Set Theory

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Glossary

Baire space: The set ω^ω of all infinite sequences of natural numbers, endowed with the topology whose basic open neighborhoods are the sets $N_s = \{x \in \omega^\omega \mid s \subseteq x\}$, for s a finite sequence of natural numbers. The Baire space is homeomorphic to the irrationals, with the topology induced by the topology of the real line.

Borel set: In any topological space, a member of the smallest family of sets containing the open sets and closed under countable unions and complements

Borel determinacy: The assertion that every game on ω with a payoff set which is Borel (as a subset of Baire space) is determined.

Cardinal number: An object associated with every set, so that two sets are equinumerous exactly when theve the same cardinal number. If the Axiom of Choice is being assumed, then the standard convention is to take the cardinal number of X to be the least ordinal number equinumerous with X.

Closed subset of κ: If κ is an ordinal, a set $C \subseteq \kappa$ is closed just in case whenever $\lambda < \kappa$ is a limit ordinal and C is unbounded in λ, then $\lambda \in C$. The closed subsets of κ can be considered to be "large".

Covering lemma: R. Jensen's theorem that either $0^\sharp$ exists, or for every uncountable set X of ordinals there is an constructible set Y such that $X \subseteq Y$ and Y is equinumerous with X.

Consistency strength (of a theory T): The logical strength of the assertion that T is consistent. T has consistency strength less than or equal to that of S just in case some weak theory (usually Peano Arithmetic) proves "if S is consistent, then T is consistent".

Constructible set: A set obtained through transfinite iteration of the process of forming sets which are definable over the universe of sets formed previously. The universe L of constructible sets is the smallest model of ZFC containing all the ordinals.

Continuum Hypothesis: The assertion that every set of real numbers is either countable or equinumerous with the set of all real numbers.

Countable set: A set which is equinumerous with some subset of the set of natural numbers.

Critical point: The critical point of an embedding j is the least ordinal κ such that $j(\kappa) \neq \kappa$.

Descriptive Set Theory: The study of sets of real numbers, with special attention to their definability. The central thesis of descriptive set theory is that definable sets of reals have useful properties which arbitrary sets of reals may not have.

Determinacy: A game is determined just in case of of its players has a winning strategy, that is, a strategy which wins no matter what the opponent plays.

Easton's theorem: A theorem which characterizes the possibilities for the cardinal exponentiation function $\kappa \mapsto 2^\kappa$, restricted to regular cardinals κ, in models of ZFC.

Elementary embedding: A map $j\colon M \to N$ is elementary iff it preserves truth in the language of set theory; that is, whenever $(M, \in) \models \varphi[x_1, ..., x_n]$, then $(N, \in) \models \varphi[j(x_1), ..., j(x_n)]$.

Forcing: A method for extending models of set theory to larger models, while controlling truth in the larger model. Forcing is the most powerful method for establishing that one theory extending ZFC is consistent relative to some other theory extending ZFC.

Forcing axiom: One of a certain family of assertions to the effect that the universe of sets is saturated with generic filters of various kinds. Examples are *Martin's axiom*, the *Proper Forcing axiom*, and *Martin's maximum.*

Generalized Continuum Hypothesis: The assertion that whenever X is an infinite set, then for any family A of subsets of X, either A is equinumerous with a subset of X, or A is equinumerous with the famiily of all subsets of X.

Generic extension: A model of set theory obtained by adding to an initial model M (usually called the *ground model*) an M-generic filter, and then closing under certain elementary operations.

Generic filter: A subset G of a partially ordered set $\mathbb{P}$ which is a filter (i.e. upward-closed, and such that any two elements of G have a common lower bound in G) is called F-generic just in case $G \cap D$ is nonempty whenever $D \subseteq \mathbb{P}$ is dense (i.e. $\forall q \in \mathbb{P} \exists \imath \leq \imath\imath(\imath \in \mathbb{D})$) and $D \in F$.

Game on X: In set theory, most commonly a game in which *two players* alternate making moves from X, with *perfect information* as to what the opponent has done at each stage. Most commonly, the game ends after ω moves, with the winner declared according to some *payoff set* A consisting of infinite sequences from X considered to be winning runs for the first player.

Inner model: A subuniverse of the universe of all sets which also satisfies the axioms of ZFC. For example, the universe L of constructible sets is an inner model.

Large cardinal hypothesis: One of a certain family of strengthenings of the Axiom of Infinity, asserting the existence of a cardinal number which is inaccessible from below in some way. Many large cardinal hypotheses assert the existence of elementary embeddings j with domain the universe V of all sets. The large cardinal associated to such a j is its critical point.

Measurable cardinal: κ is measurable just in case $\kappa > \omega$, and there is a κ-additive measure $\mu \colon P(\kappa) \to \{0, 1\}$ defined on all subsets of κ, such that μ gives singletons measure zero. Equivalently, κ is measurable iff it is the critical point of an elementary embedding from V into some transitive class.

Powerset: The powerset of a set X is the set $P(X)$ of all subsets of X. The *Powerset Axiom* of ZFC asserts that every set has a powerset.

Projective set: A set of reals, or a subset of Baire space, which can be obtained from a Borel set of reals by repeatedly taking complements and images under continuous functions. Equivalently, a set which is definable over the structure $(V_{\omega+1}, \in)$.

Projective determinacy: The assertion that every game on ω whose payoff set is projective is determined.

Singular Cardinals Hypothesis: The assertion that if κ is a singular cardinal, then $\kappa^{\mathrm{cof}(\kappa)} = (\Sigma_{\alpha<\kappa} 2^\alpha)^+$, which is the smallest value for $\kappa^{\mathrm{cof}(\kappa)}$ consistent with Konig's theorem. The negation of the Singular Cardinals Hypothesis is consistent relative to ZFC augmented by large cardinal hypotheses, but has consistency strength strictly greater than that of ZFC.

Stationary set: If κ is an ordinal, a set $S \subseteq \kappa$ is stationary just in case $S \cap C$ is nonempty, for all C which are closed and unbounded in κ.

Supercompact cardinal: κ is λ-supercompact iff there is an elementary $j: V \to M$ with critical point κ, where M is transitive and every λ-sequence of elements of M is in M. We say κ is supercompact iff κ is λ-supercompact for all λ. At this moment, the assertion that supercompact cardinals exist is one of the strongest large cardinal hypotheses to have been extensively used.

Suslin's Hypothesis: The assertion that there are no Suslin trees on ω_1.

Suslin tree: A Suslin tree on κ is a tree order on κ in which all chains and all antichains have cardinality strictly less than κ.

Tree order: A tree order on X is a partial order $\leq$ of X with the property that for any $y \in X$, $\{z \mid z \leq y\}$ is wellordered by $\leq$.

Summary All mathematical statements can be expressed in the very simple *language of set theory*, whose quantifiers are understood as ranging over sets, and whose only non-logical symbol $\in$ stands for the membership relation. All mathematical proofs to date can be carried out granted certain basic axioms about sets. In this sense, the general theory of sets is a foundation for all of mathematics.

The vast majority of mathematical proofs require no more than the axioms of Zermelo-Fraenkel set theory with Choice, or ZFC. Nevertheless, a surprising number of quite basic questions about sets in general are not decided by the axioms of ZFC; moreover many of the more abstract questions of analysis, algebra, and topology are left similarly undecided. Perhaps the most famous of the undecided questions is Cantor's Continuum Problem: what is the cardinality of the set of all real numbers?

In this article we shall describe some of the basic ideas of the general theory of sets, and then move to an exposition of the methods by which one can show that a given statement in the language of set theory is not decided by ZFC. We conclude with a discussion of extensions of ZFC obtained by strengthening the Axiom of Infinity, and show that these suffice to remove the incompleteness of ZFC in one important realm.

1 Introduction

Much of set theory is motivated by the simple question

What are the proper axioms for mathematics?

Mathematicians prove things for a living; what should they take as their common assumptions in these proofs?

In a sense, this question goes back to Euclid, although in his day, merely axiomatizing geometry was a great achievement. It was through Descartes' reduction of geometry to analysis, the reduction of analysis to arithmetic by 19^{th} century mathematicians such as Cauchy and Dedekind, and the reduction of arithmetic to set theory by Gottlob Frege, that the project of axiomatizing all of mathematics became feasible around the turn of the 20^{th} century. These reductions showed that all of the mathematics of the time could be expressed in the very simple *language of set theory*, whose quantifiers are understood as ranging over *pure sets*, and whose only non-logical symbol $\in$ is interpreted as standing for the membership relation. The pure sets are just those built up from the empty set by repeatedly forming sets of objects previously constructed in a process which is iterated into the transfinite. Since we shall not need to consider impure sets (such as the set of knives in my cabinet), henceforth we use *set* to mean pure set.

In the period 1905-1927, Russell, Zermelo, Fraenkel, and Skolem isolated an elegant list of basic statements about sets, expressed in the language of set theory, and showed that from these axioms one could derive all of the mathematics of the time. This system of axioms is now known as Zermelo-Fraenkel Set Theory with Choice, or ZFC. Most of its axioms are set-existence axioms; for example, the Axiom of Infinity asserts that there is an infinite set, and the Powerset Axiom asserts that for every set x, there is a set $P(x)$ whose members are precisely the subsets of x. (See Moschovakis for further discussion.) ZFC is a good provisional answer to the question with which we began, and it is still true today that almost everything that mathematicians have proved has been derived from its axioms.

While there is still no hint of a mathematical statement which cannot be expressed in the language of set theory, we have discovered that ZFC is incomplete in important ways. A surprising number of quite basic questions about sets in general are not decided by the axioms of ZFC, moreover, many of the more abstract questions of analysis, algebra, and topology are similarly left undecided. Here is a list of some examples.

The Continuum Hypothesis.

The first theorem of general set theory was Cantor's remarkable discovery, in 1873, that infinite sets come in different sizes. Cantor showed that for any set x, there are strictly more subsets of x than there are elements of x; in modern cardinality notation, that $|x| < |P(x)|$ for all x. He naturally asked whether for infinite x there are any sets of intermediate size. The *Generalized Continuum Hypothesis*, or GCH, asserts that there are not, that is, that for all infinite x, $|P(x)| = |x|^+$, where $|x|^+$ is the least cardinal strictly greater than $|x|$. The special case in which $|x| = \omega$ is the smallest infinite cardinal is called the *Continuum Hypothesis*, or CH, because in this case $|P(x)| = |\mathbb{R}|$. Calling a set *countable* if it is either finite or equinumerous with the set ω of all natural numbers, the Continuum Hypothesis is equivalent to the statement that every uncountable set of real numbers is equinumerous with the set $\mathbb{R}$ of all real numbers.

In 1937, Gödel produced a model of ZFC in which the GCH is true. In 1963, Cohen produced a family of models of ZFC in which the CH is false; there are, as we shall see, many possibilities for the function $|x| \mapsto |P(x)|$ in such models. By itself, ZFC has very little to say about one of the most basic problems in general set theory, the problem of counting the powerset of an infinite set.

Suslin's Hypothesis.

A partial order is a structure $\mathbb{P} = (P, \leq)$, where $\leq$ is a reflexive, antisymmetric, transitive binary relation on P. $\mathbb{P}$ is *linear* if for all $x, y \in P$, $x \leq y$ or $y \leq x$. If $\mathbb{L} = (L, \leq)$ is a linear order with no largest or smallest element such that

(1) Every $X \subseteq L$ which is bounded above has a least upper bound, and

(2) there is an $X \subseteq L$ such that $|X| = \omega$ and X is dense in $\mathbb{L}$ (i.e. $\forall a, b \in L(a < b \to \exists c \in X(a < c < b))$,

then $\mathbb{L}$ is isomorphic to $(\mathbb{R}, \leq^{\mathbb{R}})$, the real numbers with the standard order on them. In the 1920's, Suslin conjectured that this characterization of $(\mathbb{R}, \leq^{\mathbb{R}})$ can be varied by replacing (2) with the ostensibly weaker

(2a) if $\mathcal{I}$ is a family of pairwise disjoint intervals of $\mathbb{L}$, then $\mathcal{I}$ is countable.

A *Suslin line* is a linear order $\mathbb{L}$ without endpoints satisfying (1) and (2a), but not isomorphic to $(\mathbb{R}, \leq^{\mathbb{R}})$. *Suslin's Hypothesis* (SH) asserts, confusingly enough, that there are no Suslin lines.

In the late 1960's, Jech used Cohen's techniques to produce a model of ZFC is which there are Suslin lines. Not long afterward, Solovay and Tennenbaum extended Cohen's technique in an important way, and thereby produced a model of ZFC in which there are no Suslin lines.

Scales in $(\omega^\omega, \leq^*)$.

We define the partial order of eventual domination on $\omega^\omega = \{f \mid f\colon \omega \to \omega\}$ by: $f \leq^* g$ iff $\exists m \forall n \geq m(f(n) \leq g(n))$. A *scale* in $(\omega^\omega, \leq^*)$ is a set $S \subseteq \omega^\omega$ which is linearly ordered by $\leq^*$ and cofinal, in the sense that $\forall f \in \omega^\omega \exists g \in S(f \leq^* g)$. It is not hard to see that CH implies that there is a scale. There are models of ZFC constructed by Cohen's method in which CH is false and there are no scales, as well as models in which CH is false and nevertheless, there is a scale.

Lebesgue Measure.

In 1902, the analyst Lebesgue defined a natural measure μ on certain sets of reals. If A is an interval, then $\mu(A)$ is just its length. If A is open, then $A = \bigcup_{n<\omega} I_n$ where the I_n are disjoint intervals, and we set $\mu(A) = \Sigma_{n=1}^{\infty} \mu(I_n)$. If A is closed, then $\mu(A) = \Sigma_{n \in \mathbb{Z}} 1 - \mu((n, n+1) \setminus A)$. Finally, for arbitrary A, we define the outer measure

$$\mu^+(A) = \inf(\{\mu(U) \mid U \text{ is open and } A \subseteq U\}),$$

and inner measure

$$\mu^-(A) = \sup(\{\mu(F) \mid F \text{ is closed and } F \subseteq A.\}).$$

We say that A is *Lebesgue measurable* iff $\mu^+(A) = \mu^-(A)$, in which case we write $\mu(A)$ for the common value. Lebesgue's measure leads to a natural and important extension of the Riemann integral to functions which may have infinitely many discontinuities.

