LSA from least branch hod pairs

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We prove two theorems on the existence of models of the theory AD^+ + "The largest Suslin cardinal exists, and belongs to the Solovay sequence". This theory is known as LSA.

<u>Theorem I</u> Assume AD^+, and that there is an lbr hod pair (M, Σ) such that $M \models ZFC$ + "λ is a limit of Woodins, and for some Woodin cardinal $\delta < \lambda$, letting κ be the least $<\delta$-strong cardinal, κ is a limit of Woodin cardinals"; then there is a pointclass Γ such that $L(\Gamma, \mathbb{R}) \models$ LSA.

Sargsyan obtained what is probably the optimal upper bound via a hod mouse in the rigidly layered hierarchy in [1]. One can obtain what is essentially equivalent to Sargsyan's mouse by taking an M as in the theorem, letting $P = M|\delta$ and Ψ be the short-tree-component of Σ_P, and considering the structure $M_\omega^{\#,\Psi}(P)$. See [2] for a general definition of such "short-strategy stacks". ~~The existence of an~~ Our proof shows that the existence of an iterable structure of the form $M_\omega^{\#,\Psi}(P)$ yields an $L(\Gamma,\mathbb{R}) \models LSA$. This is essentially equivalent to Sargsyan's upper bound. (1A)

(Sargsyan, S.)

Theorem 2 | Assume AD^+, and that there is an lbr hod pair (M,Σ) such that $M \models ZFC +$ "there is a Woodin limit of Woodin cardinals" with infinitely many woodins above it, then there is a pointclass Γ such that

$$L(\Gamma,\mathbb{R}) \models LSA + \text{"whenever } A \subseteq \mathbb{R} \text{ is } OD(s) \text{ for some } s : \omega \to \theta, \text{ then } A \text{ is Suslin and co-Suslin"}.$$

For Γ as in the conclusion, $HOD({}^\omega\theta)^{L(\Gamma,\mathbb{R})}$ has an interesting theory. For notice that if $A \subseteq \mathbb{R}$ is Suslin-co-Suslin, then it is OD from $\bar{\mu}$, where $\bar{\mu}$ is any homogeneity system for A. So A is $OD(s)$, for some $s \in {}^\omega\theta$. So we get in $L(\Gamma,\mathbb{R})$

$$P(\mathbb{R}) \cap HOD({}^\omega\theta) = P_\kappa(\mathbb{R}),$$

where κ is the largest Suslin cardinal. With a little more work, we can arrange that the Solovay measure μ on $P_{\omega_1}(\mathbb{R})$ can be added to $L(\Gamma,\mathbb{R})$, and

$L(\Gamma,\mathbb{R},\mu) \models$ LSA + "whenever $A \subseteq \mathbb{R}$ is $OD(s,\mu)$ for some $s:\omega\to\Theta$, then A is Suslin-co-Suslin".

Then in $L(\Gamma,\mathbb{R},\mu)$

$$P(\mathbb{R}) \cap HOD_\mu({}^\omega\Theta) = P_\kappa(\mathbb{R}),$$

where κ is the largest Suslin. So

$HOD_\mu({}^\omega\Theta)|\Theta \models ZF + AD_{\mathbb{R}}$ + for all sets X, there is a fine, normal measure on $P_{\omega_1}(X)$.

It seems plausible that one can obtain models of still stronger forms of higher-type determinacy this way.

<u>Remark</u> Theorem 2 was inspired by a conjecture of Sargsyan that something much like it holds for ~~noice~~ hod mice in the rigidly layered hierarchy. In particular, the idea that the Woodin limit of Woodins is what you need is due to him. Sargsyan's proof adapts to lbr hod pairs, as verified by him. The proof we give below is different.

Proof of Theorem 1.

Let (M, Σ) be as in the hypothesis. Let δ be least as in the hypotheses, and $\kappa < \delta$ be the least $< \delta$-strong of M. Let $\lambda > \delta$ be the first limit of Woodins in M.

