Scales in $K(\mathbb{R})$

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1 Introduction

In this paper, we shall extend the fine-structural analysis of scales in $L(\mathbb{R})$ ([7]) and $L(\mu, \mathbb{R})$ ([1]) to models of the form $L(\vec{E}, \mathbb{R})$, constructed over the reals from a coherent sequence $\vec{E}$ of extenders. We shall show that in the natural hierarchy in an iterable model of the form $L(\vec{E}, \mathbb{R})$ satisfying AD , the appearance of scales on sets of reals not previously admitting a scale is tied to the verification of new Σ_1 statements about $\vec{E}$ and individual reals in exactly the same way as it is in the special case $\vec{E} = \emptyset$ of [7]. For example, we shall show:

Theorem 1.1 *Let $\mathcal{M}$ be a passive, countably iterable premouse over $\mathbb{R}$, and suppose $\mathcal{M} \models \mathsf{AD}$; then the pointclass consisting of all $\Sigma_1^{\mathcal{M}}$ sets of reals has the scale property.*

A premouse is said to be *countably iterable* if all its countable elementary submodels are (ω_1+1)-iterable. It is easy to show, using a simple Lowenheim-Skolem argument, that if $\mathcal{M}$ and $\mathcal{N}$ are ω-sound, countably iterable premice over $\mathbb{R}$ which project to $\mathbb{R}$, then either $\mathcal{M}$ is an initial segment of $\mathcal{N}$, or vice-versa. We shall write $K(\mathbb{R})$ for the "union" of all such premice over $\mathbb{R}$, regarded as itself a premouse over $\mathbb{R}$. This is a small abuse of notation, since our $K(\mathbb{R})$ is determined by its sets of reals, but since we are concerned with the scale property, sets of reals are all that matter here. In fact, as in [7] and the work of [6] and [4] on which it rests, our existence results for scales require determinacy hypotheses, and so we are really only concerned here with the longest initial segment of $K(\mathbb{R})$ satisfying AD .

Section 2 is devoted to preliminaries. In section 3 we show that for any $\mathbb{R}$-mouse $\mathcal{M}$ satisfying "θ exists", $\mathrm{HOD}^{\mathcal{M}}$ is a T-mouse, for some $T \subseteq \theta^{\mathcal{M}}$.[1] We use this representation of $\mathrm{HOD}^{\mathcal{M}}$ in the proof of 1.1, which is given in section 4. There we also extend the proof of 1.1 so as to obtain a complete description of those pointclasses which have the scale property and are definable over initial segments of $K(\mathbb{R})$ satisfying AD.

2 Preliminaries

2.1 Potential $\mathbb{R}$-premice

We shall be interested in premice built over $\mathbb{R}$, which we take to be $V_{\omega+1}$ in this context, but nevertheless refer to as the set of all reals on occasion. In most respects, the basic theory of premice built over $\mathbb{R}$ is a completely routine generalization of the theory of ordinary premice (built over $\emptyset$); however, because $\mathbb{R}$-premice do not in general satisfy the axiom of choice, one must be careful at a few points. Here are some details.

Let M be a transitive, rud-closed set, and $X \in M$. Let E be an extender over M. We say that E is (M, X)-*complete* iff whenever a is a finite subset of $\mathrm{lh}(E)$ and $f\colon X \to E_a$ and $f \in M$, then $\bigcap \mathrm{ran}(f) \in E_a$. In the contrapositive: whenever $g\colon [\mathrm{crit}(E)]^{<\omega} \to P(X)$ is in M, then

$$(E_a \text{ a.e. } u)(\exists i \in X)(i \in g(u)) \Rightarrow (\exists i \in X)(E_a a.e.u)(i \in g(u)).$$

It is clear that if E is (M, X)-complete and (M, Y)-complete, then E is $(M, X \times Y)$-complete. Thus if E is (M, X)-complete and $\alpha < \mathrm{crit}(E)$, then E is $(M, X \times \alpha)$-complete.

For any transitive set M, let $o(M)$ be the least ordinal not in M.

Definition 2.1 *Let M be transitive and $X \in M$; we say M is wellordered mod X iff $\forall Y \in M \exists \alpha \in o(M) \exists g \in M(g\colon X \times \alpha \overset{onto}{\to} Y)$.*

Our $\mathbb{R}$-premice will be wellordered mod $\mathbb{R}$, moreover, if we take an ultrapower of such a premouse M by an extender E, then E will be $(M, \mathbb{R})$-complete. In this context we have

[1] What we actually show is slightly weaker than this in some very technical respects.

Proposition 2.2 *Let M be transitive, rud-closed, and wellordered mod X, where X is transitive. Let E be an extender over M; then the following are equivalent:*

1. *E is (M,X)-complete,*

2. *ult(M,E) satisfies the Los theorem for Σ_0 formulae, and the canonical embedding from M to ult(M,E) is the identity on $X \cup \{X\}$.*

Proof. We shall just sketch (1) $\Rightarrow$ (2), which is the direction we use anyway.

The usual proof of Los's theorem works except at the point where one would invoke the axiom of choice in M. At this point we have assumed

$$(E_a a.e.u)M \models \exists v \in g(u)\varphi[v, f_1(u), ..., f_k(u)],$$

where φ is Σ_0, and $g, f_1, ..., f_k$ are in M, and we wish to find f in M such that

$$(E_a a.e.u)M \models \varphi[f(u), f_1(u), ..., f_k(u)].$$

Now since M is wellordered mod X, we can fix $h \in M$ so that $h\colon X \times \alpha \overset{onto}{\to} \bigcup \mathrm{ran}(g)$. For $u \in \mathrm{dom}(g)$ and $\beta < \alpha$, set

$$f^*(u,\beta) = \{i \in X \mid M \models \varphi[h(i,\beta), f_1(u), ..., f_k(u)]\}.$$

For $u \in \mathrm{dom}(g)$, let

$$t(u) = f^*(u,\beta_u), \text{ where } \beta_u \text{ is least s.t. } f^*(u,\beta_u) \neq \emptyset,$$

and let $t(u) = \emptyset$ if $f^*(u,\beta) = \emptyset$ for all β. Because M is rud-closed, the functions f^* and t are in M. But now for $E_a a.e.u$ there is an $i \in t(u)$, and so by (M,X)-completeness we can fix i_0 such that for E_a a.e. u, $i_0 \in t(u)$. The desired function f is then given by

$$f(u) = h(i_0, \beta_u).$$

To see that the canonical embedding j is the identity on $X \cup \{X\}$, suppose $g \in M$ maps $[\mathrm{crit}(E)]^{|a|}$ to X. We have that for $E_a a.e.u$, there is an $i \in X$

such that $i = g(u)$, hence we can fix $i \in X$ such that $g(u) = i$ for $E_a a.e.u.$ It follows that $[g] = j(i)$.

$\square$

For X transitive and appropriate $\vec{E}$, we define $J_\alpha^{\vec{E}}(X)$ by:

$$J_0^{\vec{E}}(X) = X,$$

$$J_{\alpha+1}^{\vec{E}}(X) = \text{ rud-closure of } J_\alpha^{\vec{E}}(X) \cup \{J_\alpha^{\vec{E}}(X), E_\alpha\},$$

and taking unions at limits. Here the appropriate $\vec{E}$ are those such that each E_α is either the emptyset or an extender over $J_\alpha^{\vec{E}}(X)$ which is $(J_\alpha^{\vec{E}}(X), X)$-complete. We write

$$\mathcal{J}_\alpha^{\vec{E}}(X) = (J_\alpha^{\vec{E}}(X), \in, \vec{E} \restriction \alpha, E_\alpha, X)$$

for the structure for the language of set theory expanded by predicate symbols $\dot{E}$ for $\vec{E}$, $\dot{F}$ for E_α, and a constant symbol $\dot{\mathbb{R}}$ for X (chosen because $X = \mathbb{R} \cap J_\alpha^{\vec{E}}(X)$ is the case of greatest interest).[2] This language of relativised premice we call $\mathcal{L}^*$.

Definition 2.3 *An appropriate $\vec{E}$ is X-acceptable at α iff $\forall\beta < \alpha\forall\kappa(P(J_\kappa^{\vec{E}}(X) \cap (J_{\beta+1}^{\vec{E}}(X) \setminus J_\beta^{\vec{E}}(X)) \neq \emptyset) \Rightarrow (J_{\beta+1}^{\vec{E}}(X) \models \exists f\colon J_\kappa^{\vec{E}}(X) \overset{onto}{\rightarrow} J_\beta^{\vec{E}}(X))$.*

The following proposition is a uniform, local version of the fact that that every set in $L[\vec{E}, X]$ is ordinal-definable from parameters in $X \cup \{X\}$.

Proposition 2.4 1. *There is a fixed Σ_1 formula φ_0 of our expanded language such that whenever X is transitive, $\vec{E}$ is appropriate for X, and $\alpha < lh(\vec{E})$, then φ defines over $\mathcal{J}_\alpha^{\vec{E}}(X)$ a map $h\colon (X^{<\omega} \times [\alpha]^{<\omega}) \overset{onto}{\rightarrow} J_\alpha^{\vec{E}}(X)$. We write $h_\alpha^{\vec{E},X}$ for the map h so defined.*

2. *We can (and do) take the maps $h_\alpha^{\vec{E},\mathbb{R}}$ to have domain $\mathbb{R} \times [\alpha]^{<\omega}$ (replacing φ_0 with φ_1, another Σ_1 formula).*

3. *If $\vec{E}$ is appropriate for $\mathbb{R}$, then for any α, there is a map from $\mathbb{R} \times \alpha$ onto $J_\alpha^{\vec{E}}(\mathbb{R})$ which is Σ_1 definable from parameters over $\mathcal{J}_\alpha^{\vec{E}}(\mathbb{R})$.*

[2] If $\alpha = 0$, take $\dot{\mathbb{R}}$ to name 0.

The proof is a routine extension of Jensen's [2]. See [7] for the case $\vec{E} = \emptyset$. We shall need the *uniformly* Σ_1 maps of assertion (2) later on.

Using 2.4, we can reformulate $\mathbb{R}$-acceptability with $\mathbb{R}\times\kappa$ replacing $J_\kappa^{\vec{E}}(\mathbb{R})$, etc. Let us call λ an $\mathbb{R}$-*cardinal* iff there is no map $f\colon \kappa\times\mathbb{R} \overset{onto}{\longrightarrow} \lambda$ with $\kappa < \lambda$. It is then easy to see that if $\vec{E}$ is $\mathbb{R}$-acceptable at α, and $J_\alpha^{\vec{E}}(\mathbb{R}) \models \lambda$ is an $\mathbb{R}$-cardinal, then for all $\kappa < \lambda$, $P(J_\kappa^{\vec{E}}(\mathbb{R})) \cap J_\alpha^{\vec{E}}(\mathbb{R}) \subseteq J_\lambda^{\vec{E}}(\mathbb{R})$.

Definition 2.5 Θ *is the least ordinal which is not the surjective image of* $\mathbb{R}$*.*

Let $\mathcal{M}$ be an $\mathbb{R}^{\mathcal{M}}$-premouse satisfying "Θ exists", and let $\theta = \Theta^{\mathcal{M}}$. It is easy to see, using 2.4, that θ is regular in $\mathcal{M}$. By acceptability, $\mathcal{M}|\theta$ satisfies "every set is the surjective image of $\mathbb{R}$". We can conclude then that the structure $\mathcal{M}|\theta$ is admissible.

It is easy to show, without using the axiom of choice, that for $\lambda > 1$, λ is an $\mathbb{R}$-cardinal iff λ is a cardinal and $\lambda \geq \Theta$. Thus if $\vec{E}$ is $\mathbb{R}$-acceptable at α, then $\mathcal{J}_\alpha^{\vec{E}}(\mathbb{R})$ satisfies: "whenever $\kappa \geq \Theta$ and κ^+ exists, then $P(J_\kappa^{\vec{E}}(\mathbb{R})) \subseteq J_{\kappa^+}^{\vec{E}}(\mathbb{R})$".

Definition 2.6 *Let X be transitive; then a fine extender sequence over X is a sequence $\vec{E}$ such that for each $\alpha \in dom(\vec{E})$, $\vec{E}$ is X-acceptable at α, and either $E_\alpha = \emptyset$ or E_α is a (κ, α) pre-extender over $J_\alpha^{\vec{E}}(X)$ for some κ such that $J_\alpha^{\vec{E}}(X) \models P(J_\kappa^{\vec{E}}(X))$ exists, and E_α is $(J_\alpha^{\vec{E}}(X), X)$-complete, and: E_α satisfies clauses (1), (2), and (3) of definition 2.4 of [9].*

Of course, definition 2.4 of [9] was formulated there for the case $X = \emptyset$, but it is now easy to see what its clauses should mean in the general case.

Definition 2.7 *A potential premouse over X (or X-ppm) is a structure of the form $\mathcal{J}_\alpha^{\vec{E}}(X)$, where $\vec{E}$ is a fine extender sequence over X. If $\mathcal{M} = \mathcal{J}_\alpha^{\vec{E}}(X)$ is an X-ppm, we write $\mathcal{J}_\beta^{\mathcal{M}}$, or simply $\mathcal{M}|\beta$, for the structure $\mathcal{J}_\beta^{\vec{E}}(X)$. We say $\mathcal{N}$ is an initial segment of $\mathcal{M}$, and write $\mathcal{N} \trianglelefteq \mathcal{M}$, iff $\mathcal{N} = \mathcal{M}|\beta$ for some β.*

Active potential premice are in general not amenable structures, but we can code an active X-ppm $\mathcal{M}$ by an amenable structure $\mathfrak{C}_o(\mathcal{M})$. That involves replacing the last extender $F = \dot{F}^{\mathcal{M}}$ with $F \restriction \nu(F)$ in the case $\nu(F)$ (the sup of the generators of F) is a limit ordinal, and with a certain predicate F^* coding the fragments of F in the case $\nu(F)$ is a successor ordinal.

The details are given in [9, 2.11]. The "Σ_0-code" $\mathfrak{C}_0(\mathcal{M})$ is also a structure for the language $\mathcal{L}^*$, and it really this interpretation of $\mathcal{L}^*$ which is of importance in what follows. Whenever we speak of definability over a ppm $\mathcal{M}$, we shall in reality mean definabilty over $\mathfrak{C}_0(\mathcal{M})$.

2.2 Cores, Projecta, Soundness

These notions carry over routinely to X-ppm. One need only remember that $X \cup \{X\}$ is contained in all cores of X-ppm. For example, we define the first projectum, standard parameter, and core of an X-ppm $\mathcal{M}$ by[3]:

$$\begin{array}{rcl} \rho_1(\mathcal{M}) & = & \text{least } \alpha \text{ such that for some boldface } \Sigma_1^{\mathfrak{C}_0(\mathcal{M})} \text{ set } A, \\ & & A \subseteq \mathcal{M}|\alpha \text{ and } A \notin \mathcal{M}, \\ p_1(\mathcal{M}) & = & \text{least } p \in [o(\mathcal{M})]^{<\omega} \text{ such that there is an} \\ & & A \subseteq \mathcal{M}|\rho_1(\mathcal{M}) \text{ s.t. } A \text{ is } \Sigma_1^{\mathfrak{C}_0(\mathcal{M})} \text{ in } p \text{ but } A \notin \mathcal{M}, \\ \mathfrak{C}_1(\mathcal{M}) & = & \Sigma_1 \text{ Skolem hull of } \mathcal{M} \text{ generated by } \mathcal{M}|\rho_1(\mathcal{M}) \cup p_1(\mathcal{M}). \end{array}$$

Here we order finite sets of ordinals by listing their elements in decreasing order, and comparing the resulting finite sequences lexicographically. It is possible that $\mathcal{M}|\rho_1(\mathcal{M}) = \mathcal{M}$; if this is not the case, then $\rho_1(\mathcal{M})$ is an X-cardinal of $\mathcal{M}$. In the case $\rho_1(\mathcal{M}) = 1$, we shall generally write $\rho_1(\mathcal{M}) = X$ instead. The core $\mathfrak{C}_1(\mathcal{M})$ is taken to be transitive, and a structure for the language of X-ppm, and so taken, it is in fact an X-ppm.