Lebesgue showed that the collection of Lebesgue measurable sets is closed under complements and countable unions, and therefore includes the class of Borel subsets of $[0, 1]$. On the other hand, Vitali showed with a heavy use of the Axiom of Choice that there is a non-Lebesgue-measurable set. His construction is the following: for $x, y \in [0, 1]$, let $xEy \Leftrightarrow x - y$ is rational. E is an equivalence relation, and by the Axiom of Choice there is a set $A \subseteq [0, 1]$ such that A meets each equivalence class of E in exactly one point. Suppose A were Lebesgue measurable, and consider the rational translations $T_q\colon [0, 1] \to [0, 2]$ given by

$$T_q(x) = x + q$$

for rational $q \in [0, 1]$. Our choice of A guarantees that the images $T_q(A)$ for $q \in \mathbb{Q}$ are pairwise disjoint and cover $[1, 2]$. But Lebesgue measure is translation invariant, so $\mu(A) = \mu(T_q(A))$ for all q. Since Lebesgue measure is also countably additive, we have a contradiction: if $\mu(A) = 0$, then $\mu([1, 2]) = 0$, and if $\mu(A) > 0$, then $\mu([1, 2]) = \infty$.

Vitali's example illustrates the fact that *arbitrary* sets of reals may have no geometric content. A dramatic example of this phenomenon is the famous *Banach-Tarski paradox*: there is a decomposition of the unit ball in $\mathbb{R}^3$ into finitely many pieces, which can be moved by translations and rotations to become a decomposition of the unit cube. The pieces in such a decomposition cannot be measurable with respect to the natural extension of Lebesgue measure to $\mathbb{R}^3$.

It is natural to ask whether the sets more familiar to analysts, those given by some construction or definition, can exhibit such pathology, or whether they are all Lebesgue measurable. Gödel showed in 1937 that in his model of $\mathsf{ZFC} + \mathsf{GCH}$, it is also true that there is a simply definable subset of $[0, 1]$ which is not Lebesgue measurable. On the other hand, Solovay in 1965 was able to use Cohen's technique to produce a model of ZFC in which all reasonably definable sets of reals are Lebesgue measurable.

It is also natural to ask whether there is an extension of Lebesgue measure to a measure ν defined on all $A \subseteq [0, 1]$. (Such a measure ν cannot be translation-invariant, by Vitali's argument.) In Gödel's model, there is no such extension; on the other hand, Solovay in 1966 produced a model of ZFC in which there is such an extension.

In the rest of this paper we shall outline the independence techniques of Gödel and Cohen, and show how they apply to these and several other natural problems. The sheer volume of work in this vein can lead to the impression that the set-theorist's sole function deliver to other mathematicians, often other set-theorists, the bad news that they will not be able to solve this or that problem. We hope to lead the reader to a different conclusion: that one of the most interesting and potentially useful global features of the universe V of all sets is that it has within it descriptions of many alternate universes satisfying various natural theories. These alternate universes are not so far from V in some ways; for example, they all have the same natural numbers as does V, so that any statement of number theory true in one of them is true in V. The study of such alternate universes is an inevitable and important part of the global theory of V.

We shall also introduce some plausible strengthenings of the Axiom of Infinity, and show that these strong axioms of infinity (also known as *large cardinal hypotheses*), decide many of the more concrete statements of mathematical interest which are left undecided by ZFC. This suggests that our answer to "Euclid's question" is still evolving.

2 Some elementary tools.

We begin with a quick inventory of some of the more elementary ideas and results of set theory. The reader who would like a gentler and more thorough introduction to this material should see Moschovakis .

2.1 Ordinals

Transfinite induction is one of the signature methods of set theory. Indeed, set theory began with Cantor's use of transfinte induction to study trigonometric series.

A *wellfounded relation* is a binary relation R on a set Y such that every nonempty set $X \subseteq Y$ has an R-minimal element, that is, $\exists a \in X \forall b \in Y(\neg bRa)$. Equivalently, R is wellfounded iff there is no sequence $\langle a_i \mid i < \omega \rangle$ such that $\forall i(a_{i+1} R a_i)$. One can carry out proofs and definitions by transfinite induction along a wellfounded relation; for example, to define a function f with domain Y, it suffices to define $f(a)$ from $f \restriction \{b \mid bRa\}$. The existence of a unique f with domain Y satisfying the induction clause at all $a \in Y$ can be proved in ZFC, using the Axiom of Replacement.

A *wellorder* is a linear order whose associated strict order is wellfounded. Any two wellorders $\mathbb{W}_1$ and $\mathbb{W}_2$ can be compared as to length: either $\mathbb{W}_1$ is isomorphic to a proper initial segment of $\mathbb{W}_2$, or $\mathbb{W}_1$ is isomorphic to $\mathbb{W}_2$, or $\mathbb{W}_2$ is isomorphic to a proper initial segment of $\mathbb{W}_1$, moreover these possibilities are mutually exclusive, and the isomorphism is in each case unique. These facts are proved using transfinite induction.

A set X is *transitive* if whenever $a \in b$ and $b \in X$, then $a \in X$; that is, X is an $\in$-initial segment of the universe of all sets. If $(Y, <^*)$ is a strict wellorder, we define by transfinite induction

$$\pi(a) = \{\pi(b) \mid b <^* a\}.$$

The range of π is a transitive set which is strictly wellordered by $\in$; we call such a set an *ordinal*. We have thus shown that every (strict) wellorder is isomorphic to an ordinal. It is easy to show that the ordinal and isomorphism are uniquely determined by the wellorder. If α and β are ordinals, we write $\alpha < \beta$ to mean that α is isomorphic to a proper initial segment of β. It is a pleasant consequence of this choice of representatives for isomorphism types of wellorders (which is due to Von Neumann) that $\alpha < \beta$ iff $\alpha \in \beta$. The relation $<$ wellorders the class of all ordinals, with the least upper bound of a set X being the ordinal $\bigcup X$. The least ordinal is $0 = \emptyset$, the order-type of the empty wellorder. If α is an ordinal, then $\alpha+1 = \alpha \cup \{\alpha\}$ is the least ordinal strictly greater than α. λ is a *limit ordinal* iff $\lambda > 0$ and $\forall\alpha(\lambda \neq \alpha + 1)$. The Axiom of Infinity is equivalent to the statement: there is a limit ordinal. The least limit ordinal is called ω, and its members are called natural numbers.

2.2 The wellordering theorem

The Axiom of Choice implies that wellorders exist in abundance:

Theorem 2.1 (Zermelo 1905) *Every set admits a wellorder.*

Proof. Let X be given, and let f be a choice function with domain $P(X)$, so that $f(Y) \in Y$ for all nonempty $Y \subseteq X$. We define a function π with domain the ordinals by transfinite induction as follows:

$$\pi(\alpha) = f(X \setminus \{\pi(\beta) \mid \beta < \alpha\}).$$

That is, we keep listing distinct elements of X, using f to pick the next one. The Axiom of Replacement implies that there is an α such that $\mathrm{ran}(\pi \restriction \alpha) = X$. But then the image under π of the wellorder $\in$ on α is a wellorder of X. $\square$

This combination of definition by transfinite induction and the Axiom of Choice is quite powerful. Here is another example. A *filter* on a set I is a family $\mathcal{F} \subseteq P(I)$ closed under finite intersections and superset, and not containing $\emptyset$. One should think of $\mathcal{F}$ as a notion of largeness for subsets of I. An *ultrafilter* on I is a filter $\mathcal{F}$ on I such that for all $X \subseteq I$, either $X \in \mathcal{F}$ or $I \setminus X \in \mathcal{F}$. Using Zermelo's method, we can show *every filter extends to an ultrafilter*. For let $\langle X_\alpha \mid \alpha < \theta \rangle$ enumerate $P(I)$, and let $\mathcal{F}$ be a given filter on I. We define filters $\mathcal{F}_\alpha$ for $\alpha \leq \theta$ by induction: set $\mathcal{F}_0 = \mathcal{F}$,

$$\mathcal{F}_{\alpha+1} = \text{ some filter } \mathcal{G} \text{ extending } \mathcal{F}_\alpha \text{ s.t. } X_\alpha \in \mathcal{G} \text{ or } I \setminus X_\alpha \in \mathcal{G},$$

and

$$\mathcal{F}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{F}_\alpha$$

for λ a limit ordinal. It is not hard to show that a filter $\mathcal{G}$ as above must exist, and that at limit stages, $\mathcal{F}_\lambda$ remains a filter. The desired ultrafilter is then $\mathcal{F}_\theta$.

2.3 The cumulative hierarchy; proper classes

One further definition by transfinite induction deserves mention, that of the *cumulative hierarchy.* We define

$$V_0 = \emptyset,$$

$$V_{\alpha+1} = V_\alpha \cup P(V_\alpha),$$

and for λ a limit ordinal,

$$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha.$$

Thus V_α consists of those sets which can be built up in $< \alpha$ stages by taking sets of objects previously formed. Each V_α is transitive, and $\alpha \leq \beta \Rightarrow V_\alpha \subseteq V_\beta$. It follows from the Axiom of Foundation of ZFC that every set is in some V_α. We shall also write V for the union of all the V_α, the universe of all sets. V is not itself a set, but rather what is sometimes called a *proper class.* Another useful proper class is OR, the class of all ordinals. If, as we have claimed, the language of set theory is universal, then our apparent references to proper classes should be eliminable, and indeed they are. Usually, this is because we can replace "$x \in A$" with $\varphi(x)$, where φ is a formula of the language of set theory defining membership in A. Thus instead of saying that $V_\alpha \cap \mathrm{OR} = \alpha$ (which, incidentally, is true), we could simply say $\forall x(x \in V_\alpha \wedge x \text{ is an ordinal }) \Leftrightarrow x \in \alpha$. Nevertheless, the apparent reference to proper classes can be convenient, so we shall make use of it on occasion.

2.4 Cardinals

Two sets are *equinumerous* iff there is a bijection between them. By the Wellordering Theorem 2.1, every set is equinumerous with an ordinal, and we can set

$$|x| = \text{ least } \alpha \in \mathrm{OR} \text{ s.t. } x \text{ is equinumerous with } \alpha.$$

We have that $|x| \leq |y|$ iff there is an injection from x into y iff there is a surjection from y onto x. If κ is an ordinal, then

$$\kappa \text{ is a cardinal } \Leftrightarrow \exists x(\kappa = |x|) \Leftrightarrow \kappa = |\kappa|.$$

For any cardinal κ, κ^+ is the least cardinal strictly greater than κ. For κ infinite, κ^+ is just the set of ordinals representing order-types of wellorders of κ. The sequence of infinite cardinals is given by

$$\omega_0 = \omega,$$

$$\omega_{\alpha+1} = (\omega_\alpha)^+,$$

and for λ a limit ordinal

$$\omega_\lambda = \bigcup_{\alpha<\lambda} \omega_\alpha.$$

The notation $\aleph_\alpha$ is often used for ω_α.

The arithmetic laws for addition and multiplication of infinite cardinals are trivial, as $\kappa + \lambda = \kappa \times \lambda = \max(\kappa, \lambda)$. Exponentiation is the source of some of the deepest problems of set theory. Setting

$$\kappa^\lambda = |\{f \mid f\colon \kappa \to \lambda\}|,$$

it is easy to see that $2^\kappa = |P(\kappa)|$, so that Cantor's theorem is that

$$\kappa < 2^\kappa$$

for all infinite κ. GCH is the assertion that $2^\kappa = \kappa^+$ for all $\kappa \geq \omega$, and CH is the assertion that $2^\omega = \omega_1$.

2.5 Cofinality, inaccessibility, and König's theorem

If λ is a limit ordinal, then we say a set X is *cofinal in* λ iff $\forall\alpha < \lambda\exists\beta \in X(\alpha < \beta < \lambda)$. The *cofinality* of λ is given by

$$\mathrm{cof}(\lambda) = \text{ least } \alpha \text{ s.t. } \exists f\colon \alpha \to \lambda(\mathrm{ran}(f) \text{ is cofinal in } \lambda).$$

We say λ is *regular* if $\lambda = \mathrm{cof}(\lambda)$, and *singular* otherwise. It is easy to see that every regular ordinal is a cardinal. Every infinite successor cardinal is regular, for if $\mathrm{cof}(\kappa^+) \leq \kappa$, then κ^+ can be written as a union of κ sets of cardinality $\leq \kappa$, which would imply $|\kappa \times \kappa| > \kappa$, a contradiction. There are singular cardinals; the least of these is $\aleph_\omega$, whose cofinality is ω, as witnessed by the function $f(n) = \aleph_n$. Other singular limit cardinals are $\aleph_{\omega+\omega}, \aleph_{\omega_1}, \aleph_{\aleph_{\omega_2}}, \ldots$. Indeed, the reader will be hard-pressed to name a regular limit cardinal, for it is consistent with ZFC that there are none.

κ is *weakly inaccessible* just in case κ is a regular and $\forall\alpha < \kappa(\alpha^+ < \kappa)$, and κ is (strongly) *inaccessible* iff κ is regular and $\forall\alpha < \kappa(2^\alpha < \kappa)$. The two concepts are coextensive under GCH, but diverge in some models of ZFC. Clearly, ω is inaccessible. The assertion that there is an inaccessible cardinal strictly greater than ω is our first example of a *strong axiom of infinity*, or *large cardinal axiom.* We shall discuss such principles in greater detail below. Suffice it to say for now that although ZFC does not prove there are inaccessibles (even weak ones) above ω, we believe that the evidence that this is true is about as convincing as the evidence that ω exists.

