Remarks on a hod mouse construction

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August 2014

We refine some arguments in [2]. Let $j : V \to M$ witness that $\kappa_0 = \text{crit}(j)$ is measurable. Suppose that κ_0 is a limit of Woodins. Let $\kappa_1 = j(\kappa_0)$, and h be V-generic over $\text{Col}(\omega, <\kappa_1)$, and let $g = h \upharpoonright \text{Col}(\omega, <\kappa_0)$.

So we have $\hat{j} : V[g] \to M[h]$ extending j. Let

$$\Gamma \subseteq \text{Hom}_g^*$$

be a pointclass such that $\Gamma = P(\mathbb{R}_g^*) \cap L(\Gamma, \mathbb{R}_g^*)$ and

$$L(\Gamma, \mathbb{R}_g^*) \models AD_{\mathbb{R}}.$$

Suppose that $HOD^{L(\Gamma, \mathbb{R}_g^*)}$ can be analyzed as a hod mouse by current methods — e.g., suppose there is no $\Omega \subseteq \Gamma$ such that $L(\Omega, \mathbb{R}_g^*) \models LST$. ($LST = LSA =$ "the largest Suslin is a Θ_α"), so that work of Sargsyan, and to some extent the author, does this. Let

$$H_0 = HOD^{L(\Gamma, \mathbb{R}_g^*)} | \Theta^{L(\Gamma, \mathbb{R}_g^*)}$$

and

$$\theta^0 = \Theta^{L(\Gamma, \mathbb{R}_g^*)} = o(H_0).$$

Set

$$H_0^+ = \left(L_p^{\bigoplus_{\alpha < \theta^0} \Sigma_{H_0(\alpha)}} (H_0) \right)^{\hat{J}(\Gamma)}$$

the stack of all appropriate strategy mice with iteration strategies in $\hat{J}(\Gamma)$ over H_0.

We assume

$$o(H_0^+) < \kappa_0^+.$$

Our goal is to construct, in $M[h]$, a hod pair $(\mathcal{H}, \Sigma_{\mathcal{H}})$ extending (H_0^+, Σ_{H_0}) such that either $\mathcal{H}$ reaches $O_h^{\mathbb{P}}$, or $(\mathcal{H}, \Sigma_{\mathcal{H}})$ generates $\hat{J}(\Gamma)$.

This is done in [2], but under stronger assumptions on j. We show in this note that the construction of [2] works under the weaker assumptions we have here.

$\mathcal{H}$ is constructed by induction on its full levels (i.e. strategy-activation levels), and the construction is actually done in M. We adopt the notation of [1]. So $\dot{\Gamma}$ is a canonical name for Γ, and we are constructing in M pairs $(\mathcal{N}_\xi, \dot{\psi}_\xi)$, ~~such that~~ starting with

$$\mathcal{N}_0 = H_0^+$$

$$\dot{\psi}_0 = \text{canonical name for } \Sigma_{H_0} = \bigoplus_{\alpha < \theta^0} \Sigma_{H_0(\alpha)}, \text{ extended to } M[L].$$

We have $\Vdash^{Col(\omega, <\kappa_1)} \dot{\psi}_0 \subseteq j(\dot{\Gamma})$.

We maintain by induction, for $\xi < \kappa_1$

<u>Induction hypotheses $(\dagger)_\xi$</u> In M, the following hold, for $(\mathcal{N}, \mathring{\Psi}) = (\mathcal{N}_\xi, \mathring{\Psi}_\xi)$:

(a) $\Vdash_{Col(\omega,<\kappa_1)}$ $(\check{\mathcal{N}}, \dot{\Psi})$ is a $j(\dot{\Gamma})$ hod pair such that $\dot{\Psi}$ has branch condensation and is positional, with top block beginning at θ°,

(b) $\Vdash_{Col(\omega,<\kappa_1)}$ $\forall \nu < \xi$ $(\mathcal{N}_\nu, \dot{\Psi}_\nu)$ is a full initial segment of $(\check{\mathcal{N}}, \dot{\Psi})$,

(c) for m and ℓ on $Col(\omega,<\kappa_1)$ with $M[\ell] = M[m]$, we have $\mathring{\Psi}^m = \dot{\Psi}^\ell$,

(d) $\Vdash_{Col(\omega,<\kappa_1)}$ $(\mathcal{N}, \dot{\Psi})$ is $j \restriction H_0^+$-consistent,

(e) if E is a θ°-relevant extender in the top block of $\mathcal{N}$, then $crit(E) = \theta^\circ$; and $\Vdash_{Col(\omega,<\kappa_1)}$ $\check{E}$ is $j \restriction H_0^+$-certified over $(\mathcal{N} \| lh E, \mathring{\Psi}_{\mathcal{N} \| lh E})$,

(f) $\Vdash_{Col(\omega,<\kappa_1)}$ $\mathring{\Psi} \in j(\dot{\Gamma})$

Remarks

(i) The first part of (e) says that the top block of $\mathcal{N}$ is below $\mathcal{O}_h^{\mathbb{P}}$.

(ii) The construction continues until we reach a first ξ such that $(\dagger)_\xi$ fails. At that point, we shall see that $(\dagger)_\xi$ (a)–(e) hold, and that for h on $col(\omega, <\kappa_1)$, $\dot{\psi}^h \in Hom_h^* - \hat{\jmath}(\Gamma)$. This implies $\dot{\psi}^h$ is a pointclass generator for $\hat{\jmath}(\Gamma)$, i.e. $\Gamma(\mathcal{N}_\xi, \dot{\psi}^h) = \hat{\jmath}(\Gamma)$.

(iii) if the construction does not stop for reason (ii), then $\bigcup_{\xi<\kappa_1} \mathcal{N}_\xi$ can be expanded to a hod premouse of $\mathcal{O}_h^{\mathbb{P}}$ type.

~~(iv) Notice that (a) does not imply $(\dagger)$. From (a) we only get that if $\mathcal{M}$ is a $\dot{\psi}^h$-iterate of $\mathcal{N}$, and $\mathcal{M}(\alpha)$ is a full proper initial segment of $\mathcal{M}$, then $\dot{\psi}^h_{\mathcal{M}(\alpha)} \in \hat{\jmath}(\Gamma)$.~~

(iv) From (a) we only get that if $\vec{\mathcal{T}}$ is a stack of trees by $\mathring{\psi}^h$, and $\mathcal{M}$ is a full <u>proper</u> initial segment of the last model of $\vec{\mathcal{T}}$, then $\mathring{\psi}^h_{\vec{\mathcal{T}}, \mathcal{M}} \in \hat{\mathcal{J}}(T)$.

So (a) does not imply (f).

(v) Sargsyan's theorem 2.41 of [IJ] shows that if (a) and (f) hold, then $\mathring{\psi}^h$ is positional and commuting. The pointclass generator we get in case (ii) above will also be positional and commuting.

[IJ] shows that if Ω for R is positional and has branch condensation, then Π is commuting

(5)

(vi) Clause (e) is explained in [2aJ].

Clause (d) we explain now. It is a substitute for ~~what~~

(d′) $\Vdash^{Col(\omega, <\kappa_1)} \dot{\psi} = j(\dot{\psi})^{j}$,

which was assumed in [2aJ]. The trouble with (d′) is that the pullback strategy $j(\dot{\psi})^{j}$ does not make (immediate) sense unless $j \restriction \eta \in M$. Our assumption that $o(H_0^+) < \kappa_0^+$ implies $j \restriction H_0^+ \in M$, so (e) makes sense, and so too will (d) when we get to defining it. But if $\xi \geq \kappa_0^+$, (d′) does not make sense unless j witnesses κ_0^+-supercompactness.