The definitions of solidity and universality for the standard parameter go over to X-ppm in the obvious way. We say $p_1(\mathcal{M})$ is *universal* if $P(\mathcal{M}|\rho_1(\mathcal{M})) \cap \mathcal{M} \subseteq \mathfrak{C}_1(\mathcal{M})$. We say $p = p_1(\mathcal{M})$ is *solid* if for each $\alpha \in p$, letting b be the set of Σ_1 sentences in our expanded language augmented further by names for all elements of $(p \setminus (\alpha+1)) \cup \alpha$ which are true in $\mathcal{M}$, we have $b \in \mathcal{M}$. We call such b the *solidity witnesses* for $p_1(\mathcal{M})$. If $p_1(\mathcal{M})$ is solid and universal, we go on to define $\rho_2(\mathcal{M}), p_2(\mathcal{M})$, and $\mathfrak{C}_2(\mathcal{M})$; the reader should consult [5] for all further details here regarding the $\rho_n(\mathcal{M}), p_n(\mathcal{M})$, and $\mathfrak{C}_n(\mathcal{M})$ for $n > 1$. In this paper, when we need to go into fine-structural details, we shall stick to the representative case $n = 1$.

[3]Here and elsewhere, we write $\mathcal{M}$ for the universe of the structure $\mathcal{M}$, if no confusion can come from doing so.

Definition 2.8 *An X-ppm $\mathcal{M}$ is n-solid iff $\mathfrak{C}_\mathfrak{n}(\mathcal{M})$ exists, and $p_n(\mathfrak{C}_\mathfrak{n}(\mathcal{M})$ is solid and universal. $\mathcal{M}$ is n-sound iff $\mathcal{M}$ is n-solid, and $\mathcal{M} = \mathfrak{C}_\mathfrak{n}(\mathcal{M})$. $\mathcal{M}$ is an X-premouse iff every proper initial segment of $\mathcal{M}$ is ω-sound (i.e., n-sound for all $n < \omega$).*

2.3 Ultrapowers, Iteration Trees

It is easy to adapt the material of [5, §4, §5], or [9, §2, §3], to X-premice. The definitions make sense, and the theorems continue to hold true, if one replaces "premouse" with "X-premouse" everywhere. The following propositions summarize some of the basic facts:

Proposition 2.9 *Let $\mathcal{M}$ be an n-sound premouse over X, and let E be an extender over $\mathcal{M}$ which is $(\mathcal{M}, X)$-complete; then*

1. *For any generalized $r\Sigma_n$ formula φ, functions f_i definable over $\mathcal{M}$ from parameters using Σ_n Skolem terms $\tau_i \in\ Sk_n$, and $a \in [lh(E)]^{<\omega}$ such that $dom(f_i) = [crit(E)]^{|a|}$ for all i, we have*

   $$ult_n(\mathcal{M}, E) \models \varphi[[a, f_0], ..., [a, f_k]] \Leftrightarrow \text{ for } E_a a.e.u, \mathcal{M} \models \varphi[f_0(u), ..., f_k(u)].$$

2. *The canonical embedding $i_E^\mathcal{M}$ from $\mathcal{M}$ to $ult_n(\mathcal{M}, E)$ is an n-embedding.*

3. *Suppose also E is close to $\mathcal{M}$, $\rho_{n+1}(\mathcal{M}) \leq crit(E)$, and $p_{n+1}(\mathcal{M})$ is solid and universal; then*

   $$\rho_{n+1}(\mathcal{M}) = \rho_{n+1}(ult_n(\mathcal{M}, E)),$$

   and

   $$i_E^\mathcal{M}(p_{n+1}(\mathcal{M})) = p_{n+1}(ult_n(\mathcal{M}, E)),$$

   and $p_{n+1}(ult_n(\mathcal{M}, E))$ is solid and universal.

Here the ultrapower $\mathrm{ult}_0(\mathcal{M}, E)$ is formed using the functions $f_i \in \mathcal{M}$. Thus we have already proved the $n = 0$ case of (1) of 2.9. There are no new ideas in the rest of the proof.

Proposition 2.10 *Suppose $\mathcal{T}$ is an n-maximal iteration tree on the n-sound X-premouse $\mathcal{M}$, and that $\alpha T \beta$ and $D^\mathcal{T} \cap (\alpha, \beta]_T = \emptyset$; then the canonical embedding $i^\mathcal{T}_{\alpha,\beta} \circ i^*_\alpha$ from $\mathcal{M}^*_\alpha$ to $\mathcal{M}^\mathcal{T}_\beta$ is a k-embedding, where $k = \deg^\mathcal{T}(\beta)$. Moreover, if $crit(i^*_\alpha) \geq \rho_{k+1}(\mathcal{M}^*_\alpha)$, then $i^\mathcal{T}_{\alpha,\beta} \circ i^*(p_{k+1}(\mathcal{M}^*_\alpha)) = p_{k+1}(\mathcal{M}^\mathcal{T}_\beta)$.*

The notions of [5] and [9] associated with iterability relativise to X-premice in an obvious way.

Definition 2.11 *Let $\mathcal{M}$ be a k-sound X-premouse; then we say $\mathcal{M}$ is countably k-iterable iff whenever $\bar{\mathcal{M}}$ is a countable $\bar{X}$-premouse, and there is a $\pi\colon \bar{\mathcal{M}} \to \mathcal{M}$ which is fully $\mathcal{L}^*$-elementary, then $\bar{\mathcal{M}}$ is (k, ω_1+1)-iterable. We say $\mathcal{M}$ is countably iterable just in case it is countably ω-iterable.*

The comparison process yields

Theorem 2.12 *Let $\mathcal{M}$ and $\mathcal{N}$ be X-premice. and suppose that $\mathcal{M}$ is $m+1$-sound and countably m-iterable, where $\rho_{m+1}(\mathcal{M}) = X$, and $\mathcal{N}$ is $n+1$-sound and countably n-iterable, where $\rho_{n+1}(\mathcal{N}) = X$; then either $\mathcal{M} \trianglelefteq \mathcal{N}$ or $\mathcal{N} \trianglelefteq \mathcal{M}$.*

We can therefore define

Definition 2.13 *For any transitive set X, $K(X)$ is the unique X premouse $\mathcal{M}$ whose proper initial segments are precisely all those countably iterable X-premice $\mathcal{N}$ such that $\rho_\omega(\mathcal{N}) = X$.*

3 Some local HOD's

Our analysis of scales proceeds by getting optimal closed game representations. In the relevant closed game, player I attempts to verify that a given $\mathbb{R}$-mouse $\mathcal{M}$ satisfies $\varphi(x)$, where φ is Σ_1 and $x \in \mathbb{R}$. He does so by describing an $\dot{\mathbb{R}}^{\mathcal{N}}$-premouse $\mathcal{N}$ satisfying $\varphi(x)$; one can think of him as claiming that his $\mathcal{N}$ is an elementary submodel of $\mathcal{M}$. Player II helps keep I honest about this by playing reals which I must then put into $\mathcal{N}$. In order to ensure that I is indeed being honest about what is true in $\mathcal{M}$, we must ask him in addition to verify that his $\mathcal{N}$ is iterable. Of course, the obvious way to verify this is to play an elementary $\pi\colon \mathcal{N} \to \mathcal{M}$, but this leads to a payoff condition for I which is not closed in the appropriate topology. [4] Our main new idea

[4] I may play reals as well as ordinals in our game, and the elements of $\mathrm{ran}(\pi)$ can be coded by pairs $\langle x, \alpha\rangle$ where $x \in \mathbb{R}$ and $\alpha \in \mathrm{OR}$. However, the elementarity requirement on π would not be closed in the appropriate topology on $(\mathrm{OR} \times \mathbb{R})^\omega$, which is the product of ω copies of the discrete topology on OR and the *Baire* (not discrete) topology on $\mathbb{R}$.

here is just that I can verify iterability by elementarily embedding $\mathrm{HOD}^{\mathcal{N}}$ into $\mathrm{HOD}^{\mathcal{M}}$. The key here is that $\mathrm{HOD}^{\mathcal{M}}$ is definably wellordered [5], so that the embedding is essentially an ω-sequence of ordinals, and the elementarity condition is closed in the appropriate topology.

We must consider here iterations of $\mathcal{N}$ involving Σ_n-ultrapowers of the form ult_n. We shall reduce such ultrapowers to Σ_n-ultrapowers of $\mathrm{HOD}^{\mathcal{N}}$, and to do so we need a fine-structure theory for $\mathrm{HOD}^{\mathcal{N}}$. Fortunately, we can restrict ourselves to $\mathcal{N}$ satisfying "Θ exists", and be content with a fine-structural analysis of $\mathrm{HOD}^{\mathcal{N}}$ *above* $\Theta^{\mathcal{N}}$: that is, a representation of $\mathrm{HOD}^{\mathcal{N}}$ as an X-mouse, for some $X \subseteq \Theta^{\mathcal{N}}$. Now in the case $\mathcal{N} = L(\mathbb{R})$, the following theorem of Woodin does the job:

Theorem 3.1 (Woodin) *There is a partial order P on $\Theta^{L(\mathbb{R})}$ such that*

$$HOD^{L(\mathbb{R})} = L(P),$$

and moreover $L(\mathbb{R})$ is an inner model of a P-generic extension of $HOD^{L(\mathbb{R})}$.

Here P is a modification of the Vopenka partial order designed to add a generic enumeration of $\mathbb{R}$.

We shall extend Woodin's argument so as to show that if $\mathcal{M}$ is an $\mathbb{R}$-premouse satisfying "Θ exists", then there is a $P \subseteq \Theta^{\mathcal{M}}$ such that $\mathrm{HOD}^{\mathcal{M}}$ is the universe of a P-premouse $\mathcal{H}$. The main new thing here is to show that the projecta and standard parameters of levels of $\mathcal{H}$ match those of the corresponding levels of $\mathcal{M}$, and indeed establish level-by-level intertranslatability of the theories of initial segments of $\mathcal{H}$ and $\mathcal{M}$ respectively. This we get from the fact that $\mathcal{M}$ is a symmetric[6] inner model of a P-extension of $\mathcal{H}$, using the level-by-level definability of forcing.

In turning to the details, it will be convenient to replace P with a superficially more powerful set. Let us fix for the remainder of this section an $\mathbb{R}^{\mathcal{M}}$-premouse $\mathcal{M}$ such that

- $\mathcal{M} \models$ "Θ exists".

[5] The definition which guarantees a set is in $\mathrm{HOD}^{\mathcal{M}}$ must use the language $\mathcal{L}^*$ of premice. $\dot{E}$ and $\dot{F}$ are allowed, but names for individual reals are not! Further, this definition must be interpreted over some *proper* initial segment of $\mathcal{M}$; there may be sets of ordinals in $\mathcal{M}$ which are definable over $\mathcal{M}$ itself, yet not in $\mathrm{HOD}^{\mathcal{M}}$.

[6] We shall explain the meaning of this shortly.

We set $\theta = \Theta^{\mathcal{M}}$, and we also fix an $n_0 < \omega$ such that

- $\mathcal{M}$ is n_0-sound and $\rho_{n_0}(\mathcal{M}) \geq \theta$.

Finally, letting $o(\mathcal{M}) = \omega\gamma_0$, we assume that

- for all $\langle \xi, k\rangle <^{\text{lex}} \langle \gamma_0, n_0\rangle$, $\mathcal{M}|\xi$ is countably k-iterable. [7]

We get a certain amount of condensation from these assumptions.

Lemma 3.2 *For any $\xi < \gamma_0$,*

$$\mathcal{H}_1^{\mathcal{M}|\xi}(\mathbb{R}^{\mathcal{M}}) \cong \mathcal{M}|\tau$$

for some $\tau < \theta$.

In fact, if $\langle \xi, k\rangle \leq^{\text{lex}} \langle \gamma_0, n_0\rangle$ and $k \geq 1$, then for any finite $F \subseteq \omega\xi$, $\mathcal{H}_k^{\mathcal{M}|\xi}(\mathbb{R}^{\mathcal{M}} \cup F) \in \mathcal{M}|\theta$, since the theory of the hull is in $\mathcal{M}|\theta$, and the latter is an admissible structure. However, $\mathcal{H}_k^{\mathcal{M}|\xi}(\mathbb{R}^{\mathcal{M}} \cup F)$ may not be sound, and hence may not be of the form $\mathcal{M}|\tau$.

Corollary 3.3 *$\mathcal{M}|\theta$ is a Σ_1-elementary submodel of $(J_{\gamma_0}^{\dot{E}^{\mathcal{M}}}, \in, \dot{E}^{\mathcal{M}} \restriction \gamma_0, \emptyset)$.*

Set

$$T^{\mathcal{M}} = \{\langle \varphi, \vec{\alpha}\rangle \mid \vec{\alpha} \in \theta^{<\omega} \text{ and } \mathcal{M}|\theta \models \varphi[\vec{\alpha}]\}.$$

Since θ is a cardinal in $\mathcal{M}$, we can use Gödel's pairing function to identify $T^{\mathcal{M}}$ with a subset of θ. Letting $\omega\eta = o(\mathcal{M})$, we have that

$$\text{HOD}^{\mathcal{M}|\theta} \cap V_\theta = J_\eta(T^{\mathcal{M}}) \cap V_\theta.$$

Since $\mathcal{M}|\theta$ is a Σ_1-elementary submodel of $\mathcal{M}$ with its last extender removed, we have

$$\text{HOD}^{\mathcal{M}} \cap V_\theta = J_\eta(T^{\mathcal{M}}) \cap V_\theta.$$

We can construct a $T^{\mathcal{M}}$-premouse whose universe is the whole of $\text{HOD}^{\mathcal{M}}$ by simply constructing from $T^{\mathcal{M}}$ together with the extenders from the $\mathcal{M}$-sequence having critical points above θ. More precisely, letting

$$\mathcal{M} = \mathcal{J}_{\gamma_0}^{\vec{E}}(\mathbb{R}^{\mathcal{M}}),$$

[7] The analysis of $\text{HOD}^{\mathcal{M}}$ we are developing will be used to show that $\mathcal{M}$ is countably n_0-iterable, given that $\text{HOD}^{\mathcal{M}}$ is.

we define an appropriate sequence $\vec{F}$ over $\theta \cup \{T^{\mathcal{M}}\}$ by setting

$$F_\alpha = E_{\theta+\alpha} \cap J_\alpha^{\vec{F}\restriction\alpha}(\theta \cup \{T^{\mathcal{M}}\})$$

for all α such that $\theta + \alpha \leq \gamma_0$. It is not hard to see that the sequence $\vec{F}$ is indeed appropriate for $\theta \cup \{T^{\mathcal{M}}\}$; the main point is that $\mathrm{crit}(E_{\theta+\alpha}) > \Theta^{\mathcal{M}|(\theta+\alpha)} = \theta$, which implies that F_α is sufficiently complete. We set

$$\mathcal{H}_\alpha = \mathcal{J}_\alpha^{\vec{F}}(\theta \cup \{T^{\mathcal{M}}\}),$$

for all α such that $\theta + \alpha \leq \gamma_0$.

It will be convenient to ignore the $\mathcal{H}_\alpha$ for small α. Therefore, we add to our assumptions on $\mathcal{M}$ that there is some $\lambda \leq \gamma_0$ such that $\mathcal{M}|\lambda \models \mathsf{ZF}$, and let

$$\lambda_0 = \text{ least } \lambda \text{ such that } \mathcal{M}|\lambda \models \mathsf{ZF}.$$

Notice that our new assumption on $\mathcal{M}$ holds if some $E_{\theta+\alpha} \neq \emptyset$, and in this case $\mathrm{crit}(E_{\theta+\alpha}) > \lambda_0$. Since in our closed game representation we only need the $\mathcal{H}_\alpha$ to "verify" extenders from the $\mathcal{M}$-sequence with index above θ, we can afford to ignore $\mathcal{H}_\alpha$ except when λ_0 exists and $\lambda_0 \leq \theta + \alpha$. Note that $\theta + \alpha = \alpha$ for the α we do not ignore, so that we are already rewarded for our ignorance. We set

$$\mathcal{H} = \mathcal{H}_{\gamma_0}.$$

We shall show that for $\alpha \geq \lambda_0$, $\mathcal{H}_\alpha$ is a $T^{\mathcal{M}}$-premouse and $\mathcal{M}|\alpha$ is an inner model of a generic extension of $\mathcal{H}_\alpha$.[8] The relevant partial order is the same Vopenka-like partial order used by Woodin.

