The Derived Model Theorem

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We shall exposit here one of the basic theorems leading from large cardinals to determinacy, a result of Woodin known as the *derived model theorem*. The theorem dates from the mid-80's and has been exposited in several sets of informally circulated lecture notes (e.g., [12], [14]), but we know of no exposition in print. We shall also include a number of subsidiary and related results due to various people.

We shall use very heavily the technique of *stationary tower forcing*. The reader should see Woodin's paper [15] or Larson's [3] for the basic facts about stationary tower forcing. The second main technical tool needed for a full proof of the derived model theorem is the theory of iteration trees. This is one of the main ingredients in the proof of Theorem 2.1 below, but since we shall simply take that theorem as a "black box" here, it is possible to read this paper without knowing what an iteration tree is.

The paper is organized as follows. In §1, we introduce homogeneity, weak homogeneity, and universal Baireness. The main result here is the Martin-Solovay theorem, according to which all weakly homogeneous sets are universally Baire. We give a reasonably complete proof of this theorem. In §2 and §3, we show that in the presence of Woodin cardinals, homogeneity, weak homogeneity, and universal Baireness are equivalent. We also give, in §3, an argument of Woodin's which shows that strong cardinals yield universally Baire representations after a collapse. In §4 we prove the *tree production lemma*, according to which sets admitting definitions with certain absoluteness properties are universally Baire. §5 contains a generic absoluteness theorem for $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$ statements. In §6 we state and prove the derived model theorem. In §7 we prove that in derived models, the pointclass Σ^2_1 has the Scale Property, and in §8, we use this to produce derived models which satisfy $\mathsf{AD}_{\mathbb{R}}$.

This paper was written in Fall 2002, and circulated informally. In 2004, Larson's monograph [3] on stationary tower forcing appeared. Much of the material in §2, §3, and §4 can be found in section 3.3 of [3]. Sections 3.2 and 3.4 of [3] use this machinery to prove important results of Woodin concerning generic absoluteness. In another direction, Neeman's recent paper [10] gives a proof of $\mathsf{AD}^{L(\mathbb{R})}$ which avoids stationary tower forcing entirely, relying entirely on iteration trees instead. Finally, author has published a stationary-tower-free proof of the full derived model theorem in [11].

1 Homogeneously Suslin and universally Baire sets

If Y is a set, then $Y^\omega = \{x \mid x\colon \omega \to Y\}$ is the set of infinite sequences of elements of Y. We often regard Y^ω as being equipped with the Baire topology, whose basic open sets are just those of the form $N_s = \{x \in Y^\omega \mid s \subseteq x\}$, for $s \in Y^{<\omega}$. We are most interested in the case $Y = \omega$. We call the elements of ω^ω reals, and write $\mathbb{R}$ for ω^ω. In this section we introduce three regularity properties a set $A \subseteq Y^\omega$ might have.

The first of these, homogeneity, derives from Martin [4], and was first explicitly isolated by Martin and Kechris. In a word, a set $A \subseteq Y^\omega$ is homogeneously Suslin just in case it is continuously reducible to wellfoundedness of towers of measures. Here is some further detail.

We shall use the terms *ultrafilter on I* and *measure on I* interchangeably; thus all our measures take values in $\{0, 1\}$.

Definition 1.1 *For any Z, $meas_\kappa(Z)$ is the set of all κ-additive measures on $Z^{<\omega}$. We let $meas(Z) = meas_{\omega_1}(Z)$.*

Clearly, if $\mu \in \text{meas}(Z)$, then there is exactly one $n < \omega$ such that $\mu(Z^n) = 1$. We call n the *dimension* of μ, and write $n = \dim(\mu)$.

If $\mu, \nu \in \text{meas}(Z)$, then we say that μ *projects* to ν iff for some $m \le n < \omega$, $\dim(\mu) = n$, $\dim(\nu) = m$, and for all $A \subseteq Z^m$

$$\nu(A) = \mu(\{u \mid u \restriction m \in A\}.$$

We say μ and ν are *compatible* if one projects to the other. If μ projects to ν, then there is a natural embedding

$$\pi_{\nu,\mu}\colon \text{Ult}(V, \nu) \to \text{Ult}(V, \mu)$$

given by $\pi([f]_\nu) = [f^*]_\mu$, where $f^*(u) = f(u \restriction m)$ for all $u \in Z^n$.

