_h)^{\hat{j}}$.

However, the argument of the last paragraph falls short of proving $(*)_\xi$ from $(+)_\xi$, because it only works for T in $V[g]$, so that $\hat{j}(T)$ makes sense. So it only gives $\dot{\psi}_g \subseteq (j(\dot{\psi})_h)^\alpha$, not the full $(*)_\xi$.

$\boxed{\text{Insert p. 48f.}}$

Set

$$\mathcal{N}_0 = H_0^+$$

$\dot{\psi}_0$ = canonical $\mathrm{coll}(\omega, <\kappa_1)$-name in M for $\bigoplus_{\alpha<\theta^0} \Phi_\alpha$, where

Φ_α = $H_0(\alpha)$-tail of $\hat{j}(\Lambda)$, for any and all (P, Λ) s.t. $H_0(\alpha) = M_\infty(P, \Lambda)$ in $V[g]$.

We have $o(\mathcal{N}_0) < \kappa_0^+$, so $j \upharpoonright \mathcal{N}_0 \in M$. The reader can easily check $(\dagger)_0$ (see [1] and [3]). The main thing is that $\mathcal{N}_0 \models \theta^0$ is regular. For $(*)_0$, this says in our earlier notation that Σ_{H_0}, as defined on $HC^{M[h]}$, is the

We need two further induction hypotheses:

$(\dagger)_\xi(d)$: If E is on the $\mathcal{N}$-sequence and $\mathrm{crit}(E) = \theta^0$, then in M

$$\Vdash^{Col(\omega,<\kappa_1)} \check{E} \text{ is certified by } j \upharpoonright H_0^+ \text{ over } (\mathcal{N} \| \mathrm{lh} E, \mathring{\Psi}_{\mathcal{N}\|\mathrm{lh}(E)}).$$

(See definition 2.5 below for "certifies".)

$(*)_\xi(b)$: (Absolute condensation to pullbacks)

Let $\pi : R \to V_\gamma^M$ with γ large, $\pi \in M$, R transitive, $V_{\kappa_0+1} \cup \{\mathcal{N}\} \subseteq R$, and $|R| < \kappa_1$. Suppose $\pi(\overline{j(\mathring{\Psi})}) = j(\mathring{\Psi})$. Let h be M-generic over $col(\omega, <\kappa_1)$, and h_0 be R-generic over $col(\omega, <\bar{\kappa}_1)$, with $h_0 \in M[h]$; then

$$\overline{j(\mathring{\Psi})}_{h_0} \subseteq \left(j(\mathring{\Psi})_h\right)^\pi$$

j-pullback of Σ_{H_1}, as defined on $V_{\kappa_2}^{N[h]}$ (where $j_1 : M \to N$).

That follows from the fact that $j \restriction H_0$ is the iteration map $\pi : H_0 \to H_1$ by Σ_{H_0}, and Σ_{H_1} is the H_1-tail of Σ_{H_0} wherever they are defined, and Σ_{H_0} is pullback consistent, wherever it is defined.

Because we are just shooting for $O_h^{\mathbb{P}}$, we shall never add Θ^0-relevant extenders with critical point $> \Theta^0$. As a consequence, all levels of our construction will be fully sound, and we'll never have to core down. Thus we'll set

$$\mathcal{N}_\lambda = \text{“}\bigcup_{\alpha<\lambda}\text{”}\, \mathcal{N}_\alpha$$

and

$$\dot{\Psi}_\lambda = \text{canonical name for } \bigoplus_{\alpha<\lambda} \dot{\psi}_\alpha$$

for λ limit.

Now suppose we are given $(\mathcal{N}_\xi, \mathring{\Psi}_\xi) =$ ~~[illegible]~~ $(\mathcal{N}, \mathring{\Psi})$ satisfying $(\dagger)_\xi$ and $(*)_\xi$.

We assume that $j \restriction \mathcal{N} \in M$, so that $(*)_\xi$ is not vacuous, and deal with the more general case in ~~[illegible]~~ a subsequent section. We ~~[illegible]~~ shall obtain $(\mathcal{N}_{\xi+1}, \mathring{\Psi}_{\xi+1})$ in one of two ways:

(i) close under $(\mathrm{Lp}\text{-strategy})^\Gamma$, as on p. 16 ff., or

(ii) add an extender with critical point θ^0.

The main tool in our arguments will be "j-condensation", i.e. lemma 11.15 of [1], generalized slightly as lemma 2 in [3]. The form we need is

<u>Lemma 2.4</u> Let $(\mathcal{N}, \dot{\Psi})$ satisfy $(\dagger)_\xi$ and $(*)_\xi$, and suppose $j \restriction \mathcal{N} \in M$. Let h be $Col(\omega, <\kappa_1)$ generic, and suppose we have

$$\begin{tikzcd}
\mathcal{N} \arrow[rr, "j"] \arrow[d, "i"'] & & j(\mathcal{N}) \\
P \arrow[r, "k"'] \arrow[urr, "\sigma"] & Q \arrow[ur, "\tau"'] &
\end{tikzcd}$$

commuting, with $P, Q, \sigma, \tau \in M[h]$ and countable there. Let $\Sigma_P = \left(j(\dot{\Psi})_h\right)^\sigma$ and $\Sigma_Q = \left(j(\dot{\Psi})_h\right)^\tau$ be the pullback strategies.

<u>Then</u>

(i) $\Sigma_P, \Sigma_Q \in \hat{j}(\Gamma)$

(ii) $\Theta^0 = \Theta^{L(\Gamma, \mathcal{N})}$ is regular in $L(\Gamma, \mathcal{N})$, and

(iii) for all wffs φ, all $s \in P$, all $X \in \mathcal{N}$:

$$L(\hat{j}(\Gamma), j(\mathcal{N})) \models \varphi[P, \Sigma_P, s, j(X)]$$

iff

$$L(\hat{j}(\Gamma), j(\mathcal{N})) \models \varphi[Q, \Sigma_Q, k(s), j(X)].$$

<u>Proof sketch</u> We shall prove conclusion (i). The rest is proved just as in [1] and [3], so we do not give any details here. For (i), we let $j_1 = j(j)$ and $j_1 : M \to N$, and let g, h, ℓ be on ~~the~~ $\mathrm{col}(\omega, <\kappa_0)$, $\mathrm{col}(\omega, <\kappa_1)$, $\mathrm{col}(\omega, <\kappa_2)$ respectively, with $g \subseteq h \subseteq \ell$. We have, for Q, τ as in the hypotheses

$$\Sigma_Q = \left(\text{common value of all } j(\dot{\Psi})_m \cap HC^{M[h]}, \text{ for } m \geq h,\ m \text{ on } \mathrm{col}(\omega, <\kappa_2)\right)^\tau$$

by definition. So

$$\hat{j}_1(\Sigma_Q) = \left(\begin{array}{l}\text{common value of all } j_1(j(\dot{\Psi}))_m \cap HC^{N[\ell]}, \\ \text{for } m \geq \ell \text{ on } \mathrm{col}(\omega, <j_1(\kappa_2))\end{array}\right)^{\hat{j}_1(\tau)}$$

$$= \left(\begin{array}{l}\text{common value of all } j_1(j(\dot{\Psi}))_m \cap HC^{N[\ell]}, \\ \text{for } m \geq \ell \text{ on } \mathrm{col}(\omega, <j_1(\kappa_2))\end{array}\right)^{j_1 \circ \tau}$$

(since $j_1(\tau) = j_1 \circ \tau$)

$$= \left(\left(\text{common value of all } j_1(j(\dot{\psi}))_m \restriction H(^{N[\ell J]}) \right)^{j_1} \right)^{\gamma}$$

$$= j(\dot{\psi})^{\gamma}_{\ell}\ .$$

The last step is $(*)_\xi$, moved from M ~~[illegible]~~ where it holds of $\dot{\psi}$ ~~[illegible]~~ and $j \restriction \eta$, to N ~~[illegible]~~, where it holds of $j(\dot{\psi})_*$ and $j(j \restriction \eta) = j_1 \restriction j(\eta)$.

But then

$$N[\ell J] \models j(\dot{\psi})'_{\ell} \in \hat{j}_1(\hat{j}(\Gamma)),$$

so

$$N[\ell J] \models j(\dot{\psi})^{\gamma}_{\ell} \in \hat{j}_1(\hat{j}(\Gamma)),$$

so

$$N[\ell J] \models \hat{j}_1(\Sigma_Q) \in \hat{j}_1(\hat{j}(\Gamma)),$$

so

$$M[hJ] \models \Sigma_Q \in \hat{j}(\Gamma),$$

as desired.

$\boxtimes$

We add extenders when they fit on the sequence, i.e. yield hod premice, and are "certified by j" in the following sense.

<u>Def 2.5</u> Let $(\mathcal{P}, \Lambda)$ be a $\hat{j}(\Gamma)$ hod pair in $M[h]$, and suppose $\mathcal{P}$ has a top block beginning at κ. Suppose

$$k : \mathcal{P}|(\kappa^+)^{\mathcal{P}} \longrightarrow H_1^+$$

is fully elementary. We say that <u>E is k-certified over $(\mathcal{P}, \Lambda)$</u> iff

(a) $(\mathcal{P}, E)$ is a hod premouse, and

(b) for all $a \in [lh(E)]^{<\omega}$ and $X \in \mathcal{P}|(\kappa^+)^{\mathcal{P}}$,

$$X \in E_a \text{ iff } \pi^{\Lambda}_{\mathcal{P},\infty}(a) \in k(X),$$

where $\pi^{\Lambda}_{\mathcal{P},\infty} : \mathcal{P} \longrightarrow H_1$ is the map given by $(\mathcal{P}, \Lambda)$ being in the hod-limit-system of $L(\hat{j}(\Gamma), \mathbb{R}^*_h)$.

We consider first the case in which $\mathcal{N}_{\xi+1}$ is obtained by adding an extender with critical point Θ^0 to $\mathcal{N}_\xi$.

Let h be $col(\omega, <\kappa_1)$-generic.

We have $(\mathcal{N}_\xi, \dot{\Psi}_\xi) = (\mathcal{N}, \dot{\Psi})$ our current pair satisfying $(T)_\xi$ and $(*)_\xi$.

Set
$$\Sigma = \dot{\Psi}_h .$$

So in $M[h]$, where we will be working most of the time, $(\mathcal{N}, \Sigma)$ is a $\hat{j}(\Gamma)$ hod pair.

<u>Case 1</u> There is an E that is j-certified over $(\mathcal{N}, \Sigma)$.

In this case, there is a unique such E, (so $E \in M$). We set
$$\mathcal{N}_{\xi+1} = (\mathcal{N}, E).$$

Our goal now is to construct an iteration strategy Ω for $(\mathcal{N}, E)$ such that in $M[h]$, $((\mathcal{N}, E), \Omega)$ is a $\hat{j}(\Gamma)$

hod pair, and Ω is fullness preserving and has branch condensation. The construction will give a symmetric term $\dot{\Omega}$ such that $\dot{\Omega}^h = \Omega$, and we'll show (*) holds for $((\mathcal{N}, E), \dot{\Omega})$.

<u>Notation</u> If $\mathcal{P}$ is a hod premouse having a top block, then $\kappa^{\mathcal{P}}$ is the ordinal that begins the top block of $\mathcal{P}$.

What we need about (N, E) to construct an iteration strategy is just that its top block is not too complicated.

<u>Definition 2.6</u> Let $\mathcal{P}$ be a hod premouse having a top block. We say that the <u>top block of $\mathcal{P}$ is below $O^{\mathcal{P}}$</u> iff whenever E is a $\kappa^{\mathcal{P}}$-relevant extender in the top block of $\mathcal{P}$, then $\mathrm{crit}(E) = \kappa^{\mathcal{P}}$.