Let g be $Col(\omega, <\lambda)$-generic over M, and let

$$D(M, <\lambda)^g = L(\mathbb{R}^*_g, \mathrm{Hom}^*_g)$$

be the corresponding realization of the derived model of M. For each ~~[illegible]~~ $\eta < \lambda$, $\Sigma_{M|\eta} \cap D(M, <\lambda)^g \in \mathrm{Hom}^*_g$, and is given by a term in M, uniformly over all g. Let us write $\dot{\Sigma}_{M,\eta}$ for this term. The sets $(\dot{\Sigma}_{M,\eta})^g$ for $\eta < \lambda$ are Wadge cofinal in Hom^*_g. See [3], section 7. We also have

$$(\mathrm{HOD}|\theta)^{D(M, <\lambda)} = M_\infty(M|\lambda, \Sigma).$$

from §7 of [3].

Note that $D(M, <\lambda) \models AD_{\mathbb{R}}$, so it is not the model we want. Instead, we move down within it: let

$$P_0 = M|\delta,$$

$$\psi = \Sigma_{P_0},$$

$$\psi_0 = \psi^{stc}.$$

That is, ψ_0 is the short-tree-component of ψ, defined by

$$\psi_0 = \{(\mathcal{T}, b) \mid \mathcal{T}^\frown b \text{ is a normal tree on } P_0 \text{ by } \psi \text{, and either } b \text{ drops, or } i_b^{\mathcal{T}}(\delta) > \delta(\mathcal{T}).\}$$

<u>Remark</u> ψ_0 is a partial strategy. Its domain is the set of all <u>ψ-short</u> trees. A normal $\mathcal{T} \in \mathrm{dom}(\psi)$ s.t. $\mathcal{T} \notin \mathrm{dom}(\psi_0)$ is called <u>ψ-maximal</u>. We could let ψ_0 act on stacks, but since ψ and hence ψ_0 normalize well, we don't need to here. See [2].

For (Q, Φ) in the hod-limit system of $D(M, <\lambda)^g$, let (2)

$$\pi_{(Q,\Phi),\infty} : Q \longrightarrow M_\infty(M, \Sigma)$$

be the direct limit map. It is shown in [4] that, setting

$$\psi^* = \psi \cap \mathbb{R}^*_g \quad (= \psi \cap HC^*_g \text{, of course})$$

$$\psi_0^* = \psi_0 \cap \mathbb{R}^*_g$$

that in $L(\mathbb{R}^*_g, Hom^*_g)$:

(1) ψ_0^* is $\pi_{(P_0,\psi^*),\infty}(\kappa^{P_0})$ - Suslin, and not α - Suslin for any $\alpha < \pi_{(P_0,\psi),\infty}(\kappa^{P_0})$

and

(2) ψ^* is $\pi_{(P_0,\psi^*),\infty}(\delta^{P_0})$ - Suslin, and not α - Suslin for any $\alpha < \pi_{(P_0,\psi^*),\infty}(\delta^{P_0})$.

Here κ^{P_0} = least card strong to $\delta^{P_0} = o(P_0)$ in P_0,

and

(3) $|\pi_{(P_0,\psi^*),\infty}(\delta^{P_0})|$ is at most the least Suslin cardinal that is $> \pi_{(P_0,\psi^*),\infty}(\kappa^{P_0})$.

~~Remark Probably $\pi_{(P_0,\psi^*),\infty}(\delta^{P_0})$ is the least Suslin $> \pi_{(P_0,\psi^*),\infty}(\kappa^{P_0})$.~~

It suffices now to show that

$$\pi_{(P_0,\psi^*),\infty}(\delta^{P_0}) = \theta(\psi_0^*)^{L(\mathbb{R}_g^*, \mathrm{Hom}_g^*)},$$

for then letting $N = L(P_{\theta(\psi_0^*)}(\mathbb{R}))^{L(\mathbb{R}_g^*, \mathrm{Hom}_g^*)}$,

$N \models LSA$, with its set at the largest Suslin being ψ_0^*.

