Woodin's Analysis of $\mathrm{HOD}^{L(\mathbb{R})}$

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1 Introduction

This paper is an exposition of some work of Hugh Woodin, work which identifies $\mathrm{HOD}^{L(\mathbb{R})}$ as a core model augmented with information as to how to iterate it. More precisely, Woodin shows that there is a class size mouse $\mathcal{M}$ which is ω-small, satisfies "There are ω Woodin cardinals", and a partial iteration strategy Σ for $\mathcal{M}$, such that $\mathrm{HOD}^{L(\mathbb{R})} = \mathcal{M}[\Sigma]$. The domain of Σ is the set of iteration trees in $\mathcal{M}$ which are on $J^{\mathcal{M}}_{\delta_0}$, for δ_0 the least Woodin cardinal of $\mathcal{M}$, and have $\mathcal{M}$-cardinality strictly less than the sup of the Woodin cardinals of $\mathcal{M}$.

One needs a large cardinal hypothesis for this result, of course. Woodin has shown that $\mathrm{AD}^{L(\mathbb{R})}$ suffices, but it is easier to work with a slightly stronger hypothesis, and that is what we shall do here.

Definition 1.1 *We say $M^\sharp_\omega$* **exists** *iff there is a premouse $\mathcal{P}$ which is not ω-small such that $\mathcal{P}$ is ω_1+1-iterable, not just in V, but also in $V^{Col(\omega,P(\mathbb{R}))}$. The least such premouse (in the mouse order) is called $M^\sharp_\omega$.*

If it exists, $M^\sharp_\omega$ is countable, and in fact ω-sound and projects to ω. Standard arguments show that there is in fact a *unique* ω_1+1 iteration strategy for $M^\sharp_\omega$ in V, and this strategy extends to a unique ω_1+1 iteration strategy in $V^{Col(\omega,P(\mathbb{R}))}$. $M^\sharp_\omega$ is active, and if we iterate its last extender through the ordinals, we leave behind a class size premouse which is ω-small and satisfies "There are ω Woodin cardinals". We call this premouse M_ω. (See [1, section 4].) The existence of $M^\sharp_\omega$ (in the precise sense defined above) actually follows from the existence of M_ω (in the parallel sense). This is because one can iterate M_ω so as to make a real coding it generic over the iterate for the collapse of its bottom Woodin cardinal.

Theorem 1.2 (Woodin) *If $M_\omega^\sharp$ exists, then $AD^{L(\mathbb{R})}$ holds and $\mathbb{R}^\sharp$ exists.*

Proof.(Sketch) In $V^{Col(\omega,\mathbb{R})}$ we can form an iterate $\mathcal{P}$ of $M_\omega^\sharp$, and a symmetric extension of $\mathcal{P}$ via the collapse of everything below the sup of its Woodin cardinals to ω whose set of reals is $\mathbb{R}^V$. (See [1, section 4].) By a result of Woodin, we then have $L_\alpha(\mathbb{R}^V) \models \text{AD}$, where $\alpha = \text{OR} \cap \mathcal{P}$. But we can iterate the last extender predicate of $\mathcal{P}$, which prolongs to the collapse extension because it is small forcing, through the ordinals. This yields AD in $L(\mathbb{R}^V)$; further, the iteration points for the last extender are indiscernibles for $L(\mathbb{R}^V)$, so $\mathbb{R}^\sharp$ exists in V. $\square$

One reason to work with the assumption that $M_\omega^\sharp$ exists is that it yields a natural and intuitive description of the premouse and iteration strategy which constitute $\text{HOD}^{L(\mathbb{R})}$. Let Σ be the unique ω_1+1-iteration strategy for $M_\omega^\sharp$ of $V^{Col(\omega,\mathbb{R})}$, and note that the restriction of Σ to V is in V. In V, set

$$\begin{aligned}\mathcal{F} \;=\; \{\, \mathcal{M}_\alpha^{\mathcal{T}} \mid (&\mathcal{T} \text{ is an iteration tree on } M_\omega^\sharp \text{ played}\\ &\text{according to } \Sigma, \text{ and } \alpha<\omega_1 \text{ and } D^{\mathcal{T}}\cap[0,\alpha]_T=\emptyset \text{ , and}\\ &\forall\beta\in[0,\alpha]_T(\text{crit}(i^{\mathcal{T}}_{\beta,\alpha}) < \text{ least Woodin cardinal of } \mathcal{M}_\beta^{\mathcal{T}}).\}\end{aligned}$$

That is, $\mathcal{F}$ is the set of countable, no dropping, Σ-iterates of $M_\omega^\sharp$ via iteration trees which can be regarded as being on $J_\delta^{M_\omega^\sharp}$, where δ is the least Woodin cardinal of $M_\omega^\sharp$. For $\mathcal{P},\mathcal{Q}\in\mathcal{F}$, put

$$\mathcal{P} \preceq \mathcal{Q} \text{ iff } \mathcal{Q} \text{ is a } \Sigma\text{-iterate of } \mathcal{P}.$$

It is easy to see that if $\mathcal{P} \preceq \mathcal{Q}$, then any Σ-iteration leading from $\mathcal{P}$ to $\mathcal{Q}$ does not drop, and only uses extenders from below the image of the bottom Woodin cardinal of $\mathcal{P}$. (We could have just made this part of the definition of $\preceq$.) Every $\mathcal{P}$ in $\mathcal{F}$ is δ-sound, where δ is its least Woodin cardinal, so by standard arguments the coiteration of $\mathcal{P}$'s in $\mathcal{F}$ using Σ results in iteration trees which do not drop on their main branches and which only use extenders from below the images of the bottom Woodin cardinal. It follows that $(\mathcal{F},\preceq)$ is countably directed. If $\mathcal{P}\preceq\mathcal{Q}$, then there is by Dodd-Jensen a unique Σ-iteration map from $\mathcal{P}$ to $\mathcal{Q}$ (although there may be more than one play according to Σ giving rise to this map). With these maps, $(\mathcal{F}, \preceq)$ is a direct limit system; let $\mathcal{M}_\infty$ be the direct limit.

Let X be the set of iteration trees in $\mathcal{M}_\infty$ which are on $J_\delta^{\mathcal{M}_\infty}$, for $\delta =$ least Woodin cardinal of $\mathcal{M}_\infty$, and have $\mathcal{M}_\infty$-cardinality strictly less than the sup of the Woodin cardinals of $\mathcal{M}_\infty$. In $V^{Col(\omega,\mathbb{R})}$, $\mathcal{M}_\infty$ is a countable iterate of $M_\omega^\sharp$ via Σ; we get this iteration by choosing a $\preceq$-cofinal subset of $\mathcal{F}$ of $\preceq$-order type ω and stacking the iterations between successive elements. Thus Σ acts on X, and in fact $\Sigma \restriction X$ is in V. Let $\mathcal{M}$ be the result of iterating the last extender of $\mathcal{M}_\infty$ through the ordinals.

Theorem 1.3 (Woodin) *Assume that $M_\omega^\sharp$ exists, and let $\mathcal{M}, \Sigma,$ and X be as described above; then $HOD^{L(\mathbb{R})} = \mathcal{M}[\Sigma \restriction X]$.*

Our purpose in this note is to record a proof of 1.3. The proof we give here is due to Woodin, and was communicated by him to the author. As to prerequisites, the reader who is still with us is probably able to follow the rest. Woodin's work was inspired by, and is an extension of, [2]. Although familiarity with that paper certainly wouldn't hurt, it is not essential.

2 A-iterability

The system $(\mathcal{F}, \preceq)$ is not in $L(\mathbb{R})$, but we shall "cover" it with a direct limit system which is in $L(\mathbb{R})$, and in fact definable over $L(\mathbb{R})$, and has the same direct limit .(That is, it does if we ignore the last extender.)

Definition 2.1 *A premouse $\mathcal{M}$ is* **full** *iff whenever B is an $OD^{L(\mathbb{R})}(\{\mathcal{M}\})$ set of ordinals, then $\forall \alpha \in OR^{\mathcal{M}}(B \cap \alpha \in \mathcal{M})$.*

Definition 2.2 *A premouse $\mathcal{M}$ is* **suitable** *iff $\mathcal{M}$ is ω-small, full, and $\mathcal{M} \models$ (ZFC-Powerset + "There is a Woodin cardinal whose cardinal successor is the largest cardinal").*

If $\mathcal{M}$ is suitable, then it has some finite number k of Woodin cardinals, $1 \leq k < \omega$. We write $k = k(\mathcal{M})$, and let $\delta_0^{\mathcal{M}}, ..., \delta_{k-1}^{\mathcal{M}}$ be the Woodin cardinals of $\mathcal{M}$.

Definition 2.3 *Let $\mathcal{M}$ be suitable, $k+1 = k(\mathcal{M})$, and $A \subset \mathbb{R}$; then*

1. *τ is an **$\mathcal{M}$-term** iff $\emptyset$ forces over $\mathcal{M}$ in $Col(\omega, \delta_k^{\mathcal{M}})$ that τ is a set of reals;*

2. *An $\mathcal{M}$-term τ is **invariant** iff whenever G and H are $\mathcal{M}$-generic over $Col(\omega, \delta_k^{\mathcal{M}})$ and $\mathcal{M}[G] = \mathcal{M}[H]$, then $\tau^G = \tau^H$;*

3. *An $\mathcal{M}$-term τ **captures** A (**over** $\mathcal{M}$) iff whenever G is $\mathcal{M}$-generic over $Col(\omega, \delta_k^{\mathcal{M}})$, then $\tau^G = A \cap \mathcal{M}[G]$.*

Clearly, if τ captures some A over $\mathcal{M}$, then τ is an invariant $\mathcal{M}$-term. Also, the set of invariant $\mathcal{M}$-terms is definable over $\mathcal{M}$. There may be more than one $\mathcal{M}$-term capturing a given set A of reals, but we can pick a standard one.

Definition 2.4 *Let $\mathcal{M}$ be suitable,$k+1 = k(\mathcal{M})$, and τ be an $\mathcal{M}$-term; then*

1.
$$\begin{aligned}\tau^* \;=\; & \{ (p, \sigma) \mid \sigma \subseteq Col(\omega, \delta_k^{\mathcal{M}}) \times \{ \check{n} \mid n \in \omega\} \text{ and} \\ & p \text{ forces in } Col(\omega, \delta_k^{\mathcal{M}}) \text{ that } \sigma \in \tau\}.\end{aligned}$$

2. *τ is **standard** iff $\tau = \tau^*$.*

It is clear that $\emptyset$ forces $\tau = \tau^*$, and that τ^* is standard.

Lemma 2.5 *Let σ and τ be invariant $\mathcal{M}$-terms, and $k + 1 = k(\mathcal{M})$; then the following are equivalent:*

1. *$\sigma^* = \tau^*$,*

2. *for all G which are $\mathcal{M}$-generic over $Col(\omega, \delta_k^{\mathcal{M}})$, $\sigma^G = \tau^G$,*

3. *for some G which is $\mathcal{M}$-generic over $Col(\omega, \delta_k^{\mathcal{M}})$, $\sigma^G = \tau^G$.*

We omit the easy proof of lemma 2.5.