Note that if $\mathcal{U}$ is a normal tree on a hod premouse $\mathcal{P}$, and $[0,\alpha]_{\mathcal{U}}$ does not drop, and $E_\alpha^{\mathcal{U}}$ is taken from the top block of $\mathcal{M}_\alpha^{\mathcal{U}}$ & with $\mathrm{crit}(E_\alpha^{\mathcal{U}}) = \kappa^{\mathcal{M}_\alpha^{\mathcal{U}}}$, then $E_\alpha^{\mathcal{U}}$ is applied to $\mathcal{M}_\alpha^{\mathcal{U}}$ (i.e. $\mathcal{M}_{\alpha+1}^{\mathcal{U}} = \mathrm{Ult}(\mathcal{M}_\alpha^{\mathcal{U}}, E_\alpha^{\mathcal{U}})$), and the rest of $\mathcal{U}$ is based on $\mathcal{M}_{\alpha+1}^{\mathcal{U}}$. This is because $lh(E_\alpha^{\mathcal{U}})$ is a cutpoint of $\mathcal{M}_{\alpha+1}^{\mathcal{U}}$.

Thus if $\mathcal{P}$ has a top block that is below $o^{\mathcal{P}}$, and any non-dropping iteration of $\mathcal{P}$ can be given by a sequence $\langle(\mathcal{P}_\alpha, \vec{\mathcal{T}}_\alpha) \mid \alpha<\eta\rangle$ where $\mathcal{P}_0 = \mathcal{P}$, $\vec{\mathcal{T}}_\alpha$ is a stack of normal trees with base model $\mathcal{P}_\alpha$ and last model $\mathcal{P}_{\alpha+1}$, $\mathcal{P}_\lambda = \lim_{\alpha<\lambda} \mathcal{P}_\alpha$ for λ limit, and for each α, either

(i) $\vec{\mathcal{T}}_\alpha$ uses no extenders in the top block of $\mathcal{P}_\alpha$ or its images, or

(ii) $\mathcal{P}_{\alpha+1} = \mathrm{Ult}(\mathcal{P}_\alpha, G)$, for some G in the top block of $\mathcal{P}_\alpha$, with $\mathrm{crit}(G) = \kappa^{\mathcal{P}_\alpha}$.

Definition 2.6.1 An iteration $\langle(\mathcal{P}_\alpha, \vec{\mathcal{T}}_\alpha) \mid \alpha<\eta\rangle$ satisfying (i) and (ii) is said to be in *normal form*.

~~We now describe our complete iteration strategy~~ $\cancel{\Omega}$ ~~for~~ $\cancel{(\mathcal{M}, E)}$. ~~Set~~

$$\cancel{Q_0 = \mathcal{M}}, \quad \cancel{E_0 = E},$$

$$\cancel{\mathcal{P}_0 = (Q_0, E_0)};$$

~~and~~

$$\cancel{\Omega_0 = \Sigma}.$$

Before describing our iteration strategy for $(\mathcal{N}, E)$, we make some definitions and prove a few simple things.

<u>Definition 2.7</u> (a) Let $\sigma : R \to H_1^+$ be fully elementary, and suppose (R, Λ) is a $\hat{j}(\Gamma)$-hod pair in $M[h]$. We say Λ is <u>σ-consistent</u> iff letting $\pi : R \to H_1$ be the iteration map by Λ of the $\hat{j}(\Gamma)$-hod-limit system, $\pi \restriction K^R = \sigma \restriction K^R$.

(b) Let $\sigma : R \to j(\mathcal{N})$ be fully elem., and (R, Λ) a $\hat{j}(\Gamma)$-hod-pair in $M[h]$. We say Λ is <u>locally σ-consistent</u> iff $\Lambda_{R|(K^R)^{+R}}$ is σ-consistent.

<u>Remark</u> If $\pi : R \to H_1$ is the map of R into $M_\infty(R, \Lambda)^{\hat{j}(\Gamma)}$, the hod limit, then $\pi \restriction K^R$ is the map of $R|K^R$ into $M_\infty(R|K^R, \Lambda_{R|K^R})$ (We are assuming K^R regular in R.) So really, it's $\Lambda_{R|K^R}$ that determines local σ-consistency. We call it plain

σ-consistency in (a) because then all iterations of $\mathcal{R}$ (that don't drop) are by $\Lambda_{\mathcal{R}|K^{\mathcal{R}}}$.

<u>Remark</u> If $\sigma: \mathcal{R} \to H_1^+$, and $(\mathcal{R}, \Lambda)$ is a $\hat{j}(\Gamma)$ hod pair such that Λ is σ-consistent, then $\Lambda = \Sigma_{H_1}^{\sigma}$. This is because, letting $j_1 = j(j)$, Σ_{H_1} is pullback-consistent in $\hat{j}_1(M[h])$, and $\Sigma_{H_1(\pi(K^{\mathcal{R}}))}$ is a tail of $\hat{j}_1(\Lambda)$ there. Thus $\Sigma^{\sigma}_{H_1(\pi(K^{\mathcal{R}}))} \cap HC^{M[h]} = \hat{j}_1(\Lambda) \cap HC^{M[h]}$

$= \Lambda$.

Thus σ-consistency determines Λ in this case.

<u>Remark</u> There are probably examples of $\sigma: \mathcal{R} \to j(\eta)$ and $(\mathcal{R}, \Lambda)$ locally σ-consistent, but $\Lambda \neq \Sigma^{\sigma}_{j(\eta)}$. We do not know one at the moment, but surely local σ-consistency is not enough to determine Λ.

Here are some simple facts about σ-consistency.

(61) ~~(54)~~

~~Definition 2.7 Let $\sigma : R \to H_1^+$ be fully elementary, and suppose (R, Λ) is a $\hat{j}(\Gamma)$-hod pair in $M\Sigma hJ$. We say that Λ is σ-consistent iff letting $\pi : R \to H_1$ be the iteration map, we have $\pi \upharpoonright \kappa^R = \sigma \upharpoonright \kappa^R$.~~

~~Notice that if Λ is σ-consistent, then $\Lambda = \Sigma_{H_1}^{\sigma}$. This is because $\Sigma_{H_1(\pi(\kappa^R))}$ is a tail of $\hat{j}_1(\Lambda)$, where $j_1 = j(j)$, and $\hat{j}_1(\Lambda)$ is pullback-consistent, and $\Lambda \subseteq \hat{j}_1(\Lambda)$.~~

(61)

Proposition 2.8 Let $\sigma : R \to H_1^+$ be elem., and (R, Λ) a $\hat{j}(\Gamma)$ hod pair in $M\Sigma hJ$. The following are equivalent:

(1) Λ is σ-consistent,

(2) whenever $i : R \to S$ is by Λ, then the extender of i is given by

$$E_i = E_\sigma \upharpoonright \pi^{\Lambda}_{S,\infty} \text{''} \kappa^S$$

where $\pi^{\Lambda}_{S,\infty} : S \to H_1$ is the iteration map by Λ,

(3) $E_\pi \upharpoonright \pi(\kappa^R) = E_\sigma \upharpoonright \pi(\kappa^R)$, for $\pi = \pi^{\Lambda}_{R,\infty}$.

Remark In clause (2) of 2.6, the equation
$E_i = E_\sigma \restriction \pi^{\Lambda}_{\Delta,\infty}{}''K^R$ means: for all $a \in [K^\Delta]^{<\omega}$
and all $X \in R$, $(a, X) \in E_i$ iff $(\pi^{\Lambda}_{\Delta,\infty}(a), X) \in E_\sigma$.
Equivalently

$$a \in i(X) \text{ iff } \pi^{\Lambda}_{\Delta,\infty}(a) \in \sigma(X),$$

for $a \in [K^\Delta]^{<\omega}$ and $X \in R$. Notice that i is continuous at K^R, so it suffices here to consider $X \in R \mid K^R$.

Proof of 2.8

(1) $\Rightarrow$ (2). Let $i : R \to \Delta$ by Λ, $a \in [K^\Delta]^{<\omega}$, and $X \in R \mid K^R$. Then

$$a \in i(X) \text{ iff } \pi^{\Lambda}_{\Delta,\infty}(a) \in \pi^{\Lambda}_{\Delta,\infty}(i(X)),$$

$$\text{iff } \pi^{\Lambda}_{\Delta,\infty}(a) \in \sigma(X),$$

because $\sigma \restriction (R \mid K^R) = \pi^{\Lambda}_{R,\infty} \restriction (R \mid K^R)$.

(2) $\Rightarrow$ (3) Let $X \in R \mid K^R$ and $b \in [\pi \mid (K^R)]^{<\omega}$, where $\pi = \pi^{\Lambda}_{R,\infty}$. Let $b = \pi^{\Lambda}_{\Delta,\infty}(a)$, where $i : R \to \Delta$ is by Λ. Then

$$(b, X) \in E_\sigma \text{ iff } \pi_{S,\infty}(a) \in \sigma(X)$$
$$\text{iff } a \in i(X) \quad (\text{by } (2))$$
$$\text{iff } \pi_{S,\infty}(a) \in \pi_{S,\infty}(i(X))$$
$$\text{iff } b \in \pi_{R,\infty}(X)$$
$$\text{iff } (b, X) \in E_\pi,$$

as desired.

$(3) \to (1)$ is clear.

$\boxtimes$

Let (R, Λ) be a $j(\Gamma)$-hod pair in $N[h]$, and $\sigma : R \to j(N)$ be such that Λ is locally σ-consistent. Let $i : R \to S$ be an iteration map by $\Lambda_{R|\kappa^R}$; that is, an iteration not using any extenders in the top block or its images. There is then a natural factor map τ from $S = Ult(R, E_\sigma \restriction \pi_{S,\infty}{}''\kappa^S)$ to $j(N) = Ult(R, E_\sigma)$, given by

$$\tau(i(f)(a)) = \sigma(f)(\pi_{S,\infty}(a)).$$

These maps commute with the iteration maps by $\Lambda_{R|K^R}$, yielding

$$\begin{tikzcd} R \arrow[rr, "\sigma"] \arrow[dr, "i"'] & & j(\mathcal{M}) \\ & S \arrow[ur, "\tau"'] & \end{tikzcd}$$

Remark We didn't need anything about the full Λ to get τ.

Proposition 2.9 Let $\sigma : R \to j(\mathcal{M})$ and (R, Λ) be locally σ-consistent. Let $i : R \to S$ be an iteration map by $\Lambda_{R|K^R}$, and $\tau : S \to j(\mathcal{M})$ the factor map; then the S-tail of Λ is locally τ-consistent.

Proof Let $\eta < \kappa^S$. Since i is continuous at κ^R, we have $\eta = i(f)(a)$ where $f \in R|\kappa^R$ and $a \in [\kappa^S]^{<\omega}$. Then

$$\begin{aligned} \pi_{S,\infty}(\eta) &= \pi_{S,\infty}(i(f)(a)) \\ &= \pi_{S,\infty}(i(f))(\pi_{S,\infty}(a)) \\ &= \sigma(f)(\pi_{S,\infty}(a)) \\ &= \tau(\eta), \end{aligned}$$

as desired. $\boxtimes$

Now we want to see what happens when we touch the top block.