Let us work in $\mathcal{M}$ for a while. Fix a bijection $\pi\colon \theta \to \mathcal{O}$, where $\mathcal{O}$ is the collection of all subsets of $\mathbb{R}^n = \{s \mid s\colon n \to \mathbb{R}\}$ which are definable from ordinal parameters over $\mathcal{M}|\theta$. We choose π so that it is definable over $\mathcal{M}|\theta$. We write A^* for $\pi(A)$ henceforth. Let

$$A \in \mathrm{Vop}_n \Leftrightarrow \exists n < \omega(A^* \subseteq \mathbb{R}^n \wedge A^* \neq \emptyset),$$

and

$$A \in \mathrm{Vop}_\omega \Leftrightarrow \exists n < \omega(A \in \mathrm{Vop}_n).$$

[8] The first assertion is true for smaller α as well.

For A in Vop_ω, we write $s(A)$ for the unique $n < \omega$ such that $A \in \mathrm{Vop}_n$. For $A, B \in \mathrm{Vop}_\omega$, we put

$$A \leq^v B \Leftrightarrow s(B) \leq s(A) \wedge \forall s \in A^*(s \restriction s(B) \in B^*).$$

We also use Vop_ω to denote the partial order $(\mathrm{Vop}_\omega, \leq^v)$. Clearly, Vop_ω is coded into $T^{\mathcal{M}}$ in a simple way, and hence $\mathrm{Vop}_\omega \in \mathcal{H}|2$.

The standard Vopenka argument shows that for any $n < \omega$, Vop_n is a complete Boolean algebra in $\mathcal{H}$, and each $s \in \mathbb{R}^n$ determines an $\mathcal{H}$-generic filter $G_s = \{A \in Vp_n \mid s \in A\}$ on Vop_n.[9] It is easy to see that the inclusion map is a complete embedding of Vop_n into Vop_ω. Motivated by this, we define for $h\colon \omega \to \mathbb{R}$ and $A \in \mathrm{Vop}_\omega$:

$$A \in G_h \Leftrightarrow h \restriction s(A) \in A^*.$$

Lemma 3.4 *If h is $\mathcal{M}$-generic over $Col(\omega, \mathbb{R})$, then G_h is $\mathcal{H}$-generic over Vop_ω.*

Proof. Let $\mathcal{D}$ be dense in Vop_ω, and $\mathcal{D} \in \mathcal{M}$. Let $s \in \mathbb{R}^n$ be a condition in $\mathrm{Col}(\omega, \mathbb{R})$. It will be enough to find a t extending s in $\mathrm{Col}(\omega, \mathbb{R})$ such that $G_t \cap \mathcal{D} \neq \emptyset$. Let

$$X = \{u \in \mathbb{R}^n \mid \exists B \in \mathcal{D} \exists t(u \subseteq t \wedge t \in B^*\}.$$

We want to see $s \in X$, so it will be enough to see $X = \mathbb{R}^n$. Suppose not; then since X is clearly OD in $\mathcal{M}$, there is an $A \in \mathrm{Vop}_n$ such that

$$A^* = \mathbb{R}^n \setminus X.$$

Since $\mathcal{D}$ is dense, we can find $B \in \mathcal{D}$ such that $B \leq A$. But now pick any $t \in B$, and it is clear that $t \restriction n \in X$, a contradiction. $\square$

We can recover h from G_h in a simple way. For $b \in V_\omega$ and $n < \omega$, let $A_{b,n} \in Vp_{n+1}$ be such that $A^*_{b,n} = \{s \in \mathbb{R}^{n+1} \mid b \in s(n)\}$. We assume that the map $\langle b, n\rangle \mapsto A_{b,n}$ is definable over $\mathcal{M}_\theta$, and hence in $\mathcal{H}$, as any natural such map will be.[10] Then clearly,

$$b \in h(n) \Leftrightarrow A_{b,n} \in G_h.$$

[9]Note that any subset of $\mathbb{R}$ which is $\mathrm{OD}^{\mathcal{M}}$ is actually $\mathrm{OD}^{\mathcal{M}|\theta}$, since $\mathcal{M}|\theta$ is a Σ_1 elementary submodel of $\mathcal{M}$.

[10]The $*$ map is one-one on the separative quotient of Vop_ω, so the question as to what to choose for $A_{b,n}$ disappears if we replace Vop_ω with its separative quotient.

We define Vop_ω-terms for the $h(n)$ and $\mathrm{ran}(h)$ by

$$\sigma_n = \{\langle A, \check{b}\rangle \mid A \leq^v A_{b,n}\},$$

and

$$\dot{R} = \{\langle A, \sigma_n\rangle \mid A \in \mathrm{Vop}_\omega \wedge n < \omega\}.$$

These terms are of course in $\mathcal{H}$. It is easy to see

Lemma 3.5 *1. For any $h{:}\,\omega \to \mathbb{R}$, $\sigma_n^{G_h} = h(n)$ for all n, and $\dot{R}^{G_h} = ran(h)$.*

2. If h is $\mathcal{M}$-generic for $Col(\omega, \mathbb{R}^{\mathcal{M}})$, then $\dot{R}^{G_h} = \mathbb{R}^{\mathcal{M}}$.

3. For any condition $A \in Vop_\omega$, there is an $\mathcal{H}$-generic filter G on Vop_ω such that $A \in G$ and $\dot{R}^G = \mathbb{R}^{\mathcal{M}}$.

By 3.5, truth in $\mathcal{H}(\mathbb{R}^{\mathcal{M}})$ can be reduced to truth in $\mathcal{H}$ via the forcing relation for Vop_ω. In order to see that $\mathcal{H}(\mathbb{R}^{\mathcal{M}})$ determines $\mathcal{M}$ we need to know that the extenders on $\vec{F}$ generate the corresponding extenders on $\vec{E}$. For this, we need that the forcing relation for Vop_ω is locally definable. We also need this local definability to show that the reduction of $\mathcal{M}$-truth to $\mathcal{H}$-truth is local, and thereby that $\mathcal{H}$ is a $T^{\mathcal{M}}$-premouse.

We shall use the usual Shoenfield terms for our forcing language. Besides these terms, the language of forcing over an amenable structure

$$(J_\xi^A(X), \in, A, X, B)$$

has $\in, =$, a constant symbol $\dot{R}$ for X, and predicate symbols $\dot{E}$ and $\dot{F}$ for A and B. A filter G over a poset $\mathbb{P} \in J_\xi^A(X)$ is generic over this structure just in case it meets all $\mathbb{P}$-dense sets $D \in J_\xi^A(X)$. We let $J_\xi^A(X)[G] = \{\tau^G \mid \tau \in J_\xi^A(X)\}$ be the set of G-interpretations of terms, and say

$$p \Vdash \varphi \Leftrightarrow \forall G(G \text{ is generic over } J_\xi^A(X) \Rightarrow (J_\xi^A(X)[G], \in, A, X, B) \models \varphi).$$

We use $S_\alpha^A(X)$ for the α^{th} level of Jensen's S-hierarchy on $J_\xi^A(X)$. Let $\Sigma_{0,n}$ be the collection of Σ_0 sentences of the forcing language containing at most n bounded quantifiers.

Lemma 3.6 *Let $(J^A_\xi(X), \in, B)$ be amenable, and let $\mathbb{P}$ be a poset, with $\mathbb{P} \in J^A_\nu(X)$, where $\nu < \xi$ and $J^A_\nu(X) \models \mathsf{ZFC}$. For $\nu \leq \alpha < \omega\xi$, let*

$$F_{\alpha,n} = \{\langle p, \varphi\rangle \mid p \in \mathbb{P} \wedge \varphi \in (\Sigma_{0,n} \cap S^A_\alpha(X)) \wedge p \Vdash \varphi\}.$$

Then

1. *$\forall \alpha < \omega\xi \forall n < \omega(F_{\alpha,n} \in J^A_\xi(X))$; moreover the function $\langle \alpha, n\rangle \mapsto F_{\alpha,n}$ is $\Sigma_1^{(J^A_\xi(X), \in, A, X, B)}$ in the parameter $\mathbb{P}$ (uniformly in ξ).*

2. *If $(J^A_\xi(X)[G], \in, A, X, B) \models \varphi$, where φ is Σ_0 and G is generic over $J^A_\xi(X)$, then $\exists p \in G(p \Vdash \varphi)$.*

A proof of 3.6 can be organized as follows. Let $F^*_{\alpha,n} = \{\langle p, \varphi\rangle \in F_{\alpha,n} \mid \dot{F}$ does not occur in $\varphi\}$. One first proves the lemma with $F^*_{\alpha,n}$ replacing $F_{\alpha,n}$. This amounts to observing that the standard inductive definition of forcing for Σ_0 sentences is a "local Σ_0-recursion" of the same sort that defines the function $\alpha \mapsto S^A_\alpha(X)$ itself in a Σ_1 way over $(J^A_\xi(X), \in, A, X)$. Of course, one needs that everything true is forced to verify that the inductive definition works. The reason for restricting ourselves to $\alpha \geq \nu$ is that we need a starting point for the induction, and when $\alpha = \nu$, 3.6 literally is a standard basic forcing lemma.[11] Finally, one can show that $F_{\alpha,0}$ is uniformly rudimentary in $\langle F^*_{\alpha,0}, B \cap S^A_\alpha(X)\rangle$ (since $p \Vdash \dot{F}(\tau) \Leftrightarrow \forall q \leq p \exists r \leq q \exists x \in B(r \Vdash \tau = \check{x})$). Also, $F_{\alpha,n+1}$ is uniformly rudimentary in $F_{\alpha,n}$. This completes our psuedo-proof of 3.6.

As a consequence of 3.6, we get the level-by-level adequacy of the Shoenfield terms:

Lemma 3.7 *Let G be $\mathcal{H}$-generic over Vop_ω; then for all ξ such that $\lambda_0 \leq \xi \leq \gamma_0$, $J^{\vec{F}}_\xi(T^{\mathcal{M}})[G] = J^{\vec{F}}_\xi(\langle T^{\mathcal{M}}, G\rangle)$.*

If G is a Vopenka-generic over $\mathcal{H}$ such that $\dot{R}^G = \mathbb{R}^{\mathcal{M}}$, then $\mathcal{H}[G]$ can recover $\mathcal{M}$:

Lemma 3.8 *If G is Vop_ω-generic over $\mathcal{H}$, $\dot{R}^{\mathcal{M}} = \mathbb{R}^{\mathcal{M}}$, and $\lambda_0 \leq \xi \leq \gamma_0$, then $\mathcal{M}|\xi$ is Δ_1-definable over $\mathcal{H}_\xi[G]$ from the parameter G; moreover, this definition is uniform in such G and ξ.*

[11]For small α, we have the problem that there may be sentences in $S^A_\alpha(X)$ involving terms of rank greater than α.

Proof. (Sketch.) For $\xi = \lambda_0$ this is clear. In general, what we need to see is that if $F_\xi \neq \emptyset$, then F_ξ determines the corresponding extender E_ξ on the $\mathcal{M}$-sequence in a $\Delta_1^{\mathcal{H}_\xi[G]}$ way. We may assume by induction that $J_\xi^{\vec{E}}(\mathbb{R}^{\mathcal{M}}) \subseteq \mathcal{H}_\xi[G]$. Since Vop_ω has cardinality strictly less than $\mathrm{crit}(F_\xi)$ in $\mathcal{H}_\xi$, F_ξ lifts to an extender F^* over $J_\xi^{\vec{E}}(\mathbb{R}^{\mathcal{M}})$ defined by: for $a \in [\mathrm{lh}(F_\xi)]^{<\omega}$ and $Z \subseteq [\mathrm{crit}(F_\xi)]^{|a|}$ such that $Z \in J_\xi^{\vec{E}}(\mathbb{R}^{\mathcal{M}})$,

$$Z \in F_a^* \Leftrightarrow \exists Y(Y \in (F_\xi)_a \wedge Y \subseteq Z).$$

Clearly, any extender over $J_\xi^{\vec{E}}(\mathbb{R}^{\mathcal{M}})$ whose restriction to sets in $\mathcal{H}_\xi$ is F_ξ must then be equal to F^*. Thus $E_\xi = F^*$, and hence E_ξ is Δ_1 over $\mathcal{H}_\xi[G]$ in the parameter G. The uniformity in ξ and G is obvious upon inspection of the definition we have given. (The uniformity is needed to pass through limit stages.) □

Theorem 3.9 *$\mathcal{H}$ is a $T^{\mathcal{M}}$-premouse; moreover for all $k \leq n_0$, $\mathcal{H}$ is k-sound, $\rho_k(\mathcal{H}) = \rho_k(\mathcal{M})$, and $p_k(\mathcal{H}) = p_k(\mathcal{M}) \setminus \{\theta\}$.*

Proof. We show by induction on ξ such that $\lambda_0 \leq \xi \leq \gamma_0$, that $\mathcal{H}_\xi$ is a $T^{\mathcal{M}}$-premouse, and if $0 \leq k \leq \omega$, and $k \leq n_0$ if $\xi = \gamma_0$, then $\mathcal{H}_\xi$ is k-sound, $\rho_k(\mathcal{H}_\xi) = \rho_k(\mathcal{M}|\xi)$, and $p_k(\mathcal{H}_\xi) = p_k(\mathcal{M}|\xi)$.

This is clear for $\xi = \lambda_0$. Now let $\xi > \lambda_0$. We first show that $\mathcal{H}_\xi$ is a premouse. Since all proper initial segments of $\mathcal{H}_\xi$ are ω-sound $T^{\mathcal{M}}$-premice by our induction hypotheses, it suffices to show that $\mathcal{H}_\xi$ is a $T^{\mathcal{M}}$-ppm. If $F_\xi = \emptyset$, this is trivial, so assume $F_\xi = E_\xi \cap \mathcal{H}_\xi$ where E_ξ is an extender over $\mathcal{M}|\xi$. We must verify that $\vec{F}$ has the properties of a fine extender sequence at ξ, that is, that it satisfies clauses (1)-(3) of definition 2.4 in [9]. Let us write $F = F_\xi$ and $E = E_\xi$. Notice that $\xi = \mathrm{lh}(F) = \mathrm{lh}(E) = o(\mathcal{M}|\xi) = o(\mathcal{H}_\xi)$. Set $\kappa = \mathrm{crit}(F) = \mathrm{crit}(E)$.

Claim 1. If $a \in [\xi]^{<\omega}$ and $f \in \mathcal{M}|\xi$ and $f\colon [\kappa]^{|a|} \to \mathcal{H}_\xi$, then there is a $Z \in F_a$ such that $f \restriction Z \in \mathcal{H}_\xi$.

Proof. We need to take a little care with the standard argument because $F_a \notin \mathcal{M}$ is possible. Note that by 2.4, f is definable over some $\mathcal{M}|\gamma$, where $\gamma < \xi$, from ordinals and a real x_0. We can therefore fix a term $\dot{f}$ in $\mathcal{H}_\xi$ such that whenever h is $\mathcal{M}$-generic over $\mathrm{Col}(\omega, \mathbb{R})$, and $G = G_h$ is the associated Vopenka generic, then $\dot{f}^G = f$. We claim there is an $A \in \mathrm{Vop}_\omega$ such that

$$\text{for } F_a \text{ a.e. } u, \exists\eta(A \Vdash \dot{f}(u) = \eta,$$

and

$$\exists s \in A^*(s(0) = x_0).$$

If not, then for each $A \in \mathrm{Vop}_\omega$ such that $\exists s \in A^*(s(0) = x_0)$, the set Z_A of all $u \in [\kappa]^{|a|}$ such that A forces no value for $\dot{f}(u)$ is in F_a. The local definability of Vopenka forcing implies that the function $A \mapsto Z_A$ is in $\mathcal{M}$ (in fact, in $\mathcal{H}_\xi$). But F is $(\mathcal{M}, \mathbb{R})$-complete, and hence we have a u such that $u \in Z_A$ whenever Z_A is defined. Now let h be $\mathcal{M}$-generic over $\mathrm{Col}(\omega, \mathbb{R})$ with $h(0) = x_0$. Then $\exists A \in G_h \exists\eta(A \Vdash \dot{f}(u) = \eta)$, so Z_A is defined and $u \notin Z_A$, a contradiction.