Corollary 2.6 *If σ and τ capture A over $\mathcal{M}$, then $\sigma^* = \tau^*$. Thus any A captured over $\mathcal{M}$ is captured by exactly one standard term.*

Definition 2.7 *Suppose A is captured over $\mathcal{M}$; then we let $\tau_A^{\mathcal{M}}$ be the unique standard term which captures A over $\mathcal{M}$*

Definition 2.8 *$(\mathcal{M}, A_1, ..., A_n)$ is* **good** *iff $\mathcal{M}$ is suitable, and $A_1, ..., A_n$ is a finite list of $OD^{L(\mathbb{R})}$ sets of reals which are captured over $\mathcal{M}$.*

We now introduce a notion of iterability for good $(\mathcal{M}, \vec{A})$. Recall the weak iteration game $\mathcal{WG}(\mathcal{M}, \omega)$ of length ω on $\mathcal{M}$ described in [1]. (In this game, I plays countable putative iteration trees, starting with a tree on $\mathcal{M}$. II can accept this tree if it has a last, wellfounded model; otherwise he must play a maximal wellfounded branch of I's tree. The next tree played by I must be on the last model of the tree "certified" by II in the preceding round. All of I's trees must be appropriately n-maximal : $n = \omega$ for the first tree. If both players meet their obligations at round n for all finite n, then II wins iff the unique cofinal branch of the stack of ω trees which have been certified during the play is wellfounded.) $\mathcal{WG}(\mathcal{M}, \omega)$ is a game on $\mathbb{R}$ with Π_1^1 payoff; if II has a winning strategy then one says that $\mathcal{M}$ is "G-reals Π_1^1 iterable". If $\mathcal{M}$ and $\mathcal{N}$ are ω-small and G-reals Π_1^1 iterable, then they have a successful coiteration ([1]). We shall strengthen this notion of iterability, as it applies to good $(\mathcal{M}, \vec{A})$, by requiring of II that if he has certified an embedding $i_{0,\alpha}^{\mathcal{T}}$, then $(\mathcal{M}_\alpha^{\mathcal{T}}, \vec{A})$ is good, and $i_{0,\alpha}^{\mathcal{T}}(\tau_{A_m}^{\mathcal{M}}) = \tau_{A_m}^{\mathcal{M}_\alpha^{\mathcal{T}}}$ for all m.

Definition 2.9 *Let $(\mathcal{M}, \vec{A})$ be good; then*

1. *$\mathcal{WG}(\mathcal{M}, \vec{A}, \omega)$, the* **weak $\vec{A}$-iteration game (on $\mathcal{M}$, of length ω)**, *is the game with the same rules and payoff as $\mathcal{WG}(\mathcal{M}, \omega)$, except that we require in addition of* II *that if $\mathcal{T} = \mathcal{T}_0 \oplus \mathcal{T}_1 \oplus \mathcal{T}_2 ...$ is certified by* II*, and $\alpha < lh(\mathcal{T})$ is such that $D^{\mathcal{T}} \cap [0, \alpha]_T = \emptyset$, then $(\mathcal{M}_\alpha^{\mathcal{T}}, \vec{A})$ is good, and $i_{0,\alpha}^{\mathcal{T}}(\tau_{A_m}^{\mathcal{M}}) = \tau_{A_m}^{\mathcal{M}_\alpha^{\mathcal{T}}}$ for all m.*

2. *An* **$\vec{A}$-iteration strategy for** *$\mathcal{M}$ is a winning strategy for* II *in the weak $\vec{A}$-iteration game on $\mathcal{M}$. We say that $\mathcal{M}$ is* **$\vec{A}$-iterable** *(or that $(\mathcal{M}, \vec{A})$ is iterable) iff there is such a strategy. An* **$\vec{A}$-iteration** *of $\mathcal{M}$ is a finite initial segment of a play according to some $\vec{A}$-iteration strategy for $\mathcal{M}$.*

It is worth remarking that the rules of $\mathcal{WG}(\mathcal{M},\vec{A})$ require that $\mathcal{M}^{\mathcal{T}}_{\alpha}$ be full in the case that $[0,\alpha]_T$ does not drop. An ordinary weak iteration strategy for $\mathcal{M}$ need not preserve its fullness in this way, so even $\emptyset$-iterability is stronger than G-reals Π^1_1 iterability.

It is important that the additional rules of the weak $\vec{A}$ iteration game are "closed" conditions on finite initial segments of the play. That is, we require nothing of the final direct limit along the unique cofinal branch except that it be wellfounded. Because of this, the usual proof that winning a real game with Π^1_1 payoff is absolute between V and $L(\mathbb{R})$ goes through here.

Lemma 2.10 *If $\mathcal{M}$ is $\vec{A}$-iterable, then $L(\mathbb{R}) \models$ ($\mathcal{M}$ is $\vec{A}$-iterable).*

Notice that an $\vec{A}$-iteration is a *finite* initial segment of a play according to an $\vec{A}$-iteration strategy, so that the final model is also $\vec{A}$-iterable, provided that we haven't dropped in getting there. Nonetheless, $\vec{A}$-iterations suffice for comparison of good pairs; the reason is that we only have to go another move deeper in our iteration games when we pass the image of another Woodin cardinal from one of the base models we are comparing.

Lemma 2.11 *Let b and c be distinct cofinal branches of $\mathcal{T}$ such that both $\mathcal{T} \oplus b$ and $\mathcal{T} \oplus c$ are $\emptyset$ iterations of $\mathcal{M}$; then*

1. $D^{\mathcal{T}} \cap b = D^{\mathcal{T}} \cap c = \emptyset$,
2. *for some $j < k(\mathcal{M})$, $i_b(\delta^{\mathcal{M}}_j) = i_c(\delta^{\mathcal{M}}_j) = \delta(\mathcal{T})$.*

Proof. Suppose that $D^{\mathcal{T}} \cap b \neq \emptyset$. It follows that $\mathcal{Q}(b,\mathcal{T})$ (the first initial segment of $\mathcal{M}^{\mathcal{T}}_b$ defining a failure of $\delta(\mathcal{T})$ to be Woodin) exists. (See [1].) If also $D^{\mathcal{T}} \cap c \neq \emptyset$, then $\mathcal{Q}(c,\mathcal{T})$ exists as well. Since both $\mathcal{Q}$ structures are ω-small, G-reals Π^1_1 iterable, and $\delta(\mathcal{T})$-sound, they must be the same. This is a contradiction (see [1]). So $D^{\mathcal{T}} \cap c = \emptyset$, and therefore $\mathcal{M}^{\mathcal{T}}_c$ is full. But $\mathcal{Q}(b,\mathcal{T})$ is $OD^{L(\mathbb{R})}(J^{\mathcal{M}^{\mathcal{T}}_c}_{\delta(\mathcal{T})})$ as its unique G-reals Π^1_1 iterable $\mathcal{Q}$ structure, and so $\mathcal{Q}(b,\mathcal{T}) \in \mathcal{M}^{\mathcal{T}}_c$, which is again a contradiction. This proves (1), and (2) is an easy consequence. $\square$

Lemma 2.12 *Suppose $\mathcal{M}$ is $\vec{A}$-iterable and $\mathcal{N}$ is $\vec{B}$-iterable; then there is an $\vec{A}$-iteration of $\mathcal{M}$ with last model $\mathcal{P}$ whose $\mathcal{M}$-to-$\mathcal{P}$ branch does not drop, and a $\vec{B}$-iteration of $\mathcal{N}$ with last model $\mathcal{Q}$ whose $\mathcal{N}$-to-$\mathcal{Q}$ branch does not drop, such that either $\mathcal{P} = \mathcal{Q}$, or $\mathcal{P} = J^{\mathcal{Q}}_{\kappa}$ where $\kappa = (\delta^{++}_j)^{\mathcal{Q}}$ for some $j+1 < k(\mathcal{Q})$, or $\mathcal{Q} = J^{\mathcal{P}}_{\kappa}$ where $\kappa = (\delta^{++}_j)^{\mathcal{P}}$ for some $j+1 < k(\mathcal{P})$.*

Proof. We compare $\mathcal{M}$ with $\mathcal{N}$ as in [1, Theorem 1.10], using the given iteration strategies, going another move deeper in the iteration games and dovetailing in new comparisons whenever we get distinct $\vec{A}$ or distinct $\vec{B}$ iterable branches. By 2.11, this only happens finitely often, so indeed the comparison terminates successfully with iteration trees $\mathcal{T}$ and $\mathcal{U}$ which are obtained from finite plays of $\mathcal{WG}(\mathcal{M}, \vec{A})$ and $\mathcal{WG}(\mathcal{N}, \vec{B})$ respectively. Let $\mathcal{P}$ and $\mathcal{Q}$ be the last models, and assume without loss of generality that $\mathcal{P}$ is an initial segment of $\mathcal{Q}$. From general theory we get that $\mathcal{M}$-to-$\mathcal{P}$ does not drop. However, this is an $\vec{A}$ iteration, so $\mathcal{P}$ must be full. This easily implies that $\mathcal{N}$-to-$\mathcal{Q}$ does not drop either, and that $\mathcal{P}$ is an initial segment of $\mathcal{Q}$ of the indicated type. (See [2].) $\square$

3 The existence of A-iterable premice

We shall show that for every $\mathrm{OD}^{L(\mathbb{R})}$ tuple $\vec{A}$, there is an $\vec{A}$-iterable suitable premouse. The foundation for this and the other existence results we shall prove is the iterability of $M_\omega^\sharp$.

Definition 3.1 Σ_0 *is the unique* $\omega_1 + 1$*-iteration strategy for* $M_\omega^\sharp$.

Theorem 3.2 *Let* $\mathcal{M}$ *be a suitable initial segment of a* Σ_0*-iterate of* $M_\omega^\sharp$*, and* $\vec{A}$ *a tuple of sets of reals which is definable over* $L(\mathbb{R})$ *from a finite sequence of* $\mathbb{R}$*-indiscernibles; then the weak iteration strategy for* $\mathcal{M}$ *induced by* Σ_0 *is an* $\vec{A}$*-iteration strategy.*

Proof. We first show that each A_i is captured over $\mathcal{M}$. Let $A = A_i$ be definable over $L(\mathbb{R})$ from $c_1, ..., c_n$, where we may assume that $c_1, ..., c_n$ are chosen to be uncountable regular cardinals in $V^{Col(\omega,\mathbb{R})}$. Say $x \in A \Leftrightarrow L(\mathbb{R}) \models \varphi[x, \vec{c}]$.

Let $\mathcal{N}$ be an iterate of M_ω via Σ_0 such that $\mathcal{M}$ is an initial segment of $\mathcal{N}$. Let τ be an $\mathcal{M}$-term such that, letting $k + 1 = k(\mathcal{M})$, whenever G is $\mathcal{M}$-generic (equivalently, $\mathcal{N}$-generic) for $Col(\omega, \delta_k^{\mathcal{M}})$, then for all $x \in \mathbb{R} \cap \mathcal{M}[G] (= \mathbb{R} \cap \mathcal{N}[G]), x \in \tau^G$ iff it is forced over $\mathcal{N}[G]$ in the collapse up to the sup of the ω Woodin cardinals of $\mathcal{N}[G]$ that $L(\mathbb{R}^*) \models \varphi[\check{x}, \check{c}_1, ..., \check{c}_n]$. (Here $\mathbb{R}^*$ stands for the set of reals added by initial segments of the collapse.) Clearly there is a term in $\mathcal{N}$ with the property demanded of τ, and since $\mathcal{M}$

agrees sufficiently with $\mathcal{N}$, we get a term in $\mathcal{M}$ with this property. We claim that τ captures A over $\mathcal{M}$.