<u>Lemma 2.10</u> Let $(R, \overline{\Psi})$ be a $\hat{j}(\Gamma)$-hod-pair in $M[h]$, and $\sigma : R \to j(\mathcal{N})$ be such that $\overline{\Psi}$ is locally σ-consistent. Suppose we have i such that

$$\begin{tikzcd} H_0^+ \arrow[rr, "j"] \arrow[dr, "i"'] & & H_1^+ \\ & R|(\kappa^R)^{+R} \arrow[ur, "\sigma"'] & \end{tikzcd}$$

commutes. Suppose also G is on the R-sequence, $\mathrm{crit}(G) = \kappa^R$, and G is σ-certified over $(R\|\mathrm{lh}(G), \overline{\Psi}_{R\|\mathrm{lh}(G)})$. Let $\mathcal{S} = \mathrm{Ult}(R, G)$, and $\tau : \mathcal{S} \to j(\mathcal{N})$ be given by $\tau(i_G(f)(a)) = \sigma(f)(\pi^{\overline{\Psi}_{R\|\mathrm{lh}(G)}}_{R\|\mathrm{lh}G, \infty}(a))$; then

(1) τ is well-defined,

(2) $$\begin{tikzcd} R \arrow[rr, "\sigma"] \arrow[dr, "i_G"'] & & j(\mathcal{N}) \\ & \mathcal{S} \arrow[ur, "\tau"'] & \end{tikzcd}$$ commutes,

(3) the $\mathcal{S}$-tail of $\overline{\Psi}$ is locally τ-consistent.

<u>Remark</u> The same proof yields the same conclusions if we assume only that G is an amenable predicate such that $(\mathcal{R}, G)$ is a hod premouse, and G is σ-certified over $(\mathcal{R}, \Psi)$.

<u>Proof of 2.10</u> Parts (1) and (2) follow at once from the fact that G is σ-certified over $(\mathcal{R}\|h(G), \Psi_{\mathcal{R}\|h(G)})$.

For (3), we apply j-condensation, as stated in Lemma 2 of [3]. (This is why we assumed i exists.) Let $\eta = i_G(f)(a) < \kappa^{\mathcal{S}}$. Let $Q = \mathcal{R}|(\kappa^{\mathcal{R}})^{+\mathcal{R}}$ and $W = \mathcal{S}|(\kappa^{\mathcal{S}})^{+\mathcal{S}}$.

We may assume $a \in [h(G)]^{<\omega}$, and we can write

~~(with $a \in [lh\, G]^{<\omega}$) we can write~~

$$\eta = 1_G \circ i(g)(b, a), \text{ where } b \in [K^{\Phi}]^{<\omega}.$$

Then

$$L(Hom^*_h, H_1^+) \models \pi^{\Sigma_{H_0}}_{H_0^+,\infty} \text{"} g \subseteq j(g),$$

a statement about Σ_{H_0}, g, H_0^+, and $j(g)$.

So by j-condensation

$$L(Hom^*_h, H_1^+) \models \pi^{\Phi_W}_{W,\infty} \text{"} 1_G \circ i(g) \subseteq j(g)$$

Let us write $\pi^{\Psi} = \pi^{\Psi_\varphi}_{\varphi,\infty}$, $\pi^{\Phi} = \pi^{\Phi_W}_{W,\infty}$.

Then

$$\begin{aligned}\pi^{\Phi}(\eta) &= \pi^{\Phi}(1_G \circ i(g)(b,a)) \\ &= j(g)(\pi^{\Phi}(b), \pi^{\Phi}(a)) \\ &= j(g)(\pi^{\Psi}(b), \pi^{\Psi}(a))\end{aligned}$$

(by strategy coherence for Ψ, and $lh(G)$ being a cutpoint of W --- this is why we indexed that way!)

$$\begin{aligned}&= \sigma(i(g))(\sigma(b), \pi^{\Psi}(a)) \\ &= \sigma(i(g)(b))(\pi^{\Psi}(a)) = \sigma(f)(\pi^{\Psi}(a)) \\ &= \tau(\eta),\end{aligned}$$

as desired.

☒

The next lemma deals with how certification in the top block is preserved.

<u>Lemma 2.11</u> Suppose in $M[\Sigma]$ we have R, i, σ with

$$\begin{tikzcd} M \arrow[rr, "j"] \arrow[dr, "i"'] & & j(M) \\ & R \arrow[ur, "\sigma"'] & \end{tikzcd}$$

commuting, R countable, suppose $j \restriction M \in M$. Let $\Psi = \hat{j}(\Sigma)^{\sigma}$, and suppose that Ψ is locally σ-consistent. Let G be an extender from the top block of R, with $\mathrm{crit}(G) = \kappa^R$; then G is σ-certified over $(R\|lh(G), \Psi_{R\|lh(G)})$.

<u>Proof</u> For $R = M$ and $\Psi = \Sigma = \hat{j}(\Sigma)^{i}$, this is true. We express it as a collection of statements involving parameters $j(A)$ for $A \in H_0^+$, and then apply j-condensation in the form of lemma 2.4.

Note that since j is continuous at $(\kappa^M)^{+M} = o(H_0^+)$, i is continuous at $(\kappa^R)^{+R}$.

Let $\theta^0 < \xi < o(H_0^+)$, and let

$A^\xi = \langle A^\xi_\alpha \mid \alpha < \theta^0 \rangle \in H_0^+$ be an enumeration

of $H_0^+ | \xi$. Then

$$L(\mathrm{Hom}^*_{h}, j(\mathcal{N})) \models \text{for all extenders } G \text{ on the sequence of } \mathcal{N} \text{ with } \mathrm{crit}(G) = \theta^0, \text{ for all } a \in [lh\, G]^{<\omega} \text{ and all } \alpha < \theta^0$$

$$A^\xi_\alpha \in G_a \iff \pi^{\Sigma_{\mathcal{N} \| lh G}}_{\mathcal{N} \| lh G, \infty}(a) \in j(A^\xi)_{\pi^{\Sigma_{H_0}}_{H_0, \infty}(\alpha)},$$

which is a statement φ about the parameters $\mathcal{N}$, Σ, A^ξ, and $j(A^\xi)$. By 2.4, $\varphi(R, \Psi, i(A^\xi), j(A^\xi))$ holds in $L(\mathrm{Hom}^*_h, j(\mathcal{N}))$, i.e.

$$L(\mathrm{Hom}^*_h, j(\mathcal{N})) \models \text{for all extenders } G \text{ on the sequence of } R \text{ with } \mathrm{crit}(G) = \kappa^R, \text{ for all } a \in [lh(G)]^{<\omega} \text{ and all } \alpha < \kappa^R$$

$$i(A^\xi)_\alpha \in G_a \iff \pi^{\Psi_{R \| lh G}}_{R \| lh G, \infty}(a) \in j(A^\xi)_{\pi^{\Psi_Q}_{Q,\infty}(\alpha)}$$

However, notice ~~that~~ $\sigma(i(A^\xi)_\alpha) = j(A^\xi)_{\sigma(\alpha)} = j(A^\xi)_{\pi^{\Psi_Q}_{Q,\infty}(\alpha)}$, since Ψ is locally σ-consistent.

It follows that all G on the $\mathcal{R}$-sequence such that $\mathrm{crit}(G) = \kappa^{\mathcal{R}}$ are σ-certified over $(\mathcal{R} \| lh(G), \Psi_{\mathcal{R}\| lh(G)})$, so far as sets to be measured in $i(H_0^+ | \xi)\, g^0$. But i is continuous at $o(H_0^+)$, and ξ was arbitrary.

$\boxtimes$

Virtually the same proof gives

<u>Lemma 2.12</u> Suppose in $M[h]$ we have $\mathcal{R}, i, \sigma$ with

$$\begin{tikzcd} \mathcal{N} \arrow[rr, "j"] \arrow[dr, "i"'] & & j(\mathcal{N}) \\ & R \arrow[ur, "\sigma"'] & \end{tikzcd}$$

commuting and $\mathcal{R}$ countable, and $j \restriction \mathcal{N} \in M$. Suppose $j(\Sigma)^\sigma$ is locally σ-consistent. Let E be j-certified over $(\mathcal{N}, \Sigma)$, and $F = \bigcup\{i(G) \mid G \subseteq E \wedge G \in \mathcal{N}\}$; then F is σ-certified over $(\mathcal{R}, j(\Sigma)^\sigma)$.

<u>Proof</u> We apply the proof of 2.11 to the fragments $i(G)$ of F, fragment-by-fragment. We leave the details to the reader.

$\boxtimes$

We would like to apply 2.11 and 2.12 with $i : \mathcal{N} \to R$ an iteration map by Σ, or more generally, by the iteration strategy Ω for $(\mathcal{N}, E)$ we are trying to construct. The problems are, in the case of Σ-iterations

(a) how do we know $\sigma : R \to j(\mathcal{N})$ with $j = \sigma \circ i$ exists?

(b) ~~how do we know~~ why is $j(\Sigma)^{\sigma}$ the R-tail of Σ?

There are parallel problems in the case i is by Ω, ~~and~~ our strategy for $(\mathcal{N}, E)$.

More generally, let

$$\begin{tikzcd}
\mathcal{N} \arrow[rr, "j"] \arrow[dr, "k"'] & & j(\mathcal{N}) \\
& P \arrow[ur, "l"'] \arrow[rr, "i"'] & & R
\end{tikzcd}$$

be given in $M[h]$, where $j = l \circ k$, and i is an iteration by $j(\Sigma)^{l} =_{df} \Psi$, and Ψ is locally l-consistent. We'd like to find r such that

$$\begin{tikzcd} & j(\mathcal{M}) & \\ P \arrow[ur, "\ell"] \arrow[rr, "i"'] & & R \arrow[ul, "r"'] \end{tikzcd}$$

commutes, $j(\Sigma)^r$ is the R-tail of Ψ, and $j(\Sigma)^r$ is locally r-consistent. To do this, we put our iteration in normal form (cf. Def. 2.6.1). So we have $\langle (P_\alpha, \vec{\mathcal{T}}_\alpha) \mid \alpha < \eta \rangle$ with $P = P_0$, and last model R. We write $R = P_\eta$. (So R is the last model of $\vec{\mathcal{T}}_{\eta-1}$ if $\eta - 1$ exists, and $R = \lim_{\alpha<\eta} P_\alpha$ otherwise.) Let $\ell = r_0$.

We define embeddings $r_\alpha : P_\alpha \longrightarrow j(\mathcal{M})$ by induction so that

(1) $r_\gamma = r_\alpha \circ i_{\gamma\alpha}$ for $\gamma < \alpha$, $i_{\gamma\alpha} : P_\gamma \to P_\alpha$,

(2) $j(\Sigma)^{r_\alpha}$ is the P_α-tail of Ψ,

(3) the P_α-tail of Ψ is locally r_α-consistent.

Suppose first that we have r_α such that

(1) – (3) hold, and we want $\tau_{\alpha+1}$.

<u>Case 1</u> $\mathcal{P}_{\alpha+1} = \mathrm{Ult}(\mathcal{P}_\alpha, G)$, for $\mathrm{crit}(G) = \kappa^{\mathcal{P}_\alpha}$ on the $\mathcal{P}_\alpha$-sequence.

Then G is τ_α-certified over $(\mathcal{P}_\alpha \| \mathrm{lh}\, G, j(\Sigma)^{\tau_\alpha}_{\mathcal{P}_\alpha \| \mathrm{lh}(G)})$ by 2.11. So letting

$$\tau_{\alpha+1}(i_G(f)(a)) = \tau_\alpha(f)(\pi^{\Phi}_{\mathcal{P}_\alpha \| \mathrm{lh}(G), \infty}(a))$$

for $\Phi = j(\Sigma)^{\tau_\alpha}$, we have by 2.10 that $\tau_\alpha = \tau_{\alpha+1} \circ i_{\alpha,\alpha+1}$, and $j(\Sigma)^{\tau_{\alpha+1}}$ is locally $\tau_{\alpha+1}$-consistent. So it is enough to show that $j(\Sigma)^{\tau_{\alpha+1}}$ is the $\mathcal{P}_{\alpha+1}$-tail of Ψ. $\boxed{\text{\underline{Remark} See p. 73a}}$

But this follows from theorem 3.76 in the 3/25/00 version of Sargsyan's thesis [2]. ("Branch condensation pulls back".) To jog the reader's memory, here is the briefest sketch: we get a hod mouse W

<u>Remark</u> Let $\Phi = j(\Sigma)^{\tau_\alpha}$, and Λ be the $P_{\alpha+1}$-tail of Φ. Let $W = P_{\alpha+1} | (K^{P_{\alpha+1}})^{+P_{\alpha+1}}$. Lemma 2.10 then says that Λ_Q is $\tau_{\alpha+1}$-consistent. But then $\Lambda_Q = j(\Sigma)_Q^{\tau_{\alpha+1}}$, so $j(\Sigma)^{\tau_{\alpha+1}}$ is locally $\tau_{\alpha+1}$-consistent.

with ω Woodins extending $\mathcal{N}$, and having a UB representation of Σ that is moved properly. $i_{0,\alpha+1} \circ l$ ~~acts~~ extends to act on W, yielding

$$\begin{tikzcd}
W \arrow[rr, "j"] \arrow[dr, "l"'] & & j(W) \\
& W_0 \arrow[ur, "\varphi_0"] \arrow[dr] & \\
& & i_{0,\alpha+1} \circ l(W) \arrow[uu, "\varphi"']
\end{tikzcd}$$

with $\varphi \restriction P_{\alpha+1} = \tau_{\alpha+1}$, $\varphi_0 \restriction P_0 = \tau_0$.