Now let A be as in our claim, and let Z be the set of all u such that $\exists\eta A \Vdash \dot{f}(u) = \eta$. It is clear that $f \restriction Z$ can be computed inside $\mathcal{H}_\xi$ from $A, \dot{f}$, and the forcing relation. (Note that there is an $\mathcal{M}$-generic h such that $h(0) = x_0$ and $A \in G_h$.)

□

Claim 2. F and E have the same generators.

Proof. If $\eta < \mathrm{lh}(F)$ is not a generator of F, then there is an $f \in \mathcal{H}_\xi$ and finite $a \subseteq \eta$ such that $f(u) = v$ for $(F)_{a \cup \{\eta\}}$-a.e. $u \cup \{v\}$. Since $f \in \mathcal{M}|\xi$ and $F \subseteq E$, this means that η is not a generator of E. Conversely, if η is not a generator of E, as witnessed by $f \in \mathcal{M}_\xi$ and $a \subseteq \eta$ finite, then we can apply claim 1 to see that η is not a generator of F. □

Claim 3. For all $\theta \leq \eta < \xi$, η is a cardinal of $\mathcal{H}_\xi$ iff η is a cardinal of $\mathcal{M}|\xi$.

Proof. If η is a cardinal of $\mathcal{M}|\xi$, then η is a cardinal of the smaller model $\mathcal{H}_\xi$. If $\eta > \theta$ is a cardinal of $\mathcal{H}_\xi$ and G is Vop_ω-generic over $\mathcal{H}_\xi$, then η is a cardinal of $\mathcal{H}_\xi[G]$. Choosing G so that $\dot{R}^G = \mathbb{R}^{\mathcal{M}}$, we see that η is a cardinal of $\mathcal{M}|\xi$. For $\eta = \theta$, we have that η is a cardinal of both models. □

We can now verify the first clause in the definition of fine extender sequences, that $\xi = \nu(F)^+$ in $\mathrm{ult}(\mathcal{H}_\xi, F)$. We have $\nu(F) = \nu(E)$ by claim 2, and $\xi = \nu(E)^+$ in $\mathrm{ult}(\mathcal{M}|\xi, E)$ because $\vec{E}$ is a fine extender sequence. Letting $i_E \colon \mathcal{M}|\xi \to \mathrm{ult}(\mathcal{M}|\xi, E)$ be the canonical embedding, we have $\mathrm{ult}(\mathcal{H}_\xi, F) = i_E(\mathcal{H}_\xi)$ by our first claim.[12] By claim 3 and the elementarity of i_E, $\mathrm{ult}(\mathcal{M}|\xi, E)$

[12]Where $i_E(\mathcal{H}_\xi)$ is the "union" of the $i_E(\mathcal{H}_\eta)$ for $\eta < \xi$.

has the same cardinals as $i_E(\mathcal{H}_\xi)$, so $\xi = \nu(E)^+$ in $i_E(\mathcal{H}_\xi)$. Thus $\xi = \nu(F)^+$ in $\mathrm{ult}(\mathcal{H}_\xi, F)$, as desired.

To verify clause 2, coherence, notice that $i_E(\vec{F}) \restriction \xi = F \restriction \xi$ by coherence for the $\vec{E}$ sequence and the fact that the F_α are uniformly locally definable from the E_α. Since by claim 1, $i_F(\vec{F}) = i_E(\vec{F})$, we are done.

The initial segment condition for E easily implies the initial segment condition for F; we leave the details to the reader. We have therefore shown that $\mathcal{H}_\xi$ is a $T^{\mathcal{M}}$-premouse.

We now show by induction on k such that $k \leq n_0$ if $\xi = \gamma_0$ that $\rho_k(\mathcal{H}_\xi) = \rho_k(\mathcal{M}|\xi)$, $p_k(\mathcal{H}_\xi) = p_k(\mathcal{M}|\xi) \setminus \{\theta\}$, and $\mathcal{H}_\xi$ is k-sound. This is trivial if $k = 0$. Let us first consider the case $k = 1$.

The key is that the Σ_1 theories (in the language $\mathcal{L}^*$) of $\mathcal{H}_\xi$ and $\mathcal{M}|\xi$ are intertranslatable. First, let us translate the Σ_1 theory of $\mathcal{H}_\xi$ into that of $\mathcal{M}|\xi$. Here we shall expand the latter theory by allowing a name for θ; notice that $T^{\mathcal{M}}$ is Σ_1-definable over $\mathcal{M}|\xi$ from the parameter θ. We can then see that the universe of $\mathcal{H}_\xi$, together with the interpretations of $\dot{E}$ and $\dot{F}$ in $\mathcal{H}_\xi$, are Δ_1-definable over $\mathcal{M}|\xi$ from θ. Clearly $\dot{\mu}^{\mathcal{H}_\xi} = \dot{\mu}^{\mathcal{M}|\xi} = \kappa$, and $\dot{\nu}^{\mathcal{H}_\xi} = \dot{\nu}^{\mathcal{M}|\xi}$ by claim 2 above. We leave it to the reader to show that $\dot{\gamma}^{\mathcal{H}_\xi}$ is Σ_1-definable over $\mathcal{M}|\xi$ from θ; this is a bit of a mess because of the "one ultrapower away" case in the initial segment condition, but otherwise routine.[13] These calculations constitute a proof of:

Claim 4. There is a recursive map $\varphi(v_1, ..., v_n) \mapsto \varphi^*(v_0, v_1, ..., v_n)$ associating to each Σ_1 formula of $\mathcal{L}^*$ a Σ_1 formula of $\mathcal{L}^*$ with one additional free variable, such that for all $\varphi(v_1, ..., v_n)$ and $a_1, ..., a_n$,

$$\mathcal{H}_\xi \models \varphi[a_1, ..., a_n] \Leftrightarrow \mathcal{M}|\xi \models \varphi^*[\theta, a_1, ..., a_n].$$

We translate in the other direction using the strong forcing relation for Σ_1 formulae and 3.6. Let $\mathcal{L}^{**}$ be the sublanguage of $\mathcal{L}*$ with symbols $\in, =, \dot{E}, \dot{F}$. If

$$\varphi(v_1, ..., v_n) = \exists u_1 ... \exists u_k \psi(u_1, .., u_k, v_1, ..., v_n)$$

where ψ is a Σ_0 formula of $\mathcal{L}^{**}$, then for $p \in \mathrm{Vop}_\omega$ and $\tau_1, ..., \tau_n$ Shoenfield

[13] E and F fall under the same case in the initial segment condition, and $\dot{\gamma}^{\mathcal{H}_\xi} = \dot{\gamma}^{\mathcal{M}|\xi}$ unless E and F are type II, and their last initial segments are an ultrapower away from the corresponding sequence.

terms, we put

$$p \overset{s}{\Vdash} \varphi(\tau_1, ..., \tau_n) \Leftrightarrow \exists r_1 ... \exists r_k (p \Vdash \psi(r_1, ..., r_k, \tau_1, ..., \tau_n)).$$

Now if G is generic over $\mathcal{H}_\xi$ for Vop_ω and $\dot{R}^G = \mathbb{R}^{\mathcal{M}}$, then the universe of $\mathcal{M}|\xi$ is Δ_1 definable over $\mathcal{H}_\xi[G]$ from $\dot{R}^G$; moreover, the interpretations in $\mathcal{M}|\xi$ of the symbols of $\mathcal{L}^*$ are Δ_1 definable over $\mathcal{H}_\xi[G]$ from their interpretations in $\mathcal{H}_\xi$.[14] Since strong forcing equals truth, we get

Claim 5. There is a recursive map $\varphi(v_1, ..., v_n) \mapsto \varphi^\dagger(w, x, y, z, v_1, ..., v_n)$ associating to each Σ_1 formula of $\mathcal{L}^*$ a Σ_1 formula of $\mathcal{L}^{**}$ with four additional free variables, such that whenever G is generic over $\mathcal{H}_\xi$ for Vop_ω and $\dot{R}^G = \mathbb{R}^{\mathcal{M}}$, and $\tau_1, ..., \tau_n$ are Shoenfield terms, then

$$\mathcal{M}|\xi \models \varphi[\tau_1^G, ..., \tau_n^G]$$

if and only if

$$\exists p \exists w, x, y (p \in G \wedge \langle x, y, z \rangle = \langle \dot{\gamma}^{\mathcal{H}_\xi}, \dot{\mu}^{\mathcal{H}_\xi}, \dot{\nu}^{\mathcal{H}_\xi} \rangle \wedge p \overset{s}{\Vdash} \varphi^\dagger(\dot{R}, \check{x}, \check{y}, \check{z}, \tau_1, ..., \tau_n)).$$

These translations give

Claim 6. $\rho_1(\mathcal{H}_\xi) = \rho_1(\mathcal{M}|\xi)$.

Proof. We first show $\rho_1(\mathcal{H}_\xi) \geq \rho_1(\mathcal{M}|\xi)$. This follows at once from

Subclaim 6.1 Let $S \subseteq \mathcal{H}_\xi$ be $\Sigma_1^{\mathcal{M}|\xi}$-definable from parameters in $\mathcal{H}_\xi$, and suppose $S \in \mathcal{M}|\xi$; then $S \in \mathcal{H}_\xi$.

Proof. We may as well assume S is a set of ordinals; say $S \subseteq \rho < \omega\xi$. By 2.4 we can fix a real x_0 and an ordinal δ such that $S = h_\xi^{\vec{E}, \mathbb{R}^{\mathcal{M}}}(\delta, x_0)$. Now for y a real, let

$$y \in \emptyset \leftrightarrow \exists \eta \in S \exists Z (Z = h_\xi^{\vec{E}, \mathbb{R}^{\mathcal{M}}}(\delta, y) \wedge \eta \notin Z).$$

Since S is $\Sigma_1^{\mathcal{M}|\xi}$ in parameters from $\mathcal{H}_\xi$, O is $\Sigma_1^{\mathcal{M}|\xi}$ in ordinal parameters, so by 3.2, O is ordinal definable over $\mathcal{M}|\theta$. Hence there is a condition $r \in \mathrm{Vop}_1$ such that $r^* = \mathbb{R}^{\mathcal{M}} \setminus O$. Notice that $x_0 \in r^*$.

[14]Our previous comments regarding $\dot{\gamma}$ apply here too.

Let φ_h be a Σ_1 formula defining $h_\xi^{\vec{E},\mathbb{R}^{\mathcal{M}}}$ over $\mathcal{M}|\xi$. Let $\alpha < \omega\xi$ be large enough that

$$(S_\alpha^{\vec{E}}(\mathbb{R}^{\mathcal{M}}), \in, \vec{E} \restriction \alpha^*) \models \varphi_h[\delta, x_0, S],$$

where α^* is largest such that $\omega\alpha^* \leq \alpha$. By the proof of 3.8, we can fix a term $\tau \in \mathcal{H}_\xi$ such that whenever G is Vop_ω-generic over $\mathcal{H}_\xi$ and $\dot{R}^G = \mathbb{R}^{\mathcal{M}}$, then

$$\tau^G = (S_\alpha^{\vec{E}}(\mathbb{R}^{\mathcal{M}}), \in, \vec{E} \restriction \alpha^*).$$

Subclaim 6.1.1. For any $\eta < \rho$

$$\eta \notin S \Leftrightarrow \exists p \leq^v r(p \Vdash [\tau \models (\varphi_h(\check{\delta}, \sigma_0, Z) \wedge \check{\eta} \notin Z)]).$$

Proof. Assume $\eta \notin S$. Let $f\colon \omega \overset{\text{onto}}{\to} \mathbb{R}^{\mathcal{M}}$ be $\mathcal{M}$-generic over $\mathrm{Col}(\omega, \mathbb{R}^{\mathcal{M}})$, with $f(0) = x_0$. It follows that $r \in G_f$, $\tau^{G_f} = (S_\alpha^{\vec{E}}(\mathbb{R}^{\mathcal{M}}), \in, \vec{E} \restriction \alpha^*)$, and $\sigma_0^{G_f} = x_0$. But then $\tau^{G_f} \models \exists Z(\varphi_h[\delta, \sigma_0^{G_f}, Z] \wedge \eta \notin Z)$, since S is in fact the unique such Z. Hence we have some $p \in G_f$ forcing this fact, and we may as well take $p \leq^v r$, so that p witnesses the right hand side of our equivalence.

Conversely, let p be as on the right hand side of our equivalence. By 3.5, we can find and $f\colon \omega \overset{\text{onto}}{\to} \mathbb{R}^{\mathcal{M}}$ which is $\mathcal{M}$-generic over $\mathrm{Col}(\omega, \mathbb{R}^{\mathcal{M}})$ such that $p \in G_f$. Then $\tau^{G_f} = (S_\alpha^{\vec{E}}(\mathbb{R}^{\mathcal{M}})$, and what's forced by p is true in the generic extension, so $(S_\alpha^{\vec{E}}(\mathbb{R}^{\mathcal{M}}) \models \exists Z(\varphi_h[\delta, \sigma_0^{G_f}, Z] \wedge \eta \notin Z)$. If $\eta \in S$, this implies $\sigma_0^{G_f} \in O$ by the defintion of O; however, $\sigma_0^{G_f} = f(0) \in r^* = \neg O$ because $r \in G_f$. Thus $\eta \notin S$, as desired. □

From 6.1.1 and the definability of the forcing relation for Σ_0 sentences given by 3.6, we get that that $S \in \mathcal{H}_\xi$. This yields 6.1. □

Now let $S \subseteq \rho_1(\mathcal{H}_\xi)$ be boldface $\Sigma_1^{\mathcal{H}_\xi}$ but not in $\mathcal{H}_\xi$. By claim 4 and subclaim 6.1, $S \notin \mathcal{M}|\xi$, and this implies $\rho_1(\mathcal{M}|\xi) \leq \rho_1(\mathcal{H}_\xi)$.

We now show $\rho_1(\mathcal{H}_\xi) \leq \rho_1(\mathcal{M}|\xi)$. Let $S \subseteq (\mathbb{R}^{\mathcal{M}} \times \rho)$ be boldface $\Sigma_1^{\mathcal{M}|\xi}$ but not a member of $\mathcal{M}|\xi$, where $\rho = \rho_1(\mathcal{M}|\xi)$. Let

$$\langle x, \eta \rangle \in S \Leftrightarrow \mathcal{M}|\xi \models \varphi[x, \eta, y, \beta],$$

where $y \in \mathbb{R}^{\mathcal{M}}$ and $\beta < \omega\xi$ are fixed parameters. Now consider the strong forcing relation:

$$\langle p, \eta \rangle \in F \Leftrightarrow (p \in \mathrm{Vop}_\omega \wedge p \overset{s}{\Vdash} \varphi(\sigma_0, \check{\eta}, \sigma_1, \check{\beta})).$$

F is $\Sigma_1^{\mathcal{H}_\xi}$ by 3.6, and a subset of $\theta \times \rho$. Since $\theta \leq \rho$, we will be done if we show $F \notin \mathcal{H}_\xi$. In fact, $F \notin \mathcal{M}|\xi$, for

$$\langle x, \eta \rangle \in S \Leftrightarrow \exists p[\langle p, \eta \rangle \in F \wedge (\exists s \in p^*(s(0) = x \wedge s(1) = y)],$$

so that if $F \in \mathcal{M}|\xi$, then $S \in \mathcal{M}|\xi$. Both directions of the equivalence displayed are proved by considering Vopenka generics of the form G_f, where f is $\mathrm{Col}(\omega, \mathbb{R}^{\mathcal{M}})$ generic over $\mathcal{M}|\xi$ and $f(0) = x$ and $f(1) = y$. We leave the rest to the reader. This proves claim 6. □

Claim 7. $p_1(\mathcal{H}_\xi) = p_1(\mathcal{M}|\xi) \setminus \{\theta\}$.