Cofinalities enter into an improvement of Cantor's theorem that $\kappa < 2^\kappa$ proven by J. Konig around 1900.

Theorem 2.2 (König) *Let κ be and infinite cardinal; then*

(1) $\kappa < \kappa^{\operatorname{cof}\kappa)}$,

(2) $\kappa < \operatorname{cof}(2^\kappa)$.

Proof. For (1), let $\kappa = \bigcup_{\alpha<\operatorname{cof}(\kappa)} A_\alpha$, where $|A_\alpha| < \kappa$ for all α. Let $f\colon \kappa \to \{g \mid g\colon \operatorname{cof}(\kappa) \to \kappa\}$; we must find a $g\colon \operatorname{cof}(\kappa) \to \kappa$ which is not in the range of f. This we do by Cantor's diagonal method; set

$$g(\alpha) = \text{ least } \eta \in (\kappa \setminus \{f(\xi)(\alpha) \mid \xi \in A_\alpha\}).$$

It is easy to see that g works.

(2) follows from (1) and some cardinal arithmetic: $(2^\kappa)^\kappa = 2^{\kappa\times\kappa} = 2^\kappa$, so $(2^\kappa)^\kappa < (2^\kappa)^{\operatorname{cof}(2^\kappa)}$ by (1). It follows that $\kappa < \operatorname{cof}(2^\kappa)$. □

It follows from König's theorem that 2^ω is not $\aleph_\omega$, or more generally, any cardinal of cofinality ω. We shall see that Cohen's method shows that, roughly speaking, this is the only restriction on 2^ω provable in ZFC.

2.6 Club and stationary sets

Fix an uncountable regular cardinal κ. A set $C \subseteq \kappa$ is *closed* in κ if it contains all its limit points, that is, $\forall\alpha < \kappa(C \text{ is cofinal in } \alpha \Rightarrow \alpha \in C)$. We say C is *club* in κ if it is closed and unbounded (i.e. cofinal) in κ. The intersection of $< \kappa$ club sets is still club. [Given club C_α for $\alpha < \beta$, where $\beta < \kappa$, pick $\nu^0_\alpha \in C_\alpha$, and set $\nu^0_\beta = \sup(\{\nu^0_\alpha \mid \alpha < \beta\})$. Since κ is regular, $\nu^0_\beta < \kappa$. Now pick $\nu^1_\alpha \in C_\alpha \setminus \nu^0_\beta$, and set $\nu^1_\beta = \sup(\{\nu^1_\alpha \mid \alpha < \beta\})$, etc. After ω rounds, we have $\sup(\{\nu^i_\beta \mid i < \omega\}) \in \bigcap_{\alpha<\beta} C_\alpha$.] The club sets therefore generate a natural filter on κ, and this filter is *κ-complete* (i.e. closed under intersections of size $< \kappa$).

Club sets arise as the closure points of some construction. An abstract version of this fact is that whenever $f\colon \kappa^{<\omega} \to \kappa$, then $\{\alpha < \kappa \mid f[\alpha^{<\omega}] \subseteq \alpha\}$ is club in κ. (Here $X^{<\omega} = \{s \mid \exists n < \omega(s\colon n \to X)\}$ is the set of finite sequences from X.) Thus for any finitary procedure for generating ordinals, most $\alpha < \kappa$ are not generated from below. Here it is crucial that κ be regular and uncountable. There is a model-theoretic form of this fact. We call a sequence $\langle \mathcal{A}_\alpha \mid \alpha < \kappa\rangle$ of structures a *substructure chain* if $\mathcal{A}_\alpha \subseteq \mathcal{A}_\beta$ for $\alpha < \beta$ and we call the chain *continuous* is $\mathcal{A}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{A}_\alpha$ whenever λ is a limit. The following *reflection theorem* is closely related to the Downward Lowenheim-Skolem theorem of model theory. We write $\mathcal{A} \prec \mathcal{B}$ to mean that $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$. (See Kiesler .)

Theorem 2.3 *Let κ be regular and uncountable, and let $\langle \mathcal{A}_\alpha \mid \alpha < \kappa\rangle$ be a continuous substructure chain such that $|\mathcal{A}_\alpha| < \kappa$ for all $\alpha < \kappa$. Let $\mathcal{A} = \bigcup_{\alpha<\kappa} \mathcal{A}_\alpha$ be the union of the chain; then there are club many $\alpha < \kappa$ such that $\mathcal{A}_\alpha \prec \mathcal{A}$.*

The *stationary* subsets of κ are those S which are "not small" from the point of view of the club filter, that is, such that $S \cap C \neq \emptyset$ for all club C. An often used and not too difficult to prove fact is *Fodor's Lemma*, which states that whenever $f\colon S \to \kappa$ where S is stationary, and $f(\alpha) < \alpha$ for all $\alpha \in S$, then f is constant on some stationary $T \subseteq S$.

Is the club filter an ultrafilter? The following theorem of Solovay (building on results of Ulam) shows that this is far from the case.

Theorem 2.4 (Solovay 1971) *Let κ be regular and uncountable; then every stationary subset of κ can be partitioned into κ pairwise disjoint stationary sets.*

2.7 Trees

Infinitary combinatorics extends well beyond counting problems. One structure of great interest is the *tree*. A tree is a partial order $\mathbb{T} = (T, \leq_T)$ such that for all $a \in T$, the set of $\leq_T$- predecessors of a is wellordered by $\leq_T$. The *rank* of a in $\mathbb{T}$ is the ordinal isomorphic to $(\{b \mid b <_T a\}, <_T)$. The α-th *level* of $\mathbb{T}$ is $T_\alpha = \{a \mid \text{ rank } (a) = \alpha\}$. The *height* of $\mathbb{T}$ is the least α such that $T_\alpha = \emptyset$. A *branch* through $\mathbb{T}$ is a $B \subseteq T$ such that any two elements of B are $\leq_T$-comparable, and B is closed downward under $\leq_T$.

According to the (easily proved) *Konig's lemma*, every tree of height ω all of whose levels are finite has an infinite branch. This result does not generalize to ω_1:

Theorem 2.5 (Aronszajn 1950) *There is a tree $\mathbb{T}$ of height ω_1 such that $|T_\alpha| \leq \omega$ for all α, but $\mathbb{T}$ has no uncountable branches.*

Aronszajn's construction works with an arbitrary successor cardinal playing the role of ω_1, assuming GCH. Mitchell has shown that it is consistent with ZFC that there are no Aronszajn trees at ω_2; in his model, $2^\omega = \omega_2$. The nonexistence of Aronszajn trees at some cardinal $\kappa > \omega$ has large cardinal strength; see below.

An *antichain* in a tree is a set of pairwise incomparable elements. A *Suslin tree* is a tree of height ω_1 having no uncountable branches and no uncountable antichains. Clearly, every Suslin tree is an Aronszajn tree. We have

Theorem 2.6 (Miller) *There is a Suslin tree iff there is a Suslin line.*

Thus ZFC does not decide whether there are Suslin trees.

This is just a sampling of many questions and results concerning trees. Let us mention one more: a *Kurepa tree* is a tree of height ω_1 all of whose levels are countable, but which has at least ω_2 uncountable branches. In the self-effacing tradition of Suslin, *Kurepa's Hypothesis*, or KH, is the assertion that there are no Kurepa trees. KH is not decided by ZFC.

2.8 Transitive models and the Levy hierarchy

Let us use LST for the first order language with one binary relation symbol $\in$. An interpretation, or *model*, for LST is a pair (M, E), where M serves as a universe over which the quantifiers range, and E is a binary relation on M interpreting the $\in$-symbol. We shall be able to restrict our attention to the case that M is a transitive set, and E is the true membership relation on M. Since E is then determined by M, we shall drop mention of it, writing for example

$$M \models \varphi[a_1, ..., a_n]$$

to mean that the formula $\varphi(v_1, ..., v_n)$ is true of $a_1, ..., a_n$ when its quantifiers range over M, and $\in$ stands for the membership relation on M. Although it might seem unduly restrictive to consider only *transitive models* M of ZFC, we shall see that in practice it is not.

Up to isomorphism, the transitive models of ZFC are just the wellfounded ones:

Lemma 2.6.1 *(a) Let (X, R) be a model for LST which satisfies the Axiom of Extensionality, and suppose R is wellfounded; then there is a unique transitive set M and isomorphism $\sigma\colon (X, R) \cong (M, \in)$.*

(b) Let N be transitive, and suppose $(X, \in) \prec (N, \in)$; then there is a unique transitive set M and elementary embedding $\pi\colon M \to N$ such that $X = \mathrm{ran}(\pi)$.

Proof. Part (a) is a generalization of our result that every wellorder is isomorphic to an ordinal. We let

$$\sigma(a) = \{\sigma(b) \mid bRa\},$$

and take M to be the range of σ. Since $(X, R) \models$ Extensionality, σ is one-one. Part (b) follows at once from (a) applied to $R = \in \cap (X \times X)$; the map π is just σ^{-1}. $\square$

Lemma 2.6.1 is known as the *Mostowski Collapse Lemma*, and $\pi\colon M \to N$ as in (b) is called the *transitive collapse* of X.

In order to help determine what is true in a transitive model, we introduce a hierarchy of formulae.

Definition 2.7 *A formula φ of LST is Σ_0 , or equivalently Π_0, iff all its quantifiers are* bounded*, that is, of the form $(\exists x \in y)$ or $(\forall x \in y)$. For $n < \omega$, we say φ is Σ_{n+1} iff $\varphi = \exists v_1 ... \exists v_k \psi$ where ψ is Π_n, and we say φ is Π_{n+1} iff $\varphi = \forall v_1 ... \forall v_k \psi$ where ψ is Σ_n.*

Because transitive models are closed downward under $\in$, they interpret the bounded quantifiers $(\exists x \in y)$ and $(\forall x \in y)$ correctly. This leads to

Lemma 2.7.1 *Let M, N be transitive with $M \subseteq N$. Let $\varphi(v_1, ..., v_n)$ be a formula of LST, and $a_1, ..., a_n \in M$; then*

(1) if φ is Σ_0, then $M \models \varphi[a_1, ..., a_n]$ iff $N \models \varphi[a_1, ..., a_n]$;

(2) if φ is Σ_1 and $M \models \varphi[a_1, ..., a_n]$, then $N \models \varphi[a_1, ..., a_n]$, and

(3) if φ is Π_1 and $N \models \varphi[a_1, ..., a_n]$, then $M \models \varphi[a_1, ..., a_n]$.

One says that Σ_0 formulae are *absolute* (for transitive models), while Σ_1 formulae are *upward absolute* and Π_1 formulae are *downward absolute*. This is useful because many elementary notions can be defined using such simple formulae. For example, $x \subseteq y$ iff $\forall w \in x(w \in y)$, so $x \subseteq y$ is Σ_0- definable. It is an exercise in elementary set theory to show

Lemma 2.7.2 *The following relations are Σ_0 definable: $y = \bigcup x$, $z = x \cap y, z = x \cup y$, $z = x \setminus y$, $z = \{x, y\}, z = \langle x, y\rangle$, z is an ordered pair, $z = x \times y$, z is a binary relation, z is a function, z is an injection (surjection, bijection) from x to y, x is transitive, x is an ordinal, x is a limit ordinal.*

This list is far from complete, but if the reader will think through a few of these examples, he will become significantly more adept at determining truth in transitive models. Not all elementary relations are Σ_0 definable, most notably those associated with the powerset and cardinality.

Lemma 2.7.3 *The relation $|x| \leq \kappa$ is Σ_1 definable. The relation κ is a cardinal, and the relation $y = P(x)$, are Π_1-definable.*

For example, κ is a cardinal iff $\forall f \forall \alpha \in \kappa(f$ is not a surjection from α onto κ.) This is visibly Π_1. It is therefore downward absolute: if $M \subseteq N$ are transitive, and $N \models \kappa$ is a cardinal, then $M \models \kappa$ is a cardinal. (We have abbreviated $M \models$ (v is a cardinal) $[\kappa]$ here, and we shall often use similar abbreviations in the future.)

Being a cardinal is not upward absolute, and hence not Σ_1 definable. For let $\lambda > \omega_1$, and by Downward Lowenheim-Skolem (see Kiesler), let $(X, \in) \prec (V_\lambda, \in)$ with X countable and $\mu = \omega_1 \in X$. Let $\pi \colon M \to V_\lambda$ be the transitive collapse of X, and $\pi(\kappa) = \mu$. It is easy to see that $\omega + 1 \subseteq X$, so that $\pi \restriction \omega + 1$ is the identity. But $V_\lambda \models \mu$ is a cardinal, since this is Π_1 and true in V, so $M \models \kappa$ is a cardinal, whereas in reality, $\omega < \kappa < \omega_1$.

It is not hard to see that the ordinal κ in the last paragraph is ω_1^M, by which we mean that $M \models$ (v is the least uncountable cardinal)$[\kappa]$. As an exercise, using similar notation, the reader might verify that if M is transitive and $x \in M$, then $M \models P(x)$ exists iff $P(x) \cap M \in M$, in which case $P(x)^M = P(x) \cap M$. It is quite possible that $P(x)^M \neq P(x)$; for example, for the M of the last paragraph, $P(\omega)^M$ exists, but is countable in V. Thus $y = P(x)$ is not upward absolute.