The W-representation of Σ gets moved to the W_0-representation of its P_0-tail, and ~~Sargsyan show~~ also to $j(\Sigma)^{\tau_0}$, as Sargsyan shows. It is then further moved to its $P_{\alpha+1}$-tail, and to $j(\Sigma)^{\tau_{\alpha+1}}$.

<u>Case 2</u> $\vec{\mathcal{T}}_\alpha$ involves no extenders in the top block of P_α, or its images.

In this case we get $\tau_{\alpha+1} : P_{\alpha+1} \to j(\mathcal{N})$ as in (1) - (3) by using proposition 2.9 where we used 2.10 in case 1. We leave the details to the reader.

Finally, suppose $\lambda \leq \eta$ is a limit ordinal. We define $\tau_\lambda(i_{\alpha\lambda}(x)) = \tau_\alpha(x)$, for $\alpha < \lambda$. This gives $\tau_\lambda : P_\lambda \to j(M)$, and it is clear that (1) holds. For (2), we use Sargsyan's 3.76 of [2] again.

For (3), let $\nu < \kappa^{P_\lambda}$, and $\pi : P_\lambda \to H_1$ the iteration map by $j(\Sigma)^{\tau_\lambda} = P_\lambda$-tail of Ψ. Let $\nu = i_{\alpha\lambda}(\bar{\nu})$, and ~~$\varphi = j(\Sigma)^{\tau_\alpha} = P_\alpha$-tail~~ $\varphi : P_\alpha \to H_1$ be the iteration map by $j(\Sigma)^{\tau_\alpha} = P_\alpha$-tail of Ψ. Then

$$\tau_\lambda(\nu) = \tau_\alpha(\bar{\nu}) = \varphi(\bar{\nu})$$
$$= \pi(\nu),$$

as desired.

We have shown

Lemma 2.13 Assume ~~[illegible]~~ $j \restriction \mathcal{N} \in M$. Let

$$\begin{tikzcd} \mathcal{N} \arrow[rr, "j"] \arrow[dr, "k"'] & & j(\mathcal{N}) \\ & \mathcal{P} \arrow[ur, "\ell"] \arrow[r, "i"'] & \mathcal{R} \end{tikzcd}$$

be given in $M[h]$, where $j = \ell \circ k$ and i is an iteration map by $j(\Sigma)^\ell = \Psi$. Suppose Ψ is locally ℓ-consistent. Let Φ be the $\mathcal{R}$-tail of Ψ. Then there is a unique embedding τ such that

$$\begin{tikzcd} & j(\mathcal{N}) & \\ \mathcal{P} \arrow[ur, "\ell"] \arrow[rr, "i"'] & & \mathcal{R} \arrow[ul, "\tau"'] \end{tikzcd}$$

commutes, $\Phi = j(\Sigma)^\tau$, and Φ is locally τ-consistent.

Proof All that's left is uniqueness of τ. But since the top block of $\mathcal{P}$ is below $o^{\mathcal{P}}$, any $x \in \mathcal{R}$ has the form $i(f)(a)$, where $a \in [\kappa^{\mathcal{R}}]^{<\omega}$. But then $\tau(x) = \tau(i(f)(a)) = \ell(f)(\pi^{\Phi}_{\mathcal{R},\infty}(a))$, so τ is determined by ℓ and the $\mathcal{R}$-tail of Ψ, hence by ℓ and $\mathcal{R}$. ☒

We are now ready to define our iteration strategy Ω for $(\mathcal{N}, E)$, where E is j-certified over $(\mathcal{N}, \Sigma)$.

We are assuming $j \upharpoonright \mathcal{N} \in M$. ~~witnesses hugeness~~. Let

$Q_0 = \mathcal{N}$, $E_0 = E$, and

$$P_0 = (Q_0, E_0).$$

Let $\Lambda_0 = \Sigma$, and $\tau_0 = j \upharpoonright \mathcal{N}$. Let $\langle (P_\alpha, \vec{\mathcal{T}}_\alpha) \mid \alpha < \eta \rangle$ be an iteration of P_0 in normal form, played according to the strategy Ω that we are defining. We maintain by induction that there are $\tau_\alpha : Q_\alpha \longrightarrow j(\mathcal{N})$ so that

$$\begin{tikzcd}
Q_\beta \arrow[rr, "\tau_\beta"] \arrow[dr, "i_{\beta\alpha}"'] & & j(\mathcal{N}) \\
& Q_\alpha \arrow[ur, "\tau_\alpha"'] &
\end{tikzcd}$$

commutes, when $\beta < \alpha$, where $i_{\beta\alpha} : P_\beta \to P_\alpha$ is the iteration map, and strategies Λ_α for Q_α

such that

(1) Ω_α is good, and has branch condensation,

(2) $\Omega_\alpha = \hat{j}(\Sigma)^{\tau_\alpha}$, and

(3) Ω_α is locally τ_α - consistent,

(4) if $\vec{\mathcal{T}}_\alpha$ is on $\mathcal{Q}_\alpha$, then it is by Ω_α, and $\Omega_{\alpha+1} = (\Omega_\alpha)_{\vec{\mathcal{T}}_\alpha, \mathcal{Q}_{\alpha+1}}$, and

(5) if $\mathcal{P}_{\alpha+1} = Ult(\mathcal{P}_\alpha, E_\alpha)$, then ~~that~~ $\Omega_\alpha = (\Omega_{\alpha+1})_{\mathcal{Q}_\alpha}$.

Note that $\mathcal{Q}_\alpha$ is a cutpoint ~~initial~~ full initial segment of $\mathcal{Q}_{\alpha+1}$, ~~necessarily~~ when (5) applies.

Suppose we have $\langle \tau_\beta \mid \beta \leq \alpha \rangle$.

<u>Case 1</u> $\vec{\mathcal{T}}_\alpha$ is on $\mathcal{Q}_\alpha$.

In this case, lemma 2.13, with $\mathcal{P} = \mathcal{Q}_\alpha$ and $\mathcal{R} = \mathcal{Q}_{\alpha+1}$, gives $\tau_{\alpha+1}$ and (1)-(5).

<u>Case 2</u> $\mathcal{P}_{\alpha+1} = \mathrm{Ult}(\mathcal{P}_\alpha, E_\alpha)$.

We have that E_α is τ_α-certified over $(Q_\alpha, \mathcal{R}_\alpha)$, by 2.12 and induction hypotheses (2) and (3). By ~~...~~ the remark after 2.10, if we set

$$\tau_{\alpha+1}\left(i_{E_\alpha}(f)(a)\right) = \tau_\alpha(f)\left(\pi^{\mathcal{R}_\alpha}_{Q_\alpha,\infty}(a)\right),$$

then $\tau_{\alpha+1}$ is well-defined and $\tau_\alpha = \tau_{\alpha+1} \circ i_{\alpha,\alpha+1}$.

Moreover, setting $\mathcal{R}_{\alpha+1} = j(\Sigma)^{\tau_{\alpha+1}}$, we have as in the proof of 2.10 that $\mathcal{R}_{\alpha+1}$ is good, has branch condensation, and is locally $\tau_{\alpha+1}$ consistent. (This all uses j-condensation, lemma 2.4.)

Since Q_α is a cutpoint of $Q_{\alpha+1}$

$$\pi^{(\mathcal{R}_{\alpha+1})_{Q_\alpha}}_{Q_\alpha,\infty} \restriction o(Q_\alpha) = \pi^{\mathcal{R}_{\alpha+1}}_{Q_{\alpha+1},\infty} \restriction o(Q_\alpha)$$

$$= \tau_{\alpha+1} \restriction o(Q_\alpha) \quad \mathcal{R}_{\alpha+1} \text{ locally consis.}$$

(since $\Omega_{\alpha+1}$ is locally $\mathcal{T}_{\alpha+1}$-consistent)

$$= \pi^{\Omega_\alpha}_{Q_\alpha,\infty} \restriction o(Q_\alpha)$$

(by the definition of $\mathcal{T}_{\alpha+1}$). Pullback condensation for Ω_α and $\Omega_{\alpha+1}$ then give us (5).

Finally, if λ is a limit, we define $\mathcal{T}_\lambda$ by $\mathcal{T}_\lambda(i_{\alpha\lambda}(x)) = \mathcal{T}_\alpha(x)$, and set $\Omega_\lambda = j(\Sigma)^{\mathcal{T}_\lambda}$. We leave the rest to the reader.

This then tells us how to define Ω for one more step. We only need worry about the case that $\eta = \alpha+1$, and $\vec{\mathcal{T}}_\alpha$ is on Q_α. (Otherwise, there is no choice of branch to be made.) But in this case, we let Ω choose $\Omega_\alpha(\vec{\mathcal{T}}_\alpha)$.

This completes our definition of Ω for (M, E), assuming ~~witnesses hugeness~~ that $j \restriction \eta \in M$.

Lemma 2.15 Assume that $j \restriction \mathcal{N} \in M$, ~~j witnesses that κ_0 is huge~~, and let Ω be the iteration strategy for $(\mathcal{N}, E)$ defined above; then Ω is good.

Proof That Ω is self-consistent and coherent is just clauses (4) and (5) of our induction hypotheses. That Ω is $\hat{j}(\Gamma)$-fullness preserving is also implicit in the induction hypotheses. (We get it from j-condensation.)

$\boxtimes$

Lemma 2.16 Assume that $j \restriction \mathcal{N} \in M$, ~~j witnesses that κ_0 is huge~~, and let Ω be the iteration strategy for $(\mathcal{N}, E)$ defined above; then Ω has branch condensation.

Proof. Let $\mathcal{P} = (M, E)$, and suppose branch condensation fails. We can then find a stack $\vec{\mathcal{T}}^\frown\langle\mathcal{U}\rangle$ on $\mathcal{P}$ such that $\vec{\mathcal{T}}$ is by Ω, and there is an iteration $i : \mathcal{P} \to \mathcal{Q}$ by Ω and cofinal branches b and c of $\mathcal{U}$ and $\pi : M_b^{\mathcal{U}} \to \mathcal{Q}$ such that

(1) $i = \pi \circ i_b^{\mathcal{U}} \circ i^{\vec{\mathcal{T}}}$,

(2) $c = \Omega(\vec{\mathcal{T}}^\frown\langle\mathcal{U}\rangle)$, and

(3) $b \neq c$.

Letting $\mathcal{R}$ be the base model of $\mathcal{U}$, the picture is

$$\begin{tikzcd}
\mathcal{P} \arrow[rr, "i"] \arrow[dr, "i^{\vec{\mathcal{T}}}"'] & & \mathcal{Q} \\
& \mathcal{R} \arrow[r, "i_b"] \arrow[dr, "i_c"'] & M_b^{\mathcal{U}} \arrow[u, "\pi"'] \\
& & M_c^{\mathcal{U}}
\end{tikzcd}$$

b is $i_b \circ i^{\vec{\mathcal{T}}}$-realized, while c is by Ω.

b does not drop because that is part of the hypothesis in branch condensation. We shall show that c does not drop.

