The domestic levels of K^c are iterable

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October 8, 2000

Abstract

We show that the models produced by the K^c construction before (if ever) it reaches a non-domestic premouse are all iterable. As a corollary we get that PFA plus the existence of a measurable cardinal implies the existence of a non-domestic premouse.

1 Introduction

Iterability remains the only obstacle on the way to a general construction of inner models below superstrong cardinals, and so the only obstacle to the many equiconsistency results which will follow from such a construction. A more precise description of what "iterability" means is given by (†) of 2.21, but we shall not go into this now. It is perhaps worthwhile pointing out one of the many potential applications of the construction of inner models:

Theorem 1.1 (Schimmerling–Steel) *Suppose* PFA, *the* proper forcing axiom, *holds and there exists a measurable cardinal* Ω. *Assume* $(\dagger)_{\nu,k}$ *for all* $\nu < \Omega$, $k < \omega$. *Then there exists an inner model with a superstrong cardinal.*

*Partially supported by the Italian MURST
†Research partially supported by NSF grant DMS 98-03292
‡Research partially supported by NSF grant DMS 98-03611

Theorem 1.1 is really the result of the combined work of several people over many years. It is perhaps deficient in several respects; the measurable cardinal should not be necessary, and the consistency strength of PFA is anyway supposed to be much more than a superstrong cardinal. But most annoying is our inability to prove (†). There have been several attempts to approximate a proof of (†). All have the form "failure of (†) gives a transitive model for ZFC plus large cardinal axiom (A)," where the particular large cardinal which (A) stands for has changed over the years, becoming stronger. Thus if we wish to eliminate the assumption of (†) in Theorem 1.1 we may deduce only the weaker of (A) and "there exists an inner model with a superstrong cardinal."

This paper presents the latest in the series of strengthenings of (A). In Corollary 3.3 we show that a failure of (†) implies the existence of a *non-domestic* premouse. The notion of domestic is stated precisely in Definition 3.1. A non-domestic premouse gives a model $\mathcal{M}$ with a cardinal κ so that

- κ is a limit of Woodin cardinals in $\mathcal{M}$;
- κ is a limit of cardinals strong to κ in $\mathcal{M}$; and
- κ is externally measurable.

(We refer the reader to [Kan97, §26] for the definition of strong and Woodin cardinals. A cardinal τ is said to be **strong to** κ just in case that τ is γ strong for all $\gamma < \kappa$.) This of course is substantially weaker than a superstrong cardinal. Thus combining Corollary 3.3 with Theorem 1.1 we get

Corollary 1.2 *Suppose that* PFA *holds and that there exists a measurable cardinal. Then there exists a non-domestic premouse.*

Some (conceptually) simple modifications of our proof allow substituting for (A) an axiom slightly stronger than non-domestic (specifically the existence of premice $\mathcal{M}$ with any finite number of cardinals $\kappa_0 < \ldots < \kappa_n$ such that each κ_i is: strong in $\mathcal{M}$; a limit of Woodin cardinals of $\mathcal{M}$; and a limit of strong cardinals of $\mathcal{M}$). Yet this too falls far short of an outright proof of (†). How to actually prove (†) remains one of the greatest mysteries of inner model theory.

This paper divides into two Sections. Section 2 gives a brief account of the K^c construction of [Ste96], adapted to our context. There are two main differences there. We follow Jensen's indexing of extenders, and we use a slightly different notation to deal with the fine-structural concepts of [Ste96] and [MitSt94]. Section 2 also states (†), and present a fundamental iterability theorem (2.28) which traces back to [Ste96] and [MarSt94]. Theorem 2.28 demonstrates the existence of *maximal* branches through iteration trees, but not necessarily *cofinal* branches. (†) however requires cofinal branches. In Section 3 we show how to use Theorem 2.28 so as to obtain cofinal branches through the relevant iteration trees. This argument makes strong use of the smallness assumption that the premouse considered is *domestic*. More precisely, Lemma 3.24 demonstrates that iteration trees on domestic premice have a certain property which we call semilinearity. This allows viewing the trees as compositions of "better" iteration trees — trees with *unique* realizable branches. Uniqueness then allows us to deduce that a maximal realizable branch must in fact be cofinal.

Acknowledgement: We thank Ernest Schimmerling and Martin Zeman for some fine structural help.

2 Preliminaries

Stretching our proof to work for the largest possible large cardinal seems to require use of Jensen's indexing method. Jensen, unlike [MitSt94] and [Ste96], indexes an extender E at the successor of $i_E(\kappa)$ (computed in the ultrapower by E) where $\kappa = \mathrm{crit}(E)$. Exact details of this method of indexing can be found in [Jen97], which unfortunately is unpublished. Our own approach here is a combination of the indexing of [Jen97] with the fine structure of [MitSt94]. We include in this Section a description of the basic definitions involved in this combination, continue with a particular kind of realizability for finite phalanges which we shall need, and end with the fundamental iterability theorem for such realizable phalanges. Throughout the Section we indicate how our notions correspond to those which exist in the literature; both in [Jen97] and in [MitSt94]. This Section does not contain any proofs, since all results involve only simple modifications to proofs which exist in the literature. We shall refer the reader to the location of these proofs, and occasionally remark on the modifications which must be made.

Definition 2.1 *Let $\bar{\mathcal{M}}$ be a transitive structure with a largest cardinal κ. F is said to be a* **whole**[1] **extender** *on $\bar{\mathcal{M}}$ just in case that there exist $\mathcal{N}, \pi$ such that:*

1. *$\mathcal{N}$ is transitive;*
2. *$\pi: \bar{\mathcal{M}} \to \mathcal{N}$ is Σ_0 elementary and cofinal;*
3. *$\mathrm{crit}(\pi) = \kappa$; and*
4. *F is the restriction of π to $P(\kappa)^{\bar{\mathcal{M}}}$.*

Given $\bar{\mathcal{M}}$ and a whole extender F on $\bar{\mathcal{M}}$ one can recover a *minimal* pair $\mathcal{N}, \pi$ which satisfies conditions 1–4 of Definition 2.1. Indeed, given any $\mathcal{M}$ such that $P(\kappa)^{\mathcal{M}} = P(\kappa)^{\bar{\mathcal{M}}}$ one can form a structure $\mathrm{Ult}_0(\mathcal{M}, F)$ and an embedding $\sigma: \mathcal{M} \to \mathrm{Ult}_0(\mathcal{M}, F)$ such that:

1. The restriction of σ to $P(\kappa)^{\mathcal{M}}$ is equal to F;

[1] General, non-whole, extenders are structures of the form $F|\lambda$ presented below. A precise definition can be found in [Jen97, §1]. It is customary also to define extenders as the directed system of measures derived from $F|\lambda$. See for example [Kan97, §26].

2. σ is Σ_0 elementary and cofinal; and

3. $\mathrm{Ult}_0(\mathcal{M}, F), \sigma$ is minimal, in the sense that every element of $\mathrm{Ult}_0(\mathcal{M}, F)$ has the form $\sigma(f)(\vec{a})$ for some function $f \in \mathcal{M}$, $f\colon \kappa^{<\omega} \to \mathcal{M}$, and some $\vec{a} \in F(\kappa)^{<\omega}$.

($\mathrm{Ult}_0(\mathcal{M}, F)$ need not in general be wellfounded. If it is we assume it's transitive.) $\mathrm{Ult}_0(\mathcal{M}, F)$ and σ are known as the **coarse ultrapower** of $\mathcal{M}$ by F and the **ultrapower embedding** respectively. The precise method of the construction of $\mathrm{Ult}_0(\mathcal{M}, F)$ can be found in [Kan97, §26].

$\mathrm{lh}(F)$, the **length** of F, is $\pi(\kappa)$. For $\lambda \leq \mathrm{lh}(F)$ we let $F|\lambda$ be given by

$$(F|\lambda)(X) = F(X) \cap \lambda.$$

C_F denotes the set of $\lambda \leq \mathrm{lh}(F)$ such that $F|\lambda$ is itself a whole extender. There is an implicit dependence on $\bar{\mathcal{M}}$ here, but in fact using the ultrapower construction one can see that C_F depends only on F.

Remark 2.2 $\mathrm{lh}(F)$ always belongs to C_F, and is quite often the only element of C_F. However we should point out that already at the level of sharps one can construct extenders F such that C_F contains *additional* elements other than $\mathrm{lh}(F)$. These examples, at least below a superstrong cardinal, will fail to satisfy the initial segment condition 2.4(5), see Footnote 7.

Definition 2.3 *$\mathcal{N} = \langle J_\alpha[A], F\rangle$ is* **coherent** *iff*

1. *$J_\alpha[A]$ is acceptable;*[2]

2. *For some $\bar{\alpha} < \alpha$, F is a whole extender with domain $P(\kappa)^{J_{\bar{\alpha}}[A]}$ where $\kappa = \mathrm{crit}(F)$, and κ is the largest cardinal in $J_{\bar{\alpha}}[A]$; and*

3. *$J_\alpha[A] = \mathrm{Ult}_0(J_{\bar{\alpha}}[A], F)$.*[3]

Definitions 2.1 and 2.3 are taken essentially from [Jen97]. Our next definition, that of a premouse, is similar to that of [Jen97] except that we view the height of a premouse not as an ordinal α, but as a pair $\langle \alpha, k\rangle$ where α is an ordinal and $k \leq \omega$. The main difference between our premice and those of [Jen97] comes in through condition 4 and we shall elaborate on this below.

Definition 2.4 *$\mathcal{M} = (\langle J_\alpha[\vec{E}], \in, \vec{E}, E_\alpha\rangle, k)$ is a* **premouse** *if it satisfies the following conditions:*

1. *$\alpha \in \mathrm{ON}$ and $k \leq \omega$;*

[2] A structure is **acceptable** if it satisfies a strong form of the GCH, to the effect that every subset of an ordinal γ in the structure is constructed before the successor of γ in the structure.

[3] Equivalently in this situation, taking $\bar{\mathcal{M}} = J_{\bar{\alpha}}[A]$ and $\mathcal{N} = J_\alpha[A]$ would satisfy the conditions of Definition 2.1.

2. *$J_\alpha[\vec{E}]$ is acceptable;*

3. *(Coherence) $\vec{E}$ has the form $\{\langle \nu, Z\rangle \mid \nu < \alpha \wedge Z \in E_\nu\}$. For each $\nu \leq \alpha$, E_ν is either empty or else it is a whole extender[4] so that*

   (a) *$\langle J_\nu[\vec{E}\restriction \nu], E_\nu\rangle$ is* coherent *and*

   (b) *$\mathrm{dom}(E_\nu)$ includes* all *subsets of $\mathrm{crit}(E_\nu)$ in $J_\nu[\vec{E}\restriction \nu]$ (in other words the ordinal $\bar{\alpha}$ of Definition 2.3 is the successor of κ in $J_\nu[\vec{E}\restriction \nu]$);*

4. *For $\langle \beta, l\rangle <_{\mathrm{Lex}} \langle \alpha, k\rangle$ (strictly), $\mathcal{M}\|\langle \beta, l\rangle = (\langle J_\beta[\vec{E}\restriction \beta], \in, \vec{E}\restriction \beta, E_\beta\rangle, l)$ is a premouse, is* sound, *and has a* solid standard parameter *(both concepts are explained below);*

5. *(Initial segment condition) In the case that $k = 0$, if $E_\alpha \neq \emptyset$, $\lambda \in C_{E_\alpha}$, and $\lambda < \mathrm{lh}(E_\alpha)$ (strictly) then $E_\alpha|\lambda \in \mathcal{M}$.*

The *elements* of $\mathcal{M}$ are the elements of $J_\alpha[\vec{E}]$, but $\mathcal{M}$ contains additional information which includes k and the predicates $\in$, $\vec{E}$ and E_α. We shall use $\mathcal{J}_{\alpha,k}[\vec{E}]$ to denote the structure $(\langle J_\alpha[\vec{E}], \in, \vec{E}\rangle, k)$ and write $\langle \mathcal{J}_{\alpha,k}[\vec{E}], E_\alpha\rangle$ for the premouse $\mathcal{M}$. When we wish to draw attention to k we shall refer to $\mathcal{M}$ as a k–premouse. We shall refer to α as $\alpha(\mathcal{M})$ and to k as $k(\mathcal{M})$.

Remark 2.5 [MitSt94] includes an additional condition in the definition of a premouse, saying that if $E_\alpha \neq \emptyset$ then $E_\alpha^{\mathcal{N}} = \emptyset$ where $\mathcal{N} = \mathrm{Ult}_0(\mathcal{M}, E_\alpha)$ and $\vec{E}^{\mathcal{N}}$ is the image of $\vec{E}$ under the ultrapower embedding. In the context of Jensen's indexing this extra condition follows provably from the others.

Exclusion 2.6 To avoid technical complications related to the preservation of the initial segment condition, 2.4(5), WE LIMIT OUR DISCUSSION THROUGHOUT THE PAPER TO STRUCTURES $\mathcal{M}$ SUCH THAT $\mathcal{M} \models$" THERE DO NOT EXIST SUPERSTRONG CARDINALS." We shall say more on the preservation of 2.4(5) in Remark 2.15 below. For the time being let us note that under our exclusion, 2.4(5) is equivalent to: "In the case that $k = 0$, if $E_\alpha \neq \emptyset$ then C_{E_α} does *not* contain elements other than $\mathrm{lh}(E_\alpha)$."

[Jen97] defines the notion of Σ^* elementarity which is then used throughout the notes, instead of the elementarity notions of [MitSt94]. In this paper we shall adopt an approach to elementarity which is close to that of [MitSt94] and [Ste96], although the end result, at least for the premice we shall work with here, is the same as Jensen's Σ^*. Roughly speaking, we shall follow [MitSt94] except that the index "k" which denotes the degree of elementarity is transferred, from the embedding to the premouse on which it operates. This movement was suggested

[4] We alternate between thinking of whole extenders as functions and as predicates. The *predicate* E_ν is simply the graph of the *function* E_ν. Thus $Z \in E_\nu$ iff $Z = \langle X, Y\rangle$ for some X, Y such that $E_\nu(X) = Y$.

to the authors by Sy Friedman. It does not affect the mathematical content of our statements; however the notation is simplified substantially. We have already taken the first step of this movement in our definition of a premouse. Next we explain how this movement affects the fine structure of our premice, and the notion of elementarity which it generates.

Our definitions below, and indeed Definition 2.4, are carried inductively on the lexicographic order for pairs $\langle\alpha, k\rangle$. For each premouse we define below its **true height**, $\vartheta(\mathcal{M})$; its **true domain**, $\mathfrak{D}(\mathcal{M})$; its **projectum**, $\rho(\mathcal{M})$; its **standard parameter**, $p(\mathcal{M})$; its **reduct**, $\mathfrak{R}(\mathcal{M})$; and its **core**, $\mathfrak{C}(\mathcal{M})$. We then say what it means for $\mathcal{M}$ to be **sound**, and (neglect to) say what **solidity** is. Once these concepts are defined for premice of height $\langle\alpha, k\rangle$ we can make sense of a premouse of height $\langle\alpha, k+1\rangle$ — particularly of condition 4 of Definition 2.4.

During the induction we shall make use of the following two assumptions

$(\mathrm{U})_{\mathcal{M}}$	The standard parameter, $p(\mathcal{M})$, of $\mathcal{M}$ is universal.
$(\mathrm{S})_{\mathfrak{C}(\mathcal{M})}$	The standard parameter, $p(\mathfrak{C}(\mathcal{M}))$, of $\mathfrak{C}(\mathcal{M})$ is solid.

Although we indicate below how we use $(\mathrm{U})_{\mathcal{M}}$ and $(\mathrm{S})_{\mathfrak{C}(\mathcal{M})}$ (both are used to prove the preservation of the standard parameter under embeddings), it is not within the scope of this paper to define solidity and universality. We refer the reader to [MitSt94, Definition 2.7.4] and [MitSt94, Definition 2.7.5], as well as [MitSt94, §8]. Solidity and universality for k–premice correspond to "$k+1$ solidity" and "$k+1$ universality" in the terminology of [MitSt94].

Before we begin the induction, let us briefly review the fundamental definitions of fine structure, which trace back to [Jen72]. Additional details can be found in [Jen72] or the forthcoming [Zem]. Suppose $\mathcal{S} = \langle S, \{A_i\}_{i\in I}\rangle$ is an acceptable structure with predicates $\{A_i\}_{i\in I}$. The $\mathbf{\Sigma_1}$ **projectum** of $\mathcal{S}$, denoted $\rho_1(\mathcal{S})$, is the least ordinal ρ such that for some $q \in S \cap \mathrm{ON}^{<\omega}$ and some Σ_1 formula φ, $\{\xi < \rho \mid \mathcal{S} \models \varphi[\xi, q]\} \notin S$. $\rho_1(\mathcal{S})$ may be $\mathrm{ON} \cap S$ or it may be smaller. The $\mathbf{\Sigma_1}$ **standard parameter** of $\mathcal{S}$, denoted $p_1(\mathcal{S})$, is the least q (wrt the lexicographic order on decreasing sequences of ordinals) which witnesses the above. The $\mathbf{\Sigma_1}$ **reduct** of $\mathcal{S}$ is the structure $\mathcal{R} = \langle R, \{B_i\}_{i\in I}, B^*\rangle$ where: $R = S \cap H(\rho_1(\mathcal{S}))^{\mathcal{S}}$; $B_i = A_i \cap R$; and B^* is an *additional* predicate which codes the Σ_1 theory of $\rho_1(\mathcal{S}) \cup \{p_1(\mathcal{S})\}$ in $\mathcal{S}$. (More precisely, $\langle i, \vec{\alpha}\rangle \in B^*$ just in case that $i < \omega$, $\vec{\alpha} \in \rho_1(\mathcal{S})^{<\omega}$, and $\mathcal{S} \models \varphi_i[\vec{\alpha}, p_1(\mathcal{S})]$. $\{\varphi_i\}_{i<\omega}$ here is a recursive enumeration of all Σ_1 formulae of $\mathcal{S}$.) Finally, the $\mathbf{\Sigma_1}$ **core** of $\mathcal{S}$ is the transitive collapse of the Σ_1 Skolem hull in $\mathcal{S}$ of $\rho_1(\mathcal{S}) \cup \{p_1(\mathcal{S})\}$. Of course making this precise requires a definition of Σ_1 Skolem terms in $\mathcal{S}$, but we shall not go into this. $\mathcal{S}$ is $\mathbf{\Sigma_1}$ **sound** just in case that the Σ_1 Skolem hull of $\rho_1(\mathcal{S}) \cup \{p_1(\mathcal{S})\}$ in $\mathcal{S}$ includes *all* of S.

Let us now return to the induction. We start with the case that $\mathcal{M}$ is a 0–premouse. In this case $\vartheta(\mathcal{M})$ is $\omega \cdot \alpha = \mathrm{ON} \cap \mathcal{M}$; $\mathfrak{D}(\mathcal{M}) = \mathcal{M}$; $\rho(\mathcal{M})$ is the Σ_1 projectum of $\mathcal{M}$; $p(\mathcal{M})$ is the Σ_1 standard parameter of $\mathcal{M}$; $\mathfrak{R}(\mathcal{M})$ is the Σ_1 reduct of $\mathcal{M}$; and $\mathfrak{C}(\mathcal{M})$ is its Σ_1 core. We say that $\mathcal{M}$ is sound just in case that $\mathcal{M}$ is Σ_1 sound (and in particular $\mathfrak{C}(\mathcal{M}) = \mathcal{M}$). Observe that $\mathfrak{R}(\mathcal{M})$ codes the Skolem hull which collapses to $\mathfrak{C}(\mathcal{M})$. It follows that $\mathfrak{C}(\mathcal{M})$ can be recovered from $\mathfrak{R}(\mathcal{M})$.

Remark 2.7 One can check that $\rho(\mathfrak{C}(\mathcal{M})) = \rho(\mathcal{M})$ and that the two premice agree up to $\rho(\mathcal{M})$. The universality assumption $(\mathrm{U})_{\mathcal{M}}$ implies that in fact $\mathfrak{C}(\mathcal{M})$ and $\mathcal{M}$ have the same subsets of $\rho(\mathcal{M})$. From this it follows that $p(\mathfrak{C}(\mathcal{M}))$ is the collapsed image of $p(\mathcal{M})$, and so $\mathfrak{C}(\mathcal{M})$ is *sound*. $\mathfrak{C}(\mathcal{M})$ is then the unique sound premouse whose reduct equals $\mathfrak{R}(\mathcal{M})$.

$\mathfrak{C}(\mathcal{M})$ is still a 0–premouse but since it is sound and —by $(\mathrm{S})_{\mathfrak{C}(\mathcal{M})}$— has a solid standard parameter, it can be extended to a 1–premouse. We call this a trivial extension: Given a *sound* k–premouse $\mathcal{M} = \langle \mathcal{J}_{\alpha,k}[\vec{E}], E_\alpha \rangle$ with a *solid* standard parameter, the **trivial extension** of $\mathcal{M}$ is the $k+1$–premouse $\langle \mathcal{J}_{\alpha,k+1}[\vec{E}], E_\alpha \rangle$. The requirements of soundness and solidity are needed to establish 2.4(4) for the trivial extension.

Let us next consider the case when $\mathcal{M} = \langle \mathcal{J}_{\alpha,k+1}[\vec{E}], E_\alpha \rangle$ is a $k+1$–premouse. Our notions now will correspond to the appropriate "$k+2$ notions" in the terminology of [MitSt94]. Let $\mathcal{M}^{\mathrm{trn}}$ denote the k–premouse $\langle \mathcal{J}_{\alpha,k}[\vec{E}], E_\alpha \rangle$, which we call the **immediate truncation** of $\mathcal{M}$. We define $\vartheta(\mathcal{M})$ to be $\rho(\mathcal{M}^{\mathrm{trn}})$; and $\mathfrak{D}(\mathcal{M}) = \mathfrak{R}(\mathcal{M}^{\mathrm{trn}})$. $\mathfrak{D}(\mathcal{M})$ is then a structure whose ordinal height is $\vartheta(\mathcal{M})$, and $\mathcal{M}$ is the *unique* premouse whose true domain equals $\mathfrak{D}(\mathcal{M})$.

$\rho(\mathcal{M})$ is defined to be the Σ_1 projectum of $\mathfrak{D}(\mathcal{M})$; $p(\mathcal{M})$ is the Σ_1 standard parameter of $\mathfrak{D}(\mathcal{M})$; $\mathfrak{R}(\mathcal{M})$ is the Σ_1 reduct of $\mathfrak{D}(\mathcal{M})$. Let $\bar{\mathcal{D}}$ be the Σ_1 core of $\mathfrak{D}(\mathcal{M})$. There is a unique $k+1$–premouse whose true domain equals $\bar{\mathcal{D}}$. Let $\mathfrak{C}(\mathcal{M})$ be this premouse.