To show this, let

$$f: \mathbb{R}_g^* \xrightarrow{\text{onto}} \beta$$

be $OD(\psi_0^*)$ in $L(\mathbb{R}_g^*, \mathrm{Hom}_g^*)$, say

$$f(x) = \xi \text{ iff } L_\alpha(\mathbb{R}_g^*, \mathrm{Hom}_g^*) \models \varphi[x, \xi, \psi_0^*].$$

(The ordinal parameter is part of α.) We are going to use the argument of [3], §7.2, Claim 5, which is also used many other places. The extra difficulty here is that ψ_0^* is "close" to ψ^*. This adds one layer of complexity to the proof that $\beta \leq \pi_{(P_0,\psi^*),\infty}(\delta^{P_0})$.

Let $\langle x_i \mid i \in \omega \rangle$ enumerate $\mathbb{R}_g^*$, and let $\langle (Q_i, \Lambda_i) \mid i \in \omega \rangle$ be a sequence of points in the direct limit system $\mathcal{F}(P_0^+, \Psi)^{D(M,<\lambda)g}$ converging to $M_\infty(P_0^+, \Psi)^{D(M,<\lambda)g}$.

Here $P_0^+ = M \mid (\delta^{+M})$. We construct by induction $(P_i^+, \Phi_i) \in \mathcal{F}(P_0^+, \Psi)^{D(M,<\lambda)g}$ such that, with $(P_0^+, \Phi_0) = (P_0^+, \Psi)$:

(i) (P_{i+1}^+, Φ_{i+1}) is an iterate of (P_i^+, Φ_i) and an iterate of (Q_i, Λ_i) and of (P_0^+, Φ_0), via normal trees in each case

(ii) Let $\mathcal{U}_i$ be the unique normal tree from P_0^+ to P_{i+1}^+ via Ψ; then $lh(\mathcal{U}_i) = \delta(\mathcal{U}_i) = \delta^{P_{i+1}^+} = o(P_{i+1})$, $\mathcal{U}_i$ is Ψ-maximal, and for all $x \in \{x_0, \ldots, x_i\}$, $\langle P_0, \mathcal{U}_i, x \rangle$ is $\mathbb{B}^{P_{i+1}^+}$-generic over P_{i+1}^+.

Here $\mathbb{B}^S$ is the δ^S-generator extender-algebra of S, when S is an lpm having a largest Woodin cardinal δ^S.

Concerning (ii), note that we could not ask that the branch $b = \Psi(\mathcal{U}_i)$ be $\mathbb{B}^{P^+_{i+1}}$-generic, for b singularizes $\delta(\mathcal{U}_i) = \delta^{P^+_{i+1}}$, and $\mathbb{B}^{P^+_{i+1}}$ is $\delta^{P^+_{i+1}}$-c.c. in P^+_{i+1}.

The construction of (P^+_{i+1}, Φ_{i+1}) is easy. Working in $D(M, <\lambda)^g$, we can find an inductive-like pointclass Γ with the scale property such that Λ_i, Φ_i, and Ψ have codesets in Δ. We let N^* be a coarse Γ-Woodin model with an iteration strategy Σ^* s.t. (N^*, Σ^*) Suslin-co-Suslin captures those codesets and $x_0, \ldots x_i \in N^*$. We let $\mathbb{C}$ be the lbr-hod-pair construction of N^*.

We can find

$$(S, \Omega) = (M^{\mathbb{Q}}_{\nu,k}, \Omega^{\mathbb{Q}}_{\nu,k})$$

such that (S, Ω) is an iterate of (P_0^+, Ψ).