To see this, suppose G is $\mathcal{M}$-generic for $Col(\omega, \delta_k^{\mathcal{M}})$, and let $x \in \mathbb{R} \cap \mathcal{M}[G]$. As in the proof of 1.2, we can form in $V^{Col(\omega,\mathbb{R})}$ an iteration map $i: \mathcal{N} \to \mathcal{P}$ with critical point strictly greater than $\delta_k^{\mathcal{M}}$ and a generic object for the collapse of everything below the sup of the Woodins of $\mathcal{P}$ to ω extending G whose associated set of reals is $\mathbb{R}^V$. Now i lifts to an embedding i^* from $\mathcal{N}[G]$ to $\mathcal{P}[G]$. But then $\emptyset$ forces $L(\mathbb{R}^*) \models \varphi[\check{x}, \check{c}_1, ..., \check{c}_n]$ over $\mathcal{N}[G]$ iff $\emptyset$ forces $L(\mathbb{R}^*) \models \varphi[\check{x}, i(\check{c}_1), ..., i(\check{c}_n)]$ iff $L(\mathbb{R}^V) \models \varphi[x, i(c_1), ..., i(c_n)]$. Since the symmetric collapse is homogeneous, and $i(c_j) = c_j$ for all j, we see that $x \in \tau^G$ iff $x \in A$, as desired.

Next, we must see that if $j: \mathcal{M} \to \mathcal{Q}$ comes from an iteration via Σ_0, then $j(\tau_A^{\mathcal{M}})$ captures A over $\mathcal{P}$, for all A on $\vec{A}$. But j extends to $j^*: \mathcal{N} \to \mathcal{R}$ for some $\mathcal{R}$. Since $j^*(c_i) = c_i$ for all i, we see by means of the argument of the previous paragraph that for τ the term specified there, $j(\tau)$ captures A over $\mathcal{Q}$. But $\tau_A^{\mathcal{M}}$ is equivalent to τ, so $j(\tau_A^{\mathcal{M}})$ is equivalent to $j(\tau)$. This completes the proof. $\square$

Remark 3.2.5. One can show by a variant of this argument that if $\mathcal{M}$ is suitable and $\vec{A}$ is $\mathrm{OD}^{L(\mathbb{R})}$, then $\vec{A}$ is captured over $\mathcal{M}$. For one can construct a class-size premouse $\mathcal{N}$ having ω Woodin cardinals, extending $\mathcal{M}$, and being fully iterable for iteration trees with all critical points above $\mathrm{OR}^{\mathcal{M}}$. (See 5.1 ff. and 5.3 ff.) In $V^{Col(\omega,\mathbb{R})}$ form an iterate $\mathcal{P}$ of $\mathcal{N}$ above $\mathrm{OR}^{\mathcal{M}}$ as in the proof of 3.2, but do it simultaneously for all G, so that *any* $\mathcal{M}$-generic $G \in V$ over $Col(\omega, \delta_k^{\mathcal{M}})$ can be extended to a generic for the symmetric collapse of $\mathcal{P}$ with $\mathbb{R}^V$ being the associated set of reals. The rest is as in the proof of 3.2.

Corollary 3.3 *Let $\vec{A}$ and $\vec{B}$ be $OD^{L(\mathbb{R})}$ tuples of sets of reals, and suppose $\mathcal{M}$ is $\vec{A}$-iterable; then there is an $\vec{A}$-iterate $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{N}$ is $\vec{A} \oplus \vec{B}$-iterable.*

Proof. Suppose not. By minimizing the ordinals from which it is possible to define over $L(\mathbb{R})$ counterexamples $\vec{A}$ and $\vec{B}$, we may assume that $\vec{A}$ and $\vec{B}$ are definable over $L(\mathbb{R})$ from *no* ordinals. But now fix any $\vec{A}$-iterable $\mathcal{M}$. By comparing $\mathcal{M}$ with $M_\omega^\sharp$, using Σ_0 to iterate $M_\omega^\sharp$ and an $\vec{A}$-iteration strategy to iterate $\mathcal{M}$, we find an $\vec{A}$-iterate $\mathcal{N}$ of $\mathcal{M}$ which is a suitable initial segment of an iterate of $M_\omega^\sharp$ via Σ_0. By 3.2, $\mathcal{N}$ is $\vec{A} \oplus \vec{B}$-iterable, a contradiction. $\square$

The trick of arguing that the least $\mathrm{OD}^{L(\mathbb{R})}$ counterexample to some first order assertion about $L(\mathbb{R})$ is in fact definable over $L(\mathbb{R})$ from no parameters will be used several times in the sequel.

Corollary 3.4 *Let $\mathcal{M}$ be $\vec{A}$-iterable and $\mathcal{N}$ be $\vec{B}$-iterable; then there is an $\vec{A}$-iterate $\mathcal{P}$ of $\mathcal{M}$ and a $\vec{B}$-iterate $\mathcal{Q}$ of $\mathcal{N}$ such that $\mathcal{P}$ is an initial segment of $\mathcal{Q}$ or vice-versa, and each of $\mathcal{P}$ and $\mathcal{Q}$ is $\vec{A} \oplus \vec{B}$-iterable.*

Proof. First use 3.3 to get $\vec{A}$ and $\vec{B}$ iterates $\mathcal{M}^*$ and $\mathcal{N}^*$ of $\mathcal{M}$ and $\mathcal{N}$ which are $\vec{A} \oplus \vec{B}$-iterable. Then apply 2.12. □

We shall actually need stronger existence results than these, but those stronger results are intertwined with the uniqueness properties of $\vec{A}$-iteration maps, and so we defer them to the next section.

4 The uniqueness of $\vec{A}$-iteration maps

The indices of the direct limit system we use to cover $(\mathcal{F}, \preceq)$ will be certain good pairs $(\mathcal{M}, \vec{A})$ such that $\mathcal{M}$ is $\vec{A}$-iterable. However, we cannot let the structure indexed here simply be $\mathcal{M}$, and the maps to "later" structures be those arising from $\vec{A}$-iterations, because there may be more than one $\vec{A}$-iteration leading from $\mathcal{M}$ to $\mathcal{N}$. For example, suppose that $\mathcal{M}$ is $\vec{A}$-iterable, and, for a moment, that $k(\mathcal{M}) = 1$. It is possible to have an $\vec{A}$-iteration $\mathcal{T}$ of $\mathcal{M}$ with distinct cofinal branches b and c such that both $\mathcal{T} \oplus b$ and $\mathcal{T} \oplus c$ are $\vec{A}$-iterations. Lemma 2.11 tells us that in this case neither b nor c drops and $i_b(\delta_0^{\mathcal{M}}) = i_c(\delta_0^{\mathcal{M}}) = \delta(\mathcal{T})$. Since $k(\mathcal{M}) = 1$, we get $\mathcal{M}_b^{\mathcal{T}} = \mathcal{M}_c^{\mathcal{T}}$. Thus i_b and i_c map into the same model $\mathcal{N}$, but because $b \neq c$, $i_b \neq i_c$. There is, however, a natural hull of $\mathcal{M}$ on which i_b and i_c must agree because they map the τ_{A_m}'s the same way. One of the key ideas in Woodin's proof is to let only these hulls contribute to the direct limit of the covering system.

Definition 4.1 *Let $(\mathcal{M}, \langle A_0, ..., A_n\rangle)$ be good; then*

1. $\gamma_{(\mathcal{M},\vec{A})} = sup(H^{\mathcal{M}}(\{\tau_{A_0}^{\mathcal{M}}, ..., \tau_{A_n}^{\mathcal{M}}\}) \cap \delta_0^{\mathcal{M}})$,

2. $\Delta(\mathcal{M}) = \begin{cases} \mathcal{M}, & \text{if } k(\mathcal{M}) = 1 \\ J_\eta^{\mathcal{M}}, \text{ for } \eta = (\delta_0^{++})^{\mathcal{M}}, & \text{if } k(\mathcal{M}) > 1 \end{cases}$

3. $\xi_{(\mathcal{M},\vec{A})} = \gamma_{(\Delta(\mathcal{M}),\vec{A})}$

4. $H_{(\mathcal{M},\vec{A})} = H^{\mathcal{M}}(\xi_{(\mathcal{M},\vec{A})} \cup \{\tau^{\mathcal{M}}_{A_0}, ..., \tau^{\mathcal{M}}_{A_n}\})$.

The hull of $\mathcal{M}$ referred to on the right hand side in 4.2 is the fully elementary hull. For the most part, we shall think of $H_{(\mathcal{M},\vec{A})}$ as being the *uncollapsed* hull. We have the following easy lemma, whose proof we shall omit.

Lemma 4.2 *Let $(\mathcal{M}, \vec{A})$ be good; then*

1. $\xi_{(\mathcal{M},\vec{A})} \leq \gamma_{(\mathcal{M},\vec{A})} < \delta_0^{\mathcal{M}}$*, and* $\xi_{(\mathcal{M},\vec{A})} < \gamma_{(\mathcal{M},\vec{A})}$ *iff* $k(\mathcal{M}) = 1$,

2. $\xi_{(\mathcal{M},\vec{A})} \subseteq (H_{(\mathcal{M},\vec{A})} \cap \delta_0^{\mathcal{M}}) \subseteq \gamma_{(\mathcal{M},\vec{A})}$,

3. *if* $k(\mathcal{M}) = 1$*, then* $\xi_{(\mathcal{M},\vec{A})} = H_{(\mathcal{M},\vec{A})} \cap \delta_0^{\mathcal{M}} = \gamma_{(\mathcal{M},\vec{A})}$.

(Part (1) follows from the fact that $\delta_0^{\mathcal{M}}$ is a cardinal in $L[\mathcal{M}]$; part (2) uses the fact that $\delta_0^{\mathcal{M}}$ is regular in $\mathcal{M}$.)

Life would be simpler if one could show that any two $\vec{A}$-iteration maps from $\mathcal{M}$ to $\mathcal{N}$ agree on $H_{(\mathcal{M},\vec{A})}$, but do not know whether this is in fact true. We *can* show, however, that any $\vec{A}$-iterable $\mathcal{M}$ has an $\vec{A}$-iterate $\mathcal{P}$ such that any two $\vec{A}$-iteration maps from $\mathcal{P}$ to a common $\mathcal{N}$ agree on $H_{(\mathcal{P},\vec{A})}$.

Definition 4.3 *Let $\mathcal{M}$ be suitable; then we say $\mathcal{M}$ is* **strongly $\vec{A}$-iterable** *iff $\mathcal{M}$ is $\vec{A}$-iterable, and whenever $i: \mathcal{M} \to \mathcal{N}$ and $j: \mathcal{M} \to \mathcal{N}$ are $\vec{A}$-iteration maps, then $i \upharpoonright H_{(\mathcal{M},\vec{A})} = j \upharpoonright H_{(\mathcal{M},\vec{A})}$.*

Clearly, in order to show that two $\vec{A}$-iteration maps agree on $H_{(\mathcal{M},\vec{A})}$, it suffices to show that they agree on $\xi_{(\mathcal{M},\vec{A})}$, as they agree on the $\tau^{\mathcal{M}}_{A_i}$ by hypothesis. As a first step, we consider $\vec{A}$-iterations which have just begun to diverge.