A *tower of measures* on Z is a sequence $\langle \mu_n \mid n < k\rangle$, where $k \le \omega$, such that each $\mu_n \in \text{meas}(Z)$, and whenever $m \le n < k$, then $\dim(\mu) = n$ and μ_n projects to μ_m. If $\langle \mu_n \mid n < \omega\rangle$ is an infinite tower of measures, then

$$\text{Ult}(V, \langle \mu_n \mid n < \omega\rangle) = \text{ dir lim}_{n<\omega}\text{Ult}(V, \mu_n),$$

where the direct limit is taken using the natural embeddings π_{μ_n,μ_m}, which commute with one another.[1] We say that the tower $\langle \mu_n \mid n < \omega\rangle$ is *countably complete* just in case whenever

[1] Note that $Z^0 = \{\emptyset\}$, so in any tower, μ_0 is principal, and $\text{Ult}(V, \mu_0) = V$.

$\mu_{x\restriction n}(A_n) = 1$ for all $n < \omega$, then $\exists\, f\, \forall\, n\ (f \restriction n \in A_n)$. It is easy to show that $\langle \mu_n \mid n < \omega \rangle$ is countably complete if and only if $\mathrm{Ult}(V, \langle \mu_n \mid n < \omega \rangle)$ is wellfounded, and so we shall say that a tower is wellfounded just in case it is countably complete.

Definition 1.2 *A homogeneity system over Y with support Z is a function*

$$\bar{\mu}\colon Y^{<\omega} \to \mathit{meas}(Z)$$

such that, writing $\mu_s = \bar{\mu}(s)$, we have that for all $s, t \in Y^{<\omega}$,

1. $\dim(\mu_t) = \mathrm{dom}(t)$, *and*
2. $s \subseteq t \Rightarrow \mu_t$ *projects to* μ_s.

If $\mathrm{ran}(\bar{\mu}) \subseteq \mathit{meas}_\kappa(Z)$, *then we say that* $\bar{\mu}$ *is* κ*-complete.*

Definition 1.3 *If $\bar{\mu}$ is a homogeneity system over Y with support Z, then for each $x \in Y^{<\omega}$, we let $\bar{\mu}_x$ be the tower of measures $\langle \mu_{x\restriction n} \mid n < \omega \rangle$, and set*

$$S_{\bar{\mu}} = \{ x \in Y^{<\omega} \mid \bar{\mu}_x \textit{ is countably complete } \}.$$

Definition 1.4 *Let $A \subseteq Y^{<\omega}$; then A is κ-homogeneous iff $A = S_{\bar{\mu}}$, for some κ-complete homogeneity system $\bar{\mu}$. We say A is homogeneous if it is κ-homogeneous for some κ.*

For the collection of all κ-homogeneous sets we write

$$\mathrm{Hom}^Y_\kappa = \{ A \subseteq Y^{<\omega} \mid A \text{ is } \kappa\text{-homogeneous } \}.$$

We set

$$\mathrm{Hom}^Y_{<\lambda} = \bigcap_{\kappa<\lambda} \mathrm{Hom}^Y_\kappa,$$

and

$$\mathrm{Hom}^Y_\infty = \bigcap_{\kappa\in 0R} \mathrm{Hom}^Y_\kappa\ .$$

We write Hom_κ for $\mathrm{Hom}^\omega_\kappa$, etc. It is clear that Hom^Y_κ is closed downward under continuous reducibility (also called Wadge reducibility, at least if $Y = \omega$). It is not too hard to show Hom^Y_κ is closed under countable intersections. One cannot prove closure under complementation in ZFC. We don't know whether closure under union is provable in ZFC.

Homogeneity is often considered in conjunction with trees. A *tree* on a set X is a set $T \subseteq X^{<\omega}$ such that $\forall s \in T \forall k (s \restriction k \in T)$. We let $[T] = \{ f \in X^\omega \mid \forall n < \omega (f \restriction n \in T) \}$ be the set of infinite branches of T, so that T is wellfounded (under reverse inclusion) iff $[T] = \emptyset$.

We think of a tree on $Y \times Z$ as a set of pairs $(s,t) \in Y^{<\omega} \times Z^{<\omega}$ such that $\mathrm{dom}(s) = \mathrm{dom}(t)$. If T is a tree on $Y \times Z$ and $s \in Y^{<\omega}$, then

$$T_s = \{t \mid (s,t) \in T\},$$

and for $x \in Y^\omega$,

$$T_x = \bigcup_{n<\omega} T_{x \restriction n}.$$

The projection on Y^ω of $[T]$ is given by

$$\begin{aligned} x \in p[T] &\Leftrightarrow T_x \text{ is illfounded} \\ &\Leftrightarrow \exists f \in Z^\omega \forall n((x \restriction n, f \restriction n) \in T), \end{aligned}$$

for $x \in Y^\omega$. We call T a *Suslin representation* of $p[T]$, and say that $p[T]$ is Z-*Suslin* via T.