For any set x, the *transitive closure* of x is the smallest transitive set inculding x. It is given by $\mathrm{TC}(x) = x \cup \bigcup x \cup \bigcup \bigcup x \cup ....$ The collection of sets hereditarily of cardinality $< \kappa$ is given by

$$H_\kappa = \{x \mid |\mathrm{TC}(x)| < \kappa\}.$$

One can show $H_\kappa \subseteq V_\kappa$, so H_κ is a set, rather than a proper class. We also write $\mathrm{HC} = H_{\omega_1}$ for the set of hereditarily countable sets. The reader can gain some appreciation for the restrictions ZFC imposes on its transitive models by working through a proof of the following proposition.

Proposition 2.7.1 *(a) V_ω satisfies all axioms of* ZFC *except the Axiom of Infinity. The Axiom of Infinity is false in V_ω.*

(b) $V_{\omega+\omega}$ satisfies all axioms of ZFC *except certain instances of Replacement. There is an instance of Replacement which is false in $V_{\omega+\omega}$.*

(c) HC satisfies all axioms of ZFC *except Powerset. HC $\models P(\omega)$ does not exist.*

The Levy hierarchy gives us a useful classification of the relations definable over a transitive set M.

Definition 2.8 *Let M be transitive, $X \subseteq M$, and let R be a k-ary relation on M; then R is $\Sigma_n^M(X)$ iff for some Σ_n formula φ and parameters $b_1, ..., b_i \in X$,*

$$R = \{\langle a_1, ..., a_k\rangle \mid M \models \varphi[a_1, ..., a_k, b_1, ..., b_i].$$

We write Σ_n^M for $\Sigma_n^M(\emptyset)$.

Let $n \geq 1$. A relation is $\Sigma_n^{(V_\omega,\in)}$ iff it is Σ_n^0 when coded in some natural way as a relation on ω. A relation is $\Sigma_n^{(V_{\omega+1},\in)}$ iff it is Σ_n^1 when coded in some natural way as a relation on ω^ω. A relation is Σ_n^{HC} iff it is Σ_{n+1}^1 when coded in some natural way as a relation on ω^ω. These equivalences give us a handy way to make the definability calculations of descriptive set theory. (See Moschovakis .)

2.9 Large cardinals and the consistency-strength hierarchy

Set theorists are often concerned with proving T *is consistent*, where T is some theory extending ZFC. Such results cannot be proved in ZFC, by Godel's second incompleteness theorem (see Moschovakis). What one can hope to do is prove a result of the form *if S is consistent, then T is consistent*, where S is some theory one already has reason to believe consistent. When such a *relative consistency* result is provable, it is generally provable by completely finitary reasoning about proofs (e.g. in Peano Arithmetic), all of the infinitary content having been packed into the assumption that S is consistent. However, we shall not take this last step into proof theory, and content ourselves with showing $\mathrm{Con}(S) \Rightarrow \mathrm{Con}\ (T)$, using ZFC as our background theory.

If $\mathsf{ZFC} \vdash \mathrm{Con}(S) \rightarrow \mathrm{Con}(T)$, then we say T has *consistency strength* less than or equal to that of S, and write $T \leq_{\mathrm{Con}} S$. If $T \leq_{\mathrm{Con}} S$ and $S \leq_{\mathrm{Con}} T$, that is, if $\mathsf{ZFC} \vdash \mathrm{Con}(S) \leftrightarrow$

$\mathrm{Con}(T)$, then we say that S and T have the same consistency- strength, or are *equiconsistent*. It is a remarkable fact that the consistency-strengths of the many different extensions of ZFC with which set-theorists have been concerned seem to be wellordered by $\leq_{\mathrm{Con}}$.

The standard yardstick by which one calibrates consistency-strengths is the hierarchy of large cardinal hypotheses. Such hypotheses strengthen the Axiom of Infinity. The most general way to obtain them is through the following informal *reflection principle*: suitable properties of V reflect to some V_α. For example, V is infinite, and the ordinary Axiom of Infinity asserts that some V_α shares this property. For another, V is a model of second-order ZFC (also called *Kelley-Morse set theory*), so our informal principle leads the assertion that some V_κ satisfies second-order ZFC. This is equivalent to κ being inaccessible.

The stronger large cardinal hypotheses assert the existence of elementary embeddings $j\colon V \to M$, where M is a transitive class, which are nontrivial, i.e. not the identity. For such a j, the *critical point* of j, or $\mathrm{crit}(j)$, is the least κ such that $j(\kappa) \neq \kappa$. Since j is order-preserving, this implies $\kappa < j(\kappa)$. The reflection here occurs at κ: if $\varphi(\kappa)$ holds in V, and M resembles V enough that $M \models \varphi[\kappa]$, then $M \models \exists\alpha < j(\kappa)\varphi(\alpha)$, so by the elementarity of j, there is an $\alpha < \kappa$ such that $\varphi(\alpha)$ holds in V. The more M resembles V, the more reflection we have. By a result of Kunen, $M = V$ is impossible, but weaker conditions on M lead to plausible and useful principles.

For example, κ is *measurable* iff there is any transitive M and nontrivial embedding $j\colon V \to M$ with $\mathrm{crit}(j) = \kappa$. If there is such a j, M with $V_\beta \subseteq M$, we say κ is *β-strong*, and we say κ is *strong* iff κ is β-strong for all β. If there is such a j, M with M closed under λ-sequences, then we say κ is *λ-supercompact*, and we say κ is *supercompact* iff κ is λ- supercompact for all λ. We have listed these properties in order of increasing strength, indeed, if κ is strong, then $V_\kappa \models$ "there is a measurable cardinal", and if κ is supercompact, then $V_\kappa \models$ "there is a strong cardinal".

There is a more familiar formulation of measurability: κ is measurable iff $\kappa > \omega$ and there is a κ-complete ultrafilter U on κ such that no singleton is in U. Strongness and supercompactness can also be reformulated as asseertions that certain ultrafilters exists.

The consistency-strengths of the many different large cardinal hypotheses are easily compared, and wellordered. Remarkably, it seems that every theory extending ZFC which arises naturally is equiconsistent with some large cardinal hypothesis. Most consistency-strength comparisons are proved by relating the theories in question to the large cardinal hierarchy.

For example, of the theories we have mentioned so far, the following are equiconsistent with ZFC: $\mathsf{ZFC} + \mathsf{GCH}$, $\mathsf{ZFC} + \neg\mathsf{CH}$, $\mathsf{ZFC} + \mathsf{SH}$, $\mathsf{ZFC} + \neg\mathsf{SH}$, $\mathsf{ZFC} + \neg\mathsf{KH}$. The following are equiconsistent with $\mathsf{ZFC}+$ "there is an inaccessible cardinal", and hence consistency-wise strictly stronger than ZFC: $\mathsf{ZFC}+$ "all sets of reals definable over $(\mathrm{HC}, \in)$ are Lebesgue measurable", $\mathsf{ZFC} + \mathsf{KH}$. The theory $\mathsf{ZFC}+$ "there is a total extension of Lebesgue measure" is equiconsistent with $\mathsf{ZFC}+$ "there is a measurable cardinal".

In practice, if $T \leq_{\mathrm{Con}} S$, then every purely number-theoretic consequence of T follows from S. This is because our relative consistency proof will transform a model of S into a model of T having the same natural numbers. On the other hand, in practice, if $T <_{\mathrm{Con}} S$, then $S \vdash \mathrm{Con}(T)$, so that S proves a purely number-theoretic fact not proved by T. Thus as we climb the consistency-strength hierarchy, our increased uncertainty as to the consistency of the theories we reach is compensated for by an increased ability to decide sentences in the language of number theory.

3 Constructible sets

3.1 Gödel's work on L

The hierarchy of *constructible* sets is given by

$$L_0 = \emptyset,$$

$$L_{\alpha+1} = \{R \subseteq L_\alpha \mid \exists n(R \text{ is } \Sigma_n^{L_\alpha}(L_\alpha)\},$$

and for λ a limit ordinal

$$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha.$$

That is, instead of adding arbitrary subsets of what we had previously at successor stages, as is done in the full cumulative hierarchy, we add only sets definable from parameters over the universe of previously defined sets. It is easy to see that each L_α is transitive, that $\alpha \leq \beta \Rightarrow L_\alpha \subseteq L_\beta$, that $L_\alpha \in L_{\alpha+1}$, and that $\mathrm{OR} \cap L_\alpha = \alpha$. Let

$$L = \bigcup_{\alpha \in \mathrm{OR}} L_\alpha.$$

L is a transitive proper class, and its members are called the constructible sets.

Theorem 3.1 (Gödel 1937) $L \models \mathsf{ZF}$.

Proof. We focus on the Powerset, Replacement, and Choice axioms, as the others are fairly routine.

For Powerset, let $x \in L$. It will be enough to see that $P(x) \cap L \in L$, as then $P(x) \cap L = P(x)^L$. For each $y \subseteq x$ such that $y \in L$, let α_y be the least α such that $y \in L_\alpha$. Set $\alpha = \sup_y \alpha_y$, and note that α exists (i.e. is an ordinal) by Replacement. But then $P(x) \cap L = P(x) \cap L_\alpha$ is definable from the parameter x over L_α, so $P(x) \cap L \in L_{\alpha+1}$.

For Replacement, suppose $a, t \in L$ and for all $x \in a$ there is exactly one y such that $L \models \varphi[x, y, t]$. Letting $b = \{y \mid \exists x \in a \; L \models \varphi[x, y, t]\}$, we must see that $b \in L$. Now $\langle L_\alpha \mid \alpha \in \mathrm{OR}\rangle$ is a continuous substructure chain of length OR, and (by Replacement in V), OR is enough like a regular cardinal that we can apply the proof of our reflection theorem 2.3 to see that there are arbitrarily large $\alpha \in \mathrm{OR}$ such that for all $u, v, w \in L_\alpha$,

$$L_\alpha \models \varphi[u, v, w] \Leftrightarrow L \models \varphi[u, v, w].$$

(The instances of Replacement we use here are related to the Skolem functions for subformulae of φ.) As $b \subseteq L$, and b is a set, $b \subseteq L_\alpha$ for some such α with $a, t \in L_\alpha$. But then

$$b = \{y \in L_\alpha \mid L_\alpha \models \exists x \in a \ \ \varphi[x, y, t]\},$$

so $b \in L_{\alpha+1}$. □

Definition 3.2 $\mathsf{V} = \mathsf{L}$ *is the sentence of LST which naturally expresses: every set is constructible.*

Lemma 3.2.1 *(a) The relation $R(x, \alpha) \leftrightarrow x = L_\alpha$ is Σ_1-definable.*

(b) V=L *is a Π_2 sentence.*

This lemma can be proved by looking closely at the definitions we have given, and carefully analyzing their logical structure. Very roughly, $R(x, \alpha)$ holds iff $\exists f(f$ is a function with domain $\alpha + 1$ and $f(\alpha) = x$ and $\forall \beta \in \mathrm{dom}(f)$ $(f(\beta)$ is obtained properly from $f \restriction \beta)$. Here "obtained properly" is cashed out via the defintion of L_β from $\langle L_\gamma \mid \gamma < \beta\rangle$ which we gave. A full proof of the lemma would involve many details, but no further ideas worth mentioning.

Theorem 3.3 (Gödel 1937) $L \models \mathsf{V} = \mathsf{L}$.

Proof. Let $\theta(u, v)$ be the Σ_1 formula defining the relation $x = L_\alpha$ given by 3.2.1. V=L is just the sentence

$$\forall t \exists v \exists u(\theta(u, v) \wedge t \in u).$$

To see that this is true in L, fix $t \in L$. Let $t \in L_\alpha$. The construction of the L_γ required no use of Choice, so $\mathsf{ZF} \vdash \forall v(v$ is an ordinal$\rightarrow \exists u \theta(u, v)$. Since $L \models \mathsf{ZF}$, $L \models \exists u \theta[\alpha]$. Fixing u such that $L \models \theta[u, \alpha]$, we see that because θ is Σ_1, and hence upward absolute, θ is true in V of u, α, that is, $u = L_\alpha$. Thus $t \in u$, so that $L \models \exists v \exists u(\theta(u, v) \wedge t \in u)$, as desired. □

In a nutshell, L realizes that all its sets are contructible, because it has constructions for them, and can recognize those constructions for what they are. The upward absoluteness of being constructible also yields a sort of converse to this theorem:

Lemma 3.3.1 *Let M be a transitive model of* V=L*; then $M = L_\alpha$ for $\alpha = OR \cap M$.*

Theorem 3.4 (Gödel 1937) *Assume* $\mathsf{ZF} + \mathsf{V} = \mathsf{L}$*; then*

(a) the Axiom of Choice holds,

(b) GCH *holds,*

(c) the reals admit a Σ^1_2 *wellorder,*

(d) there is a Σ^1_2 *set of reals which is not Lebesgue measurable.*

Proof. For (a), one defines by induction on α a wellorder $\leq_\alpha$ of L_α, in such a way that if $\alpha < \beta$, then $\leq_\alpha$ is *end-extended* by $\leq_\beta$, in that everything in $L_\beta \setminus L_\alpha$ comes after everything in L_α. The end-extension property guarantees that we may set $\leq_\lambda = \bigcup_{\alpha<\lambda} \leq_\alpha$, and still have a wellorder. To get $\leq_{\alpha+1}$ from $\leq_\alpha$, notice that $\leq_\alpha$ induces a wellorder of $(L_\alpha)^{<\omega}$, and this wellorder together with some natural wellorder of the formulae of LST induces a wellorder of $L_{\alpha+1}$.

For (b), notice first that if $\alpha \geq \omega$, then $|L_\alpha| = |\alpha|$. (This is easy to prove by induction on α.) The following claim then immediately implies $|P(\kappa)| \leq \kappa^+$ for $\kappa \geq \omega$.

Claim. If $\kappa \geq \omega$, then $P(\kappa) \subseteq L_{\kappa^+}$.

