

Remarks on a hod mouse construction

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August 2014

We refine some arguments in [2]. Let $j: V \rightarrow M$ witness that $K_0 = \text{crit}(j)$ is measurable. Suppose that K_0 is a limit of Woodins. Let $K_1 = j(K_0)$, and let h be \mathbb{R} -generic over $\text{Col}(\omega, < K_1)$, and let $g = h \restriction \text{Col}(\omega, < K_0)$. So we have $\hat{j}: V[g] \rightarrow M[h]$ extending j . Let

$$\Gamma \subseteq \text{Hom}_g^*$$

be a pointclass such that $\Gamma = P(\mathbb{R}_g^*) \cap L(\Gamma, \mathbb{R}_g^*)$ and

$$L(\Gamma, \mathbb{R}_g^*) \models \text{AD}_{\mathbb{R}}.$$

Suppose that $\text{HOD}^{L(\Gamma, \mathbb{R}_g^*)}$ can be analyzed as a hod mouse by current methods — e.g.,

suppose there is no $\Omega \subseteq \Gamma$ such that $L(\Omega, \mathbb{R}_g^*) \models LST$, (LST = LSA = "the largest Suslin is a Θ_α "), so that work of Sargsyan, and to some extent the author, does this. Let

$$H_0 = \text{HOD}^{L(\Gamma, \mathbb{R}_g^*)}, \Theta^{L(\Gamma, \mathbb{R}_g^*)}$$

and

$$\Theta^0 = \Theta^{L(\Gamma, \mathbb{R}_g^*)} = o(H_0).$$

Set

$$H_0^+ = \left(L_p^{\oplus_{\alpha < \Theta^0} \Sigma_{H_0(\alpha)}} (H_0) \right)^{\hat{j}(\Gamma)}$$

the stack of all appropriate strategy mice with iteration strategies in $\hat{j}(\Gamma)$ over H_0 .

We assume

$$o(H_0^+) < \kappa_0^+$$

Our goal is to construct, in $M[H]$, a hod pair $(\mathcal{P}, \Sigma_{\mathcal{H}})$ extending (H_0^+, Σ_{H_0}) such that either $\text{Nbr} \mathcal{H}$ reaches $O_k^{\mathbb{P}}$, or $(\mathcal{P}, \Sigma_{\mathcal{H}})$ generates $\hat{j}(\Gamma)$.

This is done in [22], but under stronger assumptions on j . We show in this note that the construction of [22] works under the weaker assumptions we have here.

\mathcal{H} is constructed by induction on its full levels (i.e. strategy-activation levels), and the construction is actually done in \mathcal{M} . We adopt the notation of [21]. So $\overset{\circ}{\Gamma}$ is a canonical name for Γ , and we are constructing in \mathcal{M} pairs $(\mathcal{N}_\xi, \overset{\circ}{\Psi}_\xi)$, ~~start~~ starting with

$$\mathcal{N}_0 = H_0^+$$

$$\overset{\circ}{\Psi}_0 = \text{canonical name for } \Sigma_{H_0} = \bigoplus_{\alpha < \theta^0} \Sigma_{H_0(\alpha)}, \text{ extended to } \mathcal{M}[\mathcal{L}].$$

We have $\underset{\perp}{\text{col}(\omega, \kappa)} \overset{\circ}{\Psi}_0 \in j(\overset{\circ}{F})$.

We maintain by induction; for $\xi < \kappa_1$

④

Induction hypotheses $(†)_\xi$ In \mathcal{M} , the following hold, for $(\mathcal{N}, \dot{\Psi}) = (\mathcal{N}_\xi, \dot{\Psi}_\xi)$:

- (a) $\text{Col}(\omega, \leq \kappa_1)$ $(\check{\mathcal{N}}, \dot{\Psi})$ is a $j(\check{\Gamma})$ hod pair such that $\dot{\Psi}$ has branch condensation ^{and is positional} \triangleright with top block beginning at θ° ,
- (b) $\text{Col}(\omega, \leq \kappa_1)$ $\forall \nu < \xi$ $(\mathcal{N}_\nu, \dot{\Psi}_\nu)$ is a full initial segment of $(\check{\mathcal{N}}, \dot{\Psi})$,
- (c) for m and ℓ on $\text{Col}(\omega, \leq \kappa_1)$ with $\mathcal{M} \restriction \ell \supseteq \mathcal{M} \restriction m$, we have $\dot{\Psi}^m = \dot{\Psi}^\ell$,
- (d) $\text{Col}(\omega, \leq \kappa_1)$ $(\mathcal{N}, \dot{\Psi})$ is $j \upharpoonright H_0^+$ -consistent,
- (e) if E is a θ° -relevant extender in the top block of \mathcal{N} , then $\text{crit}(E) = \theta^\circ$, and $\text{Col}(\omega, \leq \kappa_1)$ \check{E} is $j \upharpoonright H_0^+$ -certified over $(\mathcal{N} \parallel \text{lh} E, \dot{\Psi}_{\mathcal{N} \parallel \text{lh} E})$,
- (f) $\text{Col}(\omega, \leq \kappa_1)$ $\dot{\Psi} \in j(\check{\Gamma})$

Remarks

(i) The first part of (e) says that the top block of \mathcal{N} is below O_h^{IP} .

(ii) The construction continues until we reach a first ξ such that (f) $_{\xi}$ fails.

At that point, we shall see that (f) $_{\xi}$ (a) - (e) hold, and that for h on

$col(\omega, \kappa_1)$, $\dot{\psi}^h \in Hom_h^* - \hat{J}^n(\Gamma)$.

This implies $\dot{\psi}^h$ is a pointclass generator for $\hat{J}^n(\Gamma)$, i.e. $\Gamma(\mathcal{N}_{\xi}, \dot{\psi}^h) = \hat{J}^n(\Gamma)$.

(iii) if the construction does not stop for reason (ii), then $\bigcup_{\xi \in \kappa_1} \mathcal{N}_{\xi}$ can be expanded to a hod premouse of O_h^{IP} type.

(iv) Notice that (a) does not imply (f). From (a) we only get that if \mathcal{M} is a $\dot{\psi}^h$ iterate of \mathcal{N} , and $\mathcal{M}(\alpha)$ is a full proper initial segment of \mathcal{M} , then $\dot{\psi}_{\mathcal{M}(\alpha)}^h \in \hat{J}^n(\Gamma)$.

(iv) From (a) we only get that \vec{T} is a stack of trees by ψ^h , and m is a full proper initial segment of the last model of \vec{T} , then $\psi_{\vec{T}, m}^h \in \hat{f}(T)$.

So (a) does not imply (f).

(v) Saegsyar's theorem 2.41 of [15] shows that if (a) and (f) hold, then ψ^h is positional and commuting. The pointclass generator we get in case (ii) above will also be positional and commuting. [15] shows that if Ω for R is positional and has branch condensation, then it is commuting.

(vi) Clause (e) is explained in 22J.

Clause (d) we explain now. It is a substitute for ~~what~~

$$(d') \quad \underbrace{\text{Col}(\omega, \kappa_1)} \quad \dot{\Psi} = j(\dot{\Psi})^{\dot{d}},$$

which was assumed in 22J. The trouble with (d') is that the pullback strategy $j(\dot{\Psi})^{\dot{d}}$ does not make (immediate) sense unless $j \upharpoonright \mathcal{M} \in \mathcal{M}$. Our assumption that $o(H_0^+) \leq \kappa_0^+$ implies $j \upharpoonright H_0^+ \in \mathcal{M}$, so (e) makes sense, and so too will (d) when we get to defining it. But if $\xi \geq \kappa_0^+$, (d') does not make sense unless j witnesses κ_0^+ -supercompactness.

Notation Recall that if R is a hod premouse with top block, then K^R begins with that block. We also set $R_{bt} = R \upharpoonright (K^R)^{+R}$, and call it the "bottom part" of R .

(6)

Definition 4. Let (R, Ω) be a $\hat{j}(\Gamma)$ hod pair, with $\Omega \in \hat{j}(\Gamma)$ and Ω having branch condensation. Let

$$k: R_{bt} \longrightarrow H_1^+$$

be cofinal and Σ_0 elementary; then Ω is k -consistent iff there is a cofinal, Σ_0 elementary

$$\sigma: M_\infty(R, \Omega)_{bt} \longrightarrow H_1^+$$

such that

$$k = \sigma \circ \left(\prod_{R, \infty}^{\Omega} \upharpoonright R_{bt} \right)$$

and

$$\text{crit}(\sigma) = K^{M_\infty(R, \Omega)}$$

In this definition, the direct limit system giving rise to $M_\infty(R, \Omega)$ and $\prod_{R, \infty}^{\Omega}$ is that of $L(\hat{j}(\Gamma), \mathbb{R}_h^*)$. We have assumed that $\Omega \in \hat{j}(\Gamma)$, so $M_\infty(R, \Omega)$

is a proper initial segment of H_1^+ .

If Ω is a k -consistent strategy for R , then $\pi_{R, \infty}^\Omega \upharpoonright KR = k \upharpoonright KR$. Thus Ω is locally k -consistent in the sense of [2]. What was called k -consistency in [2] was essentially equivalent to local k -consistency, and thus weaker than what we shall henceforth mean by k -consistency.