Proposition 1.5 (Woodin) *Let $A = S_{\bar\mu}$, where $\bar\mu$ is a homogeneity system over Y with support Z. Suppose that $\bar\mu$ is $|Y|^+$-complete; then there is a tree T on $Y \times Z$ such that*

$$A = p[T],$$

and for all $s \in Y^{<\omega}$,

$$\mu_s(T_s) = 1.$$

Proof. For each $x \in Y^\omega \setminus A$, pick a sequence of sets B^x_n witnessing that $\bar\mu_x$ is not countably complete, so that $\mu_{x\restriction n}(B^x_n) = 1$ for all n, but $\nexists f \forall n\ (f \restriction n \in B^x_n)$. Then put

$$(s,u) \in T \Leftrightarrow u \in Z^{\mathrm{dom}\, s} \wedge\ \forall\ x \supseteq s\ (x \notin A \Rightarrow u \in B^x_{\mathrm{dom}\, s}).$$

Since the additivity of any measure is a measurable cardinal, each μ_s is sufficiently additive that $\mu_s(T_s) = 1$. If $x \in A$, then the countable completeness of $\bar\mu_x$ implies there is an f such that for all n, $f \restriction n \in T_{x\restriction n}$, so that $x \in p[T]$. On the other hand, if $x \notin A$, then the choice of the B^x_n guarantees that $x \notin p[T]$. □

The completeness hypothesis on $\bar\mu$ in 1.5 is redundant in the case Y is countable. If T is related to $\bar\mu$ as in 1.5, and $\bar\mu$ is κ-complete, then we shall say that T is a κ- *homogeneous tree*, and $\bar\mu$ is a homogeneity system for T. We also say that $p[T]$ is κ-*homogeneously Suslin.*

K. Windszus has proved a stronger version of 1.5, showing that if $A \subseteq Y^\omega$ is continuously reducible to wellfoundedness of direct limit systems on $|Y^\omega|$-closed models, then A is homogeneously Suslin.

Remark 1.6 Let $Y \subseteq \mathrm{meas}_\kappa(Z)$, and let

$$A = \{\langle \mu_n \mid n < \omega\rangle \in Y^\omega \mid \langle \mu_n \mid n < \omega\rangle \text{ is wellfounded }\}.$$

It is clear that A is κ-homogeneous, the homogeneity system over Y being simply the identity function. As one might suspect, this is not a very useful fact. If in addition $|Y| < \kappa$, then by 1.5, A is κ-homogeneously *Suslin*, which is more useful. (See remark 2.2.)

The first and most important fact about homogeneously Suslin sets is

Theorem 1.7 (Martin [4], essentially) *If $A \subseteq Y^\omega$ is $|Y|^+$-homogeneous, then the two-person game of perfect information on Y with payoff set A is determined.*

We now consider a weakening of homogeneity, also first isolated by Kechris and Martin.

Definition 1.8 *A* weak homogeneity system over Y with support Z *is an injective function $\bar{\mu}\colon Y^{<\omega} \to meas(Z)$ such that for all $s \in Y^{<\omega}$*

1. $\dim(\mu_s) \leq \mathrm{dom}(s)$, *and*

2. *if μ_s projects to ν, then $\exists i(\mu_{s\restriction i} = \nu)$.*

Definition 1.9 *If $\bar{\mu}$ is a (κ-complete) weak homogeneity system over Y, then we set*

$$W_{\bar{\mu}} = \{x \in Y^\omega \mid \exists \langle i_k \mid k < \omega \rangle \in \omega^\omega (\langle \mu_{x\restriction i_k} \mid k < \omega \rangle \text{ is a wellfounded tower })\},$$

*and say that $W_{\bar{\mu}}$ is (κ-)*weakly homogeneous *via $\bar{\mu}$.*

So a weak homogeneity system over Y associates continuously to each $x \in Y$ a countable tree of towers of measures, and x is in the set being represented iff at least one of the branches of this tree is a wellfounded tower.[2] This leads us to

Proposition 1.10 *Let $A \subseteq Y^\omega$. If there is a measurable cardinal, then the following are equivalent:*

1. *A is κ-weakly homogeneous,*

2. *there is a κ-homogeneous set $B \subseteq Y^\omega \times \omega^\omega$ such that $x \in A \Leftrightarrow \exists y(x, y) \in B$, for all x.*

We leave the proof to the reader. The measurable cardinal is needed for a minor technical reason: if there are no measurables, then $Y^\omega \times Z^\omega$ is the only homogeneous $B \subseteq Y^\omega \times Z^\omega$, while the projection of any closed $B \subseteq Y^\omega \times Z^\omega$ is weakly homogeneous. This situation could be remedied by allowing *partial* $\bar{\mu}$ as homogeneity systems, but in any case, homogeneity isn't very interesting if there are no measurable cardinals.

