Local K^c constructions

J.R. Steel

November 14, 2003

The full-background-extender K^c-construction of [2] has the property that, if it does not break down and produces a final model $L[\vec{E}]$, then

$$\delta \text{ is Woodin in } V \Rightarrow \delta \text{ is Woodin in } L[\vec{E}],$$

for all δ. It is natural to ask whether

$$\kappa \text{ is strong in } V \Rightarrow \kappa \text{ is strong in } L[\vec{E}],$$

for all κ, or even better,

$$\kappa \text{ is } \lambda\text{-strong in } V \Rightarrow \kappa \text{ is } \lambda\text{-strong in } L[\vec{E}],$$

As one might suspect, the more useful answer would be "yes".

For the K^c-construction of [2], this question is open. The problem is that the construction of [2] is not local: because of the full-background-extender demand, it may produce mice projecting to ρ at stages much greater than ρ. Because of this, there is no reason to believe that if E is a λ-strong extender of V, then $i_E(\vec{E}) \upharpoonright \lambda = \vec{E} \upharpoonright \lambda$. The natural proof only gives that if κ is Σ_2-strong, then κ is strong in $L[\vec{E}]$.[1]

Since we do not not how to get started on this question, and suspect that in fact (1) may fail for the construction of [2], we shall look instead for a modification of this construction. What we want for our applications is a construction with output $L[\vec{E}]$ such that

(1) iteration trees on $L[\vec{E}]$ can be lifted to iteration trees on V,

(2) $\forall\delta(\delta$ is Woodin $\Rightarrow \delta$ is Woodin in $L[\vec{E}])$, and

(3) (a) $\forall\kappa(\kappa$ is a strong cardinal $\Rightarrow \kappa$ is strong in $L[\vec{E}])$, and

(b) $\forall\kappa\forall\lambda(\mathrm{Lim}(\lambda) \wedge \kappa$ is λ-strong $\Rightarrow \kappa$ is λ-strong in $L[\vec{E}])$.

[1] This question, and the observations we have just recorded, are due to R. Jensen.

In §1 we shall describe a construction which satisfies these demands, but we shall need to assume that V itself is a fine-structural inner model $L[\vec{F}][x]$, built over some set x. In this case, the iterability reduction of (1) will produce iteration trees on V which may drop, though only when the tree on $L[\vec{E}]$ being lifted also drops. A reduction to non-dropping trees would have some use. In [7], we describe a construction satisfying (1), (2), and (3)(a). There are no special assumptions on V there, and so the trees on V arising in the iterability reduction are ordinary coarse trees. However, we do not see how to get (3)(b) in that situation. Of the two constructions, the one in §1 seems more natural. It seems to be a coarse version of a construction due to Itay Neeman ([3]), his "lightening" of a premouse.

In §2 we show how to lift iteration trees on $L[\vec{E}]$ to trees on $L[\vec{F}][x]$. In §3 we show that UBH holds for the non-dropping, plus-two iteration trees on $L[\vec{F}][x]$, as well as for arbitrary trees on $L[\vec{F}][x]$ which come via the iterability reduction from trees on levels of $L[\vec{E}]$ which project to ω. These facts are useful in studying the derived models associated to mice, as we show in §4.

1 A local construction in $L[\vec{F}][x]$

In this section we take as our background universe a model of the form $L[\vec{F}][x]$, where x is transitive and $\vec{F}$ is a coherent sequence of extenders with all critical points above the rank of x.[2] We shall assume that $L[\vec{F}][x]$ is fully iterable in V. We are *not* assuming that $L[\vec{F}][x]$ satisfies "I am iterable", in any sense. Indeed, in the applications we have in mind, $L[\vec{F}][x]$ satisfies "There is a Woodin cardinal", and hence by unpublished work of Woodin it satisfies "I am not fully iterable". Since we will be working inside $L[\vec{F}][x]$, what we really need are some first-order consequences of its being iterable in V. We won't bother to record these now.

In order to confuse the reader as much as possible, we shall assume $V = L[\vec{F}][x]$ for most of the rest of this section. The reader should just keep in mind that V is fully iterable in some larger universe.

We shall use the terminology of [1] with regard to premice, K^c-constructions, and resurrections, except that we use premice in the sense of [2] rather than premice in the sense of Jensen. This we must do, since a full-background-extender K^c-construction of Jensen mice requires superstrong cardinals in V, while a partial-background-extender K^c-construction does not permit one to lift iteration trees to V.

Thus for us, a premouse is a pair

$$\mathcal{M} = (\mathcal{N}, k),$$

[2] In the special case $x = \emptyset$, $L[\vec{F}][x]$ is a premouse in the sense of [2]; otherwise, it satisfies the obvious relativizations to x of the premouse conditions of [2].

where $\mathcal{N}$ is a k-sound premouse in the sense of [2], and $k \leq \omega$. We write $\mathcal{N} = (\mathcal{M})_0$ and $k = k(\mathcal{M})$, and call $\mathcal{M}$ a *k-premouse.* If

$$\mathcal{N} = (J_\alpha[\vec{E}], \in, \vec{E}, F),$$

then we write $\alpha = \alpha(\mathcal{M})$, $\dot{E}^{\mathcal{M}} = \dot{E}^{\mathcal{N}} = \vec{E}$ and $\dot{F}^{\mathcal{M}} = \dot{F}^{\mathcal{N}} = F$. We write $o(\mathcal{M}) = o(\mathcal{N})$ for the least ordinal not in $\mathcal{N}$. We say $\mathcal{M}$ is *active* iff $\mathcal{N}$ is active, that is, iff $F \neq \emptyset$. The *projectum* and *standard parameter* of $\mathcal{M}$ are $\rho(\mathcal{M}) = \rho_k(\mathcal{N})$ and $p(\mathcal{M}) = p_k(\mathcal{N})$ respectively. For other terminology, see [1]. All the notions here relativise in an obvious way, so that for X a set of ordinals, we can speak of k-premice over X, etc. The main points are that all extenders on the coherent sequence of a premouse over X must have critical point strictly greater than $\sup(X)$, and that $\sup(X) \cup \{X\}$ is contained in all hulls we take. (We restrict ourselves to the case that X is a set of ordinals so that the fine-structural Los theorem goes through as usual.)

Any K^c-construction, in the sense of [2] or [5], determines a sequence $\langle \mathcal{C}_k(\mathcal{N}_\xi) \mid \langle \xi, k\rangle \leq_{\text{lex}} \langle \theta, j\rangle\rangle$ of premice in the sense of [2]; in the present notation, we shall write $\mathcal{M}_{\xi,k}$ for the k-premouse $\langle \mathcal{C}_k(\mathcal{N}_\xi), k$. If we drop all background-extender requirements, we are left with some elementary "first-order" requirements which a sequence $\langle \mathcal{M}_{\xi,k} \mid \langle \xi, k\rangle \leq_{\text{lex}} \langle \theta, j\rangle\rangle$ must satisfy in order to fit the K^c-construction format. Following Jensen, we call such a sequence an *ms-array.*

Definition 1.1 *Let X be a set of ordinals; then an* ms-array over X *is a sequence $\langle \mathcal{M}_{\nu,k} \mid (\nu, k) <_{\text{lex}} (\theta, l)\rangle$ such that*

(1) $\mathcal{M}_{1,0} = ((rud\ (\sup(X) \cup \{X\}), \in, \emptyset), 0)$,

(2) each $\mathcal{M}_{\nu,k}$ is a k-premouse over X,

(3) $\mathcal{M}_{\nu,k+1} = (\mathcal{C}_{k+1}((\mathcal{M}_{\nu,k})_0), k+1)$, whenever $(\nu, k+1) <_{\text{lex}} (\theta, l)$,

(4) if ν is a limit, then setting [3]

$$\mathcal{M}_{<\nu} = \liminf_{\eta \to \nu} (\mathcal{M}_{\eta,0})_0,$$

we have that

$$\mathcal{M}_{\nu,0} = ((\mathcal{M}_{<\nu}, F), 0),$$

where of course $F = \emptyset$ or F is an extender fulfilling the premouse conditions, and

[3] More precisely, $\mathcal{M}_{<\nu} = (J_\alpha[\vec{E}], \in, \vec{E})$, where $\vec{E}$ is the union of all $\dot{E}^{\mathcal{M}_{\eta,0}}$ such that $\eta < \nu$ and $\mathcal{M}_{\eta,0} \trianglelefteq \mathcal{M}_{\xi,0}$ for all ξ such that $\eta \leq \xi < \nu$.

(5) *$\mathcal{M}_{\nu+1,0}$ is obtained from $\mathcal{M}_{\nu,\omega}$ by one step of rudimentary closure: letting $\vec{E} = \dot{E}^{\mathcal{M}_{\nu,\omega}}$, $F = \dot{F}^{\mathcal{M}_{\nu,\omega}}$, and $\alpha = \alpha(\mathcal{M}_{\nu,\omega})$, we have*

$$\mathcal{M}_{\nu+1,0} = ((J_{\alpha+1}[\vec{E}^\frown F], \in, \vec{E}^\frown F, \emptyset), 0).$$

The following notation will be convenient.