Proof. Let $A \subseteq \kappa$. We are assuming $\mathsf{V}{=}\mathsf{L}$, so we can fix a $\theta > \kappa$ such that $A \in L_\theta$ and $L_\theta \models \mathsf{V} = \mathsf{L}$. By Lowenheim-Skolem, we have $X \prec L_\theta$ such that $\kappa \cup \{A\} \subseteq X$ and $|X| \leq \kappa$. Let $\pi\colon M \to L_\theta$ be the transitive collapse of X, as given by 2.6.1. Since κ is already transitive, $\pi \restriction \kappa$ is the identity, which gives $\pi^{-1}(A) = A$, so $A \in M$. Since π is elementary, $M \models \mathsf{V} = \mathsf{L}$, whence $M = L_\alpha$ for some α by 3.3.1. Since $|M| \leq \kappa$, $\alpha < \kappa^+$, and we are done. □

For (c), we return to the function $\alpha \mapsto \leq_\alpha$ of (a). A somewhat more careful analysis shows that $\leq_{\omega_1}$ is Σ_1^{HC}. Thus the restriction of $\leq_{\omega_1}$ to the reals is Σ_1^{HC}, or equivalently Σ^1_2.

For (d), we show that $S = \leq_{\omega_1} \cap([0,1] \times [0,1])$ is not Lebesgue measurable in the plane. If it is, then since $\{y \mid ySx\}$ is countable, and therefore of measure 0, for all x, Fubini's theorem implies S has measure 0. But for all y, $\{x \mid ySx\}$ is co-countable, so measure 1, so that S has measure 1 by Fubini. So Fubini's theorem cannot apply to S; that is, S is not Lebesgue measurable. □

We have been somewhat sloppy in our statements of theorems 3.1 and 3.3. The satisfaction relation for proper class models is not definable in LST by results of Tarski (see Moschovakis). (Defining it involves quantification over proper classes.) However, for each particular φ, we *can* express $L \models \varphi$ in LST. For let $\theta(v)$ be the natural formula of LST expressing that v is constructible, and let φ^L be φ with all its quantifiers *relativised to* L, that is, with $\forall v \ldots$ replaced by $\forall v(\theta(v) \to \ldots)$ and $\exists v \ldots$ replaced by $\exists v(\theta(v) \wedge \ldots)$. Clearly, φ^L has the same meaning as $L \models \varphi$. Our proofs of 3.1 and 3.3 show that for each axiom φ of $\mathsf{ZF} + \mathsf{V} = \mathsf{L}$, φ^L holds. Moreover, our proofs only used that ZF holds in V, so we have shown that $\mathsf{ZF} \vdash \varphi^L$ for each axiom φ of $\mathsf{ZF} + \mathsf{V} = \mathsf{L}$. Thus any proof of $0 = 1$ in $\mathsf{ZF} + \mathsf{V} = \mathsf{L}$ can be transformed into a proof of $0 = 1$ in ZF, and we have

Theorem 3.5 (Gödel 1937) *If* ZF *is consistent, then so is* $\mathsf{ZF} + \mathsf{V} = \mathsf{L}$.

It follows, of course, that all the consequences of $\mathsf{ZF} + \mathsf{V} = \mathsf{L}$ are consistent as well.

Finally, let us note that L has a priveleged position among transitive models of ZF of a given ordinal height: it is the minimal one.

Theorem 3.6 (Gödel 1937) *Let M be a transitive model of* ZF*, and $\alpha = OR \cap M$; then $L^M = L_\alpha$, and thus $L_\alpha \subseteq M$.*

Here if M is a proper class, so that $\alpha = \mathrm{OR}$, we set $L_\alpha = L$. The proof of 3.6 is implicit in the absoluteness argument for 3.3.

3.2 Suslin trees, $\diamondsuit$, and $\Box$

It wasn't until the late 60's that Gödel's work on L was taken to a significantly deeper level. It was then that Ronald Jensen realized that the *condensation* phenomenon behind Gödel's proof that $L \models \mathsf{GCH}$, the fact that elementary submodels of an L_α collapse to some L_β with $\beta \leq \alpha$, can be exploited in a systematic way. The result is a detailed *fine structure theory* of L which seems capable of answering all questions concerning the general set theory of L.

Jensen's first result along these lines was that $L \models \neg\mathsf{SH}$. Slightly earlier, Jech had shown $\mathsf{ZFC} + \neg\mathsf{SH}$ by Cohen's forcing method, so there was a subtle but important change in focus in Jensen's result. It shifted emphasis from L as a tool for obtaining relative consistency results to L as an object of study in its own right.

We shall give the briefest sketch of Jensen's construction of a Suslin tree in L. The key is a combinatorial principle called *diamond*.

Definition 3.7 $\diamondsuit$ *is the assertion: there is a sequence $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ such that $S_\alpha \subseteq \alpha$ for all α, and for any $A \subseteq \omega_1$, there are stationarily many α such that $A \cap \alpha = S_\alpha$,*

A diamond sequence $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ can be used to anticipate arbitrary $A \subseteq \omega_1$ while doing a construction of length ω_1: at stage α, one treats S_α as if it were $A \cap \alpha$. For example, to construct a Suslin tree $\mathbb{T}$ with universe ω_1, one constructs the levels of the tree by induction; at limit steps α, if S_α is a maximal antichain or branch of the tree built so far, one adds an α-th level in such a way that S_α cannot be properly extended to a branch or antichain in $\mathbb{T}$. Suppose now that A is a maximal antichain in the final $\mathbb{T}$. By our reflection theorem 2.3, there are club many α such that $A \cap \alpha$ is a maximal antichain in $\mathbb{T} \restriction \alpha$. For some such α, $A \cap \alpha = S_\alpha$, and then our construction insures that $A \cap \alpha$ cannot be properly extended. That is, $A = A \cap \alpha$, so that A is countable. Thus $\diamondsuit$ implies there are Suslin trees.

To construct a diamond sequence in L, one defines S_α by induction on α. If α is a succcessor, set $S_\alpha = \emptyset$. If α is a limit, let (A, C) be the least (in the canonical wellorder of L) pair such that $A \subseteq \alpha$, C is club in α, and $\forall \beta \in C(A \cap \beta \neq S_\beta)$. We then set $S_\alpha = A$. Having "killed" at each stage the L-least potential reflection of a counterexample to the prediction property we wish our sequence to have, we can argue using the condensation phenomenon that in the end, there are no such counterexamples.

It is possible to turn these sketches into real mathematics, yielding

Theorem 3.8 (Jensen) *(a) $\diamondsuit \Rightarrow$ there is a Suslin tree.*

(b) $L \models \diamondsuit$.

This result opened the way to a wealth of further results in the general set theory of L. For example, Solovay soon showed that there are Kurepa trees in L. Many of these results require a closer, level-by-level analysis of the way new sets appear in L. One basic result here is a sharp, local version of Godel's theorem that $P(\kappa)^L \subseteq L_{\kappa^+}$:

Theorem 3.9 (Jensen 1971) *Let $\kappa \geq \omega$, and suppose there is a set $A \subseteq \kappa$ such that A is $\Sigma_n^{L_\alpha}(L_\alpha)$, where $n \geq 1$, but $A \notin L_\alpha$. Then there is a function f from a subset of κ onto L_α whose graph is $\Sigma_n^{L_\alpha}(L_\alpha)$.*

That is, as soon as a new subset of κ appears in L, the stage at which it appears is seen to have cardinality at most κ. One combinatorial principle that seems to require such a fine analysis is Jensen's $\Box$.

Definition 3.10 *Let κ be a cardinal; then $\Box_\kappa$ is the assertion: there is a sequence $\langle C_\lambda \mid \lambda < \kappa^+ \wedge \lambda$ a limit ordinal $\rangle$ such that for all limit $\lambda < \kappa^+$*

(a) C_λ is club in λ,

(b) C_λ has order-type $\leq \kappa$, and

(c) if ν is a limit point of C_λ, then $C_\nu = C_\lambda \cap \nu$.

Theorem 3.11 (Jensen 1971) $L \models \forall \kappa\ \Box_\kappa$.

Very roughly, the set C_λ is obtained by going to the first level in L at which it is seen that $|\lambda| \leq \kappa$, and extracting a canonical cofinal subset of λ with order-type $\leq \kappa$. The coherence property (c) in definition 3.10 comes from a fine analysis of the condensation properties of the L_α's. Turning this basic idea into a proof involves many difficulties.

$\Box$ is quite powerful. For example, using it, Jensen was able to construct in L the analog of a Suslin tree at every successor cardinal.

3.3 Canonical inner models larger than L

Are there any nonconstructible sets? One answer is provided by

Theorem 3.12 (Gaifman,Rowbottom 1964) *If there is a measurable cardinal, then $P(\omega)\cap L$ is countable.*

In fact, the existence of measurable cardinals implies that true ω_1 is an inaccessible cardinal in L. Building on 3.12, Solovay showed that the existence of measurable cardinals implies that all Σ^1_2 sets of reals are Lebesgue measurable, in contrast to the "less desirable" consequence of $\mathsf{V}{=}\mathsf{L}$ recorded in 3.4 above.

Granting that there are measurable cardinals, one can still ask whether there is any *definable*, or *canonical*, structure beyond L. In fact, there is. If there is a measurable cardinal, then there is a proper class model $L[U]$ which satisfies "there is a measurable cardinal" which is in a certain sense minimal with this property. One obtains from an ultrafilter U witnessing that κ is measurable by constructing relative to U, that is, by putting into $L_{\alpha+1}[U]$ those $X \subseteq L_\alpha[U]$ which are definable from parameters over $(L_\alpha[U], \in, U \cap L_\alpha[U])$. One can show that the model obtained depends only on κ, and not the particular U witnessing κ is measurable. The basic theory of this model resembles the theory of L.

Theorem 3.13 (Silver 1966) *If there is a measurable cardinal, then there is a proper class inner model $L[U]$ such that*

(a) $L[U] \models$ *there is a measurable cardinal,*

(b) $L[U] \models \mathsf{GCH}$*, and*

(c) $L[U] \models$ *"$\mathbb{R}$ admits a Σ^1_3 wellorder".*

Solovay showed that Jensen's fine-structural analysis of L applies to $L[U]$ as well, so that for example, $L[U] \models \forall\kappa\ \square_\kappa$. $L[U]$ cannot be unique or minimal in exactly the same sense as L is, since by measurability there must be some $L[j(U)]$ with all its first order properties. But we have

Theorem 3.14 (Kunen 1968) *Let M and N be transitive proper class models satisfying the sentence $\mathsf{V} = \mathsf{L[U]}$; then either there is an elementary embedding $j\colon M \to N$ or there is an elementary embedding $j\colon N \to M$.*

In particular, all proper class models of $\mathsf{V}{=}\mathsf{L[U]}$ have the same first-order theory, and the same reals. More is true: they are all generated from a single such model by a process called taking *iterated ultrapowers.*

The reals in (any) $L[U]$ are canonical. The first-order theory of the first level $L_\alpha[U]$ over which a nonconstructible real is definable is called $0^\sharp$. Silver showed that $0^\sharp$ is the unique theory T in LST satisfying a certain Π_1 formula in $(\mathrm{HC}, \in)$; this formula asserts that T contains certain first-order sentences characteristic of $0^\sharp$, and has a property called *iterability*. By $0^\sharp$ *exists*, we mean the assertion that there is a theory T satisfying this formula. Kunen showed that $0^\sharp$ exists iff there is a nontrivial elementary $j\colon L \to L$. One can think of $0^\sharp$ *exists* as the weakest large cardinal hypothesis stronger than those which are true in L. This brings us to one of the most beautiful theorems of set theory:

Theorem 3.15 (Covering Lemma, Jensen 1975) *Exactly one of the following is true*

(1) $0^\sharp$ *exists,*

(2) For every uncountable set $X \subseteq L$*, there is a set* $Y \in L$ *such that* $X \subseteq Y$ *and* $|Y| = |X|$*.*

In other words, either V is close to L in the sense of (2), or there is canonical large-cardinal structure beyond L. This theorem has many applications.

The construction and study of canonical inner models for large-cardinal hypotheses stronger than the existence of a measurable cardinal is central to our understanding of higher consistency-strengths. It is an ongoing effort. At the moment we have had success with hypotheses somewhat beyond the existence of strong cardinals. How to extend inner model theory to the level of supercompact cardinals is one of the main open problems in set theory.

4 Forcing

After Gödel's 1937 work, the most important problem for set theory was the construction of models of $\mathsf{ZFC} + \neg\mathsf{CH}$. (Most set-theorists believed that there were such models.) This problem was solved in 1963 by Paul Cohen. Cohen's method, which he called *forcing*, is very powerful. As a method for obtaining relative consistency results along the lines of 3.5, it is much more powerful and flexible that Godel's method of inner models. However, perhaps because of the great variety of forcing models, no one of them stands out the way L and larger canonical inner models constructed along the same lines do. The exception might be models which are constructed by doing "as much forcing as possible" in some way, that is, models of *Martin's Axiom* and its generalizations. We shall discuss these at the end of this section.

4.1 The basics of forcing

It is beyond the scope of this article to give an account of the basics of forcing as thorough as that which we have given of constructibility. We shall simply state some of the main definitions and theorems. The main ideas in this section are due to Cohen, but Solovay and several other people made important contributions in bringing them to their full, most general form.

As we discussed in §1, we need to assume at least Con(ZFC) if we seek to prove Con($\mathsf{ZFC}+\neg\mathsf{CH}$). For expository reasons, we shall make a slightly stronger assumption, namely, that ZFC has a transitive set model, and then construct a transitive set model of $\mathsf{ZFC}+\mathsf{GCH}$. A relatively minor elaboration of the argument, which we shall not describe, gives Con(ZFC) $\Rightarrow$ Con($\mathsf{ZFC}+\neg\mathsf{CH}$). Notice that by the Lowenheim-Skolem theorem (see Kiesler) and 2.6.1, if there is a transitive model of ZFC, then there is a countable transitive model of ZFC.