For trees we make the following definition:

Definition 1.11 *A tree T on $Y \times Z$ is κ-*weakly homogeneous via $\bar{\mu}$ *iff $\bar{\mu}$ is a κ-complete weak homogeneity system over Y such that*

[2] Had we taken a weak homogeneity system to be *any* function $\bar{\mu}\colon Y^{<\omega} \to \mathrm{meas}(Z)$, we would have obtained the same class of sets $W_{\bar{\mu}}$. Our restrictions on $\bar{\mu}$ make some manipulations easier. Note that they imply $\mu_\emptyset$ is principal, and that if $\langle i_k \mid k < \omega \rangle$ is a sequence witnessing $x \in W_{\bar{\mu}}$ then $i_0 = 0$ and $k < l \Rightarrow i_k < i_l$.

1. $p[T] = W_{\bar{\mu}}$, *and*

2. $\forall s \in Y^{<\omega} \exists k(\mu_s(T_{s \restriction k}) = 1$.

We say that $p[T]$ *is* κ-weakly homogeneously Suslin *in this case.*

Parallel to 1.5 we have:

Proposition 1.12 *Let* $\bar{\mu}$ *be a* $|Y|^+$*-complete weak homogeneity system over* Y *with support* Z*; then there is a tree* T *on* $Y \times Z$ *such that which is weakly homogeneous via* $\bar{\mu}$*.*

So if $A \subseteq Y^\omega$ and $\kappa > |Y|$, then A is κ-homogeneous iff A is κ-homogeneously Suslin, and A is κ-weakly homogeneous iff A is κ-weakly homogeneously Suslin. In the sequel, Y will almost always be countable, so these equivalences apply. Our characterization of weak homogeneity for trees also simplifies a bit in the case Y is countable:

Proposition 1.13 *A tree* T *on* $\omega \times Z$ *is* κ*-weakly homogeneous iff there is a countable set* $\sigma \subseteq meas_\kappa(Z)$ *so that* $\forall\ x$

$$x \in p[T] \iff \begin{array}{l} \textit{There is a tower } \langle \mu_n \mid n < \omega \rangle \textit{ of} \\ \textit{measures from } \sigma \textit{ such that} \\ \mu_n(T_{x \restriction n}) = 1 \textit{ for all } n, \textit{ and} \\ \langle \mu_n \mid n < \omega \rangle \textit{ is countably complete.} \end{array}$$

Proof. Given such a σ, let $\sigma = \{\nu_i \mid i < \omega\}$ be a one-one enumeration such that if ν_i projects to μ, then $\exists k \leq i(\nu_k = \mu)$. Setting $\mu_s = \nu_{\text{dom}(s)}$, it is clear that T is weakly homogeneous via $\bar{\mu}$. Conversely, if T is weakly homogeneous via $\bar{\mu}$, then take $\sigma = \text{ran}(\bar{\mu})$. Since $W_{\bar{\mu}} = p[T]$, if $x \in p[T]$, then there is a countably complete tower from σ concentrating on T_x. The converse is true in general, because of countable completeness. □

A still further weakening of homogeneity is the property of being κ*-universally Baire* (see [1]).

Definition 1.14 *We say* G *is* $< \kappa$-generic *over* M *iff* G *is* M*-generic for some poset* $\mathbb{P}$ *such that* $M \models |\mathbb{P}| < \kappa$*.*

Definition 1.15 *Let* T *on* $X \times Y$ *and* U *on* $X \times Z$ *be two trees; then we say* T *and* U *are* κ-absolute complements *iff whenever* G *is* $< \kappa$*-generic over* V

$$V[G] \models p[T] = X^\omega \setminus p[U].$$

We say T is κ-absolutely complemented *iff* $\exists U$ *(*T *and* U *are* κ*-absolute complements).*

If $p[T] \cap p[U] = \emptyset$ in V, then the same is true in any generic extension of V by the absoluteness of wellfoundedness. We shall use this simple observation again and again. What absolute complementation adds is that T and U are sufficiently "fat" that in the relevant $V[G]$, we have $p[T] \cup p[U] = X^\omega$.

Definition 1.16 *(1) A set $A \subseteq X^\omega$ is* κ-universally Baire*, or* κ-absolutely Suslin *iff $A = p[T]$ for some κ-absolutely complemented T.*

(2) $UB_\kappa = \{A \subseteq \omega^\omega \mid A$ is κ-universally Baire$\}$.

Every provably-in-ZFC Δ^1_2 set of reals is κ-universally Baire for all κ. This is the key to Solovay's proof that such sets are Lebesgue measurable and have the Baire property. Indeed, any $(2^{\aleph_0})^+$- universally Baire set of reals has these regularity properties ([1]).

It is one of the main results of Martin–Solovay [6] that every κ-weakly homogeneous Suslin set is κ-universally Baire. Here is a brief review of the Martin–Solovay construction.

For simplicity, we begin with a homogeneity systems.