Remark 2.8 Much is hiding in our claim that there is a $k+1$–premouse whose true domain equals $\bar{\mathcal{D}}$. To verify this we must capture the properties which make $\mathcal{M}$ into a $k+1$–premouse as Σ_1 statements over $\mathfrak{D}(\mathcal{M})$, and reflect these statements to $\bar{\mathcal{D}}$. Some of the properties are easy to capture, others are more difficult. Among the more difficult ones is the solidity of $p(\mathcal{M}^{\mathrm{trn}})$. Capturing this property seemingly requires a Σ_2 statement over $\mathfrak{D}(\mathcal{M})$. [MitSt94] avoids the problem by putting witnesses for this Σ_2 statement as additional parameters in the Skolem hull which defines $\bar{\mathcal{D}}$, see [MitSt94, pp.23–24]. In fact the solidity of $p(\mathcal{M}^{\mathrm{trn}})$ is equivalent to a Σ_1 statement over $\mathfrak{D}(\mathcal{M})$, and so additional parameters are not needed. We refer the reader to the discussion of generalized witnesses in [Jen97, §7 pp.1–5]. (The existence of generalized witnesses is Σ_1 over $\mathfrak{D}(\mathcal{M})$.)

We say that $\mathcal{M}$ is sound just in case that $\mathfrak{D}(\mathcal{M})$ is Σ_1 sound (in particular $\bar{\mathcal{D}} = \mathfrak{D}(\mathcal{M})$ and $\mathfrak{C}(\mathcal{M}) = \mathcal{M}$). Observe that $\mathfrak{R}(\mathcal{M})$ codes the Skolem hull which collapses to $\bar{\mathcal{D}}$. Thus knowledge of $\mathfrak{R}(\mathcal{M})$ suffices to determine $\mathfrak{C}(\mathcal{M})$. Remark 2.7 applies and assuming $(U)_{\mathcal{M}}$ it follows that $\mathfrak{C}(\mathcal{M})$ is the unique sound premouse whose reduct equals $\mathfrak{R}(\mathcal{M})$.

Finally, in the case of an ω–premouse $\mathcal{M} = \langle \mathcal{J}_{\alpha,\omega}[\vec{E}], E_\alpha \rangle$: By condition 4 of Definition 2.4 we know that each of $\langle \mathcal{J}_{\alpha,k}[\vec{E}], E_\alpha \rangle$ is a sound premouse. Let us refer to them as $\mathcal{M}_0, \mathcal{M}_1, \ldots$. Our definitions for these premice are such that $\vartheta(\mathcal{M}_0) \geq \rho(\mathcal{M}_0) = \vartheta(\mathcal{M}_1) \geq \rho(\mathcal{M}_1) = \vartheta(\mathcal{M}_2) \geq \ldots$. Since there are no infinite (strictly) descending chains of ordinals we see that for all sufficiently large $k < \omega$, $\vartheta(\mathcal{M}_k) = \rho(\mathcal{M}_k) = \vartheta(\mathcal{M}_{k+1})$. We let $\rho(\mathcal{M}) = \vartheta(\mathcal{M})$ be this eventual value. $\mathcal{M}$ by default is sound, and $\mathfrak{C}(\mathcal{M}) = \mathcal{M} = \mathfrak{C}(\mathcal{M}_k)$. Again for all sufficiently large $k < \omega$, the reducts $\mathfrak{R}(\mathcal{M}_k)$ all have the same *elements* (though not the same predicates; $\mathfrak{R}(\mathcal{M}_{k+1})$ has one additional predicate on top of the predicates of $\mathfrak{R}(\mathcal{M}_k)$). We let $\mathfrak{D}(\mathcal{M}) = \mathfrak{R}(\mathcal{M})$ have as its elements the elements of $\mathfrak{R}(\mathcal{M}_k)$ for some (all) sufficiently large $k < \omega$, and as its (infinitely many) predicates the accumulation of the predicates of $\mathfrak{R}(\mathcal{M}_k)$, $k < \omega$ sufficiently large. We set $p(\mathcal{M}) = \emptyset$. $p(\mathcal{M})$ by default is solid and universal.

Remark 2.9 Each element of $\mathcal{M}$ is coded as a term over $\mathfrak{D}(\mathcal{M})$. We shall not make this precise; readers who have some familiarity with fine structure of any fashion may interpret this in their favorite way. Terms can always be viewed as finite sequences of ordinals. For $t \in \mathfrak{D}(\mathcal{M})$ we use $\ulcorner t \urcorner$ to denote the element of $\mathcal{M}$ coded by t, if t has the form of a term. $t \mapsto \ulcorner t \urcorner$ is then a partial map from $\vartheta(\mathcal{M})^{<\omega}$ onto $\mathcal{M}$.

Let $\mathcal{M} = \langle \mathcal{J}_{\alpha,k}[\vec{E}], E_\alpha \rangle$ and $\mathcal{N} = \langle \mathcal{J}_{\beta,l}[\vec{F}], F_\beta \rangle$ be two premice. An embedding $\pi : \mathcal{M} \to \mathcal{N}$ is said to be **elementary** just in case that conditions 1–4 below are satisfied.

1. $k = l$;
2. $\text{crit}(\pi) < \vartheta(\mathcal{M})$;[5]

Let $\bar{\pi}$ be the restriction of π to $\mathfrak{D}(\mathcal{M})$.

3. $\bar{\pi}$ is a Σ_1 elementary embedding of $\mathfrak{D}(\mathcal{M})$ into $\mathfrak{D}(\mathcal{N})$. (Note that formulae here may make reference to the additional predicates of $\mathfrak{D}(\mathcal{M})$);
4. For $t \in \vartheta(\mathcal{M})^{<\omega}$ a term,
$$\pi(\ulcorner t \urcorner) = \ulcorner \bar{\pi}(t) \urcorner$$
where the RHS is of course interpreted in $\mathcal{N}$.

In the situation described by condition 4 above we say that π is **induced** by $\bar{\pi}$. We say that π is **precise** if in addition to 1–4 the following condition is satisfied:

5. $\text{crit}(\pi) \geq \rho(\mathcal{M})$, $\rho(\mathcal{N}) = \rho(\mathcal{M})$, and $p(\mathcal{N}) = \pi(p(\mathcal{M}))$.

The importance of precise embeddings has to do with Remarks 2.11, 2.18, and *particularly* Lemma 2.25(b). The reader who survives to the end of Section 3 will observe that Lemma 2.25(b) is essential to the proof of Claim 3.32. This use of Lemma 2.25 is typical of its general use in inner model theory.

[5] If $\pi = id$ we view this condition as vacuous. Similarly in condition 5 below.

Remark 2.10 Almost by definition $\mathfrak{C}(\mathcal{M})$ embeds elementarily into $\mathcal{M}$. The embedding is induced by the map $\bar{\pi}$ which embeds the Σ_1 core of $\mathfrak{D}(\mathcal{M})$ into $\mathfrak{D}(\mathcal{M})$. We call this elementary embedding the **anti-core** embedding. Its critical point of course is at least $\rho(\mathcal{M})$. Assuming $(\mathrm{U})_{\mathcal{M}}$ the anti-core embedding is in fact precise.

Remark 2.11 If $\pi: \mathcal{M} \to \mathcal{N}$ is precise and $\mathcal{M}$ is *sound*, then $\mathfrak{C}(\mathcal{N}) = \mathcal{M}$ and π is in fact the *anti-core* embedding.

Fact 2.12 *The composition of two elementary embeddings is elementary. Indeed, any transfinite composition of elementary embeddings is elementary.*

$\mathcal{P} = \langle \mathcal{J}_{\gamma,i}[\vec{H}], H_\gamma \rangle$ is said to be an **initial segment** of $\mathcal{M}$ just in case that $\langle \gamma, i \rangle \leq_{\mathrm{Lex}} \langle \alpha, k \rangle$, $\vec{H} = \vec{E} \restriction \gamma$, and $H_\gamma = E_\gamma$. $\mathcal{P}$ is a **strict initial segment** of $\mathcal{M}$ if strict inequality holds (even if $\gamma = \alpha$).

Fact 2.13 *Let $\pi: \mathcal{M} \to \mathcal{N}$ be elementary. Let $\bar{\mathcal{M}}$ be an initial segment of $\mathcal{M}$ and let $\bar{\mathcal{N}} = \pi(\bar{\mathcal{M}})$ (where this is understood to be $\mathcal{N}$ if $\bar{\mathcal{M}} = \mathcal{M}$, and an n-th immediate truncation of $\mathcal{N}$ if $\bar{\mathcal{M}}$ is an n-th immediate truncation of $\mathcal{M}$). Then $\pi \restriction \bar{\mathcal{M}}: \bar{\mathcal{M}} \to \bar{\mathcal{N}}$ is elementary.*

We say that $\pi: \mathcal{M} \to \mathcal{N}$ is a **weak** embedding if it satisfies the conditions of elementarity with 3 replaced by the following weaker condition:

w3. $\bar{\pi}: \mathfrak{D}(\mathcal{M}) \to \mathfrak{D}(\mathcal{N})$ is Σ_0 elementary and cardinal preserving.[6]

Any elementary embedding is also weak. Facts 2.12 and 2.13 continue to hold with elementary replaced by weak. Another quality of weak embeddings, crucial to the proof of Theorem 2.28, is given in Remark 2.14 below. As a rough guide to the future distinction between elementary and weak embeddings, let us say that embeddings given by iteration trees are elementary, while *realization* embeddings (see 2.27) are weak. The need for this distinction is explained in Remark 2.31.

Remark 2.14 Suppose $\pi: \mathcal{M} \to \mathcal{N}$ is weak. Fix $\kappa < \lambda \in \mathcal{M}$. Suppose that there exists a strict initial segment $\mathcal{P}$ of $\mathcal{N}$ such that $\alpha(\mathcal{P}) \geq \pi(\lambda)$ and $\rho(\mathcal{P}) \leq \pi(\kappa)$. Let $\mathcal{N}^-$ be the least such. Then there exists a strict initial segment $\mathcal{Q}$ of $\mathcal{M}$ such that $\alpha(\mathcal{Q}) \geq \lambda$ and $\rho(\mathcal{Q}) \leq \kappa$. Let $\mathcal{M}^-$ be the least such. Then $\pi(\mathcal{M}^-) = \mathcal{N}^-$. (As always $\pi(\mathcal{M}^-)$ is understood to be an n-th immediate truncation of $\mathcal{N}$ if $\mathcal{M}^-$ is an n-th immediate truncation of $\mathcal{M}$.)

An extender F with critical point κ is said to be an extender **over** a premouse $\mathcal{M} = \langle \mathcal{J}_{\alpha,k}[\vec{E}], E_\alpha \rangle$ just in case that:

F1. The domain of F equals $P(\kappa)^{\mathcal{M}}$;

[6] [MitSt94, 5.1.7 ff.] makes the additional requirement that $\bar{\pi}$ be Σ_1 elementary on a cofinal subset of $\mathfrak{D}(\mathcal{M})$. This gives a class of embeddings which is not closed under compositions, creating problems later on, particularly in 3.33–3.35.

F2. $\kappa < \vartheta(\mathcal{M})$; and

F3. If $k = 0$ and $E_\alpha \neq \emptyset$ then $\kappa < \mathrm{lh}(E_\alpha)$ (strictly).

The **(fine) ultrapower** of $\mathcal{M}$ by F is then defined essentially using terms. One way to state this is the following: Let $\mathcal{D} = \mathrm{Ult}_0(\mathfrak{D}(\mathcal{M}), F)$ be the coarse ultrapower of $\mathfrak{D}(\mathcal{M})$ by F, and let $\bar{\pi}$ be the coarse ultrapower embedding. Define $\mathcal{N} = \mathrm{Ult}(\mathcal{M}, F)$ to be the term structure recovered from $\mathcal{D}$ and define the **(fine) ultrapower embedding** $\pi\colon \mathcal{M} \to \mathcal{N}$ to be the embedding induced by $\bar{\pi}$. $\mathrm{Ult}(\mathcal{M}, F)$ needn't always be wellfounded. If $\mathrm{Ult}(\mathcal{M}, F)$ is wellfounded then we assume it's transitive. It is easy in this case to check that $\mathfrak{D}(\mathrm{Ult}(\mathcal{M}, F)) = \mathcal{D}$ and that π is elementary.

Conditions F1–F3 are all necessary for this to make sense. F1 and F2 allow forming the coarse ultrapower of $\mathfrak{D}(\mathcal{M})$. F3 is used to make sure that $\mathcal{N}$ continues to satisfy the initial segment condition.[7] If condition F2 fails then F can*not* be applied to $\mathcal{M}$, even if $\mathrm{crit}(F) < \mathcal{M} \cap \mathrm{ON}$ and the domain of F includes precisely all subsets in $\mathcal{M}$ of its critical point. However it is possible in this case to (trivially) truncate $\mathcal{M}$ to some k'–premouse $\mathcal{M}' = \langle \mathcal{J}_{\alpha,k'}[\vec{E}], E_\alpha \rangle$ with a true height large enough so that F2 holds for $\mathcal{M}'$. This process of trivial truncation corresponds to the drops in degree of [MitSt94]. If condition F3 fails then again we don't apply F to $\mathcal{M}$. Generally speaking when this happens we end up applying F to a different premouse altogether, usually the next premouse on the iteration tree in question.

Remark 2.15 Using the equivalence of Exclusion 2.6 one can verify that (assuming Exclusion 2.6) the initial segment condition is preserved under Skolem hulls and under ultrapowers by extenders which satisfy F1–F3. Without Exclusion 2.6 preserving 2.4(5), particularly under Skolem hulls, becomes more complicated (see [Jen98b] for details). Let us point out already here that the initial segment condition, 2.4(5), is essential later on. Without it Fact 3.28 below may fail. Fact 3.28 in turn is crucial to one of the key arguments of inner model theory, the demonstration that comparisons terminate.

For a structure $\mathcal{S}$, we use $\boldsymbol{\Sigma_1}(\mathcal{S})$ to denote the collection of subsets of $\mathcal{S}$ which are Σ_1 definable over $\mathcal{S}$ with parameters. More precisely, $X \in \Sigma_1(\mathcal{S})$ just in case that there exist $\vec{a} \in \mathcal{S}^{<\omega}$ and a Σ_1 formula φ so that $x \in X \iff \mathcal{S} \models \varphi[x, \vec{a}]$.

Definition 2.16 *Suppose F is an extender over a premouse $\mathcal{M}$. Let $\kappa = \mathrm{crit}(F)$ and let $\mathcal{N} = \mathrm{Ult}(\mathcal{M}, F)$. F is* **related** *to $\mathcal{M}$ just in case that*

1. $P(\kappa) \cap \mathcal{N} \subset P(\kappa) \cap \mathcal{M}$; and

[7] In this respect F3 is a necessary assumption: If $\mathrm{crit}(F) = \mathrm{lh}(E_\alpha)$ one may still attempt to form the ultrapower $\mathcal{N} = \mathrm{Ult}(\mathcal{M}, F)$ and ultrapower embedding π. Letting $E^* = \pi(E_\alpha)$ be the final extender predicate of $\mathcal{N}$, one can see easily that $\mathrm{lh}(E_\alpha) \in C_{E^*}$. In particular C_{E^*} contains additional elements other than $\mathrm{lh}(E^*)$, and may *fail* to satisfy the initial segment condition.

2. $P(\kappa) \cap \Sigma_1(\mathfrak{D}(\mathcal{N})) \subset P(\kappa) \cap \Sigma_1(\mathfrak{D}(\mathcal{M}))$.

Note that the converse inclusions to 1 and 2 hold automatically. Thus F is related to $\mathcal{M}$ iff equality holds in both. We shall use these equalities later in conjunctions with Remark 2.18 below.

Fact 2.17 *Suppose* $\mathcal{M}$ *and* $\mathcal{P} = \langle \mathcal{J}_{\beta,l}[\vec{F}], F_\beta \rangle$ *are premice, fix* $\eta \leq \beta$*, and suppose* $F_\eta \neq \emptyset$*. Let* $\kappa = \mathrm{crit}(F_\eta)$ *and let* τ *be the successor of* κ *in* $\mathcal{P} \parallel \langle \eta, 0 \rangle$*. Assume that (a)* F_η *is an extender over* $\mathcal{M}$*; (b)* $\mathcal{M}$ *and* $\mathcal{P} \parallel \langle \eta, 0 \rangle$ *agree up to* τ*; and (c)* $P(\tau) \cap \Sigma_1(\mathcal{P} \parallel \langle \eta, 0 \rangle) \subset \Sigma_1(\mathfrak{D}(\mathcal{M}))$.[8] *Then* F_η *is related to* $\mathcal{M}$.

Fact 2.17 is our main tool for establishing that some extenders are related to $\mathcal{M}$. Its proof is a direct computation using our definitions of Ult and Ult_0. See [Jen97, §1 Lemma 8] or the final part in the proof of [MitSt94, Lemma 4.5].

Remark 2.18 Suppose $\pi: \mathcal{M} \to \mathcal{N}$ is elementary. Let $\kappa = \mathrm{crit}(\pi)$. Suppose (a) $P(\kappa) \cap \mathcal{M} = P(\kappa) \cap \mathcal{N}$; (b) $P(\kappa) \cap \Sigma_1(\mathfrak{D}(\mathcal{M})) = P(\kappa) \cap \Sigma_1(\mathfrak{D}(\mathcal{N}))$; and (c) $\kappa \geq \rho(\mathcal{M})$. Then $(\mathrm{S})_\mathcal{M} \Rightarrow [(\mathrm{S})_\mathcal{N}$ and π is precise].

Remark 2.18 is a restatement to our context of [Jen97, §7 Lemma 2.1]. Conditions a–c by themselves suffice to establish that $\rho(\mathcal{N}) = \rho(\mathcal{M})$. Solidity is used to make the additional claim that $p(\mathcal{N}) = \pi(p(\mathcal{M}))$. We will only use Remark 2.18 in the special case where π is a composition of ultrapower embeddings. For the proof in this special case see [MitSt94, Lemmas 4.6,4.7].

This ends our introduction to fine structure. Readers familiar with Jensen's Σ^* theory should take comfort in the fact that our elementary embeddings are in practice almost always Σ^* elementary, and our precise embeddings *are* Σ^* elementary. Really the difference between our approach here and that of [Jen97] is in the *level of generality*. Our premice satisfy strong demands of soundness. The fine structure of [Jen97] applies in much more general settings to structures which do not satisfy these soundness demands and therefore are not, by our definition, premice. Readers familiar with [MitSt94] and [Ste96] should note that our approach here differs only linguistically. For example, a normal, non-overlapping, maximal iteration tree on a k–premouse $\mathcal{M} = \langle \mathcal{J}_{\alpha,k}[\vec{E}], E_\alpha \rangle$ is simply a normal, non-overlapping, k–maximal iteration tree on $\langle J_\alpha[\vec{E}], E_\alpha \rangle$ in the terminology of [MitSt94].

[Ste96] and [Jen97, §11] both include a description of the construction of a model known as K^c. The two accounts differ in the way they index extenders and in their concepts of elementary embeddings. We include below a rough description of this construction, adapted to our own context: Our indexing of extenders follows [Jen97], while our concept of elementarity is closer to that of [Ste96].

[8] This is similar to but slightly weaker than the definitions of **close** extenders in [MitSt94, Definition 4.4.1] and $\mathbf{\Sigma_1}$ **amenable** extenders in [Jen97, §1 pp.12].

Fix throughout a cardinal Ω. We shall later assume that Ω is measurable, but for the time being this is not necessary. We define a sequence of premice, $\langle \mathcal{M}_{\nu,k} \mid \nu < \Omega, k \leq \omega \rangle$. These premice will "converge" to a model $K^c = \mathcal{M}_{\Omega,0}$. For each ν and k, $\mathcal{M}_{\nu,k}$ will be a k–premouse, of height some $\alpha(\nu,k)$, and with predicates $\vec{E}^{\nu,k}, E^{\nu,k}$. We abbreviate this by saying that $\mathcal{M}_{\nu,k}$ has the form $\langle \mathcal{J}_{\alpha(\nu,k),k}[\vec{E}^{\nu,k}], E^{\nu,k}\rangle$.

For limit ν, define $\mathcal{M}_{<\nu}$ to be the *liminf* of the premice $\mathcal{M}_{\bar{\nu},\omega}$ for $\bar{\nu} < \nu$. $\mathcal{M}_{<\nu}$ then has the form $\langle \mathcal{J}_{\alpha(<\nu),0}[\vec{E}^{<\nu}], \emptyset\rangle$ where $\vec{E}^{<\nu}$ contains extenders $E^{<\nu}_{\gamma}$ for $\gamma < \alpha(<\nu)$. Each proper initial segment of $\mathcal{M}_{<\nu}$ is an initial segment of $\mathcal{M}_{\bar{\nu},\omega}$ for all sufficiently large $\bar{\nu} < \nu$; and $\mathcal{M}_{<\nu}$ is the longest premouse with this property. The fact that this is well defined is explained further in Remark 2.19.

Most important is the definition of $\mathcal{M}_{\nu,0}$ when ν is a limit ordinal. This is divided into two cases.

Case 1a: If there exists a unique extender F so that $\langle \mathcal{J}_{\alpha(<\nu),0}[\vec{E}^{<\nu}], F\rangle$ is a premouse and is furthermore *certifiable* (see Definition 2.20), then set $\mathcal{M}_{\nu,0} = \langle \mathcal{J}_{\alpha(<\nu),0}[\vec{E}^{<\nu}], F\rangle$ for this F.

Case 1b: If there are extenders $F_1 \neq F_2$ so that both $\langle \mathcal{J}_{\alpha(<\nu),0}[\vec{E}^{<\nu}], F_1\rangle$ and $\langle \mathcal{J}_{\alpha(<\nu),0}[\vec{E}^{<\nu}], F_2\rangle$ are certifiable, then pick your favorite such F_1, F_2 and set $\mathcal{M}_{\nu,0} = \langle \mathcal{J}_{\alpha(<\nu),0}[\vec{E}^{<\nu}], F_1, F_2\rangle$. If this happens the construction *ends* at $\nu, 0$, and $\mathcal{M}_{\nu,0}$ is a pre-*bicephalus* rather than a premouse (see [MitSt94, Definition 9.1.1] or [Jen97, §6]).[9] Given enough iterability it can be shown (Remark 3.4) that Case 1b in fact *never occurs.*

Case 2: Otherwise, set $\mathcal{M}_{\nu,0} = \langle \mathcal{J}_{\alpha(<\nu),0}[\vec{E}^{<\nu}], \emptyset\rangle$.