Definition 4.4 *Let $\mathcal{M}$ be suitable, and $\mathcal{T}$ be an iteration tree on $\mathcal{M}$; then $\mathcal{T}$* **lives below** $\delta_0^{\mathcal{M}}$ *iff there is an iteration tree $\mathcal{U}$ on $J^{\mathcal{M}}_{\delta_0^{\mathcal{M}}}$ such that $lh(\mathcal{U}) = lh(\mathcal{T})$, and $E^{\mathcal{U}}_\alpha = E^{\mathcal{T}}_\alpha$ for all $\alpha < lh(\mathcal{T})$.*

Lemma 4.5 *Let $\mathcal{T}$ be an ω-maximal iteration tree of limit length on $\mathcal{M}$, and b and c distinct cofinal branches of $\mathcal{T}$ such that $\mathcal{T} \oplus b$ and $\mathcal{T} \oplus c$ are $\vec{A}$-iterations. Suppose also that $\mathcal{T}$ lives below $\delta_0^{\mathcal{M}}$; then $i_b^{\mathcal{T}} \restriction \gamma_{(\mathcal{M},\vec{A})} = i_c^{\mathcal{T}} \restriction \gamma_{(\mathcal{M},\vec{A})}$.*

Proof. By 2.11, neither b nor c drops. By 2.11 and the fact that $\mathcal{T}$ lives below $\delta_0^{\mathcal{M}}$, $i_b(\delta_0^{\mathcal{M}}) = \delta(\mathcal{T}) = i_c(\delta_0^{\mathcal{M}})$. Both $\mathcal{M}_b^{\mathcal{T}}$ and $\mathcal{M}_c^{\mathcal{T}}$ are full and $\vec{A}$-iterable, so by 2.12 there are $\vec{A}$-iteration maps $k: \mathcal{M}_b^{\mathcal{T}} \to \mathcal{N}$ and $l: \mathcal{M}_c^{\mathcal{T}} \to \mathcal{N}$, and from the proof of 2.12 we get that $\mathrm{crit}(k), \mathrm{crit}(l) > \delta(\mathcal{T})$. We have then that $k \circ i_b$ and $l \circ i_c$ agree on all $\tau_{A_m}^{\mathcal{M}}$, and hence on a cofinal subset of $\gamma(\mathcal{M}, \vec{A})$. Thus i_b and i_c agree on a cofinal subset of $\gamma_{(\mathcal{M},\vec{A})}$.

Let α be largest in $b \cap c$, and let κ be the smaller of $\mathrm{crit}(i_{\alpha,b})$ and $\mathrm{crit}(i_{\alpha,c})$. If $\kappa \leq \eta \leq \delta(\mathcal{T})$, then there is an extender E used of one of b and c such that $\mathrm{crit}(E) \leq \eta < lh(E)$, and so if $\eta < \delta(\mathcal{T})$ is in both $\mathrm{ran}(i_b)$ and $\mathrm{ran}(i_c)$, then $\eta < \kappa$. Now pick any $\beta < \gamma_{(\mathcal{M},\vec{A})}$, and let $\beta < \xi < \gamma_{(\mathcal{M},\vec{A})}$ where $i_b(\xi) = i_c(\xi)$. We have then that $i_b(\xi) < \kappa$, and hence $i_{0,\alpha}(\beta) < \kappa$, and therefore $i_b(\beta) = i_{0,\alpha}(\beta) = i_c(\beta)$, as desired. $\square$

Theorem 4.6 *Let $\mathcal{M}$ be $\vec{A}$-iterable; then there is an $\vec{A}$-iterate $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{N}$ is strongly $\vec{A}$-iterable.*

Proof. Suppose not. By minimizing the ordinals from which a counterexample $\vec{A}$ can be defined over $L(\mathbb{R})$, we find a counterexample $\vec{A}$ definable over $L(\mathbb{R})$ from no parameters. Fix $\mathcal{M}$ which witnesses that $\vec{A}$ is a counterexample. Now let $\mathcal{N}$ be an $\vec{A}$-iterate of $\mathcal{M}$ such that $\mathcal{N}$ is an initial segment of Σ_0-iterate of $M_\omega^\sharp$; we get $\mathcal{N}$ from the comparison process. It will be enough to show that $\mathcal{N}$ is strongly $\vec{A}$-iterable.

Let $i: \mathcal{N} \to \mathcal{P}$ and $j: \mathcal{N} \to \mathcal{P}$ be the maps arising from two $\vec{A}$-iterations. In order to show that i and j agree on $H_{(\mathcal{N},\vec{A})}$, it will suffice to show that they agree on $\xi_{(\mathcal{N},\vec{A})}$. But this part of i and j is given by iteration trees living below $\delta_0^{\mathcal{N}}$ and acting on $\Delta(\mathcal{N})$, and in fact these trees represent $\vec{A}$-iterations of $\Delta(\mathcal{N})$ (since the i/j image of $\tau_A^{\Delta(\mathcal{N})}$ is determined by these trees, and i/j moves $\tau_A^{\Delta(\mathcal{N})}$ properly because it moves $\tau_A^{\mathcal{N}}$ properly). So the relevant part of i and j is given by $\vec{A}$-iterations from $\Delta(\mathcal{N})$ to $\Delta(\mathcal{P})$.

It will be enough to show that there is a iteration map $k: \Delta(\mathcal{N}) \to \Delta(\mathcal{P})$ associated to a play according to Σ_0 such that i and k agree on $\xi_{(\mathcal{N},\vec{A})}$, and a second such iteration map agreeing similarly with j. This is because, by Dodd-Jensen, there is at most one iteration map from $\Delta(\mathcal{N})$ to $\Delta(\mathcal{P})$ associated to iterations according to Σ_0.

We now produce k. Set $\mathcal{R}_0 = \Delta(\mathcal{N})$. We can find a finite sequence $\mathcal{T}_0, ..., \mathcal{T}_n$ of ω-maximal iteration trees, $\mathcal{T}_e$ being on $\mathcal{R}_e$ and having last model $\mathcal{R}_{e+1}$ and iteration map $i_e: \mathcal{R}_e \to \mathcal{R}_{e+1}$, with $\mathcal{R}_{n+1} = \Delta(\mathcal{P})$ and $i = i_n \circ ... \circ i_0$. Each i_e is moreover an $\vec{A}$-iteration map. For each e, if λ is a limit ordinal and $\lambda + 1 < lh(\mathcal{T}_e)$, then $[0, \lambda]_{T_e}$ is the unique cofinal branch of $\mathcal{T}_e \restriction \lambda$ having a G-reals iterable Q-structure. (The lengths of the extenders used in $\mathcal{T}_e$ are increasing, and as $E_\lambda^{\mathcal{T}_e}$ exists, we haven't yet reached the Woodin cardinal.) This means that either $\mathcal{T}_0$ is according to Σ_0, or $lh(\mathcal{T}_0) = \lambda_0 + 1$, where λ_0 is a limit, $\mathcal{T}_0 \restriction \lambda_0$ is according to Σ_0, and $\delta(\mathcal{T}_0 \restriction \lambda_0) = i_0(\delta_0^{\mathcal{N}})$. Let us define $\mathcal{S}_0$ to be $\mathcal{T}_0$ in the first case, and to be the unique one-model extension of $\mathcal{T}_0 \restriction \lambda_0$ which is according to Σ_0 in the second case. Notice that in the second case we have reached the unique Woodin cardinal, so that the last model of $\mathcal{S}_0$, being suitable, is still $\mathcal{R}_1$. Let $k_0: \mathcal{R}_0 \to \mathcal{R}_1$ be the iteration map from $\mathcal{S}_0$. Notice that by 4.4, i_0 agrees with k_0 on $\xi_{(\mathcal{N}, \vec{A})} = \gamma_{(\mathcal{R}_0, \vec{A})}$. Proceeding inductively in this fashion we define $\mathcal{S}_e$, which is according to Σ_0 and gives rise to $k_e: \mathcal{R}_e \to \mathcal{R}_{e+1}$ agreeing with i_e on $\gamma_{(\mathcal{R}_e, \vec{A})}$. Clearly, the desired k is $k_n \circ ... \circ k_0$. □

We are now ready to define (over $L(\mathbb{R})$) our direct limit system "covering" $\mathcal{F}$. The index set for the system is

$$I = \{ (\mathcal{M}, \vec{A}) \mid \mathcal{M} \text{ is strongly } \vec{A}\text{-iterable} \}.$$

The ordering on indices is defined by

$$(\mathcal{M}, \vec{A}) \leq_I (\mathcal{N}, \vec{B}) \quad \text{iff} \quad \text{there is an } \vec{A}\text{-iteration from } \mathcal{M} \text{ to an initial segment of } \mathcal{N}, \text{ and } \exists \vec{C}(\vec{B} = \vec{A} \oplus \vec{C}).$$

Lemma 4.7 *$(I, \leq_I)$ is a directed partial order.*

Proof. Reflexivity and antisymmetry are clear. For transitivity, use the fact that a $\vec{B}$-iteration of $\mathcal{N}$ to $\mathcal{P}$ determines a $\vec{B}$-iteration of $J_\eta^{\mathcal{N}}$ to $J_\xi^{\mathcal{P}}$ whenever for some $k < k(\mathcal{N})$, $\eta = (\delta_k^{++})^{\mathcal{N}}$ and $\xi = (\delta_k^{++})^{\mathcal{P}}$.

For directedness, let $(\mathcal{M}, \vec{A}), (\mathcal{N}, \vec{B}) \in I$. By 3.4 we have $\mathcal{P}$ and $\mathcal{Q}$ which are $\vec{A}$ and $\vec{B}$ iterates of $\mathcal{M}$ and $\mathcal{N}$ such that one is an initial segment of the other, and each is $\vec{A} \oplus \vec{B}$-iterable. By 4.6, there is an $\vec{A} \oplus \vec{B}$- iterate of the longer of $\mathcal{P}$ and $\mathcal{Q}$ which is strongly $\vec{A} \oplus \vec{B}$-iterable. This provides a common upper bound for $(\mathcal{M}, \vec{A})$ and $(\mathcal{N}, \vec{B})$ in I. $\square$

If $(\mathcal{M}, \vec{A}) \in I$, then $(\mathcal{M}, \vec{A})$ indexes the structure $H_{(\mathcal{M},\vec{A})}$ in our covering system. The maps are the unique $\vec{A}$-iteration maps.

Definition 4.8 *For $(\mathcal{M}, \vec{A}) \leq_I (\mathcal{N}, \vec{B})$, we let $\pi_{(\mathcal{M},\vec{A}),(\mathcal{N},\vec{B})}$ be the common restriction to $H_{(\mathcal{M},\vec{A})}$ of all $\vec{A}$-iteration maps from $\mathcal{M}$ to an initial segment of $\mathcal{N}$.*

The composition of $\vec{A}$-iterations is an $\vec{A}$-iteration, so the maps commute as required. This defines our covering system.

The index poset $(I, \leq_I)$ is not countably directed, so it is not clear at the moment that the direct limit of the covering system is even wellfounded. We shall show this in the next section, by showing the direct limit is isomorphic to that of the system $(\mathcal{F}, \preceq)$.

5 The isomorphism of the two direct limits

We begin by showing that Theorem 3.2 characterizes the weak iteration strategy induced by Σ_0; that is, if $\mathcal{P}$ is a suitable initial segment of an iterate via Σ_0 of $M_\omega^\sharp$, then the weak strategy for $\mathcal{P}$ induced by Σ_0 is the unique weak strategy which is simultaneously an $\vec{A}$-iteration strategy for all $\vec{A}$ definable over $L(\mathbb{R})$ from $\mathbb{R}$-indiscernibles.