Proof. The proof of claim 6 actually shows that for any finite $F \subseteq \omega\xi$ and any α such that $\theta < \alpha$,

$$\mathrm{Th}_1^{\mathcal{H}_\xi}(\alpha \cup F) \in \mathcal{H}_\xi \Leftrightarrow \mathrm{Th}_1^{\mathcal{M}|\xi}(\mathbb{R}^{\mathcal{M}} \cup \alpha \cup F) \in \mathcal{M}|\xi,$$

where $\mathrm{Th}_1^{\mathcal{P}}(X)$ denotes the Σ_1-theory in $\mathcal{P}$ of parameters in X. Letting

$$p_1(\mathcal{H}_\xi) = \langle \alpha_0, ..., \alpha_n \rangle,$$

we have by the solidity of $p_1(\mathcal{H}_\xi)$ that for $i \leq n$,

$$\alpha_i = \text{ least } \beta \text{ s.t. } \mathrm{Th}_1^{\mathcal{H}_\xi}(\beta \cup \{\alpha_0, ..., alpha_{i-1}\}) \notin \mathcal{H}_\xi.$$

Using the equivalence displayed above and the solidity of $p_1(\mathcal{M}|\xi)$, we then get by induction on i that α_i is the i^{th} member of $p_1(\mathcal{M}|\xi)$. Thus $p_1(\mathcal{H}_\xi) \subseteq p_1(\mathcal{M}|\xi) \setminus \{\theta\}$. (Note that $\theta \notin p_1(\mathcal{H}_\xi)$, since θ is easily definable over $\mathcal{H}_\xi$.) A similar argument shows $p_1(\mathcal{M}_\xi) \setminus \{\theta\} \subseteq p_1(\mathcal{H}_\xi)$. □

Claim 8. $\mathcal{H}_\xi$ is 1-sound.

Proof. Let $\eta < \omega\xi$; we must show η is $\Sigma_1^{\mathcal{H}_\xi}$-definable, as a point, from parameters in $\rho \cup p$, where $\rho = \rho_1(\mathcal{H}_\xi)$ and $p = p_1(\mathcal{H}_\xi)$. Since $\mathcal{M}|\xi$ is 1-sound, we can find a finite $F \subseteq \rho \cup p$ and a real x and a Σ_1 formula $\varphi(t, u, v, w)$ such that

$$\eta = \text{ unique } \beta \text{ s.t. } \mathcal{M}|\xi \models \varphi[x, F, \theta, \beta].$$

We may assume that φ has been "uniformised", so that over any premouse it defines the graph of a partial function of its first three variables. Now, letting

f be $\mathrm{Col}(\omega, \mathbb{R}^{\mathcal{M}})$-generic over $\mathcal{M}|\xi$ with $f(0) = x$, we can find a $A \in G_f$ such that

$$A \overset{s}{\Vdash} \varphi(\sigma_0, \check{F}, \check{\theta}, \check{\eta}).$$

Since φ has been uniformised, this gives us a $\Sigma_1^{\mathcal{H}_\xi}$ definition of η from A, σ_0, F, and θ. But $A < \theta$, and $\sigma_0 \subseteq \theta$ is coded into $T^{\mathcal{M}}$ in a simple way. Thus η is $\Sigma_1^{\mathcal{H}_\xi}$ definable from F, as desired. $\square$

This finishes the $k = 1$ case in our induction on $\langle \xi, k \rangle$. The case $k > 1$ can be handled quite similarly, using the master code structures. For example, let $\mathcal{P} = (J_\rho^{\vec{F}}(T^{\mathcal{M}}), \in, \vec{F} \restriction \rho, A)$ and $\mathcal{Q} = (J_\rho^{\vec{E}}(\mathbb{R}^{\mathcal{M}}), \in, \vec{E} \restriction \rho, B)$ be the first master code structures of $\mathcal{H}_\xi$ and $\mathcal{M}|\xi$ respectively. The arguments above show that $\alpha \mapsto A \cap \alpha$ is $\Delta_1^{\mathcal{Q}}$ and total on $\mathcal{Q}$, and this can be used to show as above that $\rho_1(\mathcal{Q}) \leq \rho_1(\mathcal{P})$, that is, $\rho_2(\mathcal{M}|\xi) \leq \rho_2(\mathcal{H}_\xi)$. In the other direction, one can show that $\mathcal{Q}$ is $\Delta_1^{\mathcal{P}[G]}$, uniformly in all Vopenka-generic G such that $\dot{R}^G = \mathbb{R}^{\mathcal{M}}$ (as in 3.8), and using the definability of forcing over $\mathcal{P}$ given by 3.6, this implies $\rho_1(\mathcal{P}) \leq \rho_1(\mathcal{Q})$, that is $\rho_2(\mathcal{H}_\xi) \leq \rho_2(\mathcal{M}|\xi)$. We leave the remaining details to the reader.

This completes the proof of 3.9 $\square$

If $\mathcal{M}$ is a model of ZFC minus the powerset axiom, and $\mathcal{H}$ is the $T^{\mathcal{M}}$-premouse we have defined above, then it is easy to see that the universe of $\mathcal{H}$ is just $\mathrm{HOD}^{\mathcal{M}}$. Indeed, $\mathcal{H} \subseteq \mathrm{HOD}^{\mathcal{M}}$ is clear, and $\mathrm{HOD}^{\mathcal{M}} \subseteq \mathcal{H}$ follows at once from subclaim 6.1. In general, for arbitrary $\mathcal{M}$ satisfying the assumptions behind 3.9,

$$x \in \mathcal{H} \Leftrightarrow \exists \alpha(\omega\alpha < o(\mathcal{M}) \wedge \forall y \in \ \mathrm{TC}(x) \cup \{x\}(y \in \mathrm{OD}^{\mathcal{M}|\alpha}).$$

It is therefore tempting to write $\mathcal{H} = \mathrm{HOD}^{\mathcal{M}}$ in general, as this would be a reasonable general meaning for $\mathrm{HOD}^{\mathcal{M}}$. However, we shall stick to

$$\mathcal{H} = \mathcal{H}(\mathcal{M}) = \mathcal{H}^{\mathcal{M}}.$$

Finally, we come to the main reason we have isolated $\mathcal{H}$.

Theorem 3.10 *Let $\mathcal{M}$ be n_0-sound $\mathbb{R}^{\mathcal{M}}$-premouse, and satisfy "Θ exists". Suppose $\rho_{n_0}(\mathcal{M}) \geq \Theta^{\mathcal{M}}$, and $\mathcal{M}|\lambda \models \mathsf{ZF}$, for some $\lambda \geq \Theta^{\mathcal{M}}$. Finally, suppose $\mathcal{M}|\xi$ is countably k-iterable, for all $\langle \xi, k \rangle <^{lex} \langle \gamma_0, n_0 \rangle$ such that $\Theta^{\mathcal{M}} \leq \xi$,* *where γ_0 is such that $\omega\gamma_0 = o(\mathcal{M})$. Then, letting $\mathcal{H} = \mathcal{H}(\mathcal{M})$, we have for all τ*

$$\mathcal{H} \textit{ is } (n_0, \tau)\textit{-iterable } \Rightarrow \mathcal{M} \textit{ is } (n_0, \tau)\textit{-iterable above } \Theta^{\mathcal{M}}.$$

Proof.(Sketch) Let Σ be an (n_0, τ)-iteration strategy for $\mathcal{H}$. We define an (n_0, τ)-iteration strategy Γ for $\mathcal{M}$ which operates on trees all of whose extenders have critical points above $\Theta^{\mathcal{M}}$. Given such a tree $\mathcal{T}$ played according to Γ, the construction insures that there is a tree $\mathcal{T}^*$ on $\mathcal{H}$ which is according to Σ with the same tree order, drop, and degree structure as $\mathcal{T}$, and such that

- $\mathcal{T}^*$ is according to Σ,
- $\mathcal{M}_\alpha^{\mathcal{T}^*} = \mathcal{H}(\mathcal{M}_\alpha^{\mathcal{T}})$, for all $\alpha < \mathrm{lh}(\mathcal{T})$, and
- $E_\alpha^{\mathcal{T}^*} = E_\alpha^{\mathcal{T}} \cap \mathcal{M}_\alpha^{\mathcal{T}^*}$.

Because $\mathcal{M}_\alpha^{\mathcal{T}}$ is an inner model of a generic extension of $\mathcal{M}_\alpha^{\mathcal{T}^*}$ via a poset (i.e. $\mathrm{Vop}_\omega^{\mathcal{H}}$ of size $\Theta^{\mathcal{M}}$, these conditions imply that

- $i_{\alpha,\beta}^{\mathcal{T}^*} = i_{\alpha,\beta}^{\mathcal{T}} \upharpoonright \mathcal{M}_\alpha^{\mathcal{T}^*}$.

Given that $\mathcal{T}$ has limit length $< \tau$, and we have a $\mathcal{T}^*$ as above, we simply define $\Gamma(\mathcal{T}) = \Sigma(\mathcal{T}^*)$. Setting $b = \Gamma(\mathcal{T})$, it is routine to verify that the b-extensions of $\mathcal{T}$ and $\mathcal{T}^*$ satisfy the conditions above. The main point is that for $\alpha \in b$ sufficiently large,

$$\mathcal{M}_b^{\mathcal{T}^*} = i_{\alpha,b}^{\mathcal{T}^*}(\mathcal{M}_\alpha^{\mathcal{T}^*}) = i_{\alpha,b}^{\mathcal{T}}(\mathcal{H}(\mathcal{M}_\alpha^{\mathcal{T}})) = \mathcal{H}(\mathcal{M}_b^{\mathcal{T}}),$$

where we have applied the embeddings to classes of their domain models in the usual way.

It is also clear that the existence of $\mathcal{T}^*$ as above propagates through successor steps in the construction of $\mathcal{T}$. This completes our sketch. $\square$

4 The scale property in $K(\mathbb{R})$

Using the local HOD's of the last section to verify iterability, in the same way that the ordinals were used to verify wellfoundedness in [7], we shall construct closed game representations of minimal complexity for sets in $K(\mathbb{R})$. As explained in [7], an argument due to Moschovakis ([6]) converts these closed game representations to scales of minimal complexity. Modulo the use of the local HOD's to verify iterability, everything goes pretty much as it did in [7]. We shall therefore keep our notation as close as possible to that of [7], and omit many of the details treated more carefully there.

4.1 Scales on $\Sigma_1^{\mathcal{M}}$ sets, for $\mathcal{M}$ passive

In this subsection we prove 1.1. In fact, we prove the slightly stronger

Theorem 4.1 *Let $\mathcal{M}$ be a passive, countably iterable premouse over $\mathbb{R}^{\mathcal{M}}$, and suppose $\mathcal{M} \models \mathsf{AD}$; then*

$$\mathcal{M} \models \text{ “the pointclass } \Sigma_1^{\mathcal{M}} \text{ has the scale property.”}$$

It should be clear what it means for $\mathcal{M}$ to believe that a pointclass of $\mathcal{M}$-definable sets of reals has the scale property: the norms of the putative scale, which are $\mathcal{M}$-definable, must have the limit and lower semi-continuity properties of a scale with respect to all sequences of reals $\langle x_i \mid i < \omega \rangle \in \mathcal{M}$. If $\mathbb{R}^{\mathcal{M}} = \mathbb{R}$ and $\mathcal{M}$ believes that the pointclass $\Sigma_1^{\mathcal{M}}$ has the scale property, then indeed $\Sigma_1^{\mathcal{M}}$ does have the scale property, as every ω-sequence of reals is in $\mathcal{M}$. Thus 4.1 implies 1.1.

Proof. Let us fix a passive, countably 0-iterable $\mathbb{R}^{\mathcal{M}}$-premouse $\mathcal{M}$ such that $\mathcal{M} \models \mathsf{AD}$. We want to show that $\mathcal{M}$ satisfies a certain sentence, so by taking a Skolem hull we may assume that $\mathcal{M}$ is countable.

For $x \in \mathbb{R}^{\mathcal{M}}$, let

$$P(x) \Leftrightarrow \mathcal{M} \models \varphi_0[x],$$

where φ_0 is a Σ_1 formula of $\mathcal{L}^*$. Since $\mathcal{M}$ is passive, we may (and do) assume that φ_0 does not contain $\dot{F}, \dot{\mu}, \dot{\nu}$, or $\dot{\gamma}$; that is, it contains only $\in$ and $\dot{E}$. We want to show that $\mathcal{M}$ believes that there is $\Sigma_1^{\mathcal{M}}$ scale on P.

Let us first assume that $o(\mathcal{M}) = \omega\alpha$, where α is a limit ordinal, and deal with the general case later. If $\mathcal{M}$ satisfies “Θ exists”, then set $\alpha^* = \Theta^{\mathcal{M}}$, and otherwise set $\alpha^* = \alpha$. For $\beta < \alpha^*$ and $x \in \mathbb{R}^{\mathcal{M}}$, let

$$P^\beta(x) \Leftrightarrow \mathcal{M}|\beta \models \varphi_0[x],$$

so that $P = \bigcup_{\beta<\alpha*} P^\beta$. Here we use 3.2 to see that the union out to α^* suffices. [15]

For each $\beta < \alpha^*$, we will construct a closed game representation $x \mapsto G^\beta_x$ of P^β. Letting

$$P^\beta_k(x, u) \;\Leftrightarrow\; u \text{ is a position of length } k \text{ from which}$$
$$\text{I has a winning quasi-strategy in } G^\beta_x,$$

we shall arrange that each $P^\beta_k \in \mathcal{M}$, and that the map $\langle\beta, k\rangle \mapsto P^\beta_k$ is $\Sigma^{\mathcal{M}}_1$. As explained in [7], Moschovakis' argument then gives that $\mathcal{M}$ believes there is a scale on P of the desired complexity.[16]

So let β and x be given; we want to define G^β_x. Our plan is to force Player I to describe a model of $V = K(\mathbb{R}) + \varphi_0(x) + \forall\gamma(\mathcal{J}^{\vec{E}}_\gamma(\mathbb{R}) \not\models \varphi[x])$ which includes all the reals played by II. Player I will verify that his model is well-founded by embedding its ordinals into $\omega\beta$, and verify his model is iterable by embedding its local HOD's into the local HOD's of $\mathcal{M}|\beta$ corresponding to them under his embedding of the ordinals.

Player I describes his model in the language $\mathcal{L}$, which is $\mathcal{L}^*$ together with new constant symbols $\dot{x}_i$ for $i < \omega$. He uses $\dot{x}_i$ to denote the i^{th} real played in the course of G^β_x. Let us fix recursive maps

$$m, n \colon \{\sigma \mid \sigma \text{ is an } \mathcal{L}\text{-formula }\} \to \{2n \mid 1 \leq n < \omega\}$$

which are one-one, have disjoint recursive ranges, and are such that whenever $\dot{x}_i$ occurs in σ, then $i < \min(m(\sigma), n(\sigma))$. These maps give stages sufficiently late in G^β_x for I to decide certain statements about his model.

Let us call an ordinal ξ of an $\mathbb{R}^{\mathcal{P}}$-premouse $\mathcal{P}$ *relevant* iff $\mathcal{P}|\xi$ satisfies "Θ exists, and there is a $\lambda > \Theta$ such that $J^{\dot{E}}_\lambda(\mathbb{R}) \models \mathsf{ZF}$." That is, the relevant ξ are just those for which, under the additional assumption that $\mathcal{P}$ is countably iterable, we have have defined and proved the existence of $\mathcal{H}^{\mathcal{P}}_\xi$. Also, "$v$ is relevant" is the $\mathcal{L}$-formula which expresses that v is relevant viv-a-vis the universe as $\mathcal{P}$. Similarly for the $\mathcal{L}$-formula "$\mathcal{H}_v$ exists".

Player I's description must extend the following $\mathcal{L}$-theory T. The axioms of T include

[15] We have restricted ourselves to $\beta < \alpha^*$ for a minor technical reason connected to the definability of "honesty".

[16] Moschovakis uses the "second periodicity" method to construct scales on the P^β_k. It is here that one needs $\mathcal{M} \models \mathsf{AD}$.

(1) Extensionality plus $V = K(\mathbb{R})$

(2) $\forall v(v$ is relevant $\Rightarrow \mathcal{H}_v$ exists)

$(3)_\varphi$ $\exists v \varphi(v) \Rightarrow \exists v(\varphi(v) \wedge \forall u \in v \neg\varphi(u))$

$(4)_i$ $\dot{x}_i \in \mathbb{R}$.