Having defined $\mathcal{M}_{\nu,k}$ we let $\mathcal{M}_{\nu,k+1}$ be the trivial extension of $\mathfrak{C}(\mathcal{M}_{\nu,k})$. Observe that then $\vartheta(\mathcal{M}_{\nu,k}) \geq \rho(\mathcal{M}_{\nu,k}) = \vartheta(\mathcal{M}_{\nu,k+1})$. It follows that for all sufficiently large $k < \omega$ we have $\vartheta(\mathcal{M}_{\nu,k}) = \rho(\mathcal{M}_{\nu,k})$ and hence trivially $\mathfrak{C}(\mathcal{M}_{\nu,k}) = \mathcal{M}_{\nu,k}$. Thus $\alpha(\nu,k)$, $\vec{E}^{\nu,k}$, and $E^{\nu,k}$ are *constant* for all sufficiently large k. We let $\mathcal{M}_{\nu,\omega} = \langle \mathcal{J}_{\alpha(\nu,k),\omega}[\vec{E}^{\nu,k}], E^{\nu,k}\rangle$ for some (any) $k < \omega$ large enough to stabilize these objects.

Remark 2.19 Using Remark 2.7, $\mathcal{M}_{\nu,k+1}$ and $\mathcal{M}_{\nu,k}$ agree up to the successor of $\rho(\mathcal{M}_{\nu,k})$ in $\mathcal{M}_{\nu,k}$. Inductively one can now verify (in the case of limit ν) that if $\kappa \leq \rho(\mathcal{M}_{\bar{\nu},\omega})$ for all sufficiently large $\bar{\nu} < \nu$, then the premice $\mathcal{M}_{\bar{\nu},\omega}||\langle \kappa^+, 0\rangle$ (where κ^+ is computed inside $\mathcal{M}_{\bar{\nu},\omega}$) are *increasing* (in the initial segment order) as $\bar{\nu} \to \nu$, for $\bar{\nu}$ large enough. Their common values up to their κ^+-s are then put as initial segments of $\mathcal{M}_{<\nu}$, and all initial segments of $\mathcal{M}_{<\nu}$ are of this form.

In the successor case we simply add one more level of constructibility. Thus $\mathcal{M}_{\nu+1,0} = \langle \mathcal{J}_{\alpha(\nu,\omega)+1,0}[\vec{E}^{\nu+1,0}], \emptyset\rangle$ where $\vec{E}^{\nu+1,0} = \vec{E}^{\nu,\omega}{}^\frown E^{\nu,\omega}$. The construction starts with the premouse $\mathcal{M}_{1,0} = \langle \mathcal{J}_{1,0}[\emptyset], \emptyset\rangle$.

[9] Bicephali are slightly simpler in the context of Jensen's indexing, since there is no need to distinguish types.

Definition 2.20 *Let $\mathcal{M} = \langle \mathcal{J}_{\alpha,0}[\vec{E}], F\rangle$ be a premouse such that $F \neq \emptyset$. Let $\kappa = \mathrm{crit}(F)$. $\langle N, G\rangle$ is a* **certificate** *for $\mathcal{M}$ iff*

1. *N is a transitive ZFC^- model and $\mathrm{V}_\kappa \in N$;*
2. *G is an extender on N with critical point κ. Note that G need* not *measure all subsets of κ in V, but only the ones in N;*
3. *Let $\tilde{N} = \mathrm{Ult}(N, G)$. Then $\mathrm{V}_{\lambda+1} \subset \tilde{N}$, where $\lambda = F(\kappa)$;*
4. *$F(X) = G(X) \cap \lambda$ for $X \in P(\kappa) \cap N \cap \mathcal{M}$.*

$\mathcal{M}$ is **certifiable** *iff (in V) for every $A \subset \kappa$ there exists a certificate $\langle N, G\rangle$ such that $A \in N$.*

Definition 2.20 is taken from [Jen97, §11]. Note that condition 4 above is stronger than the parallel condition, Definition 1.1(b), of [Ste96]. [Ste96] only requires $F(X) \cap \nu = G(X) \cap \nu$ where ν is the supremum of generators of F. This strengthening is essential to handling premice indexed according to Jensen.

During the construction we have made constant use of the assumptions (U) and (S). To be more precise, following stage $\langle \nu, k\rangle$ we appeal to $(\mathrm{U})_{\mathcal{M}_{\nu,k}}$ to secure the soundness of $\mathfrak{C}(\mathcal{M}_{\nu,k})$ (see Remark 2.7), and then appeal to $(\mathrm{S})_{\mathfrak{C}(\mathcal{M}_{\nu,k})}$ to secure solidity, so that we may take the trivial extension of $\mathfrak{C}(\mathcal{M}_{\nu,k})$. We again use $(U)_{\mathcal{M}_{\nu,k}}$ in Remark 2.19. For $k = 0$ we assumed further that Case 1b didn't occur $(\neg 1b)_{\nu,k}$, so that we don't have to end the construction at a stage before Ω. Officially the construction of K^c is an induction during which we prove $(\mathrm{U})_{\mathcal{M}_{\nu,k}}$, $(\mathrm{S})_{\mathfrak{C}(\mathcal{M}_{\nu,k})}$, and $(\neg 1b)_{\nu,k}$ once $\mathcal{M}_{\nu,k}$ has been constructed. A substantial step in this direction is given by the following Theorem of Mitchell–Steel:

Theorem 2.21 *Assume $\mathcal{M}_{\nu,k}$ has been constructed, and the following statement holds:*

$(\dagger)_{\nu,k}$ *All countable elementary substructures $\bar{\mathcal{M}}$ of $\mathcal{M}_{\nu,k}$ are $\omega_1 + 1$ iterable (for normal, maximal, non-overlapping iteration trees, see below). Moreover, for any given enumeration $\vec{e}$ of $\bar{\mathcal{M}}$ there exists an iteration strategy which has the weak Dodd–Jensen property relative to $\vec{e}$.*

Then $(\mathrm{U})_{\mathcal{M}_{\nu,k}}$, $(\mathrm{S})_{\mathfrak{C}(\mathcal{M}_{\nu,k})}$, and $(\neg 1b)_{\nu,k}$ are true.

The reader may consult [NS99] for an explanation of the weak Dodd–Jensen property and some remarks on the proof of Theorem 2.21, which traces back to [MitSt94, §8]. Let us remind the reader that an iteration tree is said to be **normal** if the indices of extenders it uses are increasing. It is **non-overlapping** if extenders are always applied to the earliest possible model. An iteration tree is **maximal** if extenders are always applied to the largest possible initial segment. More precisely, if ζ is the $<^{\mathcal{T}}$ predecessor of $\epsilon+1$ then $E^{\mathcal{T}}_\epsilon$ is applied to some $\mathcal{N}^*_{\epsilon+1}$

which is an initial segment of $\mathcal{N}_\zeta^\mathcal{T}$. In the case of a maximal tree one requires that $\mathcal{N}_{\epsilon+1}^*$ be the *largest* initial segment over which $E_\epsilon^\mathcal{T}$ is an extender. Thus $\mathcal{N}_{\epsilon+1}^*$ is either equal to $\mathcal{N}_\zeta^\mathcal{T}$ or else $\rho(\mathcal{N}_{\epsilon+1}^*) \leq \mathrm{crit}(E_\epsilon^\mathcal{T})$ [10] and there is a subset of $\mathrm{crit}(E_\epsilon^\mathcal{T})$, definable over $\mathcal{N}_{\epsilon+1}^*$, which is not measured by $E_\epsilon^\mathcal{T}$.

Theorem 2.21 reduces $(\mathrm{U}, \mathrm{S}, \neg 1b)$ to $(\dagger)$ but unfortunately no general proof of $(\dagger)$ is known. Much of the current research in inner model theory attempts to obtain increasingly more general proofs of $(\dagger)$ and our paper is another step along this line. We shall prove ultimately that $(\dagger)_{\nu,k}$ is true assuming that $\mathcal{M}_{\nu,k}$ is *domestic*. As is usual with iterability proofs, our proof relies heavily on a fundamental iterability theorem which produces maximal branches through iteration trees. Several concepts are needed before we can formulate this theorem precisely. These concepts (resurrections, easy phalanges, sturdy iteration trees, and realizability) are defined below, and are followed by the fundamental iterability theorem (2.28). This theorem is essentially taken from [Ste96]. [Ste96] relies partly on the iterability proof of [MitSt94] which in turn draws on the results of [MarSt94].

Through the construction we define **resurrections** which trace initial segments of our current stage back to the stage where they appeared in the construction. The resurrection at stage $\langle \nu, k\rangle$ is a pair $Res_{\nu,k}, \sigma_{\nu,k}$. Both are functions which are defined on all initial segments of $\mathcal{M}_{\nu,k}$. For any such initial segment $\mathcal{N}$, $Res_{\nu,k}[\mathcal{N}]$ is a stage $\langle \eta, l\rangle \leq_{\mathrm{Lex}} \langle \nu, k\rangle$, and $\sigma_{\nu,k}[\mathcal{N}]$ is an elementary embedding of $\mathcal{N}$ into $\mathcal{M}_{\eta,l}$.

The resurrection is defined inductively over the lexicographic order for pairs $\langle \nu, k\rangle$. Let us start with the case of $\langle \nu, 0\rangle$ for limit ν. If $\mathcal{N}$ is a strict initial segment of $\mathcal{M}_{\nu,0}$ we let $Res_{\nu,0}[\mathcal{N}]$ and $\sigma_{\nu,0}[\mathcal{N}]$ be the eventual values of $Res_{\bar\nu,\omega}[\mathcal{N}]$ and $\sigma_{\bar\nu,\omega}[\mathcal{N}]$ as $\bar\nu \to \nu$. (That this makes sense follows from our definition of the resurrection in the successor cases, particularly case B.) If $\mathcal{N} = \mathcal{M}_{\nu,0}$ then $Res_{\nu,0}[\mathcal{N}] = \langle \nu, 0\rangle$ and $\sigma_{\nu,0}[\mathcal{N}]$ is the identity embedding.

Next, let us define $Res_{\nu,k+1}$ and $\sigma_{\nu,k+1}$ assuming that these are defined for $\langle \nu, k\rangle$. We distinguish three cases:

A. If $\mathcal{N} = \mathcal{M}_{\nu,k+1}$: we let $Res_{\nu,k+1}[\mathcal{N}] = \langle \nu, k+1\rangle$ and let $\sigma_{\nu,k+1}[\mathcal{N}]$ be the identity embedding.

B. If $\mathcal{N}$ is an initial segment of $\mathcal{M}_{\nu,k+1}$, cut below the successor of $\rho(\mathcal{M}_{\nu,k})$ in $\mathcal{M}_{\nu,k}$: By Remark 2.19 $\mathcal{M}_{\nu,k+1}$ and $\mathcal{M}_{\nu,k}$ agree up to this successor, so that $\mathcal{N}$ is an initial segment of $\mathcal{M}_{\nu,k}$. We let $Res_{\nu,k+1}[\mathcal{N}] = Res_{\nu,k}[\mathcal{N}]$ and $\sigma_{\nu,k+1}[\mathcal{N}] = \sigma_{\nu,k}[\mathcal{N}]$.

C. Otherwise: Since case A fails, $\mathcal{N}$ is an initial segment of the immediate truncation of $\mathcal{M}_{\nu,k+1}$, which by definition is $\mathfrak{C}(\mathcal{M}_{\nu,k})$. Let $\pi\colon \mathfrak{C}(\mathcal{M}_{\nu,k}) \to \mathcal{M}_{\nu,k}$ be the anti-core embedding. Let $\tilde{\mathcal{N}} = \pi(\mathcal{N})$. By Fact 2.13, $\pi \restriction \mathcal{N}\colon \mathcal{N} \to \tilde{\mathcal{N}}$ is

[10] So that $E_\epsilon^\mathcal{T}$ is no longer an extender over the trivial extension of $\mathcal{N}_{\epsilon+1}^*$, because of F2.

elementary. $\tilde{\mathcal{N}}$ is an initial segment of $\mathcal{M}_{\nu,k}$ and so $Res_{\nu,k}[\tilde{\mathcal{N}}]$ and $\sigma_{\nu,k}[\tilde{\mathcal{N}}]$ are defined. Let $Res_{\nu,k+1}[\mathcal{N}] = Res_{\nu,k}[\tilde{\mathcal{N}}]$ and let $\sigma_{\nu,k+1}[\mathcal{N}] = \sigma_{\nu,k}[\tilde{\mathcal{N}}] \circ \pi$. This last definition produces an elementary embedding by Fact 2.12.

In the case of the stage $\langle \nu, \omega \rangle$: If $\mathcal{N} = \mathcal{M}_{\nu,\omega}$ we let $Res_{\nu,\omega}[\mathcal{N}] = \langle \nu, \omega \rangle$ and let $\sigma_{\nu,\omega}[\mathcal{N}]$ be the identity embedding. Otherwise: Our construction ensures the existence of j large enough that $\mathcal{N}$ is an initial segment of $\mathcal{M}_{\nu,j}$ and $\rho(\mathcal{M}_{\nu,k}) = \vartheta(\mathcal{M}_{\nu,k})$ for all $k \geq j$. We let $Res_{\nu,\omega}[\mathcal{N}] = Res_{\nu,k}[\mathcal{N}]$ and $\sigma_{\nu,\omega}[\mathcal{N}] = \sigma_{\nu,k}[\mathcal{N}]$ for some/any such j. (It doesn't matter which of these j we pick; note that by choice of j, $k \geq j \Rightarrow$ the anti-core embedding from $\mathcal{M}_{\nu,k+1}$ into $\mathcal{M}_{\nu,k}$ is the identity.)

Finally we must consider the stage $\langle \nu + 1, 0 \rangle$. Again if $\mathcal{N} = \mathcal{M}_{\nu+1,0}$ we let $Res_{\nu+1,0}[\mathcal{N}] = \langle \nu + 1, 0 \rangle$ and $\sigma_{\nu+1,0}[\mathcal{N}]$ be the identity embedding. Otherwise, $\mathcal{N}$ is an initial segment of $\mathcal{M}_{\nu,\omega}$ and we let $Res_{\nu+1,0}[\mathcal{N}] = Res_{\nu,\omega}[\mathcal{N}]$ and $\sigma_{\nu+1,0}[\mathcal{N}] = \sigma_{\nu,\omega}[\mathcal{N}]$.

In the interest of saving ink, we shall from now on use ν (and occasionally η) to range not over ordinals but over pairs $\langle \nu_0, \nu_1 \rangle$ of ordinals such that $\nu_1 \leq \omega$. Thus the stages in the construction of K^c will be denoted as $\mathcal{M}_\nu$. By $\eta < \nu$ we shall mean $\langle \eta_0, \eta_1 \rangle <_{\text{Lex}} \langle \nu_0, \nu_1 \rangle$.

Definition 2.22 *An* **easy phalanx** *(of length $n+1$) is a finite sequence of premice $\langle \mathcal{N}_i \mid i \leq n \rangle$ such that the following conditions hold:*

1. *For $i < n$, $\mathfrak{C}(\mathcal{N}_{i+1})$ is a strict initial segment of $\mathcal{N}_i$;*
2. *For $i < j < n$, $\rho(\mathcal{N}_{i+1}) < \rho(\mathcal{N}_{j+1})$; and*
3. *For $i < n$, $p(\mathcal{N}_{i+1})$ is solid.*

We shall use $\bar{\mathcal{N}}_i$ to denote $\mathfrak{C}(\mathcal{N}_{i+1})$, which by 1 is an initial segment of $\mathcal{N}_i$.

For notational convenience we shall often index the premice of an easy phalanx by transfinite ordinals rather than natural numbers. Thus we shall talk of the easy phalanx consisting of $\mathcal{N}_\zeta$ for $\zeta \in I$, where I is some finite *set of ordinals. We continue to require that the core of each premouse in the phalanx is an initial segment of its predecessor in the phalanx, etc.*

Given an easy phalanx, set $\lambda_i = \rho(\mathcal{N}_{i+1}) = \rho(\bar{\mathcal{N}}_i)$. The λ_i-s are then increasing, and since $\bar{\mathcal{N}}_i$ embeds into $\mathcal{N}_{i+1}$ via an embedding whose critical point is at least λ_i one sees that $\mathcal{N}_i$ and $\mathcal{N}_{i+1}$ *agree* up to λ_i. Thus the pair of sequences $\langle \mathcal{N}_i \mid i \leq n \rangle$, $\langle \lambda_i \mid i < n \rangle$ is a phalanx in the usual sense (see for example [Ste96, Definition 6.5]). We shall refer to this phalanx as $\vec{\mathcal{N}}$. Iteration trees on easy phalanges are formed as always, using the λ_i-s as exchange ordinals: The first extender used, $E_n^{\mathcal{T}}$, must have length greater than λ_{n-1}; and if $E_\epsilon^{\mathcal{T}}$ — the ϵ-th extender used — has critical point smaller than λ_i then the $<^{\mathcal{T}}$ predecessor of $\epsilon + 1$ is allowed to be i. (Observe in this case that $\mathcal{N}_\epsilon^{\mathcal{T}}$ and $\bar{\mathcal{N}}_i$ have the same subsets of $\text{crit}(E_\epsilon^{\mathcal{T}})$. Of course $\mathcal{N}_i$ may have more subsets of $\text{crit}(E_\epsilon^{\mathcal{T}})$, in which case $E_\epsilon^{\mathcal{T}}$ is applied to a strict initial segment of $\mathcal{N}_i$.) We shall refer to the premice of $\vec{\mathcal{N}}$ and to their indices as **roots** of $\mathcal{T}$, and will say for example that m is the root of ξ if $m \leq n$ and the branch of $\mathcal{T}$ which leads to $\mathcal{N}_\xi^\mathcal{T}$ starts with m. We shall follow similar terminology when indexing the premice of $\vec{\mathcal{N}}$ by transfinite ordinals. Those ζ-s so that $\mathcal{N}_\zeta$ is a premouse of $\vec{\mathcal{N}}$ are referred to as **roots** of $\mathcal{T}$. Other premice of $\mathcal{T}$ are indexed starting from $\xi + 1$, where ξ is the largest root. We shall refer to the smallest root of $\mathcal{T}$ as the **primordial root**, denoted $\mathrm{pr}_\mathcal{T}$.

Let $\mathcal{T}$ be an iteration tree on an easy phalanx $\vec{\mathcal{N}}$ of length $n+1$. Let b be a branch of $\mathcal{T}$, either leading to a premouse on $\mathcal{T}$ or else leading to a wellfounded direct limit which we refer to as the last premouse of b. Consider some $\epsilon + 1 \in b$ and let $\zeta \in b$ be the $<^\mathcal{T}$ predecessor of $\epsilon + 1$. $\mathcal{N}_{\epsilon+1}^\mathcal{T}$ is an ultrapower either of $\mathcal{N}_\zeta^\mathcal{T}$, or else of $\mathcal{N}_{\epsilon+1}^*$ which is a strict initial segment of $\mathcal{N}_\zeta^\mathcal{T}$. If the latter is true we say that ζ is a **truncation** point of b.[11] For notational convenience we shall denote $\mathcal{N}_{\epsilon+1}^*$ by $\mathcal{N}_\zeta^-$ when ζ is a truncation point of b and $\epsilon+1$ is the successor of ζ in b. This is an abuse of notation, since $\mathcal{N}_\zeta^-$ depends on the branch b to determine ϵ. However b will generally be clear from the context. At any rate $\mathcal{N}_\zeta^-$ is then a strict initial segment of $\mathcal{N}_\zeta^\mathcal{T}$. For uniformity we define $\mathcal{N}_{\epsilon+1}^*$ and $\mathcal{N}_\zeta^-$ also if ζ is not a truncation point. In this case $\mathcal{N}_{\epsilon+1}^*$ and $\mathcal{N}_\zeta^-$ equal $\mathcal{N}_\zeta^\mathcal{T}$. Thus if $\zeta <^\mathcal{T} \xi$ and there are no truncations on the segment $(\zeta, \xi)_\mathcal{T}$ then $\mathcal{T}$ defines an embedding $i_{\zeta,\xi}^\mathcal{T} : \mathcal{N}_\zeta^- \to \mathcal{N}_\xi^\mathcal{T}$. We would like under certain circumstances to argue that this embedding is *precise* (Lemma 2.25). We shall do this by securing the conditions of Remark 2.18.

Definition 2.23 *($\mathcal{T}$ an iteration tree on an easy phalanx $\vec{\mathcal{N}}$.) $\gamma < \mathrm{lh}(\mathcal{T})$ is* **simple** *if (a) the root of γ is primordial; and (b) there are no truncations on the branch $[\mathrm{pr}_\mathcal{T}, \gamma)_\mathcal{T}$. Otherwise γ is* **non-simple**.

To motivate Definition 2.23, let us point out that our easy phalanges will always be induced by some branch through a past iteration tree. (Definition 2.26 makes this precise.) The models of $\vec{\mathcal{N}}$, except for $\mathcal{N}_{\mathrm{pr}}$, will correspond to failures of 2.23(b) on this past iteration tree. Thus in defining simplicity it is natural to demand both 2.23(a) and 2.23(b).

Definition 2.24 *An iteration tree $\mathcal{T}$ on an easy phalanx $\vec{\mathcal{N}}$ is* **sturdy** *if:*

1. *$\mathcal{T}$ is normal;*
2. *$\mathcal{T}$ is maximal;*
3. *For $\epsilon + 1 < \mathrm{lh}(\mathcal{T})$ and γ the immediate $<^\mathcal{T}$ predecessor of $\epsilon+1$, $E_\epsilon^\mathcal{T}$ is related to $\mathcal{N}_\gamma^- = \mathcal{N}_{\epsilon+1}^*$;*

[11] The corresponding terminology in [MitSt94] is **drops**. Note that [MitSt94] would say the drop occurs at $\epsilon+1$, rather than ζ. [MitSt94] also distinguishes between **proper drops**, where $\alpha(\mathcal{N}_{\epsilon+1}^*) < \alpha(\mathcal{N}_\zeta)$, and **drops in degree**, where $\alpha(\mathcal{N}_{\epsilon+1}^*) = \alpha(\mathcal{N}_\zeta)$ and $k(\mathcal{N}_{\epsilon+1}^*) < k(\mathcal{N}_\zeta)$. Our terminology does not make this distinction.

4. (ϵ, γ as above) if γ is non-simple then $\mathrm{crit}(E^{\mathcal{T}}_{\epsilon}) \geq \rho(\mathcal{N}^*_{\epsilon+1})$.

Note that if γ is a truncation point on the branch to $\epsilon + 1$ then $\mathrm{crit}(E_\epsilon) \geq \rho(\mathcal{N}^-_\gamma)$ follows by the maximality of $\mathcal{T}$. Thus 2.24(4) can only place an additional constraint when γ is not a truncation point. The additional constraint in this case is simply $\mathrm{crit}(E_\epsilon) \geq \rho(\mathcal{N}^{\mathcal{T}}_\gamma)$.