Proposition 2 Let (R, Ω) be a $\hat{J}(\Gamma)$ hod pair, Ω have branch condensation, and $\Omega \in \hat{J}(\Gamma)$. Let $k: R_{bt} \rightarrow H_1^+$. Equivalences are:

- (1) Ω is k -consistent
- (2) $\pi_{R, \infty}^\Omega$ and k generate the same extender of length $K^{M_{\infty}(R, \Omega)}$ over R_{bt}
- (3) for all $A \subseteq KR$ in R , $\pi_{R, \infty}^\Omega(A) = k(A) \cap K^{M_{\infty}(R, \Omega)}$,
- (4) if $i: R \rightarrow S$ is by Ω , then $E_i \upharpoonright K^S = E_k \upharpoonright \pi_{S, \infty}^\Omega \upharpoonright K^S$
- (5) if $A \subseteq KR$, $A \in R$, and $i: R \rightarrow S$ by Ω , then $\pi_{S, \infty}^\Omega \upharpoonright i(A) = k(A) \cap \pi_{S, \infty}^\Omega \upharpoonright K^S$.

(8)

Proof (3) is just a re-statement of (2), and (5) is a re-statement of (4).
 The ~~other~~ equivalences of (3) with (5) is easy, as is the equivalence of (1) with (2).

□

Notice that $\overline{\pi}_{R, \infty}^{\Omega}$ is given completely by its extender of length $K^{M_{\infty}(R, \Omega)}$ that is, $M_{\infty}(R, \Omega) = \text{Ult}_0(R, \overline{\pi}_{R, \infty}^{\Omega} \uparrow K^{M_{\infty}(R, \Omega)})$.

Notice also that the factor map σ of definition I is uniquely determined by k and Ω ; σ is just the factor map from $\text{Ult}_0(R_{bt}, \overline{E}_k \uparrow K^{M_{\infty}(R, \Omega)})$ into $H_1^+ = \text{Ult}_0(R_{bt}, \overline{E}_k)$.

σ is determined by k and $k^{M_\infty}(R, \Omega)$. (9)

We stress that it is $M_\infty(R, \Omega)_{bt}$ and not the ^{potentially} smaller $M_\infty(R_{bt}, \Omega_{R_{bt}})$, that is relevant in definition 1. In general, if $k: R_{bt} \rightarrow H_1^+$ and Ω for R is k -consistent, and P is a full initial segment of R with $k^P = k^R$, then Ω_P is k -consistent too, this is because there is a natural

$$\psi: M_\infty(P, \Omega_P) \rightarrow \pi_{R, \infty}^\Omega(P)$$

such that $\pi_{R, \infty}^\Omega(P) = \psi \circ \pi_{P, \infty}^{\Omega_P}$, and

$\text{crit}(\psi) \cong k^{M_\infty(P, \Omega_P)}$. So if σ witnesses k -consistency for Ω , $\sigma \circ \psi$ witnesses k -consistency for Ω_P . But there is no reason k -consistency for Ω_P should imply k -consistency for Ω .

$$k \uparrow (R_{bt} | \xi) = \sigma_{\xi} \circ \pi_{R, \infty}^{\Omega} \uparrow (R_{bt} | \xi). \text{ Taking}$$

$\sigma = \bigcup_{\xi} \sigma_{\xi}$, we get the required factor map?

(9)

(10)

Tails of a k -consistent strategy are consistent with respect to the appropriate "extension" of k :

Proposition 3. Let (R, Ω) be a $\hat{j}(\Gamma)$ -hod pair in MCHJ. Let $k: R_{bt} \rightarrow H_1^+$ be such that Ω is k -consistent. Let $i: R \rightarrow S$ be an iteration by Ω . Then there is a unique $\tau: S_{bt} \rightarrow H_1^+$ such that

$$\begin{array}{ccc} R_{bt} & \xrightarrow{k} & H_1^+ \\ & \searrow i & \nearrow \tau \\ & S_{bt} & \end{array}$$

commutes, and $\tau \uparrow K^S = \pi_{S, \infty}^{\Omega_S} \uparrow K^S$.

Moreover, Ω_S is τ -consistent.

Proof Just let

$$\gamma(i(f)(a)) = k(f)(\pi_{S,\infty}(a)),$$

for all $a \in [K^S]^{<\omega}$. If $\sigma: M_\infty(R, \Omega)_{bt} \rightarrow H_1^+$ witnesses k -consistency for Ω , then σ also witnesses γ -consistency for Ω_S .

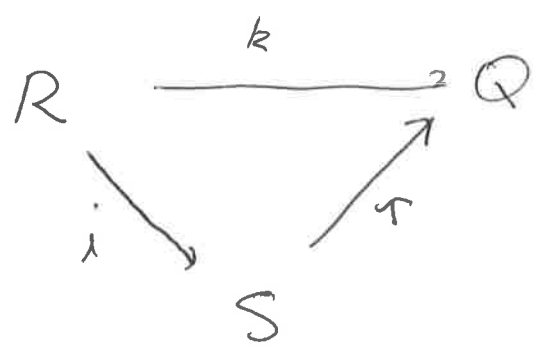


The next proposition concerns extensions of our embeddings to the whole top block.

Proposition 4. Let (R, Ω) be a $j(\Gamma)$ -hod pair, and $k: R \rightarrow Q$ where $H_1^+ \trianglelefteq Q$, and suppose Ω is $k \upharpoonright R_{bt}$ -consistent.

Let $i: R \rightarrow S$ be by Ω , or let $S = M_\infty(R, \Omega)$ and $i = \pi_{R,\infty}^\Omega$. Then

there is a unique $\tau: S \rightarrow Q$ such that



commutes, and $\tau \circ k^S = \pi_{S, \infty}^{\Omega_S} \circ k^S$ if $S \neq M_\infty(R, \Omega)$, while $\tau \circ k^S = id$ if $S = M_\infty(R, \Omega)$. In the former case, Ω_S is $\tau \circ S_{be}$ -consistent.

Proof Suppose first $S = M_\infty(R, \Omega)$.
 Let $S_0 = Ult_0(R, E_{\pi_{R, \infty}} \circ k^{M_\infty(R, \Omega)})$
 $= Ult_0(R, E_k \circ k^{M_\infty(R, \Omega)})$. Let

$$\sigma : Ult_0(R, E_k \circ k^{M_\infty(R, \Omega)}) \rightarrow Ult_0(R, E_k)$$

be the factor map. So $\sigma : M_\infty(R, \Omega) \rightarrow Q$,
 and $\sigma \circ k^{M_\infty(R, \Omega)} = id$, and $k = \sigma \circ i$.
 Thus σ has the properties required of τ .
 If instead, S is a point in the

direct limit system leading to $M_\infty(R, \Omega)$, we can just let

$$\gamma = \sigma \circ \pi_{S, \infty}^{\Omega_S},$$

where $\sigma: M_\infty(R, \Omega) \rightarrow \mathcal{Q}$ is as above.

Remark See p. 12b.



The following uniqueness theorem is one of the main points. It strengthens the uniqueness-of-pullback lemma 2.17.1 from [2], Some form of it was also known to Sangsyan. The ~~new~~ proof traces back to Sangsyan's earlier work.

Theorem 5: (Uniqueness of k -consistent strategies)
Let $k: R_{bt} \rightarrow H_1^+$, and let Φ and $\Psi \in \hat{\mathcal{J}}(T)$ be k -consistent strategies for R (so both have branch condensation, are positional, and are $\hat{\mathcal{J}}(T)$ -fullness preserving); then $\Phi = \Psi$.

Remark Clause (e) of (†), actually follows from the others. For let $\Sigma = \Psi_\xi^{\circ h}$, and let E be on the π -sequence with $\text{crit}(E) = \theta^\circ$. Let $S = \text{Ult}(\pi, E)$.

Since Σ was $j \upharpoonright H_0^+$ -consistent, proposition 3 gives that the tail Σ_S is τ -consistent, for $\tau: S_{bt} \rightarrow H_1^+$

given by $\tau(i_E(f)(a)) = j(f)(\pi_{S, \infty}^{\Sigma_S}(a))$.

But $\pi_{S, \infty}^{\Sigma_S}(a) = \pi_{\text{Ult}(\pi, E), \infty}^{\Sigma}(a)$ for $a \in \text{lh } E^{< \omega}$

by strategy coherence. Thus $a \in i_E(A)$ iff $\pi_{\text{Ult}(\pi, E), \infty}^{\Sigma}(a) \in j(A)$.

This remark is more motivational than useful, because when we put E onto the π sequence, we'll want to know it is $j \upharpoonright H_0^+$ -certified ^{by the strategy of $\text{Ult}(\pi, E)$} before we construct a $j \upharpoonright H_0^+$ -consistent strategy for $(\text{Ult}(\pi, E), E)$.

Proof. This is essentially shown in 2.17 of [2]. Here is the argument again.