<u>Notation</u> Recall that if R is a hod premouse with at top block, then κ^R begins ~~each~~ that block. We also set $R_{bt} = R|(\kappa^R)^{+R}$, and call it the "bottom part" of R.

<u>Definition I.</u> Let (R, Ω) be a $\hat{\jmath}(\Gamma)$ hod pair, with $\Omega \in \hat{\jmath}(\Gamma)$ and Ω having branch condensation. Let

$$k : R_{bt} \longrightarrow H_1^+$$

be cofinal and Σ_0 elementary; then Ω is <u>k-consistent</u> iff there is a cofinal, Σ_0 elementary

$$\sigma : M_\infty(R, \Omega)_{bt} \longrightarrow H_1^+$$

such that

$$k = \sigma \circ (\pi^{\Omega}_{R,\infty} \restriction R_{bt})$$

and

$$\mathrm{crit}(\sigma) = \kappa^{M_\infty(R, \Omega)} .$$

In this definition, the direct limit system giving rise to $M_\infty(R, \Omega)$ and $\pi^{\Omega}_{R,\infty}$ is that of $L(\hat{\jmath}(\Gamma), \mathbb{R}^*_h)$. We have assumed that $\Omega \in \hat{\jmath}(\Gamma)$, so $M_\infty(R, \Omega)$

is a proper initial segment of H_1^+.

If Ω is a k-consistent strategy for R, then $\pi^{\Omega}_{R,\infty} \restriction \kappa^R = k \restriction \kappa^R$. Thus Ω is locally k-consistent in the sense of [2]. What was called k-consistency in [2] was essentially equivalent to local k-consistency, and thus weaker that what we shall henceforth mean by k-consistency.

<u>Proposition 2</u> Let (R, Ω) be a $\hat{\jmath}(\Gamma)$ hod pair, Ω have branch condensation, and $\Omega \in \hat{\jmath}(\Gamma)$. Let $k : R_{bt} \to H_1^+$. Equivalent are:

(1) Ω is k-consistent

(2) $\pi_{R,\infty}$ and k generate the same extender of length $\kappa^{M_\infty(R,\Omega)}$ over R_{bt}

(3) for all $A \subseteq \kappa^R$ in R, $\pi^{\Omega}_{R,\infty}(A) = k(A) \cap \kappa^{M_\infty(R,\Omega)}$,

(4) if $i : R \to S$ is by Ω, then $E_i \restriction \kappa^S = E_k \restriction \pi_{S,\infty}{}'' \kappa^S$

(5) if $A \subseteq \kappa^R$, $A \in R$, and $i : R \to S$ by Ω, then $\pi_{S,\infty}{}'' i(A) = k(A) \cap \pi_{S,\infty}{}'' \kappa^S$.

Proof (3) is just a re-statement of (2), and (5) is a re-statement of (4). The ~~other~~ equivalence of (3) with (5) is easy, as is the equivalence of (1) with (2).

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Notice that $\pi^{\Omega}_{R,\infty}$ is given completely by its extender of length $\kappa^{M_\infty(R,\Omega)}$, that is, $M_\infty(R,\Omega) = \mathrm{Ult}_0(R, E_{\pi^{\Omega}_{R,\infty}} \upharpoonright \kappa^{M_\infty(R,\Omega)})$. Notice also that the factor map σ of definition I is uniquely determined by k and Ω; σ is just the factor map from $\mathrm{Ult}_0(R_{bt}, E_k \upharpoonright \kappa^{M_\infty(R,\Omega)})$ into $H_1^+ = \mathrm{Ult}_0(R_{bt}, E_k)$.

~~σ is determined by k and $k^{M_\infty(R,\Omega)}$.~~

We stress that it is $M_\infty(R,\Omega)_{bt}$ and not the potentially smaller $M_\infty(R_{bt}, \Omega_{R_{bt}})$, that is relevant in definition I. In general, if $k : R_{bt} \longrightarrow H_1^+$ and Ω for R is k-consistent, and P is a full initial segment of R with $\kappa^P = \kappa^R$, then Ω_P is k-consistent too. This is because there is a natural

$$\psi : M_\infty(P, \Omega_P) \longrightarrow \pi^{\Omega}_{R,\infty}(P)$$

such that $\pi^{\Omega}_{R,\infty} \restriction P = \psi \circ \pi^{\Omega_P}_{P,\infty}$, and $\mathrm{crit}(\psi) \geq \kappa^{M_\infty(P,\Omega_P)}$. So if σ witnesses k-consistency for Ω, $\sigma \circ \psi$ witnesses k-consistency for Ω_P. But there is no reason k-consistency for Ω_P should imply k-consistency for Ω.

~~$k \restriction (R_{bt}|\xi) = \sigma_\xi \circ \pi^{\Omega}_{R,\infty} \restriction (R_{bt}|\xi)$. Taking $\sigma = \bigcup_\xi \sigma_\xi$, we get the required factor map.~~ 9

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Tails of a k-consistent strategy are consistent with respect to the appropriate "extension" of k:

<u>Proposition 3</u>. Let (R, Ω) be a $\hat{j}(\Gamma)$-hod pair in $M[L]$. Let $k : R_{bt} \to H_1^+$ be such that Ω is k-consistent. Let $i : R \to S$ be an iteration by Ω. Then there is a unique $\tau : S_{bt} \to H_1^+$ such that

$$\begin{tikzcd} R_{bt} \arrow[rr, "k"] \arrow[dr, "i"'] & & H_1^+ \\ & S_{bt} \arrow[ur, "\tau"'] & \end{tikzcd}$$

commutes, and $\tau \restriction K^S = \pi^{\Omega_S}_{S,\infty} \restriction K^S$. Moreover, Ω_S is τ-consistent.

Proof Just let

$$\tau(i(f)(a)) = k(f)(\pi_{S,\infty}(a)),$$

for all $a \in [K^S]^{<\omega}$. If $\sigma : M_\infty(R,\Omega)_{bt} \to H_1^+$ witnesses k-consistency for Ω, then σ also witnesses τ-consistency for Ω_S.

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The next proposition concerns extensions of our embeddings to the whole top block.

Proposition 4. Let (R,Ω) be a $\hat{j}(\Gamma)$-hod pair, and $k : R \to Q$ where $H_1^+ \trianglelefteq Q$, and suppose Ω is $k \restriction R_{bt}$-consistent. Let $i : R \to S$ be by Ω, or let $S = M_\infty(R,\Omega)$ and $i = \pi^\Omega_{R,\infty}$. Then there is a unique $\tau : S \to Q$ such that

$$\begin{tikzcd} R \arrow[rr, "k"] \arrow[dr, "i"'] & & Q \\ & S \arrow[ur, "\Upsilon"'] & \end{tikzcd}$$

commutes, and $\Upsilon \restriction K^S = \pi^{\Omega_S}_{S,\infty} \restriction K^S$ if $S \neq M_\infty(R, \Omega)$, while $\Upsilon \restriction K^S = id$ if $S = M_\infty(R, \Omega)$. In the former case, Ω_S is $\Upsilon \restriction S_{bt}$-consistent.

Proof Suppose first $S = M_\infty(R, \Omega)$. ~~Then~~ So $S = \text{Ult}_0(R, E_{\pi_{R,\infty}} \restriction K^{M_\infty(R,\Omega)}) = \text{Ult}_0(R, E_k \restriction K^{M_\infty(R,\Omega)})$. Let

$$\sigma : \text{Ult}_0(R, E_k \restriction K^{M_\infty(R,\Omega)}) \longrightarrow \text{Ult}_0(R, E_k)$$

be the factor map. So $\sigma : M_\infty(R, \Omega) \longrightarrow Q$, and $\sigma \restriction K^{M_\infty(R,\Omega)} = id$, and $k = \sigma \circ i$. Thus σ has the properties required of Υ.