<u>Claim I.</u> $M_b^{\mathcal{U}} \models \delta(\mathcal{U})$ is Woodin.

<u>Proof</u> If η is a cardinal of a hod premouse $\mathcal{M}$, then $\mathcal{M}|\eta$ is a full level of $\mathcal{M}$. Thus $\mathcal{M}(\mathcal{U})$ is a limit of full levels of $M_b^{\mathcal{U}}$ and $M_c^{\mathcal{U}}$.

By minimizing our counterexample to branch condensation, we can arrange (letting $i = i^{\vec{w}}$)

(*) Let $\mathcal{A}$ be a full proper initial segment of $\mathcal{M}(\mathcal{U})$; then

$$\left(\Omega_{\vec{w},\pi(\mathcal{A})}\right)^{\pi} = \left(\Omega_{\vec{a}^\frown\langle\mathcal{U}\rangle^\frown c}\right)_{\mathcal{A}} .$$

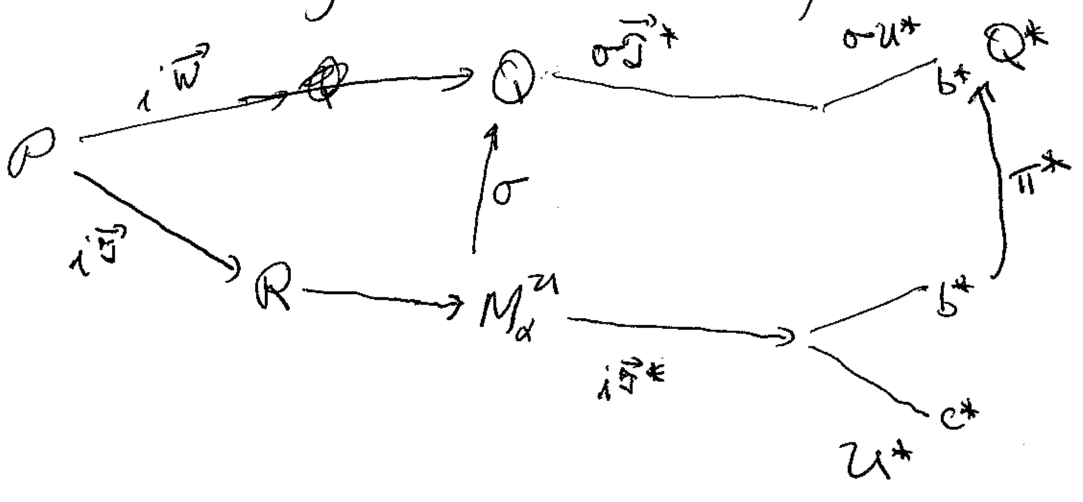
<u>Proof</u> Note that $\left(\Omega_{\vec{a}^\frown\langle\mathcal{U}\rangle^\frown c}\right)_{M(\mathcal{U})} \in \hat{J}(\Gamma)$ by our construction. (All tails of Ω, projected to proper initial segments of their base models, are in $\hat{J}(\Gamma)$.) Suppose we have chosen our

counterexample $\vec{\mathcal{T}}, \mathcal{U}, b, c, \pi, \vec{\mathcal{W}}$ so that $\left(\Omega_{\vec{\mathcal{T}}^\frown\langle\mathcal{U}\rangle^\frown c}\right)_{M(\mathcal{U})}$ has minimal possible Wadge rank. We claim that (*) holds.

For let $\mathcal{S}$ be a counterexample to (*), and let $\alpha \in b$ be such that $\mathcal{S}$ is a proper initial segment of $M_\alpha^{\mathcal{U}} \,|\, \mathrm{crit}(i^{\mathcal{U}}_{\alpha b})$. Then we have that for $\sigma = \pi \circ i_{\alpha b}$, $\sigma \restriction \mathcal{S} \cup \{\mathcal{S}\} = \pi$, and so $\left(\Omega_{\vec{\mathcal{W}},Q}\right)^\sigma_{\mathcal{S}} = \left(\Omega_{\vec{\mathcal{W}},\pi(\mathcal{S})}\right)^\pi$, so

$\left(\Omega_{\vec{\mathcal{W}},Q}\right)^\sigma_{\mathcal{S}} \neq \Omega_{\vec{\mathcal{T}}^\frown\langle\mathcal{U}|\alpha+1\rangle,\mathcal{S}}$, since the latter is just $\Omega_{\vec{\mathcal{T}}^\frown\langle\mathcal{U}\rangle^\frown c,\mathcal{S}}$ by strategy coherence.

This means we get a counterexample of the form



with $\vec{\mathcal{T}}^{*\frown}\mathcal{U}^*$ based on $\mathcal{S}$ being according to both $(\Omega_{\vec{w},Q})^\sigma_\Delta$ and $\Omega_{\vec{d}^\frown u\restriction\alpha+1,\Delta}$,

but the former choosing b^*, the latter choosing c^*, and $b^* \neq c^*$. But now

$\Omega_{\vec{d}^\frown\langle u\restriction\alpha+1\rangle^\frown \vec{d}^{*\frown}\langle u^*\rangle^\frown c^*, M(u^*)}$ is a tail

of $\Omega_{\vec{d}^\frown\langle u\rangle^\frown c,\Delta}$, so is projective in

$\Omega_{\vec{d}^\frown\langle u\rangle^\frown c,\Delta}$, so has Wadge rank

strictly less than that of $\Omega_{\vec{d}^\frown\langle u\rangle^\frown c, M(u)}$.

This contradiction yields (*).

$\boxtimes$

~~Let Φ be the join of~~

<u>Remark</u> Our notation from §1 is such that if ν is a limit of full levels of $\mathcal{M}$, where $(\mathcal{M},\Sigma)$ is a hod pair, then Σ_ν is just the join of the $\Sigma_{\nu'}$ for ν' a proper full initial segment of ν. So another way to write (*) is: $(\Omega^\pi_{\vec{w},Q})_{M(u)} = \Omega_{\vec{d}^\frown\langle u\rangle^\frown c, M(u)}$

Now let $\Phi = \left(\Omega^{\pi}_{\vec{w},q}\right)_{M(\mathcal{U})} = \Omega_{\vec{\mathcal{I}}^\frown\langle\mathcal{U}\rangle^\frown c,\, M(\mathcal{U})}$.

We claim

> **Remark** The remainder of this proof, up to Claim 2 on p. 89, is written out in more generality as 2.3.4, p. 45a, etc.

(**) $L_p^{\Phi}(M(\mathcal{U}))^{\hat{j}(\Gamma)}$ is the next full level of $M_b^{\mathcal{U}}$ after $M(\mathcal{U})$.

This follows from j-condensation, i.e. 2.4. For recall that $\mathcal{S}^-$ is $\mathcal{S}$ with its last extender predicate removed. We then have

$$\begin{tikzcd}
\mathcal{N} = P^- \arrow[r, "i"'] \arrow[rr, bend left, "j"] \arrow[d, "k"'] & Q^- \arrow[r, "\tau"] & j(\mathcal{N}) \\
(M^{\mathcal{U}}_b)^- \arrow[ur, "\pi"'] & &
\end{tikzcd}$$

where $k = i_b \circ i^{\vec{\mathcal{I}}}$. Here τ is the "realization map" that is part of the definition of Ω. We have $\Omega_{\vec{w},Q^-} = j(\Sigma)^{\tau}$, so $\Phi = j(\Sigma)^{\tau\circ\pi}_{M(\mathcal{U})}$.

This gives us, via 2.4, that $(M^{\mathcal{U}}_b)^-$ is $L_p^{\Phi}(M(\mathcal{U}))^{\hat{j}(\Gamma)}$ - full, as desired.

We then get

(***) $Lp^{\Phi}(M(\mathcal{U}))^{\hat{j}(\Gamma)}$ is the next full level of $M_c^{\mathcal{U}}$ after $\mathcal{M}(\mathcal{U})$.

If not, since Ω is $\hat{j}(\Gamma)$-fullness preserving, we must have that c dropped, and $M_c^{\mathcal{U}}$ is the first level of $Lp^{\Phi}(\mathcal{M}(\mathcal{U}))^{\hat{j}(\Gamma)}$ that is not in $M_c^{\mathcal{U}}$, and this level projects strictly accross $\delta(\mathcal{U})$. But that means $\delta(\mathcal{U})$ is not a cardinal in $M_b^{\mathcal{U}}$, contradiction.

By (**) and (***), $Lp^{\Phi}(M(\mathcal{U}))^{\hat{j}(\Gamma)} \models \delta(\mathcal{U})$ is Woodin. Let κ be the least cardinal strong to $\delta(\mathcal{U})$ in $M(\mathcal{U})$. Let $\mu = o(Lp^{\Phi}(M(\mathcal{U}))^{\hat{j}(\Gamma)})$. Let $\Psi_b = (\Omega^{\pi}_{\vec{w},Q})_{M_b|\mu}$ and $\Psi_c = \Omega_{\mathcal{T}^\frown\langle\mathcal{U}\rangle^\frown c, M_c|\mu}$.

It is possible that $\Psi_b \neq \Psi_c$. Nevertheless, because we are below O_n^P, ~~also~~ in both

M_b and M_c the κ-blocks end with ω ~~[illegible]~~ more full levels above ~~[illegible]~~ $M_c | \mu = M_b | \mu$, and moreover $\delta(\mathcal{U})$ remains Woodin in M_b and M_c. (But $\mu < \delta(\mathcal{U})^{+M_b}$ and $\mu < \delta(\mathcal{U})^{+M_c}$.) [~~For example~~ κ is not a limit of Woodins in $M(\mathcal{U})$, as otherwise we are past O_h^P. By (ZJ then, $Lp^{\Psi_b}(M_b \restriction \mu)^{M_b} \models \delta(\mathcal{U})$ is Woodin. But then the κ-block ends in M_b with ω more Lp's, unless ~~[illegible]~~ $o(Lp^{\Psi_b}(M_b | \mu))^{M_b}$ is an index on M_b of an extender with critical point κ. Since κ is not a limit of Woodins, $o(Lp^{\Psi_b}(M_b|\mu))^{M_b}$ is not such an index. Similarly on the c-side.]

This finishes our proof of claim I.

We leave it to the reader to check that we actually proved:

<u>Claim 2</u> $\delta(\mathcal{U})$ is a Woodin cutpoint in both $M_b^{\mathcal{U}}$ and $M_c^{\mathcal{U}}$, moreover neither of b and c drops, and $\left(\Omega^{\mathcal{T}}_{\vec{\nu},Q}\right)_{M(\mathcal{U})} = \Omega_{\vec{\mathcal{I}}^\frown\langle\mathcal{U}\rangle^\frown c,\, M(\mathcal{U})}$.

Let $\eta < \delta(\mathcal{U})$ be the strict sup of the Woodins of $M(\mathcal{U})$. We may and do assume that all critical points in $\mathcal{U}$ are $> \eta$. Thus we have $\delta \in R$ s.t.

$$\delta(\mathcal{U}) = i_b'(\delta) = i_c(\delta).$$

By tracing back to where δ came from in $\vec{\mathcal{I}}$, we get that

$$\{ i^{\vec{\mathcal{I}}}(f)(a) \mid f \in H_0^+ \wedge a \in [\eta]^{<\omega} \}$$

is cofinal in δ.