Definition 5.1 *Φ_n is the type of the first n $\mathbb{R}$-indiscernibles; that is, it is the set of formulae with real parameters satisfied in $L(\mathbb{R})$ by n-tuples of $\mathbb{R}$-indiscernibles.*

Of course, Φ_i is easily definable from Φ_n whenever $i \leq n$, so $H_{(\mathcal{M},\Phi_1,\ldots,\Phi_n)} = H_{(\mathcal{M},\Phi_n)}$, and similarly for $\gamma_{(\mathcal{M},\Phi_n)}$ and $\xi_{(\mathcal{M},\Phi_n)}$.

We shall need to relate the Φ_n's to the types of n-tuples of indiscernibles over M_ω. First, notice that our assumption that $M_\omega^\sharp$ exists implies that for all reals x, $M_\omega^\sharp(x)$ exists; that is, there is a non-ω-small premouse built over x which is $\omega_1 + 1$-iterable in $V^{Col(\omega,\mathbb{R})}$. This follows from the fact that we can iterate $M_\omega^\sharp$ in V so as to make x generic over the (countable) iterate for the Woodin poset at its bottom Woodin cardinal. (There is a familiar paradox here, which is resolved by realizing that "$M_\omega^\sharp$ exists" means not just that a certain real exists, but also that an iteration strategy for it exists.) Let $M_\omega(x)$ be the result of iterating the last extender of $M_\omega^\sharp(x)$ through the ordinals.

Definition 5.2 *For any $n < \omega$, we set $\Psi_n = \{ (x, \varphi) \mid x \in \mathbb{R}$ and $M_\omega(x) \models \varphi[x, c_1^x, ..., c_n^x]\}$, where for any real x, c_i^x is the i^{th} indiscernible of $M_\omega(x)$.*

It is clear from the proof of 1.2 that Φ_n is Wadge reducible to Ψ_n via a recursive function, for all n. On the other hand :

Lemma 5.3 *For any $n < \omega$, Ψ_n is definable over $L(\mathbb{R})$ from the first $n + 1$ $\mathbb{R}$-indiscernibles.*

Proof.(*Sketch.*) One can show that Ψ_n is G-reals-$\omega(n + 1) - \Pi_1^1$, and from this the lemma follows by considering the auxiliary closed game definable over $L(\mathbb{R})$ from $n + 1$ indiscernibles. Given x and φ, our $\omega(n + 1) - \Pi_1^1$ real game, which I is to win iff $(x, \varphi) \in \Psi_n$, requires that I play an x-premouse with ω Woodin cardinals on his first move. In the later moves the players play out the weak iteration game of length ω on this premouse (with I in the role of II of that game). Simultaneously, they play integer-by-integer $\omega(n + 1)$ many codes for ordinals, with the wellfoundedness burden going back and forth as usual. These sup to ordinals $\alpha_1, ..., \alpha_{n+1}$, and I must insure that if $\mathcal{P}$ is the iterate of his premouse they produce, and the iteration doesn't drop, that $J_{\alpha_{n+1}}^{\mathcal{P}} \models \varphi[x, \alpha_1, ..., \alpha_n]$. □

If $\mathcal{P}$ is suitable, then we can build a premouse $M_\omega^\sharp(\mathcal{P})$ extending $\mathcal{P}$ and fully iterable above $\mathcal{P}$ by simply doing the $L[\vec{E}]$ construction of [1] with critical points above $\mathrm{OR}^{\mathcal{P}}$ inside $M_\omega^\sharp(x)$, for some $x \in \mathbb{R}$ coding $\mathcal{P}$. Let $M_\omega(\mathcal{P})$ be the result of iterating away the last extender.

Definition 5.4 *Let $\mathcal{P}$ be suitable and $k + 1 = k(\mathcal{P})$; then $T_n(\mathcal{P})$ is the type over $M_\omega(\mathcal{P})$ of the first n indiscernibles of $M_\omega(\mathcal{P})$, with parameters for all ordinals strictly less than $(\delta_k^+)^{\mathcal{P}}$.*

We have the following obvious corollary of 5.4:

Corollary 5.5 *For any $n < \omega$, the function $\mathcal{P} \mapsto T_n(\mathcal{P})$ is definable over $L(\mathbb{R})$ from the first $n+1$ $\mathbb{R}$-indiscernibles.*

In general, an elementary submodel of a suitable premouse will not be suitable itself, as it will not in general be full. But we do have the following important condensation lemma.

Lemma 5.6 *Let $\mathcal{Q}$ be suitable, and $\pi: \mathcal{P} \to \mathcal{Q}$ be elementary and such that $T_n(\mathcal{Q}) \in \operatorname{ran} \pi$ for all $n < \omega$; then $\mathcal{P}$ is suitable.*

Proof.(*Sketch.*) Show that the pre-image of $T_n(\mathcal{Q})$ under π is just $T_n(\mathcal{P})$. This implies that we can build an iterable ω-Woodin sharp over $\mathcal{P}$ without constructing any new bounded subsets of $\mathcal{P}$, so $\mathcal{P}$ is full. $\square$

Corollary 5.7 *Let $\mathcal{P}$ be suitable; then*

1. *for any $n < \omega$, $T_n(\mathcal{P}) \in \mathcal{P}$,*
2. *$H^{\mathcal{P}}(\{T_n(\mathcal{P}) \mid n < \omega\})$ is unbounded in $\delta_i^{\mathcal{P}}$, for all $i < k(\mathcal{P})$.*
3. *$\delta_0^{\mathcal{P}} = \sup(\{\xi_{(\mathcal{P},\Phi_1,\ldots,\Phi_n)} \mid n < \omega\})$.*

Proof. By 5.6, $T_n(\mathcal{P})$ is $\mathrm{OD}^{L(\mathbb{R})}$ from $\mathcal{P}$. Since $\mathcal{P}$ is full, we get $T_n(\mathcal{P}) \in \mathcal{P}$. For the second item, suppose $H^{\mathcal{P}}(\{T_n(\mathcal{P}) \mid n < \omega\}) \cap \delta_i^{\mathcal{P}} = \alpha$, where $\alpha < \delta_i^{\mathcal{P}}$. Let $\mathcal{Q}$ be the transitive collapse of $H^{\mathcal{P}}(\alpha \cup \{T_n(\mathcal{P}) \mid n < \omega)$. As in the proof of 4.2.2, α is the image of $\delta_i^{\mathcal{P}}$ in the collapse, so α is Woodin in $\mathcal{Q}$. But $\mathcal{P}$ agrees with $\mathcal{Q}$ below α, and α is not Woodin in $\mathcal{P}$, and $\mathcal{P}$ is ω-small and G-reals Π^1_1 iterable. It follows that $\mathcal{Q}$ is not full, contrary to 5.7.

The third item follows from the second, using 5.5 to see that $T_n(\Delta(\mathcal{P})) \in H_{(\Delta(\mathcal{P}),\Phi_{n+1})}$. $\square$

Corollary 5.8 *Let $\mathcal{M}$ be a suitable initial segment of a countable iterate of $M_\omega^\sharp$ by Σ_0, let $\mathcal{T}$ be an ω-maximal iteration tree on $\mathcal{M}$ of countable limit length which is according to Σ_0, and let $b = \Sigma_0(\mathcal{T})$; then b is the unique cofinal branch of $\mathcal{T}$ such that for all n, $\mathcal{T} \oplus b$ is (or can be regarded as) a Φ_n-iteration.*

Proof. We get this from 4.5 and 5.7 for trees $\mathcal{T}$ which live below $\delta_0^{\mathcal{M}}$, but the reader who has followed the proofs will see easily that they give 5.8 in full. (The case that $\mathcal{T}$ is below $\delta_0^{\mathcal{M}}$ is all we need in the sequel, anyway.) $\square$

The results we have proved so far in this section can be used to show that the direct limit of $(\mathcal{F}, \preceq)$ embeds naturally in that of our covering system. The key to getting the embedding in the other direction is the following lemma.

Lemma 5.9 *Let $\mathcal{M}$ be $\vec{A}$-iterable; then there is an $\vec{A}$-iterate $\mathcal{P}$ of $\mathcal{M}$ such that $\mathcal{P}$ is an initial segment of an iterate of $M_\omega^\sharp$, and for some $n < \omega$ such that $\mathcal{P}$ is strongly $\vec{A} \oplus \langle \Phi_n \rangle$-iterable we have that $\tau_A^{\mathcal{P}} \in H_{(\mathcal{P}, \Phi_n)}$ for all A on $\vec{A}$.*

Proof. This follows easily from the following claim.

Claim 1. Let $\mathcal{M}$ be $\vec{A}$-iterable and $\mathcal{N}$ be $\vec{B}$-iterable. Suppose $k(\mathcal{M}) \leq k(\mathcal{N})$, and fix A on $\vec{A}$ and B on $\vec{B}$ such that A is Wadge reducible to B; then there is a $\mathcal{P}$ such that

1. $\mathcal{P}$ is a $\vec{B}$-iterate of $\mathcal{N}$, and some initial segment of $\mathcal{P}$ is a $\vec{A}$-iterate of $\mathcal{M}$,

2. $\mathcal{P}$ is strongly $\vec{A} \oplus \vec{B}$-iterable, and

3. $\tau_A^{\mathcal{P}} \in H^{\mathcal{P}}(\gamma_{(\mathcal{P},B)} \cup \{\tau_B^{\mathcal{P}}\})$.

Proof. Suppose not. By minimizing our counterexample, we may assume $\vec{A}$ and $\vec{B}$ are definable over $L(\mathbb{R})$. Let $\mathcal{M}$ and $\mathcal{N}$ be the offending premice. We may as well assume that they are initial segments of iterates of $M_\omega^\sharp$ via Σ_0, since by 2.12 we can $\vec{A}$ and $\vec{B}$ iterate them to such a point. Let $\mathcal{Q}$ result from the coiteration of $\mathcal{M}$ and $\mathcal{N}$, using Σ_0 on both sides. Clearly, $\mathcal{Q}$ is a $\vec{B}$-iterate of $\mathcal{N}$, and has an initial segment which is an $\vec{A}$-iterate of $\mathcal{M}$.

Let x be a real coding a Wadge reduction of A to B. By Woodin's every-real-generic theorem, we can find an iterate $\mathcal{P}$ of $\mathcal{Q}$ via Σ_0 such that x is generic over $\mathcal{P}$ for the Woodin poset at the bottom Woodin cardinal of $\mathcal{P}$. Let us call this poset $\mathbb{Q}$. We claim that $\mathcal{P}$ satisfies items (1) and (3) of the claim. Item (1) is clear, so we turn to (3).