Definition 1.2 *For H and G transitive sets, and $\rho \subseteq H \cap G$:*

$$H \overset{\rho}{\sim} G \Leftrightarrow \forall\eta < \rho(P(\eta)^H = P(\eta)^G).$$

We shall use this notation when H and G are (the universes of) premice, and H is an initial segment of G (that is, $H \trianglelefteq G$), and there is a least $\tau \geq \rho$ such that $H \models \tau$ is a cardinal. In this case, $H \overset{\rho}{\sim} G$ if and only if τ remains a cardinal in G.

We now define *certificates* for ms-arrays. The main idea is just to make the certification of an $\mathcal{M}_{\nu,k}$ projecting to ρ local by allowing size ρ Skolem hulls of certificates in the sense of [2] to count as certificates as well. One must "close off", so that size ρ Skolem hulls of our new certificates are also certificates. Recall that we are working in the universe $L[\vec{F}][x]$. By condensation, we can restrict our certificates to Skolem hulls which are initial segments of $L[\vec{F}][x]$.

Let μ^{+n} be the n^{th} cardinal $> \mu$.

Definition 1.3 *Let $\vec{\mathcal{M}} = \langle \mathcal{M}_{\nu,k} \mid (\nu, k) \leq_{\text{lex}} (\eta_0, j_0)\rangle$ be an ms-array (with last element) over Y. A* plus-n certificate *for $\vec{\mathcal{M}}$ is a pair*

$$\vec{C} = (\langle H_\nu \mid \nu < \eta_0\rangle, \langle F^*_\nu \mid \nu \leq \eta_0\rangle)$$

of sequences such that for all $\nu \leq \eta_0$,

(1) *$H_\nu \trianglelefteq L[\vec{F}][x]$ and $H_\nu \models \mathsf{ZF}^- +$ "μ^{+n} exists", where $\mu = \sup(\{o(\mathcal{M}_{\xi,i} \mid (\xi, i) \leq_{\text{lex}} (\nu, k)\})$.*

(2) *If $\mathcal{M}_{\nu,0}$ is passive, then $F^*_\nu = \emptyset$. If $\mathcal{M}_{\nu,0}$ is active, then $F^*_\nu \cap ([s(\dot{F}^{\mathcal{M}_{\nu,0}})]^{<\omega} \times \mathcal{M}_{\nu,0}) = \dot{F}^{\mathcal{M}_{\nu,0}} \restriction s(\dot{F}^{\mathcal{M}_{\nu,0}})$[4], and*

 (a) *if $\nu < \eta_0$, then F^*_ν is an extender from the sequence of H_ν, and*

$$H_\nu \models \text{lh}(F^*_\nu) \geq s(\dot{F}^{\mathcal{M}_{\nu,0}})^{+n}$$

 and

 (b) *if $\nu = \eta_0$, then F^*_ν is an extender from $\vec{F}$, and*

$$L[\vec{F}][x] \models \text{lh}(F^*_\nu) \geq s(\dot{F}^{\mathcal{M}_{\nu,0}})^{+n}.$$

(3) *If $(\nu, k) <_{\text{lex}} (\eta_0, j_0)$, then $\vec{\mathcal{M}} \restriction (\nu, k), \vec{C} \restriction \nu \in H_\nu$.*[5]

[4]If F is an extender, then $s(F)$ is its *support*, also called its *natural length* and denoted $\nu(F)$ in [2] and [5].

[5]Here $\vec{\mathcal{M}} \restriction (\nu, k) = \langle \mathcal{M}_{\xi,n} \mid (\xi, n) <_{\text{lex}} (\nu, k)\rangle$, and $\vec{C} \restriction \nu = (\langle H_\xi \mid \xi < \nu\rangle, \langle F^*_\xi \mid \xi \leq \nu\rangle)$.

(4) If $\gamma < \nu < \eta_0$ and

$$\rho = \inf\{\rho(\mathcal{M}_{\xi,n} \mid (\gamma, 0) \leq_{\text{lex}} (\xi, n) \leq_{\text{lex}} (\nu, 0)\},$$

and $\rho > \sup(Y)$, then

$$H_\gamma \overset{\mu}{\sim} H_\nu,$$

where

$$\mu = (\rho^{+n})^{H_\gamma} = (\rho^{+n})^{H_\nu}.$$

(5) If $\gamma < \eta_0$, and

$$\rho = \inf\{\rho(\mathcal{M}_{\xi,n}) \mid (\gamma, 0) \leq_{\text{lex}} (\xi, n) <_{\text{lex}} (\eta_0, 0)\},$$

and $\rho > \sup(Y)$, then

$$H_\gamma \overset{\mu}{\sim} L[\vec{F}][x],$$

where

$$\mu = (\rho^{+n})^{H_\gamma} = (\rho^{+n})^{L[\vec{F}][x]}.$$

Remark 1.4 *Adopting the notation of 1.3, notice that if $\nu < \eta_0$, then for all $k \varepsilon \omega$, $\mathcal{M}_{\nu,k} \in H_\nu$. (This follows easily from clause (3) if $k > 0$, and clause (2) if $k = 0$.)Moreover*

$$H_\nu \models (\vec{\mathcal{M}} \restriction \langle \nu, k \rangle)^\frown \langle \mathcal{M}_{\nu,k} \rangle \text{ is certified by } \vec{C} \restriction \nu.$$

We shall say that H_ν is the background universe for $\mathcal{M}_{\nu,k}$ provided by $\vec{C}$. *We also say that $L[\vec{F}][x]$ itself is the background universe for the $\mathcal{M}_{\eta_0,i}$.*

In the next section, we shall prove:

Theorem 1.5 *Let $\mathcal{P}$ be the last model of an ms-array which has a plus-1 certificate; then iteration trees on $\mathcal{P}$ can be lifted to iteration trees on $L[\vec{F}][x]$, and therefore $\mathcal{P}$ is fully iterable.*

Working still in $L[\vec{F}][x]$, and given Y a set of ordinals, we now construct a maximal plus-n certified ms-array over Y. The construction is organized as an induction on the cardinals of the model to which our array converges. For a given such cardinal κ, we extend our current array so as to add as many mice projecting to κ as possible, then move on to the next cardinal. More precisely, we define, by induction on α, $\vec{\mathcal{M}}_\alpha$, $\vec{C}_\alpha$, $\mathcal{P}_\alpha$, and κ_α such that

(i) $\vec{\mathcal{M}}_\alpha$ is an ms-array over Y which is plus-n certified by $\vec{C}_\alpha$, and $\mathcal{P}_\alpha$ is the last model of $\vec{\mathcal{M}}_\alpha$,

(ii) $k(\mathcal{P}_\alpha) = \omega$ and $\kappa_\alpha = \rho(\mathcal{P}_\alpha)$,

(iii) if $\vec{\mathcal{N}}$ is an ms-array extending $\vec{\mathcal{M}}_\alpha$, and $\vec{\mathcal{N}}$ is plus-n certified by some $\vec{D}$ which extends $\vec{C}_\alpha$, and $\mathcal{Q}$ is the last model of $\vec{\mathcal{N}}$, then $\kappa_\alpha \leq \rho(\mathcal{Q})$, and

(iv) if $\alpha < \beta$, then $\vec{\mathcal{M}}_\beta$ extends $\vec{\mathcal{M}}_\alpha$, $\vec{C}_\beta$ extends $\vec{C}_\alpha$, and $\kappa_\beta \geq \kappa_\alpha$.

Conditions (i),(iii), and (iv) together imply that if $\alpha \leq \beta$, then $\mathcal{P}_\alpha \trianglelefteq \mathcal{P}_\beta$. The union of the $\mathcal{P}_\alpha$'s is the model we are building.

We begin by setting

$$\begin{aligned}
\vec{\mathcal{M}}_0 &= \langle((\text{ rud }(\sup(Y) \cup \{Y\}), \in, \emptyset, \emptyset), i) \mid i \leq \omega\rangle, \\
\mathcal{P}_0 &= ((\text{ rud }(\sup(Y) \cup \{Y\}), \in, \emptyset, \emptyset), \omega), \\
\vec{C}_0 &= \langle \emptyset, \langle \emptyset \rangle \rangle, \\
\kappa_0 &= \sup(Y).
\end{aligned}$$

Given $\vec{\mathcal{M}}_\alpha$, $\vec{C}_\alpha$,$\mathcal{P}_\alpha$, and κ_α, we set

$$\begin{aligned}
\vec{\mathcal{M}}_{\alpha+1} &= \text{unique } \omega\text{-term extension of } \vec{\mathcal{M}}_\alpha \text{ determined} \\
& \text{ by constructing one step and taking cores,} \\
\vec{C}_{\alpha+1} &= L[\vec{F}][x]\text{-least plus-}n \text{ certificate for } \vec{\mathcal{M}}_{\alpha+1} \text{ extending } \vec{C}_\alpha, \\
\mathcal{P}_{\alpha+1} &= \text{last model of } \vec{\mathcal{M}}_{\alpha+1}, \\
\kappa_{\alpha+1} &= \kappa_\alpha.
\end{aligned}$$

Note here that there is such a certificate $\vec{C}_{\alpha+1}$, for if $\vec{C}_\alpha = \langle\langle H_\nu \mid \nu < \eta\rangle, \langle F^*_\nu \mid \nu \leq \eta\rangle\rangle$, then we can set $F^*_{\eta+1} = \emptyset$ and let H_η be and initial segment of $L[\vec{F}][x]$ such that $H_\eta \overset{\kappa_\alpha^{+n}}{\sim} L[\vec{F}][x]$ be such that $\langle \vec{\mathcal{M}}_\alpha, \vec{C}_\alpha, F^*_\eta\rangle \in H_\eta$ and $H_\eta \models \mathsf{ZF}^-$. This determines $\vec{C}_{\alpha+1}$.