This is true not just in N^*, but in $D(M, < \lambda)^g$ if we prolong $\Omega^{\mathbb{Q}}_{\nu,k}$ using $\mathcal{E}^*$.

By Dodd-Jensen, (S, Ω) is also an iterate of (Q_i, Λ_i) and (P_i^+, ϕ_i).

We have to check (ii).

All is straightforward except the genericity claim. So let E be an extender on the $P_{i+1} | \delta^{P_{i+1}}$ sequence, with $\kappa = \text{crit}(E)$ and $\nu = \text{strength}(E)$ an inaccessible cardinal of P_{i+1}. Let

$$\bigvee_{\alpha < \kappa} \varphi_\alpha \longleftrightarrow i_E\left(\bigvee_{\alpha < \kappa} \varphi_\alpha\right) \restriction \nu(E)$$

be an axiom induced by E. This implies that $i_E\left(\bigvee_{\alpha<\kappa}\varphi_\alpha\right)\restriction\nu(E)$ is a subset of $P_{i+1}|\nu(E)$, and in particular, ~~no~~ only atomic statements about $(P_0, \mathcal{U}|\nu(E), x)$ are relevant to its truth.

E occurs on the $M^{\mathbb{C}}_{\nu,k}$-sequence. Let $\sigma: M^{\mathbb{C}}_{\nu,k}|\mathrm{lh}\,E \longrightarrow M^{\mathbb{C}}_{\eta,0}$ be the resurrection map, and E^* the background extender for $\sigma(E) = \dot{F}^{M_{\eta,0}}$ given by $\mathbb{C}$. Note $\sigma\restriction\left(M^{\mathbb{C}}_{\nu,k}|\nu(E)\right) = \text{identity}$. It follows that

$$i_E\left(\bigvee_{\alpha<\kappa}\varphi_\alpha\right)\restriction\nu(E) = i_{\sigma(E)}\left(\bigvee_{\alpha<\kappa}\varphi_\alpha\right)\restriction\nu(E)$$

$$= i_{E^*}\left(\bigvee_{\alpha<\kappa}\varphi_\alpha\right)\restriction\nu(E).$$

Fix $x\in\{x_0 \ldots x_i\}$. Assume that $\langle P_0, \mathcal{U}, x\rangle \models i_E\left(\bigvee_{\alpha<\kappa}\varphi_\alpha\right)\restriction\nu(E)$. We must show $\langle P_0, \mathcal{U}, x\rangle \models \bigvee_{\alpha<\kappa}\varphi_\alpha$. But we have $\langle P_0, \mathcal{U}, x\rangle \models i_{E^*}\left(\bigvee_{\alpha<\kappa}\varphi_\alpha\right)\restriction\nu(E)$.

Further

$$i_{E^*}(\mathcal{U}) \restriction \nu(E) = \mathcal{U} / \nu(E).$$

This is because $i_{E^*}(\varphi) \leq \varphi$ by absoluteness, and $i_{E^*}(M^{\Phi}_{\nu,k}) / \nu(E) = M^{\Phi}_{\nu,k} / \nu(E)$. Thus

$$i_{E^*}(\langle P_0, \mathcal{U}, x \rangle) \not\models i_{E^*}\left(\bigvee_{\alpha<\kappa} \varphi_\alpha\right),$$

so

$$\langle P_0, \mathcal{U}, x \rangle \not\models \bigvee_{\alpha<\kappa} \varphi_\alpha,$$

as desired.

<u>Remark</u> Our lpms use Jensen indexing, but it would not be correct to put axioms of the form $\bigvee_{\alpha<\kappa} \varphi_\alpha \leftrightarrow i_E\left(\bigvee_{\alpha<\kappa} \varphi_\alpha\right)$ into $\mathcal{B}$. That is, we need to cut off at $\nu(E)$, not $\lambda(E)$.

Similarly, genericity iterations in Jensen indexing seem to need MS-trees; otherwise we might have to drop ~~[illegible]~~ when we don't want to.