(5) $\varphi_0(\dot{x}_0) \wedge \forall\delta(\mathcal{J}_\delta^{\dot{E}}(\mathbb{R}) \not\models \varphi_0[\dot{x}_0])$

Finally, T has axioms which guarantee that in any model, the definable closure of the interpretations of the $\dot{x}_i$ constitute an elementary submodel. Recall from 2.4 the uniformly definable maps $h_\gamma\colon [\omega\gamma]^{<\omega} \stackrel{onto}{\rightarrow} \mathcal{M}|\gamma$; let $\sigma_0(v_0, v_1, v_2)$ be a Σ_1 formula which for all γ defines the graph of h_γ over $\mathcal{M}|\gamma$. Now, for any $\mathcal{L}$-formula $\varphi(v)$ of one free variable, T has axioms

(6) $\forall v_0 \forall v_1 \forall y \forall z(\sigma_0(v_0, v_1, y) \wedge \sigma_0(v_0, v_1, z) \Rightarrow y = z)$

$(7)_\varphi$ $\exists v\varphi(v) \Rightarrow \exists v \exists F \in [\mathrm{OR}]^{<\omega}(\varphi(v) \wedge \sigma_0(F, \dot{x}_{m(\varphi)}, v))$

$(8)_\varphi$ $\exists v(\varphi(v) \wedge v \in \mathbb{R}) \Rightarrow \varphi(\dot{x}_{n(\varphi)})$.

This completes the axioms of T.

A typical run of G_x^β has the form

$$\begin{array}{llll} \mathrm{I} & i_0, x_0, \eta_0 & i_1, x_2, \eta_1 & \\ & & & \cdots \\ \mathrm{II} & & x_1 & \quad x_3 \end{array}$$

where for all k, $i_k \in \{0,1\}$, $x_k \in \mathbb{R}$, and $\eta_k < \omega\beta$. If $u = \langle (i_k, x_{2k}, \eta_k, x_{2k+1}) \mid k < n\rangle$ is a position of length n, then we set

$$T^*(u) = \{\sigma \mid \sigma \text{ is a sentence of } \mathcal{L} \ \wedge i_{n(\sigma)} = 0\},$$

and if p is a full run of G_x^β,

$$T^*(p) = \bigcup_{n<\omega} T^*(p \restriction n).$$

Now let $p = \langle (i_k, x_{2k}, \eta_k, x_{2k+1}) \mid k < \omega\rangle$ be a run of G_x^β; we say that p is winning for I iff

(a) $x_0 = x$,

(b) $T^*(p)$ is a complete, consistent extension of T such that for all i, m, n, "$\dot{x}_i(n) = m$" $\in T^*(p)$ iff $x_i(n) = m$,

(c) if φ and ψ are $\mathcal{L}$-formulae of one free variable, and "$\iota v\varphi(v) \in \mathrm{OR} \wedge \iota v\psi(v) \in \mathrm{OR}$" $\in T^*(p)$, then "$\iota v\varphi(v) \leq \iota v\psi(v)$" $\in T^*(p)$ iff $\eta_{n(\varphi)} \leq \eta_{n(\psi)}$, and

(d) if ψ is an $\mathcal{L}$-formula of one free variable, and "$\iota v\psi(v) \in \mathrm{OR} \wedge \iota v\psi(v)$ is relevant" $\in T^*(p)$, then $\eta_{n(\psi)}$ is relevant vis-a-vis $\mathcal{M}$; moreover if $\sigma_1, ..., \sigma_n$ are $\mathcal{L}$ formulae of one free variable such that for all k, "$\iota v\sigma_k(v) < (\iota v\psi(v))$" $\in T^*(p)$, then for any $\mathcal{L}^*$ formula $\theta(v_1, ..., v_n)$,

$$\text{“}\mathcal{H}_{\iota v\psi(v)} \models \theta[\iota v\sigma_1(v), ..., \iota v\sigma_n(v)]\text{”} \in T^*(p)$$

if and only if

$$\mathcal{H}^{\mathcal{M}}_{\eta_{n(\psi)}} \models \theta[\eta_{n(\sigma_1)}, ..., \eta_{n(\sigma_n)}].$$

Clearly, G^β_x is a game on $\mathbb{R} \times \omega\beta$ whose payoff is continuously associated to x. It remains to show that the winning positions for I in G^β_x are those in which he has been honest. More precisely, let us call a position $u = \langle (i_k, x_{2k}, \eta_k, x_{2k+1}) \mid k < n \rangle$ (β, x)-*honest* iff $\mathcal{M}|\beta \models \varphi_0[x]$, and letting $\gamma \leq \beta$ be least such that $\mathcal{M}|\gamma \models \varphi_0[x]$, we have

(i) $n > 0 \Rightarrow x_0 = x$,

(ii) if we let $I_u(\dot{x}_i) = x_i$ for $i < 2n$, then all axioms of $T^*(u) \cup T$ thereby interpreted in $(\mathcal{M}|\gamma, I_u)$ are true in this structure, and

(iii) if $\sigma_0, ..., \sigma_m$ enumerates those $\mathcal{L}$-formulae σ of one free variable such that $n(\sigma) < n$ and

$$(\mathcal{M}|\gamma, I_u) \models \iota v\sigma(v) \in \mathrm{OR},$$

and if $\delta_i < \omega\gamma$ is such that

$$(\mathcal{M}|\gamma, I_u) \models \iota v\sigma_i(v) = \delta_i,$$

then the map

$$\delta_i \mapsto \eta_{n(\sigma_i)}$$

is well-defined and extendible to an order preserving map $\pi\colon \omega\gamma \to \omega\beta$ with the additional property that whenever δ_i is relevant vis-a-vis $\mathcal{M}$, so that $\eta_{n(\sigma_i)}$ is as well, then $\pi \restriction \omega\delta_i$ is extendible to an elementary embedding from $\mathcal{H}^{\mathcal{M}}_{\delta_i}$ to $\mathcal{H}^{\mathcal{M}}_{\eta_{n(\sigma_i)}}$. [17]

It is not immediately clear that the set of (β, x)-honest positions even belongs to $\mathcal{M}$, because of condition (iii).

Claim 1. Letting $Q^{\beta}_k(x, u)$ iff u is a (β, x)-honest position of length k, we have that $Q^{\beta}_k \in \mathcal{M}$ for all β, and the map $(\beta, k) \mapsto Q^{\beta}_k$ is $\Sigma^{\mathcal{M}}_1$.

Proof.(Sketch.) It is enough to see that the truth of clause (iii) can be determined within $\mathcal{M}$. But note that (iii) is equivalent to the existence of a winning strategy for the closed player in a certain "embedding game" on $\omega\beta$. It is enough then to see that if the closed player wins the embedding game in V, then he wins it in $\mathcal{M}$. (The converse is obvious.) So suppose the closed player wins the embedding game in V. Let $A \in \mathcal{M}$ be a set of ordinals which codes up this game; since $\beta < \Theta^{\mathcal{M}}$, and $\mathcal{M} \models \mathsf{AD}$, we can find a model N of ZFC such that $A \in N$, and $(N, \in)$ is coded by a set of reals $B_N \in \mathcal{M}$. (E.g., let $N = L_\alpha[A]$, where α is the supremum of the order types of the $\Delta^{\mathcal{M}|\gamma}_n$ prewellorders of $\mathbb{R}^{\mathcal{M}}$, for an appropriate n and γ.) Since $N \models \mathsf{ZFC}$, the closed player wins the embedding game via a strategy $\Sigma \in N$. For $\gamma < \omega\beta$, let $f(\gamma) = \{z \in \mathbb{R}^{\mathcal{M}} \mid z \text{ codes } \gamma \text{ via } B_N\}$. We can arrange that $f \in \mathcal{M}$, and use f to show that $\Sigma \in \mathcal{M}$. $\square$

We now show that the winning positions are the honest ones.

Claim 2. For any position u in G^{β}_x, Player I has a winning quasi-strategy starting from u iff u is (β, x)-honest; that is, $P^{\beta}_k(x, u) \Leftrightarrow Q^{\beta}_k(x, u)$ for all k.

Proof. It is easy to see that I can win from honest positions u by continuing to tell the truth, while continuing to play his η's according to some map π satisfying (iii) in the definition of honesty for u.

Conversely, let Σ be a winning quasi-strategy for I in G^{β}_x from u. Since $\mathbb{R}^{\mathcal{M}}$ is countable, we can easily construct a complete run

$$p = \langle (i_k, x_{2k}, \eta_k, x_{2k+1}) \mid k < \omega \rangle$$

[17] Notice that this extension is determined by π, since every point in $\mathcal{H}_{\delta_i}$ is definable from an ordinal.

of G^β_x according to Σ such that

$$\mathbb{R}^{\mathcal{M}} = \{x_i \mid i < \omega\}.$$

Since $T^*(p)$ is consistent, it has a model $\mathcal{A}$, and by the axioms in groups (3), (6), (7), and (8), the $\mathcal{L}$-definable points of $\mathcal{A}$ constitute an elementary submodel $\mathcal{N} \prec \mathcal{A}$. Let

$$\pi(\iota v\psi(v)^{\mathcal{N}}) = \eta_{n(\psi)}$$

whenever "$\iota v\psi(v) \in \mathrm{OR}$" $\in T^*(p)$. Then π witnesses that $\mathcal{N}$ is wellfounded, and so we may assume $\mathcal{N}$ is transitive. By axiom (1) of T, $\mathcal{N}$ is a premouse. Note that $\mathcal{N}$ is a premouse over $\{\dot{x}_i^{\mathcal{A}} \mid i < \omega\}$, and this set is $\mathbb{R}^{\mathcal{M}}$ since $\dot{x}_i^{\mathcal{A}} = x_i$ by (ii). The main thing we need to show is that $\mathcal{N}$ is an initial segment of $\mathcal{M}|\beta$. Since π guarantees $o(\mathcal{N}) \leq \omega\beta$, it suffices to show $\mathcal{N}$ is an initial segment of $\mathcal{M}$.

We show by induction on γ that if $\rho_\omega(\mathcal{N}|\gamma) = 1$, then $\mathcal{N}|\gamma = \mathcal{M}|\gamma$. This is clear if $\gamma = 1$.

Suppose first that $\mathcal{N}|\gamma \not\models \Theta$ exists . Note that then γ is not active in $\mathcal{N}$. If γ is a limit ordinal, then there are arbitrarily large $\xi < \gamma$ such that $\rho_\omega(\mathcal{N}|\xi) = 1$, and by induction $\mathcal{N}|\xi = \mathcal{M}|\xi$ for such ξ. This implies γ is not active in $\mathcal{M}$, and $\mathcal{N}|\gamma = \mathcal{M}|\gamma$, as desired. If $\gamma = \xi + 1$, then $\rho_\omega(\mathcal{N}|\xi) = 1$, so $\mathcal{N}|\xi = \mathcal{M}|\xi$. Since successor ordinals are not active, we get $\mathcal{N}|\gamma = \mathcal{M}|\gamma$, as desired. (Note that if $\omega\gamma = o(\mathcal{N})$, then it falls under this last case by axiom (5) of T, so we may assume $\omega\gamma < o(\mathcal{N})$ henceforth.)

Next, suppose $\mathcal{N}|\gamma \models \Theta$ exists, and let $\theta = \Theta^{\mathcal{N}|\gamma}$. Note $\mathcal{N}|\theta = \mathcal{M}|\theta$ by the argument of the last paragraph.

If there is no $\xi \in (\theta, \gamma)$ such that $\mathcal{N}|\xi \models \mathsf{ZF}$, then $\mathcal{N}|\gamma$ is just the constructible closure of $\mathcal{N}|\theta$ through γ steps. Also, no $\xi \in (\theta, \gamma]$ can be active in $\mathcal{M}$, and so $\mathcal{N}|\gamma = \mathcal{M}|\gamma$. So we may assume that there is a $\xi \in (\theta, \gamma)$ such that $\mathcal{N}|\xi \models \mathsf{ZF}$; that is, γ is relevant in $\mathcal{N}$. Let n_0 be the largest $n < \omega$ such that $\rho_n(\mathcal{N}|\gamma) \geq \theta$. By rule (iii), π determines an elementary embedding from $\mathcal{H}(\mathcal{N}|\xi)$ to $\mathcal{H}(\mathcal{M}|\pi(\xi))$, for each relevant ξ of $\mathcal{N}$. Using 3.10 and a simple induction, we can then see that $\mathcal{N}|\gamma$ is countably n_0-iterable. Now $\beta < \alpha*$, and hence we can find a $\xi \geq \beta$ such that $\rho_\omega(\mathcal{M}|\xi) = 1$. Applying the comparison theorem 2.12 to $\mathcal{N}|\gamma$ and $\mathcal{M}|\xi$, we get that $\mathcal{N}|\gamma = \mathcal{M}|\gamma$, as desired.

Thus $\mathcal{N}$ is an initial segment of $\mathcal{M}|\beta$. Clearly, $\mathcal{N} = \mathcal{M}|\gamma$, where γ is least such that $\mathcal{M}|\gamma \models \varphi_0[x]$. The theory $T^*(u)$ is true in $\mathcal{N} = \mathcal{M}|\gamma$. The remainder of (β, x) honesty for u is witnessed by π. This completes the proof of claim 2. $\square$

Claims 1 and 2 yield the desired scale, as we have explained. This completes the proof of 4.1 in the case that $o(\mathcal{M}) = \omega\alpha$ for α a limit. The case that α is a successor ordinal can be handled similarly, using Jensen's S-hierarchy. See [7]. $\square$

4.2 Σ_1 gaps

Definition 4.2 *Let $\mathcal{M}$ and $\mathcal{N}$ be X-premice; then we write $\mathcal{M} \prec_1 \mathcal{N}$ iff $\mathcal{M}$ is an initial segment of $\mathcal{N}$, and whenever $\varphi(v_1, ..., v_n)$ is a Σ_1 formula of the language $\mathcal{L}^*$ in which $\dot{F}, \dot{\mu}, \dot{\nu}$, and $\dot{\gamma}$ do not occur, then for any $a_1, ..., a_n \in X \cup \{X\}$,*

$$\mathcal{N} \models \varphi[a_1, ..., a_n] \Rightarrow \mathcal{M} \models \varphi[a_1, ..., a_n].$$

Notice here that such Σ_1 formulae go up from $\mathcal{M}$ to $\mathcal{N}$ simply because $\mathcal{M}$ is an initial segment of $\mathcal{N}$. This uses our restriction that the symbols of $\mathcal{L}^*$ which have to do with the last extender of a premouse do not occur in φ.

Definition 4.3 *Let $\mathcal{M}$ be an X-premouse, and suppose $\omega\alpha \leq \omega\beta \leq o(\mathcal{M})$; then we call the interval $[\alpha, \beta]$ a Σ_1-gap of $\mathcal{M}$ iff*

1. *$\mathcal{M}|\alpha \prec_1 \mathcal{M}|\beta$,*
2. *$\forall\gamma < \alpha(\mathcal{M}|\gamma \not\prec_1 \mathcal{M}|\alpha)$, and*
3. *$\forall\gamma > \beta(\mathcal{M}|\beta \not\prec_1 \mathcal{M}|\gamma)$.*

That is, a Σ_1-gap is a maximal interval of ordinals in which no new Σ_1 facts about members of $X \cup \{X\}$ and the extender sequence $\vec{E}$ are verified. If $[\alpha, \beta]$ is a Σ_1-gap, we say α begins the gap and β ends it. Notice that we allow $\alpha = \beta$. It is easy to see

Lemma 4.4 *Let $o(\mathcal{M}) = \omega\alpha$; then the Σ_1-gaps of $\mathcal{M}$ partition $\alpha + 1$.*

We shall use the Σ_1-gaps of $K(\mathbb{R})$ to characterize the levels of the Levy hierarchy in the initial segment of $K(\mathbb{R})$ satisfying AD which have the Scale Property. Until we get to the end-of-gap case, the proofs are quite easy, and completely parallel to those of [7], so we shall omit them.

If $\mathcal{M}$ is an $\mathbb{R}^{\mathcal{M}}$-premouse, then by *the pointclass* $\boldsymbol{\Sigma}_n^{\mathcal{M}}$ we mean the collection of all $A \subseteq \mathbb{R}^{\mathcal{M}}$ such that A is Σ_n definable over $\mathfrak{C}_{\mathfrak{n}-\mathfrak{1}}(\mathcal{M})$ from arbitrary parameters in $\mathfrak{C}_{\mathfrak{n}-\mathfrak{1}}(\mathcal{M})$, using the language $\mathcal{L}^*$.