It is easy to verify that our induction hypotheses continue to hold. Notice here that $\rho(\mathcal{P}_{\alpha+1}) < \kappa_{\alpha+1}$ is impossible by induction hypothesis (iii).

Now let λ be a limit ordinal.

Case 1. κ_α is not eventually constant as $\alpha \to \lambda$.

In this case, we set

$$\kappa_\lambda = \sup_{\alpha<\lambda} \kappa_\alpha,$$

and

$$\mathcal{P}_\lambda = ((J_\nu[\vec{E}], \in, \vec{E}, \emptyset), \omega)),$$

where

$$\vec{E} = \bigcup_{\alpha<\lambda} \dot{E}^{\mathcal{P}_\alpha}, \text{ and } \nu = \sup_{\alpha<\lambda} o(\mathcal{P}_\alpha).$$

Notice here that $(J_\nu[\vec{E}], \in, \vec{E}, \emptyset)$ is produced by a certified construction extending all $(\vec{\mathcal{M}}_\alpha, \vec{C}_\alpha)$, for $\alpha < \lambda$, and hence $\rho_\omega((J_\nu[\vec{E}], \in, \vec{E}, \emptyset)) \geq \kappa_\alpha$, for all $\alpha < \lambda$, by induction hypothesis (iii). Thus

$$\rho_\omega((J_\nu[\vec{E}], \in, \vec{E}, \emptyset)) = \kappa_\lambda = \nu,$$

so that $(J_\nu[\vec{E}], \in, \vec{E}, \emptyset)$ is indeed ω-sound. We set

$$\vec{\mathcal{M}}_\lambda = (\bigcup_{\alpha<\lambda} \vec{\mathcal{M}}_\alpha)^\frown\langle((J_\nu[\vec{E}], \in, \vec{E}, \emptyset), k) \mid k \leq \omega\rangle,$$

and let $\vec{C}_\lambda$ be the unique certificate for $\vec{\mathcal{M}}_\lambda$ which extends all $\vec{C}_\alpha$ for $\alpha < \lambda$ (essentially, the union of the $\vec{C}_\alpha$ for $\alpha < \lambda$). It is easy to check the induction hypotheses.

Case 2. κ_α is eventually constant, with value κ, as $\alpha \to \lambda$.

Let $\vec{\mathcal{M}}_{<\lambda} = \bigcup_{\alpha<\lambda} \vec{\mathcal{M}}_\alpha$, and $\vec{C}_{<\lambda} = \bigcup_{\alpha<\lambda} \vec{C}_\alpha$. Let

$$\vec{E} = \bigcup_{\alpha<\lambda} \dot{E}^{\mathcal{P}_\alpha}, \eta = \sup_{\alpha<\lambda} o(\mathcal{P}_\alpha),$$

and let

$$\mathcal{P} = (J_\eta[\vec{E}], \in, \vec{E}).$$

Note that κ is the largest cardinal of $\mathcal{P}$.

Subcase 2a. There are F and F^* such that $(\mathcal{P}, F)$ is an active premouse (in the sense of [2]), $F^* \cap ([s(F)]^{<\omega} \times \mathcal{P}) = F$, and $L[\vec{F}][x] \models$ "F^* is an extender from my sequence, and $\mathrm{lh}(F^*) \geq s(F)^{+n}$".

We pick such F, F^* with $s(F)$ as small as possible, and F^* being $L[\vec{]}[x]$-least for that value of $s(F)$. Note that $\rho_1((\mathcal{P}, F)) = \kappa = \rho_\omega((\mathcal{P}, F))$, and that $(\mathcal{P}, F)$ is ω-sound. We set

$$\begin{aligned}
\mathcal{P}_\lambda &= ((\mathcal{P}, F), \omega), \\
\vec{\mathcal{M}}_\lambda &= (\vec{\mathcal{M}}_{<\lambda})^\frown\langle((\mathcal{P}, F), k) \mid k \leq \omega\rangle, \\
\kappa_\lambda &= \kappa, \\
\vec{C}_\lambda &= \text{ extension of } \vec{C}_{<\lambda} \text{ determined by adding } F^*.
\end{aligned}$$

Subcase 2b. Subcase 2a does not hold, and the is a proper extension $(\vec{\mathcal{N}}, \vec{D})$ of $(\vec{\mathcal{M}}_{<\lambda}, \vec{C}_{<\lambda})$ such that $\vec{\mathcal{N}}$ has last entry $(\mathcal{Q}, \omega)$, where $\rho_\omega(\mathcal{Q}) = \kappa$. In this case, we pick the $L[\vec{F}][x]$-least such $(\vec{\mathcal{N}}, \vec{D})$, and set

$$\vec{\mathcal{M}}_\lambda = \vec{\mathcal{N}}, \vec{C}_\lambda = \vec{D}, \mathcal{P}_\lambda = Q, \text{ and } \kappa_\lambda = \kappa.$$

It is easy to check induction hypotheses (i)-(iv).

Subcase 2c. Otherwise.

In this case, we are done with adding all the mice which project to κ. Let $\vec{E} = \bigcup_{\alpha<\lambda} \dot{E}^{\mathcal{P}_\alpha}$. We set

$$\begin{aligned}
\kappa_\lambda &= \sup_{\alpha<\lambda} o(\mathcal{P}_\alpha), \\
\vec{\mathcal{M}}_\lambda &= \langle ((J_{\kappa_\lambda}[\vec{E}], \in, \vec{E}, \emptyset), k) \mid k \leq \omega \rangle, \\
\mathcal{P}_\lambda &= (J_{\kappa_\lambda}[\vec{E}], \in, \vec{E}, \emptyset), \omega), \\
\vec{C}_\lambda &= \vec{C}_{<\lambda}
\end{aligned}$$

It is easy to check the induction hypotheses.

This completes our construction. Let

$$\vec{E} = \bigcup_\alpha \dot{E}^{\mathcal{P}_\alpha}.$$

Granted our iterability lemma 1.5, $L[\vec{E}][Y]$ is a proper class premouse over Y. The κ_α such that $\kappa_\alpha > \sup(Y)$ are precisely the cardinals of $L[\vec{E}][Y]$ which are $> \sup(Y)$, enumerated in nondecreasing order and without leaving gaps. If κ is a cardinal of $L[\vec{E}][Y]$ such that $\kappa > \sup(Y)$, then the $\mathcal{P}_\alpha$ such that $\kappa_\alpha = \kappa$ are precisely those levels of $L[\vec{E}][Y]$ which project to κ, enumerated in strictly increasing order and without leaving gaps.[6]

Because we always extended our current certified array in an $L[\vec{F}][x]$-minimal way, our construction is uniformly locally definable in $L[\vec{F}][x]$ in the following sense.

Lemma 1.5.1 *For any $\kappa \geq \sup(Y)$, letting $\mu = (\kappa^{+(n+1)})^{L[\vec{F}][x]}$, we have*

(a) Whenever $\kappa_\alpha \leq \kappa$, then $\vec{\mathcal{M}}_\alpha, \vec{C}_\alpha \in L[\vec{F}][x]|\mu$[7], and

(b) the sequence $\langle (\vec{\mathcal{M}}_\alpha, \vec{C}_\alpha, \mathcal{P}_\alpha) \mid \kappa_\alpha \leq \kappa \rangle$ is Σ_1 definable over $L[\vec{F}][x]|\mu$.

Proof sketch. We prove (a) by induction on α, using the condensation properties of the $L[\vec{F}][x]$-hierarchy. Part (b) then follows immediately. □

We can now show that Woodiness and the version of λ-strength in which the witnessing extenders must come from the coherent sequence go down to $L[\vec{E}][Y]$.

[6] In these respects, the organization of our construction is more natural than that of the construction of [2].

[7] If $\mathcal{N}$ is a premouse, then $\mathcal{N}|\xi$ is its ξ^{th} level, so that $\mathcal{Q} \trianglelefteq \mathcal{N}$ iff $\exists \xi (\mathcal{Q} = \mathcal{N}|\xi)$.