Suppose $\Phi \neq \Psi$. We can find a stack of trees \vec{T} on R with last model W such that \vec{T} is by both strategies, and a normal tree U on W such that

letting
$$b = \Phi(\vec{T} \frown U)$$

and
$$c = \Psi(\vec{T} \frown U),$$

we have $b \neq c$. We choose this disagreement so as to minimize the minimum of the Wadge ranks of the tails of Φ and Ψ acting on $M(U)$. This gives

$$\Phi_{\vec{T} \frown U, M(U)} = \Psi_{\vec{T} \frown U, M(U)}. \quad (M(U) \text{ is a limit of full levels in } M_b^U \text{ and } M_c^U, \text{ not itself full, so these are really join-of-strategies.})$$

Letting

$$\Lambda = \bar{\Phi}_{\vec{J}^{\mathcal{U}}, M(\mathcal{U})} = \bar{\Psi}_{\vec{J}^{\mathcal{U}}, M(\mathcal{U})},$$

we have

$$L_p^{\mathcal{U}}(M(\mathcal{U})) \leq M_b^{\mathcal{U}}, M_c^{\mathcal{U}}$$

and

$$L_p^{\mathcal{U}}(M(\mathcal{U})) \neq \delta(\mathcal{U}) \text{ is Woodin.}$$

(~~Let~~ So $M_b^{\mathcal{U}} \neq \delta(\mathcal{U})$ is Woodin, as does $M_c^{\mathcal{U}}$.) Let

$$\eta = \sup \{ \xi < \delta(\mathcal{U}) \mid M(\mathcal{U}) \neq \xi \text{ is Woodin} \}:$$

Since we are below LST^- , $\eta < \delta(\mathcal{U})$,

moreover η is a cutpoint of $M(\mathcal{U})$.

We may assume that all critical points in \mathcal{U} are above η . Letting

$$\delta = \text{least Woodin of } W > \eta,$$

we then have

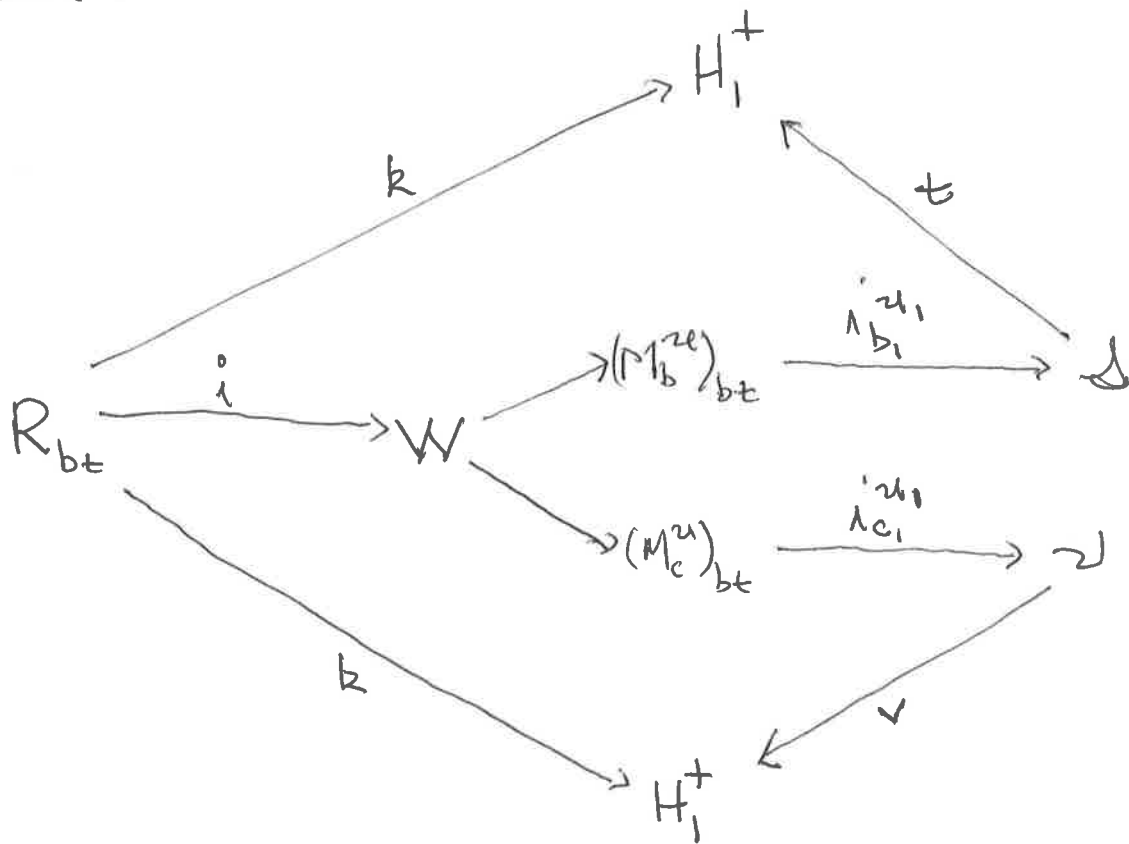
(15)

$$i_b'(\delta) = i_c'(\delta) = \delta(\mathcal{U})$$

Finally, looking at how "successor Woodins" can be produced by iteration, we see that

$$\delta = \sup \{ i_{\vec{\sigma}}(f)(a) \mid a \in \mathcal{U} \}^{\mathcal{W}} \text{ and } f \in R_{bt} \}$$

The rest of the proof is in the following diagram. Set $i = i_{\vec{\sigma}}$



Let

(16)

$$P = L_p^{\wedge} (M(\mathcal{U})).$$

We may have that $\Phi_{\mathcal{U}, P} \neq \Psi_{\mathcal{U}, P}$.

But we can compare these two strategies via a single normal tree \mathcal{U}_1 that is above η , with branches b_1 and c_1 chosen by Φ and Ψ . We get that

$$\lambda_{b_1}^{\mathcal{U}_1}(P) = \lambda_{c_1}^{\mathcal{U}_1}(P),$$

and

$$\Phi_{\mathcal{U} \wedge \mathcal{U}_1, P} = \Psi_{\mathcal{U} \wedge \mathcal{U}_1, P}.$$

Letting $\mathcal{A} = (M_{b_1}^{\mathcal{U}_1})_{bt}$ and $\mathcal{B} = (M_{c_1}^{\mathcal{U}_1})_{bt}$,

the maps $u: \mathcal{A} \rightarrow H_1^+$ and

$v: \mathcal{B} \rightarrow H_1^+$ witness k -consistency,

as in proposition 3.

But then we got, for any $a \in \mathbb{Z}_q J^{sw}$, (17)

$$\tau(i_{b_1} \circ i_b \circ i^{\vec{f}}(f)(a))$$

$$= k(f)(\pi_{\delta, \infty}(a))$$

$$= \mathcal{K}(i_{c_1} \circ i_c \circ i^{\vec{f}}(f)(a)),$$

for all $f \in R_{bt}$. If $i^{\vec{f}}(f)(a) < \delta$,

then $i_{b_1} \circ i_b \circ i^{\vec{f}}(f)(a) < \delta(u_1)$. But

$$u \uparrow \delta(u_1) = v \uparrow \delta(u_1)$$

because we have lined up $\Phi_{\vec{f} \uparrow u \uparrow u, \uparrow b_1}$
with $\Psi_{\vec{f} \uparrow u \uparrow c \uparrow u, \uparrow c_1}$. So

$$i_{b_1} \circ i_b \circ i^{\vec{f}}(f)(a)$$

$$= i_{c_1} \circ i_c \circ i^{\vec{f}}(f)(a)$$

for these f and a . Since there are
continually many points of this form

below $\delta(U_1)$, $b_1 = c_1$. But then $i_b \circ i_a^{-1}(f)(a) = i_c \circ i_a^{-1}(f)(a)$ for those f and a , so $b = c$. Contradiction.

(18)



The same proof gives the following generalization, used by Sargsyan in [3] in the case $R = R_{bt}$.

Theorem 6 (Uniqueness of \mathcal{F} -consistent strategies)

Let (R, Ω_0) and (R, Ω_1) be $\hat{\delta}(\Gamma)$ hod pairs ^{in $\hat{\delta}(\Gamma)$} such that Ω_i has branch condensation and is positional, for $i = 0, 1$. Let $k_i : R_{bt} \rightarrow H_{1,1}^+$ and suppose Ω_i is k_i -consistent, for $i = 0, 1$. Suppose that for

$$\mathcal{F} = \text{ran}(k_0) \cap \text{ran}(k_1)$$

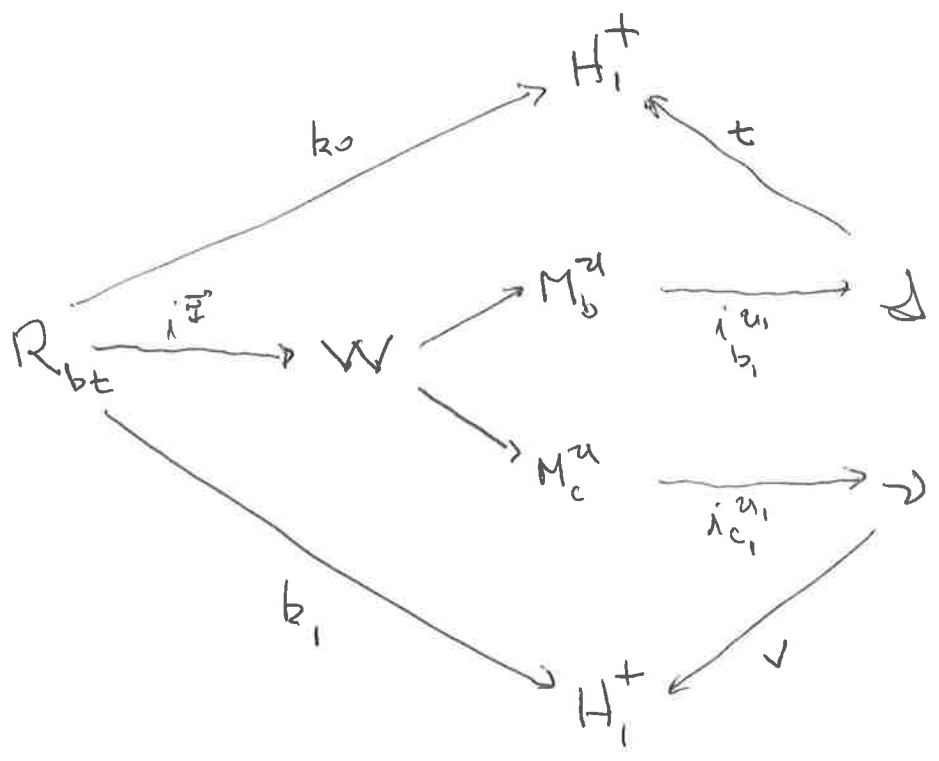
we have for $i = 0, 1$:

- (a) $R_{bt} = \{f(a) \mid a \in [K^R]^{<\omega} \wedge k_i(f) \in \mathcal{F}\}$,
- (b) if $R_{bt} \neq \delta$ is Woodin, then $\exists \eta < \delta$ s.t. $\sup(\{f(a) \mid a \in [\eta]^{<\omega} \wedge k_i(f) \in \mathcal{F}\} \cap \delta) = \delta$.

Then $\Omega_0 = \Omega_1$.

Proof sketch

The proof here is the same. We get



where \vec{J} is by both ~~the~~ Ω_0 and Ω_1 , as before. Let $\mathcal{L}_i = k_i^{-1} \vec{F}$. Using our notation above, (a) and (b) of theorem 6 propagate to W , so points of the form $i_{\vec{F}}(f)(a)$ for $a \in \mathbb{Z}\eta J^{<w}$ and $f \in \mathcal{L}_i$ are critical in δ , for $i=0,1$.

For $f \in \mathcal{L}_0$ and $a \in \mathbb{Z}\eta J^{<w}$, we can write

$$\begin{aligned} & \in (i_{b_1} \circ i_b \circ i^{\vec{d}}(f)(a)) \\ & = k_0(f)(\pi_{\mathcal{D}, \mathcal{A}}(a)) \end{aligned}$$

But $\pi_{\mathcal{D}, \mathcal{A}}(a) = \pi_{\mathcal{V}, \mathcal{A}}(a)$, and moreover we have g s.t. $k_1(g) = k_0(f)$. So

$$\begin{aligned} k_0(f)(\pi_{\mathcal{D}, \mathcal{A}}(a)) &= k_1(g)(\pi_{\mathcal{V}, \mathcal{A}}(a)) \\ &= v(i_{c_1} \circ i_c \circ i^{\vec{d}}(g)(a)) \end{aligned}$$

So $i_{b_1} \circ i_b \circ i^{\vec{d}}(f)(a) = i_{c_1} \circ i_c \circ i^{\vec{d}}(g)(a)$.

This gives $b_1 = c_1$, and then $b = c$, as before.



If $j \in \mathcal{N} \in \mathcal{M}$, then $j(\Sigma)^{\vec{d}} \cong \Sigma$,

~~is consistent~~ as we shall show, ~~and it is~~ now.

~~$j(\Sigma)^{\vec{d}} = \Sigma$ so (d) of (f) follows from~~

so our new (f)_ε(d) implies the old one,

when $j \uparrow \pi \in M$.

Lemma 7 Assume $(+)_\xi$, and let $(\pi, \Sigma) = (\pi_\xi, \dot{\Psi}_\xi^h)$, where $h \in \text{col}(\omega, < \kappa_1)$. Suppose $k: \pi \rightarrow j(\pi)$, $k \uparrow H_0^+ = j \uparrow H_0^+$, and $k \in M[h]$; then $\Sigma \subseteq \hat{j}(\Sigma)^k$.

~~(21)~~

Proof Let $j_1 = j(j)$, and $j_1: M \rightarrow N$. Let $\kappa_2 = j_1(\kappa_1) = j(\kappa_1)$ and $H_2^+ = j_1(H_1^+) = j(H_1^+)$. Let \mathcal{Q} be \mathcal{V} -generic on $\text{col}(\omega, < \kappa_2)$, with $g \subseteq h \subseteq \mathcal{Q}$. So we have $\hat{j}_1: M[h] \rightarrow N[\mathcal{Q}]$.

We should explain " $\hat{j}(\Sigma)$ ". By invariance, letting $\dot{\Psi} = \dot{\Psi}_g$, we have

$$\Sigma \cap (V_{K_2})^{V \Sigma g J} = \left(\begin{array}{l} \text{common value of all } j(\dot{\Psi})_h \\ \text{on } \text{col}(w, < K_1), \\ g \subseteq h \end{array} \right) \cap V \Sigma g J.$$

So $\Sigma \cap V \Sigma g J \in V \Sigma g J$, and

$\hat{j}(\Sigma \cap V \Sigma g J)$ makes sense,

$$\hat{j}(\Sigma \cap V \Sigma g J) = \text{common value of all } j(\dot{\Psi})_h, \\ \text{on } \text{col}(w, < K_2), h \in \mathcal{L}, \\ \text{intersected with } \mathcal{M}[h].$$

By $\hat{j}(\Sigma)$ we mean $\hat{j}(\Sigma \cap V \Sigma g J)$. It

is a strategy for $j(n)$ in $\mathcal{M}[h]$ acting on trees of size $< K_2$ there.

It extends to $\mathcal{H}C$ of $\mathcal{M}[h]$, and satisfies

the hypotheses in (†) $j(\Sigma)$ in \mathcal{N} .

Thus $\hat{j}(\Sigma)^k$ makes sense,

as $k \in \mathbb{N} \setminus \{h\}$. It acts on trees of size $< k_2$ in $\mathbb{N} \setminus \{h\}$. We claim that it agrees with Σ on trees that are countable in $\mathbb{N} \setminus \{h\}$.

If not, we have a countable stack \vec{J} on \mathbb{N} in $\mathbb{N} \setminus \{h\}$ that is according to both Σ and $\hat{J}(\Sigma)^k$, yielding

$$i^{\vec{J}} : \mathbb{N} \longrightarrow W,$$

and such that the tails of the two strategies differ on how to iterate W_{bt} , i.e.

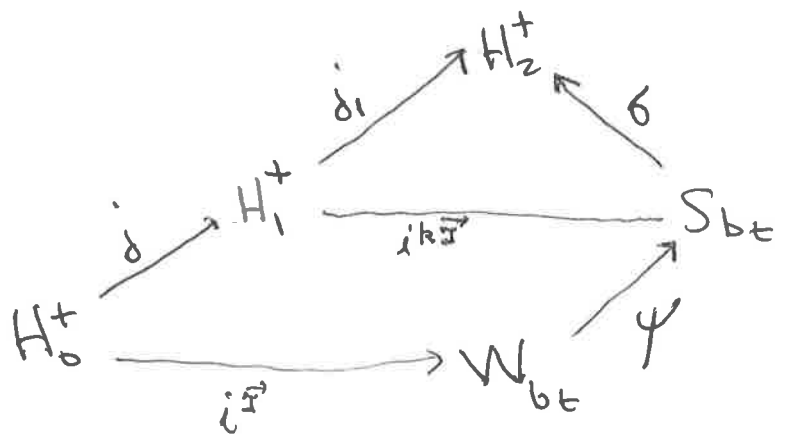
$$\Sigma_{\vec{J}, W_{bt}} \neq \hat{J}(\Sigma)^k_{\vec{J}, W_{bt}}$$

(Σ is positional, but we don't yet know $\hat{J}(\Sigma)^k$ is positional.)

~~This holds in $\mathbb{N} \setminus \{h\}$, so moving by J_1 to $\mathbb{N} \setminus \{h\}$ we get~~ We have that kI is by $\hat{J}(\Sigma)$.

Let S be the last model of $k\mathcal{J}$.

Viewing things from the point of view of $N[\mathcal{J}]$, where $\hat{j}(\Sigma)$ is extended by $j(\hat{\Psi})_2$ satisfying $(\dagger)_{j(\xi)}$, we have



Since we have restricted our maps to bottom parts, and $j|_{H_0^+} \in M$, the whole diagram exists in $N[\mathcal{J}]$. Here our δ is given by the fact that $(\dagger)_{j(\xi)} (d)$ holds in N , so that $\hat{j}(\Sigma)$ is $j|_{H_1^+}$ - consistent in $N[\mathcal{J}]$.

Clearly

(25)

$$\hat{\delta}(\Sigma)^k_{\vec{J}, W_{bt}} = \left(\hat{\delta}(\Sigma)_{k\vec{a}, S_{bt}} \right)^\Psi.$$

Thus $\hat{\delta}(\Sigma)^k_{\vec{J}, W_{bt}} \in \hat{\delta}_1(\hat{j}(\Gamma))$. Writing

$$\Omega = \hat{\delta}(\Sigma)^k_{\vec{J}, W_{bt}} \text{ on } HC^{N_{\text{red}}},$$

we have $\Omega \in \hat{\delta}_1(\hat{j}(\Gamma))$, and

$$\Omega = \left(\left(\Sigma_{H_2^+} \right)^\sigma \right)^\Psi$$

so Ω is ~~in~~ $\hat{\delta}_1(\hat{j}(\Gamma))$ -fullness

preserving and has branch condensation,

by $\hat{\delta}_1(\hat{j})$ -condensation in $N[\mathcal{L}]$.