If instead, S is a point in the

direct limit system leading to $M_\infty(R,\Omega)$, we can just let

$$\varphi = \sigma \circ \pi^{\Omega_s}_{s,\infty},$$

where $\sigma : M_\infty(R,\Omega) \to Q$ is as above.

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Remark See p. 12b.

The following uniqueness theorem is one of the main points. It strengthens the uniqueness-of-pullback lemma 2.17.1 from [?]. Some form of it was also known to Sargsyan. The ~~technique~~ proof traces back to Sargsyan's earlier work.

Theorem 5. (Uniqueness of k-consistent strategies.) Let $k : R_{bt} \to H_1^+$, and let Φ and Ψ in $\hat{j}(\Gamma)$ be k-consistent strategies for R (so both have branch condensation, are positional, and are $\hat{j}(\Gamma)$-fullness preserving); then $\Phi = \Psi$.

Remark Clause (e) of $(\dagger)_\xi$ actually follows from the others. For let $\Sigma = \mathring{\varphi}_\xi^h$, and let E be on the $\mathcal{N}$-sequence with $\mathrm{crit}(E) = \theta^0$. Let $S = \mathrm{Ult}(\mathcal{N}, E)$. Since Σ was $j \restriction H_0^+$-consistent, proposition 3 gives that the tail Σ_S is τ-consistent, for $\tau : S_{b+} \to H_1^+$ given by $\tau(i_E(f)(a)) = j(f)(\pi^{\Sigma_S}_{S,\infty}(a))$. But $\pi^{\Sigma_S}_{S,\infty}(a) = \pi^{\Sigma}_{\mathcal{N}\|\mathrm{lh}E,\infty}(a)$ for $a \in \mathrm{lh}E^{<\omega}$, by strategy coherence. Thus $a \in i_E(A)$ iff $\pi_{\mathcal{N}\|\mathrm{lh}E,\infty}(a) \in j(A)$.

This remark is more motivational than useful, because when we put E onto the $\mathcal{N}$ sequence, we'll want to know it is $j \restriction H_0^+$-certified by the strategy of $\mathcal{N}\|\mathrm{lh}E$ before we construct a $j \restriction H_0^+$-consistent strategy for $(\mathcal{N}\|\mathrm{lh}E, E)$.

Proof. This is essentially shown in 2.17 of [2]. Here is the argument again. Suppose $\Phi \neq \Psi$. We can find a stack of trees $\vec{\mathcal{T}}$ on R with last model W such that $\vec{\mathcal{T}}$ is by both strategies, and a normal tree $\mathcal{U}$ on W such that letting

$$b = \Phi(\vec{\mathcal{T}}^\frown \mathcal{U})$$

and

$$c = \Psi(\vec{\mathcal{T}}^\frown \mathcal{U}),$$

we have $b \neq c$. We choose this disagreement so as to minimize the minimum of the Wadge ranks of the tails of Φ and Ψ acting on $M(\mathcal{U})$. This gives

$\Phi_{\vec{\mathcal{T}}^\frown \mathcal{U}, M(\mathcal{U})} = \Psi_{\vec{\mathcal{T}}^\frown \mathcal{U}, M(\mathcal{U})}$. ($M(\mathcal{U})$ is a limit of full levels in $M_b^{\mathcal{U}}$ and $M_c^{\mathcal{U}}$, not itself full, so these are really joins-of-strategies.)

Letting

$$\Lambda = \Phi_{\vec{\mathcal{T}}^\frown \mathcal{U}, M(\mathcal{U})} = \Psi_{\vec{\mathcal{T}}^\frown \mathcal{U}, M(\mathcal{U})},$$

we have

$$L_p^{\Lambda}(M(\mathcal{U})) \trianglelefteq M_b^{\mathcal{U}}, M_c^{\mathcal{U}}$$

and

$$L_p^{\Lambda}(M(\mathcal{U})) \models \delta(\mathcal{U}) \text{ is Woodin.}$$

(So $M_b^{\mathcal{U}} \models \delta(\mathcal{U})$ is Woodin, as does $M_c^{\mathcal{U}}$.) Let

$$\eta = \sup\{\xi < \delta(\mathcal{U}) \mid M(\mathcal{U}) \models \xi \text{ is Woodin}\}:$$

Since we are below LST^-, $\eta < \delta(\mathcal{U})$, moreover η is a cutpoint of $M(\mathcal{U})$. We may assume that all critical points in $\mathcal{U}$ are above η. Letting

$$\delta = \text{least Woodin of } W > \eta,$$

we then have

$$i_b(\delta) = i_c(\delta) = \delta(\mathcal{U}).$$

Finally, looking at how "successor Woodins" can be produced by iteration, we see that

$$\delta = \sup\{ i^{\vec{\mathcal{T}}}(f)(a) \mid a \in [\eta]^{<\omega} \text{ and } f \in R_{bt} \}.$$

The rest of the proof is in the following diagram. Set $i = i^{\mathcal{T}}$

$$\begin{tikzcd}
 & & & H_1^+ \\
 & & (M_b^{\mathcal{U}})_{bt} \arrow[r, "i_{b_1}^{\mathcal{U}_1}"] & \mathcal{S} \arrow[u, "t"'] \\
R_{bt} \arrow[uurrr, "k"] \arrow[r, "i"] \arrow[ddrrr, "k"'] & W \arrow[ur] \arrow[dr] & & \\
 & & (M_c^{\mathcal{U}})_{bt} \arrow[r, "i_{c_1}^{\mathcal{U}_1}"] & \mathcal{J} \arrow[d, "v"] \\
 & & & H_1^+
\end{tikzcd}$$

Let

$$P = L_p^{\wedge}(M(\mathcal{U})).$$

We may have that $\Phi_{\vec{s}^\frown\mathcal{U}, P} \neq \Psi_{\vec{s}^\frown\mathcal{U}, P}$.

But we can compare these two strategies via a single normal tree $\mathcal{U}_1$ that is above η, with branches b_1 and c_1 chosen by Φ and Ψ. We get that

$$i_{b_1}^{\mathcal{U}_1}(P) = i_{c_1}^{\mathcal{U}_1}(P),$$

and

$$\Phi_{s^\frown\mathcal{U}^\frown\mathcal{U}_1^\frown b_1, P} = \Psi_{s^\frown\mathcal{U}^\frown\mathcal{U}_1^\frown c_1, P}.$$

Letting $\mathcal{S} = (M_{b_1}^{\mathcal{U}_1})_{bt}$ and $\mathcal{V} = (M_{c_1}^{\mathcal{U}_1})_{bt}$,

the maps $t: \mathcal{S} \to H_1^+$ and $v: \mathcal{V} \to H_1^+$ witness k-consistency, as in proposition 3.

But then we get, for any $a \in [\eta]^{<\omega}$,

$$t\left(i_{b_1} \circ i_b \circ i^{\vec{J}}(f)(a)\right)$$

$$= k(f)\left(\pi_{\delta,\infty}(a)\right)$$

$$= v\left(i_{c_1} \circ i_c \circ i^{\vec{J}}_{\sim}(f)(a)\right),$$

for all $f \in R_{bt}$. If $i^{\vec{J}}(f)(a) < \delta$, then $i_{b_1} \circ i_b \circ i^{\vec{J}}(f)(a) < \delta(\mathcal{U}_1)$. But

$$t \restriction \delta(\mathcal{U}_1) = v \restriction \delta(\mathcal{U}_1)$$

because we have lined up $\Phi_{\vec{J}^\frown \mathcal{U}^{b\frown}\mathcal{U}_1{}^\frown b_1}$ with $\Psi_{\vec{J}^\frown \mathcal{U}^\frown c^\frown \mathcal{U}_1{}^\frown c_1}$. So

$$i_{b_1} \circ i_b \circ i^{\vec{J}}(f)(a)$$

$$= i_{c_1} \circ i_c \circ i^{\vec{J}}(f)(a)$$

for these f and a. Since there are cofinally many points of this form below $\delta(\mathcal{U}_1)$, $b_1 = c_1$. But then $i_b \circ i^{\vec{\mathcal{T}}}(f)(a) = i_c \circ i^{\vec{\mathcal{T}}}(f)(a)$ for those f and a, so $b = c$. Contradiction.