This finishes the construction of $\langle (P_i^+, \Phi_i) \mid i \in \omega \rangle$

Let

$$M_0 = M_\infty$$

$$M_{i+1} = \text{iterate of } M \text{ obtained by lifting } P_0^+\text{-to-}P_{i+1}^+ \text{ so that it acts on } M.$$

We then dovetail $\mathbb{R}_g^+$-genericity iterations as usual, iterating M_i strictly above P_i^+ to realize $L(\mathbb{R}_g^*, \mathrm{Hom}_g^*)$ as $D(Q_i, <\lambda^{Q_i})$.

The diagram is

$$\begin{tikzcd}
Q_0 \arrow[r] & Q_1 \arrow[r, dashed, no head] & Q_k \arrow[r, "\sigma_{k,k+1}"] & Q_{k+1} \arrow[r, dashed, no head] & Q_\omega \\
M_0 \arrow[u] \arrow[r] \arrow[d, phantom, "\triangledown"] & M_1 \arrow[u] \arrow[r, dashed] \arrow[d, phantom, "\triangledown"] & M_k \arrow[u, "j_k"] \arrow[r, "\pi_{k,k+1}"] \arrow[d, phantom, "\triangledown"] & M_{k+1} \arrow[u, "j_{k+1}"'] \arrow[r, dashed, no head] \arrow[d, phantom, "\triangledown"] & {} \\
P_0^+ \arrow[r] & P_1^+ \arrow[r, dashed, no head] & P_k^+ \arrow[r] & P_{k+1}^+ \arrow[r, dashed, no head] & P_\omega^+ \arrow[d, phantom, "\shortparallel"] \\
 & & & & M_\infty(P_0, \psi^*)
\end{tikzcd}$$

We have l_k on $Col(\omega, <\lambda^{Q_k})$ s.t.

$\mathbb{R}^*_{l_k} = \mathbb{R}^*_g$ and $Hom^*_{l_k} = Hom^*_g$, for

all $k < \omega$.

~~Claim There is a fixed term $\rho \in Q_0$~~
~~such that for all k~~

$$\cancel{\sigma_{0,k}(\rho)^{l_k} = \Psi_0^*}.$$

~~Claim There is a fixed formula ρ such that~~
~~for all k~~

Claim There is a fixed term $\rho \in Q_0^{Col(\omega,<\lambda^{Q_0})}$ ~~such~~
such that for all k

$$\Psi_0 \cap Q_k[h_k][l] = \left(\pi_{0,k}(\rho)\left(\pi_{0,k}(\tau)^{h_k}\right)\right)^l.$$

whenever h on $\mathbb{B}_{k+1}$ is such that $\pi_{0,k}(\tau)^h = \langle P_0, \mathcal{U}_k, x\rangle$ for some x.

Proof It is enough to ~~determine~~ define
$\Psi_0 \cap Q_k[h_k]$ from $\langle P_0, \mathcal{U}_k\rangle$ and the
extender sequence of Q_k. Since short-tree

components determine themselves on generic extensions, we then get $\varphi_0 \cap Q_k[h_k][l]$. The definition of $\varphi_0 \cap Q_k[h_k]$ will be uniform in $\langle P_0, \mathcal{U}_k \rangle$, giving the claim.

Here's the definition. Consider some l that is $Col(\omega, <\lambda_1^{Q_k})$ gen / $Q_k[h_k]$. Let $\mathcal{T}$ on P_0 be by φ_0, and short. In $Q_k[h_k][l]$, we have the diagram

$$\begin{tikzcd}
 & P_{k+1} \arrow[rr, "\sigma"] & & S \\
P_0 \arrow[ur, "\mathcal{U}_k^\frown c"] \arrow[urrr, "X"] \arrow[rr, "i^{\mathcal{T}}_b"'] & & M_b^{\mathcal{T}} \arrow[ur, "\nu"'] &
\end{tikzcd}$$