Let $k+1 = k(\mathcal{P})$, and set $\mathbb{P} = Col(\omega, \delta_k^{\mathcal{P}})$. If G is $\mathbb{Q}$-generic over $\mathcal{P}$ and H is $\mathbb{P}$-generic over $\mathcal{P}[G]$, then we use $K(G,H)$ for the $\mathbb{P}$-generic object equivalent to $G * H$. Now for each standard, invariant $\mathcal{P}$-term τ, pick a condition $r_\tau \in \mathbb{Q}$, if there is one, such that r_τ forces in $\mathbb{Q}$ that if G is the generic and x_G the real it determines, then in $\mathcal{P}[G]$ it is forced by $\emptyset$ in $\mathbb{P}$ that if H is the generic and $K = K(G,H)$, then τ^K is Wadge reducible to $(\tau_B^{\mathcal{P}})^K$ via x_G. Notice that if $\sigma \neq \tau$ and r_σ, r_τ are defined, then r_σ is incompatible with r_τ.(Otherwise, let G extend them both, and let H be $\mathbb{P}$-generic over $\mathcal{P}[G]$ and $K = K(G,H)$; then $\tau^K = \sigma^K$, contrary to 2.5.) Now $\{ r_\tau \mid r_\tau \text{ is defined} \}$ is definable over $\mathcal{P}$, provided we pick, say, the least such r in the canonical wellorder for r_τ. Each r_τ is in $J^{\mathcal{P}}_{\delta_0^{\mathcal{P}}}$, and since $\mathbb{Q}$ is $\delta_0^{\mathcal{P}}$-c.c. in $\mathcal{P}$, we can find $\gamma < \delta_0^{\mathcal{P}}$ such that $r_\tau \in J_\gamma^{\mathcal{P}}$ whenever r_τ is defined. The least such γ is definable over $\mathcal{P}$ from $\tau_B^{\mathcal{P}}$, and hence strictly less than $\gamma_{(\mathcal{P},B)}$. But notice that for $\tau = \tau_A^{\mathcal{P}}$, r_τ is defined because x is $\mathbb{Q}$-generic over $\mathcal{P}$. Since τ is definable over $\mathcal{P}$ from r_τ, we are done with (3) of the claim.

We can arrange that $\mathcal{P}$ is strongly $\vec{A} \oplus \vec{B}$-iterable by applying 4.6. Notice that (3) is preserved by any $A \oplus B$-iteration. This proves the claim.

Notice that if $\mathcal{P}$ satisfies the conclusion of Claim 1, then so too does any $\vec{A} \oplus \vec{B}$-iterate of $\mathcal{P}$. Thus we can apply Claim 1 repeatedly, and obtain:

*Claim 2.*Let $\mathcal{M}$ be $\vec{A}$-iterable and $\mathcal{N}$ be $\vec{B}$-iterable, and suppose $k(\mathcal{M}) \leq k(\mathcal{N})$; then there is a $\mathcal{P}$ such that

1. $\mathcal{P}$ is a $\vec{B}$-iterate of $\mathcal{N}$, and some initial segment of $\mathcal{P}$ is an $\vec{A}$-iterate of $\mathcal{M}$,
2. $\mathcal{P}$ is strongly $\vec{A} \oplus \vec{B}$-iterable
3. for all A on $\vec{A}$ and B on $\vec{B}$, if A is Wadge reducible to B, then $\tau_A^{\mathcal{P}} \in H^{\mathcal{P}}(\gamma_{(\mathcal{P},B)} \cup \{\tau_B^{\mathcal{P}}\})$

We now prove 5.9. Let $\mathcal{M}$ be $\vec{A}$-iterable, and pick n so large that every A on $\vec{A}$ is Wadge reducible to Φ_{n-2}. (Every set of reals in $L(\mathbb{R})$ is Wadge reducible to some Φ_k.) Let $\mathcal{N}$ be a suitable initial segment of $M_\omega^\sharp$ such that $k(\mathcal{N}) = k(\mathcal{M})$, and let $\vec{B} = \langle \Phi_{n-2}, \Phi_n \rangle$. Let $\mathcal{P}*$ satisfy the conclusion of Claim 2 vis a vis this choice of $\mathcal{M}$,$\vec{A}$, $\mathcal{N}$, and $\vec{B}$. Now $\mathcal{P}*$ may not be an iterate of $M_\omega^\sharp$ via Σ_0, but we can find an iterate $\mathcal{Q}$ of $\mathcal{N}$ via Σ_0 such that $\Delta(\mathcal{P}*) = \Delta(\mathcal{Q})$ by the approximation process from the proof of 4.6.

Comparing $\mathcal{Q}$ with $\mathcal{P}*$ using Σ_0 and an $\vec{A} \oplus \langle \Phi_{n-2}, \Phi_n \rangle$-iteration strategy respectively, we get a common iterate $\mathcal{P}$. $\mathcal{P}$ still satisfies the conclusion of Claim 2, and is in addition an iterate of $\mathcal{N}$ via Σ_0. In order to see that $\mathcal{P}$ satisfies the conclusion of 5.9, we have only to deal with the difference between $\xi_{(\mathcal{M},\Phi_n)}$ and $\gamma_{(\mathcal{M},\Phi_n)}$. But notice that $\gamma_{(\mathcal{P},\Phi_{n-2})} < \xi_{(\mathcal{P},\Phi_n)}$. (We have $T_{n-1}(\Delta(\mathcal{P})) \in H^{\Delta(\mathcal{P})}(\{\tau_{\Phi_n}^{\Delta(\mathcal{P})}\})$. But from $T_{n-1}(\Delta(\mathcal{P})$ we can reconstruct, in a simple way, a structure elementarily equivalent to $(\mathcal{P}, \tau^{\mathcal{P}}_{\Phi_{n-2}})$ with respect to parameters below $\delta_0^{\mathcal{P}}$.) This observation completes the proof of 5.9. $\square$

We are now ready to prove that $(\mathcal{F}, \preceq)$ and its covering system have isomorphic direct limits.

Theorem 5.10 *The direct limit of the covering system is isomorphic to $J_\lambda^{\mathcal{M}_\infty}$, where $\mathcal{M}_\infty$ is the direct limit of $(\mathcal{F}, \preceq)$, and λ is the supremum of the Woodin cardinals of $\mathcal{M}_\infty$.*

Proof.Let N be the direct limit of the covering system. We define the desired isomorphism σ from N to $J_\lambda^{\mathcal{M}_\infty}$ as follows. Let $z \in N$. Pick any $(\mathcal{M}, \vec{A}) \in I$ and $x \in H_{(\mathcal{M},\vec{A})}$ such that $\pi_{(\mathcal{M},\vec{A}),\infty}(x) = z$, where of course $\pi_{(\mathcal{M},\vec{A}),\infty}$ is the map into the direct limit of the covering system. By 5.9 we can find $(\mathcal{P}, \vec{A} \oplus \Phi_n) \geq_I (\mathcal{M}, \vec{A})$ such that $\pi_{(\mathcal{M},\vec{A}),(\mathcal{P},\vec{A}\oplus\Phi_n)}(x) \in H_{(\mathcal{P},\Phi_n)}$ and $\mathcal{P}$ is an iterate of a suitable initial segment of $M_\omega^\sharp$ via Σ_0. Clearly, there is an iterate $\mathcal{Q}$ of this initial segment of $M_\omega^\sharp$ via Σ_0 such that $\Delta(\mathcal{Q}) = \Delta(\mathcal{P})$ and the iteration lives below the bottom Woodin cardinal of $M_\omega^\sharp$, and an elementary $j: \mathcal{Q} \to \mathcal{P}$ such that $\mathrm{crit}(j) > \delta_0^{\mathcal{Q}}$ and $j(y) = \pi_{(\mathcal{M},\vec{A}),(\mathcal{P},\vec{A}\oplus\Phi_n)}(x)$ for some y. Now $\mathcal{Q} \in \mathcal{F}$; we set $\sigma(x)$ to be the image of y under the direct limit map of $(\mathcal{F}, \preceq)$.

Our main problem is to show that σ is well-defined, and we turn to this now. Let $x_1 \in H_{(\mathcal{M},\vec{A})}$ and $x_2 \in H_{(\mathcal{N},\vec{B})}$, where $(\mathcal{M}, \vec{A}), (\mathcal{N}, \vec{B}) \in I$, be such that $\pi_{(\mathcal{M},\vec{A}),\infty}(x_1) = \pi_{(\mathcal{N},\vec{B}),\infty}(x_2)$. Let $(\mathcal{P}_1, \vec{A} \oplus \Phi_n) \geq_I (\mathcal{M}, \vec{A})$ and $(\mathcal{P}_2, \vec{B} \oplus \Phi_m) \geq_I (\mathcal{N}, \vec{B}))$ with $u_1 = \pi_{(\mathcal{M},\vec{A}),(\mathcal{P}_1,\vec{A}\oplus\Phi_n)}(x_1) \in H_{(\mathcal{P}_1,\Phi_n)}$ and $u_2 = \pi_{(\mathcal{N},\vec{B}),(\mathcal{P}_2,\vec{B}\oplus\Phi_m)}(x_2) \in H_{(\mathcal{P}_2,\Phi_m)}$. Let $j_1: \mathcal{Q}_1 \to \mathcal{P}_1$ and $j_2: \mathcal{Q}_2 \to \mathcal{P}_2$ be as in the definition of σ, with $j_1(y_1) = u_1$ and $j_2(y_2) = u_2$. We must see that y_1 and y_2 are mapped to the same thing in the system $(\mathcal{F}, \preceq)$. For this it is enough to find iteration maps $k_1: \mathcal{Q}_1 \to \mathcal{S}$ and $k_2: \mathcal{Q}_2 \to \mathcal{S}$ produced by iterations according to Σ_0 such that $k_1(y_1) = k_2(y_2)$.

Let $(\mathcal{W}, \vec{A} \oplus \vec{B} \oplus \langle \Phi_n, \Phi_m \rangle)$ be a common upper bound in $(I, \leq_I)$ for $(\mathcal{P}_1, \vec{A} \oplus \Phi_n)$ and $(\mathcal{P}_2, \vec{B} \oplus \Phi_2)$. Let $i_1 : \mathcal{P}_1 \to \mathcal{W}$ and $i_2 : \mathcal{P}_2 \to \mathcal{W}$ be the maps arising from some iterations of the appropriate sort ($\vec{A} \oplus \Phi_n$ in the first case, $\vec{B} \oplus \Phi_m$ in the second). We have $i_1(u_1) = i_2(u_2)$ by our choice of x_1 and x_2. Now the part of the iteration producing i_1 which led from $\Delta(\mathcal{P}_1)$ to $\Delta(\mathcal{W})$ can be approximated, as in 4.6, by an iteration which is according to Σ_0 and lives below $\delta_0^{\mathcal{P}_1}$. Since $\Delta(\mathcal{Q}_1) = \Delta(\mathcal{P}_1)$, we can take this iteration to be one of $\mathcal{Q}_1$. In this way we get an iteration map $l_1 : \mathcal{Q}_1 \to \mathcal{R}_1$, produced by an iteration according to Σ_0 living below $\delta_0^{\mathcal{Q}_1}$, with $\Delta(\mathcal{R}_1) = \Delta(\mathcal{W})$. We also get $l_2 : \mathcal{Q}_2 \to \mathcal{R}_2$ with the parallel properties on the other side.