Definition 4.1 *A* ctm *is a countable transitive M such that $M \models \mathsf{ZFC}$.*

In contrast to Godel's method, which starts with a transitive M and produces an *inner model* $L^M \subseteq M$, Cohen's method starts with a ctm M and produces an *outer model* $N \supseteq M$. This is inevitable, as we might be starting with $M = L_\alpha$ for α least such that $L_\alpha \models \mathsf{ZFC}$, and in that case, M has no proper transitive inner models. In fact, N will be generated by adding a single set G to M. It will be important that we have approximations to G, also called *conditions* on G, in M. These conditions have the structure of a partially ordered set in M, with $p \leq q$ meaning intuitively that p gives at least as much information about G as does q. Our G will then be an *M-generic filter* on this partial order, a concept which we now define precisely.

Definition 4.2 *Let $\mathbb{P} = (P, \leq)$ be a partial order.*

(a) G is a filter on $\mathbb{P}$ *iff $\forall p \in G \forall q \geq p(q \in G)$, and $\forall p \in G \forall q \in G \exists r \in G(r \leq p \wedge r \leq q)$.*

(b) $D \subseteq P$ is dense in $\mathbb{P}$ *iff $\forall p \in P \exists q \in D(q \leq p)$.*

(c) G is F-generic *iff G is a filter on $\mathbb{P}$ such that for all dense $D \in F$, $G \cap D \neq \emptyset$.*

The idea is that generic filter contain no special information, because by (c), anything that always can happen eventually does happen.

We are considering countable M because in that case, we can prove M-generic filters exist.

Lemma 4.2.1 (Existence of generics) *Let $\mathbb{P}$ be a partial order, and F be countable; then for any $p \in P$, there is an F-generic filter G on $\mathbb{P}$ such that $p \in G$.*

Proof. Let $\langle D_n \mid n < \omega \rangle$ enumerate the dense sets in F. Define $p_0 = p$, and

$$p_{n+1} = \text{ some } q \leq p_n \text{ such that } q \in D_n.$$

(Since D_n is dense, there is such a q.) It is easy to see that $G = \{r \mid \exists n(p_n \leq r)\}$ is an F-generic filter. $\square$

As N will be generated from the elements of M together with G, there will be names in M for the elements of N. This is important, because we must use that $M \models \mathsf{ZFC}$ to show that $N \models \mathsf{ZFC}$, and the natural way to do this is to somehow reduce truth in N to truth in M. There are many naming systems possible, but the simplest uses all elements of M as names. The interpretation τ^G of the name τ relative to G is defined by $\in$-induction on τ.

Definition 4.3 *Let M be transitive, $\mathbb{P}$ a partial order in M, and let G be M-generic on $\mathbb{P}$; then*

(a) for any $\tau \in M$, $\tau^G = \{\sigma^G \mid \exists p(p \in G \wedge \langle p, \sigma\rangle \in \tau)\}$;

(b) $M[G] = \{\tau^G \mid \tau \in M\}$.

Any $x \in M$ has the name

$$\check{x} = P \times \{\check{y} \mid y \in x\}.$$

It is easy to see that $\check{x}^G = x$ for all G. We also have a canonical name for G,

$$\dot{G} = \{\langle p, \check{p}\rangle \mid p \in P\}.$$

It is easy to see that $\dot{G}^H = H$ for all M-generic H on $\mathbb{P}$.

One of the main results is

Theorem 4.4 (Generic model theorem) *Let M be transitive, $\mathbb{P}$ a partial order in M, and let G be M-generic on $\mathbb{P}$; then*

(1) $M[G] \models \mathsf{ZFC}$,

(2) $M \cup \{G\} \subseteq M[G]$,

(3) $OR \cap M[G] = OR \cap M$, and

(4) if $N \models \mathsf{ZF}$ and $M \cup \{G\} \subseteq N$, then $M[G] \subseteq N$.

The heart of this theorem is that $M[G] \models \mathsf{ZFC}$. The rest is elementary, and we leave it to the reader. We shall prove $M[G] \models \mathsf{ZFC}$ under the assumption that M is countable; a simple Lowenheim-Skolem argument lets one reduce to this case.

The main idea is to reduce truth in $M[G]$ to truth in M through what is called the *forcing relation.*

Definition 4.5 *Let M be a ctm, $\mathbb{P}$ a partial order in M, $\varphi(v_1, ..., v_n)$ a formula of LST, $\tau_1, ..., \tau_n \in M$, and $p \in P$; then*

$$p \Vdash \varphi(\tau_1, ..., \tau_n) \text{ iff } \forall G(G \text{ is } M\text{-generic and } p \in G \Rightarrow M[G] \models \varphi[\tau_1^G, ..., \tau_n^G].)$$

The symbol $\Vdash$ is read "forces".

The main lemma about the forcing relation is

Lemma 4.5.1 *Let M be a ctm, $\mathbb{P}$ a partial order in M, and $\varphi(v_1, ..., v_n)$ a formula of LST; then*

(a) (Definability of Forcing) Letting

$$F_\varphi(p, \tau_1, ..., \tau_n) \text{ iff } p \Vdash \varphi(\tau_1, ..., \tau_n),$$

the relation F_φ is definable over M (from the parameter $\mathbb{P}$), and

(b) (Forcing = truth) letting G be M-generic on $\mathbb{P}$,

$$M[G] \models \varphi[\tau_1^G, ..., \tau_n^G] \text{ iff } \exists p \in G(p \Vdash \varphi(\tau_1, ..., \tau_n)).$$

Note that one direction of (b) is trivial, but the other definitely has content. Parts (a) and (b) of Lemma 4.5.1 are proved simultaneously by induction on the logical complexity of φ. In the case that φ is atomic, there is a subinduction on the $\in$-rank of the terms τ_i involved. The argument is complicated, but not difficult. Here are some equivalences which are useful in defining forcing for compound formulae:

- $p \Vdash \psi \wedge \theta$ iff $p \Vdash \psi$ and $p \Vdash \theta$,
- $p \Vdash \forall v\psi$ iff $\forall \tau \in M(p \Vdash \psi(\tau))$,
- $p \Vdash \neg\psi$ iff $\forall q \leq p(q \nVdash \psi)$,
- $p \Vdash \psi \vee \theta$ iff $\forall q \leq p \exists r \leq q(r \Vdash \psi$ or $r \Vdash \theta)$.

The reader might try to produce an equivalent for $p \Vdash \exists v\psi$.

It is now reasonably straightforward to prove $M[G] \models \mathsf{ZFC}$. We illustrate by proving an instance of the Axiom of Separation. Let $a, t \in M[G]$, let $\varphi(v_1, v_2)$ be a formula of LST, and let $b = \{u \in a \mid M[G] \models \varphi[u, t]\}$. We must see that $b \in M[G]$, and for this, we need a name for b in M. Let $a = \sigma^G$ and $t = \tau^G$, and set

$$\rho = \{\langle p, \dot{u}\rangle \in P \times \mathrm{TC}(\sigma) \mid p \Vdash \varphi(\dot{u}, \tau)\}.$$

Note that $\rho \in M$, because M satisfies Separation. Lemma 4.5.1 easily implies that $b = \rho^G$, so that $b \in M[G]$, as desired. Similarly, the other axioms of ZFC hold in $M[G]$ because they hold in M, because what is true is forced, and because forcing is definable, formula-by-formula, over M.

Theorem 4.4 is quite powerful, because of the wide variety of partial orders to which it may be applied. We now look at some examples.

4.2 $\neg\mathsf{CH}$ via adding Cohen reals

Let M be a ctm. The simplest way to generically extend M is by adding a new subset of ω, and the simplest way to approximate this new set within M is via finite sets of atomic membership conditions on the set.

Definition 4.6 *For any sets I, J and cardinal $\kappa \geq \omega$, $\mathrm{Fn}_\kappa(I, J) = \{p \mid p \text{ is a function } \wedge \mathrm{dom}(p) \subseteq I \wedge \mathrm{ran}(p) \subseteq J \wedge |p| < \kappa\}$. We regard $\mathrm{Fn}_\kappa(I, J)$ as a partial order, with the order $p \leq q$ iff $p \supseteq q$.*

The simplest nontrivial partial order, from the point of view of forcing, is $\mathrm{Fn}_\omega(\omega, 2)$. Let G be M-generic over $\mathrm{Fn}_\omega(\omega, 2)$, and let $A = \{n \mid \bigcup G(n) = 0\}$. It is easy to see that $A \notin M$. (For $B \subseteq \omega$ such that $B \in M$, let $D = \{p \in P \mid \exists n \in \mathrm{dom}(p)(p(n) = 1 \leftrightarrow n \in B\}$. Then D is dense, and $D \in M$. Thus $G \cap D \neq \emptyset$, which implies $A \neq B$.) But now $M[G]$ has the same ordinals as M, so $L^{M[G]} = L^M \subseteq M$. This means $M[G] \models A$ is not constructible, and we have shown that ZFC is consistent with the existence of a non-constructible subset of ω.

In fact, we can add λ distinct subsets of ω to M, by forcing with $\mathrm{Fn}_\omega(\omega \times \lambda, 2)$. (The α-th set added by G is $\{n \mid \bigcup G(n, \alpha) = 0\}$.) With some further analysis, one obtains

Theorem 4.7 (Cohen, Solovay) *Let M be a ctm, and suppose κ is a cardinal of M such that $M \models \kappa^\omega = \kappa$; then there is an M-generic G such that*

(a) for all η, $M[G] \models \eta$ is a cardinal iff $M \models \eta$ is a cardinal, and $\mathrm{cof}(\eta)^{M[G]} = \mathrm{cof}(\eta)^M$, and

(b) $M[G] \models 2^\omega = \kappa$.

Part (a) is sumarized by saying that M and $M[G]$ *have the same cardinals and cofinalities.* Clearly, if we want $M[G]$ to satisfy $\neg\mathsf{CH}$, it is not enough just to add ω_2^M distinct subsets of ω to M; one must also arrange that $\omega_2^M = \omega_2^{M[G]}$.

By Theorem 4.7, it is consistent with ZFC that $2^\omega = \omega_{17}$, or $2^\omega = \aleph_{\omega_1}$, or 2^ω is the least weakly inaccessible cardinal. In fact, anything not ruled out by König's theorem is possible.

4.3 Easton's theorem

Suppose M is a ctm, and $M \models \kappa$ is a regular cardinal. One can add λ new subsets of κ to M by forcing with $\mathrm{Fn}_\kappa(\kappa \times \lambda, 2)^M$. This forcing does not add any new bounded subsets of κ to M. By forcing with an intelligently constructed product of such partial orders, one can prove

Theorem 4.8 (Easton 1964) *Let M be a ctm, and suppose $M \models \mathsf{GCH}$. Let F be an M-definable function such that $\mathrm{dom}(F) = \{\kappa \mid M \models \kappa \text{ is regular }\}$, and suppose that*

(1) $\kappa \leq \mu \Rightarrow F(\kappa) \leq F(\mu)$*,*

(2) $\kappa < \mathrm{cof}(F(\kappa))$*;*

then there is a generic extension $M[G]$ *of* M *such that*

(a) $M[G]$ *had the same cardinals and cofinalities as* M*, and*

(b) for all $\kappa \in \mathrm{dom}(F)$*,* $M[G] \models 2^\kappa = F(\kappa)$*.*

Since F is allowed to be a proper class from the point of view of M, we must force over M with an M-definable partial order which is a proper class from the point of view of M. In general, the basic theorems of forcing do not generalize to forcing with proper classes of conditions, but they do apply to the conditions used to prove Theorem 4.8.

Theorem 4.8 is a fairly definitive statement of the possibilities for the function $\kappa \mapsto 2^\kappa$ on regular κ. Roughly, anything not ruled out by König's theorem is consistent with ZFC.

4.4 The singular cardinals problem

The possibilities for 2^κ when κ is *singular* are much more difficult to describe. There are deep limitations beyond König's theorem, and large cardinals enter the picture in an essential way. Here are just a few of the many results in this area.

The first indication of the limitations beyond those imposed by König's theorem was the following result, which came as quite a shock to set-theorists grown accustomed to the power of forcing.

Theorem 4.9 (Silver 1974) *Let* κ *be a singular cardinal of uncountable cofinality, and suppose* $2^\alpha = \alpha^+$ *for all* $\alpha < \kappa$*; then* $2^\kappa = \kappa^+$*.*

In the late 80's, Shelah developed a powerful analysis of reduced products at singular cardinals known as *pcf theory*, and used it to prove

Theorem 4.10 (Shelah) *If* $2^{\aleph_n} = \aleph_{n+1}$ *for all* $n < \omega$*, then* $2^{\aleph_\omega} < \aleph_{\omega_4}$*.*

It is possible that GCH first fails at $\aleph_\omega$, however, the assertion that this is true has large cardinal strength. The following result builds on earlier work of many people.

Theorem 4.11 (Gitik 1989) *The following theories are equiconsistent:*

(a) $\mathsf{ZFC} + \forall n < \omega(2^{\aleph_n} = \aleph_{n+1}) + 2^{\aleph_\omega} = \aleph_{\omega+2}$*,*

(b) $\mathsf{ZFC} + \exists \kappa(o(\kappa) = \kappa^{++})$*.*

Here $o(\kappa) = \kappa^{++}$ is a certain large cardinal hypothesis intermediate in strength between the existence of measurables and the existence of strong cardinals. The proof of the theorem involves some of the deeepest results in set theory. In one direction, one starts with a model of (b), and by forcing produces a model of (a). In the other direction, one starts with a model of (a), and produces a canonical inner model of (b). An extension of the Covering lemma (3.15) to inner models larger than L plays a key role here.

4.5 A model where the Axiom of Choice fails

$M[G]$ always satisfies the Axiom of Choice, so in order to get a model of ZF where Choice fails,, one must pass to an inner model of $M[G]$. One possibility is $L(\mathbb{R})$, the constructible closure of the reals. The following is a variant of a theorem of Cohen.