Here we assume for simplicity that b does not drop, where $b = \varphi_0(\mathcal{T})$. Then σ, ν come from comparing. (Really, we have $(P_{k+1}, \Sigma_{P_{k+1}})$ and $(M_b^{\mathcal{T}}, (\Sigma_{P_0})_{\mathcal{T}^\frown b})$ being compared to some (S, Λ) here.) In the diagram, $c = \Sigma_{P_0}(\mathcal{U}_k)$. Then

if $\mathcal{S}$ is the tree for P_{k+1}-to-S,
and $\mathcal{U}$ is the tree for $M_b^{\vec{a}}$-to-S,
we have

$$X = X(\mathcal{U}_k{}^\frown c, \mathcal{S})$$
$$= X(\vec{a}^\frown b, \mathcal{U}),$$

as the common full normalization.

But because $\vec{a}^\frown b$ is short, we get that $X(\vec{a}^\frown b, \mathcal{U})$ is weakly pseudo-hull embedded into $X(\mathcal{U}_k \restriction \eta+1, \mathcal{S})$, where $\mathcal{U}_k \restriction \eta+1$ is the proper initial segment where extenders of length $< i_c^{\mathcal{U}_k}(\kappa^{P_0})$ are used. $\mathcal{S}$ can ~~be justified using the extender seq found uniformly~~ be justified by $(\Sigma^{Q_k})_{h_k, l}$. So $Q_k[h_k]$ can define φ_0 by looking for weak pseudo-hull embeddings into trees it knows how to justify using $\mathcal{U}_k$ and itself.

Claim $\boxtimes$

11

With the claim, we can complete the usual argument. Let $Y_k \in Q_k$ be defined by

$\xi \in Y_k$ iff there is a $b \in \mathbb{B}_{k+1}$ s.t. for all s, w, x if $\langle \cdots \rangle = \pi_{0k}(\tau)^h$, then ~~some~~ $b \Vdash^{\mathbb{B}_{k+1}}$

$$\left(1 \Vdash^{Col(\omega, <\lambda)^{Q_k}} L_\alpha(\mathbb{R}_g^*, Hom_g^*) \models \varphi[\rho(s,w), x, \xi] \right)$$

~~for some $\ldots \in \mathbb{N}$~~.

~~So Q_k~~ (We can arrange that every $b \in \mathbb{B}_k$ forces there are ~~exactly one finitely many~~ at most one ξ s.t. for $\langle s, w, x\rangle = \pi_{0k}(\tau)^h$

$$1 \Vdash^{Col(\omega,<\lambda)^{Q_k}} L_\alpha(\mathbb{R}_g^*, Hom_g^*) \models \varphi[\rho(s,w), x, \xi] \Big)$$

Rmk In other words, $\xi \in Y_k$ iff $\exists h$ on $\mathbb{B}_{k+1}$ s.t. $\xi \in \mathring{Y}_k^h$, where for all h,

$$|\mathring{Y}_k^h| \leq 1.$$

$\mathrm{otp}(Y_k) < \delta_{k+1}$ by the chain condition. Let for $\xi \in Y_k$

$$\xi = \gamma^k_\xi \text{ 'the elt of } Y_k.$$

For $\xi \in \mathrm{ran}(f)$, letting k be large enough that $\sigma_{\ell,\ell+1}$ fixes α and ξ for all $\ell \geq k$, and $\xi = f(x_i)$ for some $i \leq k$), we have $\xi \in Y^\ell$ for all $\ell \geq k$, and

$$\pi_{\ell,\ell+1}(\gamma^\ell_\xi) = \gamma^{\ell+1}_\xi$$

for all $\ell \geq k$. (α being fixed guarantees $\sigma_{\ell,\ell+1}(Y^\ell) = Y^{\ell+1}$. & $\pi_{\ell,\ell+1} \upharpoonright \delta_\ell = \sigma_{\ell,\ell+1} \upharpoonright \delta_\ell$.)