Definition 1.17 *Let $\bar{\mu}$ be a homogeneity system over Y. For any ordinal θ, we define the Martin-Solovay tree $ms(\bar{\mu}, \theta)$ on $Y \times OR$ by*

$$(s, \langle \alpha_n \mid n < e \rangle) \in ms(\bar{\mu}, \theta) \quad \Leftrightarrow \quad s \in Y^e \wedge \alpha_0 < \theta \wedge \forall n(n+1 < e \Rightarrow \pi_{\mu_{s\restriction n}, \mu_{s \restriction (n+1)}}(\alpha_n) > \alpha_{n+1}).$$

That is, $ms(\bar{\mu}, \theta)_x$ searches for a proof that $\mathrm{Ult}(V, \bar{\mu}_x)$ is illfounded below the image of θ. (This last restriction makes it a set, rather than a proper class.) It is not hard to see that if $\bar{\mu}$ has support Z and is illfounded, then it is illfounded below the image of $|Z|^+$. It follows easily that for any $\theta \geq |Z|^+$,

$$p[ms(\bar{\mu}, \theta)] = Y^\omega \setminus S_{\bar{\mu}}.$$

Thus if $\bar{\mu}$ is a homogeneity system for T, then $ms(\bar{\mu}, \theta)$ and T complement each other in V.

Now suppose $\bar{\mu}$ is κ-complete, and G is $< \kappa$-generic over V. The measures μ_s extend to measures μ^*_s in $V[G]$, where

$$\mu^*_s(A) = 1 \Leftrightarrow \exists B \subseteq A(\mu_s(B) = 1).$$

(We shall use this *-notation in this way without much comment in the future.) Moreover, for every function $f \colon Z^{<\omega} \to V$ such that $f \in V[G]$ there is a function $g \in V$ such that $f(u) = g(u)$ for μ^*_s-a.e. u. Thus $\bar{\mu}^*$ is a homogeneity system in $V[G]$, whose associated embeddings, when restricted to V, are those of $\bar{\mu}$. It follows that $ms(\bar{\mu}, \theta)^V = ms(\bar{\mu}^*, \theta)^{V[G]}$, and that $p[ms(\bar{\mu}, \theta)^V] = Y^\omega \setminus S_{\bar{\mu}^*}$ in $V[G]$.

Finally, suppose in addition that $\bar{\mu}$ is a homogeneity system for T in V. In order to see that $ms(\bar{\mu}, \theta)$ is a κ-absolute complement for T, it is enough to show $p[T] = S_{\bar{\mu}^*}$ in

$V[G]$.[3] Now if $x \in S_{\bar{\mu}^*}$, then the tower $\bar{\mu}^*_x$ is countably complete, and since its measures concentrate on T_x, we get that T_x is illfounded, so $x \in p[T]$. Conversely, if $x \in p[T]$, then $x \notin p[ms(\bar{\mu}, \theta)^V]$, because the projections of these two trees were disjoint in V. As we showed above, this implies $x \in S_{\bar{\mu}^*}$ in $V[G]$, as desired.

We can extend this construction to weak homogeneity systems.

Definition 1.18 *Let $\bar{\mu}$ be a weak homogeneity system over Y, and θ be an ordinal. We define the Martin-Solovay tree $ms(\bar{\mu}, \theta)$ on $Y \times OR$ by*

$$(s, \langle \alpha_e \mid e < n \rangle) \in ms(\bar{\mu}, \theta) \quad \Leftrightarrow \quad s \in Y^n \wedge \alpha_0 < \theta \wedge \\ \forall e < k < n(\mu_{s \restriction k} \text{ projects to } \mu_{s \restriction e} \Rightarrow \pi_{\mu_{s\restriction e}, \mu_{s \restriction k}}(\alpha_e) > \alpha_k).$$

Lemma 1.19 *Suppose that $\bar{\mu}$ is a κ-complete weak homogeneity system over Y with support Z, and $\theta \geq |Z|^+$. Let G be $< \kappa$-generic over V; then $ms(\bar{\mu}, \theta)^V = ms(\bar{\mu}^*, \theta)^{V[G]}$, moreover $p[ms(\bar{\mu}, \theta)] = Y^\omega \setminus W_{\bar{\mu}^*}$ in $V[G]$.*