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The same proof gives the following generalization, used by Sargsyan in [3] in the case $R = R_{bt}$.

<u>Theorem 6</u> (Uniqueness of $\mathcal{F}$-consistent strategies)
Let (R, Ω_0) and (R, Ω_1) be $\hat{\jmath}(\Gamma)$ hod pairs in $\hat{\jmath}(\Gamma)$ such that Ω_i has branch condensation and is positional, for $i = 0, 1$. Let $k_i : R_{bt} \to H^+_{i}$, and suppose Ω_i is k_i-consistent, for $i = 0, 1$. Suppose that for

$$\mathcal{F} = \mathrm{ran}(k_0) \cap \mathrm{ran}(k_1)$$

we have for $i = 0, 1$:

(a) $R_{bt} = \{ f(a) \mid a \in [\kappa^R]^{<\omega} \wedge k_i(f) \in \mathcal{F} \}$,

(b) if $R_{bt} \models \delta$ is Woodin, then $\exists \eta < \delta$ s.t. $\sup(\{ f(a) \mid a \in [\eta]^{<\omega} \wedge k_i(f) \in \mathcal{F} \} \cap \delta) = \delta$.

<u>Then</u> $\Omega_0 = \Omega_1$.

Proof sketch

The proof here is the same. We get

$$\begin{tikzcd}
& & H_1^+ & \\
& & M_b^{\mathcal{U}} \arrow[r, "i^{\mathcal{U}_1}_{b_1}"] & \mathcal{P} \arrow[ul, "t"'] \\
R_{bt} \arrow[uurr, "k_0"] \arrow[r, "i^{\vec{\mathcal{T}}}"] \arrow[ddrr, "k_1"'] & W \arrow[ur] \arrow[dr] & & \\
& & M_c^{\mathcal{U}} \arrow[r, "i^{\mathcal{U}_1}_{c_1}"] & \mathcal{Q} \arrow[dl, "v"] \\
& & H_1^+ &
\end{tikzcd}$$

where $\vec{\mathcal{T}}$ is by both Ω_0 and Ω_1, as before. Let $\mathcal{G}_i = k_i^{-1}$ " $\mathcal{F}$. Using our notation above, (a) and (b) of theorem 6 propagate to W, so points of the form $i^{\vec{\mathcal{T}}}(f)(a)$ for $a \in [\eta]^{<\omega}$ and $f \in \mathcal{G}_i$ are cofinal in δ, for $i = 0, 1$.

For $f \in \mathcal{G}_0$ and $a \in [\eta]^{<\omega}$, we can write

$$t(i_{b_1} \circ i_b \circ i^{\vec{\jmath}}(f)(a))$$

$$= k_0(f)(\pi_{s,\infty}(a)).$$

But $\pi_{s,\infty}(a) = \pi_{v,\infty}(a)$, and moreover we have g s.t. $k_1(g) = k_0(f)$, ~~and~~ so

$$k_0(f)(\pi_{s,\infty}(a)) = k_1(g)(\pi_{v,\infty}(a))$$

$$= v(i_{c_1} \circ i_c \circ i^{\vec{\jmath}}(g)(a)).$$

So $i_{b_1} \circ i_b \circ i^{\vec{\jmath}}(f)(a) = i_{c_1} \circ i_c \circ i^{\vec{\jmath}}(g)(a)$.

This gives $b_1 = c_1$, and then $b = c$, as before.

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If $j \restriction \eta \in M$, then $j(\Sigma)^j = \Sigma$, ~~is consistent~~ as we shall show. ~~and then~~ now.

~~$j(\Sigma)^j = \Sigma \wedge \wedge$ So (d) of (†) follows from~~

So our new $(\dagger)_\Sigma(d)$ implies the old one, when $j \upharpoonright \mathcal{N} \in M$.

Lemma 7 Assume $(+)_\xi$, and let $(\mathcal{N}, \Sigma) = (\mathcal{N}_\xi, \mathring{\varphi}_\xi^h)$, where h is on $\mathrm{Col}(\omega, <\kappa_1)$. Suppose $k : \mathcal{N} \to j(\mathcal{N})$, $k \upharpoonright H_0^+ = j \upharpoonright H_0^+$, and $k \in M[h]$; then $\Sigma \subseteq \hat{j}(\Sigma)^k$.

$j(\mathcal{N})$

Proof Let $j_1 = j(j)$, and $j_1 : M \to N$. Let $\kappa_2 = j_1(\kappa_1) = j(\kappa_1)$ and $H_2^+ = j_1(H_1^+) = j(H_1^+)$. Let l be V-generic on $\mathrm{col}(\omega, <\kappa_2)$, with $g \subseteq h \subseteq l$. So we have

$$\hat{j}_1 : M[h] \to N[l].$$

We should explain "$\hat{j}(\Sigma)$". By invariance, letting $\mathring{\psi} = \mathring{\varphi}_\xi$, we have

$$\Sigma \cap (V_{\kappa_2})^{V[g]} = \left(\begin{array}{l}\text{common value of all} \\ \mathring{\psi}_h,\ h \text{ on } col(\omega, <\kappa_1), \\ g \subseteq h\end{array}\right)$$

$$\cap\ V[g].$$

So $\Sigma \cap V[g] \in V[g]$, and
$\hat{j}(\Sigma \cap V[g])$ makes sense,

$\hat{j}(\Sigma \cap V[g]) =$ common value of all $j(\mathring{\psi})_\ell$,
ℓ on $col(\omega, <\kappa_2)$, $h \subseteq \ell$,
intersected with $M[h]$.

By $\hat{j}(\Sigma)$ we mean $\hat{j}(\Sigma \cap V[g])$. It is a strategy for $j(\mathcal{M})$ in ~~M[l]~~ $M[h]$ acting on trees of size $< \kappa_2$ there. It extends to HC of $N[\ell]$, and satisfies the hypotheses in $(\dagger)_{j(\xi)}$ in N.

Thus $\hat{j}(\Sigma)^k$ makes sense,

as $k \in M[h]$. It acts on trees of size $< \kappa_2$ in $M[h]$. We claim that it agrees with Σ on trees that are countable in $M[h]$.

If not, we have a countable stack $\vec{\mathcal{J}}$ on $\mathcal{N}$ in $M[h]$ that is according to both Σ and $\hat{j}(\Sigma)^k$, yielding

$$i^{\vec{\mathcal{J}}} : \mathcal{N} \longrightarrow \mathcal{W},$$

and such that the tails of the two strategies differ on how to iterate $\mathcal{W}_{bt}$, i.e.

$$\Sigma_{\vec{\mathcal{J}}, \mathcal{W}_{bt}} \neq \hat{j}(\Sigma)^k_{\vec{\mathcal{J}}, \mathcal{W}_{bt}} .$$

(Σ is positional, but we don't yet know $\hat{j}(\Sigma)^k$ is positional.) This holds ~~in $M[h]$, so moving by~~ $\hat{j}_1$ to $N[\ell]$ ~~we get~~ We have that $k\mathcal{J}$ is by $\hat{j}(\Sigma)$.

Let S be the last model of $k\mathcal{T}$.