Claim For any $A \in \theta^0$ in H_0^+ ,

$$\pi_{W_{bt}, \infty}^{\Omega} (i^{\vec{s}}(A)) = j_1(j(A)) \cap K^{M_0}(W_{bt}, \Omega)$$

Proof It is enough to see that if

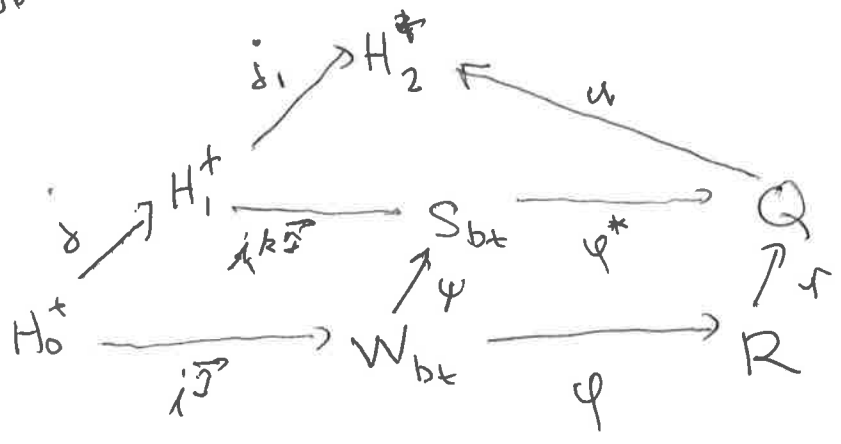
$\varphi: W_{bt} \rightarrow R$ is by Ω , then

$$\alpha \in \varphi(i^{\vec{s}}(A)) \iff \pi_{R, \infty}^{\Omega}(\alpha) \in j_1(j(A))$$

But we can lift φ to

$$\varphi^*: S_{bt} \rightarrow Q \text{ by } \hat{j}(\Sigma). \text{ We}$$

have



where τ is the copy map, and α witnesses $j_1 \uparrow H_1^+$ - consistency. But then

$$\alpha \in \varphi \circ i^{\vec{D}}(A) \text{ iff } \tau(\alpha) \in \varphi^* \circ i^{k\vec{D}} \circ j(A)$$

$$\text{iff } \pi_{\mathbb{Q}, \infty}^{\hat{j}(\Sigma)}(\tau(\alpha)) \in j_1(j(A))$$

$$\text{iff } \pi_{\mathbb{R}, \infty}^{\Omega}(\alpha) \in j_1(j(A)).$$

The second line is $j_1 \wedge H_1^+$ consistency for $\hat{j}(\Sigma)$. The last line is an application of j_1 -condensation (in the form stated in [3]), using

$$\text{that } \hat{j}(\Sigma)_{\mathbb{Q}} = \left(\Sigma_{H_2^+}\right)^{\wedge}, \text{ and}$$

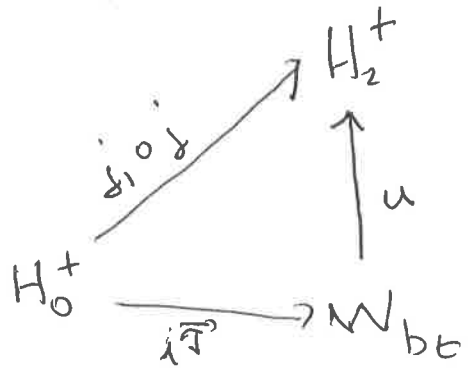
$$\Omega_{\mathbb{R}} = \left(\hat{j}(\Sigma)_{\mathbb{Q}}\right)^{\wedge}.$$



The claim implies that if we define $u: \mathcal{D}_{bE} \rightarrow H_2^+$ by

$$u(i^{\vec{D}}(t)(a)) = j_1(j(t)) \left(\pi_{\mathcal{D}_{bE}, \infty}^{\Omega}(a) \right),$$

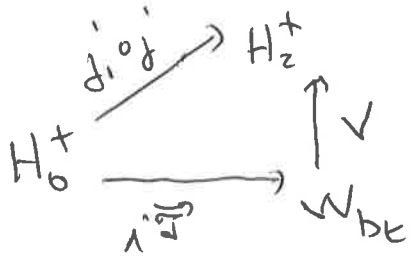
then



commutes, and Ω is u -consistent, But also

$$\text{NZZ} \hat{=} \hat{\Delta}_1(\Sigma) \text{ is } j_1(j_1^* H_0^+) = j_1 \circ j^T H_0^+ - \text{consistent,}$$

so by proposition 4, there is a $t: W_{bt} \rightarrow H_2^+$ in NZZ such that



commutes, and $\hat{\Delta}_1(\Sigma)$ is v -consistent.

Since u and v agree on $\text{ran}(i^T)$, we can apply theorem 6 in NZZ.

We get $\hat{\Delta}_1(\Sigma)_{W_{bt}} = \Omega$, But

$\Sigma \subseteq \hat{\Delta}_1(\Sigma)$, so this is a contradiction.

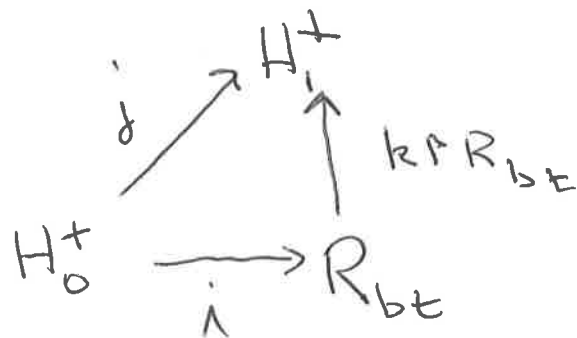


We now look at pulling back $\hat{j}(\Sigma)$ to higher points.

Lemma 8 Assume $(\dagger)_\xi$, and let $(\pi, \Sigma) = (\pi_\xi^\circ, \Psi_\xi^h)$, where h is on $\text{col}(w, \tau k_i)$.

Suppose $k: R \rightarrow j(\pi)$, $k \in M[h]$, R is countable in $M[h]$, and we

have



commuting, for some $i \in M[h]$. Suppose also that the pullback strategy $(\Sigma_{H_0^+})^{k \circ R_{bt}}$ for R_{bt} is $k \circ R_{bt}$ -consistent. Then

- (1) $(R, j(\Sigma)^k)$ is a $\hat{j}(\Gamma)$ mod pair, $j(\Sigma)^k$ having branch condensation, and $\hat{j}(\Sigma)^k \in \hat{j}(\Gamma)$, and
- (2) $\hat{j}(\Sigma)^k$ is $k \circ R_{bt}$ -consistent.

Remark It was part of our $(\dagger)_\xi$ that $\dot{f}^\circ \pi_\xi \leq \omega$; that is, π_ξ declares itself to be full. We let $\pi_\xi^0 = \pi_\xi$ unless there is a last full proper initial segment of π_ξ , and $\dot{f}^\circ \pi_\xi = 0$. In that case we set $\pi_\xi^0 = \pi_\xi$, except that $\dot{f}^\circ \pi_\xi^0 = \omega$. In other words, if π_ξ was declaring itself to be ~~the~~ full $L_p^{\Sigma_Q}(\mathcal{Q})^\Gamma$, then π_ξ^0 stops declaring this.

This has the effect of weakening the Γ -fullness-preserving clause in " Γ -hod pair". It is easy to see, using strategy coherence, that if $(\pi, \Sigma) = (\pi_\xi^0, \dot{\Psi}_\xi^h)$, then Σ is $\dot{f}^\circ(\Gamma)$ -fullness preserving iff whenever R is a Σ -iterate of π , then via $\dot{\mathcal{I}}$, then $\Sigma_{\dot{\mathcal{I}}}, R_{\text{be}}$ is $\dot{f}^\circ(\Gamma)$

R is as in the statement of lemma 8,
 then $\hat{j}(\Sigma)^k$ is $\hat{j}(\Gamma)$ fullness-preserving
 iff whenever W is an iterate of
 R by $\hat{j}(\Sigma)^k$, then say by $\vec{\alpha}$,
 then $\hat{j}(\Sigma)^k \vec{\alpha}, W_{bt}$ is $\hat{j}(\Gamma)$ -fullness-
 preserving as a strategy for W_{bt} .

There is nothing in the hypotheses
 of lemma 8 to guarantee that R is
 $\hat{j}(\Gamma)$ -full "at the very top". When
 we come to use lemma 8, we will

~~have an extender E certified by k
 such that (R, E) is a hod premouse!~~
 get fullness at the top from the existence
 of a next extender.