Proof. We claim first that $p[ms(\bar{\mu}, \theta)] = Y^\omega \setminus W_{\bar{\mu}}$ in V. It is clear that if $x \in p[ms(\bar{\mu}, \theta)]$ then $x \notin W_{\bar{\mu}}$, since a branch through $ms(\bar{\mu}, \theta)_x$ illfounds all the relevant towers, and in fact does so *continuously*. Conversely, suppose $x \notin W_{\bar{\mu}}$. All the relevant towers are then illfounded, but we must see this is true continuously, and below the image of θ. For that, pick, for each increasing $t \colon \omega \to \omega$ such that $\langle \mu_{x \restriction t(n)} \mid n < \omega \rangle$ is a tower, a sequence A^t_n witnessing the countable incompleteness of this tower. (So $A^t_n \subseteq Z^n$ and $\mu_{x \restriction t(n)}(A^t_n) = 1$.) For any k, let B_k be the intersection over all t, n such that $t(n) = k$ of the A^t_n. (n is in fact determined by k, since $\mu_{x \restriction k}(Z^n) = 1$.) Then letting

$$(k, u)R(l, v) \Leftrightarrow k > l \wedge u \in B_k \wedge v \in B_l \wedge v \subseteq u,$$

we have that R is wellfounded. Set

$$f_k(u) = \text{ rank of } (k, u) \text{ in } R$$

and let

$$\alpha_k = [f_k]_{\mu_{x \restriction k}}.$$

It is easy to check that $\langle \alpha_k \mid k < \omega \rangle$ is a branch through $ms(\bar{\mu}, \theta)_x$, as desired.

The remainder of the lemma is proved as it was for homogeneity systems. □

Theorem 1.20 (Martin, Solovay [6]) *Let T be κ-weakly homogeneous via $\bar{\mu}$, and $\theta > |T|^+$; then T and $ms(\bar{\mu}, \theta)$ are κ-absolute complements.*

[3] This also shows that $\bar{\mu}^*$ is a homogeneity system for T in $V[G]$.

The proof of 1.20 is just as it was for homogeneity systems, so we omit further detail. The proof also shows T remains weakly homogeneous via $\bar{\mu}^*$ in $< \kappa$-generic extensions.

Corollary 1.21 *If $A \subseteq \mathbb{R}$ is κ-weakly homogeneous, then A is κ-universally Baire.*

To summarize: Any κ-homogeneous set is κ-weakly homogeneous, and any κ-weakly homogeneous set is κ-universally Baire.

2 Weak homogeneity to homogeneity

In the next two sections we show that the implications above have converses, in a certain sense. Namely, if δ is Woodin, then any δ^+-universally Baire set is $< \delta$-weakly homogeneous, and any δ^+-weakly homogeneous set is $< \delta$-homogeneous. Thus if λ is a limit of Woodins, then $\mathrm{Hom}_{<\lambda} = UB_\lambda$.

Theorem 2.1 (Martin, Steel [7]) *Let δ be Woodin, and let $\bar{\mu}$ be a δ^+-complete weak homogeneity system over Y, where $|Y| < \delta$; then for all sufficiently large θ, $ms(\bar{\mu}, \theta)$ is κ-homogeneous for all $\kappa < \delta$.*

Proof. Omitted. □

Remark 2.2 Our hypothesis that $|Y|$ is strictly less than the completeness of $\bar{\mu}$ implies that there is a tree T on some $Y \times Z$ which is weakly homogeneous via $\bar{\mu}$. Given T *in advance*, the construction of [7] homogeneity system for some $ms(\bar{\mu}, \theta)$ in a way which is *continuous in $\bar{\mu}$*, in that finite bits $\bar{\mu} \restriction i$ are needed to determine the next measure in the homogeneity system for $ms(\bar{\mu}, \theta)$.

Woodin has observed that this has the following nice consequence. Let $Y \subseteq \mathrm{meas}_\gamma(Z)$, and $|Y| < \delta < \gamma$ for some Woodin cardinal δ. Let

$$I = \{\langle t \in Y^\omega \mid t \text{ is an illfounded tower } \}.$$

We observed in remark 1.6 (essentially) that $Y^\omega \setminus I$ is γ-homogeneous.[4] It then follows from the continuity implicit in the construction of [7] that I is κ-homogeneous for all $\kappa < \delta$. One should compare this with 1.6.

If $A \subseteq Q \times S$, we write $\exists^Q A$ for $\{s \mid \exists q (q, s) \in A\}$, and $\forall^Q A$ for $\{s \mid \forall q \in Q (q, s) \in A\}$. If $B \subseteq Q$, then we write $\neg B$ for $Q \setminus B$, when Q is clear from context.

[4]One must intersect with the set of all towers in order to apply 1.6, but since the set of all towers is closed in Y^ω, this is not a problem.