Viewing things from the point of view of $N[\ell\mathcal{T}]$, where $\hat{j}(\Sigma)$ is extended by $j(\mathring{\psi})_\ell$ satisfying $(\dagger)_{j(\xi)}$, we have

$$\begin{tikzcd}
 & & H_2^+ & \\
 & H_1^+ \arrow[ur, "\dot{j}_1"] \arrow[rr, dash, "i^{k\vec{\mathcal{T}}}"'] & & S_{bt} \arrow[ul, "\sigma"'] \\
H_0^+ \arrow[ur, "\dot{j}"] \arrow[rr, "i^{\vec{\mathcal{T}}}"'] & & W_{bt} \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Since we have restricted our maps to bottom parts, and $j \restriction H_0^+ \in M$, the whole diagram exists in $N[\ell\mathcal{T}]$. Here ~~our~~ σ is given by the fact that $(\dagger)_{j(\xi)}(d)$ holds in N, so that $\hat{j}(\Sigma)$ is $j_1 \restriction H_1^+$ - consistent in $N[\ell\mathcal{T}]$.

Clearly

$$\hat{\jmath}(\Sigma)^{k}_{\vec{J}, W_{bt}} = \left(\hat{\jmath}(\Sigma)_{k\vec{J}, S_{bt}} \right)^{\psi}.$$

Thus $\hat{\jmath}(\Sigma)^{k}_{\vec{J}, W_{bt}} \in \hat{\jmath}_1(\hat{\jmath}(\Gamma))$. Writing

$$\Omega = \hat{\jmath}(\Sigma)^{k}_{\vec{J}, W_{bt}} \text{ on } HC^{N[\ell]},$$

we have Ω in $\hat{\jmath}_1(\hat{\jmath}(\Gamma))$, and

$$\Omega = \left((\Sigma_{H_2^+})^{\sigma} \right)^{\psi}$$

so Ω is ~~in~~ $\hat{\jmath}_1(\hat{\jmath}(\Gamma))$-fullness preserving and has branch condensation, by $\hat{\jmath}_1(\hat{\jmath})$-condensation in $N[\ell]$.

Claim For any $A \subseteq \Theta^0$ in H_0^+,

$$\pi^{\Omega}_{W_{bt},\infty}(i^{\vec{J}}(A)) = j_1(j(A)) \cap K^{M_\infty(W_{bt},\Omega)}.$$

Proof It is enough to see that if $\varphi : W_{bt} \to R$ is by Ω, then $\alpha \in \varphi(i^{\vec{J}}(A))$ iff $\pi^{\Omega}_{R,\infty}(\alpha) \in j_1(j(A))$.

But we can lift φ to $\varphi^* : S_{bt} \to Q$ by $j(\Sigma)$. We have

$$\begin{tikzcd}
 & & H_2^* & \\
 & H_1^+ \arrow[ur, "j_1"] \arrow[r, "i^{k\vec{J}}"'] & S_{bt} \arrow[r, "\varphi^*"'] & Q \arrow[ull, "u"'] \\
H_0^+ \arrow[ur, "j"] \arrow[rr, "i^{\vec{J}}"'] & & W_{bt} \arrow[u, "\psi"'] \arrow[r, "\varphi"'] & R \arrow[u, "\sigma"']
\end{tikzcd}$$

where σ is the copy map, and u witnesses $j_1 \restriction H_1^+$ -consistency. But then

$$\alpha \in \varphi \circ i^{\vec{T}}(A) \text{ iff } \tau(\alpha) \in \varphi^* \circ i^{k\vec{T}} \circ j(A)$$

$$\text{iff } \pi^{\hat{j}(\Sigma)}_{Q,\infty}(\tau(\alpha)) \in j_1(j(A))$$

$$\text{iff } \pi^{\Omega}_{R,\infty}(\alpha) \in j_1(j(A)).$$

The second line is $j_1 \restriction H_1^+$ consistency for $\hat{j}(\Sigma)$. The last line is an application of $j_1(j)$-condensation (in the form stated in [3]), using that $\hat{j}(\Sigma)_Q = (\Sigma_{H_2^+})^u$, and $\Omega_R = (\hat{j}(\Sigma)_Q)^\tau$.

$\boxtimes$

The claim implies that if we define $u : W_{bt} \longrightarrow H_R^+$ by

$$u(i^{\vec{T}}(f)(a)) = j_1(j(f))(\pi^{\Omega}_{W_{bt},\infty}(a)),$$

then

$$\begin{tikzcd} & H_2^+ \\ H_0^+ \arrow[ur, "j_1 \circ j"] \arrow[r, "i^{\vec{I}}"'] & W_{bt} \arrow[u, "u"'] \end{tikzcd}$$

commutes, and Ω is u-consistent. But also

$N[\ell J] \models$ "$\hat{j}_1(\Sigma)$ is $j_1(j \restriction H_0^+) = j_1 \circ j \restriction H_0^+$ — consistent",

so by proposition 4, there is a $t : W_{bt} \longrightarrow H_2^+$ in $N[\ell J]$ such that

$$\begin{tikzcd} & H_2^+ \\ H_0^+ \arrow[ur, "j_1 \circ j"] \arrow[r, "i^{\vec{I}}"'] & W_{bt} \arrow[u, "v"'] \end{tikzcd}$$

commutes, and $\hat{j}_1(\Sigma)$ is v-consistent.

Since u and v agree on $\mathrm{ran}(i^{\vec{I}})$, we can apply theorem 6 in $N[\ell J]$. We get $\hat{j}_1(\Sigma)^*_{W_{bt}} = \Omega$. But $\Sigma \subseteq \hat{j}_1(\Sigma)$, so this is a contradiction.

$\boxtimes$

We now look at pulling back $\hat{j}(\Sigma)$ to higher points.

<u>Lemma 8</u> Assume $(\dagger)_\xi$, and let $(\mathcal{N}, \Sigma) = (\mathcal{N}^\circ_\xi, \dot{\psi}^h_\xi)$, where h is on $\mathrm{col}(\omega, <\kappa_1)$. Suppose $k : R \to j(\mathcal{N})$, $k \in M[h]$, R is countable in $M[h]$, and we have

$$\begin{tikzcd} & H_1^+ \\ H_0^+ \arrow[ur, "j"] \arrow[r, "i"'] & R_{bt} \arrow[u, "k \restriction R_{bt}"'] \end{tikzcd}$$

commuting, for some $i \in M[h]$. Suppose also that the pullback strategy $(\Sigma_{H_1^+})^{k \restriction R_{bt}}$ for R_{bt} is $k \restriction R_{bt}$-consistent. Then

(1) $(R, j(\Sigma)^k)$ is a $\hat{j}(\Gamma)$ hod pair, $j(\Sigma)^k$ having branch condensation, and $\hat{j}(\Sigma)^k \in \hat{j}(\Gamma)$, and

(2) $\hat{j}(\Sigma)^k$ is $k \restriction R_{bt}$-consistent.

Remark It was part of our $(+)_\xi$ that $\mathring{f}^{\mathcal{N}_\xi} \leq \omega$; that is, $\mathcal{N}_\xi$ declares itself to be full. We let $\mathcal{N}_\xi^0 = \mathcal{N}_\xi$ unless there is a last full proper initial segment of $\mathcal{N}_\xi$, and $\mathring{f}^{\mathcal{N}_\xi} = 0$. In that case we set $\mathcal{N}_\xi^0 = \mathcal{N}_\xi$, except that $\mathring{f}^{\mathcal{N}_\xi^0} = \omega$. In other words, if $\mathcal{N}_\xi$ was declaring itself to be the full $L_p^{\Sigma_Q}(Q)^\Gamma$, then $\mathcal{N}_\xi^0$ stops declaring this.