Sturdiness is exactly what we need to prove Lemma 2.25 below. This Lemma is needed later; case (b) in particular is essential for Claim 3.32. We therefore restrict our attention throughout this paper to sturdy iteration trees. The trees we really care about (for the sake of Theorem 2.21) are normal, maximal, and non-overlapping. We shall verify later (Claim 3.8 and Lemma 3.9) that such trees are sturdy.

Lemma 2.25 *Suppose $\mathcal{T}$ is a sturdy iteration tree on $\vec{\mathcal{N}}$. Let $\zeta <^{\mathcal{T}} \xi$ be such that there are no truncations on the segment $(\zeta, \xi)_{\mathcal{T}}$. Assume further that either (a) ζ is non-simple; or (b) ζ is a truncation point on the branch leading to ξ.*

Then the embedding $i^{\mathcal{T}}_{\zeta,\xi}\colon \mathcal{N}^-_\zeta \to \mathcal{N}^{\mathcal{T}}_\xi$ is precise, and $p(\mathcal{N}^{\mathcal{T}}_\xi)$ is solid. In case (b) moreover $\mathfrak{C}(\mathcal{N}^{\mathcal{T}}_\xi) = \mathcal{N}^-_\zeta$ and $i^{\mathcal{T}}_{\zeta,\xi}$ is exactly equal to the anti-core embedding.

Proof (Sketch). Precision and solidity follows from Remark 2.18, by induction on ξ. Conditions (a,b) of Remark 2.18 are given automatically by 2.24(3). To apply 2.18 one has to verify further that all extenders used on $(\zeta, \xi]_{\mathcal{T}}$ have critical points at least $\rho(\mathcal{N}^-_\zeta)$, and that $p(\mathcal{N}^-_\zeta)$ is solid. If ζ is a non-primordial root then solidity is given by condition 3 of Definition 2.22. If ζ is a truncation point then solidity is given by 2.4(4) since $\mathcal{N}^-_\zeta$ is a strict initial segment of a premouse. In all other cases solidity follows by induction. Also by induction one verifies that $\rho(\mathcal{N}^{\mathcal{T}}_\gamma) = \rho(\mathcal{N}^-_\zeta)$ for γ in $(\zeta, \xi)_{\mathcal{T}}$. This together with 2.24(2,4) allows verifying that all extender used in $(\zeta, \xi]_{\mathcal{T}}$ have critical point at least $\rho(\mathcal{N}^-_\zeta)$, as needed. In case (b) we know also that $\mathcal{N}^-_\zeta$ is *sound*, since it is a strict initial segment of a premouse. Applying Remark 2.11 then completes the proof of 2.25. $\square$

Lemma 2.25 is perhaps the most important application of the methods of fine structure to future results in this paper. It allows us under certain circumstances to pinpoint precisely an iteration tree embedding.

Definition 2.26 *($\mathcal{T}$ a sturdy iteration on an easy phalanx $\vec{\mathcal{N}}$, b a branch through $\mathcal{T}$ either leading to a premouse of $\mathcal{T}$ or to a wellfounded direct limit.) The* **easy phalanx induced by the branch b** *consists of the premice $\mathcal{N}_0, \cdots, \mathcal{N}_{m-1}$, where m is the root of b; followed in increasing order by the premice $\mathcal{N}_\zeta$ for $m \leq \zeta < \mathrm{lh}(b)$ a truncation point of b; and ending with the last premouse of b. We denote this phalanx by $\vec{\mathcal{N}}^b$.*

The reader may note that the premice of $\vec{\mathcal{N}}^b$ are indexed by (possibly) *transfinite* ordinals. $\vec{\mathcal{N}}^b$ is indeed an easy phalanx. The conditions of Definition 2.22 can be verified using (among other things) Lemma 2.25.

Definition 2.27 *A* **realization** *of an easy phalanx $\vec{\mathcal{N}}$ consists of a lexicographically decreasing sequence of stages $\langle \nu_i \mid i \leq n\rangle$ together with embeddings $\langle \pi_i \mid i \leq n\rangle$ so that the following conditions hold:*

1. *For $i \leq n$, $\pi_i : \mathcal{N}_i \to \mathcal{M}_{\nu_i}$ is a weak embedding;*
2. *For $i < n$, $\nu_{i+1} = Res_{\nu_i}[\pi_i(\bar{\mathcal{N}}_i)]$;*[12] *and*
3. *For $i < n$, the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{M}_{\nu_i} & \\
\vee & \\
\pi_i(\bar{\mathcal{N}}_i) \arrow[r, "\sigma_{\nu_i}[\pi_i(\bar{\mathcal{N}}_i)]"] & \mathcal{M}_{\nu_{i+1}} \\
\bar{\mathcal{N}}_i \arrow[u, "\pi_i\restriction\bar{\mathcal{N}}_i"] \arrow[r, "a.c."] & \mathcal{N}_{i+1} \arrow[u, "\pi_{i+1}"']
\end{tikzcd}$$

   a.c. denotes the anti-core embedding and $\vee$ illustrates the fact that $\pi_i(\bar{\mathcal{N}}_i)$ is a strict initial segment of $\mathcal{M}_{\nu_i}$.

We shall refer to such a realization as $\vec{\nu}, \vec{\pi}$.

Theorem 2.28 *Let $\vec{\mathcal{N}}$ be an easy phalanx, all of whose premice are countable, and let $\vec{\nu}, \vec{\pi}$ be a realization of $\vec{\mathcal{N}}$. (We implicitly assume that $\mathcal{M}_{\nu_0}$ has been constructed.) Let $\mathcal{T}$ be a sturdy iteration tree of countable length on $\vec{\mathcal{N}}$.*

Then there exists a maximal *branch b through $\mathcal{T}$ and a realization $\vec{\nu}^b, \vec{\pi}^b$ of the easy phalanx $\vec{\mathcal{N}}^b$ which satisfies the following:*

1. *$\vec{\pi}^b \restriction m = \vec{\pi} \restriction m$ and $\vec{\nu}^b \restriction m = \vec{\nu} \restriction m$, where m is the root of b; and*
2. *Let ζ be the first truncation point of b, or if there aren't any let ζ stand for b. (In either case $\mathcal{N}^{\mathcal{T}}_\zeta$ is the first premouse of $\vec{\mathcal{N}}^b$ following $\mathcal{N}_{m-1}$.*[13]*) Note that the iteration tree $\mathcal{T}$ defines an elementary embedding $i^{\mathcal{T}}_{m,\zeta}$. We require that $\nu^b_\zeta = \nu_m$ and that the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{M}_{\nu_m} & \\
\mathcal{N}_m \arrow[u, "\pi_m"] \arrow[r, "i^{\mathcal{T}}_{m,\zeta}"] & \mathcal{N}^{\mathcal{T}}_\zeta \arrow[ul, "\pi^b_\zeta"']
\end{tikzcd}$$

[12] We remind the reader that $\bar{\mathcal{N}}_i = \mathfrak{C}(\mathcal{N}_{i+1})$ is a strict initial segment of $\mathcal{N}_i$.
[13] This may be $\mathcal{N}_m$ if m is a truncation point of b.

Let us remark that strong forms of commutativity hold also between any truncation points of b, not just between m and the first truncation point. This commutativity follows from our requirements in Definition 2.27 and from Lemma 2.25 (which relies on the assumption that $\mathcal{T}$ is sturdy). A version of Theorem 2.28 can be stated for trees which are not sturdy. In this case one has to make allowance for the fact that the embeddings between truncation points of $\mathcal{T}$ are not always anti-core embeddings, and consider phalanges where the embeddings of $\bar{\mathcal{N}}_i$ into $\mathcal{N}_{i+1}$ are not necessarily anti-core embeddings. This has mainly the effect of complicating notation. Since anyway we only care about sturdy trees, we shall not go into this greater generality.

Definition 2.29 *In the situation described by conditions 1,2 of Theorem 2.28 we say that $\vec{\nu}, \vec{\pi}$ and $\vec{\nu}^b, \vec{\pi}^b$* **commute***. We refer to $\vec{\nu} \restriction m+1$ as the* **meet** *of the two realizations. (This is a slight abuse of notation since m depends also on b.) A branch b for which there exist $\vec{\nu}^b$ and $\vec{\pi}^b$ which commute with $\vec{\nu}, \vec{\pi}$ is called* **super-realizable** *(wrt $\vec{\nu}, \vec{\pi}$).*

Note that by 2.28(1,2), the meet $\vec{\nu} \restriction m+1$ is an initial segment of $\vec{\nu}^b$: $\nu_i = \nu_i^b$ for $i < m$ and ν_m is equal to ν_ζ^b where ζ is either the first truncation on b, or the final model of b if there are no truncations.

Remark 2.30 Theorem 2.28 applies also to trees of length ω_1, except that then the branch b is found not in V but in V$[G]$ where G is generic over V for $\mathrm{col}(\omega, \omega_1)$.

Theorem 2.28 is essentially a reformulation of [Ste96, Theorem 9.14], but with several differences. First, Theorem 2.28 is stated in the context of Jensen's indexing, and its proof requires use of condition 4 in Definition 2.20 rather than the parallel [Ste96, Definition 1.1(b)]. The reader interested in the modifications which must as a result be made to the proof of [Ste96, Theorem 9.14] may find those in [Jen98a, §3]. Secondly, [Ste96, Theorem 9.14] is stated for arbitrary phalanges, while Theorem 2.28 is more restrictive. With respect to this second difference our Theorem is easier. Thirdly, our weak embeddings are weaker than those of [Ste96]. The proof of [Ste96, Theorem 9.14] adapts with no serious modifications. Let us only point out that the property of weak embeddings described in Remark 2.14 is essential to the proof. The reader familiar with copying constructions may get a hint of the importance of Remark 2.14 by noting its use in copying. Finally, Theorem 2.28 places *stronger* demands on the branch b than do [Ste96, Theorem 9.14] and [Jen98a, §3]. The extra strength is in the case that there are truncations along b. Standard realizability demands would state only that the *final* premouse of b embeds into the construction. Our demands in Theorem 2.28 state not only that the final premouse embeds into the construction, but that every premouse on the branch which stands at a truncation point along b embeds into the construction. We further demands that these embeddings cohere in the way stated in Definition 2.27. With respect to this last difference the proof of [Ste96, Theorem 9.14] must of course be strengthened. In the proof of [Ste96, Theorem 9.14] Steel defines coarse ZFC^- models $\mathcal{R}_\beta$ for $\beta < \sup(\mathcal{T})$. Inside each $\mathcal{R}_\beta$ he maintains a $\mathcal{Q}_\beta$, which is a stage of the K^c construction relativized to $\mathcal{R}_\beta$, and an embedding π_β of $\mathcal{N}_\beta^\mathcal{T}$ into $\mathcal{Q}_\beta$. To prove Theorem 2.28 one must instead maintain in $\mathcal{R}_\beta$ a realization (in the sense of Definition 2.27) of the entire easy phalanx induces by the branch of $\mathcal{T}$ leading to β. Similarly the "tree of attempts" which Steel defines must be revised to search for a branch b and a realization of the entire easy phalanx induced by b, not just the last premouse. Checking that this modification can be carried through is a simple matter of commuting all the relevant diagrams. The careful reader will no doubt wish to do this herself.[14]

Remark 2.31 We should point out that the proof of Theorem 2.28 does not produce elementary embeddings. More precisely, even if one started with a realization $\vec{\nu}, \vec{\pi}$ whose embeddings are elementary, the commuting realization $\vec{\nu}^b, \vec{\pi}^b$ given by Theorem 2.28 may include embeddings which are just weak.

3 Domestic premice

This Section centers on the proof of Theorem 3.2, stated below. Once proved, Theorem 3.2 immediately implies that (†) holds for the domestic levels of K^c (Corollary 3.3).

Definition 3.1 *A premouse $\mathcal{N} = \langle \mathcal{J}_{\alpha,k}[\vec{E}], E_\alpha \rangle$ is said to be* **domestic** *if there does* not *exist $\nu \leq \alpha$ which indexes an extender E_ν satisfying:*

1. *$\kappa = \mathrm{crit}(E_\nu)$ is a limit of cardinals δ so that $J_\nu[\vec{E} \restriction \nu] \models$ "δ is a Woodin cardinal;" and*
2. *$\kappa = \mathrm{crit}(E_\nu)$ is a limit of cardinals τ so that $J_\nu[\vec{E} \restriction \nu] \models$ "τ is strong to κ."*

Being domestic is a Π_1 property and is therefore preserved by elementary embeddings.

Theorem 3.2 *Let $\mathcal{N}$ be a countable premouse which embeds weakly into an existing level of the K^c construction. Assume that $\mathcal{N}$ is domestic. Let ν be the least stage such that $\mathcal{N}$ embeds weakly into $\mathcal{M}_\nu$ and let $\pi: \mathcal{N} \to \mathcal{M}_\nu$ be the left most weak embedding (wrt a fixed enumeration $\vec{e} = \langle e_i \mid i < \omega \rangle$ of $\mathcal{N}$).*[15]

Then for any normal, maximal, non-overlapping iteration tree $\mathcal{T}$ on $\mathcal{N}$ of countable length, there exists at most one *cofinal branch of $\mathcal{T}$ which is super-realizable (wrt ν, π).*

[14] To help the careful reader, let us point out that the proof of Theorem 2.28 starts with a single coarse model $\mathcal{R}_0 = \mathrm{V}$ and the realization of $\vec{\mathcal{N}}$ given by $\vec{\nu}, \vec{\pi}$, *not* with n distinct $\mathcal{R}_i$-s and individual realization of the premice $\mathcal{N}_i$.

[15] By **left most** we mean that for any weak $\sigma: \mathcal{N} \to \mathcal{M}_\nu$, either $\sigma = \pi$ or else $\pi(e_i)$ is less than $\sigma(e_i)$ in the order of construction on $\mathcal{M}_\nu$, where $i < \omega$ is least such that $\pi(e_i) \neq \sigma(e_i)$.

Corollary 3.3 *Suppose that $\mathcal{M}_\nu$ has been constructed and that $\mathcal{M}_\nu$ is domestic. Let $\bar{\mathcal{M}}$ be a countable elementary substructure of $\mathcal{M}_\nu$. Then $\bar{\mathcal{M}}$ is $\omega_1 + 1$ iterable.*[16] *Furthermore, if $\vec{e}$ is an enumeration of $\bar{\mathcal{M}}$ then there exists an iteration strategy of $\bar{\mathcal{M}}$ which has the weak Dodd–Jensen property relative to $\vec{e}$.*

Proof. Reducing ν as needed let us assume it is the least stage such that $\bar{\mathcal{M}}$ embeds weakly into $\mathcal{M}_\nu$. Let π be the left most weak embedding. Define an iteration strategy Γ for $\bar{\mathcal{M}}$ by letting $\Gamma(\mathcal{T})$ be the unique maximal super-realizable branch through $\mathcal{T}$. (Existence and uniqueness are given by Theorems 2.28 and 3.2.) If $\mathcal{T}$ is itself built according Γ then Theorems 3.2 applied to initial segments of $\mathcal{T}$ guarantees that the maximal branch is in fact cofinal, as needed. If $\mathrm{lh}(\mathcal{T}) = \omega_1$, collapse ω_1 and obtain $\Gamma(\mathcal{T})$ in the generic extension. Theorem 3.2 continues to apply in the generic extension. The uniqueness given by 3.2 and the homogeneity of the collapse imply that this branch in fact exists in V.

It can be seen that Γ has the weak Dodd–Jensen property relative to $\vec{e}$. The reason is that ν was chosen least and π was chosen left most. We refer the reader to [NS99, Section 3]. $\square$

Corollary 3.3 establishes (†) for the domestic levels of K^c, demonstrating that the construction of K^c cannot break down before it reaches a non-domestic premouse. Assuming that Ω is measurable we can now apply Theorem 1.1 and so deduce Corollary 1.2.

Remark 3.4 Theorem 1.1 itself is the combination of three results. The first, due to Todorcevic, shows that PFA implies $\square_{\kappa,<\omega}$ fails for all uncountable cardinals κ. The second, due to Steel, shows that (assuming (†) holds and Ω is measurable) either $(\kappa^+)^{K^c} = (\kappa^+)^{\mathrm{V}}$ for measure 1 many $\kappa < \Omega$, or else K^c reaches a superstrong cardinal. The third, due to Schimmerling, shows that below superstrong and assuming (†), K^c satisfies $\square_{\kappa,<\omega}$ for all κ.[17] If $(\kappa^+)^{K^c} = (\kappa^+)^{\mathrm{V}}$ then $\square_{\kappa,<\omega}$ reflects from K^c to V. Thus combing the three results gives Theorem 1.1.

Steel's proof that K^c computes κ^+ correctly for measure 1 many $\kappa < \Omega$ requires the uniqueness of F given by Case 1a in the construction of K^c (Section 2). It is for this reason that we cannot in Case 1b simply choose one of the extenders F_i and continue constructing. Instead we have to know that Case 1b never occurs.

Now [MitSt94, Theorem 9.2] (or [Jen97, Lemma 6.1] in the context of Jensen's indexing) demonstrates that *iterable* pre-bicephali do not in fact exist. The iterability given by (†) then implies ($\neg 1b$). Thus to prove Corollary 1.2 we must prove Corollary 3.3 not only for pre*mice* which embed into levels of the K^c construction, but also (potentially) for pre-*bicephali* which embed into levels of the K^c construction. (We then conclude that there are no pre-bicephali on the K^c construction, but this can only be done after the fact.) The iterability proof for pre-bicephali is essentially the same as the one which we give below (for premice), and we shall not trouble the reader with the tedious repetition.

[16] For normal, maximal, non-overlapping iteration trees.

[17] There are stronger results; see [SZ].

Let us start working towards a proof of Theorem 3.2. Assuming the existence of two super-realizable cofinal branches we will derive a contradiction by comparing the easy phalanges induced by these branches. Thus the key to our proof is the ability to *compare* easy phalanges which contain only domestic premice. There are two conflicting requirements in any attempt to conduct such a comparison. In forming the iteration trees of the comparison process we must first and foremost make sure that *cofinal wellfounded branches exist* at limit stages of the comparison. The easiest way to secure this would be to use linear iterations. Secondly we must make sure that the comparison *terminates* (more precisely we must secure condition $(*)$ of Claim 3.29). The standard way to secure this is to use non-overlapping iteration trees so that generators are not moved. Our approach is to balance the two requirements. We shall use iteration trees which do move generators, but in a way limited enough that we can still secure 3.29$(*)$ and show that the comparison terminates. This moving of generators will allow us to structure the iteration trees so that it is easier to obtain cofinal wellfounded branches.

The precise structure of the trees we intend to use (balanced iteration trees) is stated in Definition 3.13. To have recourse to the results of Section 2 we must check that these trees are sturdy. This is done in 3.5–3.14. 3.16–3.23 prove the existence of cofinal branches through iteration trees in various special cases. Lemma 3.24 is our main tool in showing that *balanced* iteration trees which arise in comparison of *domestic* premice can be reduced to these special cases. The rest of the Section is a comparison process which proves Theorem 3.2.

Definition 3.5 *Let $\mathcal{R}$ be an iteration tree on a premouse $\mathcal{N}$. Fix an extender E and an ordinal $\gamma < \mathrm{lh}(\mathcal{R})$.*

We say that E **conflicts with generators** *at ϵ if $\mathrm{crit}(E)$ lies in the interval $[\mathrm{crit}(E^{\mathcal{R}}_\epsilon), \mathrm{lh}(E^{\mathcal{R}}_\epsilon))$ where $E^{\mathcal{R}}_\epsilon$ is the ϵ-th extender on $\mathcal{R}$. $g(E)$ is the least ϵ such that E conflicts with generators at ϵ if there are conflicts, and ∞ otherwise.*

We say that the pair E, γ **conflicts with projecta** *at ϵ if ϵ is a truncation point on the branch of $\mathcal{R}$ leading to γ, and $\mathrm{crit}(E) < \rho(\mathcal{N}^-_\epsilon)$. $p(E,\gamma)$ is the least ϵ such that E, γ conflicts with projecta at ϵ if there are conflicts, and γ otherwise.*

We say that E, γ is **conflict free** *if (a) $\gamma \leq g(E)$; and (b) E, γ does not conflict with projecta.*

Claim 3.6 *Suppose $\mathcal{R}$ is maximal, and E, γ conflicts with projecta at ϵ. Then $\mathrm{crit}(E) < \mathrm{lh}(E^{\mathcal{R}}_\epsilon)$.*

Proof. By assumption $\epsilon <^{\mathcal{R}} \gamma$ is a truncation point on the branch leading to γ, and (a) $\mathrm{crit}(E) < \rho(\mathcal{N}^-_\epsilon)$. Let $\zeta + 1$ be the successor of ϵ in $[\epsilon, \gamma]_{\mathcal{R}}$. Since $\mathcal{R}$ is maximal, (b) $\rho(\mathcal{N}^-_\epsilon) \leq \mathrm{crit}(E^{\mathcal{R}}_\zeta)$. Since the $<^{\mathcal{R}}$ predecessor of $\zeta + 1$ is ϵ, (c) $\mathrm{crit}(E^{\mathcal{R}}_\zeta) < \mathrm{lh}(E^{\mathcal{R}}_\epsilon)$. Combining a,b,c proves the Claim. $\square$

Definition 3.7 *An iteration tree $\mathcal{R}$ on a premouse $\mathcal{N}$ is* **suitable** *if*

1. *$\mathcal{R}$ is normal;*

2. *$\mathcal{R}$ is maximal;*

3. *For $\xi+1 < \mathrm{lh}(\mathcal{R})$ and γ the $<^{\mathcal{R}}$ predecessor of $\xi+1$, $E_\xi^{\mathcal{R}}, \gamma$ is conflict free.*

Claim 3.8 *If $\mathcal{R}$ is normal, maximal, and non-overlapping, then $\mathcal{R}$ is suitable.*

Proof. For $\gamma+1 < \mathrm{lh}(\mathcal{R})$ let λ_γ denote the length of $E_\gamma^{\mathcal{R}}$. Remember that $\mathcal{R}$ is non-overlapping just in case that (NOL) for all $\xi+1 < \mathrm{lh}(\mathcal{R})$, the immediate $<^{\mathcal{R}}$ predecessor of $\xi+1$ is the *least* γ such that $\mathrm{crit}(E_\xi^{\mathcal{R}}) < \lambda_\gamma$.

Claim 3.8 follows immediately from the minimality of γ given by NOL, using the definition of conflict with generators and Claim 3.6. $\square$

Lemma 3.9 *Suitable iteration trees are sturdy.*

Proof. We work by induction on the length of the tree. The limit case is clear, as is the case $\mathrm{lh}(\mathcal{R}) = \xi+1$ where ξ is a limit. So let us assume $\mathrm{lh}(\mathcal{R}) = \xi+2$. By induction 2.24(3,4) hold for $\epsilon < \xi$. We must verify these conditions for $\epsilon = \xi$. Let γ be the immediate $<^{\mathcal{R}}$ predecessor of $\xi+1$.