Theorem 4.12 *Let M be a ctm, and let G be M-generic over $\mathrm{Fn}_\omega(\omega\times\omega_1)^M$; then $L(\mathbb{R})^{M[G]} \models \mathsf{ZF} + \mathbb{R}$ has no wellorder .*

The key to this and many results concerning the definable sets of some generic extension is the *homogeneity* of the partial order one forces with.

Definition 4.13 $\mathbb{P}$ *is* homogeneous *iff* $\forall p, q \in P \exists \pi (\pi \colon \mathbb{P} \cong \mathbb{P} \wedge \pi(p)$ *is compatible with* $q)$.

Here two conditions p, q are *compatible* if they have a common lower bound, i.e. $\exists s (s \leq_P p \wedge s \leq_P q)$. If $M \models \mathbb{P}$ is homogeneous, then $\mathbb{P}$ is symmetric enough that for any $p, q \in P$, any formula $\varphi(v)$ of LST, and any $x \in M$,

$$p \Vdash \varphi(\check{x}) \text{ iff } q \Vdash \varphi(\check{x}).$$

It follows that the first order theory of $M[G]$, with parameters from M, is independent G. It also follows that if $x \in M[G]$ and $x \subseteq M$, and x is definable in $M[G]$ from parameters in M, then $x \in M$, as x can be defined inside M using the forcing relation.

All of the partial orders we have met so far are homogeneous.

4.6 Cardinal collapsing and Solovay's model

While preservation of cardinals is important in many arguments, such as those above, it is also important to be able to destroy them. The most commonly used partial orders for doing this are due to Levy; for $\omega \leq \kappa < \lambda$, we set

$$\mathrm{Col}(\kappa, \lambda) = \mathrm{Fn}_\kappa(\kappa, \lambda).$$

Forcing over M with $\mathrm{Col}(\kappa, \lambda)$ adds a function from κ onto λ, so that in $M[G]$, $|\lambda| \leq \kappa$. In this way, for example, one can show that it is consistent with ZFC that there are only countably many constructible subsets of ω. (Force over a ctm M with $\mathrm{Col}(\omega, \omega_1)^M$. Since $M \models |P(\omega \cap L| \leq \omega_1$, and $L^M = L^{M[G]}$, $P(\omega) \cap L$ is countable in $M[G]$.)

A very important model arises if we collapse all cardinals strictly less that some strongly inaccessible cardinal λ to be countable. The partial order $\mathrm{Col}(\omega, < \lambda)$ for doing this is the finite-support product of the $\mathrm{Col}(\omega, \alpha)$ for $\alpha < \lambda$. That is, conditions are $p \in \mathrm{Fn}_\omega(\omega \times \lambda, \lambda)$ such that for all $n, \alpha \in \mathrm{dom}(p)$, $p(n, \alpha) < \alpha$, and the order on conditions is reverse inclusion.

Theorem 4.14 *Let M be a ctm, and $M \models \lambda$ is strongly inaccessible . Let G be M-generic over* $\mathrm{Col}(\omega, < \lambda)$*; then*

(1) (Levy) $\lambda = \omega_1^{M[G]}$,

(2) (Solovay) $M[G] \models$ *every set of reals in* $L(\mathbb{R})$ *is Lebesgue measurable.*

By part (1), it is consistent that true ω_1 is an inaccessible cardinal in L. Part (2) easily implies that $L(\mathbb{R})^{M[G]}$ is a model of ZF in which all sets of reals are Lebesgue measurable. Although the full Axiom of Choice must therefore fail in $L(\mathbb{R})^{M[G]}$, a form of Choice useful in analysis, *countable dependent choice*, does hold.

A model $M[G]$ obtained as in Theorem 4.14 is called a *Solovay model*, because part (2) is much more difficult to prove than part (1). The homogeneity of $\mathrm{Col}(\omega, < \lambda)$ and an analysis of the way it can be factored play key roles. So too does *random real forcing*, which was discovered by Solovay in this context, and now has many important applications.

Theorem 4.14 is the first example we have seen in which we needed a large cardinal hypothesis in our ground model M. It is necessary:

Theorem 4.15 (Shelah 1984) *If all projective sets of reals are Lebesgue measurable, then true ω_1 is an inaccessible cardinal in L.*

Putting Solovay's and Shelah's theorems together, and adding the metamathematical refinements which enable one to work with arbitrary models, rather than merely transitive ones, one has: ZFC + " all sets of reals in $L(\mathbb{R})$ are Lebesgue measurable" is equiconsistent with ZFC + "there is an inaccessible cardinal". This is one example of the remarkable phenomenon that we alluded to earlier, that the large cardinal hierarchy seems to capture and calibrate all natural consistency strengths. The way we computed the consistency-strength of ZFC+ "all sets of reals in $L(\mathbb{R})$ are Lebesgue measurable" is quite typical. The upper bound is obtained by forcing over a model satisfying the large cardinal hypothesis in question, while the lower bound is obtained by constructing a canonical inner model (in this case L) of the large cardinal hypothesis in question.

4.7 Suslin's Hypothesis and Martin's Axiom

There are many forcing models, some more interesting and useful than others. Among them all, it is natural to look at those which are as saturated with generic objects as possible. *Martin's Axiom* is one way of saying that V is such a universe.

An *antichain* in a partial order is a family of pairwise incompatible elements. We say $\mathbb{P}$ has the *countable chain condition*, or is *ccc*, iff every antichain in $\mathbb{P}$ is countable. Forcing with a ccc partial order preserves all cardinals and cofinalities, and can generally be considered "mild".

Definition 4.16 Martin's Axiom, *or* MA, *is the assertion that whenever* $\mathbb{P}$ *is ccc and* $|F| < 2^\omega$, *then there is an* F*-generic filter on* $\mathbb{P}$.

By 4.2.1, CH implies MA, so we are really interested in models of $\mathsf{MA}+\neg\mathsf{CH}$. The restriction on $|F|$ is necessary, as even in the simplest of all $\mathbb{P}$, namely $\mathrm{Fn}_\omega(\omega, 2)$, the dense sets guaranteeing the generic filter is not in the ground model are 2^ω in number. Some structural restriction on $\mathbb{P}$ ruling out $\mathbb{P} = \mathrm{Col}(\omega, \omega_1)$ is necessary as well, as letting $F = \{D_\alpha \mid \alpha < \omega_1\}$, where $D_\alpha = \{p \in \mathrm{Col}(\omega, \omega_1) \mid \alpha \in \mathrm{ran}(p)\}$, any F-generic filter yields a counting of ω_1. The ccc is one structural restriction which suffices.

Theorem 4.17 (Solovay, Tennenbaum 1971) *Let* M *be a ctm, and suppose* $M \models \mathsf{GCH}$. *Let* κ *be an uncountable regular cardinal of* M*; then there is an* M*-generic* G *such that*

(a) M *and* $M[G]$ *have the same cardinals and cofinalities, and*

(b) $M[G] \models \mathsf{MA} + 2^\omega = \kappa$.

Solovay and Tennenbaum developed the method of *iterated forcing* in order to prove this result. The idea is just that G should be determined by a sequence $\langle G_\alpha \mid \alpha < \kappa\rangle$, where each G_α is generic for some ccc partial order over $M[\langle G_\beta \mid \beta < \alpha\rangle]$. By using all partial orders of size $< \kappa$ which show up in any intermediate model in a systematic way, one gets the desired conclusion. Cardinals and cofinalities are preserved because the partial order representing the iteration as a whole is ccc.

Shelah and others have developed the method of iterated forcing into a very powerful tool in independence results.

MA has many consequences, which is why Martin isolated it. Among them are that 2^ω is regular, that $2^\alpha = 2^\omega$ for al $\alpha < 2^\omega$, and that there is a scale. The original application that Solovay and Tennenbaum had in mind is given by

Theorem 4.18 *Assume* MA *and* $\neg\mathsf{CH}$*; then Suslin's hypothesis holds.*

The proof is easy. Suppose $(T, \leq_T)$ were a Suslin tree. Let $\mathbb{P} = (T, \geq_T)$; that is, conditions are attempts to build a branch of T, with stronger conditions being higher on the branch being built. $\mathbb{P}$ is ccc because antichains in $\mathbb{P}$ are antichains in $(T, \leq_T)$. We can assume that T has been pruned in such a way that the set of nodes of level $\geq \alpha$ is dense in $\mathbb{P}$, for each $\alpha < \omega_1$. A filter that meets these ω_1-many dense sets will yield an uncountable branch of T.

4.8 Martin's Maximum

Martin's Axiom is a *forcing axiom*, that is, an axiom asserting that the universe is closed under adding generic filters of some kind. If we stick to axioms asserting that there are filters meeting any collection of ω_1-many dense sets, but try to loosen the requirement of MA that $\mathbb{P}$ be ccc, then there is a maximal closure one can demand.

Definition 4.19 *(a) A partial order* $\mathbb{P}$ *is* stationary-preserving *iff whenever S is stationary in ω_1 and $p \in P$, then $p \Vdash \check{S}$ is stationary.*

(b) Martin's Maximum*, or* MM*, is the assertion: whenever* $\mathbb{P}$ *is stationary-preserving and* $|F| \leq \omega_1$*, then there is an F-generic filter on* $\mathbb{P}$*.*

The restriction to stationary-preserving partial orders in MM cannot be weakened. MM was discovered by Foreman, Magidor, and Shelah, who showed

Theorem 4.20 (Foreman, Magidor, Shelah 1984) *Let M be a ctm, and suppose $M \models$ there is a supercompact cardinal; then there is an M-generic G such that $M[G] \models$* MM*.*

The proof of this theorem relies heavily on Shelah's earlier contributions to the method of iterated forcing.

MM is quite powerful, as illustrated by the fact that we seem to need a model with a supercompact in order to get a model of MM. Whether such a model is truly necessary is open; it is tied up with the big problem of constructing canonical inner models with supercompacts. It *is* known at present that MM implies the existence of canonical inner models satisfying large cardinal hypotheses significantly stronger that the existence of strong cardinals.

One of the consequences of MM which does not follow from MA is that $2^{\aleph_0} = \aleph_2$.

5 Descriptive set theory

5.1 Godel's program

Given that ZFC fails to decide some statement φ, should we conclude that it is impossible to decide φ by rational means? Godel's second incompleteness theorem shows that the answer is *no*, for ZFC does not decide Con(ZFC), while it is irrational to accept ZFC while rejecting Con(ZFC). This leads us to

Godel's program: *Decide mathematically interesting questions which are independent of* ZFC *in well-justified extensions of* ZFC.

Unfortunately, merely adding Con(ZFC) as an axiom will not decide any of the questions we have discussed in this article. But we can consider Con(ZFC) as the first step up the hierarchy of reflection principles, the large cardinal hierarchy. As Godel suggested, more powerful principles of this sort might be fruitful. Although we now know that nothing like our current large cardinal hypotheses will decide the statement Godel had foremost in mind, the CH, these hypotheses have had great success with more concrete statements, such as those concerning the Lebesgue measurability of definable sets of reals. These more concrete statements belong to *descriptive set theory*.

To what extent are our large cardinal hypotheses well-justified? They do have a certain intrinsic plausibility, especially the weaker ones, as maximality principles on the universe of sets. They fit together into a finely-graded, wellordered hierarchy, so that it is hard to say "this is the rational stopping point". Most importantly, they are part of a powerful and coherent theory which seems suitable as part of a most-general framework for mathematics. As this theory grows, the evidence for our current large cardinal hypotheses only seems to get stronger. It is not possible to say more than that here, but the philosophically inclined reader is encouraged to pursue this matter further, for it involves basic questions as to how we come to know mathematical truths.

5.2 Classical descriptive set theory

Descriptive set theory originated around 1900 with the work of analysts like Borel, Baire, and Lebesgue who were interested in the foundations of their subject. Its basic thesis is that *definable* sets of reals are free of the pathologies that arbitrary sets may exhibit.

Recall that a relation on reals is $\mathbf{\Sigma^1_n}$ if it is definable from parameters over $(V_{\omega+1}, \in)$ by a Σ_n formula. A set of reals is *projective* iff it is $\mathbf{\Sigma^1_n}$ for some n. Between 1900 and 1935, descriptive set theorists proved many theorems which bore out their thesis for sets which are low in the projective hierarchy, for example $\mathbf{\Sigma^1_1}$ or Σ^1_2. For example

Theorem 5.1 *(1) [Lusin 1917] Every* $\mathbf{\Sigma^1_1}$ *set of reals is Lebesgue measurable.*

(2) [Novikov 1935] Every $\mathbf{\Sigma^1_2}$ *relation can be uniformized by a* $\mathbf{\Sigma^1_2}$ *function.*

Here we say that the function f *uniformizes* the relation R iff $\forall x(x \in \mathrm{dom}(f) \Leftrightarrow \exists y R(x,y))$ and $\forall x \in \mathrm{dom}(f) R(x, f(x))$. Part (1) of the theorem is typical of several *regularity properties* shown to hold for low-level projective sets at that time. Part (2) is the deepest of the *structural* results. One can think of the definition of $f(x)$ from x as encapsulating a general method for constructing an element of the $\mathbf{\Sigma^1_2}$ set $\{y \mid R(x,y)\}$. Read this way, (2) gives us a general method for constructing elements of nonempty $\mathbf{\Sigma^1_2}$ sets which is logically as simple as possible.

Classical descriptive set theorists tried to extend these results and other like them to the higher levels of the projective hierarchy, but without success. We now know why: ZFC decides very little about the regularity and structural properties of higher-level projective sets. For example, by theorems 3.4 and 4.14 above, ZFC does not decide whether every $\mathbf{\Sigma^1_2}$ set is Lebesgue measurable. One can use L and forcing to show that ZFC does not decide whether every $\mathbf{\Sigma^1_3}$ relation can be uniformized by a projective function.