Corollary 2.3 *If $A \subseteq \mathbb{R}^2$ is δ^+-homogeneous, where δ is Woodin, then $\neg\exists^{\mathbb{R}} A$ is κ-homogeneous for all $\kappa < \delta$.*

Proof. By 1.10, $\exists^{\mathbb{R}} A$ is δ^+-weakly homogeneous, so its complement is $< \delta$-homogeneous by 2.1. $\square$

Corollary 2.4 *If λ is a limit of Woodin cardinals, then* $\mathrm{Hom}_{<\lambda}$ *is closed under $\exists^{\mathbb{R}}$, negation, and continuous (i.e., "Wadge") reducibility.*

So if λ is a limit of Woodins, then projective determinacy (PD) holds, and in fact holds in all $< \lambda$-generic extensions (since λ remains a limit of Woodins in such an extension). In fact, we get projective generic absoluteness for such extensions, even with names for sets of reals in $\mathrm{Hom}_{<\lambda}$, as we now show.

The following lemma gives us our names.

Lemma 2.5 *Let (T, U) and (R, S) be pairs of κ-absolute complements, and suppose $p[T] = p[R]$ in V. Then for any $< \kappa$-generic G, $p[T] = p[R]$ in $V[G]$.*

Proof. Say $x \in V[G]$ and $x \in p[T]$ but $x \notin p[R]$. Since S complements R, we have $x \in p[S]$. Thus $p[T] \cap p[S] \neq \emptyset$ in $V[G]$, hence in V, a contradiction. $\square$

So we can think of a κ-absolutely complemented pair (T, U) as a name for $p[T]$, and we have that two names which agree on V also agree on any $< \kappa$-generic $V[G]$. If $A = p[T]$ for such a (T, U), then we sometimes write $A^{V[G]}$ for $p[T]^{V[G]}$ if G i $< \kappa$- generic. There is no ambiguity because $A^{V[G]}$ does not depend on which absolutely complemented name we choose.

Theorem 2.6 (Woodin) *Let λ be a limit of Woodin cardinals, and $A \in \mathrm{Hom}_{<\lambda}$. Let G be $< \lambda$-generic over V. Then*

$$(HC^V, \in, A) \equiv (HC^{V[G]}, \in, A).$$

Notice that if x is a real, then $\{x\}$ is $\mathrm{Hom}_{<\lambda}$, and any sequence of $\mathrm{Hom}_{<\lambda}$ sets can be coded by a single one. Thus 2.6 implies a superficially stronger version of itself.

Proof. Fix $A \in \mathrm{Hom}_{<\lambda}$. To each formula $\varphi(\vec{v}, \dot{A})$ in the language of second order arithmetic with additional predicate symbol $\dot{A}$, and each $\kappa < \lambda$, we associate a κ-homogeneous tree $T_{\varphi,\kappa}$ such that whenever G is $< ka$- generic

$$V[G] \models p[T_{\varphi,\kappa}] = \{\vec{y} \in \mathbb{R}^{<\omega} \mid \varphi(\vec{y}, A^{V[G]})\}.$$

The absoluteness of wellfoundedness and the Tarski-Vaught criterion then imply that V is φ-elementary in $V[G]$, as desired.

The trees $T_{\varphi,\kappa}$ are constructed (for all κ) by induction on φ. For $\varphi\Sigma^1_1$, we use the given trees for the A. Now suppose $\varphi = \neg\exists\ w\psi$, and $\kappa < \lambda$. Let $\kappa < \delta < \lambda$, where δ is Woodin. Now T_{ψ,δ^+} is δ^+-homogeneous as a tree on $(\omega \times \omega) \times Z$, and hence δ^+-weakly homogeneous as a tree S on $\omega \times (\omega \times Z)$. Of course, $p[S] = \exists^{\mathbb{R}} p[T_{\varphi,\delta^+}]$ in all generic extensions. Let $\bar{\mu}$ be a δ^+-complete weak homogeneity system for S, and θ be sufficiently large, and set $T_{\varphi,\kappa} = ms(\bar{\mu}, \theta)$. This works by the Martin–Solovay and Martin–Steel theorems.

Up to logical equivalence, all Σ^1_n formulae can be built up from Σ^1_1 using $\neg\exists\ w$, so we are done. □

Here is a sometimes useful observation about $\mathrm{Hom}_{<\lambda}$. It is due independently to Woodin and the author.

Theorem 2.7 *Let λ be a limit of Woodin cardinals; then there is a $\kappa < \lambda$ such that* $\mathrm{Hom}_\kappa = \mathrm{Hom}_{<\lambda}$.

Proof. Clearly, $\alpha < \beta \Rightarrow \mathrm{Hom}_\beta \subseteq \mathrm{Hom}_\alpha$. Each Hom_α is a boldface pointclass, that is, it is closed downward under Wadge reducibility $\leq_w$. Thus if the theorem fails, we have an infinite descending sequence $A_0 >_w A_1 >_w \ldots$ in the Wadge order. But also, we have projective-in-A_0 determinacy, and so Martin's proof that $<_w$ is wellfounded yields a contradiction. □

3 Universally Baire to weakly homogeneous

Our method for obtaining weak homogeneity originated in Martin's unpublished proof that $\mathsf{AD}_{\mathbb{R}}$ implies that every tree on $\omega \times \kappa$ is weakly homogeneous. Woodin extended the method to the context of large cardinals with Choice, and eventually obtained the following remarkable results.