Set

$$g(\xi) = \pi_{k,\omega}(\gamma^k_\xi)$$

for this k.

One can check $\xi < \eta \Rightarrow g(\xi) < g(\eta)$, as desired.

Corollary Assume $AD_{\mathbb{R}}$ + HPC, and let $\delta < \Theta$ be such that δ is a cutpoint Woodin cardinal in HOD. Then $\delta = \Theta_0$, or $\delta = \Theta_{\alpha+1}$ for some α.

Proof Let $HOD | \delta$ = $M_\infty(P, \Sigma)$. Then $\delta = \Theta(B)$, where B codes Σ^{STC}, as shown above.

$\boxtimes$

Remarks

(1) Sargsyan [1] got a model of LSA from optimal hypotheses on the existence of hybrid mice in the rigidly layered hierarchy.

(2) We can get a better upper bound for LSA as follows: we have an lbr hod pair (P, Σ) with κ^P = least ~~##~~ strong to $\delta^P = o(P)$ s.t. $P \models \kappa^P$ is a limit of Woodins.

We then leave the lpm hierarchy to form $M_\omega^{\#,\Sigma}(P)$, the first Σ mouse over P with ω Woodins. We assume $o(P)$ remains Woodin in $M_\omega^{\#,\Sigma}(P)$, and Σ moves the $M_\omega^{\#,\Sigma}$ part correctly.

$M_\omega^{\#,\Sigma}(P)$ is not literally an lpm, but you can generalize the lbr-hod-pair theory to it routinely. (See Siskind and Steel [5].)

It is close to ~~probably~~ the minimal mouse past LSA.

The proof above shows that it is indeed past LSA.

(3) One can take another step down and look at $M_\omega^{\#,\Sigma^{stc}}(P)$. This requires making sense of short-tree-strategy mice in general, which can be done. This is essentially equivalent to Sargsyan's upper bound for LSA.

(4) Going up from our hypothesis: an active lbr hod pair (P,Φ) s.t. $P \models$ there is a Woodin cardinal $> \mathrm{crit}(\dot{F}^P)$ yields an (M,Σ) as in the hypothesis of the theorem.

(5) At the moment, we do not see how to prove that mice as in (2) above exist from a large cardinal or determinacy hypotheses.

Proof of Theorem 2

Let (M, Σ) be as in the hypotheses, with δ a Woodin limit of Woodins in M, ~~and~~ and $\lambda > \delta$ the sup of the next ω Woodin cardinals. Let

$$D(M, <\lambda)^g = L(\mathbb{R}^*_g, \mathrm{Hom}^*_g)$$

be a derived model of M below λ.

Let

$$P_0 = M|\delta,$$

$$\psi = \Sigma_{M|\delta},$$

$$\psi_0 = \Sigma^{stc}.$$

Let κ^{P_0} be the least $<\delta$-strong cardinal of P_0. We have that in $D(M,<\lambda)^g$:

$$\pi^{\psi}_{P_0,\infty}(\delta) = o(M_\infty(P_0, \psi))$$
$$= \theta(\psi_0)$$

and

$$\pi^{\Psi}_{P_0,\infty}(\kappa) = \text{largest Suslin cardinal} < \theta(\Psi_0).$$

Setting $\kappa_\infty = \pi^{\Psi}_{P_0,\infty}(\kappa)$ and $\delta_\infty = \pi^{\Psi}_{P_0,\infty}(\delta)$,

we have that ~~Ψ_0 is~~ Code (Ψ_0) is κ_∞-Suslin, and Code (Ψ) is δ_∞-Suslin, and neither is α-Suslin for a smaller α.

We ~~now~~ take

$$\Gamma = P_{\delta_\infty}(\mathbb{R})^{D(m,<\lambda)^g}.$$

So $L(\Gamma, \mathbb{R}) \models LSA$, by theorem 4's proof.