Now we compare the $\hat{j}(\tau)$ hod pairs (M_b^-, Λ_b) and (M_c^-, Λ_c), where

Λ_b and Λ_c are the π-pullback and Ω-tail strategies respectively. We can do this because we've added the "minus", and because $\Lambda_b = j(\Sigma)^{\pi \circ \tau}$ and $\Lambda_c = j(\Sigma)^{\sigma}$ for σ and τ that make 2.4 apply, so that they are good and have branch condensation. In fact, it's enough just to compare $(M_b^- | \mu, \Psi_b)$ with $(M_c^- | \mu, \Psi_c)$, where $M_b^- | \mu = M_c^- | \mu = Lp^{\Phi}(M(\mathcal{U}))^{\hat{j}(\Gamma)}$, and Ψ_b and Ψ_c are the strategies induced by Λ_b and Λ_c. Letting the comparison trees act on M_b and M_c, we get the ~~diagram~~ following diagram. Notice that the comparison trees are normal (you compare with a backgrounded construction, and no strategy disagreements show up), moreover they can be written as $\mathcal{T} ^\frown \mathcal{U}_1 ^\frown b_1$ and $\mathcal{T} ^\frown \mathcal{U}_1 ^\frown c_1$, where $\mathcal{T}$ is normal on $M(\mathcal{U}) | \eta$ and by both strategies, and all crits in $\mathcal{U}_1$ are $> i^{\mathcal{T}}(\eta)$.

$$
\begin{tikzcd}
 & & & j(\mathcal{N}) \\
\mathcal{N}=\mathcal{P}^- \arrow[urrr, bend left, "j"] \arrow[rr] \arrow[dr, "i^{\mathcal{I}}"'] & & \mathcal{Q}^- \arrow[ur, "\rho"] \arrow[r] & \mathcal{W} \arrow[u, "\lambda"'] \\
 & \mathcal{R}^- \arrow[r] \arrow[dr] & M_b^- \arrow[u, "\pi"] \arrow[r, "k \quad i_{b_1}"] & \mathcal{A} \arrow[u, "\varphi"'] \\
 & & M_c^- \arrow[r, "\ell \quad i_{c_1}"] & \nu \arrow[d, "\sigma"] \\
 & & & j(\mathcal{N})
\end{tikzcd}
$$

Here k and ℓ are the embeddings of $\mathcal{I}$. They agree on $M_b^- | \mu$. Let

$$\mathcal{L} = i^{\mathcal{U}_1}_{b_1}(k(M_b^- | \mu)) = i^{\mathcal{U}_1}_{c_1}(\ell(M_c^- | \mu))$$

be the common lined up part of $\mathcal{A}$ on ν, and

$$\psi = \text{common } \mathcal{L}\text{-tail of } \psi_b \text{ and } \psi_c.$$

The map λ of our diagram is given by 2.13, which also tells us that the W-tail of Ω is $j(\Sigma)^{\lambda}$. So

$$\Psi = \left(j(\Sigma)^{\lambda \circ \varphi}\right)_{\mathcal{L}}.$$

The map σ is part of the definition of Ω, and we therefore have

$$\Psi = \left(j(\Sigma)^{\sigma}\right)_{\mathcal{L}}.$$

Let $\Lambda_{\mathcal{S}} = j(\Sigma)^{\lambda\circ\varphi}$ and $\Lambda_{\mathcal{V}} = j(\Sigma)^{\sigma}$. These are $j(\Gamma)$ fullness preserving, positional, and have branch condensation, by 2.4. $\mathcal{L}$ is a cutpoint inside both $\mathcal{S}$ and $\mathcal{V}$. So the maps $\pi^{\Lambda_{\mathcal{S}}}_{\mathcal{S},\infty} : \mathcal{S} \longrightarrow H_1$ and $\pi^{\Lambda_{\mathcal{V}}}_{\mathcal{V},\infty} : \mathcal{V} \longrightarrow H_1$ agree on $\mathcal{L}$ with $\pi^{\Psi}_{\mathcal{L},\infty} : \mathcal{L} \longrightarrow H_1$. We write

$$\pi^{\Psi} = \pi^{\Psi}_{\mathcal{L},\infty} = \pi^{\Lambda_{\mathcal{S}}}_{\mathcal{S},\infty} \restriction \mathcal{L} = \pi^{\Lambda_{\mathcal{V}}}_{\mathcal{V},\infty} \restriction \mathcal{L}.$$

Claim 3 Let $\gamma < \delta$ and $\gamma = i^{\vec{\mathcal{I}}}(f)(a)$, where $a \in [\eta]^{<\omega}$ and $f \in H_0^+$, ~~f: [\theta]~~ with $f : [\Theta^0]^{|a|} \to \Theta^0$. Then

$$i_{b_1} \circ k \circ i_b(\gamma) = i_{c_1} \circ l \circ i_c(\gamma).$$

Proof i_b does not move a, and i_{b_1} does not move $k(a)$, so

$$i_{b_1} \circ k \circ i_b(\gamma) = i_{b_1} \circ k \circ i_b \circ i^{\vec{\mathcal{I}}}(f)(k(a)).$$

Similarly

$$i_{c_1} \circ l \circ i_c(\gamma) = i_{c_1} \circ l \circ i_c \circ i^{\vec{\mathcal{I}}}(f)(l(a)).$$

But $k(a) = l(a)$. Let $a^* = k(a) = l(a)$.

Now $\pi^{\Sigma_{H_0}}_{H_0,\infty}(f) \subseteq j(f)$, so by j-condensation (lemma 2 of [3] is enough here),

$$\pi^{\Lambda_\mathcal{S}}_{\mathcal{S},\infty}(i_{b_1} \circ k \circ i_b \circ i^{\vec{\mathcal{I}}}(f)) \subseteq j(f)$$

and

$$\pi^{\Lambda_\mathcal{V}}_{\mathcal{V},\infty}(i_{c_1} \circ l \circ i_c \circ i^{\vec{\mathcal{I}}}(f)) \subseteq j(f).$$

But note that $i_{b_1} \circ k \circ i_b(\gamma) \in \mathcal{L}$ and $i_{c_1} \circ l \circ i_c(\gamma) \in \mathcal{L}$. This is where $\pi^{\Lambda_\Delta}_{\Delta,\infty}$ and $\pi^{\Lambda_\to}_{\to,\infty}$ agree with $\pi^{\Psi}_{\mathcal{L},\infty}$, so we get

$$\pi^{\Psi}(i_{b_1} \circ k \circ i_b(\gamma)) = j(f)(\pi^{\Psi}(a^*))$$
$$= \pi^{\Psi}(i_{c_1} \circ l \circ i_c(\gamma)).$$

This implies $i_{b_1} \circ k \circ i_b(\gamma) = i_{c_1} \circ l \circ i_c(\gamma)$, as desired.

$\boxtimes$

But then $\operatorname{ran} i_{b_1} \cap \operatorname{ran} i_{c_1}$ is cofinal in $\delta(\mathcal{U}_1)$, so $b_1 = c_1$. This gives

<u>Claim 4</u>. Let $\gamma = i^{\mathcal{J}}(f)(a) < \delta$, where $a \in [\eta]^{<\omega}$ and $f \in H_0^+$; then $k \circ i_b(\gamma) = l \circ i_c(\gamma)$.

But $k \restriction \delta(\mathcal{U}) = l \restriction \delta(\mathcal{U})$, so we have $i_b(\gamma) = i_c(\gamma)$ for all γ as in claim 4. But such γ are cofinal in δ, so $b = c$, a contradiction.

Lemma 2.16 $\boxtimes$

A very similar proof yields

<u>Lemma 2.17</u> Assume that $j \restriction \mathcal{N} \in M$, and let Ω be the iteration strategy for $(\mathcal{N}, E)$ defined above; then Ω is positional.

<u>Proof</u> Let (Q, F) be an Ω-iterate of $(\mathcal{N}, E)$ via two different stacks $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$. Let $i = i^{\vec{\mathcal{T}}}$ and $k = i^{\vec{\mathcal{U}}}$. Suppose that $\Omega_{\vec{\mathcal{T}},(Q,F)} \neq \Omega_{\vec{\mathcal{U}},(Q,F)}$. Clearly there is no ambiguity about how to iterate via the top extender predicate, so by perhaps iterating further, we may assume

$$\Omega_{\vec{\mathcal{T}},Q} \neq \Omega_{\vec{\mathcal{U}},Q}.$$

We may also assume $\vec{\mathcal{I}}$ and $\vec{\mathcal{U}}$ were in normal form, so that we have realization maps

$$\tau : Q \longrightarrow j(\mathcal{N})$$

and

$$\sigma : Q \longrightarrow j(\mathcal{N})$$

such that

$$\tau \circ i = \sigma \circ k = j \restriction \mathcal{N},$$

and

$$\Omega_{\vec{\mathcal{I}}, Q} = j(\Sigma)^{\tau}$$

and

$$\Omega_{\vec{\mathcal{U}}, Q} = j(\Sigma)^{\sigma}.$$

Moreover, $\Omega_{\vec{\mathcal{I}}, Q}$ is τ-consistent, and $\Omega_{\vec{\mathcal{U}}, Q}$ is σ-consistent. We'd like to see $\sigma = \tau$, a contradiction. Note that $Q = \{i(f)(a) \mid f \in \mathcal{N}$ and $a \in [K^Q]^{<\omega}\} = \{k(g)(b) \mid g \in \mathcal{N}$ and $b \in [K^Q]^{<\omega}\}$. So $\tau''Q = \{j(f)(\tau(a)) \mid f \in \mathcal{N} \wedge a \in [K^Q]^{<\omega}\}$ and $\sigma''Q = \{j(f)(\sigma(a)) \mid f \in \mathcal{N} \wedge a \in [K^Q]^{<\omega}\}$. It is enough to show $\tau''Q = \sigma''Q$, so it is enough to show $\tau \restriction K^Q = \sigma \restriction K^Q$. But these are the iteration maps by $\Omega_{\vec{J}, Q \restriction K^Q}$ and $\Omega_{\vec{U}, Q \restriction K^Q}$ respectively. So it is enough to show that $\Omega_{\vec{J}, Q|K^Q} = \Omega_{\vec{U}, Q|K^Q}$.

Suppose not. Then we can get a stack $\vec{W}$ on Q by both strategies with last model R and $i^{\vec{W}} = l : Q \to R$, and a normal tree $\mathcal{U}$ on R such that for some cofinal $b \neq c$,

~~b is by $\Omega_{\vec{J}}$~~

$\vec{W} {}^\frown \langle \mathcal{U}, b \rangle$ is by $\Omega_{\vec{J}, Q|K^Q}$,

and

$\vec{W} {}^\frown \langle \mathcal{U}, c \rangle$ is by $\Omega_{\vec{U}, Q|K^Q}$.

Let Ψ_b be the $M_b^{\mathcal{U}}$-tail of $\Omega_{\vec{J}, Q|K^Q}$ (which is positional), and Ψ_c the $M_c^{\mathcal{U}}$-tail of $\Omega_{\vec{J}, Q|K^Q}$. By minimizing, as in the proof of 2.16, we may assume

$$(\Psi_b)_{M(\mathcal{U})} = (\Psi_c)_{M(\mathcal{U})}.$$

Now using 2.3.4 on p. 45a and following (re-written in the proof of 2.16) we ~~gam~~ may assume

(i) neither b nor c drops, and $\delta(\mathcal{U})$ is a Woodin cutpoint in both M_b and M_c,

(ii) there are $\eta < \delta \in R$ such that

$$i_b(\delta) = i_c(\delta) = \delta(\mathcal{U})$$

and there are no Woodins of R in (η, δ), and all crits in $\mathcal{U}$ are $> \eta$

(iii) $\delta = \sup\{ l \circ i(f)(a) \mid a \in [\eta]^{<\omega} \wedge f \in \mathcal{N} \}$

$= \sup\{ l \circ k(f)(a) / a \in [\eta]^{<\omega} \wedge f \in \mathcal{N} \}$.