Definition 1.6 *For any premouse $\mathcal{M}$ and any $\kappa \in \mathcal{M}$,*

$$o(\kappa)^{\mathcal{M}} = \sup(\{s(F) \mid \mathrm{crit}(F) = \kappa \wedge \ F \textit{ is on the } \mathcal{M}\textit{-sequence }\}).$$

It is easy to see that if $o(\kappa)^{\mathcal{M}} \geq \lambda$, where λ is a cardinal of $\mathcal{M}$, then κ is η-strong in $\mathcal{M}$, for all η such that $|V_\eta|^{\mathcal{M}} < \lambda$. We don't know whether one can prove anything in the converse direction. For example, if $\kappa < \lambda$, and λ is inaccessible in $\mathcal{M}$, and κ is λ-strong in $\mathcal{M}$, does it follow that $o(\kappa)^{\mathcal{M}} \geq \lambda$? (One should assume here that $\mathcal{M}$ is iterable; otherwise it seems hopeless. One *can* show that if κ is Woodin in an iterable $\mathcal{M}$, then it is Woodin in $\mathcal{M}$ via extenders on the $\mathcal{M}$-sequence.)

Theorem 1.7 *(1) Let $\delta > \sup(Y), \sup(X)$ be Woodin in $L[\vec{F}][x]$; then δ is Woodin in $L[\vec{E}][Y]$.*

(2) Let $\sup(Y), \sup(X) < \kappa < \lambda$, and suppose λ is a limit cardinal in $L[\vec{F}][x]$, and $o(\kappa)^{L[\vec{F}][x]} \geq \lambda$; then $o(\kappa)^{L[\vec{E}][Y]} \geq \lambda$.

Proof. Both proofs follow closely the proof of [2, Lemma 11.4]. We shall prove (2) first; a small embellishment of the argument then yields (1). As in [2], we appeal to the iterability of certain bicephali and psuedo-premice related to our construction. The iterabilty of these structures follows from arguments much like the proof of 1.5.

Let $\kappa < \nu < \lambda$, and ν be a cardinal of $L[\vec{E}][Y]$. We want to show that there is an extender F on $\vec{E}$ such that $\mathrm{crit}(F) = \kappa$ and $s(F) \geq \nu$. Fix F^* on $\vec{F}$ such that

$$\mathrm{crit}(F^*) = \kappa \text{ and } s(F^*) \geq (\nu^{+(n+1)})^{L[\vec{F}][x]},$$

and let

$$F = F^* \cap ([\nu^+]^{<\omega} \times L[\vec{E}][Y].$$

Claim. For any $\rho \leq \nu$ such that $(\kappa^+)^{L[\vec{E}][Y]} \leq \rho$, then trivial completion of $F \restriction \rho$ is either on $\vec{E}$, or an ultrapower away, or of type Z.

Proof. See [5] for the definition of type Z extenders. We prove the claim by induction on ρ, following very closely the proof of [2, 11.4]. The proof breaks into various cases according to where ρ sits among the generators of F, exactly parallel to the cases in [2, 11.4]. We handle here just one of the more complicated cases, called case C in [2].

Case C. ρ is a limit of generators of F, ρ is itself a generator of F, and $E_\rho = \emptyset$.

Proof. Let $j\colon L[\vec{F}][x] \to \mathrm{Ult}(L[\vec{F}][x], F^*$ be the canonical embedding, and notice that

$$j(\vec{E}) \restriction \nu^+ = \vec{E} \restriction \nu^+$$

by the locality of our construction (1.5.1). This implies that

$$L[\vec{E}][Y]|\nu^+ \trianglelefteq \mathrm{Ult}(L[\vec{E}][Y], F).$$

Let

$$E = \text{ trivial completion of } F \restriction \rho,$$

and $\gamma = \mathrm{lh}(E)$, and let

$$\sigma\colon \mathrm{Ult}(L[\vec{E}][Y], E) \to \mathrm{Ult}(L[\vec{E}][Y], F)$$

be the canonical embedding. Notice that $\mathrm{crit}(\sigma) = \rho$, because ρ is a generator of F, and thus ρ is a cardinal of $\mathrm{Ult}(L[\vec{E}][Y], E)$.

By the condensation theorem [2, 8.2], we have that if $\rho < \eta < \gamma$ and $\rho(\mathrm{Ult}(L[\vec{E}][Y], E)|\eta) = \rho$, then $\mathrm{Ult}(L[\vec{E}][Y], E)|\eta \trianglelefteq \mathrm{Ult}(L[\vec{E}][Y], F)|\eta$, and thus $\mathrm{Ult}(L[\vec{E}][Y], E)|\eta \trianglelefteq L[\vec{E}][Y]$. Since γ is a limit of such η, we have $\mathrm{Ult}(L[\vec{E}][Y], E)|\gamma \trianglelefteq L[\vec{E}][Y]$. It follows that

$$(J_\gamma[\vec{E}], \in, \vec{E} \restriction \gamma, E) \text{ is a premouse}$$

in the sense of [2]. (We have just verified the coherence condition, and the initial segment condition follows from our induction hypothesis.)

Subcase C1. ρ is a cardinal of $L[\vec{E}][Y]$.

The levels of $L[\vec{E}][Y]$ projecting to ρ are then all of the form $\mathcal{P}_\xi$, so letting $\lambda = \sup(\{\alpha \mid \mathcal{P}_\alpha \triangleleft \mathrm{Ult}(L[\vec{E}][Y], E)|\gamma\})$, we have that $\kappa_\alpha = \rho$ for all sufficiently large $\alpha < \lambda$, so that case 2 applied at λ, with $\mathcal{P} = (J_\gamma[\vec{E}], \in, \vec{E} \restriction \gamma)$ in the notation there. But then E and F^* witness that subcase 2a applies at stage λ, and so

$$\mathcal{P}_\lambda = (J_\gamma[\vec{E}], \in, \vec{E} \restriction \gamma, G),$$

where $s(G)$ is as small as possible. Now if $s(G) < \rho$, then ρ is not a cardinal in $L[\vec{E}][Y]$, so $s(G) = \rho$. A bicephalus argument now gives $G = E$, and we are done. $\square$

Subcase C2. Otherwise.

Let τ be the largest cardinal of $L[\vec{E}][Y]$ which is less than ρ. If $\tau < \eta < \rho$, then

$$\mathrm{Ult}(L[\vec{E}][Y], F) \models \rho \text{ is not a cardinal },$$

because this structure agrees sufficiently with $L[\vec{E}][Y]$. But $\sigma(\eta) = \eta$, so we get

$$\mathrm{Ult}(L[\vec{E}][Y], E) \models \eta \text{ is not a cardinal },$$

so by acceptability and coherence,

$$J_\gamma[\vec{E}][Y] \models \eta \text{ is not a cardinal.}$$

Hence ρ is the cardinal succcesor of τ in $J_\gamma[\vec{E}][Y]$. (Note that ρ is a cardinal in $J_\gamma[\vec{E}][Y]$ because it is a limit of generators of E.) Let

$$\psi \colon \mathrm{Ult}(L[\vec{E}][Y], F \upharpoonright (\rho + 1)) \to \mathrm{Ult}(L[\vec{E}][Y], F)$$

be the canonical embedding, and let

$$G = \text{ trivial completion of } F \upharpoonright (\rho + 1),$$

and $\xi = \mathrm{lh}(G)$. Since $\psi(\rho) = \rho$, we have $\psi \upharpoonright \xi =$ identity , and thus τ is the largest cardinal in $J_\xi[\vec{E}][Y]$, using coherence again. By induction hypothesis, the trivial completion of $E \upharpoonright \tau$ is on $\vec{E}$ or an ultrapower away. (It is not of type Z because τ is a cardinal of $L[\vec{E}][Y]$.) It follows then that

$$(J_\xi[\vec{E}][Y], \in, \vec{E} \upharpoonright \xi, G) \text{ is a psuedo-premouse.}$$

We can show it is iterable by a proof like the one for 1.5 sketched in the next section. But then, by [2, 10.1], $(J_\xi[\vec{E}][Y], \in, \vec{E} \upharpoonright \xi, G)$ is a premouse, so that E is on $\vec{E}$ or an ultrapower away, as desired. □

This finishes case C, and thereby our sketch of the proof of Theorem 1.7. □

2 Iterabilty

We now sketch a proof of 1.5.

Theorem 2.1 *Let $\mathcal{P}$ be the last model of a plus-n certified ms-array; then $\mathcal{P}$ is fully iterable.*

Proof. Let Γ be an iteration strategy for $L[\vec{F}][x]$. Our iteration strategy Σ for $\mathcal{P}$ is a pullback of Γ: we lift the evolving iteration tree $\mathcal{T}$ on $\mathcal{P}$ to a tree $\mathcal{U}$ on $L[\vec{F}][x]$ with the same tree order, and branch embeddings which commute with the lifting maps, then choose the branch of $\mathcal{T}$ which is chosen by upstairs in $\mathcal{U}$ by Γ. The details of the lifting process are very close to those in [2]. The only real difference is that $\mathcal{U}$ will be allowed to drop.