Claim 3.10 *2.24(4) holds at $\epsilon = \xi$.*

Proof. Assume γ is non-simple. Since we are dealing with a tree on a single premouse, non-simplicity can only be caused by a failure of 2.23(b). So there are truncations on the branch of $\mathcal{R}$ leading to γ. Let $\bar\gamma$ be the largest such truncation. By Lemma 2.25 (which we can access through our induction hypothesis) $\rho(\mathcal{N}_{\bar\gamma}^-) = \rho(\mathcal{N}_\gamma^{\mathcal{R}})$. Now $E_\xi^{\mathcal{R}}, \gamma$ does not conflict with projecta by 3.7(3). Since $\bar\gamma$ is a truncation point on the branch to γ we conclude that $\mathrm{crit}(E_\xi^{\mathcal{R}}) \not< \rho(\mathcal{N}_{\bar\gamma}^-)$. In other words $\mathrm{crit}(E_\xi^{\mathcal{R}}) \geq \rho(\mathcal{N}_\gamma^{\mathcal{R}})$ as required. $\square$(Claim 3.10.)

Claim 3.11 *2.24(3) holds for $\epsilon = \xi$.*

Proof. Let $\mathcal{P} = \mathcal{N}_\xi^{\mathcal{R}}$. $E_\xi^{\mathcal{R}}$ is an extender on the sequence of $\mathcal{P}$. Let η be its index. Let $\kappa = \mathrm{crit}(E_\xi^{\mathcal{R}})$, and let τ be the successor of κ in $\mathcal{P} \parallel \langle \eta, 0\rangle$. Let $\mathcal{M}$ denote $\mathcal{N}_\gamma^- = \mathcal{N}_{\xi+1}^*$. Note that by standard agreement of premice along an iteration tree, $\mathcal{M}$ and $\mathcal{P} \parallel \langle \eta, 0\rangle$ agree to τ and $E_\xi^{\mathcal{R}}$ is an extender over $\mathcal{M}$ (else γ would not be the predecessor of ξ). It is enough to prove

$$\text{(A)} \qquad P(\tau) \cap \Sigma_1(\mathcal{P} \parallel \langle \eta, 0\rangle) \quad \subset \quad \Sigma_1(\mathfrak{D}(\mathcal{M})).$$

An appeal to Fact 2.17 would then complete the proof of the Claim. Suppose (A) fails. By reducing ξ if needed we may assume that (0) for any $\bar\xi, \bar\eta$, if (a) $\gamma \leq \bar\xi < \xi$, (b) $\bar\eta$ indexes an extender on $\bar{\mathcal{P}} = \mathcal{N}_{\bar\xi}^{\mathcal{R}}$ whose critical point equal κ, and (c) the successor of κ in $\bar{\mathcal{P}} \| \langle\bar\eta, 0\rangle$ is τ, then $P(\tau) \cap \Sigma_1(\bar{\mathcal{P}} \| \langle \bar\eta, 0\rangle) \subset \Sigma_1(\mathfrak{D}(\mathcal{M}))$.
Step 1. If $\xi = \gamma$ then $\mathcal{P} \parallel \langle \eta, 0\rangle$ is an initial segment of $\mathcal{M}$ and (A) is clear. So let us for the remaining steps assume (1) $\xi > \gamma$.

Step 2. If $\eta < \alpha(\mathcal{P})$ then $P(\tau) \cap \Sigma_1(\mathcal{P} \parallel \langle \eta, 0\rangle) \subset P(\tau) \cap \mathcal{P}$. But standard agreement between premice on iteration trees (together with 1) imply that $P(\tau) \cap \mathcal{P} \subset \mathcal{M}$, and (A) follows. So let us assume (2) $\eta = \alpha(\mathcal{P})$. In particular $E^{\mathcal{R}}_{\xi}$ is the last extender predicate of $\mathcal{P}$, so κ is Σ_1 definable over $\mathcal{P}$.
Step 3. Let μ be the largest element of $[0, \xi]_{\mathcal{R}} \cap (\gamma + 1)$. If there are truncations on $[\mu, \xi)_{\mathcal{R}}$, let $\bar{\xi} <^{\mathcal{R}} \xi$ be the largest such. Otherwise set $\bar{\xi} = \mu$. Note that $\mathrm{lh}(E^{\mathcal{R}}_{\gamma}) > \kappa$ (else γ couldn't be the predecessor of $\xi + 1$). By 1 and the normality of $\mathcal{R}$, it follows that the length(s) of the extender(s) used on $[\bar{\xi}, \xi]_{\mathcal{R}}$ must be above κ. By 2 κ belongs to the range of $i^{\mathcal{R}}_{\bar{\xi},\xi}$. Thus the extenders used on $[\bar{\xi}, \xi]_{\mathcal{R}}$ cannot overlap κ. It follows that $\mathrm{crit}(i^{\mathcal{R}}_{\bar{\xi},\xi}) > \kappa$. Since τ is the successor of κ in $\mathcal{P}$ we conclude that (3) $\mathrm{crit}(i^{\mathcal{R}}_{\bar{\xi},\xi}) \geq \tau$.
Step 4. If $\rho(\mathcal{P} \parallel \langle \eta, 0\rangle) > \tau$ then $P(\tau) \cap \Sigma_1(\mathcal{P} \parallel \langle \eta, 0\rangle) \subset \mathcal{P}$. But then standard agreement between $\mathcal{P}$ and $\mathcal{M}$ would give (A). So assume (4) $\rho(\mathcal{P} \parallel \langle \eta, 0\rangle) \leq \tau$.
Step 5. If $k(\mathcal{P}) \geq 1$ then by 4, $\vartheta(\mathcal{P}) \leq \tau$. But this would contradict 3, since the way we take ultrapowers (see Section 2) is such that the critical points are below the true height. So $k(\mathcal{P}) = 0$ and by 2 we conclude that (5) $\mathcal{P} = \mathcal{P} \parallel \langle \eta, 0\rangle$.

Let $\bar{\mathcal{P}} = \mathcal{N}^{-}_{\bar{\xi}}$. By induction we know that $\mathcal{R} \restriction \xi + 1$ is sturdy. This and 3 imply that $P(\tau) \cap \Sigma_1(\mathfrak{D}(\mathcal{P})) \subset P(\tau) \cap \Sigma_1(\mathfrak{D}(\bar{\mathcal{P}}))$. Given 5 it is thus enough to show

(B) $$P(\tau) \cap \Sigma_1(\mathfrak{D}(\bar{\mathcal{P}})) \quad \subset \quad \Sigma_1(\mathfrak{D}(\mathcal{M})).$$

Step 6. If $\bar{\xi} \geq \gamma$ then (B) follows from 0 with $\bar{\eta} = \alpha(\bar{\mathcal{P}})$. (Note that $\bar{\mathcal{P}} \parallel \langle \bar{\eta}, 0\rangle$ then equals $\bar{\mathcal{P}}$.) Thus we may assume that (6) $\bar{\xi} < \gamma$.

Remark 3.12 Our proof so far followed in the footsteps of [MitSt94, Lemma 6.1.5]. [MitSt94, Lemma 6.1.5] is stated for *non-overlapping* trees, and goes on to directly argue that 6 is impossible, using NOL. We must proceed differently since we are working with the more general concept of suitable iteration trees. We shall use our assumption that $\mathcal{R}$ is suitable in 7c and 9 below.

Step 7. Let ζ be least such that $\mathrm{lh}(E^{\mathcal{R}}_{\zeta}) > \kappa$. Extenders with critical point κ or greater cannot be applied to models indexed before ζ. Since $\mathrm{crit}(i^{\mathcal{R}}_{\bar{\xi},\xi}) \geq \tau$ it follows that (7a) $\zeta \leq \bar{\xi}$. Using 6, $\zeta < \gamma$. By normality, (7b) $\mathrm{lh}(E^{\mathcal{R}}_{\iota}) > \kappa$ for $\iota \in [\zeta, \gamma)$. By assumption $\mathcal{R}$ is suitable. Since γ is the predecessor of $E^{\mathcal{R}}_{\xi}$ it follows that $\gamma \leq g(E^{\mathcal{R}}_{\xi})$. Using 7b it follows that (7c) $\mathrm{crit}(E^{\mathcal{R}}_{\iota}) > \kappa$ for $\iota \in [\zeta, \gamma)$. In particular none of these extenders can be applied to models indexed before ζ. Thus (7d) $\zeta <^{\mathcal{R}} \iota$ for all $\iota \in (\zeta, \gamma]$.
Step 8. Let $\mathcal{Q} = \mathcal{N}^{\mathcal{R}}_{\zeta}$. Let $\bar{\mathcal{Q}}$ be the least initial segment of $\mathcal{Q}$ so that $\alpha(\bar{\mathcal{Q}}) \geq \mathrm{lh}(E^{\mathcal{R}}_{\zeta})$ and $\rho(\bar{\mathcal{Q}}) \leq \kappa$ if there is such a segment. Let $\bar{\mathcal{Q}} = \mathcal{Q}$ otherwise. Using 7c we see that (8) for any $\iota \in (\zeta, \gamma]$, $\mathcal{N}^{-}_{\zeta}$, as computed relative to the branch $[\zeta, \iota]_{\mathcal{R}}$, is an initial segment of $\bar{\mathcal{Q}}$.
Step 9. Suppose $\mathcal{N}^{-}_{\zeta}$, as computed relative to $[\zeta, \gamma]_{\mathcal{R}}$, is a strict initial segment of $\bar{\mathcal{Q}}$. (In particular $\zeta < \gamma$ is a truncation point on $[\zeta, \gamma]_{\mathcal{R}}$.) From the definition of

$\bar{\mathcal{Q}}$ it follows that $\rho(\mathcal{N}_\zeta^-) > \kappa$. But then $E_\xi^{\mathcal{R}}, \gamma$ conflicts with projecta at ζ. This is impossible, since γ is the $<^{\mathcal{R}}$ predecessor of $\xi+1$ and $\mathcal{R}$ is *suitable*. We conclude that $\mathcal{N}_\zeta^-$ cannot be a strict initial segment of $\bar{\mathcal{Q}}$. Using 8 it follows that (9a) $\mathcal{N}_\zeta^-$, as computed relative to $[\zeta,\gamma]_{\mathcal{R}}$, is equal to $\bar{\mathcal{Q}}$. Suppose next there are truncations on $(\zeta,\gamma)_{\mathcal{R}}$. Let $\bar{\gamma} < \gamma$ be the first such. We have $i_{\zeta,\bar{\gamma}}^{\mathcal{R}}\colon \bar{\mathcal{Q}} \to \mathcal{N}_{\bar{\gamma}}^{\mathcal{R}}$ elementary, and $\mathcal{N}_{\bar{\gamma}}^-$ is a strict initial segment of $\mathcal{N}_{\bar{\gamma}}^{\mathcal{R}}$. It follows again that $\rho(\mathcal{N}_{\bar{\gamma}}^-) > \kappa$ and so $E_\xi^{\mathcal{R}}, \gamma$ conflicts with projecta at $\bar{\gamma}$. Again this contradicts our initial assumption in Lemma 3.9 that $\mathcal{R}$ is suitable. We conclude that (9b) there are no truncations on $(\zeta,\gamma)_{\mathcal{R}}$. Suppose finally that γ is a truncation point on the branch leading to $\xi+1$. Remember that $\mathcal{M} = \mathcal{N}_\gamma^-$, as computed relative to the branch leading to $\xi+1$. If there is a truncation then $\mathcal{M}$ is a strict initial segment of $\mathcal{N}_\gamma^{\mathcal{R}}$ and as before we conclude that $\rho(\mathcal{M}) > \kappa$. But this is impossible; if $E_\xi^{\mathcal{R}}$ causes a truncation at γ then $\rho(\mathcal{M}) \leq \mathrm{crit}(E_\xi^{\mathcal{R}}) = \kappa$ by the maximality of $\mathcal{R}$. Thus we conclude that (9c) $\mathcal{M} = \mathcal{N}_\gamma^{\mathcal{R}}$.

Step 10. Combining 9a,b,c we see that $\mathcal{R}$ defines an embedding $i_{\zeta,\gamma}^{\mathcal{R}}\colon \bar{\mathcal{Q}} \to \mathcal{M}$. Through our inductive assumption we know that the extenders used on $[\zeta,\gamma]_{\mathcal{R}}$ satisfy 2.24(3). By 7c, these extenders have critical points at least τ. It follows that (10) $P(\tau) \cap \Sigma_1(\mathfrak{D}(\bar{\mathcal{Q}})) = P(\tau) \cap \Sigma_1(\mathfrak{D}(\mathcal{M}))$.

Step 11. Suppose $\bar{\xi} = \zeta$. Remember that $\mathrm{crit}(i_{\bar{\xi},\xi}^{\mathcal{R}}) \geq \tau$ by 3. From this and the definition of $\bar{\mathcal{Q}}$ it follows that $\bar{\mathcal{P}}$, which is equal to $\mathcal{N}_{\bar{\xi}}^-$ as computed relative to the branch leading to ξ, is an initial segment of $\bar{\mathcal{Q}}$. It follows that $P(\tau) \cap \Sigma_1(\mathfrak{D}(\bar{\mathcal{P}})) \subset \Sigma_1(\mathfrak{D}(\bar{\mathcal{Q}}))$ (with equality if $\bar{\mathcal{P}} = \bar{\mathcal{Q}}$). But this and 10 give (B), completing the proof of Claim 3.11. So let us assume that (11) $\bar{\xi} \neq \zeta$.

Step 12. By 11, 7a, and 7d, $\zeta <^{\mathcal{R}} \bar{\xi}$. $\mathcal{N}_\zeta^-$, as computed relative to the branch leading to $\bar{\xi}$, is an initial segment of $\bar{\mathcal{Q}}$ by 8. By 7c the extenders used on $(\zeta,\bar{\xi})_{\mathcal{R}}$ have critical points at least τ. These extenders, through our inductive assumption that $\mathcal{R}\restriction\xi+1$ is sturdy, satisfy 2.24(3). From all this it follows that $P(\tau) \cap \Sigma_1(\mathfrak{D}(\bar{\mathcal{P}})) \subset \Sigma_1(\mathfrak{D}(\bar{\mathcal{Q}}))$. This together with 10 gives (B). $\Box$(Claim 3.11, Lemma 3.9.)

Definition 3.13 *An iteration tree* $\mathcal{R}$ *on a premouse* $\mathcal{N}$ *is* **balanced above** η *just in case that*

1. $\mathcal{R}$ *is normal;*
2. $\mathcal{R}$ *is maximal; and*
3. *For* $\xi+1 < \mathrm{lh}(\mathcal{R})$, $\xi \geq \eta$*: the* $<^{\mathcal{R}}$ *predecessor of* $\xi+1$ *is equal to* $p\left(E_\xi^{\mathcal{R}}, \min\{\xi, g(E_\xi^{\mathcal{R}})\}\right)$.

$\mathcal{R}$ *is said to be* **balanced** *if it is balanced above* 0.

Observe that balanced iteration trees may move generators in a limited way. For $\zeta+1 <^{\mathcal{R}} \xi+1$ it *is* possible that $E_\xi^{\mathcal{R}}$ overlaps $E_\zeta^{\mathcal{R}}$, i.e., $\mathrm{crit}(E_\xi^{\mathcal{R}}) < \mathrm{lh}(E_\zeta^{\mathcal{R}})$. However $\mathrm{crit}(E_\xi^{\mathcal{R}})$ can*not* lie in $[\mathrm{crit}(E_\zeta^{\mathcal{R}}), \mathrm{lh}(E_\zeta^{\mathcal{R}}))$.

Lemma 3.14 *Suppose $\mathcal{R}$ is balanced above η, and $\mathcal{R}\restriction\eta+1$ is non-overlapping. Then $\mathcal{R}$ is sturdy.*

Proof. Let us check that $\mathcal{R}$ is suitable. (By Lemma 3.9 this is enough.) Conditions 3.7(1,2) follow trivially from the corresponding conditions in Definition 3.13. 3.7(3) for $\xi < \eta$ follows from Claim 3.8. 3.7(3) for $\xi \geq \eta$ follows trivially from 3.13(3). □

Lemma 3.14 may seem insignificant, but it is essential to our argument. It gives us access to the results of Section 2, particularly Lemma 2.25(b) which will be crucial later in Claim 3.32. It is our need to prove Lemma 3.14 that forced us to include conflicts with projecta in Definition 3.13; Claims 3.10 and 3.11 both required that we avoid conflicts with projecta.

Remark 3.15 Suppose $\mathcal{R}$ is normal and maximal, E is an extender of $\mathcal{N}^{\mathcal{R}}_{\xi}$, and ϵ is equal to $p(E, \min\{\xi, g(E)\})$. If $\epsilon < \xi$ then $\mathrm{crit}(E) < \mathrm{lh}(E^{\mathcal{R}}_{\epsilon})$. (This follows using Claim 3.6.) Thus $\mathcal{N}^{\mathcal{R}}_{\epsilon}$ and $\mathcal{N}^{\mathcal{R}}_{\xi}$ are in *sufficient agreement* that E can be applied to $\mathcal{N}^{\mathcal{R}}_{\epsilon}$ (or to the appropriate initial segment of this premouse).

Definition 3.16 *Let $\mathcal{R}$ be a sturdy iteration tree on a premouse $\mathcal{N}$ and let α be smaller than the length of $\mathcal{R}$. For $\gamma <^{\mathcal{R}} \alpha$ a truncation point of $[0,\alpha]_{\mathcal{R}}$, let λ_γ denote $\rho(\mathcal{N}^{-}_{\gamma})$, where $\mathcal{N}^{-}_{\gamma}$ is computed relative to the branch $[\gamma,\alpha]_{\mathcal{R}}$.*

$\mathcal{R}$ is said to be **semilinear at** *α just in case that for every ξ such that $\alpha < \xi + 1 < \mathrm{lh}(\mathcal{R})$:*

1. *The $<^{\mathcal{R}}$ predecessor, ζ, of $\xi+1$ is either (a) greater than or equal to α or (b) a truncation point of $[0,\alpha]_{\mathcal{R}}$; and*
2. *If (b) holds then $\mathrm{crit}(E^{\mathcal{R}}_{\xi}) < \lambda_\zeta$.*

If $\mathcal{R}$ is semilinear at α then $\mathcal{R}$ can in fact be viewed as an iteration tree on the easy phalanx $\vec{\mathcal{N}}^{[0,\alpha]_{\mathcal{R}}}$ induced by the branch of $\mathcal{R}$ leading to α. (Condition 2 of Definition 3.16 represents the limitations imposed by the exchange ordinals of the easy phalanx $\vec{\mathcal{N}}^{[0,\alpha]_{\mathcal{R}}}$.) We denote this iteration tree by $\mathcal{R}^*$. Observe that $\mathcal{R}^*$ is sturdy: Conditions 2.24(1–3) reflect trivially from $\mathcal{R}$ to $\mathcal{R}^*$. The non-primordial roots of $\mathcal{R}^*$ correspond to failures of 2.23(b) on $\mathcal{R}$. Thus an index γ of $\mathcal{R}^*$ is non-simple (as an index in $\mathcal{R}^*$) iff it is non-simple as an index in $\mathcal{R}$. It follows that 2.24(4) too reflects from $\mathcal{R}$ to $\mathcal{R}^*$.

Definition 3.17 *Let $\mathcal{R}$ be a sturdy iteration tree on a premouse $\mathcal{N}$, and assume that $\mathcal{R}$ is semilinear at α. Let $\vec{\nu}, \vec{\pi}$ be a realization of the easy phalanx $\vec{\mathcal{N}}^{[0,\alpha]_{\mathcal{R}}}$.*

We say that (wrt $\vec{\nu}, \vec{\pi}$) $\mathcal{R}$ **picks unique realizable branches above** *α if for any γ strictly between α and $\mathrm{lh}(\mathcal{R})$,*

1. *The branch of $\mathcal{R}$ leading to γ is super-realizable when viewed as a branch through $\mathcal{R}^*$ and with respect to the realization $\vec{\nu}, \vec{\pi}$ of $\vec{\mathcal{N}}^{[0,\alpha]_{\mathcal{R}}}$; and*

2. *The branch of $\mathcal{R}$ leading to γ is furthermore the* only *cofinal branch of $\mathcal{R}\restriction\gamma$ which satisfies 1.*

Lemma 3.18 *Let $\mathcal{R}$ be a sturdy iteration tree of countable length θ on a premouse $\mathcal{N}$. Let $\alpha < \theta$. Assume that*

1. *$\vec{\mathcal{N}}^{[0,\alpha]_\mathcal{R}}$ is realizable. Let $\vec{\nu}, \vec{\pi}$ witness this;*

2. *$\mathcal{R}$ is semilinear at α; and*

3. *(wrt $\vec{\nu}, \vec{\pi}$) $\mathcal{R}$ picks unique realizable branches above α.*

Then there exists a cofinal *branch b through $\mathcal{R}$ which (wrt $\vec{\nu}, \vec{\pi}$) is super-realizable when viewed as a branch through $\mathcal{R}^*$.*

Proof. By Theorem 2.28 there is a *maximal* super-realizable branch b through $\mathcal{R}^*$. Viewed as a branch through $\mathcal{R}$, b satisfies all requirements of the conclusion of Lemma 3.18. (It is cofinal because of assumption 3.) □

Lemma 3.19 *Work under the assumptions of Lemma 3.18, except that instead of assuming $\theta = \mathrm{lh}(\mathcal{R})$ is countable, assume that $\theta = \omega_1$. Then the conclusion of Lemma 3.18 still holds.*

Proof. Let us for a while work in $\mathrm{V}[G]$ where G is generic over V for $\mathrm{col}(\omega, \omega_1)$. By Remark 2.30 the proof of Lemma 3.18 goes through, producing in $\mathrm{V}[G]$ a branch b which satisfies the conclusion of Lemma 3.18. We claim that in fact b is an element of V. Given the homogeneity of the forcing $\mathrm{col}(\omega, \omega_1)$, it is enough to argue that in $\mathrm{V}[G]$ the branch b is the *unique* branch satisfying the conclusion of Lemma 3.18. Assume for contradiction that some other branch b' satisfies the same. Let H be a countable elementary substructure of V_μ for some large regular μ, and throw all relevant objects into H. Let $\bar{\mathrm{V}}$ be the transitive collapse of H and $k\colon \bar{\mathrm{V}} \to H$ the anti-collapse embedding. Let $\bar{G}$ be $\bar{\mathrm{V}}$-generic over $\mathrm{col}(\omega, \omega_1^{\bar{\mathrm{V}}})$. $\bar{G}$ can be found inside V. Let $\dot{b}$ and $\dot{b}'$ be names for b and b'. Let $\bar{b}$ and $\bar{b}'$ be the interpretations of $k^{-1}(\dot{b})$ and $k^{-1}(\dot{b}')$ using the generic $\bar{G}$. Both belong to V.