We must show that if $s : \omega \to \delta_\infty$ is in $L(\Gamma,\mathbb{R})$ (i.e. ran s is bounded in δ_∞), then (P_0, Ψ_0) is not $OD(s)^{L(\Gamma,\mathbb{R})}$.

Suppose otherwise. Let $\eta < \delta$ be such that $\mathrm{ran}(s) \subseteq \pi^{\psi}_{P_0,\infty}(\eta)$. Let P_1 be a ψ-iterate of P_0 such that $o(P_0) <$ least measurable of P_1, and

$$\mathrm{ran}(s) \subseteq \pi^{\phi}_{P_1,\infty}{}''\,\eta_1,$$

where ϕ is the P_1-tail of ψ and $\eta_1 = \pi^{\psi}_{P_0,P_1}(\eta)$. Setting

$$Z = \pi^{\phi}_{P_1,\infty}{}''\,\eta_1$$

we can find a real x_0 such that s is very simply definable from $\langle x_0, Z\rangle$.

Let

$$\eta_1 < \gamma < \delta^{P_1}$$

be such that

$$P_1 \models \gamma \text{ is Woodin}.$$

(This uses our hypothesis!) Let

Let (P_2, Σ_2) come from a genericity iteration above η_1 and below γ to make x_0 generic at ~~the~~ $\pi^{\phi}_{P_1,P_2}(\gamma)$.

The iterations above extend to M, so we have

$$\begin{tikzcd} M \arrow[r] \arrow[d, phantom, "\triangledown"] & M_1 \arrow[r] \arrow[d, phantom, "\triangledown"] & M_2 \arrow[d, phantom, "\triangledown"] \\ P_0 \arrow[r] & P_1 \arrow[r] & P_2 \end{tikzcd}$$

By iterating above the P's, we can assume that $D(M_2, <\lambda^{M_2})^h = D(M, <\lambda)^g$ for some h.

But now in $M_2[x_0]$ we can define $\mathcal{Z} = \pi''_{P_2,\infty}\,\eta_1$ from our own strategy for M_2. Thus

$$\mathcal{Z} \in M_2[x_0].$$

So (P_0, ψ_0) is OD over $D(M_2[x_0], <\lambda^{M_2})$ from parameters in $M_2[x_0]$, so $\psi_0 \restriction M_2[x_0] \in M_2[x_0]$. However every cardinal of $M_2[x_0]$ between $\pi^{\phi}_{P_1,P_2}(\gamma)$ and δ^{P_2} is collapsed by (P_0, ψ_0), contradiction. $\boxtimes$

Remark In the proof above, note that ~~the~~ $D(M, <\lambda)^g \models AD_{\mathbb{R}}$. So the Solovay measure μ on $P_{\omega_1}(\mathbb{R})$ is definable in $D(M, <\lambda)^g = D(M_2[x_0], <\lambda^{M_2})^h$. So the proof shows (P_0, ψ_0) is not $OD(s, \mu)$ in $L(\Gamma, \mathbb{R}, \mu)$ — in fact, it is not $OD(s, \mu)$ in the full $D(M, <\lambda)^g$, for any $s \in \omega \to \Theta^{L(\Gamma, \mathbb{R})}$ with ran(s) bounded.

~~The~~ A model of "$AD_{\mathbb{R}} + \forall X$ (ω_1 is X-supercompact)" is $L_\Theta({}^{\omega}\Theta \cup \{\mu_\alpha \mid \alpha < \Theta\})^{L(\Gamma, \mathbb{R})}$, where μ_α = Solovay measure transferred to $P_{\omega_1}({}^{\omega}\alpha)$. $HOD({}^{\omega}\Theta) \restriction \Theta^D$, for $\Theta = \Theta^{L(\Gamma, \mathbb{R})}$, also works.

References

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