Let us now simultaneously compare $\mathcal{R}_1, \mathcal{R}_2$, and $\mathcal{W}$. We use Σ_0 for $\mathcal{R}_1$ and $\mathcal{R}_2$, and a $\vec{A} \oplus \vec{B} \oplus \langle \Phi_n, \Phi_m \rangle$-iteration strategy for $\mathcal{W}$. The comparison is above the common δ_0 of the models, since they agree that far. Let $p_1 : \mathcal{R}_1 \to \mathcal{S}$, $p_2 : \mathcal{R}_2 \to \mathcal{S}$ and $q : \mathcal{W} \to \mathcal{S}$ be the iteration maps. Now $p_1 \circ l_1$ and $q \circ i_1$ are both Φ_n-iterations, and it follows that the two composite maps agree on $\xi_{(\mathcal{P}_1, \Phi_n)}$. From this we see that $(p_1 \circ l_1)(y_1) = (q \circ i_1)(u_1)$. In a similar way, we have $(p_2 \circ l_2)(y_2) = (q \circ i_2)(u_2)$. Since $i_1(u_1) = i_2(u_2)$, this implies $(p_1 \circ l_1)(y_1) = (p_2 \circ l_2)(y_2)$. Thus we can take $k_1 = p_1 \circ l_1$ and $k_2 = p_2 \circ l_2$.

Thus σ is well-defined. It is easy to see that σ preserves membership. We are left with showing σ is onto. Let $w \in J_\lambda^{\mathcal{M}_\infty}$. Let $\mathcal{R} \in \mathcal{F}$ and $x \in \mathcal{R}$ be such that x maps to w under the direct limit map from $\mathcal{R}$ to $\mathcal{M}_\infty$. Let $\mathcal{P}$ be a suitable initial segment of $\mathcal{R}$ such that $x \in \mathcal{P}$. Now M_ω is the Skolem closure of its indiscernibles, and since $\mathcal{R}^*$ ($\mathcal{R}$ with its last extender iterated away) was obtained from M_ω via an iteration tree living below $\delta_0^{M_\omega}$, we have that $\mathcal{R}^*$ is the Skolem closure of $\delta_0^{\mathcal{R}^*}$ together with its indiscernibles. But this means that for some n, x is definable over $\mathcal{P}$ from $T_n(\mathcal{P})$ and ordinals below $\delta_0^{\mathcal{P}}$. By 5.5 and 5.7.3, we can therefore find a k large enough that $x \in H_{(\mathcal{P}, \Phi_k)}$. But this means that if we let z be the image of x up in N, then $\sigma(z) = w$. $\square$

6 Proof of the main theorem

We proceed to the proof of Theorem 1.3. Let $\mathcal{M}$ be the result of iterating the last extender of $\mathcal{M}_\infty$ through the ordinals, and let X be the set of (finite compositions of ω-maximal) iteration trees in $\mathcal{M}$ which are on $J_\delta^{\mathcal{M}}$, for δ the least Woodin cardinal of $\mathcal{M}$, and have $\mathcal{M}$-cardinality strictly less than the sup of the Woodin cardinals of $\mathcal{M}$.In $V^{Col(\omega,\mathbb{R})}$, $\mathcal{M}$ is an iterate of M_ω via a countable iteration tree which is according to Σ_0. Thus Σ_0 acts on X, and $\Sigma_0 \restriction X$ is in V. We wish to show that $\mathrm{HOD}^{L(\mathbb{R})} = \mathcal{M}[\Sigma_0 \restriction X]$.

By 5.10, $\mathcal{M}$ is definable over $L(\mathbb{R})$. Clearly, X is definable over $L(\mathbb{R})$. We now show that $\Sigma_0 \restriction X$ is definable over $L(\mathbb{R})$.

Let $\mathbb{R}^*$ be the set of reals added by initial segments of the Levy collapse over $\mathcal{M}$ of everything below the sup of the Woodin cardinals of $\mathcal{M}$ to ω. For A an $\mathrm{OD}^{L(\mathbb{R})}$ set of reals, we define an $\mathrm{OD}^{L(\mathbb{R}^*)}$ set $A^* \subseteq \mathbb{R}^*$ as follows. First, let $\mathcal{N}$ the suitable initial segment of $\mathcal{M}$ such that $k(\mathcal{N}) = k+1$; we define an $\mathcal{N}$-term $\sigma_{A,k}$ by picking any $(\mathcal{P}, A) \in I$ such that $k(\mathcal{P}) = k+1$, and setting $\sigma_{A,k} = \pi_{(\mathcal{P},A),\infty}(\tau_A^{\mathcal{P}})$. If $\mathcal{Q}$ is a suitable initial segment of $\mathcal{P}$, then $\tau_A^{\mathcal{Q}}$ "coheres" with $\tau_A^{\mathcal{P}}$, and this is a first order property of $\tau_A^{\mathcal{Q}}, \tau_A^{\mathcal{P}}$ over $\mathcal{P}$. It follows that if G is the generic object producing $\mathbb{R}^*$, and $G_k = G \cap Col(\omega, \delta_k^{\mathcal{M}})$, then $(\sigma_{A,k})^{G_k} = (\sigma_{A,l})^{G_l} \cap \mathcal{M}[G_k]$ whenever $k \leq l$. We define A^* by setting $A^* \cap \mathcal{M}[G_k] = (\sigma_{A,k})^{G_k}$. A moment's reflection shows that because the $\sigma_{A,k}$ are invariant, A^* is independent of the particular decomposition $\langle G_k \mid k < \omega\rangle$; it depends only on $\mathbb{R}^*$. It is not clear at the moment that A^* is $\mathrm{OD}^{L(\mathbb{R}^*)}$, but that is part of the following claims.

Claim 1. For all n, Φ_n^* is the type of the first n indiscernibles of $L(\mathbb{R}^*)$.

Proof. In $V^{Col(\omega,\mathbb{R})}$, we have an iteration with models $\mathcal{Q}_i \in \mathcal{F}$ for all $i < \omega$ whose direct limit is $\mathcal{M}$. (Actually, $\mathcal{Q}_i$ comes from a member of $\mathcal{F}$ by iterating away the last extender.) Let $j \colon \mathcal{Q}_0 \to \mathcal{M}$ be the direct limit map. Let $\vec{c}$ be an n-tuple of $\mathcal{Q}_0$-indiscernibles such that $j(\vec{c}) = \vec{c}$; thus $\vec{c}$ is also a tuple of $\mathcal{M}$-indiscernibles, and hence a tuple of $L(\mathbb{R}^*)$ -indiscernibles. Let $\mathcal{P}$ be the suitable initial segment of $\mathcal{Q}_0$ such that $k(\mathcal{P}) = k+1$; then it is a first order over $\mathcal{Q}_0$ fact about $\tau = \tau_{\Phi_n}^{\mathcal{P}}$ and $\vec{c}$ that if you collapse $\delta_k^{\mathcal{P}}$ via G, then τ^G is the intersection with $\mathcal{Q}_0[G]$ of the type of $\vec{c}$ in the $L(\mathbb{R})$ of the symmetric collapse over $\mathcal{Q}_0[G]$. This fact is therefore true in $\mathcal{M}$ of $j(\tau)$ and $\vec{c}$. But the isomorphism of the two systems provided by 5.10 shows us that $j(\tau) = \sigma_{\Phi_n,k}$. This proves claim 1.

Claim 2. For all n,

$$L(\mathbb{R}^*) \models \text{ every } \emptyset\text{-iterate of } \Delta(\mathcal{M} \text{ is } \Phi_n^*\text{-iterable.}$$

Proof.Let $\mathcal{P}$ be an $\emptyset$-iterate of $\Delta(\mathcal{M})$ in the sense of $L(\mathbb{R}^*)$. By the approximation argument of 4.6, we see that $\mathcal{P}$ is an iterate of $\Delta(\mathcal{M}$ via

Σ_0. (The models of this iteration are in $L(\mathbb{R}^*)$, but the iteration itself may only be in $V^{Col(\omega,\mathbb{R})}$. But then $\mathcal{P}$ is Φ_n^{**}-iterable in $V^{Col(\omega,\mathbb{R})}$,where Φ_n^{**} is the type of the first n indiscernibles over the reals of that universe. The fact that $\Phi_n^{**} \cap \mathbb{R}^* = \Phi_n^*$ and the absoluteness argument of 2.10 give the desired conclusion.

Let us use $\vec{A}^*$ for the tuple of all A^* such that A is on $\vec{A}$.

Claim 3. For any $\mathrm{OD}^{L(\mathbb{R})}$ tuple $\vec{A}$, $\vec{A}^*$ is $\mathrm{OD}^{L(\mathbb{R}^*)}$; moreover

$$L(\mathbb{R}^*) \models \text{ every } \emptyset\text{-iterate of } \Delta(\mathcal{M}) \text{ is } \vec{A}^*\text{-iterable.}$$

Proof. If not, choose the minimal counterexample $\vec{A}$; we then have that $\vec{A}$ is definable over $L(\mathbb{R})$. Let $\mathcal{R}$ be a suitable initial segment of $\mathcal{Q}_0$. For any A on $\vec{A}$, definable over $L(\mathbb{R})$ by φ, it is a first order over $\mathcal{Q}_0$ fact about $\tau_A^{\mathcal{R}}$ and $\tau_{\Phi_1}^{\mathcal{R}}$ that for G $\mathcal{R}$-generic over the collapse of the largest Woodin, $(\tau_A^{\mathcal{R}})^G = \{\, x \mid L(\mathbb{R}) \models \varphi[x]\} \cap \mathcal{R}[G]$. These countably many first order facts (one for each suitable initial segment of $\mathcal{Q}_0$ are preserved by $j\colon \mathcal{Q}_0 \to \mathcal{M}$, and this implies that A^* is definable over $L(\mathbb{R}^*)$.It follows that every Φ_1^*-iterable $\mathcal{P}$ is $\vec{A}^*$-iterable, so we can apply claim 2 to get a contradiction.

Claim 4. Let $\vec{\mathcal{T}} = \langle \mathcal{T}_0, ..., \mathcal{T}_k \rangle$ be a stack of ω-maximal trees, with $\mathcal{T}_0$ on $\Delta(\mathcal{M})$ and $\mathcal{T}_{i+1}$ on the last model of $\mathcal{T}_i$ for all i. Suppose $\vec{\mathcal{T}} \in X$, and $\vec{\mathcal{T}}$ is according to Σ_0, and $\mathcal{T}_k$ has limit length. Let b be the maximal branch of $\mathcal{T}_k$ chosen by Σ_0; then

$$b = \bigcup_{\vec{A} \in \mathrm{OD}^{L(\mathbb{R})}} \bigcap \{\, c \mid L(\mathbb{R}^*) \models \vec{\mathcal{T}} \oplus c \text{ is an } \vec{A}^*\text{-iteration}\}.$$

Proof. This follows from claim 3 and 5.8 (the latter applied in $V^{Col(\omega,\mathbb{R})}$).

Since anything true in $L(\mathbb{R}^*)$ about parameters in $\mathcal{M}$ is forced over $\mathcal{M}$ by $\emptyset$, claim 4 implies that $\Sigma_0 \restriction X$ is definable over $L(\mathbb{R})$.

We turn to the proof that $\mathrm{HOD}^L(\mathbb{R}) \subseteq \mathcal{M}[\Sigma_0 \restriction X]$. First, let us show that the least Woodin cardinal of $\mathcal{M}$ is reasonably large.