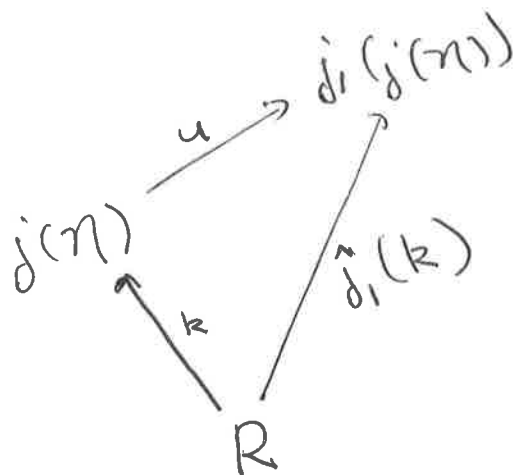
Remark $(\sum_{H_i^+})^{k \uparrow R_{bt}} = (\sum_{H_i / \text{sup } k \uparrow K^R})^{k \uparrow R_{bt} \uparrow K^R}$ (29)

$= (\hat{d}(Z)^k)_{R_{bt}}$. To say that is
 is k -consistent is just to say that
 its iteration map and k agree up to K^R .

Proof of lemma 8

Note that $k = \uparrow_h$, where \uparrow is a term coded by a set of ordinals of size $< k$, $i \in M$. $k, N \subseteq N$ in M , so $\tau \in N$, so $k \in N \cap \mathbb{Q}$.

An easy absoluteness argument then shows that there is a $u \in N \cap \mathbb{Q}$ such that



commutes, and $u \uparrow H_1^+ = \hat{j}_1 \uparrow H_1^+$.

Note here $\hat{j}_1(k)$ and $\hat{j}_1 \uparrow H_1^+$ are in $N[\mathbb{Q}]$, and $j(\mathcal{T})$ is countable there.

Then in $N[\mathbb{Q}]$ we have

$$\begin{aligned} \hat{j}_1(\hat{j}(\Sigma)^k) &= \hat{j}_1(\hat{j}(\Sigma))^{\hat{j}_1(k)} \\ &= (\hat{j}_1(\hat{j}(\Sigma))^u)^k \\ &= \hat{j}(\Sigma)^k \end{aligned}$$

where we have tacitly restricted both strategies to $HC[N[\mathbb{Q}]]$. The last equality comes from lemma 7, used in $N[\mathbb{Q}]$. But

$$N[\mathbb{Q}] \models \hat{j}(\Sigma) \in \hat{j}_1(\hat{j}(\Gamma))$$

so

$$N[\mathbb{Q}] \models \hat{j}(\Sigma)^k \in \hat{j}_1(\hat{j}(\Gamma))$$

so

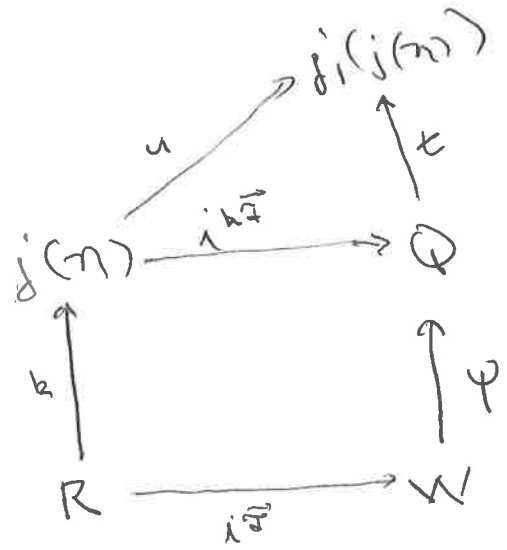
$$N[\mathbb{Q}] \models \hat{j}_1(\hat{j}(\Sigma)^k) \in \hat{j}_1(\hat{j}(\Gamma))$$

so

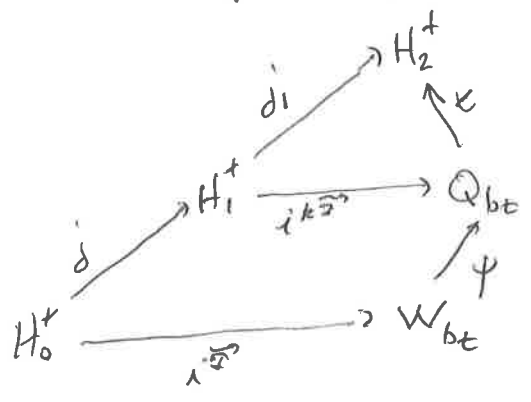
$$M[\mathbb{R}] \models \hat{j}(\Sigma)^k \in \hat{j}(\Gamma).$$

Next we show that $\hat{j}(\Sigma)^k$ is $\hat{j}(\Gamma)$ -fullness preserving in $M[\mathbb{R}]$. Let $\tilde{\mathcal{I}}$ be a countable

stack of trees on R in $N[\Sigma h]$ with last model W , and according to $\hat{j}(\Sigma)^k$. Moving to $N[\Sigma]$, we have the diagram



Here k_1^{Σ} is by $\hat{j}(\Sigma)$, ψ is the copy map, and t is the map we get from $\hat{j}(\Sigma)$ being u -consistent in $N[\Sigma]$. Restricting to bottom parts, we have



We can now apply $d_1(j)$ -condensation in $N[\Sigma]$ to see that $(W_{bt}, (\Sigma_{H_2^+}^{\#})^{\psi \circ t})$ is ~~full~~ $\hat{j}_1(\hat{j}(\Sigma))$ -full. So $(W_{bt}, \hat{j}(\Sigma)_{\vec{\Sigma}, W_{bt}}^k)$ is $\hat{j}_1(\hat{j}(\Sigma))$ full in

NZLJ, and hence $j^n(\Sigma)$ -full in $M[\Sigma]J$.

To finish the proof of part (1), we must show that $j(\Sigma)^k$ has branch condensation. The proof is quite similar to the proof of 2.16 of [ZJ], so we just sketch it. Let

$$\Omega = j(\Sigma)^k.$$

If branch condensation fails, we get

~~\vec{J} by Ω by Ω , with Ω normal~~

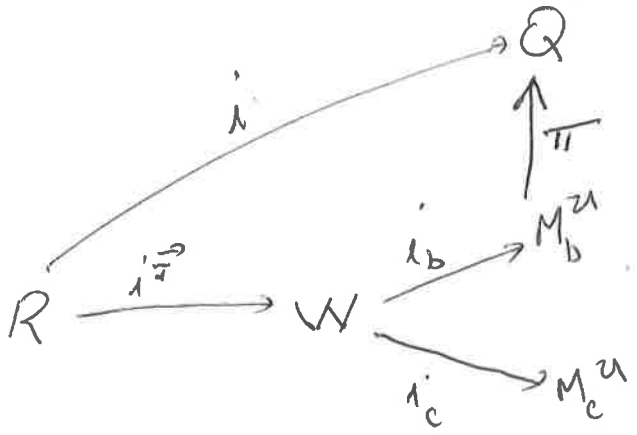
\vec{J} on R with last model W , and Ω normal on W_{bt} with distinct cotinal branches b and c such that

$$c = \Omega(\vec{J} \wedge \Omega)$$

and there is an iteration map $f_0: R \rightarrow Q$ via Ω and $\pi: M_b^{\Omega} \rightarrow Q$ such that

$$f_0 = \pi \circ j_b^{\Omega} \circ j_{\vec{J}}.$$

The picture is



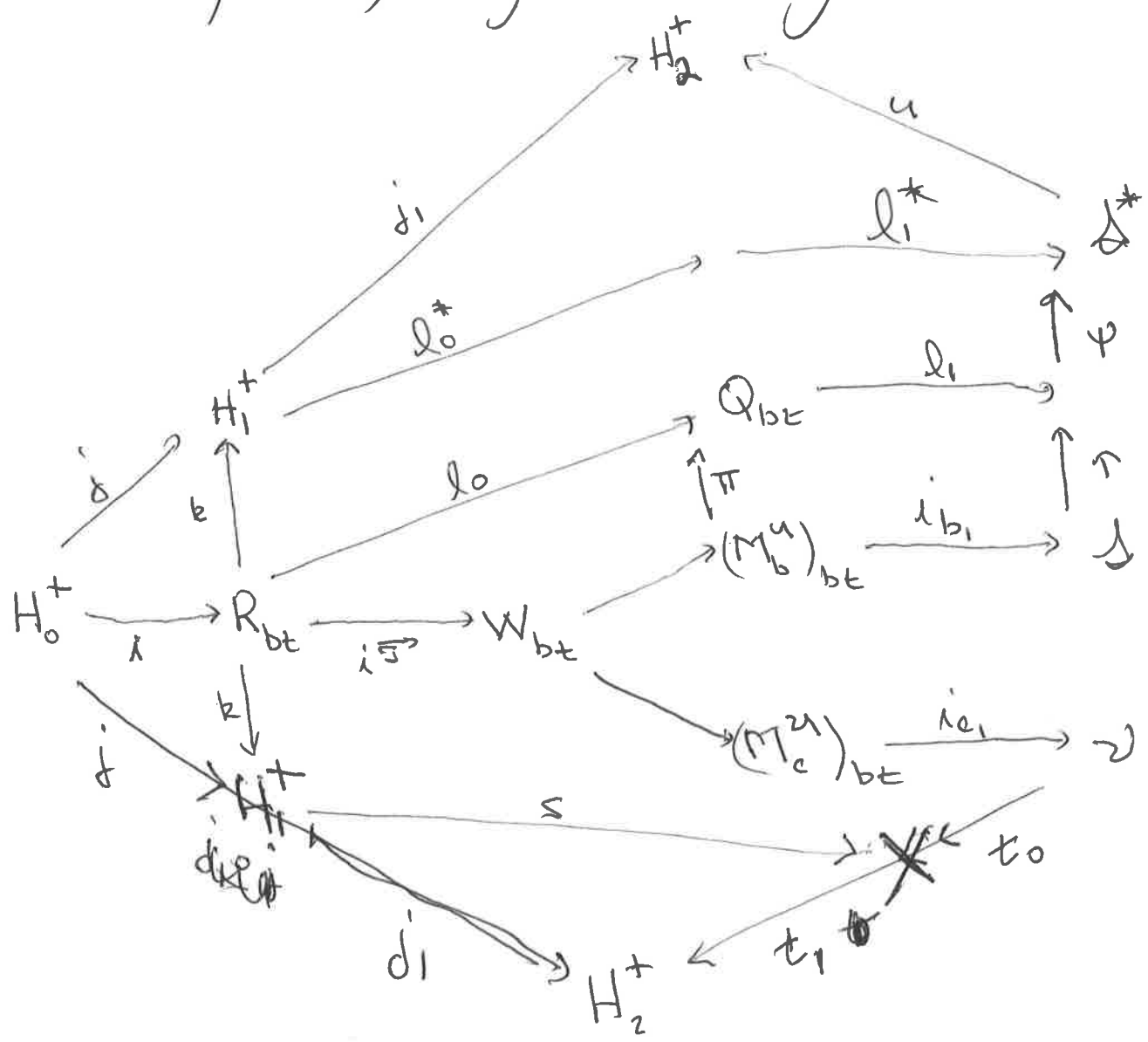
We may assume that $\delta(u) = i_b(\delta) = i_c(\delta)$, and there is $\eta < \delta$ such that η is a cusp point of W , all critical points of u are above η , and by minimizing, that the tails $(\Omega_{\vec{\sigma}_0, Q})_{M(u)}^\pi$ and