We make use of the notions of *dropdown sequence* and *(partial) resurrection* from [2, §12]. (See also [1] for a somewhat cleaner presentation.) Notice that resurrection makes sense as applied to arbitrary ms-arrays. Staying as close as possible to the notation of [2, §12], we shall use $\mathcal{P}_\alpha$ and $\mathcal{R}_\alpha$ for the α^{th} models of $\mathcal{T}$ and $\mathcal{U}$ respectively. The tree $\mathcal{U}$ will be allowed to drop; in fact we shall have $D^{\mathcal{U}} = D^{\mathcal{T}}$. Nevertheless, $\mathcal{U}$ will not be fine structural, in that it will not be maximal, since we will drop further than we have to. In fact, each $\mathcal{R}_\alpha$ will satisfy ZF^-, and hence all the embeddings of $\mathcal{U}$ will be fully elementary.

We shall maintain inductively hypotheses **H0-H7** below:

H0. $\mathcal{R}_\alpha \models \mathsf{ZF}^-$, moreover there are $\vec{\mathcal{M}}_\alpha, \vec{C}_\alpha, \mathcal{Q}_\alpha \in \mathcal{R}_\alpha$ such that

$$\mathcal{R}_\alpha \models \vec{\mathcal{M}}_\alpha \text{ is an ms-array with last model } \mathcal{Q}_\alpha \text{ and plus-}n \text{ certificate } \vec{C}_\alpha.$$

H1. There is a weak embedding $\pi_\alpha : \mathcal{P}_\alpha \to \mathcal{Q}_\alpha$. (That is, π_α is a weak k-embedding, where $k = k(\mathcal{P}_\alpha) = k(\mathcal{Q}_\alpha) = \deg^{\mathcal{T}}(\alpha)$.)

H2. If $\beta T \alpha$ and $D^{\mathcal{T}} \cap (\alpha, \beta]_T = \emptyset$, then $\pi_\beta \circ i^{\mathcal{T}}_{\alpha,\beta} = i^{\mathcal{U}}_{\alpha,\beta} \circ \pi_\alpha$.

Next, we have some agreement of models and embeddings to maintain. Let $(\sigma^\beta, \mathrm{Res}^\beta)$ be the complete resurrection in the sense of $\vec{\mathcal{M}}_\beta$ of $\pi_\beta(E^{\mathcal{T}}_\beta)$ from $\mathcal{Q}_\beta$[8] Let $\nu_\beta = s(E^{\mathcal{T}}_\beta)$ and $\lambda_\beta = \mathrm{lh}(E^{\mathcal{T}}_\beta)$. Let $\nu^*_\beta = s(\dot{F}^{\mathrm{Res}^\beta})$ and $\lambda^*_\beta = \mathrm{OR} \cap \mathrm{Res}^\beta = \mathrm{lh}(\dot{F}^{\mathrm{Res}^\beta})$. (Note that $\dot{F}^{\mathrm{Res}^\beta}$ is the extender which completely resurrects $\pi_\beta(E^{\mathcal{T}}_\beta)$.) Finally, let μ_β be $(\nu^*_\beta)^{+n}$ in the background unniverse for Res^β provided by $\vec{C}_\beta$.

H3. If $\beta < \alpha$ and Res^β is of types 1 or 2, then $\mathcal{Q}_\alpha$ agrees with Res^β below ν^*_β , moreover

$$\pi_\alpha \restriction \nu_\beta = \sigma^\beta \circ \pi_\beta \restriction \nu_\beta \text{ and } \pi_\alpha(\nu_\beta) \geq \nu^*_\beta.$$

H4. If $\beta < \alpha$ and Res^β is of type 2, then $\mathcal{Q}_\alpha$ agrees with Res^β below λ^*_β, and

$$\pi_\alpha \restriction \lambda_\beta = \sigma^\beta \circ \pi_\beta \restriction \lambda_\beta, \text{ and } \pi_\alpha(\lambda_\beta) \geq \lambda^*_\beta.$$

H5. If $\beta < \alpha$, then μ_β is a cardinal in $\mathcal{R}_\alpha$, and$\mathcal{R}_\alpha$ agrees with the background universe for Res^β provided by $\vec{C}_\beta$ below μ_β.

Finally, we can simplify H6 and H7 of [2] a bit:

H6. $\alpha \in D^{\mathcal{T}} \Leftrightarrow \alpha \in D^{\mathcal{U}}$.

H7. If $i^{\mathcal{U}}_{\beta,\alpha}$ exists, then $i^{\mathcal{U}}_{\beta,\alpha}((\vec{\mathcal{M}}_\beta, \vec{C}_\beta)) = (\vec{\mathcal{M}}_\alpha, \vec{C}_\alpha)$.

[8] In the language of [2], from stage (j, τ), where Q_β has index (j, τ) in $\vec{\mathcal{M}}_\beta$.

This completes our list of induction hypotheses. We now give the inductive definition. In the base step, we set $\mathcal{P}_0 = \mathcal{Q}_0 = \mathcal{P}$, $\pi_0 =$ identity, $\mathcal{R}_0 = L[\vec{F}][x]$, and let $(\vec{\mathcal{M}}_0, \vec{C}_0)$ be any plus-n certified array with last model $\mathcal{P}$.

Now suppose we have $\mathcal{P}_\beta, \mathcal{Q}_\beta, \pi_\beta, \mathcal{R}_\beta, \vec{\mathcal{M}}_\beta, \vec{C}_\beta$ defined for all $\beta \leq \alpha$, and satisfying H1-H7 with respect to $\mathcal{T} \upharpoonright (\alpha+1)$ and $\mathcal{U} \upharpoonright (\alpha+1)$. We want to define these objects at $\alpha+1$. We are given $\mathcal{T} \upharpoonright (\alpha+2)$, that is, we are given $E^{\mathcal{T}}_\alpha$ and $\beta = \mathrm{pd}_T(\alpha+1)$.

Claim 1. If $\gamma < \alpha$, then $\sigma^\alpha \upharpoonright \pi_\alpha(\mathrm{lh}(E^{\mathcal{T}}_\gamma) =$ identity.

See [2, p. 116] for the proof of claim 1. As in [2], we then have that $\nu^*_\gamma < \nu^*_\alpha$, for all $\gamma < \alpha$.

Set

$$F = \sigma^\alpha \circ \pi_\alpha(E^{\mathcal{T}}_\alpha) = \dot{F}^{\mathrm{Res}_\alpha},$$

and

$$E^{\mathcal{U}}_\alpha = F^* = \text{ background extender for } F \text{ provided by } \vec{C}_\alpha.$$

We are going to apply F^* to the appropriate initial segment of $\mathcal{R}_\beta$ in order to get $\mathcal{R}_{\alpha+1}$. To identify it, we must consider the appropriate dropdown sequence of $\mathcal{Q}_\beta$. Let $j = \deg^{\mathcal{T}}(\beta)$ and $\lambda = \mathrm{lh}(E^{\mathcal{T}}_\beta$, and let

$$\langle(\eta_0, k_0), ..., (\eta_e, k_e)\rangle = \ (j, \lambda) \text{ dropdown sequence of } \mathcal{P}_\beta,$$

and set

$$\kappa_i = \rho_{k_i}(\mathcal{P}_\beta | \eta_i)$$

for $0 \leq i \leq e$. The $(j, \pi_b(\lambda))$ dropdown sequence of $\mathcal{Q}_\beta$ is essentially the image under π_b of $\langle(\eta_0, k_0), ..., (\eta_e, k_e)\rangle$, although there are some fine structural complications. The exact situation is spelled out in [2, claim 2, p. 117], which goes through here.

Let $\mu = \mathrm{crit}(E^{\mathcal{T}}_\alpha)$, and let

$$i = \begin{cases} e+1 & \text{if } \mu < \kappa_e \\ \text{least } j \text{ s.t.} \kappa_j \leq \mu & \text{if } \kappa_e \leq \mu. \end{cases}$$

Notice that $i > 0$ since $\kappa_0 = \lambda > \mu$. Because $\mathcal{T}$ is maximal

$$\mathcal{P}^*_{\alpha+1} = \begin{cases} \mathcal{P}_\beta | \eta_i & \text{if } i \leq e \\ \mathcal{P}_\beta & \text{if } i = e+1, \end{cases}$$

and

$$\deg^{\mathcal{T}}(\alpha+1) = \begin{cases} k_i - 1 & \text{if } i \leq e \\ j & \text{if } i = e+1. \end{cases}$$

Let $(\sigma_i^\beta, \mathrm{Res}_i^\beta)$ be the i^{th} partial resurrection of $\pi_\beta(E_\beta^{\mathcal{T}})$ from $\mathcal{Q}_\beta$ inside $\vec{\mathcal{M}}_\beta$, if this resurection is defined. The resurrection is undefined if $i = e+1$, defined if $i < e$, and may or may not be defined when $i = e$. (See [2, p. 118] for a precise description.) Let

$$\mathcal{Q}^*_{\alpha+1} = \begin{cases} \mathrm{Res}_i^\beta & \text{if } \mathrm{Res}_i^\beta \text{ is defined} \\ \mathcal{Q}_\beta & \text{otherwise,} \end{cases}$$

$$\sigma = \begin{cases} \sigma_i^\beta & \text{if } \mathrm{Res}_i^\beta \text{ is defined} \\ \text{identity} & \text{otherwise.} \end{cases}$$

As in [2], we have that $\sigma \circ \pi_\beta \restriction \mathcal{P}^*_{\alpha+1}$ is a weak $\deg^{\mathcal{T}}(\alpha+1)$-embedding from $\mathcal{P}^*_{\alpha+1}$ to $\mathcal{Q}^*_{\alpha+1}$. Also, σ is a factor of the full resurrection map σ^β, in that we have

$$\psi\colon \mathcal{Q}^*_{\alpha+1}|(\sigma \circ \pi_\beta(\lambda)) \to \mathrm{Res}^\beta$$

such that

$$\sigma^\beta = \psi \circ (\sigma \restriction \mathcal{Q}_\beta|(\pi_\beta(\lambda)).$$

Moreover

$$\psi \restriction (\sup(\sigma \circ \pi_\beta\text{''}\kappa_{i-1})) = \text{ identity },$$

as the reader can see in [2, p. 119].