Theorem 4.5 *Let $\mathcal{M}$ be a countably 0-iterable $\mathbb{R}^{\mathcal{M}}$-premouse, and suppose α begins a Σ_1 gap of $\mathcal{M}$, and that $\mathcal{M}|\alpha \models \mathsf{AD}$; then $\mathcal{M}$ believes that the pointclass $\boldsymbol{\Sigma}_1^{\mathcal{M}|\alpha}$ has the Scale Property.*

Theorem 4.6 *Let $\mathcal{M}$ be a countably 0-iterable $\mathbb{R}^{\mathcal{M}}$-premouse, and suppose α begins a Σ_1-gap of $\mathcal{M}$, and that $\mathcal{M}|(\alpha+1) \models \mathsf{AD}$, and that $\mathcal{M}|\alpha$ is not an admissible structure. Then for all $n < \omega$,*

(a)

$$\boldsymbol{\Sigma}_{n+2}^{\mathcal{M}|\alpha} = \exists^{\mathbb{R}}(\boldsymbol{\Pi}_{n+1}^{\mathcal{M}|\alpha}),$$
$$\boldsymbol{\Pi}_{n+2}^{\mathcal{M}|\alpha} = \forall^{\mathbb{R}}(\boldsymbol{\Sigma}_{n+1}^{\mathcal{M}|\alpha}),$$

and

(b) $\mathcal{M}$ believes that the pointclasses $\boldsymbol{\Sigma}_{2n+1}^{\mathcal{M}|\alpha}$ and $\boldsymbol{\Pi}_{2n+2}^{\mathcal{M}|\alpha}$ have the Scale Property.

As in $L(\mathbb{R})$, our negative results on the Scale Property are localizations of the fact that the relation "x is not ordinal definable from y" has no ordinal definable uniformization.

Definition 4.7 *If $\mathcal{M}$ is an $\mathbb{R}^{\mathcal{M}}$-premouse and $o(\mathcal{M}) = \omega\alpha$, then for $x, y \in \mathbb{R}^{\mathcal{M}}$, we put*

$$C^{\mathcal{M}}(x,y) \Leftrightarrow \exists\gamma < \alpha(\{y\} \text{ is } \mathcal{M}|\gamma\text{-definable from parameters in } \{x\} \cup \omega\gamma).$$

We also set $\neg C^{\mathcal{M}} = (\mathbb{R}^{\mathcal{M}} \times \mathbb{R}^{\mathcal{M}}) \setminus C^{\mathcal{M}}$.

It is clear that $C^{\mathcal{M}}$ is $\Sigma_1^{\mathcal{M}}$, and indeed, it is so via a formula which does not refer to the last extender $\dot{F}^{\mathcal{M}}$.

Theorem 4.8 (Martin, [3]) *Let $\mathcal{M}$ be a countably 0-iterable $\mathbb{R}^{\mathcal{M}}$-premouse, and suppose that α begins a Σ_1-gap of $\mathcal{M}$, that $\mathcal{M}|(\alpha+1) \models \mathsf{AD}$, and that the structure $\mathcal{M}|\alpha$ is admissible. Then*

(a) *there is a $\Pi_1^{\mathcal{M}|\alpha}$ relation on $\mathbb{R}^{\mathcal{M}}$, namely $\neg C^{\mathcal{M}|\alpha}$, which has no uniformization in $\mathcal{M}|(\alpha+1)$, and hence*

(b) *$\mathcal{M}$ believes that none of the pointclasses $\boldsymbol{\Sigma}_n^{\mathcal{M}|\alpha}$ or $\boldsymbol{\Pi}_n^{\mathcal{M}|\alpha}$, for $n > 1$, have the Scale Property.*

In the interior of a Σ_1-gap, we find no new scales.

Theorem 4.9 (Kechris, Solovay) *Let $\mathcal{M}$ be a countably 0-iterable $\mathbb{R}^{\mathcal{M}}$-premouse, and suppose $[\alpha, \beta]$ is a Σ_1-gap of $\mathcal{M}$, and that $\mathcal{M}|\alpha \models \mathsf{AD}$. Then*

(a) *the relation $\neg C^{\mathcal{M}|\alpha}$ has no uniformizing function f such that $f \in \mathcal{M}|\beta$, and hence*

(b) *if $\alpha < \gamma < \beta$, then $\mathcal{M}$ believes that none of the pointclasses $\boldsymbol{\Sigma}_n^{\mathcal{M}|\gamma}$ or $\boldsymbol{\Pi}_n^{\mathcal{M}|\gamma}$, for $n < \omega$, have the Scale Property.*

4.3 Scales at the end of a gap

We are left with the question as to which, if any, of the pointclasses $\boldsymbol{\Sigma}_n^{\mathcal{M}|\beta}$ and $\boldsymbol{\Pi}_n^{\mathcal{M}|\beta}$ have the Scale Property in the case that β ends a Σ_1 gap $[\alpha, \beta]$ of $\mathcal{M}$, and $\alpha < \beta$. As in [7], the answer turns on the following reflection property of β.

Definition 4.10 *For $\mathcal{M}$ a relativised premouse and $1 \leq n < \omega$ and $a \in \mathcal{M}$, we let $\Sigma_a^{n,\mathcal{M}}$ be the Σ_n-type realized by a in $\mathcal{M}$; that is*

$$\Sigma_a^{n,\mathcal{M}} = \{\theta(v) \mid \theta \text{ is either } \Sigma_n \text{ or } \Pi_n \text{ and } \mathfrak{C}_{n-1}(\mathcal{M}) \models \theta[a].$$

We are allowing formulae of the full language $\mathcal{L}^*$ of relativised premice, so that the last extender $\dot{F}^{\mathcal{M}}$ is (partially) described in $\Sigma_a^{n,\mathcal{M}}$.

Definition 4.11 *An ordinal β is strongly Π_n-reflecting in $\mathcal{M}$ iff every Σ_n-type realized in $\mathfrak{C}_{n-1}(\mathcal{M}|\beta)$ is realized in $\mathfrak{C}_{n-1}(\mathcal{M}|\xi)$ for some $\xi < \beta$; that is*

$$\forall a \in \mathfrak{C}_{n-1}(\mathcal{M}|\beta) \exists \xi < \beta \exists b \in \mathfrak{C}_{n-1}(\mathcal{M}|\xi)(\Sigma_a^{n,\mathcal{M}|\beta} = \Sigma_b^{n,\mathcal{M}|\xi}).$$

Definition 4.12 *Let $[\alpha, \beta]$ be a Σ_1 gap of $\mathcal{M}$, with $\alpha < \beta$; then we call $[\alpha, \beta]$ strong iff β is strongly Π_n-reflecting in $\mathcal{M}$, where n is least such that $\rho_n(\mathcal{M}|\beta) = \mathbb{R}^{\mathcal{M}}$. Otherwise, $[\alpha, \beta]$ is weak.*

Martin's reflection argument of [3] yields

Theorem 4.13 (Martin) *Let $\mathcal{M}$ be a countably 0-iterable $\mathbb{R}^{\mathcal{M}}$-mouse which satisfies* AD *, and let $[\alpha, \beta]$ be a strong Σ_1-gap of $\mathcal{M}$ such that $\omega\beta < o(\mathcal{M})$; then*

(a) there is a $\Pi_1^{\mathcal{M}|\alpha}$ relation on $\mathbb{R}^{\mathcal{M}}$ which has no uniformization which is definable over $\mathcal{M}|\beta$, and hence

(b) $\mathcal{M}$ believes that none of the pointclasses $\boldsymbol{\Sigma}_n^{\mathcal{M}|\beta}$ or $\boldsymbol{\Pi}_n^{\mathcal{M}|\beta}$ have the Scale Property.

Thus at the end of strong gaps $[\alpha, \beta]$, the Scale Property first re-appears with the pointclass $\Sigma_1^{\mathcal{M}|(\beta+1)}$. The weak gap case is settled, under stronger determinacy hypotheses than should be necessary, by

Theorem 4.14 *Let $\mathcal{M}$ be a countably ω-iterable $\mathbb{R}^{\mathcal{M}}$-mouse which satisfies* AD*, and $[\alpha, \beta]$ a weak gap of $\mathcal{M}$ and $\omega\beta < o(\mathcal{M})$; then letting n be least such that $\rho_n(\mathcal{M}|\beta) = \mathbb{R}^{\mathcal{M}}$, we have that $\mathcal{M}$ believes that $\boldsymbol{\Sigma}_n^{\mathcal{M}|\beta}$ has the Scale Property.*

Remark. The hypothesis that $\omega\beta < o(\mathcal{M})$ should not be necessary. The proof below needs it because at a certain point we apply the Coding Lemma to a bounded subset of $\Theta^{\mathcal{M}|\beta}$ such that A is merely definable over $\mathcal{M}|\beta$, and we need enough determinacy to do this. (See below.) Unfortunately, adding this determinacy as a hypothesis in 4.14 makes the theorem significantly less useful in core model induction arguments than it would be otherwise. We can eliminate the additional determinacy hypothesis in one case:

Theorem 4.15 *Let $\mathcal{M}$ be a countably ω-iterable $\mathbb{R}^{\mathcal{M}}$-mouse which satisfies* AD*, and $[\alpha, \beta]$ a weak gap of $\mathcal{M}$. Suppose that either $\Theta^{\mathcal{M}}$ does not exist, or there are no extenders on the $\mathcal{M}$-sequence with index above $\Theta^{\mathcal{M}}$; then letting n be least such that $\rho_n(\mathcal{M}|\beta) = \mathbb{R}^{\mathcal{M}}$, we have that $\mathcal{M}$ believes that $\boldsymbol{\Sigma}_n^{\mathcal{M}|\beta}$ has the Scale Property.*

One can combine Theorem 4.15 with the work of [8], and thereby obtain a construction of scales at the end of a weak gap in $K(\mathbb{R})$ which is more useful in a core model induction.

Proof of 4.14. (Sketch.) One gets a proof by integrating our use of the local $\mathrm{HOD}^{\mathcal{M}}$'s into the proof of the corresponding result (theorem 3.7) of [7]. This is fairly routine, yet involves many details. We shall therefore just sketch one case in which some care with the details is needed. We want also to point out the place where the additional determinacy hypothesis is used.[18]

The case we consider is $n = 1$ and $\mathcal{M}|\beta$ is active of type II. Let us make these assumptions.

Let F^* be the amenable-to-$\mathcal{M}$ predicate coding $\dot{F}^{\mathcal{M}}$ which is descibed in [9]. For $\gamma < \beta$, let

$$\mathcal{M}||\gamma = (J_\gamma^{\dot{E}^{\mathcal{M}}}, \in, \dot{E}^{\mathcal{M}|\gamma}, F^* \cap J_\gamma^{\dot{E}^{\mathcal{M}}}).$$

The $\mathcal{M}||\gamma$ are just the initial segments of the Σ_0-code $\mathfrak{C}_\mathfrak{o}(\mathcal{M})$. They are structures for the language $\mathcal{L}^*$ of $\mathfrak{C}_\mathfrak{o}(\mathcal{M})$, and for φ a Σ_1 formula of $\mathcal{L}^*$ and $x \in \mathcal{M}$, we have

$$\mathfrak{C}_0(\mathcal{M}) \models \varphi[x] \Leftrightarrow \exists \gamma < \beta(\mathcal{M}||\gamma \models \varphi[x]).$$

Further, $\mathcal{M}||\gamma \in \mathcal{M}$ for all $\gamma < \beta$.

Let $\Sigma = \Sigma_a^{1,\mathcal{M}}$ be our nonreflecting Σ_1-type. We may assume $a = \langle G, w_1 \rangle$, where G is a finite subset of β and $w_1 \in \mathbb{R}$, and that G is Brouwer-Kleene minimal, in the sense that whenever $H \in [\beta]^{<\omega}$ and $H <_b G$ then $\langle H, w_1 \rangle$ does not realize Σ in $\mathfrak{C}_\mathfrak{o}(\mathcal{M})$. (Here $H <_b G$ iff $\max\,(H \triangle G) \in G$.) Let $\kappa = \mathrm{crit}(\dot{F}^{\mathcal{M}})$.

We define a canonical sequence of initial segments $\mathcal{M}||\beta_i$ of $\mathfrak{C}_\mathfrak{o}(\mathcal{M})$. Let $\beta_0 = \nu(\dot{F}^{\mathcal{M}})$. Given $\beta_i < \beta$, let

$$Y_i = \{a \mid a \text{ is definable over } \mathcal{M}||\beta_i \text{ from parameters in } \mathbb{R}^{\mathcal{M}} \cup \{G, \beta_0, ..., \beta_{i-1}\}\},$$

and

$$\xi_i = \sup(Y_i \cap (\kappa^+)^{\mathcal{M}}).$$

Letting

$$\psi_i = \text{ least } \psi \in \Sigma \text{ s.t. } \mathcal{M}||\beta_i \not\models \psi[\langle G, w_1 \rangle],$$

we set

$$\beta_{i+1} = \text{ least } \gamma \text{ s.t. } \mathcal{M}||\gamma \models \psi_i[\langle G, w_1 \rangle] \wedge \dot{F}^{\mathcal{M}||\gamma} \text{ measures all sets } A \in \mathcal{M}|\xi_i.$$

[18]This hypothesis is needed in the other cases not covered by 4.15 as well.

Note that $Y_i \in \mathcal{M}$, so $\xi_i < (\kappa^+)^{\mathcal{M}}$, and ψ_i exists and $\beta_{i+1} < \beta$, for all i.

Claim. $\bigcup_{i<\omega} Y_i = \mathcal{M}$.

Proof. Let $\pi\colon N \to \bigcup_i Y_i$ be the transitive collapse map. Let $\vec{E} = \bigcup_i \pi^{-1}(\dot{E}^{\mathcal{M}||\beta_i})$ and $W = \bigcup_i \pi^{-1}(\dot{F}^{\mathcal{M}||\beta_i})$. It is not hard to see that $(N, \in, \vec{E}, W) = \mathfrak{C}_{\mathfrak{o}}(\mathcal{N})$ for some premouse $\mathcal{N}$; note here that our construction insures that W measures all subsets of its critical point which lie in N. Further, $\mathcal{N}$ is $(0, \omega_1+1)$-iterable because we can lift trees on it to trees on $\mathcal{M}$. (Here we need that π is a weak 0-embedding. That is true because π is Σ_1-elementary on the collapses of the β_i, and that in turn is true because each β_i is Σ_1-definable over $\mathfrak{C}_{\mathfrak{o}}(\mathcal{M})$ from $\langle G, w_1\rangle$, so that Σ_1 facts about β_i in $\mathfrak{C}_{\mathfrak{o}}(\mathcal{M})$ get recorded in the type Σ.) Thus $\mathcal{N}$ can be compared with $\mathcal{M}$, and since $\rho_1(\mathcal{N}) = \rho_1(\mathcal{M}) = \mathbb{R}$, and $\mathcal{N}$ realizes Σ, we get $\mathcal{N} = \mathcal{M}$. The $<_b$-minimality of G then implies $\pi^{-1}(G) = G$, and the minimality of the β_i implies $\pi^{-1}(\beta_i) = \beta_i$. From this the claim follows easily. $\square$

It follows that $\sup(\{\beta_i \mid i < \omega\}) = \beta$ and $\sup(\{\xi_i \mid i < \omega\}) = (\kappa^+)^{\mathcal{M}}$.

Let $P \subseteq \mathbb{R}$ be $\mathbf{\Sigma}_1^{\mathfrak{C}_{\mathfrak{o}}(\mathcal{M})}$. Fix a Σ_1 formula φ_0 of $\mathcal{L}^*$ and a parameter b such that

$$P(x) \Leftrightarrow \mathfrak{C}_{\mathfrak{o}}(\mathcal{M}) \models \varphi_0[x, b].$$

By the claim, we may assume

$$b = \langle G, \beta_0, ..., \beta_e, w_2\rangle,$$

for some e and some real w_2. For $e < i < \omega$, let

$$P^i(x) \Leftrightarrow \mathcal{M}||\beta_i \models \varphi_0[x, b].$$

We shall construct closed game representations $i \mapsto G^i_x$ of the P^i in such a way that if

$$P^i_k(x, u) \;\Leftrightarrow\; u \text{ is a winning position for I in } G^i_x \text{ of length } k\ ,$$

then P^i_k is first order definable over $\mathcal{M}||\beta_{\sup(i,k)}$. Such a closed game representation yields scales on each P^i each of whose norms belongs to $\mathcal{M}$, and hence a $\mathbf{\Sigma}_1^{\mathcal{M}}$ scale on P.