Let $\bar{\mathcal{R}}$ be the pre-image under k of $\mathcal{R}$. Note this is equal to $\mathcal{R}\restriction\gamma$, where $\gamma = \omega_1 \cap H$. Let $\bar{\mathcal{R}}^*$ be the pre-image under k of $\mathcal{R}^*$. This again is $\mathcal{R}^*\restriction\gamma$. ($\alpha$ of course is not moved by k.) The elementarity of k guarantees that in $\bar{\mathrm{V}}$, $\bar{b}$ and $\bar{b}'$ are two *distinct* super-realizable cofinal branches through $\bar{\mathcal{R}}^*$. Composing the embeddings which witness this with k demonstrates that in V, $\bar{b}$ and $\bar{b}'$ are distinct super-realizable cofinal branches through $\mathcal{R}^*\restriction\gamma$. But this contradicts the uniqueness of assumption 3 in Lemma 3.18. □

Let $\mathcal{R}$ be a sturdy iteration tree on a premouse $\mathcal{N}$. Let $\alpha < \mathrm{lh}(\mathcal{R})$ and let $\vec{\nu}, \vec{\pi}$ be a realization of $\vec{\mathcal{N}}^{[0,\alpha]_\mathcal{R}}$. Assume that $\mathcal{R}$ is semilinear at α and that b is either a branch through $\mathcal{R}$ or a branch of $\mathcal{R}$ with supremum greater than α. A realization $\vec{\nu}^b, \vec{\pi}^b$ of $\vec{\mathcal{N}}^b$ is said to **commute** with $\vec{\nu}, \vec{\pi}$ just in case that it commutes with

$\vec{\nu}, \vec{\pi}$ when viewed as a realization of a branch through $\mathcal{R}^*$. Similarly the **meet** of $\vec{\nu}, \vec{\pi}$ and $\vec{\nu}^b, \vec{\pi}^b$ is defined according to Definition 2.29 applied to $\mathcal{R}^*$ — it is $\vec{\nu} \restriction \epsilon + 1$ where ϵ is the largest element of $b \cap (\alpha + 1)$.

Lemma 3.20 *Let $\mathcal{R}$ be a sturdy iteration tree on $\mathcal{N}$ of limit length θ. Let $\langle \alpha_k \mid k < \omega \rangle$ be a sequence of ordinals cofinal in θ. Assume that*

1. *For each $k < \omega$, $\mathcal{R}$ is semilinear at α_k.*

Let $\vec{\mathcal{N}}^k$ denote the easy phalanx $\vec{\mathcal{N}}^{[0,\alpha_k]_{\mathcal{R}}}$ induced by the branch of $\mathcal{R}$ leading to α_k. Let $\vec{\nu}^k, \vec{\pi}^k$ be realizations of the phalanges $\vec{\mathcal{N}}^k$, and assume

2. *For $k < k' < \omega$, the realizations $\vec{\nu}^k, \vec{\pi}^k$ and $\vec{\nu}^{k'}, \vec{\pi}^{k'}$ commute.*

Then there exists a cofinal branch b of $\mathcal{R}$, and a realization $\vec{\nu}^b, \vec{\pi}^b$ of $\vec{\mathcal{N}}^b$, so that for each $k < \omega$ the realizations $\vec{\nu}^k, \vec{\pi}^k$ and $\vec{\nu}^b, \vec{\pi}^b$ commute.

Proof. Let us say that a finite descending sequence of stages $\vec{\nu}$ is a **permanent resident** if there exists $k < \omega$ such that

- $\vec{\nu}$ is an initial segment of $\vec{\nu}^k$; and
- For all $k' > k$, $\vec{\nu}$ is an initial segment of the meet of the k-th and the k'-th realizations.

Non-trivial permanent residents certainly exist: $\vec{\nu}^k$ all start with the same first stage. This stage is a permanent resident of length 1.

Let $\vec{\eta}$ be the minimal permanent resident, using the lexicographic (Brouwer–Kleene) order on descending sequences of ordinals. Let k witness that $\vec{\eta}$ is permanent. Let η be the last (smallest) stage in $\vec{\eta}$. For $l \geq k$ let ϵ_l be the unique index of a premouse of $\vec{\mathcal{N}}^l$ such that $\nu^l_{\epsilon_l} = \eta$. Such an index exists by the demands of commutativity and the fact that $\vec{\eta}$ is permanent. The demands of commutativity also imply that $\epsilon_l \leq^{\mathcal{R}} \epsilon_{l'}$ for $l < l'$, that there are no truncations on the branch $[\epsilon_l, \epsilon_{l'})_{\mathcal{R}}$, and that the following diagram commutes:

$$\begin{tikzcd} \mathcal{M}_\eta & \\ \mathcal{N}^{\mathcal{R}}_{\epsilon_l} \arrow[u, "\pi^l_{\epsilon_l}"] \arrow[r, "i^{\mathcal{R}}_{\epsilon_l,\epsilon_{l'}}"] & \mathcal{N}^{\mathcal{R}}_{\epsilon_{l'}} \arrow[ul, "\pi^{l'}_{\epsilon_{l'}}"'] \end{tikzcd}$$

Claim 3.21 *For every $l \geq k$, there exists $j > l$ such that $\epsilon_j > \alpha_l$.*

Proof. Fix $l \geq k$ and assume for contradiction that $\epsilon_j \leq \alpha_l$ for all $j > l$. It follows that $\epsilon_l \leq \epsilon_j$ are both indices in $\vec{\mathcal{N}}^l$. Since there are no truncations on $[\epsilon_l, \epsilon_j)_{\mathcal{R}}$ we conclude that (1) $\epsilon_j = \epsilon_l$. Since $\vec{\nu}^l \restriction \epsilon_l + 1 = \vec{\eta}$ is the *minimal* permanent resident, there exist arbitrarily large $j > l$ such that (2) the meet of the j-th and $(j+1)$-st realizations is precisely $\vec{\nu}^l \restriction \epsilon_l + 1$. By thinning $\langle \alpha_k \mid k < \omega \rangle$

if needed we may assume that 2 holds for all $j > l$. By 1 it follows that ϵ_l is a truncation point of $[0, \alpha_j]_{\mathcal{R}}$ for all $j > l$. Let $\xi_j + 1$ be the successor of ϵ_l in $[0, \alpha_j]_{\mathcal{R}}$, let $\kappa_j = \mathrm{crit}(E^{\mathcal{R}}_{\xi_j})$, and let $\lambda_j = \rho(\mathcal{N}^*_{\xi_j+1})$. We have (3) $\lambda_j \leq \kappa_j$ since $\mathcal{R}$ is maximal. Note that $\xi_{j+1} \geq \alpha_j$ by 2 and the definition of the meet. The demands of semilinearity at α_j, specifically 3.16(2), therefore imply that (4) $\kappa_{j+1} < \lambda_j$. But 3 and 4 together produce an infinite descending chain of ordinals, giving the desired contradiction. □(Claim 3.21.)

It follows from Claim 3.21 that the ϵ_l-s converge to θ, and therefore define a cofinal branch of $\mathcal{R}$. Let b be this branch. It is easy to see that $\vec{\mathcal{N}}^b$ is realizable. The realization includes $\vec{\nu}^k \restriction \epsilon_k, \vec{\pi}^k \restriction \epsilon_k$, followed by η and π, where π is the direct limit of the embeddings $\pi^l_{\epsilon_l}$, $l \geq k$ (well defined using the commuting diagram above). Clearly this realization commutes with all previous ones. □(Lemma 3.20.)

Lemma 3.22 *Work under the assumptions of Lemma 3.20. Then there is only* one *branch b which satisfies the conclusion of Lemma 3.20.*

Proof. Let b be the branch defined in the proof of Lemma 3.20, and assume for contradiction that c is distinct from b and satisfies the conclusion of the Lemma. Fix $\vec{\nu}^c, \vec{\pi}^c$ which witness this.

We claim to begin with that $\vec{\nu}^c$ is permanent. Indeed, let k be large enough that α_k is bigger than all truncation points of c. The fact that $\mathcal{R}$ is semilinear at α_k easily implies that $\vec{\nu}^c$ is an initial segment of $\vec{\nu}^k$. The fact that $\mathcal{R}$ is semilinear at $\alpha_{k'}$ then implies that $\vec{\nu}^c$ is an initial segment of the meet of the k-th and k'-th realizations, for $k' > k$.

Work now with $k < \omega$ large enough that all truncations on c occur before α_k, and large enough to witness that $\vec{\eta}$ (of Lemma 3.20) is permanent. For $l \geq k$, let ζ_l be the largest element of c which is less than or equal to α_l. It is enough to verify that for arbitrarily large $l < \omega$, ζ_l is equal to ϵ_l (of Lemma 3.20). Since c is cofinal there certainly are arbitrarily large $l < \omega$ for which $c \cap (\alpha_l, \alpha_{l+1}]$ is not empty. Let us work with such an l and aim to prove that $\zeta_l = \epsilon_l$.

The semilinearity of $\mathcal{R}$ at α_l implies that ζ_l is an index of a premouse of $\vec{\mathcal{N}}^l$. Further, the commutativity satisfied by c implies that $\vec{\nu}^c$ and $\vec{\nu}^l \restriction \zeta_l + 1$ are the same. Now the fact that $\vec{\nu}^c$ is permanent together with the minimality of $\vec{\eta}$ implies that $\zeta_l \leq \epsilon_l$. Thus it is enough to eliminate the possibility that $\zeta_l < \epsilon_l$ strictly. Assume for contradiction that this is the case.

Before proceeding further let us note that the above argument can be repeated for $l + 1$, demonstrating that $\zeta_{l+1} \leq \epsilon_{l+1}$ (so in fact $\zeta_{l+1} \leq^{\mathcal{R}} \epsilon_{l+1}$). Our choice of l is such that $c \cap (\alpha_l, \alpha_{l+1}] \neq \emptyset$ and so by definition we have $\zeta_{l+1} > \alpha_l \geq \epsilon_l$. Combining this with our assumption for contradiction we see that $\zeta_l < \epsilon_l < \zeta_{l+1}$.

Remember that by choice of k there are no truncations on $[\zeta_l, \zeta_{l+1})_{\mathcal{R}}$. In particular ζ_l is not a truncation point of this branch. Now ζ_l, being an index below ϵ_l of a premouse of $\vec{\mathcal{N}}^l$, is a truncation point on $[\zeta_l, \epsilon_l)_{\mathcal{R}}$. As ζ_l is not a truncation on $[\zeta_l, \zeta_{l+1})_{\mathcal{R}}$, ϵ_l can*not* belong to $(\zeta_l, \zeta_{l+1})_{\mathcal{R}}$. Since we know that $\zeta_l < \epsilon_l < \zeta_{l+1}$ and that $\zeta_{l+1} \leq^{\mathcal{R}} \epsilon_{l+1}$ we conclude that $\epsilon_l \not\leq^{\mathcal{R}} \epsilon_{l+1}$. But this is a contradiction. □

Lemma 3.23 *Let $\mathcal{R}$ be a sturdy iteration tree on $\mathcal{N}$ of length ω_1. Let $\langle \alpha_\iota \mid \iota < \omega_1 \rangle$ be an increasing sequence cofinal in ω_1. Assume that*

1. *For each $\iota < \omega_1$, $\mathcal{R}$ is semilinear at α_ι.*

Let $\vec{\mathcal{N}}^\iota$ denote the easy phalanx $\vec{\mathcal{N}}^{[0,\alpha_\iota]_\mathcal{R}}$ induced by the branch of $\mathcal{R}$ leading to α_ι. Let $\vec{\nu}^\iota, \vec{\pi}^\iota$ be realizations of the phalanges $\vec{\mathcal{N}}^\iota$ and assume that

2. *For $\iota < \iota' < \omega_1$, the realizations $\vec{\nu}^\iota, \vec{\pi}^\iota$ and $\vec{\nu}^{\iota'}, \vec{\pi}^{\iota'}$ commute.*

Then there exists a cofinal branch b of $\mathcal{R}$ and a realization $\vec{\nu}^b, \vec{\pi}^\nu$ of $\vec{\mathcal{N}}^b$, so that for each $\iota < \omega_1$ the realizations $\vec{\nu}^\iota, \vec{\pi}^\iota$ and $\vec{\nu}^b, \vec{\pi}^b$ commute.

Proof. Apply Lemma 3.20 in V$[G]$ where G is generic for $\mathrm{col}(\omega, \omega_1)$. Let b and $\vec{\nu}^b, \vec{\pi}^b$ be the branch and realization given by Lemma 3.20. Observe that $\vec{\nu}^b, \vec{\pi}^b$ in fact commutes with $\vec{\nu}^\iota, \vec{\pi}^\iota$ for *all* $\iota < \omega_1$ (not only those ι-s which belong to the particular ω–sequence of the generic extension used in applying Lemma 3.20). The commutativity condition satisfied by $b, \vec{\nu}^b, \vec{\pi}^b$ is therefore independent of the choice of G. The uniqueness given by Lemma 3.22 together with the homogeneity of the collapse now imply that b is in V. $\square$

The previous Lemmas demonstrate that it is possible to find cofinal realizable branches through iteration trees with many points of semilinearity. The next Lemma shows that if one works with *domestic* premice, and uses *balanced* iteration trees, then enough points of semilinearity do indeed exist.

Lemma 3.24 *Let $\mathcal{R}$ be a sturdy iteration tree on a* domestic *premouse $\mathcal{N}$. For $\xi + 1 < \mathrm{lh}(\mathcal{R})$ let γ_ξ denote the index (in $\mathcal{N}_\xi^\mathcal{R}$) of the ξ-th extender used on $\mathcal{R}$. Let $\alpha < \mathrm{lh}(\mathcal{T})$ be a limit, let $\delta_\alpha = \sup\{\gamma_\xi \mid \xi < \alpha\}$, and assume that for all $\xi \geq \alpha$,*

$$(*) \qquad \mathcal{N}_\xi^\mathcal{R} \| \langle \gamma_\xi, 0 \rangle \models \text{“}\delta_\alpha \textit{ is a Woodin cardinal.”}$$

Assume finally that $\mathcal{R}$ is balanced above α. Then $\mathcal{R}$ is semilinear at α.

Proof. Assume for contradiction that $\mathcal{R}$ is not semilinear at α, and let $\xi \geq \alpha$ be the least counter example to semilinearity. Since $\mathcal{R}$ is balanced, the $<^\mathcal{R}$ predecessor of $\xi + 1$ is $p(E_\xi^\mathcal{R}, \min\{\xi, g(E_\xi^\mathcal{R})\})$. Since ξ witnesses failure of semilinearity, certainly $p(E_\xi^\xi, \min\{\xi, g(E_\xi^\mathcal{R})\}) < \alpha$.

We claim that in fact $g(E_\xi^\mathcal{R}) < \alpha$: Assume otherwise for contradiction. Then $\alpha \leq g(E_\xi^\mathcal{R}) \leq \xi$. By the minimality of ξ, the largest element, γ, of $[0, g(E_\xi^\mathcal{R})]_\mathcal{R} \cap (\alpha + 1)$ is either (a) α or (b) a truncation point of $[0, \alpha]_\mathcal{R}$. If (b) holds then (c) the first extender used on $[\gamma, g(E_\xi)]_\mathcal{R}$ must have critical point smaller than $\lambda_\gamma = \rho(\mathcal{N}_\gamma^-)$ (where $\mathcal{N}_\gamma^-$ is computed relative to the branch $[0, \alpha]_\mathcal{R}$). Let $\epsilon = p(E_\xi^\mathcal{R}, \min\{\xi, g(E_\xi^\mathcal{R}\})$. Since $\epsilon < \alpha$ is on the branch leading to $g(E_\xi^\mathcal{R})$ we must have $\epsilon \leq^\mathcal{R} \gamma$. Suppose first that $\epsilon = \gamma$. Then b holds. By Definition 3.5, (d) ϵ is a truncation point of $[0, g(E_\xi^\mathcal{R})]_\mathcal{R}$. It follows by maximality of $\mathcal{R}$ that (e)

$\rho = \rho(\mathcal{N}_\epsilon^-)$ (where $\mathcal{N}_\epsilon^-$ is computed relative to the branch $[0, g(E_\xi^{\mathcal{R}})]_{\mathcal{R}}$) is at most the critical point of the first extender used on $[\epsilon, g(E_\xi^{\mathcal{R}})]_{\mathcal{R}}$. Again by Definition 3.5, (f) $\mathrm{crit}(E_\xi^{\mathcal{R}}) < \rho$. Combining f, e, and c we get $\mathrm{crit}(E_\xi^{\mathcal{R}}) < \lambda_\gamma$. But this implies that ϵ satisfies conditions 1b and 2 of Definition 3.16, so ξ is *not* a counter example to semilinearity. Suppose next that $\epsilon <^{\mathcal{R}} \gamma$ strictly. Since γ is a point on $[0, g(E_\xi^{\mathcal{R}})]_{\mathcal{R}}$ above ϵ we may in d and the definition of ρ for f replace $[0, g(E_\xi^{\mathcal{R}})]_{\mathcal{R}}$ with $[0, \gamma]_{\mathcal{R}}$. Since $\gamma \leq^{\mathcal{R}} \alpha$ we may then replace $[0, \gamma]_{\mathcal{R}}$ with $[0, \alpha]_{\mathcal{R}}$. But then ϵ satisfies conditions 1b and 2 of Definition 3.16, so again ξ is *not* a counter example to semilinearity. Thus in either case we obtain a contradiction.

Let $\zeta = g(E_\xi^{\mathcal{R}})$. We have $\zeta < \alpha$ by the above, and $\mathrm{crit}(E_\xi^{\mathcal{R}})$ lies in the interval $[\mathrm{crit}(E_\zeta^{\mathcal{R}}), \mathrm{lh}(E_\zeta^{\mathcal{R}}))$ by Definition 3.5. We shall show that $\mathcal{N}_\zeta^{\mathcal{R}}$ is not domestic, thereby deriving a contradiction to our assumption that $\mathcal{N}$ is domestic. The witness that $\mathcal{N}_\zeta^{\mathcal{R}}$ is not domestic will be the extender $E_\zeta^{\mathcal{R}}$.

Since $g(E_\xi^{\mathcal{R}}) < \alpha$ and $\xi \geq \alpha$ we have $\mathrm{crit}(E_\xi^{\mathcal{R}}) < \delta_\alpha \leq \mathrm{lh}(E_\xi^{\mathcal{R}})$. Our assumption $(*)$ and the elementarity of the embedding coded by $E_\xi^{\mathcal{R}}$ thus imply that $\mathrm{crit}(E_\xi^{\mathcal{R}})$ is a limit of Woodin cardinals in $\mathcal{N}_\xi^{\mathcal{R}}$ cut at the successor of $\mathrm{crit}(E_\xi^{\mathcal{R}})$. Since $\mathcal{N}_\zeta^{\mathcal{R}} \| \langle \gamma_\zeta, 0\rangle$ and $\mathcal{N}_\xi^{\mathcal{R}}$ have the same subsets of $\mathrm{crit}(E_\xi^{\mathcal{R}})$ it follows that $\mathrm{crit}(E_\xi^{\mathcal{R}})$ is a limit of Woodin cardinals in $\mathcal{N}_\zeta^{\mathcal{R}} \| \langle \gamma_\zeta, 0\rangle$. This, our knowledge that $\mathrm{crit}(E_\xi^{\mathcal{R}}) \in [\mathrm{crit}(E_\zeta^{\mathcal{R}}), \mathrm{lh}(E_\zeta^{\mathcal{R}}))$, and the elementarity of the embedding coded by $E_\zeta^{\mathcal{R}}$, imply that $\mathrm{crit}(E_\zeta^{\mathcal{R}})$ is a limit of Woodin cardinals in $\mathcal{N}_\zeta^{\mathcal{R}} \| \langle \gamma_\zeta, 0\rangle$. Condition 1 in definition 3.1 therefore holds, and it remains only to verify condition 2. Assume for contradiction that condition 2 fails, and fix some $\beta < \mathrm{crit}(E_\zeta^{\mathcal{R}})$ so that there are no cardinals of $\mathcal{N}_\zeta^{\mathcal{R}}$ which *lie between* β and $\mathrm{crit}(E_\zeta^{\mathcal{R}})$ and are *strong* to $\mathrm{crit}(E_\zeta^{\mathcal{R}})$. Using the elementarity of the embedding coded by $E_\zeta^{\mathcal{R}}$ we have

$(**)$ No $\kappa \in [\mathrm{crit}(E_\zeta^{\mathcal{R}}), \mathrm{lh}(E_\zeta^{\mathcal{R}}))$ is strong to $\mathrm{lh}(E_\zeta^{\mathcal{R}})$ in $\mathcal{N}_\zeta^{\mathcal{R}} \| \langle \gamma_\zeta, 0\rangle$.

Remark 3.25 Note our use of the fact that the embedding coded by $E_\zeta^{\mathcal{R}}$ sends $\mathrm{crit}(E_\zeta^{\mathcal{R}})$ to $\mathrm{lh}(E_\zeta^{\mathcal{R}})$. We are thus making implicit use of the method of indexing explained in Section 2; if the length of an extender were defined differently we would not be able to claim $(**)$ as stated, and the interaction with condition 3 below would be ruined.

We have at our disposal ξ and an extender E on the sequence of $\mathcal{N}_\xi^{\mathcal{R}}$ (namely $E_\xi^{\mathcal{R}}$), with the property that

1. $\xi > \zeta$;
2. $\mathrm{lh}(E) \geq \mathrm{lh}(E_\zeta^{\mathcal{R}})$; and
3. $\mathrm{crit}(E)$ lies in the interval $[\mathrm{crit}(E_\zeta^{\mathcal{R}}), \mathrm{lh}(E_\zeta^{\mathcal{R}}))$.

(Let us point out that 3 is a direct consequence of the definition of $g(E)$ in 3.5.) To derive a contradiction to $(**)$ it is enough to show that the restriction of E

to any ordinal $\tau < \mathrm{lh}(E^{\mathcal{R}}_\zeta)$ belongs to $\mathcal{N}^{\mathcal{R}}_\zeta \| \langle \gamma_\zeta, 0\rangle$; E is indexed above γ_ζ and so these restrictions certainly are strong extenders of $\mathcal{N}^{\mathcal{R}}_\zeta \| \langle \gamma_\zeta, 0\rangle$. In light of 3 these restrictions would witness that $\kappa = \mathrm{crit}(E)$ contradicts $(**)$. The argument we give below doesn't quite do this. Instead we will either prove that the above restrictions belong to $\mathcal{N}^{\mathcal{R}}_\zeta \| \langle \gamma_\zeta, 0\rangle$, or else will produce $\tilde{\xi} < \xi$ and an extender $\tilde{E}$ which satisfy 1–3 above. Applying this argument inductively (reducing ξ as necessary) gives a contradiction to $(**)$, thereby proving that condition 2 in Definition 3.1 holds.