This has the effect of weakening the Γ-fullness-preserving clause in "Γ-hod pair". It is easy to see, using strategy coherence, that if $(\mathcal{N}, \Sigma) = (\mathcal{N}_\xi^0, \psi_\xi^h)$, then Σ is $\hat{j}(\Gamma)$-fullness preserving iff whenever R is a Σ-iterate of $\mathcal{N}$, via $\vec{\mathcal{J}}$, then $\Sigma_{\vec{\mathcal{J}}, R_{bt}}$ is $\hat{j}(\Gamma)$

R is as in the statement of lemma 8, then $\hat{j}(\Sigma)^k$ is $\hat{j}(\Gamma)$ fullness-preserving iff whenever W is an iterate of R by $\hat{j}(\Sigma)^k$, ~~then~~ say by $\vec{\mathcal{T}}$, then $\hat{j}(\Sigma)^k_{\vec{\mathcal{T}}, W_{bt}}$ is $\hat{j}(\Gamma)$-fullness-preserving as a strategy for W_{bt}.

There is nothing in the hypotheses of lemma 8 to guarantee that R is $\hat{j}(\Gamma)$-full "at the very top". When we come to use lemma 8, we will ~~have an extender E certified by k such that (R, E) is a hod premouse~~ get fullness at the top from the existence of a next extender.

<u>Remark</u> $\left(\Sigma_{H_1^+}\right)^{k \upharpoonright R_{bt}} = \left(\Sigma_{H_1 | \sup k'' \kappa^R}\right)^{k \upharpoonright R_{\cancel{bt}} \upharpoonright \kappa^R}$

$= \left(\hat{j}(\Sigma)^k\right)_{R_{bt}}$. To say that is is k-consistent is just to say that its iteration map and k agree ~~up~~ to κ^R.

Proof of lemma 8

Note that $k = \tau_h$, where τ is a term coded by a set of ordinals of size $< \kappa_1$ in M. ${}^{\kappa_1}N \subseteq N$ in M, so $\tau \in N$, so $k \in N[l]$.

An easy absoluteness argument then shows that there is a $u \in N[l]$ such that

$$\begin{tikzcd}
 & & j_1(j(\eta)) \\
j(\eta) \arrow[urr, "u"] & & \\
 & R \arrow[ul, "k"] \arrow[uur, "\hat{j}_1(k)"'] &
\end{tikzcd}$$

commutes, and $u \restriction H_1^+ = \hat{j}_1 \restriction H_1^+$.

Note here $\hat{j}_1(k)$ and $\hat{j}_1 \restriction H_1^+$ are in $N[\mathcal{l}]$, and $j(\eta)$ is countable there.

Then in $N[\mathcal{l}]$ we have

$$\hat{j}_1(\hat{j}(\Sigma)^k) = \hat{j}_1(\hat{j}(\Sigma))^{\hat{j}_1(k)}$$

$$= \left(\hat{j}_1(\hat{j}(\Sigma))^u\right)^k$$

$$= \hat{j}(\Sigma)^k$$

where we have tacitly restricted both strategies to $HC^{N[\mathcal{l}]}$. The last equality comes from lemma 7, used in $N[\mathcal{l}]$. But

$$N[\mathcal{l}] \models \hat{j}(\Sigma) \in \hat{j}_1(\hat{j}(\Gamma))$$

so

$$N[\mathcal{l}] \models \hat{j}(\Sigma)^k \in \hat{j}_1(\hat{j}(\Gamma))$$

so

$$N[\mathcal{l}] \models \hat{j}_1(\hat{j}(\Sigma)^k) \in \hat{j}_1(\hat{j}(\Gamma))$$

so

$$M[h] \models \hat{j}(\Sigma)^k \in \hat{j}(\Gamma).$$

Next we show that $\hat{j}(\Sigma)^k$ is $\hat{j}(\Gamma)$-fullness preserving in $M[h]$. Let $\tilde{\mathcal{J}}$ be a countable stack of trees on R in $M[h]$ with last model W, and according to $\hat{j}(\Sigma)^k$. Moving to $N[\ell]$, we have the diagram

$$\begin{tikzcd}
 & & j_1(j(R)) \\
j(R) \arrow[rru, "u"] \arrow[rr, "i^{k\vec{I}}"] & & Q \arrow[u, "t"'] \\
R \arrow[u, "k"] \arrow[rr, "i^{\vec{I}}"'] & & W \arrow[u, "\psi"']
\end{tikzcd}$$

Here $k\vec{I}$ is by $\hat{j}(\Sigma)$, ψ is the copy map, and t is the map we get from $\hat{j}(\Sigma)$ being u-consistent in $N[\ell]$. Restricting to bottom parts, we have

$$\begin{tikzcd}
 & & H_2^+ \\
 & H_1^+ \arrow[ru, "j_1"] \arrow[r, "i^{k\vec{I}}"'] & Q_{bt} \arrow[u, "t"'] \\
H_0^+ \arrow[ru, "j"] \arrow[rr, "i^{\vec{I}}"'] & & W_{bt} \arrow[u, "\psi"']
\end{tikzcd}$$

We can now apply $j_1(j)$-condensation in $N[\ell]$ to see that $(W_{bt}, (\Sigma^*_{H_2^+})^{\psi\circ t})$ is $\hat{j}_1(\hat{j}(\Gamma))$-full.

So $(W_{bt}, \hat{j}(\Sigma)^k_{\vec{I}, W_{bt}})$ is $\hat{j}_1(\hat{j}(\Gamma))$ full in

$N[\ell]$, and hence $\hat{j}(\Gamma)$-full in $M[h]$.

To finish the proof of part (1), we must show that $\hat{j}(\Sigma)^k$ has branch condensation. The proof is quite similar to the proof of 2.16 of [2], so we just sketch it. Let

$$\Omega = \hat{j}(\Sigma)^k .$$

If branch condensation fails, we get ~~$\vec{\mathcal{T}}$ by $\mathcal{U}$ by Ω, with $\mathcal{U}$ normal~~ $\vec{\mathcal{T}}$ on R with last model W, and $\mathcal{U}$ normal on W_{bt} with distinct cofinal branches b and c such that

$$c = \Omega(\vec{\mathcal{T}}^\frown \mathcal{U})$$

and there is an iteration map $\ell_0 : R \to Q$ via $\vec{\mathcal{T}}_0$ by Ω and $\pi : M_b^{\mathcal{U}} \to Q$ such that $\ell_0 = \pi \circ i_b^{\mathcal{U}} \circ i^{\vec{\mathcal{T}}}$. The picture is

$$\begin{tikzcd}
 & & & Q \\
 & & & M_b^{\mathcal{U}} \arrow[u, "\pi"] \\
R \arrow[rrruu, bend left, "i"] \arrow[r, "i_{\vec{T}}"] & W \arrow[rru, "i_b"] \arrow[rrd, "i_c"'] & & \\
 & & & M_c^{\mathcal{U}}
\end{tikzcd}$$

We may assume that $\delta(\mathcal{U}) = i_b(\delta) = i_c(\delta)$, and there is $\eta < \delta$ such that η is a cutpoint of W, all critical points of $\mathcal{U}$ are above η, and by minimizing, that the tails $\left(\Omega_{\vec{T}_0, Q}\right)^{\pi}_{M(\mathcal{U})}$ and

$\Omega_{\vec{T}^\frown \mathcal{U}^\frown c,\, M(\mathcal{U})}$ are equal; say to Λ.

$P = Lp^{\Lambda}(M(\mathcal{U}))^{\Gamma}$ is an initial segment of both $M_b^{\mathcal{U}}$ and $M_c^{\mathcal{U}}$. Let Φ_b and Φ_c be the two tails, acting on P.