$\Omega_{\vec{\sigma}^{\text{unc}}, M(u)}$ are equal; say to Λ .

$P = L_P^\Lambda(M(u))^\Gamma$ is an initial segment of both M_b^u and M_c^u . Let Φ_b and Φ_c be the two tails, acting on P .

We line them up, then look at everything

from the point of view of $NLLJ$, as we did before. Restricting maps to bottom parts, we get the diagram



Here $u_1, b_1,$ and c_1 come from a single normal tree on P lining up the images (rails) of Φ_b and Φ_c ; b_1 is by Φ_b and c_1 is by Φ_c . l_1 is by $\Omega_{Q_{bt}}$, which

is a tail of $\hat{j}(\Sigma)^k$, and $l_1^* \circ l_0^*$ are the k -lifts of l_0 and l_1 , hence by $\hat{j}(\Sigma)$ in $N[\Sigma]$, u is the map we get by $j_1 \uparrow H_1^+$ -consistency of $\hat{j}(\Sigma)$ in $N[\Sigma]$. τ is the map we get by lifting $\vec{a} \wedge u \wedge c \wedge u_1 \wedge c_i$ to a tree on $j(\pi)$ by $\hat{j}(\Sigma)$, and t_0 is the copy map. t_1 is the map we get by j_1 -consistency of $\hat{j}(\Sigma)$ in $N[\Sigma]$.

Let $\gamma_1 = i_{b_1} \circ i_{b_1} \circ i_{\vec{a}} \circ i$ and $\gamma_2 = i_{c_1} \circ i_{c_1} \circ i_{\vec{a}} \circ i$. It will be enough to show

$$\gamma_1(f)(a) = \gamma_2(f)(a)$$

whenever $f \in H_0^+$ and $a \in [Y]^{<\omega}$ and $\gamma_1(f)(a) < \delta(u_1)$.

But let

$$\langle \alpha, \beta \rangle \in \gamma_1(f),$$

then

$$\varphi \circ \tau(\langle \alpha, \beta \rangle) \in \ell_1^* \circ \ell_0^*(j(f))$$

so

$$\pi_{j, \infty}(\varphi \circ \tau(\langle \alpha, \beta \rangle)) \in \delta_1(j(f))$$

(by δ_1 -consistency of $f^n(\tau)$),

$$\pi_{j, \infty}(\langle \alpha, \beta \rangle) \in \delta_1(j(f)).$$

by ~~the~~ $\delta_1(j)$ -condensation in $N[\mathbb{Q}J]$, so

$$\pi_{2, \infty}(\langle \alpha, \beta \rangle) \in \delta_1(j(f)),$$

$$\text{so } \pi_{X, \infty}(\tau_0(\langle \alpha, \beta \rangle)) \in \delta_1(j(f))$$

$$\text{so } \tau_0(\langle \alpha, \beta \rangle) \in s(j(f))$$

$$\text{so } \langle \alpha, \beta \rangle \in \gamma_2(f)$$

as desired.



Finally, we show $j(\Sigma)^k$ is $k \cap R_{bt}$ -consistent. Let in $M[\mathcal{H}]$

$$t: R \longrightarrow S$$

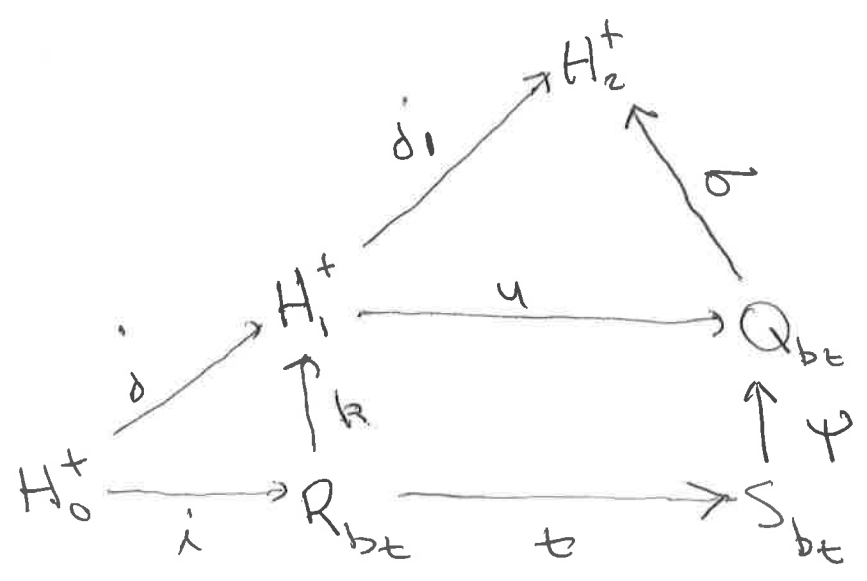
be by $j(\Sigma)^k$. We must see that for $B \subseteq K^R$ in R , and $\alpha \in K^S$,

$$\alpha \in t(B) \text{ iff } \pi_{S_{bt}, \infty}^{\Omega}(\alpha) \in k(B),$$

where $\Omega = j(\Sigma)^k_{S_{bt}}$. We first consider the case that $B = i(A)$, where $A \in H_0^+$.

Claim For $A \in H_0$ with $A \in H_0^+$, and $\alpha \in K^S$, $\alpha \in t(i(A))$ iff $\pi_{S_{bt}, \infty}^{\Omega}(\alpha) \in j(A)$.

Proof Moving over by \hat{j} , we have in $N[\mathcal{Q}]$ the diagram



Here u is by $\hat{j}(\Sigma)$ lifting t (so $u \in \text{NLEJ}$), ψ is the copy map, and σ comes from $\delta_1 \uparrow H_1^+$ - consistency of $\hat{j}(\Sigma)$ in NLEJ. We have

$$\hat{j}_1(\Omega) = \left(\Sigma_{H_2} \right)^{\sigma \circ \psi}$$

in NLEJ. (This is because $\hat{j}_1(\Omega) = \hat{j}_1 \left(\hat{j}(\Sigma)_{S_{bt}}^k \right) = \hat{j}_1 \left(\hat{j}(\Sigma)_{S_{bt}}^k \right)_{S_{bt}} = \hat{j}(\Sigma)_{S_{bt}}^k$ on HC NLEJ, as shown above. \mathcal{D})

But $\hat{j}(\Sigma)_{S_{bt}}^k = \left(\hat{j}(\Sigma)_{Q_{bt}} \right)^\psi = \left(\left(\Sigma_{H_2} \right)^\sigma \right)^\psi$?

the last step holding because by $\delta_1 \uparrow H_1^+$ consistency of $\hat{j}(\Sigma)$ in NIEL.

So it is enough to show that in NIEL,

$$\alpha \in t(i(A)) \text{ iff } \prod_{S_{bt, \infty}} \Lambda^\psi(\alpha) \in \delta_1(j(A)),$$

where

$$\Lambda = \hat{j}(\Sigma)_{Q_{bt}} = \sum_{H_2} \sigma$$

However

$$\alpha \in t(i(A)) \text{ iff } \psi(\alpha) \in u(j(A))$$

$$\text{iff } \prod_{Q_{bt, \infty}} \Lambda(\psi(\alpha)) \in \delta_1(j(A))$$

$$\text{iff } \prod_{S_{bt, \infty}} \Lambda^\psi(\alpha) \in \delta_1(j(A)),$$

by $\delta_1 \uparrow H_1^+$ consistency and $\delta_1(j)$ -condensation in NIEL. This proves our claim.