Let H be the background universe for $\mathcal{Q}^*_{\alpha+1}$ which is provided by $\vec{C}_\beta$. If Res_i^β is undefined, so that $\mathcal{Q}^*_{\alpha+1} = \mathcal{Q}_\beta$, this means that $H = \mathcal{R}_\beta$. Otherwise, $\mathcal{Q}^*_{\alpha+1} = \mathrm{Res}_i^\beta$ is a model $\mathcal{M}_{\tau,0}$ inside $\vec{\mathcal{M}}_\beta$, and H is the background universe with index τ inside $\vec{C}_\beta$. Let also N be the background universe for Res^α provided by $\vec{C}_\alpha$ in a similar way. (Thus F^* is an extender on the sequence of N, and $\mathrm{lh}(F^*) \geq s(F)^{+n}$ holds in N.

Claim 2. $(\mathrm{crit}(F)^{+n})^H = (\mathrm{crit}(F)^{+n})^N$ (and in particular, both exist), and H agrees with N below $(\mathrm{crit}(F)^{+n})^H$.

Proof. Let $\theta = \sup(\sigma \circ \pi_\beta\text{''}\kappa_{i-1})$. Note that $\mathrm{crit}(F) < \theta$. To see this, assume first $\beta = \alpha$. Since $\mathrm{crit}(E_\alpha^{\mathcal{T}} = \mu < \kappa_{i-1}$, we have

$$\mathrm{crit}(F) = \sigma^\alpha \circ \pi_\alpha(\mu) = \sigma \circ \pi_\beta(\mu) < \theta,$$

because ψ is the identity on θ. If $\beta < \alpha$, then

$$\begin{aligned} \sigma^\beta \circ \pi_\beta \restriction \mu+1 &= \sigma^\beta \circ \pi_\beta \restriction \mu+1 \\ &= \pi_\alpha \restriction \mu+1 \\ &= \sigma^\alpha \circ \pi_\alpha \restriction \mu+1. \end{aligned}$$

The first identity holds because ψ is the identity on θ, the second by induction hypothesis H3 or H4 (noting $\mu < s(E^{\mathcal{T}}_\beta)$), and the third by claim 1. These identities clearly imply $\mathrm{crit}(F) < \theta$.

It is easy to see that $\theta \leq o(\mathcal{Q}^*_{\alpha+1})$, and thus $(\theta^{+n})^H$ exists by the definition of certificates. Letting G be the background universe for Res^β provided by $\vec{C}_\beta$, we see from (4) or (5) of 1.3 that $(\theta^{+n})^G = (\theta^{+n})^H$, and $G \overset{\eta}{\sim} H$, for $\eta = (\theta^{+n})^H$. If $\beta = \alpha$, then $G = N$, and we are done. Suppose $\beta < \alpha$. Then $\mathrm{crit}(F) = \sigma^\beta \circ \pi_\beta(\mu) < \nu^*_\beta$, so by induction hypothesis H5, G and $\mathcal{R}_\alpha$ agree below their common value for $\mathrm{crit}(F)^{+n}$. However, by claim 1 and the definition of certificates, $\mathcal{R}_\alpha$ agrees with N below their common value for $\mathrm{crit}(F)^{+n}$. Thus H has the desired agreement with N. $\square$

It follows that F^* measures all subsets of μ in H, and we can set

$$\mathcal{R}^{\mathcal{U}}_{\alpha+1} = \mathrm{Ult}(H, F^*),$$

and let $j \colon H \to \mathcal{R}_{\alpha+1}$ be the canonical (fully elementary) emdedding. We set

$$\mathcal{Q}_{\alpha+1} = j(\mathcal{Q}^*_{\alpha+1}).$$

Letting (ξ, k) be the index of $\mathcal{Q}^*_{\alpha+1}$ in $\vec{\mathcal{M}}_\beta$, we let

$$(\vec{\mathcal{M}}_{\alpha+1}, \vec{C}_{\alpha+1}) = j((\vec{\mathcal{M}} \restriction (\xi, k+1), \vec{C}_\beta \restriction \xi)),$$

noting that this pair is in the domain of j by remark 1.4. Finally, we define $\pi_{\alpha+1}$ from $\mathcal{P}_{\alpha+1} = \mathrm{Ult}(\mathcal{P}^*_{\alpha+1}, E^{\mathcal{T}}_\alpha)$ to $\mathcal{Q}_{\alpha+1} = \mathrm{Ult}(\mathcal{Q}^*_{\alpha+1}, F)$ by applying the shift lemma to the maps $\sigma \circ \pi_\beta \colon \mathcal{P}^*_{\alpha+1} \to \mathcal{Q}^*_{\alpha+1}$ and $\sigma^\alpha \circ \pi_\alpha \colon \mathcal{P}_\alpha | \mathrm{lh}(E^{\mathcal{T}}_\alpha) \to \mathrm{Res}_\alpha$, then embedding $\mathrm{Ult}(\mathcal{Q}^*_{\alpha+1}, F)$ into $j(\mathcal{Q}^*_{\alpha+1})$ in the natural way. That is,

$$\pi_{\alpha+1}([a, f]) = j(\sigma \circ \pi_b(f))(\sigma^\alpha \circ \pi_\alpha(a)).$$

It is completely routine now to check that H1-H7 continue to hold at $\alpha + 1$. The reader can see [2, p. 119 ff.] for the calculations.

At limit steps λ, we set

$$\Sigma(\mathcal{T} \restriction \lambda) = \Gamma(\mathcal{U} \restriction \lambda).$$

Using H7, we see that $D^{cT} \cap [0, \lambda]_T$ is finite. By H2, the direct limit $\mathcal{P}_\lambda$ of the $\mathcal{P}_\alpha$ for $\alpha \in b$ sufficiently large embeds into the corresponding direct limit $\mathcal{Q}_\lambda$ of the $\mathcal{Q}_\alpha$. We let π_λ be the natural map. The rest is equally routine.

$\square$

3 UBH

We begin by showing that in any iterable $L[\vec{E}]$-model, UBH holds for all plus-2, nondropping, nonoverlapping iteration trees on V. This result was first proved by Woodin for tame $L[\vec{E}]$, without the plus-2 restriction. In the form just stated, it is due to the author.

Definition 3.1 *An iteration tree $\mathcal{T}$ is a* plus-2 tree *iff $\mathcal{T}$ is ω-maximal, and whenever $\alpha+1 < \mathrm{lh}(\mathcal{T})$, then there is an ordinal ρ such that*

$$\mathcal{M}_\alpha^\mathcal{T}|\,\mathrm{lh}(E_\alpha^\mathcal{T} \models \rho^{++} \text{ exists and } |s(E_\alpha^\mathcal{T})| = \rho^{++},$$

and

$$(\rho^{++})^{\mathcal{M}_\alpha^\mathcal{T}|\,\mathrm{lh}(E_\alpha^\mathcal{T})} = (\rho^{++})^{\mathcal{M}_\alpha^\mathcal{T}}.$$

We write $\rho_\alpha^\mathcal{T}$ for the least such ordinal ρ.