Let us begin the argument. We follow a line of reasoning similar to one used in [Jen97, §6]. By 1, γ_ζ is a cardinal of $\mathcal{N}^{\mathcal{R}}_\xi$. If E is *not* the last extender predicate of $\mathcal{N}^{\mathcal{R}}_\xi$ then the extender E *belongs* to $\mathcal{N}^{\mathcal{R}}_\xi$. By acceptability, for any $\tau < \mathrm{lh}(E^{\mathcal{R}}_\zeta)$ the restriction $E|\tau$ belongs to $\mathcal{N}^{\mathcal{R}}_\xi \| \langle \gamma_\zeta, 0\rangle$. But $\mathcal{N}^{\mathcal{R}}_\xi$ and $\mathcal{N}^{\mathcal{R}}_\zeta$ agree up to γ_ζ. Thus $E|\tau$ belongs to $\mathcal{N}^{\mathcal{R}}_\zeta \| \langle \gamma_\zeta, 0\rangle$ and we are done.

Consider then the remaining case, that E is the *last* extender predicate of $\mathcal{N}^{\mathcal{R}}_\xi$. First let us suppose that on the branch of $\mathcal{R}$ leading to ξ there *are* truncation points which are greater than ζ. Let $\tilde{\xi}$ be the last such truncation point, so that $\mathcal{R}$ defines an elementary embedding $i = i^{\mathcal{R}}_{\tilde{\xi},\xi}: \mathcal{N}^{-}_{\tilde{\xi}} \to \mathcal{N}^{\mathcal{R}}_\xi$, and $\mathcal{N}^{-}_{\tilde{\xi}}$ is a strict initial segment of $\mathcal{N}^{\mathcal{R}}_{\tilde{\xi}}$. Let $\tilde{E}$ be the last extender predicate of $\mathcal{N}^{-}_{\tilde{\xi}}$. The map i thus sends $\tilde{E}$ to E. Observe that $\mathrm{crit}(i)$ cannot lie in $[\mathrm{crit}(E^{\mathcal{R}}_\zeta), \mathrm{lh}(E^{\mathcal{R}}_\zeta))$ since any extender whose critical point lies in this interval would be applied to $\mathcal{N}^{\mathcal{R}}_\zeta$ or a premouse before it, and therefore would not be used on the branch $[\tilde{\xi}, \xi]_{\mathcal{R}}$. $\mathrm{crit}(i)$ also cannot be smaller than $\mathrm{crit}(E^{\mathcal{R}}_\zeta)$; if it were then the range of i would contain no elements of the interval $[\mathrm{crit}(E^{\mathcal{R}}_\zeta), \mathrm{lh}(E^{\mathcal{R}}_\zeta)]$, but we know that the range of i does contain such an element, namely $\mathrm{crit}(E) = i(\mathrm{crit}(\tilde{E}))$. Thus we conclude that $\mathrm{crit}(i) \geq \mathrm{lh}(E^{\mathcal{R}}_\zeta)$. It follows that $\mathrm{lh}(\tilde{E})$ must be at least $\mathrm{lh}(E^{\mathcal{R}}_\zeta)$, and that $\mathrm{crit}(\tilde{E})$ equals $\mathrm{crit}(E)$. Thus $\tilde{\xi} < \xi$ and $\tilde{E}$ satisfy 1–3.

Let us finally consider the case that E is the last extender predicate of $\mathcal{N}^{\mathcal{R}}_\xi$, and there are *no* truncation above ζ on the branch of $\mathcal{R}$ leading to ξ. Let $\bar{\xi}$ be the largest element of $[0, \xi]_{\mathcal{R}}$ which is smaller than or equal to ζ, and let $\tilde{\xi}+1$ be its successor on this branch. Then $\tilde{\xi}+1 \leq \xi$ so that certainly $\tilde{\xi} < \xi$. Let $\bar{\mathcal{N}}$ denote either $\mathcal{N}^{\mathcal{R}}_{\bar{\xi}}$ if $\bar{\xi}$ is not a truncation point of $[\bar{\xi}, \tilde{\xi}]_{\mathcal{R}}$, or $\mathcal{N}^{-}_{\bar{\xi}}$ if it is. In either case, the iteration tree $\mathcal{R}$ defines an elementary embedding $i = i^{\mathcal{R}}_{\bar{\xi},\xi}: \bar{\mathcal{N}} \to \mathcal{N}^{\mathcal{R}}_\xi$. The first extender used to produce i is the extender $E^{\mathcal{R}}_{\tilde{\xi}}$. Let $\tilde{E}$ be $E^{\mathcal{R}}_{\tilde{\xi}}$.

Let $\bar{E}$ be the last extender predicate of $\bar{\mathcal{N}}$, so that $i(\bar{E}) = E$. Thus $\mathrm{crit}(E) = i(\mathrm{crit}(\bar{E}))$ and therefore belongs to the range of i. Now *no* elements of the interval $[\mathrm{crit}(\tilde{E}), \mathrm{lh}(\tilde{E}))$ can belong to the range of i. The normality of $\mathcal{R}$ implies that $\mathrm{lh}(\tilde{E}) \geq \mathrm{lh}(E^{\mathcal{R}}_\zeta) > \mathrm{crit}(E)$. Since $\mathrm{crit}(E)$ belongs to the range of i we conclude that $\mathrm{crit}(\tilde{E}) > \mathrm{crit}(E)$. On the other hand, since $\tilde{E}$ is applied to a premouse of $\mathcal{R}$ indexed at or before ζ, certainly $\mathrm{crit}(\tilde{E})$ is smaller than $\mathrm{lh}(E^{\mathcal{R}}_\zeta)$. $\mathrm{crit}(\tilde{E})$ therefore belongs to $(\mathrm{crit}(E^{\mathcal{R}}_\zeta), \mathrm{lh}(E^{\mathcal{R}}_\zeta))$ and 3 holds. Since $\mathrm{crit}(\tilde{E}) > \mathrm{crit}(E^{\mathcal{R}}_\zeta)$

certainly $\tilde{E} \neq E_\zeta^{\mathcal{R}}$ so $\tilde{\xi} \neq \zeta$ and 1 is satisfied. 2 also is satisfied because of the normality of $\mathcal{R}$. Thus $\tilde{\xi} < \xi$ and $\tilde{E}$ satisfy 1–3 as required. $\square$

Equipped with the previous Lemmas we can begin the proof of Theorem 3.2. Work with $\mathcal{N}$ and ν, π as given by Theorem 3.2 and assume towards a contradiction that $\mathcal{T}$ has two distinct super-realizable branches b and c. Let $\vec{\mathcal{P}}$ and $\vec{\mathcal{Q}}$ be the easy phalanges induced by these branches, and let $\vec{\tau}$ and $\vec{\chi}$ be the super-realizations of these phalanges.[18] We view the premice of $\vec{\mathcal{P}}$ and $\vec{\mathcal{Q}}$ as indexed by their index on $\mathcal{T}$. Thus the last premouse on both $\vec{\mathcal{P}}$ and $\vec{\mathcal{Q}}$ is indexed by $\mathrm{lh}(\mathcal{T})$. Let η denote the length of $\mathcal{T}$.

We shall (roughly speaking) compare $\vec{\mathcal{P}}$ and $\vec{\mathcal{Q}}$ using *balanced* iteration trees, appealing to Lemmas 3.18, 3.19, 3.20, and 3.23 to obtain branches through the iteration trees produced. We shall make sure for each α either that both trees pick unique realizable branches at α, or else both trees are semilinear at α. To secure this second possibility we shall use Lemma 3.24: At α-s where there are two or more realizable branches we will take advantage of the distinct branches to make sure that the hypothesis of Lemma 3.24 —the condition $(*)$ that is— is satisfied. Our argument in these circumstances is similar to the multiple board comparison used by [Ste93], though we shall avoid actually adding boards to the comparison.

Though our construction is really a comparison of $\vec{\mathcal{P}}$ and $\vec{\mathcal{Q}}$, it is perhaps better viewed as the construction of two iteration trees which extend $\mathcal{T}$. On one side we construct an extension $\mathcal{U}$ which uses b as the branch $[0,\eta]_{\mathcal{U}}$, while on the other side we construct an extension $\mathcal{V}$ which uses c as the branch $[0,\eta]_{\mathcal{V}}$.

We construct these iteration trees by induction on $\alpha > \eta$. At stage α we will have on the two sides of the comparison:

A. A (padded) balanced iteration tree $\mathcal{U}\restriction\alpha$ which extends $\mathcal{T}$ and so that $[0,\eta]_{\mathcal{U}} = b$; and similarly $\mathcal{V}\restriction\alpha$ which extends $\mathcal{T}$ and so that $[0,\eta]_{\mathcal{V}} = c$;

B. A set of ordinals $B \cap \alpha$ which is a *closed* subset of α and so that $\eta \in B$;

For $\bar{\alpha} \in B \cap \alpha$ let $b_{\bar{\alpha}}$ denote the branch of $\mathcal{U}$ leading to the $\bar{\alpha}$-th premouse of this tree. Let $\vec{\mathcal{P}}^{\bar{\alpha}}$ be the easy phalanx induced by this branch of $\mathcal{U}$. Define $c_{\bar{\alpha}}$ and $\vec{\mathcal{Q}}^{\bar{\alpha}}$ similarly on the $\mathcal{Q}$ side.

C. Realizations $\vec{\tau}^{\bar{\alpha}}$ of $\vec{\mathcal{P}}^{\bar{\alpha}}$ for $\bar{\alpha} \in B \cap \alpha$, and similarly realizations $\vec{\chi}^{\bar{\alpha}}$ of $\vec{\mathcal{Q}}^{\bar{\alpha}}$ for $\bar{\alpha} \in B \cap \alpha$.

We shall maintain the following conditions

[18] The realization of $\vec{\mathcal{N}}^b$ includes a finite sequence of ordinals in addition to $\vec{\tau}$. The first ordinal in this sequence is ν, and subsequent ordinals in the sequence are determined inductively by condition 2 of Definition 2.27, given $\vec{\mathcal{N}}^b$ and $\vec{\tau}$. Since this finite sequence of ordinals is uniquely determined by $\nu, b, \vec{\tau}$ we suppress its mention below. We similarly suppress mention of the ordinals involved in the realization $\vec{\chi}$ of c, and in all realizations of future branches, in the trees on both sides.

1. Both the iteration trees constructed are balanced above η. Note then that by Lemma 3.14 both trees are sturdy;

2. Above η the iteration trees arise in comparison. In other words, for $\alpha \geq \eta$ the index of the α-th extender used on $\mathcal{U}$ and $\mathcal{V}$ is always the least γ_α which represents a disagreement between the premice $\mathcal{P}_\alpha$ and $\mathcal{Q}_\alpha$. If there are no disagreements we end the construction;

3. For $\bar{\alpha} \in B \cap \alpha$, $\mathcal{U}$ and $\mathcal{V}$ are both semilinear at $\bar{\alpha}$;

4. Any two of the realizations $\vec{\tau}^{\bar{\alpha}}$, $\bar{\alpha} \in B \cap \alpha$ commute. Similarly on the $\mathcal{Q}$ side;

5. If $B \cap \alpha$ has a maximal element, β, then the tree $\mathcal{U}$ picks unique realizable branches above β (wrt $\vec{\tau}^{\beta}$). Similarly on the $\mathcal{Q}$ side.

Having specified conditions 1–5, the construction itself is straightforward. We begin the construction at stage $\alpha = \eta + 1$ with the obvious iteration trees. On the $\mathcal{P}$ side it is the iteration tree which extends $\mathcal{T}$ using b, and on the $\mathcal{Q}$ side it is the iteration tree which extends $\mathcal{T}$ using c. The realizations $\vec{\tau}^{\eta}$ and $\vec{\chi}^{\eta}$ are set equal to $\vec{\tau}$ and $\vec{\chi}$ respectively. We set $B \cap \eta + 1 = \{\eta\}$. By [Ste93, Theorem 2.1] —or rather [Jen97, §6, Lemma 2], as the proof for premice with Jensen's indexing is slightly more complicated— δ_η is a Woodin cardinal of $\mathcal{P}_\alpha \cap \mathcal{Q}_\alpha$, where δ_η is the supremum of indices of extenders used in $\mathcal{T}$. (For convenience let us refer to these indices as γ_ξ, $\xi < \eta$.) It then follows from condition 2 that for any $\tilde{\alpha} \geq \eta$,

$$\mathcal{P}_{\tilde{\alpha}} || \gamma_{\tilde{\alpha}} = \mathcal{Q}_{\tilde{\alpha}} || \gamma_{\tilde{\alpha}} \models \text{“}\delta_\eta \text{ is a Woodin cardinal.”}$$

By Lemma 3.24 it follows that condition 3 holds for $\tilde{\alpha} > \eta$, with $\bar{\alpha} = \eta$.

Suppose now α has been reached, where α is a successor ordinal. We let $\gamma_{\alpha-1}$ be the least index of a disagreement between $P_{\alpha-1}$ and $Q_{\alpha-1}$ if there is a disagreement. On each side we either pad —if $\gamma_{\alpha-1}$ does not index an extender— or extend the iteration tree using the extender indexed at $\gamma_{\alpha-1}$. We extend both iteration trees in a balanced fashion. This is possible by Remark 3.15. We then set $B \cap \alpha + 1 = B \cap \alpha$. This completes the construction in the successor case.

Suppose next α has been reached, where α is a limit ordinal.

Claim 3.26 *There exists a cofinal branch b_α through $\mathcal{U} \restriction \alpha$ and a realization $\vec{\tau}^{\alpha}$ of the easy phalanx induced by this branch, so that $\vec{\tau}^{\alpha}$ commutes with $\vec{\tau}^{\bar{\alpha}}$ for every $\bar{\alpha} \in B \cap \alpha$. Similarly on the $\mathcal{Q}$ side.*

Proof. If $B \cap \alpha$ is bounded in α use Lemma 3.18 (or Lemma 3.19 if $\alpha = \omega_1$) together with condition 5. If $B \cap \alpha$ is cofinal in α use Lemma 3.20 (or Lemma 3.23 if $\alpha = \omega_1$) together with conditions 3 and 4. □

Of the branches given by Claim 3.26, pick cofinal branches b_α and c_α in such a way that either

- There is no disagreement between the direct limits $\mathcal{P}_{b_\alpha}$ and $\mathcal{Q}_{c_\alpha}$; or if this is not possible,

- pick b_α, c_α so as to *minimize* the index γ_α of the first disagreement between $\mathcal{P}_{b_\alpha}$ and $\mathcal{Q}_{c_\alpha}$.

Extend the iteration tree $\mathcal{U}\restriction\alpha$ by letting $\mathcal{P}_\alpha$ be the direct limit along the branch b_α. Work similarly on the $\mathcal{Q}$ side.

If $B\cap\alpha$ is cofinal in α we set by necessity $B\cap\alpha+1 = B\cap\alpha\bigcup\{\alpha\}$. Our choice of b_α together with the fact that all future extensions of $\mathcal{U}\restriction\alpha$ are semilinear at points in $B\cap\alpha$ easily imply that these future extensions are semilinear at α too. The same goes for the $\mathcal{Q}$ side. This secures condition 3 for future extensions, and the construction is then completed for this stage.

Let us then assume finally that $B\cap\alpha$ is bounded in α and thus has a maximal element β. If the branches b_α and c_α given by Claim 3.26 are the *unique* super-realizable (wrt $\vec{\tau}^\beta$ and $\vec{\chi}^\beta$) cofinal branches of $\mathcal{U}\restriction\alpha$ and $\mathcal{V}\restriction\alpha$, then we set $B\cap\alpha+1 = B\cap\alpha$. Condition 5 is maintained and the construction is completed for this stage. If uniqueness fails on either side, we must set $B\cap\alpha+1 = B\cap\alpha\bigcup\{\alpha\}$ in order to maintain 5. The fact that condition 3 continues to hold for future $\tilde{\alpha}$ (with $\bar{\alpha}=\alpha$) follows from Lemma 3.24 together with the following Claim:

Claim 3.27 *Assume that in stage α uniqueness fails on either side of the comparison (or on both). Let $\delta_\alpha = \sup\{\gamma_\xi \mid \xi<\alpha\}$. Then for all $\tilde{\alpha}\geq\alpha$,*

$$\mathcal{P}_{\tilde{\alpha}}\|\gamma_{\tilde{\alpha}} = \mathcal{Q}_{\tilde{\alpha}}\|\gamma_{\tilde{\alpha}} \models \text{“}\delta_\alpha \textit{ is a Woodin cardinal.”}$$

Proof. Assume for definitiveness that uniqueness fails on the $\mathcal{P}$ side, so that there is a cofinal realizable branch b' of $\mathcal{U}\restriction\alpha$ which is distinct from b_α. Note that both $\mathcal{P}_{b_\alpha}$ and $\mathcal{P}_{b'}$ disagree with $\mathcal{Q}_\alpha$ for otherwise the comparison would have ended at stage α. By [Jen97, §6, Lemma 2] the fact that b_α and b' are *distinct* implies that δ_α is a Woodin cardinal in $\mathcal{P}_{b_\alpha}\cap\mathcal{P}_{b'}$.

Consider first the possibility that δ_α fails to be a Woodin cardinal in $\mathcal{P}_\alpha = \mathcal{P}_{b_\alpha}$. It follows that $\mathcal{P}_{b_\alpha}$ and $\mathcal{P}_{b'}$ must disagree. Let γ be the index of the first disagreement between these premice. Then either

i. $\mathcal{P}_{b_\alpha}$ and $\mathcal{Q}_\alpha$ disagree before $\gamma+1$; or else

ii. $\mathcal{P}_{b'}$ and $\mathcal{Q}_\alpha$ disagree before $\gamma+1$.

(For otherwise $\mathcal{P}_{b_\alpha}\|\gamma+1$ and $\mathcal{P}_{b'}\|\gamma+1$ would both be initial segments of $\mathcal{Q}_\alpha$, and would hence agree.) Our choice of b_α minimized the first disagreement γ_α. From i,ii it therefore follows that

$$\gamma_\alpha \leq \gamma.$$

But then δ_α is Woodin in $\mathcal{P}_\alpha\|\gamma_\alpha$ and so the conclusion of Claim 3.27 holds for $\tilde{\alpha}=\alpha$. With regard to greater $\tilde{\alpha}$: γ_α is a cardinal of $\mathcal{P}_{\tilde{\alpha}}$ and $\mathcal{P}_{\tilde{\alpha}}\|\gamma_\alpha = \mathcal{P}_\alpha\|\gamma_\alpha$. Thus δ_α is a Woodin cardinal in $\mathcal{P}_{\tilde{\alpha}}$ and so certainly a Woodin cardinal in $\mathcal{P}_{\tilde{\alpha}}\|\gamma_{\tilde{\alpha}}$.

This completes the proof of Claim 3.27 under the assumption that δ_α is not a Woodin cardinal of $\mathcal{P}_\alpha$. Considering the other possibility: If δ_α is a Woodin cardinal of $\mathcal{P}_\alpha$ then it is a Woodin cardinal of $\mathcal{P}_{\tilde{\alpha}}$ for all $\tilde{\alpha} \geq \alpha$, and so certainly a Woodin cardinal of $\mathcal{P}_{\tilde{\alpha}} \| \gamma_{\tilde{\alpha}}$. $\square$

We continue the construction until reaching a stage where there are no further disagreements, or until we reach $\alpha = \omega_1 + 1$, which ever comes first. As always in the case of comparisons we have the following:

Fact 3.28 *Two extenders E and F are said to be* **compatible** *if they have the same critical point κ; the same domain; and for each $A \subset \kappa$ in their domain, $E(A) \cap \lambda = F(A) \cap \lambda$ where $\lambda = \min\{\mathrm{lh}(E), \mathrm{lh}(F)\}$.*

Let E, F be two extenders which were used in the construction, on $\mathcal{U}$ and on $\mathcal{V}$ respectively. If E and F are compatible then they are in fact equal*, and were used at the* same *stage on the trees,* before *η. In other words the extender $E = F$ was used on $\mathcal{T}$.*

Proof (Sketch). Say $E = E^{\mathcal{U}}_\zeta$ and $F = E^{\mathcal{V}}_\xi$. If $\zeta = \xi$ then compatibility implies $E = F$ and the nature of our construction is such that this can only happen before η. So suppose $\zeta \neq \xi$; say $\zeta < \xi$ for definitiveness. Note γ_ζ is a cardinal of $\mathcal{N}^{\mathcal{V}}_\xi$, $\mathrm{lh}(E) < \mathrm{lh}(F)$, and (by *compatibility*) $F|\mathrm{lh}(E) = E$. One can show $E = F|\mathrm{lh}(E) \in \mathcal{N}^{\mathcal{V}}_\xi$. (This may require use of the *initial segment condition*, 2.4(5).) But $E^{\mathcal{U}}_\zeta \in \mathcal{N}^{\mathcal{V}}_\xi$ implies γ_ζ is not a cardinal of $\mathcal{N}^{\mathcal{V}}_\xi$, a contradiction. $\square$

Fact 3.28 is an inner model staple. As always it can be used to argue that the comparison terminates at a *countable* α. We must however take some care, since our iteration trees are balanced rather than non-overlapping.