Further, we have j-realizations

$\varphi : M_b \to j(\mathcal{N})$ and $\rho : M_c \to j(\mathcal{N})$
such that $\Psi_b = j(\Sigma)^{\varphi}$ and $\Psi_c = j(\Sigma)^{\rho}$.

The picture is

$$\begin{tikzcd}
\mathcal{N} \arrow[rrrr, "j"] \arrow[dddd, "j"'] \arrow[dr, bend left, "k"] \arrow[dr, bend right, "i"'] & & & & j(\mathcal{N}) \\
& Q \arrow[dr, "\theta"] & & & \\
& & R \arrow[r, "i_b"] \arrow[dr, "i_c"'] & M_b^{\mathcal{U}} \arrow[uur, "\varphi"] \arrow[dr, "i_{b_1}^{\mathcal{U}_1}"] & \\
& & & M_c^{\mathcal{U}} \arrow[dlll, "\rho"] \arrow[dr, "i_{c_1}^{\mathcal{U}_1}"] & \mathcal{S} \arrow[uuu, "\pi"'] \\
j(\mathcal{N}) & & & & \mathcal{W} \arrow[llll, bend left, "\psi"]
\end{tikzcd}$$

Here, as in 2.16, $\mathcal{U}_1$ is a single normal tree used to compare $(\mathcal{N}, (\Psi_b)_{\mathcal{N}})$

with $(\mathcal{N}, (\Psi_c)_{\mathcal{N}})$, where $\mathcal{N} = L_p^{\Lambda}(M(u))^{\Gamma}$, for $\Lambda = (\Psi_b)_{M(u)} = (\Psi_c)_{M(u)}$. ~~[illegible]~~

Remark The reader may notice that the maps k and l of the proof of 2.16 (see diagram p. 91) have become the identity. That can be arranged by comparing $(\mathcal{N}, (\Psi_b)_{\mathcal{N}})$ and $(\mathcal{N}, (\Psi_c)_{\mathcal{N}})$ with the $\Lambda_{R|\eta}$ - hod-mouse construction of a sufficiently rich $N_x^{\#}$. It's not necessary to do so, it just simplifies the diagram.

Claim 7 Let $\gamma < \delta$ and $\gamma = l \circ k(f)(a)$, where $f \in H_0^+$ and $a \in [\eta]^{<\omega}$; then $i_{b_1} \circ i_b(\gamma) \in$ ~~[illegible]~~ $\operatorname{ran}(i_{c_1} \circ i_c)$

~~Proof. Just as in claim 3 of 2.16~~

<u>Proof</u> We have to arrange things a little differently than we did in the branch condensation proof, because the single embedding $i^{\vec\sigma}$ of that proof has been replaced by two embeddings, k and i.

Let

$$a^* = i_{b_1} \circ i_b(a)$$
$$= i_{c_1} \circ i_c(a),$$

and

$$a^{**} = \pi(a^*) = \varphi(a^*).$$

We have

$$\pi(i_{b_1} \circ i_b(\cancel{\ }\, l \circ k(f)(a)) =$$
$$\pi(i_{b_1} \circ i_b \circ l \circ k(f)(a^*))$$
$$= j(f)(a^{**}).$$

By $\bar{j}$-condensation

Also

$$\begin{aligned} \psi(i_{c_1} \circ i_c (l \circ i(f)(a))) &= \psi(i_{c_1} \circ i_c \circ l \circ i(f)(a^*)) \\ &= j(f)(a^{**}). \end{aligned}$$

We are using here that π and ψ agree on the common lined up part of $\mathcal{S}$ and $\mathcal{V}$, where they are the iteration map, to see that $\psi(a^*) = \pi(a^*)$. Moreover, $j(f)(a^{**}) = \pi(i_{b_1} \circ i_b(\gamma))$ is in the part of $\mathcal{H}_1$ where π^{-1} and ψ^{-1} agree. So applying π^{-1} and ψ^{-1},

$$i_{b_1} \circ i_b (l \circ k(f)(a)) = i_{c_1} \circ i_c (l \circ i(f)(a)).$$

Thus $i_{b_1} \circ i_b(\gamma) \in \operatorname{ran}(i_{c_1} \circ i_c)$. $\boxtimes$

Remark We are grateful to Nam Trang for pointing out that the branch condensation argument was not quite enough.

~~Claim~~ As before, Claim I gives $b_1 = c_1$, and then $b = c$. That contradiction completes the proof of 2.17.

$\boxtimes$

Let $\dot{\Omega}$ be the natural name in M for $\Omega = \dot{\Omega}_h$. Let $(\mathcal{N}_{\xi+1}, \dot{\Psi}_{\xi+1}) = ((\mathcal{N}, E), \dot{\Omega})$. We have now verified all of $(\dagger)_{\xi+1}$ (a)–(d). We turn to $(*)_{\xi+1}$.

<u>Lemma 2.18</u> Assume $j \restriction \mathcal{N} \in M$. Then $M \models \Vdash_{\mathrm{Col}(\omega, <\kappa_1)} \dot{\Omega} = j(\dot{\Omega})^{j \restriction \mathcal{N}}$.

<u>Proof</u>. Let h on $\mathrm{col}(\omega, <\kappa_1)$ be arbitrary. Let $j_1 = j(j)$, with $j_1 : M \to N$. Let $\kappa_2 = j_1(\kappa_1)$, and l be $\mathrm{col}(\omega, <\kappa_2)$ generic with $h \subseteq l$. We want to see

It's worth abstracting the strategy-uniqueness result behind the proofs of 2.16 and 2.17.

<u>Lemma 2.17.1</u> (Uniqueness of pullbacks)
Assume $(*)_\xi$ and $(\dagger)_\xi$, for $(M, \mathring{\Psi}) = (N_{\xi+1}, \mathring{\Psi}_{\xi+1})$. Let h be M-generic over $\mathrm{col}(\omega, <\kappa_1)$, and suppose in $M[h]$ we have

$$\begin{tikzcd}
M \arrow[rr, "j"] \arrow[dr, "i"] \arrow[d, "k"] \arrow[dd, bend right, "j"'] & & j(M) \\
Q \arrow[d, "\tau"] & P \arrow[ur, "\sigma"'] & \\
j(M) & &
\end{tikzcd}$$

commuting, with P and Q countable. Suppose

$$P|\mu = Q|\mu = \mathcal{W},$$

where $\mathcal{W}$ is a full cutpoint in both P and Q.

Suppose that whenever $\delta \in \mathcal{W}$ is Woodin, then there is a cutpoint $\eta < \delta$ of $\mathcal{W}$ such that $\delta = \sup\{ i(f)(a) \mid f \in H_0^+ \wedge a \in [\eta]^{<\omega} \}$; then

$$\left(j(\mathring{\Psi})_h^{\sigma} \right)_{\mathcal{W}} = \left(j(\mathring{\Psi})_h^{\tau} \right)_{\mathcal{W}}$$

We leave the proof of 2.17.1 to the reader; it's right there in the proofs of 2.16 and 2.17.

<u>Remark</u> Because of the demand that $\mathcal{W}$ be a cutpoint, either we have $\mathcal{P} = \mathcal{Q} = \mathcal{W}$, or $\mathcal{W}$ is below the top blocks of $\mathcal{P}$ and $\mathcal{Q}$.

In the case $\mathcal{W}$ is below the top blocks, only $(*)_0$ and $(+)_0$ matter. We have

$$\begin{tikzcd} H_0^+ \arrow[rr, "j"] \arrow[dr, "i"] \arrow[d, "k"'] \arrow[dd, bend right, "j"'] & & H_1^+ \\ Q \arrow[d, "\tau"] & \mathcal{P} \arrow[ur, "\sigma"'] & \\ H_1^+ & & \end{tikzcd}$$

with $\mathcal{P}|\mu = \mathcal{Q}|\mu = \mathcal{W}$, and we conclude that the two ~~the~~ pullbacks of Σ_{H_1} to $\mathcal{W}$ are the same.

that $\left(j(\mathring{\Omega})_\ell\right)^{j\restriction\mathcal{M}} \cap HC^{M[h]} = \mathring{\Omega}_h$.

Let I be an iteration of $P_0 = (\mathcal{M}, E)$ by $\mathring{\Omega}_h$ that is in normal form. Say $I = \langle (P_\alpha, \vec{T}_\alpha) \mid \alpha < \beta \rangle$. We show by induction on β that the j-lift jI is by $j(\mathring{\Omega})_\ell$. (Note here that $j \restriction \mathcal{M} \in N$, because $j\restriction\mathcal{M} = j_1^{-1} \circ j_1(j\restriction\mathcal{M})$. So ~~then~~ $jI \in N[\ell]$.)

The induction is obvious if β is a limit, so let $\beta = \alpha+1$. Let $P_\nu = (Q_\nu, E_\nu)$. Let $R_\nu = (S_\nu, F_\nu)$ be the νth model of jI. We then have the diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "j"] \arrow[d, "i"'] & j(\mathcal{M}) \arrow[r, "j_1"] \arrow[d, "k"] & j_1(j(\mathcal{M})) \\
Q_\alpha \arrow[r, "\sigma"'] & S_\alpha \arrow[ur, "\tau"'] &
\end{tikzcd}$$

Here i and k are the iteration maps of $\mathcal{I}$ and $j\mathcal{I}$, σ is the copy map, and τ is the j_1-realization map that we get out of our definition of $j(\mathring{\Omega})_\ell$ in $N[\ell]$, from $j(j\restriction M) = j_1 \restriction j(M)$. ($\tau$ is uniquely determined by $j\mathcal{I}$.)

We may assume $\vec{\mathcal{T}}_\alpha$ is on Q_α, as otherwise $P_{\alpha+1} = Ult(P_\alpha, E_\alpha)$, $R_{\alpha+1} = Ult(R_\alpha, F_\alpha)$, and $j\mathcal{I}$ is by $j(\mathring{\Omega})_\ell$. Let

$$\Phi = (P_\alpha\text{-tail of } \mathring{\Omega}_h)_{Q_\alpha}$$

and

$$\Lambda = (R_\alpha\text{-tail of } j(\mathring{\Omega})_\ell)_{S_\alpha}.$$

We must see that $\Phi \subseteq \Lambda^\sigma$. Recall that $\mathring{\Psi}$ is our name for the strategy of M. We have that $\Lambda = j_1(j(\mathring{\Psi}))_\ell^\tau$. Φ is a pullback of $j(\mathring{\Psi})_h$. Namely, Let $\varphi : Q_a \longrightarrow j(N)$, $\varphi \in M[h]$, be the j-realization such that $\Phi = (j(\mathring{\Psi})_h)^{\varphi}$, and Φ is φ-consistent.

Using $\hat{j}_1 : M[h] \longrightarrow N[\ell]$, we get the diagram

$$\begin{tikzcd}
 & N \arrow[r, "j"] \arrow[dl, "j"'] \arrow[dd, "i"] & j(N) \arrow[r, "j_1"] \arrow[dd, "k"] & j_1(j(N)) \\
j(N) \arrow[dd, "j_1"'] & & & \\
 & Q_\alpha \arrow[ul, "\varphi"] \arrow[r, "\sigma"'] \arrow[dl, "\hat{j}_1(\varphi)"] & S_\alpha \arrow[uur, "r"'] & \\
j_1(j(N)) & & &
\end{tikzcd}$$

From the point of view of $N[\ell]$, Φ is the $\hat{j}_1(\varphi)$-pullback of $j_1(j(\mathring{\varphi}))_\ell$ (intersected with $H^{M[h]}$). This is because, by $(*)_\xi(a)$ moved over, $j(\mathring{\varphi})_\ell = (j_1(j(\mathring{\varphi}))_\ell)^{j_1 \restriction j(N)}$. On the other hand, $\underline{\Lambda}$ is the $\tau \circ \sigma$ - pullback of $\dot{\delta}_1(j(\dot{\varphi}))_\ell$. We can now use the method of 2.16 and 2.17 to show those two pullback strategies are consistent with each other. We omit further detail for now.