Lemma 6.1 *The least Woodin cardinal of $\mathcal{M}$ is $\theta^{L(\mathbb{R})}$.*

Proof.We first show that $\theta^{L(\mathbb{R})} \leq \delta_0^{\mathcal{M}}$. Assume not. Let $\mathcal{P}$ be the suitable initial segment of $\mathcal{M}$ such that $k(\mathcal{P}) = 1$; our assumption implies that $\mathrm{OR}^{\mathcal{P}} < \theta^{L(\mathbb{R})}$. Now for any $\alpha < \theta^{L(\mathbb{R})}$, if $B_n \subseteq \alpha$ and $B_n \in L(\mathbb{R})$ for all $n < \omega$, then $\langle B_n \mid n < \omega \rangle \in L(\mathbb{R})$, simply because any ω sequence of reals can be coded by a single real. It follows that $\langle T_n(\mathcal{P}) \mid n < \omega \rangle \in L(\mathbb{R})$. But this implies that the function on reals $x \mapsto M_\omega^\sharp(x)$ is in $L(\mathbb{R})$.[We can construct in $L(\mathbb{R})$ a tree U such that for all x, $p[U_x] = \{ M_\omega^\sharp(x)\}$. The tree U builds x on its first coordinate, and a candidate y for (a real coding) $M_\omega^\sharp(x)$ on its second coordinate. On its third coordinate it builds a real coding a premouse $\mathcal{N}$, and on its fourth it elementarily embeds $\mathcal{N}$ into $M_\omega^\sharp(\mathcal{P}$. Finally, on its fifth coordinate U builds a real proving the Σ_1^1 fact that x is generic over $\mathcal{N}$ for the Woodin poset at its bottom Woodin cardinal, and that y codes the version of $M_\omega^\sharp(x)$ we get by doing the $L[\vec{E}]$ construction over x inside $\mathcal{N}[x]$. We have $U \in L(\mathbb{R})$ because $M_\omega^\sharp(\mathcal{P}) \in L(\mathbb{R})$. If $y \in p[U_x]$, then the x-premouse coded by y is fully iterable, because the premouse $\mathcal{N}$ from which it comes is embeddable in $M_\omega^\sharp(\mathcal{P})(= \mathcal{M}_\infty)$. On the other hand, the fact that there is an $\mathcal{N} \in \mathcal{F}$ such that x is generic over $\mathcal{N}$ guarantees that $M_\omega^\sharp(x) \in p[U_x]$.] Now the reals in $M_\omega^\sharp(x)$ are precisely those which are $\mathrm{OD}^{L(\mathbb{R})}(x)$, so if $x \mapsto M_\omega^\sharp(x)$ is in $L(\mathbb{R})$, then we can uniformize $y \notin \mathrm{OD}^{L(\mathbb{R})}(x)$ in $L(\mathbb{R})$. This is well known to be impossible.

In order to see that $\delta_0^{\mathcal{M}} \leq \theta^{L(\mathbb{R})}$, it is useful to consider the fragments of our covering system we get by fixing an $\mathrm{OD}^{L(\mathbb{R})}$ set A. For $A \subseteq \mathbb{R}$ such that $A \in \mathrm{OD}^{L(\mathbb{R})}$, let

$$I_A = \{ \mathcal{P} \mid (\mathcal{P}, A) \in I\},$$

and for $\mathcal{P}, \mathcal{Q} \in I_A$, define

$$\mathcal{P} \leq_A \mathcal{Q} \Longleftrightarrow (\mathcal{P}, A) \leq_I (\mathcal{Q}, A).$$

For $\mathcal{P} \in I_A$, let $H_{\mathcal{P}} = H_{(\mathcal{P},A)}$, and for $\mathcal{P} \leq_A \mathcal{Q}$, let $\pi_{\mathcal{P},\mathcal{Q}} = \pi_{(\mathcal{P},A),(\mathcal{Q},A)}$. Since I_A is directed, and the $\pi_{\mathcal{P},\mathcal{Q}}$ commute, we have a direct limit system. We call it the A-fragment of the covering system. There is a natural embedding j_A from the direct limit of the A-fragment to the direct limit of the full system, given by

$$j_A(\pi_{\mathcal{P},\infty}(x) = \pi_{(\mathcal{P},A),\infty}(x).$$

Claim 5. For any $\mathrm{OD}^{L(\mathbb{R})}$ set $A \subseteq \mathbb{R}$, $\mathrm{ran}(j_A)$=$(H_{(\Delta(\mathcal{M}),A^*)})^{L(\mathbb{R}^*)}$.

Proof. The main point is that $\tau_{A^*}^{\Delta(\mathcal{M})} = \sigma_{A,1}$ (by the definition of A^*)$=$ $\pi_{\mathcal{P},\infty}(\tau_A^{\mathcal{P}})$ for all $\mathcal{P} \in I_A$. We leave the rest to the reader.

Now the direct limit of an A-fragment is the surjective image of $\mathbb{R}$ in $L(\mathbb{R})$. Since $\delta_0^{\mathcal{M}} = \sup\{\, \xi_{(\Delta(\mathcal{M}),\Phi_n^*)} \mid n \in \omega\}$, we have $\delta_0^{\mathcal{M}} \leq \theta^{L(\mathbb{R})}$, as desired. This proves the lemma. $\square$

We shall need the following fact about the elementary equivalence of symmetric collapse models. It is due to Woodin and the author (see [1, section 4]).

Lemma 6.2 *Let N_1 and N_2 be countable iterates of $M_\omega^\sharp$ by its unique ω_1+1-iteration strategy, and let $\mathbb{R}_i$ be the set of reals added by initial segments of a collapse over N_i of everything below the sup of its Woodin cardinals. Let $x \in \mathbb{R}_1 \cap \mathbb{R}_2$, and let ψ_i be the type of the first n indiscernibles of $L(\mathbb{R}_i)$, where $n < \omega$; then*

$$(L(\mathbb{R}_1), \epsilon, x, \psi_1) \equiv (L(\mathbb{R}_2), \epsilon, x, \psi_2).$$

We are ready to show that $\mathrm{HOD}^{L(\mathbb{R})} \subseteq \mathcal{M}[\Sigma_0 \restriction X]$.

Let $\mathcal{M}^*$ be the direct limit of the covering system of $L(\mathbb{R}^*)$. Let $i_n \colon H_{(\Delta(\mathcal{M}),\Phi_n^*)} \to \Delta(\mathcal{M}^*)$ be the direct limit map for the covering system of $L(\mathbb{R}^*)$; this exists by Claim 2 above. Let $i = \cup_{n<\omega}$, so that $i \colon \Delta(\mathcal{M}) \to \Delta(\mathcal{M}^*)$. Although each $i_n \in L(\mathbb{R}^*)$, and in fact is definable from Φ_n^* over $L(\mathbb{R}^*)$, $i \notin L(\mathbb{R}^*)$. But we do have

Claim 6. $i \in \mathcal{M}[\Sigma_0 \restriction X]$.

Proof.(Sketch.) It is enough to show that the iterates of $\Delta(\mathcal{M})$ using Σ_0 and trees in X are cofinal in the $\emptyset$-iterates of $\Delta(\mathcal{M})$ in the sense of $L(\mathbb{R}^*)$. Let τ be a term such that for some $p \in Col(\omega, \delta_k^{\mathcal{M}})$, $p \vdash \tau$ is an $\emptyset$-iterate of $\Delta(\mathcal{M})$. Let $G \in L(\mathbb{R}^*)$ be $Col(\omega, \delta_k^{\mathcal{M}})$-generic over $\mathcal{M}$, and for each $q \leq p$ let G_q be the variant of G defined by: $r \in G_q$ iff r is compatible with q and $r \restriction (\mathrm{dom}(r) - \mathrm{dom}(q)) \in G$. In $L(\mathbb{R}^*)$ we simultaneously compare all τ^{G_q} and $\Delta(\mathcal{M})$. The tree on $\Delta(\mathcal{M})$ which we obtain thereby is in $\mathcal{M}$.

The following claim therefore completes the proof.

Claim 7. $\mathrm{HOD}^{L(\mathbb{R})} \subseteq \mathcal{M}[i]$.

Proof. Let $\mathrm{HOD}^{L(\mathbb{R})} = L[S]$, where $\beta \in S \Leftrightarrow L(\mathbb{R}) \models \varphi[\beta]$, and $S \subseteq \theta^{L(\mathbb{R})}$. We claim that for any $\beta < \theta^{L(\mathbb{R})}$,

$$L(\mathbb{R}) \models \varphi[\beta] \Leftrightarrow L(\mathbb{R}^*) \models \varphi[i(\beta)].$$

(It is a result of Woodin that $\mathrm{HOD}^{L(\mathbb{R})} = L[S]$ for some $S \subseteq \theta^{L(\mathbb{R})}$; S is a souped up version of the Vopenka algebra.) Since $\mathcal{M}[i]$ can compute the answer to the right hand side of the displayed equivalence (by consulting i and the forcing relation), it suffices to prove this claim. So fix $\beta < \theta^{L(\mathbb{R})}$. Let $\mathcal{P}$ be a suitable initial segment of a premouse in $\mathcal{F}$ such that $k(\mathcal{P}) = 1$, and for some $n < \omega$ and $\bar{\beta} \in H_{(\mathcal{P},\Phi_n)}$, we have $\pi_{(\mathcal{P},\Phi_n),\infty}(\bar{\beta}) = \beta$. Fix such an n and $\bar{\beta}$.

Let $\varphi^*(u, v, w)$ be the formula in the language of set theory:

$$\varphi^*(u, v, w) = \text{"}(u, v) \in I \text{ and } w \in u \text{ and } \varphi(\pi_{(u,v),\infty}(w))\text{"}.$$

Let $\mathbb{R}^{**}$ be the set of reals of $V^{Col(\omega,\mathbb{R})}$, and Φ_n^{**} the type of the first n indiscernibles of $L(\mathbb{R}^{**})$. In $V^{Col(\omega,P(\mathbb{R}))}$, each of $\mathbb{R}$, $\mathbb{R}^*$, and $\mathbb{R}^{**}$ is the set of reals of a symmetric collapse of an iterate of $M_\omega^\sharp$ via Σ_0, so by 6.2 we have

$$\begin{aligned}
L(\mathbb{R}) \models \varphi[\beta] &\Leftrightarrow L(\mathbb{R}) \models \varphi^*[\mathcal{P}, \Phi_n, \bar{\beta}] \\
&\Leftrightarrow L(\mathbb{R}^{**}) \models \varphi^*[\mathcal{P}, \Phi_n^{**}, \bar{\beta}] \\
&\Leftrightarrow L(\mathbb{R}^{**}) \models \varphi^*[\Delta(\mathcal{M}), \Phi_n^{**}, \beta] \\
&\Leftrightarrow L(\mathbb{R}^{*}) \models \varphi^*[\Delta(\mathcal{M}), \Phi_n^{*}, \beta] \\
&\Leftrightarrow L(\mathbb{R}^{*}) \models \varphi[i(\beta)].
\end{aligned}$$

(We obtain the third equivalence from the second because in $L(\mathbb{R}^{**})$, $(\mathcal{P}, \Phi_n^{**}) \leq_I (\Delta(\mathcal{M}), \Phi_n^{**})$ and $\pi_{(\mathcal{P},\Phi_n^{**}),(\Delta(\mathcal{M}),\Phi_n^{**})}(\bar{\beta}) = \beta$.)

This finishes the proof of 1.3. It is perhaps worth noting that the proof also showed $\mathrm{HOD}^{L(\mathbb{R})} = \mathcal{M}[i]$, where i is the embedding described above.

References

[1] John Steel, "Inner models with many Woodin cardinals", *Annals of Pure and Applied Logic*, vol. 65 (1993) 185-209

[2] John Steel, "$\mathrm{HOD}^{L(\mathbb{R})}$ is a core model below Θ, *Bulletin of Symbolic Logic*, vol. 1 (1995) 75-84