Claim 3.29 *The above construction must terminate at a* countable *stage α.*

Proof. Assume otherwise. Let $\tilde{b}, \tilde{c}$ be the final branches (of length ω_1) on the trees $\mathcal{U}$ and $\mathcal{V}$. Note that both $\tilde{b}$ and $\tilde{c}$ are closed unbounded subsets of ω_1. Since both $\mathcal{Q}$ and $\mathcal{P}$ consist of countable premice, there must exist a club $C \subset \omega_1 - \eta$ of limit ordinals so that $C \subset \tilde{b}$; $C \subset \tilde{c}$; and $i^{\mathcal{U}}_{\xi,\omega_1}(\xi) = i^{\mathcal{V}}_{\xi,\omega_1}(\xi) = \omega_1$ for each $\xi \in C$. For each $\xi \in C$ let α_ξ be the *largest* α such that $\alpha + 1 \in \tilde{b}$ and $\mathrm{crit}(E^{\mathcal{U}}_\alpha) \leq \xi$. (Note that there can be only finitely many such α-s, since the direct limit along $\tilde{b}$ is wellfounded.) Define β_ξ similarly on the $\mathcal{Q}$ side. By thinning C if needed, we may assume that for $\xi \in C$ in fact $\mathrm{crit}(E^{\mathcal{U}}_{\alpha_\xi}) = \xi$ and $\mathrm{crit}(E^{\mathcal{V}}_{\beta_\xi}) = \xi$. Since $\mathcal{U}$ is balanced, $\mathrm{crit}(i^{\mathcal{U}}_{\alpha_\xi+1,\omega_1})$ cannot lie in $[\mathrm{crit}(E^{\mathcal{U}}_{\alpha_\xi}), \mathrm{lh}(E^{\mathcal{U}}_{\alpha_\xi}))$. The maximality in our choice of α_ξ implies further that $\mathrm{crit}(i^{\mathcal{U}}_{\alpha_\xi+1,\omega_1})$ is strictly greater than $\mathrm{crit}(E^{\mathcal{U}}_{\alpha_\xi}) = \xi$. Combining these two statements we conclude that

$$(*) \qquad \mathrm{crit}(i^{\mathcal{U}}_{\alpha_\xi+1,\omega_1}) \quad \geq \quad \mathrm{lh}(E^{\mathcal{U}}_{\alpha_\xi})$$

and similarly on the $\mathcal{Q}$ side. The rest of the proof follows standard lines: By Fact 3.28 there must exist $X_\xi \subset \xi$ so that

$$E^{\mathcal{U}}_{\alpha_\xi}(X_\xi) \cap \lambda_\xi \quad \neq \quad E^{\mathcal{V}}_{\beta_\xi}(X_\xi) \cap \lambda_\xi$$

for each $\xi \in C$, where $\lambda_\xi = \min\{\mathrm{lh}(E^{\mathcal{U}}_{\alpha_\xi}), \mathrm{lh}(E^{\mathcal{V}}_{\beta_\xi})\}$. Using $(*)$ one sees that

$$i^{\mathcal{U}}_{\xi,\tilde{\xi}}(X_\xi) \neq i^{\mathcal{V}}_{\xi,\tilde{\xi}}(X_\xi)$$

for any $\tilde{\xi} > \max\{\alpha_\xi, \beta_\xi\}$ in C. But now a standard pressing down argument produces a contradiction. $\square$

Readers familiar with the argument used in the case of non-overlapping trees will note that the only difference between this standard argument and our proof of Claim 3.29 is in the choice of $\alpha_\xi + 1$ and $\beta_\xi + 1$. In the case of non-overlapping trees these ordinals are the successors of ξ in $\tilde{b}$ and $\tilde{c}$ respectively, and this is enough to secure $(*)$. In our case securing $(*)$ requires going a bit further, to the last points where an extender with critical point ξ (or smaller) is used on $\tilde{b}, \tilde{c}$.

Let $\theta < \omega_1$ be the end stage of the comparison, so that there are no disagreements between $\mathcal{P}_\theta$ and $\mathcal{Q}_\theta$. What follows is a short discussion with one recurring motif. We shall argue under various circumstances that embeddings induced by the two iteration trees are *equal*. From this we shall conclude that the extenders giving rise to these embeddings are *compatible*, and thus derive a contradiction using Fact 3.28. We shall consider various cases, but in all cases we end with a contradiction. Thus we shall finally obtain a contradiction to our initial assumption that there are distinct realizable branches through $\mathcal{T}$, completing the proof of Theorem 3.2. Before starting with this discussion, let us make the *recurrent motif* precise:

Claim 3.30 *Let α be a point on $[0,\theta]_{\mathcal{U}}$ which is greater than or equal to the last truncation point, if there are truncation points on this branch. Recall that $\mathcal{P}^-_\alpha$ is an initial segment of $\mathcal{P}_\alpha$ (it is strict iff α is a truncation point) and $\mathcal{U}$ defines $i^{\mathcal{U}}_{\alpha,\theta}\colon \mathcal{P}^-_\alpha \to \mathcal{P}_\theta$. Pick β on $[0,\theta]_{\mathcal{V}}$ similarly.*

Assume that $\mathcal{P}_\theta = \mathcal{Q}_\theta$, that $\mathcal{P}^-_\alpha = \mathcal{Q}^-_\beta$, and that $i^{\mathcal{U}}_{\alpha,\theta} = i^{\mathcal{V}}_{\beta,\theta}$. Then there exists an ordinal $\xi < \eta$ such that

- *ξ is greater than both α and β;*
- *ξ belongs to both $[0,\theta]_{\mathcal{U}}$ and $[0,\theta]_{\mathcal{V}}$; and*
- *$i^{\mathcal{U}}_{\xi,\theta}\colon \mathcal{N}_\xi \to \mathcal{P}_\theta$ and $i^{\mathcal{V}}_{\xi,\theta}\colon \mathcal{N}_\xi \to \mathcal{Q}_\theta$ are* equal.

Proof. Let $\tilde{\mathcal{N}}$ denote the common value of $\mathcal{P}_\theta, \mathcal{Q}_\theta$, let j denote the embedding $i^{\mathcal{U}}_{\alpha,\theta} = i^{\mathcal{V}}_{\beta,\theta}$, and let κ be the critical point of j.

Let $\tilde{\alpha}+1$ be the last point on $[\alpha,\theta]_{\mathcal{U}}$ for which $\mathrm{crit}(E^{\mathcal{U}}_{\tilde{\alpha}}) = \kappa$ (there can be at most finitely many such points). The fact that $\mathcal{U}$ is balanced above η and non-overlapping up to η implies that $\mathrm{crit}(i^{\mathcal{U}}_{\tilde{\alpha}+1,\theta}) \geq \mathrm{lh}(E^{\mathcal{U}}_{\tilde{\alpha}})$. Similarly on the $\mathcal{Q}$ side let $\tilde{\beta}+1$ be the last point on $[\beta,\theta]_{\mathcal{V}}$ so that $\mathrm{crit}(E^{\mathcal{V}}_{\tilde{\beta}}) = \kappa$, and obtain $\mathrm{crit}(i^{\mathcal{V}}_{\tilde{\beta}+1,\theta}) \geq \mathrm{lh}(E^{\mathcal{V}}_{\tilde{\beta}})$. Thus both $i^{\mathcal{U}}_{\tilde{\alpha}+1,\theta}$ and $i^{\mathcal{V}}_{\tilde{\beta}+1,\theta}$ have critical points at least

λ where $\lambda = \min\{\mathrm{lh}(E^{\mathcal{U}}_{\tilde{\alpha}}), \mathrm{lh}(E^{\mathcal{V}}_{\tilde{\beta}})\}$. Since $i^{\mathcal{U}}_{\alpha,\theta} = i^{\mathcal{V}}_{\beta,\theta}$ we now conclude that $E^{\mathcal{U}}_{\tilde{\alpha}}$ and $E^{\mathcal{V}}_{\tilde{\beta}}$ are *compatible*.

Applying Fact 3.28 we are left with the knowledge that $\tilde{\alpha} = \tilde{\beta} < \eta$ and $E^{\mathcal{U}}_{\tilde{\alpha}} = E^{\mathcal{V}}_{\tilde{\beta}}$. In particular both extenders have length equal to λ. Let ξ be the common value of $\tilde{\alpha}+1$ and $\tilde{\beta}+1$, and let F be the common value of $E^{\mathcal{U}}_{\tilde{\alpha}}$ and $E^{\mathcal{V}}_{\tilde{\beta}}$. Then $\xi < \eta$ belongs to both $(\alpha,\theta]_{\mathcal{U}}$ and $(\beta,\theta]_{\mathcal{V}}$ as required. Note that in particular we have $\alpha = \beta$ and $i^{\mathcal{U}}_{\alpha,\xi} = i^{\mathcal{U}}_{\beta,\xi}$, as both embeddings are equal to $i^{\mathcal{T}}_{\alpha=\beta,\xi}$. To complete the proof of the Claim it remains to verify that $i^{\mathcal{U}}_{\xi,\theta} = i^{\mathcal{V}}_{\xi,\theta}$. To that end note that we have on the two sides the following diagrams:

$$\begin{tikzcd} & \tilde{\mathcal{N}} \\ \mathcal{P}^-_\alpha \arrow[ur, "i^{\mathcal{U}}_{\alpha,\theta}"] \arrow[r, "i^{\mathcal{U}}_{\alpha,\xi}"] & \mathcal{N}_\xi \arrow[u, "i^{\mathcal{U}}_{\xi,\theta}"'] \end{tikzcd} \qquad \begin{tikzcd} & \tilde{\mathcal{N}} \\ \mathcal{Q}^-_\beta \arrow[ur, "i^{\mathcal{V}}_{\beta,\theta}"] \arrow[r, "i^{\mathcal{V}}_{\alpha,\xi}"] & \mathcal{N}_\xi \arrow[u, "i^{\mathcal{V}}_{\xi,\theta}"'] \end{tikzcd}$$

Since the critical points of the vertical maps on both sides are greater than $\lambda = \mathrm{lh}(F)$ it follows immediately that the vertical maps are the *factor* embeddings. To be more precise (say on the $\mathcal{P}$ side), any element of $\mathcal{N}_\xi$ has the form $\left(i^{\mathcal{U}}_{\alpha,\xi}(f)\right)(\vec{\mu})$ where $\vec{\mu}$ is some tuple of ordinals below λ. (We are using here the fact that $\mathcal{U}$ is normal; all extenders on $[\alpha,\xi]_{\mathcal{U}}$ have length at most λ.) $i^{\mathcal{U}}_{\xi,\theta}$ *must* send this element to $\left(i^{\mathcal{U}}_{\alpha,\theta}(f)\right)(\vec{\mu})$. The same reasoning applies on the $\mathcal{Q}$ side. Thus in both diagrams the "vertical" maps can be recovered from the diagonal and horizontal maps. Since the diagonal and the horizontal maps are the same on both sides, we conclude that the vertical maps are the same on both sides, as required. $\square$

Corollary 3.31 *Let $\bar{\alpha}$ be a point on $[0,\theta]_{\mathcal{U}}$ which is greater than or equal to the last truncation point (if there are truncations on this branch). Pick $\bar{\beta}$ on $[0,\theta]_{\mathcal{V}}$ similarly.*

*Assume that $\mathcal{P}_\theta = \mathcal{Q}_\theta$. Then it can*not *be the case that $\mathcal{P}^-_{\bar{\alpha}} = \mathcal{Q}^-_{\bar{\beta}}$ and $i^{\mathcal{U}}_{\bar{\alpha},\theta} = i^{\mathcal{V}}_{\bar{\beta},\theta}$.*

Proof. Assume otherwise. We can thus apply Claim 3.30 with $\alpha = \bar{\alpha}$ and $\beta = \bar{\beta}$. Let $\xi_0 < \eta$ be the ordinal produced by Claim 3.30. Working inductively we construct an increasing sequence of ordinals $\xi_\iota < \eta$ so that (a) ξ_ι belongs to both $[0,\theta]_{\mathcal{U}}$ and $[0,\theta]_{\mathcal{V}}$; and (b) $i^{\mathcal{U}}_{\xi_\iota,\theta} = i^{\mathcal{V}}_{\xi_\iota,\theta}$. We have already defined ξ_0. To define $\xi_{\iota+1}$, apply Claim 3.30 with $\alpha = \beta = \xi_\iota$. In the case of limit ι let $\xi_\iota = \sup\{\xi_{\bar{\iota}} \mid \bar{\iota} < \iota\}$. That ξ_ι belongs to $[0,\theta]_{\mathcal{U}}$ and $[0,\theta]_{\mathcal{V}}$ follows from the induction and the fact that both these branches are *closed* sets of ordinals.

Continue this construction until the sequence $\langle \xi_\iota \rangle$ is cofinal in η. Let d be the cofinal branch through $\mathcal{T}$ which contains the ordinals of this sequence. Then $d \subset [0,\theta]_{\mathcal{U}}$ and also $d \subset [0,\theta]_{\mathcal{V}}$. Since both branches are closed we have $\eta \in [0,\theta]_{\mathcal{U}}$

and also $\eta \in [0,\theta]_{\mathcal{V}}$. From this we conclude that both $[0,\eta]_{\mathcal{U}}$ and $[0,\eta]_{\mathcal{V}}$ are equal to d. But this contradicts our initial assumption that $[0,\eta]_{\mathcal{U}} = b$ and $[0,\eta]_{\mathcal{V}} = c$ are *distinct* branches through $\mathcal{T}$. $\square$

Equipped with Corollary 3.31 let us proceed with the discussion. Our aim is to obtain a final contradiction.

Claim 3.32 *At least one of the following two possibilities holds:*

1. *There are no truncations on the branch of $\mathcal{U}$ leading to $\mathcal{P}_\theta$; or*
2. *There are no truncations on the branch of $\mathcal{V}$ leading to $\mathcal{Q}_\theta$.*

Proof. Assume for contradiction that there are truncations on both sides. Then by necessity both $\mathcal{P}_\theta$ and $\mathcal{Q}_\theta$ are not sound. In particular neither one can be a strict initial segment of the other and so $\mathcal{P}_\theta = \mathcal{Q}_\theta$. Let us use $\tilde{\mathcal{N}}$ to denote their common value. Let $\bar{\alpha}$ be the last truncation point on $[0,\theta]_{\mathcal{U}}$ and let $\bar{\beta}$ be the last truncation point on $[0,\theta]_{\mathcal{V}}$. Since $\mathcal{U}$ is *sturdy* we may appeal to Lemma 2.25(b). Thus $\mathcal{P}^-_{\bar{\alpha}} = \mathfrak{C}(\tilde{\mathcal{N}})$ and $i^{\mathcal{U}}_{\bar{\alpha},\theta}\colon \mathcal{P}^-_{\bar{\alpha}} \to \tilde{\mathcal{N}}$ is the anti-core embedding. Similarly on the $\mathcal{Q}$ side, $\mathcal{Q}^-_{\bar{\beta}} = \mathfrak{C}(\tilde{\mathcal{N}})$ and $i^{\mathcal{V}}_{\bar{\beta},\theta}\colon \mathcal{Q}^-_{\bar{\beta}} \to \tilde{\mathcal{N}}$ is the anti-core embedding. In particular $i^{\mathcal{U}}_{\bar{\alpha},\theta}$ is *equal* to $i^{\mathcal{V}}_{\bar{\beta},\theta}$, but this is a contradiction by Corollary 3.31. $\square$

Claim 3.33 *There are in fact no truncations on* either *of $[0,\theta]_{\mathcal{U}}$, $[0,\theta]_{\mathcal{V}}$.*

Proof. By Claim 3.32 there are no truncations on at least one of the two sides. Assume for definitiveness that there are no truncations on $[0,\theta]_{\mathcal{U}}$, so that $i^{\mathcal{U}}_{0,\theta}$ embeds $\mathcal{N}$ into $\mathcal{P}_\theta$.

Assume for contradiction that there are truncations on $[0,\theta]_{\mathcal{V}}$. Then $\mathcal{Q}_\theta$ is not sound and so by necessity $\mathcal{P}_\theta$ is an initial segment of $\mathcal{Q}_\theta$. Our construction of $\mathcal{V}$ was such that $[0,\theta]_{\mathcal{V}}$ is super-realizable (wrt the original realization ν, π of $\mathcal{N}$). Since there are truncations on this branch, super-realizability implies that $\mathcal{Q}_\theta$ embeds weakly into $\mathcal{M}_{\bar{\nu}}$ for some $\bar{\nu}$ which is strictly *smaller* than ν. Let $\tilde{\chi}\colon \mathcal{Q}_\theta \to \mathcal{M}_{\bar{\nu}}$ witness this. Let $\hat{\mathcal{P}} = \tilde{\chi}(\mathcal{P}_\theta)$. Then $\hat{\mathcal{P}}$ is an initial segment of $\mathcal{M}_{\bar{\nu}}$ and $\tilde{\chi}\restriction\mathcal{P}_\theta\colon \mathcal{P}_\theta \to \hat{\mathcal{P}}$ is weak. Let $\bar{\bar{\nu}} = Res_{\bar{\nu}}[\hat{\mathcal{P}}]$, and let $\tilde{\pi} = \sigma_{\bar{\nu}}[\hat{\mathcal{P}}] \circ (\tilde{\chi}\restriction\mathcal{P}_\theta) \circ i^{\mathcal{U}}_{0,\theta}$.

Then $\tilde{\pi}\colon \mathcal{N} \to \mathcal{M}_{\bar{\bar{\nu}}}$ is weak, and $\bar{\bar{\nu}} \leq \bar{\nu} < \nu$. But this contradicts our initial assumption in Theorem 3.2 regarding the minimality of ν. $\square$

Given Claim 3.33, the fact that $[0,\theta]_{\mathcal{U}}$ and $[0,\theta]_{\mathcal{V}}$ are super-realizable says simply that there exist weak embeddings $\tilde{\tau}\colon \mathcal{P}_\theta \to \mathcal{M}_\nu$ and $\tilde{\chi}\colon \mathcal{Q}_\theta \to \mathcal{M}_\nu$ so that $\pi = \tilde{\tau} \circ i^{\mathcal{U}}_{0,\theta}$ (on the $\mathcal{P}$ side) and $\pi = \tilde{\chi} \circ i^{\mathcal{V}}_{0,\theta}$ (on the $\mathcal{Q}$ side).

Claim 3.34 $\mathcal{P}_\theta = \mathcal{Q}_\theta$.

Proof. Assume for contradiction that this is not the case — say $\mathcal{P}_\theta$ is a *strict* initial segment of $\mathcal{Q}_\theta$. Let $\hat{\mathcal{P}} = \tilde{\chi}(\mathcal{P}_\theta)$, so that $\hat{\mathcal{P}}$ is a *strict* initial segment of $\mathcal{M}_\nu$ and there is a weak embedding $\tilde{\chi}\restriction\mathcal{P}_\theta\colon \mathcal{P}_\theta \to \hat{\mathcal{P}}$. Let $\bar{\nu} = Res_\nu[\hat{\mathcal{P}}]$. As $\hat{\mathcal{P}}$ is a strict initial segment of $\mathcal{M}_\nu$ we have $\bar{\nu} < \nu$ strictly.

Let $\tilde{\pi} = \sigma_\nu[\hat{\mathcal{P}}] \circ (\tilde{\chi}\restriction\mathcal{P}_\theta) \circ i^{\mathcal{U}}_{0,\theta}$. Then $\tilde{\pi}\colon \mathcal{N} \to \mathcal{M}_{\bar{\nu}}$ is weak and $\bar{\nu} < \nu$. As before this contradicts the minimality of ν, assumed in Theorem 3.2. $\square$

Claim 3.35 *The embeddings $i^{\mathcal{U}}_{0,\theta}$ and $i^{\mathcal{V}}_{0,\theta}$ are equal.*

Proof. Assume for contradiction that they are not. Let $x \in \mathcal{N}$ be least, with respect to $\vec{e}$, such that $i^{\mathcal{U}}_{0,\theta}(x) \neq i^{\mathcal{V}}_{0,\theta}(x)$. Assume for definitiveness that $i^{\mathcal{U}}_{0,\theta}(x) <_{\mathrm{L}} i^{\mathcal{V}}_{0,\theta}(x)$, where $<_{\mathrm{L}}$ is the order of constructibility (in $\mathcal{P}_\theta = \mathcal{Q}_\theta$). Let $\tilde{\pi} = \tilde{\chi} \circ i^{\mathcal{U}}_{0,\theta}$. Then

$$\begin{aligned} \tilde{\pi}(x) &= \tilde{\chi}\left(i^{\mathcal{U}}_{0,\theta}(x)\right) \\ &<_{\mathrm{L}} \tilde{\chi}\left(i^{\mathcal{V}}_{0,\theta}(x)\right) \\ &= \pi(x). \end{aligned}$$

Similarly for $y \in \mathcal{N}$ which is enumerated before x in $\vec{e}$ we compute

$$\begin{aligned} \tilde{\pi}(y) &= \tilde{\chi}\left(i^{\mathcal{U}}_{0,\theta}(y)\right) \\ &= \tilde{\chi}\left(i^{\mathcal{V}}_{0,\theta}(y)\right) \\ &= \pi(y). \end{aligned}$$

Thus $\tilde{\pi}: \mathcal{N} \to \mathcal{M}_\nu$ is a weak embedding which is *to the left* of π. But this contradicts our assumption in Theorem 3.2 that π is left most. $\Box$

Finally we reached the conclusion that $\mathcal{P}_\theta = \mathcal{Q}_\theta$, there are no truncations on $[0,\theta]_{\mathcal{U}}$ and $[0,\theta]_{\mathcal{V}}$, and $i^{\mathcal{U}}_{0,\theta} = i^{\mathcal{V}}_{0,\theta}$. But this contradicts Corollary 3.31, applied with $\bar{\alpha} = \bar{\beta} = 0$. This final contradiction completes the proof of Theorem 3.2. $\Box$(Theorem 3.2)

The reader familiar with [NS99] may note a certain general pattern in our argument. Roughly speaking it follows the lines "Comparison plus Dodd–Jensen $\Rightarrow$ Uniqueness." This implication was noticed by the third author, and it seems to hold in many general settings.

Before closing let us take note of our use of the assumption that $\mathcal{T}$ is normal, maximal, and *non-overlapping* in Theorem 3.2. We have certainly throughout the argument used the fact that $\mathcal{U}, \mathcal{V}$ are sturdy. (This was especially important for Claim 3.32.) The sturdiness of $\mathcal{U}$ and $\mathcal{V}$ traces back to Lemma 3.14 with $\mathcal{R} \restriction \eta = \mathcal{T}$, and to Lemma 3.9. The argument there requires some assumptions on $\mathcal{T}$. It is not enough to assume that $\mathcal{T}$ is sturdy (the problem is in Steps 7 and 9 of Claim 3.11) but it is enough to assume that $\mathcal{T}$ is suitable. The only other occurrence of $\mathcal{T}$ is in the proof of Claim 3.30. After defining $\tilde{\alpha}+1$ (and similarly $\tilde{\beta}+1$) in this proof we argue that $i^{\mathcal{U}}_{\tilde{\alpha}+1,\theta}$ has critical point equal to or *greater* than λ. Our definition of $\tilde{\alpha}+1$ makes this argument possible, but only with some assumption limiting the way extenders on $\mathcal{U}$ may overlap $E^{\mathcal{U}}_{\tilde{\alpha}}$. For extenders indexed above η the fact that $\mathcal{U}$ is balanced above η suffices. However some additional assumption is needed to cover extenders indexed before η (i.e. extenders on $\mathcal{T}$). Again it is enough to assume that $\mathcal{T}$ is suitable. Thus Theorem 3.2 continues to hold if "normal, maximal, and non-overlapping" is weakened to "suitable." Similarly Corollary 3.3 continues to hold if the restriction of Footnote 16 is weakened to "suitable."

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