In G^i_x, player I describes $\mathfrak{C}_\mathfrak{o}(\mathcal{M})$ as the union of the $\mathcal{M}||\beta_k$. The language $\mathcal{L}$ in which he does this has $\in, =$, and constant symbols $\dot{G}$, and $\dot{\mathcal{M}}_k$, $\dot{\beta}_k$, and $\dot{x}_k$ for all $k < \omega$. If φ is an $\mathcal{L}$-formula involving no constants $\dot{\mathcal{M}}_k$ or $\dot{\beta}_k$ for $k \geq m$, then we say φ has support m. Player I will produce a Σ_0-complete theory in $\mathcal{L}$, restricting himself at move m to Σ_0 sentences with support m. Let B_0 be the collection of Σ_0 formulae of $\mathcal{L}$, and let $n\colon B_0 \overset{1-1}{\to} \omega$ be such that any $\theta \in B_0$ has support $n(\theta)$ and involves no $\dot{x}_k$ for $k \geq n(\theta)$.

A typical run of G^i_x has the form

$$\begin{array}{llllll} \mathrm{I} & T_0, s_0, \eta_0, m_0 & & T_1, s_2, \eta_1, m_1 & & \\ & & & & & \cdots \\ \mathrm{II} & & s_1 & & s_3 & \end{array}$$

where for all k, T_k is a finite set of sentences in B_0, all of which have support k, $s_k \in \mathbb{R}^{<\omega}$, $\eta_k < \omega\beta$, and $m_k \in \omega$. Given such a run of G^i_x, let

$$\langle x_k \mid k < \omega\rangle = \text{ concatenation of } \langle s_k \mid k < \omega\rangle,$$

and

$$T^* = \bigcup_k T_k.$$

Let S_0 be the set of sentences in B_0 which involve no constants of the form $\dot{x}_i$ for $i \notin \{1, 2\}$, and are true in the interpretation under which $\dot{x}_1$ denotes w_1, $\dot{x}_2$ denotes w_2, and $\dot{\beta}_k$ and $\dot{\mathcal{M}}_k$ denote β_k and $\mathcal{M}||\beta_k$ for all $k < \omega$. S_0 will enter as a real parameter in the payoff condition for G^i_x, and hence in the definition of our scale on P. We could avoid this by replacing S_0 with an appropriate finitely axiomatized subtheory, but since real parameters will enter elsewhere, there is no point in doing so. Notice that it is part of S_0 that each $\dot{\mathcal{M}}_k$ is an $\mathcal{L}^*$-structure. For θ any formula of $\mathcal{L}^*$, let $\theta^{\dot{M}_k}$ be the natural B_0-formula expressing that $\dot{\mathcal{M}}_k \models \theta$.

We say that the run of G^i_x displayed above is a win for I iff the following conditions hold:

(1) $x_0 = x, x_1 = w_1$, and $x_2 = w_2$.

(2) T^* is a consistent extension of S_0 such that for all k, m, n, "$\dot{x}_k(n) = m$" $\in T^*$ iff $x_k(n) = m$.

(3) If $\theta \in B_0$ is a sentence, then either $\theta \in T_{n(\theta)}$ or $(\neg\theta) \in T_{n(\theta)}$.

(4) If $\exists v(v \in \mathbb{R} \wedge \sigma) \in T_k$, then for some j, $\sigma(\dot{x_j}) \in T_{k+1}$.

(5) $(\varphi_0(\dot{x_0}, \langle \dot{G}, \dot{\beta}_0, ..., \dot{\beta}_e, \dot{x_2}\rangle))^{\dot{\mathcal{M}}_i} \in T_{i+1}$.

(6) If $\theta(v_1, ..., v_{n+2})$ is an $\mathcal{L}^*$-formula, and $\sigma_1, ..., \sigma_n$ are B_0 formulae of one free variable with support k, and "$\iota v \sigma_m(v) \in \mathrm{OR}$"$\in T^*$ for all $m \leq n$, then

$$\theta^{\dot{\mathcal{M}}_k}(\iota v\sigma_1(v), ..., \iota v\sigma_n(v), \dot{x}_1, \dot{x}_2) \in T^* \Leftrightarrow \mathcal{M}||\beta_k \models \theta[\eta_{n(\sigma_1)}, ..., \eta_{n(\sigma_n)}, w_1, w_2].$$

(8) If $(\iota v\sigma(v) <_b \dot{G}) \in T_k$, then either

   (a) there is a Π_1 formula $\varphi \in \Sigma$ such that

$$(\neg\varphi(\langle \iota v\sigma(v), \dot{w_1}\rangle))^{\dot{\mathcal{M}_{m_k}}} \in T_{m_k+1},$$

   or

   (b) there is a Σ_1 formula φ which is one of the first m_k elements of Σ such that for all j and l,

$$(\varphi(\langle \iota v\sigma(v), \dot{w_1}\rangle))^{\dot{\mathcal{M}}_l} \notin T_j.$$

This completes the description of the payoff set for I in G^i_x. We now show that I wins G^i_x iff $\mathcal{M}||\beta_i \models \varphi_0[x, b]$, and that the P^i_k are appropriately definable. Both claims follow from the fact that $P^i_k(x, u)$ iff u is honest. Honesty is defined as follows: let I_u be the interpretation of $\mathcal{L}$ under which $\dot{x_j}$ denotes x_j whenever x_j is the j^{th} real determined by u, and $\dot{G}$, $\dot{\beta}_k$ and $\dot{\mathcal{M}}_k$ denote G, β_k, and $\mathcal{M}||\beta_k$ for all k. For u a position in G^i_x, we say u is *x-honest* iff

(i) $T^*(u)$ is true in $(|\mathcal{M}|, \in, I_u)$,

(ii) $\mathcal{M}||\beta_i \models \varphi_0[x, b]$,

(iii) $x_0 = x, x_1 = w_1$,and $x_2 = w_2$, if u determines x_0, x_1, and x_2.

(iv) the commitments represented by the m_k can be kept,

(v) if $\sigma_0, ..., \sigma_n$ enumerates those B_0-formulae σ of one free variable such that $n(\sigma) \in \mathrm{dom}(u)$ and

$$(|\mathcal{M}|, \in, I_u) \models \iota v \sigma(v) \in \mathrm{OR},$$

and if $\delta_m < o(\mathcal{M})$ is such that

$$(|\mathcal{M}|, \in, I_u) \models \iota v \sigma_m(v) = \delta_m,$$

for all $m \leq n$, then the map

$$\delta_m \mapsto \eta_{n(\sigma_m)}$$

is well-defined and extendible to an order preserving map $\pi \colon o(\mathcal{M}) \to o(\mathcal{M})$ such that for all k, all formulae θ of $\mathcal{L}^*$, and all tuples $\bar{\gamma}$ of ordinals from $\mathcal{M}||\beta_k$, $\pi \restriction \mathcal{M}||\beta_k \subseteq \mathcal{M}||\beta_k$ and

$$\mathcal{M}||\beta_k \models \theta[\bar{\gamma}, w_1, w_2] \Leftrightarrow \mathcal{M}||\beta_k \models \theta[\pi(\bar{\gamma}), w_1, w_2].$$

Claim 1. For any position u of G^i_x, I wins G^i_x from u iff u is x-honest.

Proof. If u is x-honest, then I can win G^i_x from u by continuing to tell the truth, while using the map π given by condition (v) to play further η's.

Now suppose I wins G^i_x from u. Let p be a run of G^i_x by such a strategy, with $u \subseteq p$, such that the associated sequence of reals $\langle x_k \mid k < \omega \rangle$ enumerates $\mathbb{R}^{\mathcal{M}}$. Let $T^* = T^*(p)$ be the B_0-theory played by I. Let $\mathcal{A}$ be the unique model of T^* which is pointwise definable from parameters in $\mathbb{R}^{\mathcal{M}}$. (There is such a model by rule (4).) By rule (6) of G^i_x, $\mathcal{A}$ is wellfounded, and so we assume it is transitive. Let

$$\dot{\mathcal{M}}^{\mathcal{A}}_k = (\mathcal{N}_k, F^*_k),$$

$$\beta^*_k = \dot{\beta}^{\mathcal{A}}_k,$$

and

$$G^* = \dot{G}^{\mathcal{A}}.$$

Since $S_0 \subseteq T^*$, the $\mathcal{N}_k$ are premice, and $\mathcal{N}_k \trianglelefteq \mathcal{N}_{k+1}$ for all k. Let $\mathcal{N}$ be the union of the $\mathcal{N}_k$, and F^* the union of the F^*_k. We can define $\pi \colon o(\mathcal{N}) \to o(\mathcal{N})$ by

$$\pi(\iota v \sigma(v)^{\mathcal{A}}) = \eta_{n(\sigma)},$$

for all B_0-formulae σ such that $\iota v\sigma(v) \in \mathrm{OR}$ is in T^*. Clearly, π is well-defined, and for any tuple $\bar{\gamma}$ of ordinals from $\mathcal{N}_k$ and any formula θ of $\mathcal{L}^*$,

$$(\mathcal{N}_k, F_k^*) \models \theta[\bar{\gamma}, w_1, w_2] \Leftrightarrow \mathcal{M}||\beta_k \models \theta[\pi(\bar{\gamma}), w_1, w_2].$$

As in the proof of 4.1, this implies that $\mathcal{N}$ is countably iterable, and that π extends to an embedding, which we also call π, such that

$$\pi\colon \mathcal{H}^{\mathcal{N}} \to \mathcal{H}^{\mathcal{M}}$$

is Σ_1-elementary (in $\mathcal{L}^*$).

Because S_0 is true in $\mathcal{A}$, F_{k+1}^* measures all subsets of its critical point in $\mathcal{H}_1^{(\mathcal{N}_k, F_k^*)}(\mathbb{R}^{\mathcal{M}} \cup \{\beta_0^*, ...\beta_{k-1}^*, G^*\})$. But the union of these hulls has the same universe as $\mathcal{N}$, and thus F^* is an extender over $\mathcal{N}$. Similarly, we get

$$F_k^* \cap \mathcal{H}^{\mathcal{N}_k} \in \mathcal{H}^{\mathcal{N}_{k+1}}$$

for all k, and because π is sufficiently elementary,

$$\pi(F_k^* \cap \mathcal{H}^{\mathcal{N}_k}) = F \cap \mathcal{H}^{\mathcal{M}|\beta_k}.$$

Letting $E^* = F^* \cap \mathcal{H}^{\mathcal{N}}$ and $E = F \cap \mathcal{H}^{\mathcal{M}}$, we then have that

$$\pi\colon (\mathcal{H}^{\mathcal{N}}, E^*) \xrightarrow{\Sigma_1} \mathcal{H}^{\mathcal{M}}$$

is Σ_1 elementary. It follows that $(\mathcal{H}^{\mathcal{N}}, E^*)$ is a countably iterable premouse, and hence that $(\mathcal{N}, F^*)$ is a countably iterable premouse.

It is part of S_0 that our non-reflecting type Σ is realized for the first time, and thus we have $(\mathcal{N}, F^*) = \mathcal{M}$. Because I has kept the commitments he made according to rule (7) of G_x^i, we have $G^* = G$, and $\beta_k^* = \beta_k$ and $(\mathcal{N}_k, F_k^*) = \mathcal{M}||\beta_k$ for all k. It is now easy to verify that u was x-honest; the map π witnesses that condition (v) of x-honesty is met. □

Claim 2. Let $k \leq i$; then $\{u \mid u \text{ is an } x\text{-honest position of length } k\}$ is a member of $\mathcal{M}|\beta$.

Proof. It is clear that the set of u satisfying conditions (i)-(iii) of x-honesty is definable over $\mathcal{M}||\beta_k$, and hence in $\mathcal{M}|\beta$. The Coding Lemma argument of [7] shows that the set of u satisfying condition (iv) is also in $\mathcal{M}|\beta$. Here, as in [7], we can apply the Coding Lemma to sets belonging to $\mathcal{M}|\beta$, so we don't actually need determinacy beyond the sets in $\mathcal{M}|\beta$.

For (v), let

$$s = (\iota v\sigma(v)^{\mathcal{M}||\beta_k} \mapsto \eta_{n(\sigma)})$$

be a finite map coded into a position u of length k satisfying (i)-(iv). Note that $\mathrm{dom}(s) \subseteq Y_k$, so that if s can be extended to a π as demanded in (v), then as Y_k is Σ_1-definable over $\mathcal{M}||\beta_{k+1}$ from $\beta_0, ..., \beta_{k-1}, G$, $\mathrm{ran}(s) \subseteq Y_k$ as well. (Note here that by the proof of Claim 1, π must fix G and the β_m for $m < \omega$.) So if we let

$$Z_k = \{t \colon Y_k \to Y_k \mid |t| < \omega \wedge \exists \pi \supseteq t(\pi \text{ is as in (v) }\},$$

then it suffices to show that Z_k is definable over $\mathcal{M}||\beta_{\sup(i,k)}$.

We proceed as in the proof of Claim 1 of 4.1. $\mathcal{M}$. Note that $t \in Z_k$ iff the closed player has a winning strategy in a certain "embedding game" on $\theta^{\mathcal{M}|\beta}$. We claim that if the closed player wins the embedding game in V, then he wins it in $\mathcal{M}$. (The converse is obvious.) So suppose the closed player wins the embedding game in V. Let $A \in \mathcal{M}$ be a set of ordinals which codes up the payoff of game; since $\theta^{\mathcal{M}|\beta} < \theta^{\mathcal{M}}$, and $\mathcal{M} \models \mathsf{AD}$, we can find a model N of ZFC such that $A \in N$, and $(N, \in)$ is coded by a set of reals $B_N \in \mathcal{M}$. (E.g., let $N = L_\alpha[A]$, where α is the supremum of the order types of the $\Delta_n^{\mathcal{M}|\beta}$ prewellorders of $\mathbb{R}^{\mathcal{M}}$, for an appropriate n.) Since $N \models \mathsf{ZFC}$, the closed player wins the embedding game via a strategy $\Sigma \in N$. For $\gamma < \theta^{\mathcal{M}|\beta}$, let $f(\gamma) = \{z \in \mathbb{R}^{\mathcal{M}} \mid z \text{ codes } \gamma \text{ via } B_N\}$. We can arrange that $f \in \mathcal{M}$, and use f to show that $\Sigma \in \mathcal{M}$.

The argument of the last paragraph actually shows that there is a fixed $n < \omega$ such that for all t, $t \in Z_k$ iff II has a $\boldsymbol{\Delta}_{\mathbf{n}}^{\mathcal{M}|\beta}$ winning strategy in the embedding game associated to t. It follows that

$$Z_k \in \mathcal{M}.$$

But $\mathcal{M} \models \mathsf{AD}$, and Z_k can be identified with a bounded subset of $\theta^{\mathcal{M}|\beta}$, since $Y_k \in \mathcal{M}|\beta$ and is the surjective image of $\mathbb{R}$ by a map in $\mathcal{M}|\beta$. It follows from the Coding Lemma that $Z_k \in \mathcal{M}|\beta$, and in fact Z_k is definable over $\mathcal{M}||\beta_{\sup(i,k)}$.[19] □

[19] All we really need to get the desired scale is that $Z_k \in \mathcal{M}|\beta$.

Claim 2 completes the proof of 4.14. $\square$

As the reader can see, we use the determinacy of sets *definable over* $\mathcal{M}|\beta$ in the proof of Claim 2 above. The determinacy of sets belonging to $\mathcal{M}|\beta$ is not enough for the proof because the payoff set A for the embedding game may not be a member of $\mathcal{M}|\beta$. One can get by with the determinacy of sets in $\mathcal{M}|\beta$ in the proof of 4.15 because in that case, the only "global" role of the ordinals played by I in G^i_x is to verify that the model he is playing is wellfounded. This aspect of honesty can be explicitly defined; I needs only to have spaced his ordinals adequately. See [7] for the details. It is still true that I will have to verify that the $\mathcal{H}^{\mathcal{A}}_\nu$ for $\nu < o(\mathcal{A})$ are iterable, by embedding them into a corresponding $\mathcal{H}^{\mathcal{M}|\beta}_\mu$, but these embeddings no longer need to fit together into a single embedding, and thus this aspect of honesty does not lead out of $\mathcal{M}|\beta$. We leave the further details of the proof of 4.15 to the reader.

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