We line them up, then look at everything from the point of view of $N \| J$, as we did before. Restricting maps to bottom parts, we get the diagram

$$
\begin{tikzcd}
 & & & H_2^+ & \\
 & & & {} \arrow[r, "l_1^*"] & \Delta^* \arrow[ul, "u"'] \\
 & H_1^+ \arrow[uurr, "j_1"] \arrow[urr, "l_0^*"] & & Q_{bt} \arrow[r, "l_1"] & \bullet \arrow[u, "\psi"'] \\
 & & & (M_b^{\mathcal{U}})_{bt} \arrow[u, "\pi"'] \arrow[r, "i_{b_1}"] & \mathcal{S} \arrow[u, "r"'] \\
H_0^+ \arrow[uur, "\dot{j}"] \arrow[r, "i"'] \arrow[ddr, "\dot{j}"'] & R_{bt} \arrow[uu, "k"] \arrow[dd, "k"] \arrow[uurr, "l_0"] \arrow[r, "i_{\vec{J}}"'] & W_{bt} \arrow[ur] \arrow[dr] & & \\
 & & & (M_c^{\mathcal{U}})_{bt} \arrow[r, "i_{c_1}"] & \nu \arrow[dl, "t_0"] \\
 & H_1^+ \arrow[rr, "s"] \arrow[drr, "\dot{d}_1"'] & & {} \arrow[d, "t_1"] & \\
 & & & H_2^+ &
\end{tikzcd}
$$

Here $\mathcal{U}_1$, b_1, and c_1 come from a single normal tree on P lining up the images (tails) of Φ_b and Φ_c; b_1 is by Φ_b and c_1 is by Φ_c. l_1 is by $\Omega_{Q_{bt}}$, which is a tail of $\hat{j}(\Sigma)^k$, and $\ell_1^* \circ \ell_0^*$ are the k-lifts of ℓ_0 and ℓ_1, hence by $\hat{j}(\Sigma)$ in $N[\ell]$. u is the map we get by $j_1 \upharpoonright H_1^+$-consistency of $\hat{j}(\Sigma)$ in $N[\ell]$. s is the map we get by lifting $\vec{\mathcal{I}}^\frown \mathcal{U}^\frown c^\frown \mathcal{U}_1^\frown c_1$ to a tree on $j(\eta)$ by $\hat{j}(\Sigma)$, and t_0 is the copy map. t_1 is the map we get by j_1-consistency of $\hat{j}(\Sigma)$ in $N[\ell]$.

Let $\gamma_1 = i_{b_1} \circ i_b \circ i^{\vec{\mathcal{I}}} \circ i$ and $\gamma_2 = i_{c_1} \circ i_c \circ i^{\vec{\mathcal{I}}} \circ i$. It will be enough to show

$$\gamma_1(f)(a) = \gamma_2(f)(a)$$

whenever $f \in H_0^+$ and $a \in [\eta]^{<\omega}$ and $\gamma_1(f)(a) < \delta(\mathcal{U}_1)$.

But let

$$\langle a, \beta \rangle \in \gamma_1(f),$$

then

$$\varphi \circ \gamma(\langle a, \beta \rangle) \in l_1^* \circ l_0^* (j(f))$$

so

$$\pi_{y^*,\infty}(\varphi \circ \gamma(\langle a, \beta \rangle)) \in j_1(j(f))$$

(by j_1-consistency of $\overset{n}{j}(\tau)$),

so

$$\pi_{\mathcal{S},\infty}(\langle a, \beta \rangle) \in j_1(j(f)).$$

by ~~j~~ $j_1(j)$-condensation in $N[Q]$, so

$$\pi_{\mathcal{N},\infty}(\langle a, \beta \rangle) \in j_1(j(f)),$$

so

$$\pi_{X,\infty}(t_0(\langle a, \beta \rangle)) \in j_1(j(f))$$

so

$$t_0(\langle a, \beta \rangle) \in s(j(f))$$

so

$$\langle a, \beta \rangle \in \gamma_2(f)$$

as desired.

Branch cond. $\boxtimes$

Finally, we show $j(\mathcal{E})^k$ is $k \restriction R_{bt}$ - consistent. Let in $M[h]$

$$t : R \longrightarrow S$$

be by $j(\mathcal{E})^k$. We must see that for $B \subseteq \kappa^R$ in R, and $\alpha < \kappa^S$,

$$\alpha \in t(B) \text{ iff } \pi^{\Omega}_{S_{bt},\infty}(\alpha) \in k(B),$$

where $\Omega = j(\mathcal{E})^k_{S_{bt}}$. We first consider the case that $B = i(A)$, where $A \in H_0^+$.

<u>Claim</u> For $A \subseteq H_0$ with $A \in H_0^+$, and $\alpha < \kappa^S$, $\alpha \in t(i(A))$ iff $\pi^{\Omega}_{S_{bt},\infty}(\alpha) \in j(A)$.

<u>Proof</u> Moving over by $\hat{j}'$, we have in $N[g]$ the diagram

$$\begin{tikzcd}
 & & H_2^+ & \\
 & H_1^+ \arrow[ur, "j_1"] \arrow[rr, "u"] & & Q_{bt} \arrow[ul, "\sigma"'] \\
H_0^+ \arrow[ur, "j"] \arrow[r, "i"'] & R_{bt} \arrow[u, "k"] \arrow[rr, "t"'] & & S_{bt} \arrow[u, "\psi"']
\end{tikzcd}$$

Here u is by $\hat{j}(\Sigma)$ lifting t (so $u \in M[h]$), ψ is the copy map, and σ comes from $j_1 \restriction H_1^+$ - consistency of $\hat{j}(\Sigma)$ in $N[\ell]$. We have

$$\hat{j}_1(\Omega) = \left(\Sigma_{H_2}\right)^{\sigma \circ \psi}$$

in $N[\ell]$. (This is because $\hat{j}_1(\Omega) = \hat{j}_1\left(\hat{j}(\Sigma)^k_{S_{bt}}\right) = \hat{j}_1\left(\hat{j}(\Sigma)^k\right)_{S_{bt}} =$ $\hat{j}(\Sigma)^k_{S_{bt}}$ on $HC^{N[\ell]}$, as shown above. ⊠)

But $\hat{j}(\Sigma)^k_{S_{bt}} = \left(\hat{j}(\Sigma)_{Q_{bt}}\right)^{\psi} = \left(\left(\Sigma_{H_2}\right)^{\sigma}\right)^{\psi}$, the last step holding because ~~by~~ of $j_1 \restriction H_1^+$ consistency of $\hat{j}(\Sigma)$ in $N[\ell]$.)

So it is enough to show that in $N[\ell]$,

$$\alpha \in t(i(A)) \text{ iff } \pi^{\Lambda^\psi}_{S_{bt},\infty}(\alpha) \in j_1(j(A)),$$

where

$$\Lambda = \hat{j}(\Sigma)_{Q_{bt}} = \Sigma^\sigma_{H_2}.$$

However

$$\begin{aligned} \alpha \in t(i(A)) &\text{ iff } \psi(\alpha) \in u(j(A)) \\ &\text{ iff } \pi^{\Lambda}_{Q_{bt},\infty}(\psi(\alpha)) \in j_1(j(A)) \\ &\text{ iff } \pi^{\Lambda^\psi}_{S_{bt},\infty}(\alpha) \in j_1(j(A)), \end{aligned}$$

by $j_1 \restriction H_1^+$ consistency and $j_1(j)$-condensation in $N[\ell]$. This proves our claim.

$\boxtimes$

Now let $B = i(f)(a)$ be an arbitrary subset of κ^R in R, with

$a \in (\kappa^R)^{<\omega}$ and $f : \theta^{|a|} \to H_0^+$ in H_0^+.