We are interested here in the case that the base model of $\mathcal{T}$ is a premouse. Notice that if $\mathcal{T}$ is a plus-2 iteration tree, and $\mathcal{U}$ is an arbitrary ω-maximal iteration tree extending $\mathcal{T}$, and $\mathrm{pd}_U(\beta+1) = \alpha$, where $\alpha < \mathrm{lh}(\mathcal{T})$, then $\mathrm{crit}(E_\beta^\mathcal{U}) \leq \rho_\alpha^\mathcal{T}$. This is true simply because $\mathrm{crit}(E_\beta^\mathcal{U})$ is a limit cardinal of $\mathcal{M}_\alpha^\mathcal{T}|\,\mathrm{lh}(E_\alpha^\mathcal{T})$. So to lift extensions of $\mathcal{T}$, we need embeddings of the models of $\mathcal{T}$ whose agreement is given by the $\rho_\alpha^\mathcal{T}$. This motivates

Definition 3.2 *Let $\mathcal{T}$ be a plus-2 iteration tree; then the* (weak) phalanx of $\mathcal{T}$ *is the phalanx*

$$\Phi^-(\mathcal{T}) = (\langle \mathcal{M}_\alpha^\mathcal{T} \mid \alpha < \mathrm{lh}(\mathcal{T})\rangle, \langle((\rho_\alpha^\mathcal{T})^+)^{\mathcal{M}_\alpha^\mathcal{T}} \mid \alpha+1 < \mathrm{lh}(E_\alpha^\mathcal{T})\rangle).$$

Theorem 3.3 *Let $L[\vec{E}][Y]$ be fully iterable proper mouse over Y satisfying* ZF. *Then there are no $\mathcal{T}, b, c \in L[\vec{E}][Y]$ such that*

$$L[\vec{E}][Y] \models \mathcal{T} \text{ is a plus-2 iteration tree on } V \text{ of limit length,}$$

and $D^\mathcal{T} = \emptyset$, and

$$L[\vec{E}][Y] \models \text{ b and c are distinct, cofinal, wellfounded branches of } \mathcal{T}.$$

Proof. Let us assume $Y = \emptyset$ in order to save ink. Assume the theorem fails for $L\vec{E}]$, and let $(\mathcal{T}, b, c)$ be the $L[\vec{E}]$-least counterexample. Let μ be a regular cardinal of $L[\vec{E}]$ such that $(\mathcal{T}, b, c) \in L[\vec{E}]|\mu$.

It is clear that $\mathcal{T}$ is a nondropping tree on $L[\vec{E}]|\mu$, and that b and c are distinct cofinal branches which are wellfounded when the base model is regarded as $L[\vec{E}]|\mu$. On the other hand, if $(\mathcal{S}, d, e)$ is satisfied to be a counterexample in $L[\vec{E}]|\mu$, then $\mathcal{S}$ can be regarded as an iteration tree on the whole $L[\vec{E}]$, and a standard Lowenheim-Skolem argument shows that d and e are wellfounded when the base model is taken to be $L[\vec{E}]$. Thus

$$L[\vec{E}]|\mu \models \ (\mathcal{T}, b, c) \text{ is the } L[\vec{E}]\text{-least counterexample to the theorem.}$$

Now let

$$\mathcal{N} = \mathcal{H}_\omega^{L[\vec{E}]|\mu}(\emptyset)$$

be the transitive collapse of smallest elementary submodel of $L[\vec{E}]|\mu$. Thus $\mathcal{N}$ is pointwise definable, and we have a fully elementary $\pi: \mathcal{N} \to L[\vec{E}]|\mu$. Let

$$\pi((\bar{\mathcal{T}}, \bar{b}, \bar{c})) = (\mathcal{T}, b, c).$$

Let us write $\mathcal{M}_0 = L[\vec{E}]|\mu$, and write $\mathcal{M}_\alpha$ for the α^{th} model of $\mathcal{T}$, regarded as a tree on $\mathcal{M}_0$, and $\mathcal{M}^*_\alpha$ for the α^{th} model of $\mathcal{T}$ regarded as a tree on the whole of $L[\vec{E}]$. Let us use $\mathcal{N}_\alpha$ for the α^{th} model of $\bar{\mathcal{T}}$, which is a tree on $\mathcal{N}$. It is clear that for any $\alpha < \text{lh}(\bar{\mathcal{T}})$,

$$\pi \restriction \mathcal{N}_\alpha : \mathcal{N}_\alpha \to \mathcal{M}_{\pi(\alpha)} = \mathcal{M}^*_{\pi(\alpha)}|\mu,$$

noting here that $i^*_{0,\beta}(\mu) = \mu$ for all $\beta < \text{lh}(\mathcal{T})$. Also, $\pi \restriction \mathcal{N}_{\bar{b}}$ is fully elementary from $\mathcal{N}_{\bar{b}}$ to $\mathcal{M}_b$.

It is enough for a contradiction to show that the phalanxes $\Phi(\bar{\mathcal{T}}^\frown\bar{b}$ and $\Phi(\bar{\mathcal{T}}^\frown\bar{c})$ are iterable. For in that case, we can compare them, and use the pointwise definability of $\mathcal{N}$ to reach the usual contradiction. (See for example the proof of uniqueness for iteration strategies with the weak Dodd-Jensen property in [4].) Of course, $\bar{b}$ and $\bar{c}$ are indistinguishable at this point, so it is enough to show $\Phi(\bar{\mathcal{T}}^\frown\bar{b})$ is iterable. This we shall do by realizing the phalanx $\Phi^-(\bar{\mathcal{T}}^\frown\bar{b})$ into a phalanx Ψ whose models are initial segments of $L[\vec{E}]$. Since $L[\vec{E}]$ is iterable, Ψ will be iterable, and since phalanx realized in an iterable phalanx is itself iterable (see [6, §9]), we get that $\Phi^-(\bar{\mathcal{T}}^\frown\bar{b})$ is iterable. But as we remarked above, iterating $\Phi(\bar{\mathcal{T}}^\frown\bar{B})$ is the same as iterating $\Phi^-(\bar{\mathcal{T}}^\frown\bar{b})$, so we are done.

In order to get Ψ, it suffices to show

Claim.

$$\begin{aligned} \mathcal{M}^*_b \ \models \ & \exists\Psi(\ \Psi \text{ is a phalanx whose models are initial segments of me} \\ & \text{and } \Phi^-(\bar{\mathcal{T}}^\frown\bar{b}) \text{ is realized in } \Psi. \end{aligned}$$

For granted the claim, the elementarity of $i^{\mathcal{T}}_b : L\vec{E}] \to \mathcal{M}^*_b$ implies that there is a realization of $\Phi^-(\bar{\mathcal{T}}^\frown\bar{b})$ whose models are initial segments of $L[\vec{E}]$. (Notice here that $i^{\mathcal{T}}_b$ fixes $\Phi^-(\bar{\mathcal{T}}^\frown\bar{b})$, since $\Phi^-(\bar{\mathcal{T}}^\frown\bar{b})$ is hereditarily countable in $L[\vec{E}]$.)

The claim follows by an easy absoluteness argument from:

Subclaim. There is a phalanx $\Psi^* = (\langle \mathcal{P}_\alpha \mid \alpha < \text{lh}(\bar{\mathcal{T}})\rangle, \langle \nu_\alpha \mid \alpha < \text{lh}(\bar{\mathcal{T}})\rangle)$ such that

(a) For all α, $\mathcal{P}_\alpha \trianglelefteq \mathcal{M}^*_b$, and ν_α is a cardinal of $\mathcal{M}^*_b$, and

(b) $\Phi^-(\bar{\mathcal{T}})$ is realizable in Ψ via maps σ_α such that $\sigma_\alpha \restriction ((\rho^{\bar{\mathcal{T}}}_\alpha)^+)^{\mathcal{N}_\alpha} = \pi \restriction ((\rho^{\bar{\mathcal{T}}}_\alpha)^+)^{\mathcal{N}_\alpha}$.

Proof. We could choose $\mathcal{P}_\alpha = \mathcal{M}_{\pi(\alpha)}$ and $\sigma_\alpha = \pi$, but the $\mathcal{M}_\alpha$ are not initial segments of $\mathcal{M}^*_b$. However, we can "internalize" as in the proof of iterability, replacing $\mathcal{M}_{\pi(\alpha)}$ by a Skolem hull taken inside $\mathcal{M}^*_{\pi(\alpha)}$, and using condensation to see that the resulting $\mathcal{P}_\alpha$ is an initial segment of $\mathcal{M}_{\pi(\alpha)}$ which is small enough that it is also an initial segment of $\mathcal{M}^*_b$.

To execute this plan, we need a small embellishment of a standard condensation result.