Now let $B = i(f)(a)$ be an arbitrary subset of K^R in R , with

$a \in (KR)^{<\omega}$ and $f: \theta^{|\alpha|} \rightarrow H_0^+$ in H_0^+ . (40)

Notice that $\pi_{S_{bt}, \infty}^{\Omega} \circ t \uparrow KR$ is the iteration map by $\tilde{j}(\Sigma)^k$, so by our assumption on k ,

$$k(a) = \pi_{S_{bt}, \infty}^{\Omega}(t(a)).$$

But then, for $\alpha \in K^S$,

$$\alpha \in t(B) \text{ iff } \alpha \in t(i(f))(t(a))$$

$$\text{iff } \pi_{S_{bt}, \infty}^{\Omega}(\alpha) \in \pi_{S_{bt}, \infty}^{\Omega}(t(i(f)))(k(a))$$

$$\text{iff } \pi_{S_{bt}, \infty}^{\Omega}(\alpha) \in \tilde{j}(f)(k(a))$$

$$\text{iff } \pi_{S_{bt}, \infty}^{\Omega}(\alpha) \in k(i(f))(k(a))$$

$$\text{iff } \pi_{S_{bt}, \infty}^{\Omega}(\alpha) \in k(B),$$

as desired.

Lemma 8. \square

We can do a little better than lemma 8 if we have another extra to add.

Lemma 9 Assume $(+)_\xi$, and all the other hypotheses and notation of lemma 8. Suppose that in addition

$$k: (R, F) \longrightarrow (j(\pi), G)$$

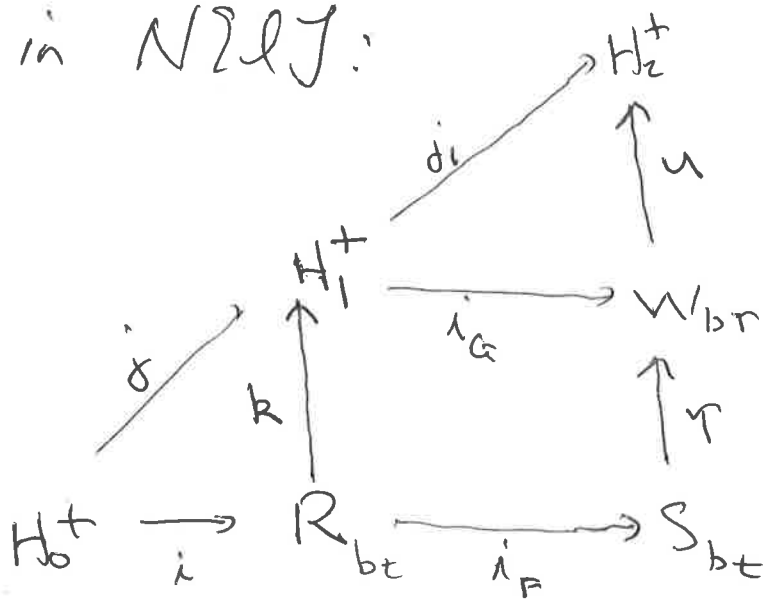
is \mathcal{I}_1 , where G is $j_! M_{\mathcal{H}_1}^+$ -centrifuged over $\hat{j}(\Sigma)$ in $\mathcal{N}[\mathcal{L}]\mathcal{J}$; then $\hat{j}(\Sigma)^k$ is $\hat{j}(\Gamma)$ -fullness preserving in $\mathcal{N}[\mathcal{L}]\mathcal{J}$.

Proof. It is enough to show $\hat{j}(\Sigma)^k$ is $\hat{j}_!(\hat{j}(\Gamma))$ -fullness preserving in $\mathcal{N}[\mathcal{L}]\mathcal{J}$.

Let $\mathcal{O}/\pi(R, F) = S$ and $\mathcal{O}/\pi(j(\pi), G) = W$, with i_F and i_G the canonical embeddings.

Restricting maps to bottom parts, we have

in $N2LJ$:



By coherence, R is an initial segment of S_{bt} , and $j(\pi)$ is an initial segment of W_{bt} . Moreover, $r \upharpoonright R = k$,

and $\hat{j}(\Sigma) = \left(\sum_{H_2}^u \right) \upharpoonright_{j(\pi)}$, because

u agrees with the revision map by $\hat{j}(\Sigma)$ on $j(\pi)$. Thus

$$j(\Sigma)^k = \left(\left(\sum_{H_2}^u \right) \upharpoonright \right)_R.$$

We have that $\left(\sum_{H_2}^u \right) \upharpoonright$ is $\hat{j}_1(j(\Gamma))$ -fullness preserving by $\hat{j}_1(j)$ -condensation, so we're done.

This leads to

Lemma 10, Assume $(\dagger)_\xi$, and let $(\pi, \Sigma) = (\pi_\xi, \Psi_\xi^h)$. Suppose that $(j(\pi), G)$ is a hod premouse, and G is $j_1 \upharpoonright H_1^+$ -certified over $j(\Sigma)$ in $N \mathcal{L} \mathcal{E} \mathcal{J}$. Suppose that in $M \mathcal{L} \mathcal{E} \mathcal{H} \mathcal{J}$, we have

$$k: (R, F) \rightarrow (j(\pi), G)$$

Σ , elementary and $i: H_0^+ \rightarrow R_{bt}$ such that $k \circ i = j \upharpoonright H_0^+$. Suppose also that $j(\Sigma)_{R_{bt}}^k$ is k -consistent; then

~~(a)~~ in $M \mathcal{L} \mathcal{E} \mathcal{H} \mathcal{J}$:

(a) F is $k \upharpoonright R_{bt}$ -certified over $j(\Sigma)^k$,

and

(b) Let $(S, E) = \text{Ult}_0((R, F), F)$,

and $\tau: (S, E) \rightarrow (j(\pi), G)$ be given by (a), that is, $\tau(i_F(g)(a)) = k(g)(\pi_{R,0}^\Omega(a))$, where $\Omega = j(\Sigma)^k$. Then

(i) $j(\Sigma)^\tau$ is τ -consistent,

(ii) $(j(\Sigma)^\tau)_R = j(\Sigma)^k$.

Proof. For (a): let $\Omega = j(\Sigma)^k$.

Let $B \subseteq K^R$ and $B \in R$, and let

$a \in [HG]^{<\omega}$. We must show that

$a \in i_F(B)$ iff $\pi_{R, \infty}^\Omega(a) \in k(B)$. Moving

over to $N \setminus \{ \emptyset \}$, we must show

$$a \in i_F(B) \text{ iff } \pi_{R, \infty}^{\Omega^*} \in j_1(k(B)),$$

where $\Omega^* = \hat{j}_1(\Omega)$ is the extension of Ω on ~~$N \setminus \{ \emptyset \}$~~ $HC^{M \setminus \{ \emptyset \}}$ to $HC^{N \setminus \{ \emptyset \}}$.

Fix $c \in (K^R)^{<\omega}$ and $f \in H_0^+$ so that $B = i(f)(c)$. Such c exists because i must be cofinal in R_{bt} , as j was cofinal in H_1^+ . Then

$$\begin{aligned} k(B) &= k(i(f)(c)) \\ &= j(f)(k(c)) \\ &= j(f)(\pi_{R_{bt}, \infty}^\Omega(c)). \end{aligned}$$

Thus

$$j_1(k(B)) = j_1(j(t)) \left(\pi_{R_{bt}, \infty}^{\Omega^*}(c) \right).$$

We then get, referring to its diagram in the proof of lemma 9,

$$a \in i_F(B) \text{ iff } (a, c) \in i_F(i(t))$$

$$\text{iff } (\tau(a), \tau(c)) \in i_G(j(t))$$

~~iff~~

$$\text{iff } \pi_{W_{bt}, \infty}^{\Sigma^u}(\tau(a), \tau(c)) \in j_1(j(t))$$

$$\text{iff } \pi_{R_{bt}, \infty}^{\Omega^*}(a, c) \in j_1(j(t))$$


$$\text{iff } \pi_{R_{bz}, \infty}^{\Omega^*}(a) \in j_1(j(t)) \left(\pi_{R_{bt}, \infty}^{\Omega^*}(c) \right)$$

$$\text{iff } \pi_{R_{bt}, \infty}^{\Omega^*}(a) \in j_1(j(B)),$$

as desired. This proves (a).

Part (i) of (b) follows at once from part (2) of lemma 8, with the k of lemma 8 taken to be τ .

Since τ agrees with ~~the iteration~~ the iteration map by $j(\Sigma)^k$ on R , and $j(\Sigma)^\tau$ is τ -consistent, $j(\Sigma)^\tau_R = j(\Sigma)^k$.

Lemma 10. 

Suppose now we have $(\tau)_{\xi}$, and $(\mathcal{M}_{\xi}, \dot{\psi}_{\xi}^h) = (\mathcal{M}, \Sigma)$, and we have E certified by j so that (\mathcal{M}, E) is a hod premouse. (And \mathcal{M} was extender-ready in $M[hJ$].) We can easily use the lemmas above to get an iteration strategy for (\mathcal{M}, E) in $M[hJ$. We maintain that for (R, F) the current iterate, we have the situation of lemma 10, with $G = j(E)$.

References

- [1] Sargsyan, A tale of hybrid mice
- [2] Steel, Hod mice below LST
- [3] Steel, Remarks on a paper of Sargsyan