<u>Remark</u> One can apply 2.17.1 in NZLJ, with $P = Q = Q_\alpha$.

$\boxtimes$

<u>Remark</u> In the situation above, we won't have $\sigma \circ \tau = \hat{\dot{\delta}}_1(\varphi)$ unless i happened to be an iteration of $\mathcal{M} | \kappa^{\eta}$. Otherwise $\sigma \circ \tau$ moves things up further than $\hat{\dot{\delta}}_1(\varphi)$, so $\underline{\Lambda}$ is not $\sigma \circ \tau$ - consistent.

<u>Lemma 2.19</u> $(*)_{\xi+1}$ (b) holds.

<u>Proof</u> Let $\pi : R \to V_\gamma^M$ where R is transitive, $V_{\kappa_0+1} \cup \{\mathcal{M}\} \subseteq R$, $|R| < \kappa_1$, and $\pi \in M$. Let h be $\mathrm{col}(\omega, <\kappa_1)$-generic over M, and $h_0 \in M[h]$ be $\mathrm{col}(\omega, <\bar{\kappa}_1)$ generic over R. Here $\pi(\bar{\kappa}_1) = \kappa_1$. Let $\pi(\overline{j(\dot{\Omega})}) = j(\dot{\Omega})$. We must show that $\overline{j(\dot{\Omega})}_{h_0} \subseteq j(\dot{\Omega})_h^\pi$.

If not then we have an iteration

$$i : (\overline{j(\mathcal{M})}, \overline{j(E)}) \longrightarrow (P, F)$$

that is by both $\overline{j(\dot{\Omega})}_{h_0}$ and $j(\dot{\Omega})_h^\pi$, and such that the projected strategies $(\overline{j(\dot{\Omega})}_{h_0})_P$ and $(j(\dot{\Omega})_h^\pi)_P$ disagree on some normal tree $\mathcal{U}$ on P, with $\mathcal{U} \in R[h_0]$.

Let $\Lambda = \left(\overrightarrow{j(\mathring{\Omega})_{h_0}}\right)_P$ and $\Phi = \left(j(\mathring{\Omega})_h^{\pi}\right)_P$. Let

$$\hat{j}_1 : M[h] \longrightarrow N[\ell],$$

with ℓ on $\mathrm{col}(\omega, <\kappa_2)$. Note that ~~that~~ $\pi \in N$. So in $N[\ell]$, we have the diagram

$$\begin{tikzcd}
M \arrow[r, "\bar{j}"] & \overline{j(M)} \arrow[r, "\pi"] \arrow[dl, "\bar{j}_1"'] \arrow[d, "i"] & j(M) \arrow[r, "j_1"] \arrow[d] & j_1(j(M)) \\
\overline{j_1(j(M))} \arrow[d, "\pi"] & P \arrow[l, "\varphi"] \arrow[r, "\sigma"'] & {} \arrow[ur, "\tau"'] & \\
j_1(j(M)) & & &
\end{tikzcd}$$

Here φ is the realization we get from the definition of Ω, σ is the copy map, and τ is the realization we get from the definition of $j(\mathring{\Omega})_\ell$.

Let

$$\Sigma = \mathring{\Psi}_h ,$$

so that in $N[\ell]$

$$\Lambda \subseteq \hat{j}_1(\hat{j}(\Sigma))^{\pi \circ \varphi}$$

and

$$\Phi \subseteq \hat{j}_1(\hat{j}(\Sigma))^{\tau \circ \sigma}.$$

The first inclusion takes a bit of proof, as we must see that $\overline{j_1(j(\mathring{\Psi}))_{h_0}} \subseteq$

$\hat{j}_1(\hat{j}(\Sigma))^{\pi}$. But this just follows from $(*)_\xi(b)$ holding in N of $j(\mathring{\Psi})$.

Note here that $R \cap N \in N$, so $R \cap N \in N$, (assume here N is closed under κ_1 sequences in M, which we may), and that we can extend h_0 to a $\mathrm{col}(\omega, <\bar{\kappa}_2)$-generic over $R \cap N$ in $N[\ell]$.

But now 2.17.1 applied in $N[\alpha]$ gives that the two pullbacks of $\hat{j}_1(\hat{j}(\Sigma))$, by $\pi \circ \varphi$ and $\sigma \circ \sigma$ respectively, are equal. So they agree on $\mathcal{U}$, contradiction.

$\boxtimes$

We have now verified all of $(\dagger)_{\xi+1}$ and $(*)_{\xi+1}$. The motivation for $(*)_{\xi+1}(b)$ is brought out by the following.

<u>Lemma 2.20</u> Let h be $\mathrm{col}(\omega, <\mathring{\kappa}_1)$ generic over M; then $\Omega_h \in \mathrm{Hom}^*_h$.

<u>*Proof*</u> Let h_0 be $\mathrm{col}(\omega, \{\mathring{\Sigma}\})^{V[\alpha \mathbb{Z}\,0]}$-generic with $h_0 \subseteq h$. We show that Ω_h ~~class~~ ~~$\alpha\ M[h_0]$~~ has a $<\kappa_1$-UB code in $M[h_0]$. ~~For this, we need there~~

Let $\varphi(v_0, \dots, v_4)$ be the formula:

"it is forced in $col(\omega, < v_0)$ that v_1 is a stack of iteration trees on v_2 that is according to the v_3-pullback of $(v_4)_g$".

thus for γ reasonably closed and any size $< \kappa_1$ - generic n over $M[h_0]$, and any $\vec{\mathcal{T}} \in M[h_0, n]$,

$$\vec{\mathcal{T}} \text{ is by } \dot{\Omega}_{h_0 \frown n} \text{ iff } V_\gamma^{M[h_0,n]} \models \varphi[\kappa_1, \vec{\mathcal{T}}, n, j \restriction M, j(\dot{\Omega})]$$

It is enough to see that for any such n, and any

$$\sigma : S \to V_\gamma^{M[h_0,n]}$$

with $\sigma, S \in M[h_0, n]$ and countable there, S transitive, enough in $ran(\sigma)$, we have for all $\vec{\mathcal{T}} \in S = R[h_0, n]$

$$\vec{\mathcal{T}} \text{ is by } \dot{\Omega}_{h_0 \frown n} \text{ iff } R[h_0,n] \models \varphi[\bar{\kappa}_1, \vec{\mathcal{T}}, n, \overline{j \restriction M}, \overline{j(\dot{\Omega})}]$$

Here $\sigma(\langle\bar{\kappa}_1, j\restriction\bar{\eta}, j(\bar{\dot{\Omega}})\rangle) = \langle\kappa_1, j\restriction\eta, j(\dot{\Omega})\rangle$.

We may assume $R \in M$ and $\sigma\restriction R = \pi \in M$.
Suppose $R[h_0, n] \models \varphi[\bar{\kappa}_1, \vec{I}, \eta, j\restriction\eta, j(\dot{\Omega})]$.
Let $m \in M[h]$ be $R[h_0, n]$-generic over $\mathrm{col}(\omega, <\bar{\kappa}_1)$. Note that

$$\overline{j(\dot{\Omega})_m} \subseteq \left(j(\dot{\Omega})_h\right)^{\bar{\pi}}$$

by $(*)_{\xi+1}(b)$. Thus $\vec{I}$ is by $j(\dot{\Omega})_h^{\pi \circ \overline{j\restriction\eta}} = j(\dot{\Omega})_h^{j}$, as $\pi \circ \overline{j\restriction\eta} = j\restriction\eta$. But then $\vec{I}$ is by $\dot{\Omega}_h$ by $(*)_{\xi+1}(a)$.

This gives the $\Leftarrow$ direction of our desired equivalence, and the $\Rightarrow$ direction has a similar proof.

$\boxtimes$

Appendix to §2

Let $(\mathcal{M}, \Phi)$ be a $\hat{j}(\Gamma)$-hod-pair in $M[h]$, with Φ being the $\hat{j}(\Gamma)$-fullness preserving, and having branch condensation.

Let

$$\tau : \mathcal{M} \longrightarrow \mathcal{H}$$

where $\mathcal{M}_\infty(\mathcal{M}, \Phi) \trianglelefteq \mathcal{H}_1$ and $\mathcal{H}_1^+ \trianglelefteq \mathcal{H}$.

We say <u>Φ is τ-consistent</u> iff there is an embedding k such that

$$\begin{tikzcd}
\mathcal{M} \arrow[rr, "\tau"] \arrow[dr, "\pi^\Phi"'] & & \mathcal{H} \\
& \mathcal{M}_\infty(\mathcal{M}, \Phi) \arrow[ur, "k"'] &
\end{tikzcd}$$

commutes, with π^Φ the natural iteration map, and

$$k \restriction \pi^\Phi(\kappa^{\mathcal{M}}) = \text{identity}.$$

(So we are assuming here that $\mathcal{M}$ has a top block.)

Notice that k is determined by Φ and τ, if $\mathcal{M}$ is below O^P_h. That is because elements of $\mathcal{M}_\infty(\mathcal{M}, \Phi)$ are then of the form $\pi^\Phi(f)(a)$ for $a \in \pi^\Phi(\kappa^{\mathcal{M}})^{<\omega}$, and $k(\pi^\Phi(f)(a)) = \tau(f)(a)$ for such a.

Now let $i : \mathcal{M} \to Q$ be an iteration map by Φ. Let $l : Q \to \mathcal{M}_\infty(\mathcal{M}, \Phi)$ be the iteration map by the Q-tail of Φ. Let $\sigma = k \circ l$. So we have

$$\begin{tikzcd}
\mathcal{M} \arrow[rr, "\tau"] \arrow[d, "i"'] & & \mathcal{H} \\
Q \arrow[urr, "\sigma"] \arrow[rr, "l"'] & & \mathcal{M}_\infty(\mathcal{M}, \Phi) \arrow[u, "k"']
\end{tikzcd}$$

We then have, for

$$\Lambda = Q\text{-tail of } \Phi,$$

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(1) Λ is locally σ-consistent, and

(2) If E is on the Q-sequence, with $\mathrm{crit}(E) = \kappa^Q$, then E is σ-certified over $(Q\|lh(E), \Lambda_{Q\|lhE})$.

Part (1) comes from $k \restriction \ell(\kappa^Q) =$ identity.

Part (2) comes from considering

$$\begin{tikzcd}
\mathcal{M} \arrow[rr, "\pi"] \arrow[dr, "i"'] & & \mathcal{H} \\
& Q \arrow[ur, "\sigma"] \arrow[r, "\ell"] \arrow[dr, "i_E"'] & \mathcal{M}_\infty(\mathcal{M}, \Phi) \arrow[u, "k"'] \\
& & \mathrm{Ult}(Q,E) \arrow[u, "t"']
\end{tikzcd}$$

where t is the iteration map. We have $t \restriction lh(E)$ is the iteration map by $\Lambda_{Q\|lh(E)}$ by strategy coherence and the fact that $lh(E)$ is a cutpoint of $\mathrm{Ult}(Q,E)$. Moreover, for $X \subseteq [\kappa^Q]^{|a|}$ and $a \in [lh E]^{<\omega}$,

$$a \in i_E(X) \text{ iff } t(a) \in \ell(X)$$
$$\text{iff } t(a) \in \sigma(X)$$

because $k(t(a)) = t(a)$.

What we have done in §2 is construct a j-consistent strategy Ω for $(\mathcal{M}, E)$. This we did by maintaining (1) and (2) as we went along, with respect to inductively determined j-realization maps σ.

The argument also showed that all the $(\overset{\circ}{\psi}_\gamma)_h$ for $\gamma \leq \xi$ were also j-consistent. This also follows at once from the j-consistency of Ω.