Lemma 3.3.1 *Let $\mathcal{M}$ be a fully iterable mouse, and let ν be a cardinal of $\mathcal{M}$, and suppose $\mathcal{M} \models |\eta| = \nu$. Let $\mathcal{H} = \mathcal{H}^{\mathcal{M}}_1(\eta+1)$ be the transitive collapse of the Σ_1 Skolem hull of $\eta+1$ in $\mathcal{M}$; then $\mathcal{H}$ is an initial segment of $\mathcal{M}$.*

Proof. By the remarks on p. 88 of [2], it is enough to show that $\mathcal{H}$ is 1-sound. Clearly $\rho_1(\mathcal{H}) = \nu$, and $|\eta| = \nu$ in $\mathcal{H}$. Since the first standard parameter $p_1(\mathcal{H})$ is universal ([2, 8.1]), $\eta + 1 \subseteq \mathcal{H}^{\mathcal{H}}_1(\nu \cup p_1(\mathcal{H}))$. But then $\mathcal{H} = \mathcal{H}^{\mathcal{H}}_1(\nu \cup p_1(\mathcal{H}))$, as desired. □

Now fix $\alpha < \mathrm{lh}(\bar{\mathcal{T}})$, and let

$$\nu = \pi(((\rho^{\bar{\mathcal{T}}}_\alpha)^+)^{\mathcal{N}_\alpha}) = ((\rho^{\mathcal{T}}_{\pi(\alpha)})^+)^{\mathcal{M}^*_{\pi(\alpha)}}).$$

Standard arguments, using the fact that $\mathcal{T}$ is nonoverlapping, yield that

$$\mathrm{ran}(\pi \restriction \mathcal{N}_\alpha) \subseteq \mathcal{H}^{\mathcal{M}_{\pi(\alpha))}}_\omega(\nu \cup i^*_{0,\alpha}(\mathrm{ran}(\pi \restriction \mathcal{N}_0))).$$

Notice that $\mathcal{H}^{\mathcal{M}_{\pi(\alpha))}}_\omega(\nu \cup i^*_{0,\alpha}(\mathrm{ran}(\pi \restriction \mathcal{N}_0)))$ has cardinality ν inside $\mathcal{M}^*_{\pi(\alpha)}$, and let $\mathcal{H}$ be its transitive collapse, and X_H its first order theory, coded as a subset of ν. Let X_H be the η^{th} subset of ν in $\mathcal{M}^*_{\pi(\alpha}$.

Now let

$$\mathcal{G} = \mathcal{H}_1^{\mathcal{M}^*_{\pi(\alpha)}|\mu+1}(\eta+1)$$

be the transitive collapse of the Skolem hull. By 3.3.1, $\mathcal{G} \trianglelefteq \mathcal{M}^*_{\pi(\alpha)}$. But since $\mathcal{G}$ has cardinality ν in $\mathcal{M}^*_{\pi(\alpha)}$, and $\mathcal{T}$ is a plus-2 tree, we get that $\mathcal{G} \trianglelefteq \mathcal{M}^*_b$. But now $\mathcal{H} \in cG$, and

$$\mathcal{G} \models \exists \xi, \sigma(\sigma : \mathcal{H} \xrightarrow{\Sigma_\omega} \mathcal{G}|\xi \wedge \sigma \restriction (\nu+1) = \text{ identity }.$$

Thus we can find $\mathcal{P} \trianglelefteq \mathcal{G}$ and $\sigma : \mathcal{H} \xrightarrow{\Sigma_\omega} \mathcal{P}$ such that $\sigma \restriction (\nu+1) = \text{identity}$.

Let $\mathcal{P}_\alpha = \mathcal{P}$ and $\nu_\alpha = \nu$. Letting $\tau : \mathcal{H} \to \mathcal{M}_{\pi(\alpha)}|\mu$ be the collapse map, set $\sigma_\alpha = \sigma \circ \tau^{-1} \circ \pi$. The desired phalanx is then $\Psi = (\langle \mathcal{P}_\alpha \mid \alpha < \mathrm{lh}(\mathcal{T}) \rangle, \langle \nu_\alpha \mid \alpha < \mathrm{lh}(\mathcal{T}) \rangle)$, as the σ_α's realize $\Phi^-(\bar{\mathcal{T}})$ into Ψ. This proves our subclaim. □

We can now construct a tree U (in the sense of descriptive set theory) in $\mathcal{M}_b^*$ which searches for Ψ and realization maps of $\Phi^-(\bar{\mathcal{T}}^\frown \bar{b})$ into Ψ witnessing the claim. Then $(\Psi^*)^\frown\langle\mathcal{M}_b\rangle$ and $\langle\sigma_\alpha \mid \alpha < \mathrm{lh}(\bar{\mathcal{T}})\rangle^\frown\langle pi\rangle$ determine a branch through U, so U has a branch in $\mathcal{M}_b^*$, and we have proved the claim. □

This proves 3.3. □

The hypothesis of 3.3 that $\mathcal{T}$ does not drop is necessary. There are iteration trees $\mathcal{T}$ on $M_1^\sharp$ which have distinct wellfounded branches such that $\mathcal{T} \in L[M_1^\sharp]$, and of course $L[M_1^\sharp]$ is an $L[\vec{E}]$-model. However, there is an extension of 3.3 to trees on $L[\vec{F}][x]$ induced by the construction of §2 which are allowed to drop. This extension has an application, so we give it.

Theorem 3.4 *Let $L[\vec{F}][x]$ be fully iterable, and, in $L[\vec{F}][x]$, let $\vec{\mathcal{M}}$ be an ms-array over Y with plus-2 certificate $\vec{C}$, and last model $\mathcal{P}$ such that $\mathcal{P}$ is sound and $\rho(\mathcal{P}) = \sup(Y)$. Let $\mathcal{U}$ be an iteration tree on $L[\vec{F}][x]$ which arises from an iteration tree $\mathcal{T} \in L[\vec{F}][x]$ on $\mathcal{P}$ via the construction described in §2; then $\mathcal{U}$ has at most one cofinal wellfounded branch.*

Proof. Let $\mathcal{T}$ be a counterexample. An easy Lowenheim-Skolem argument shows that we may assume $\mathcal{T}$ is countable in $L[\vec{F}][x]$. Note that $\mathcal{T}$ and $\mathcal{U}$ have the same tree order.

Claim. If b is a cofinal, wellfounded branch of $\mathcal{U}$, then $\Phi(\mathcal{T}^\frown b)$ is iterable.

Proof sketch. Let us adopt the notation from the proof of theorem 1.5. So for $\alpha < \mathrm{lh}(\mathcal{T})$ we have $(\vec{\mathcal{M}}_\alpha, \vec{C}_\alpha)$ a plus-2 certified array in the sense of $\mathcal{R}_\alpha$, the α^{th} model of $\mathcal{U}$. We also have embeddings $\pi_\alpha\colon \mathcal{P}_\alpha \to \mathcal{Q}_\alpha$, where $\mathcal{Q}_\alpha$ is the last model of $\vec{\mathcal{M}}_\alpha$. Hypotheses H1-H7 hold.

As in the proof of 3.3, we can internalize a version of this whole apparatus to $\mathbb{R}_b$, at the price of weakening the agreement condition H5 by replacing $(\nu_\beta^*)^{++}$ with $(\nu_\beta^*)^+$, and weakening the theory the $\mathcal{R}_\alpha$ must satisfy to some reasonable fragment of ZF^-. We can then use elementarity to pull back the existence of this apparatus to $\mathcal{R}_0 = L[\vec{F}][x]$. (Notice that b may drop, so that there may be no embedding $i_b^{\mathcal{U}}\colon \mathcal{R}_0 \to \mathcal{R}_b$. Nevertheless, Σ_1 statements about parameters below the level of truncation in the $\mathcal{R}_\alpha$ for $\alpha \in b$ pull back across the drops in b, and this is what we need.) This gives us $\mathcal{R}_\alpha^*, \vec{\mathcal{M}}_\alpha^*, \vec{C}_\alpha^*, \mathcal{Q}_\alpha^*$, and π_α^*, for $\alpha < \mathrm{lh}(\mathcal{T})$, together with similar objects defined for $\alpha = b$, such that

- H0, H1, H3, H4, and H5 hold, except that ZF^- is appropriately weakened in H1, , and μ_β is replaced by $((\nu_\beta^*)^+)^{\mathcal{R}_\beta^*}$ in H5, and

- $\mathcal{R}_\alpha^*$ is an initial segment of $L[\vec{F}][x]$, for $\alpha < \mathrm{lh}(\mathcal{T})$ or $\alpha = b$.

The system $\mathcal{R}^*_\alpha, \vec{\mathcal{M}}^*_\alpha$, etc., can therefore serve as the initial position for a construction like that in the proof of 1.5 which lifts iteration trees on the phalanx $\Phi(\mathcal{T} \frown b)$ to iteration trees on $L[\vec{F}][x]$. Since $L[\vec{F}[x]$ is iterable, we are done. $\Box$

But now, since $\mathcal{P}$ is sound and projects to $\sup(Y)$, $\mathcal{T}$ has at most one cofinal branch b such that $\Phi(\mathcal{T}^\frown b)$ is iterable. This completes the proof. $\Box$

References

[1] A. Andretta, I. Neeman, and J. R. Steel, *The domestic levels of K^c are iterable*, Israel Journal of Mathematics, **125** (2001), pp. 157-201

[2] W.J. Mitchell and J.R. Steel, *Fine structure and iteration trees* , Lecture Notes in Logic 3, Springer-Verlag, Berlin 1994.

[3] I. Neeman, unpublished manuscript.

[4] I. Neeman and J.R. Steel, *A weak Dodd-Jensen lemma*, Journal of Symbolic Logic **64** (1999), pp. 1285-1294.

[5] J.R. Steel, *An outline of inner model theory*, Handbook of Set Theory, to appear.

[6] J.R. Steel, *The core model iterability problem* , Lecture Notes in Logic 8, Springer-Verlag, Berlin 1996.

[7] J.R. Steel, *A coarse-structural local K^c-construction*, in preparation.