Notice that $\pi^{\Omega}_{S_{bt},\infty} \circ t \restriction \kappa^R$ is the iteration map by $\tilde{j}(\mathcal{E})^k$, so by our assumption on k,

$$k(a) = \pi^{\Omega}_{S_{bt},\infty}(t(a)).$$

But then, for $\alpha < \kappa^S$,

$$\begin{aligned}
\alpha \in t(B) \text{ iff } & \alpha \in t(i(f))(t(a)) \\
\text{iff } & \pi^{\Omega}_{S_{bt},\infty}(\alpha) \in \pi_{S_{bt},\infty}(t(i(f)))(k(a)) \\
\text{iff } & \pi^{\Omega}_{S_{bt},\infty}(\alpha) \in \cancel{\ldots} \; j(f)(k(a)) \\
\text{iff } & \pi^{\Omega}_{S_{bt},\infty}(\alpha) \in k(i(f))(k(a)) \\
\text{iff } & \pi^{\Omega}_{S_{bt},\infty}(\alpha) \in k(B),
\end{aligned}$$

as desired.

Lemma 8. $\boxtimes$

We can do a little better than lemma 8 if we have another extender to add.

<u>Lemma 9</u> Assume $(\dagger)_\xi$, and all the other hypotheses and notation of lemma 8. Suppose that in addition

$$k : (R, F) \longrightarrow (j(\mathcal{M}), G)$$

is Σ_1, where G is $j_1{}^\wedge H_1^+$-certified over $\hat{j}(\Sigma)$ in $N[\ell]$; then $\hat{j}(\Sigma)^k$ is $\hat{j}(\Gamma)$-fullness preserving in $M[h]$.

<u>Proof</u>. It is enough to show $\hat{j}(\Sigma)^k$ is $\hat{j}_1(\hat{j}(\Gamma))$-fullness preserving in $N[\ell]$. Let $Ult(R, F) = S$ and $Ult(j(\mathcal{M}), G) = W$, with i_F and i_G the canonical embeddings. Restricting maps to bottom parts, we have

in $N\{\ell J$:

$$\begin{tikzcd}
 & & H_2^+ \\
 & H_1^+ \arrow[ur, "j_1"] \arrow[r, "i_G"'] & W_{bt} \arrow[u, "u"'] \\
H_0^+ \arrow[ur, "j"] \arrow[r, "i"'] & R_{bt} \arrow[u, "k"] \arrow[r, "i_F"'] & S_{bt} \arrow[u, "r"']
\end{tikzcd}$$

By coherence, R is an initial segment of S_{bt}, and $j(\eta)$ is an initial segment of W_{bt}. Moreover, $r \restriction R = k$, and $\hat{j}(\Sigma) = \left(\Sigma_{H_2}^{u}\right)_{j(\eta)}$, because u agrees with the iteration map by $\hat{j}(\Sigma)$ on $j(\eta)$. Thus

$$j(\Sigma)^k = \left(\left(\Sigma_{H_2}^{u}\right)^{r}\right)_R.$$

We have that $\left(\Sigma_{H_2}^{u}\right)^{r}$ is $\hat{j}_1(\hat{j}(\Gamma))$-fullness preserving by $j_1(j)$-condensation, so we're done.

This leads to

<u>Lemma 10</u>. Assume $(\dagger)_\xi$, and let $(\mathcal{N}, \Sigma) = (\mathcal{N}_\xi, \dot{\Psi}^h_\xi)$. Suppose that $(j(\mathcal{N}), G)$ is a hod premouse, and G is $j_1 \restriction H_1^+$-certified over $\hat{j}(\Sigma)$ in $N[g]$. Suppose that in $M[h]$, we have

$$k : (R, F) \longrightarrow (j(\mathcal{N}), G)$$

Σ_1 elementary and $i : H_0^+ \longrightarrow R_{bt}$ such that $k \circ i = j \restriction H_0^+$. Suppose also that $\dot{j}(\Sigma)^k_{R_{bt}}$ is k-consistent; then

in $M[h]$:

(a) F is $k \restriction R_{bt}$-certified over $\dot{j}(\Sigma)^k$,

and

(b) Let $(S, E) = \mathrm{Ult}_0((R, F), F)$, and $\tau : (S, E) \longrightarrow (j(\mathcal{N}), G)$ be given by (a), that is, $\tau(i_F(g)(a)) = k(g)(\pi^{\Omega}_{R,\infty}(a))$, where $\Omega = \dot{j}(\Sigma)^k$. Then

(i) $\dot{j}(\Sigma)^\tau$ is τ-consistent,

(ii) $(\dot{j}(\Sigma)^\tau)_R = \dot{j}(\Sigma)^k$.

Proof. For (a): let $\Omega = j(\Sigma)^k$.
Let $B \subseteq \kappa^R$ and $B \in R$, and let $a \in [\text{lh}\,G]^{<\omega}$. We must show that $a \in i_F(B)$ iff $\pi^{\Omega}_{R,\infty}(a) \in k(B)$. Moving over to $N[\ell]$, we must show

$$a \in i_F(B) \text{ iff } \pi^{\Omega^*}_{R,\infty} \in j_1(k(B)),$$

where $\Omega^* = \hat{j}_1(\Omega)$ is the extension of Ω on ~~[illegible]~~ $HC^{M[h]}$ to $HC^{N[\ell]}$.

Fix $c \in (\kappa^R)^{<\omega}$ and $f \in H_0^+$ so that $B = i(f)(c)$. Such c exists because i must be cofinal in R_{bt}, as j was cofinal in H_1^+. Then

$$\begin{aligned} k(B) &= k(i(f)(c)) \\ &= j(f)(k(c)) \\ &= j(f)(\pi^{\Omega}_{R_{bt},\infty}(c)). \end{aligned}$$

Thus

$$j_1(k(B)) = j_1 j(f)\left(\pi_{R_{bt,\infty}}^{\Omega^*}(c)\right).$$

We then get, referring to the diagram in the proof of lemma 9,

$$\begin{aligned}
a \in i_F(B) \text{ iff } & (a,c) \in i_F(i(f)) \\
\text{iff } & (r(a), r(c)) \in i_G(j(f)) \\
& \cancel{\text{iff } \pi^{\Omega^*}} \\
\text{iff } & \pi_{W_{bt,\infty}}^{\Sigma^u_{H_2}}(r(a), r(c)) \in j_1(j(f)) \\
\text{iff } & \pi_{R_{bt,\infty}}^{\Omega^*}(a,c) \in j_1(j(f)) \\
\text{iff } & \pi_{R_{bt,\infty}}^{\Omega^*}(a) \in j_1(j(f))\left(\pi_{R_{bt,\infty}}^{\Omega^*}(c)\right) \\
\text{iff } & \pi_{R_{bt,\infty}}^{\Omega^*}(a) \in j_1(j(B)),
\end{aligned}$$

as desired. This proves (a).

Part (i) of (b) follows at once from part (2) of lemma 8, with the k of lemma 8 taken to be τ.

Since τ agrees with ~~[illegible]~~ the iteration map by $j(\Sigma)^k$ on R, and $j(\Sigma)^\tau$ is τ-consistent,

$$j(\Sigma)^\tau_R = j(\Sigma)^k .$$

Lemma 10. $\boxtimes$

Suppose now we have $(\dagger)_\xi$, and $(\mathcal{M}_\xi, \dot{\psi}^h_\xi) = (\mathcal{N}, \Sigma)$, and we have E certified by j so that $(\mathcal{N}, E)$ is a hod pmmouse. (And $\mathcal{N}$ was extender-ready in $M[h]$.) We can easily use the lemmas above to get an iteration strategy for $(\mathcal{N}, E)$ in $M[h]$. We maintain that for (R, F) the current iterate, we have the situation of lemma 10, with $G = j(E)$.

References

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[2] Steel, Hod mice below LST$^-$

[3] Steel, Remarks on a paper of Sargsyan