5.3 Determinacy

Determinacy is the fundamental regularity property of definable sets of reals.

Let X be any set, and let $A \subseteq X^\omega$ be a collection of ω-sequences whose entries come from X. We define an infinite, two-person game of perfect information with payoff set A, which we call G_A. In G_A, the two players I and II alternate making moves from X, with I making the first move. After ω moves, the game is over, and one has a *run* of the game

$$\begin{array}{llllll} \mathrm{I} & a_0 & & a_2 & & \\ & & & & & \cdots \\ \mathrm{II} & & a_1 & & a_3 & \end{array}$$

Player I wins this run iff $\langle a_i \mid i < \omega\rangle \in A$, and otherwise player II wins this run. A *strategy* for one of the players is a function $\sigma\colon X^{<\omega} \to X$, giving him the instructions to play $\sigma(s)$ if s is the sequence of his opponent's moves made so far. (The perfect information is given by s.) We say σ is *winning* for one of the players if every run of G_A in which he has played according to σ is a win for him. Clearly, it cannot be that both players have winning strategies, since we could play these strategies against each other. We say A is *determined* iff one of the two players has a winning strategy in G_A.

The determinacy of the set A can be expressed as an infinitary generalization of De Morgan's law:

$$\neg(\exists a_0 \forall a_1 \exists a_2 \forall a_3 \ldots \langle a_i \mid i < \omega\rangle \in A) \Leftrightarrow \forall a_0 \exists a_1 \forall a_2 \exists a_3 \ldots \langle a_i \mid i < \omega\rangle \notin A.$$

Here the infinite string of quantifiers is interpreted using Skolem functions, so that the left hand side says that I has no winning strategy in G_A, and the right hand side says that II does have a winning strategy in G_A. This way of looking at determinacy is useful in some applications. It also gives a hint that the question as to which A are determined might be important.

As with Lebesgue measurability and other regularity properties, simply definable games are determined, while arbitrary games are not. Some of the simplest games are those with *open* payoff, where we say $A \subseteq X^\omega$ is open iff for some $B \subseteq X^{<\omega}$,

$$f \in A \Leftrightarrow \exists n < \omega(f \upharpoonright n \in B),$$

for all $f \in X^\omega$. The open sets are a topology on X^ω, and if $|X| = \omega$, then X^ω with this topology is homeomorphic to the irrationals.

Theorem 5.2 (Gale, Stewart 1951) *(a) For any X, all open $A \subseteq X^\omega$ are determined.*

(b) There is an $A \subseteq \omega^\omega$ which is not determined.

Proof. For (a), suppose that I has no winning strategy in G_A. Call a position in G_A *losing* (for player II) if I has a winning strategy for the game as begun from that position. The following is then a winning strategy for II in G_A: avoid losing positions. By assumption, the initial position is not losing, and it is easy to see that II can maintain this property. Because A is open, any run f of G_A which is lost by II must reach a position $f \upharpoonright j$ which is not just losing, but already lost. Thus avoiding losing positions will actually win for II.

Part (b) is a simple diagonal argument using transfinite induction and the Axiom of Choice. We wellorder all strategies $\langle \sigma_\alpha \mid \alpha < 2^\omega \rangle$, and then at step $\alpha < 2^\omega$ decide two more membership facts about A in such a way as to prevent σ_α from winning for either player in G_A. $\square$

Theorem 5.2 leads to the question whether more complicated definable sets are determined.

Definition 5.3 PD *is the assertion that all projective $A \subseteq \omega^\omega$ are determined.* $\mathsf{AD}^{L(\mathbb{R})}$ *is the assertion that all $A \subseteq \omega^\omega$ such that $A \in L(\mathbb{R})$ are determined.*

The fundamental role of determinacy in descriptive set theory emerged in the 60's and 70's, with work of Kechris, Martin, Moschovakis, Solovay, and others, who showed that the assumption that all definable $A \subseteq \omega^\omega$ are determined leads to a powerful theory of definable sets of reals extending classical descriptive set theory in a natural way. For example

Theorem 5.4 *If* PD *holds, then*

(1) [Mycielski, Swierczkowski 1964] Every projective set of reals is Lebesgue measurable,

(2) [Moschovakis 1971] for all $n \geq 1$, every $\boldsymbol{\Sigma^1_{2n}}$ relation can be uniformized by a $\boldsymbol{\Sigma^1_{2n}}$ function.

In retrospect, much of classical descriptive set theory can be seen as based on the principle that all open $A \subseteq \omega^\omega$ are determined.

5.4 Large cardinals and determinacy

Given the usefulness of definable determinacy, it is natural to try to derive it from more fundamental principles concerning sets in general.

If we allow ourselves only the axioms of ZFC, the following is the best we can do.

Theorem 5.5 (Martin 1975) *All Borel $A \subseteq \omega^\omega$ are determined.*

(Martin also proved a version of this theorem for $A \subseteq X^\omega$, for arbitrary X, which we shall not state.) The proof of 5.5 is especially interesting in that it uses much more of the strength of ZFC than is ordinarily used to prove statements about objects low in the cumulative hierarchy. (It needs the existence of V_α for all $\alpha < \omega_1$.) H. Friedman had earlier shown that this strength would be necessary.

By a refinement of part (1) of theorem 5.4, Σ^1_1-determinacy implies all Σ^1_2 sets of reals are Lebesgue measurable, and hence by 3.4 Σ^1_1-determinacy is false in L. Inspired perhaps by Solovay's result that measurable cardinals imply all $\boldsymbol{\Sigma^1_2}$ sets are Lebesgue measurable, Martin showed

Theorem 5.6 (Martin 1968) *If there is a measurable cardinal, then all $\boldsymbol{\Sigma^1_1}$ sets $A \subseteq \omega^\omega$ are determined.*

Martin's proof of 5.6 proceeds by associating to G_A, where $A \subseteq \omega^\omega$ is $\boldsymbol{\Sigma^1_1}$, an *auxiliary game* G^*_A. In G^*_A, the players play out G_A, while player II makes as additional moves ordinals $< \kappa$, where κ is measurable. These ordinals prove that he is winning G_A. The notion of proof here is such that if II's proof breaks down, it does so at some finite stage of the game, and thus the payoff of G^*_A is open in $(\omega \times \kappa)^\omega$. By Gale-Stewart, G^*_A is determined. Martin then shows how to convert a winning strategy σ for G^*_A into one for G_A. This is easy if σ is for II; he plays G_A by using σ, but skipping the extra moves it makes which prove that he is winning. If σ is for I, then in order to use σ in G_A we must imagine additional moves $< \kappa$ for II. This we can do: because κ is measurable, additional moves which give no special information can be imagined.

Martin's method of associating an open auxiliary game on some larger space lies behind most proofs of determinacy.

Σ^1_1 determinacy is in fact *equivalent* to the existence of canonical inner models approximating $L[U]$:

Theorem 5.7 (Martin 1968, Harrington 1975) *The following are equivalent*

(1) Σ^1_1*-determinacy,*

(2) $0^\sharp$ *exists.*

Σ^1_2-determinacy implies that all Σ^1_3 sets are Lebesgue measurable, and therefore is false in $L[U]$. Thus one needs more than the existence of a measurable cardinal to prove Σ^1_2-determinacy. Martin showed in 1978 that a certain very strong large cardinal hypothesis would suffice, and in 1984 Woodin showed that something still stronger, but of the same variety, implies PD. At about the same time, the work of Foreman, Magidor, and Shelah on Martin's maximum showed that very probably, much weaker large cardinal hypotheses than those used by Martin and Woodin would suffice. The upshot was

Theorem 5.8 *(1) [Martin, Steel 1985] If there are infinitely many Woodin cardinals, then* PD *holds.*

(2) [Woodin, 1986] If there is a measurable cardinal κ such that there are infinitely many Woodin cardinals $< \kappa$, then $\mathsf{AD}^{L(\mathbb{R})}$ *holds.*

Here δ is *Woodin* iff $\forall f\colon \delta \to \delta\ \exists j\colon V \to M\ \ \exists\kappa\ \ (\kappa = \mathrm{crit}(j) \wedge f[\kappa] \subseteq \kappa \wedge V_{j(f)(\kappa)} \subseteq M)$. The existence of a Woodin cardinal is intermediate in strength between the existence of strong cardinals and the existence of supercompact cardinals. The following analog of the Martin-Harrington theorem shows that Woodin cardinals are in some sense needed for PD.

Theorem 5.9 (Martin, Steel, Woodin 1987) *The following are equivalent*

(1) PD*,*

(2) for every $n < \omega$ and sentence φ of LST, there is transitive M which satisfies "there are n Woodin cardinals" such that: $V^M_{\omega+1} \models \varphi$ iff $V_{\omega+1} \models \varphi$,

(3) for all $n < \omega$ and all sentences φ of LST, if ZFC+ *"there are n Woodin cardinals" proves that $V_{\omega+1} \models \varphi$, then $V_{\omega+1} \models \varphi$.*

Roughly speaking, PD is the "trace" left by Woodin cardinals in the first-order theory of $V_{\omega+1}$.

The connection between large cardinals and determinacy is under active investigation now, and much more is known. The theory of canonical inner models with Woodin cardinals plays an important role here.

5.5 Generic absoluteness and CH

Large cardinal axioms seem to decide all natural questions in the first-order theory of $V_{\omega+1}$. This is a nontrivial acheivement, since these include all the main questions of classical descriptive set theory concerning projective sets. There is metamathematical evidence of this "completeness in practice" in the fact that no sentence of the form $V_{\omega+1} \models \varphi$ can be shown independent of existence of arbitrarily large Woodin cardinals by forcing.

Theorem 5.10 (Woodin 1985) *Let M be a transitive, and suppose $M \models$"there are arbitrarily large Woodin cardinals"; then for any M-generic G,*

$$V_{\omega+1}^{M} \prec V_{\omega+1}^{M[G]},$$

In particular, $V_{\omega+1}^{M}$ and $V_{\omega+1}^{M[G]}$ satisfy the same sentences. We say that the theory of $V_{\omega+1}$ is *generically absolute*, granted that there are arbitrarily large Woodin cardinals. Woodin actually proved a stronger result, that the theory of $L(\mathbb{R})$ is generically absolute under the same large cardinal hypothesis.

If we regard Theorem 5.10 as a key indicator of the practical completeness of large cardinal axioms in the realm of first-order questions about $V_{\omega+1}$, then it is natural to ask whether this this kind of completeness extends to questions in the first-order theory of $V_{\omega+2}$. There we immediately encounter CH, which is equivalent to an assertion of the form $V_{\omega+2} \models \varphi$, where φ is Σ_1. None of our current large cardinal axioms decide CH, and consequently they do not yield a generically absolute theory of $V_{\omega+2}$.

Theorem 5.11 (Levy, Solovay 1967) *Let ψ be one of the current large cardinal axioms, and suppose M is a ctm such that $M \models \psi$; then there are M-generic G and H such that $M[G] \models \psi + \mathsf{CH}$ and $M[H] \models \psi + \neg\mathsf{CH}$.*

The reason is that our current large cardinal axioms are preserved by "small" forcing: if κ is the large cardinal witnessing the truth of ψ in M, and G is M-generic for a partial order of size $< \kappa$ in M, then κ continues to witness the truth of ψ in $M[G]$. Since we can make CH fail by forcing with $\mathrm{Fn}_{\omega}(\omega \times \omega_2)^M$, and make it hold by forcing with $\mathrm{Col}(\omega_1, 2^{\omega})^M$, both of which are smaller than the large cardinal witnessing ψ, we get the Levy-Solovay result. Similar arguments show that our current large cardinal axioms do not decide SH,KH,$\diamondsuit$, and many other natural questions in the first-order theory of $V_{\omega+2}$.

This brings us near the frontier of current research in Gödel's program. We have had great success in deciding mathematically important questions dealing with objects more concrete than arbitrary sets of reals. On the other hand, our current large cardinal axioms, the foundation for this success, leave undecided CH, SH, and many other questions about arbitrary sets of reals. Although the attempt to decide these questions by rational means continues to generate new results and questions, the area is in sufficient flux that we shall not attempt to describe it here.

6 Other topics

Here are a few of the many important themes and results in set theory which have been too far from our theme to permit much discussion.

The *partition calculus* deals with infinitary generalizations of Ramsey's theorem from finite combinatorics. A typical, basic result is the *Erdös-Rado theorem*: if κ is an infinite cardinal, and the unordered pairs of ordinals $< (2^\kappa)^+$ are colored with two colors, then there is a set $X \subseteq (2^\kappa)^+$ such that $|X| = \kappa^+$, and all unordered pairs from X have the same color. There are many results and open questions in this vein.

The theory of *cardinal characteristics of the continuum* deals with the possible sizes of various cardinal numbers related to Lebesgue measure, Baire category, and other natural structures on sets of size 2^ω. For example, letting b be the minimal cardinality of an unbounded set in $(\omega^\omega, \leq^*)$, d the minimal cardinality of a cofinal set in $(\omega^\omega, \leq^*)$, and n the minimal cardinality of a set of positive outer Lebegue measure, one can ask for the possible relationships between b, d and n. In general, such questions cannot be decided by ZFC, but there are some surprises, and some important open questions.

Finally, there is a more applied side to modern descriptive set theory. This area might be called the theory of *Borel equivalence relations and definable cardinality.* A typical question here is: given two equivalence relations E and H on ω^ω, is there a Borel function f such that for all $x, y \in \omega^\omega$, $xEy \Leftrightarrow f(x)Hf(y)$. A theory of such *Borel reducibility* relations has been developed, and used to shed light on complexity many classification problems arising naturally in analysis and elsewhere. Although most of this theory can be developed in ZFC, $\mathbf{\Sigma^1_1}$-determinacy is sometimes useful.