Theorem 3.1 (Woodin) *Let δ be Woodin, and let T and U be δ^+-absolutely complementing trees on $\omega \times Z$; then T is κ-weakly homogeneous for all $\kappa < \delta$.*

Proof. We shall actually just use that $p[T] = \mathbb{R} \setminus p[U]$ in $V[G]$, whenever G is generic for the "countable" stationary tower $\mathbb{Q}_{<\delta}$. (Conditions are stationary $a \subseteq P_{\omega_1}(V_\alpha)$, for $\alpha < \delta$. See the appendix.) Names for reals modulo $\mathbb{Q}_{<\delta}$ are elements of V_δ, so if we let T^* and U^* be the subtrees of T and U consisting of all nodes definable over V_η from T, δ, U, and parameters in V_δ (where $\eta \gg \delta$ and $T, U \in V_\eta$), then T^* and U^* have size δ and are forced in $\mathbb{Q}_{<\delta}$ to project to complements. Rearranging and renaming, we may assume T and U are on $\omega \times \delta$.

Let $\kappa < \delta$ be given; we wish to show T is κ-weakly homogeneous. We may assume κ is T-reflecting in δ, since there are arbitrarily large such $\kappa < \delta$. That is, for each λ such that $\kappa < \lambda < \delta$, there is $j : V \to M$ with $\text{crit}(j) = \kappa$ such that $V_\lambda \subseteq M$ and $j(T) \cap V_\lambda = T \cap V_\lambda$.

Let $\kappa < \lambda < \delta$ and j be as above. We define a continuous function $\Sigma_{\lambda,j}$ on the branches of $T \cap V_\lambda$ whose outputs are towers of measures on $T \cap V_\kappa$. Namely, for $(s, u) \in T \cap V_\lambda$, and $X \subseteq V_\kappa$, let

$$\Sigma_{\lambda,j}(s, u)(X) = 1 \text{ iff } u \in j(X).$$

Writing $\Sigma = \Sigma_{\lambda,j}$, we see

(a) $\Sigma(s, u)$ is a κ-complete measure concentrating on $T_s \cap V_\kappa$.

(b) $(s, u) \subseteq (t, v) \Rightarrow \Sigma(s, u)$ is compatible with $\Sigma(t, v)$.

(c) If $(x, f) \in [T \cap V_\lambda]$, then the tower $\Sigma(x, f) \underset{\text{df.}}{=} \langle \Sigma(x \restriction n, f \restriction n) \mid n < \omega \rangle$ is wellfounded. (Note that its ultrapower embeds into M.)

(d) In any generic extension $V[H]$ of V, if $(x, f) \in [T \cap V_\lambda]$, then $\text{Ult}(V, \Sigma(x, f))$ is wellfounded. (Here the ultrapower is formed using functions in V.) This is because the tree of attempts to produce $(x, f, \overrightarrow{\alpha})$ such that $(x, f) \in [T \cap V_\lambda]$ and $\overrightarrow{\alpha}$ is a descending sequence in $\text{Ult}(V, Z(x, f))$ is wellfounded in V, hence in $V[H]$.

Now let G be $\mathbb{Q}_{<\delta}$ generic over V, and

$$i : V \to N \subseteq \text{Ult}(V, G)$$

be the generic embedding. It suffices to show that $i(T)$ is $i(\kappa)$-weakly homogeneous in N. In fact, we show

Claim. $\sigma = i''\text{meas}_\kappa(\kappa^{<\omega})$ is a witness that $i(T)$ is $i(\kappa)$-weakly homogeneous in N.

Proof. Since ${}^\omega N \subseteq N$ in $V[G]$, we have $\sigma \in N$ and is countable in N. Now let $x \in p[i(T)]$ and $x \in N$. Since T and U are absolutely complementing, either $x \in p[T]$ or $x \in p[U]$. But

$$x \in p[U] \Rightarrow x \in p[i(U)] \Rightarrow p[i(U)] \cap p[i(T)] \neq \emptyset,$$

whereas $p[i(U)] \cap p[i(T)] = \emptyset$ because this is true in N, which is a wellfounded model. Thus $x \in p[T]$.

Let $\lambda < \delta$ be such that $x \in p[T \cap V_\lambda]$; note here $\delta = \omega_1^{V[G]}$, so there is such a λ. Choose $\lambda < \kappa$, and let j be such that $\Sigma = \Sigma_{\lambda,j}$ exists in V. Letting $(x, f) \in [T \cap V_\lambda]$, we have

$$N \models \text{every tower in } i(\Sigma)''[i(T \cap V_\lambda)] \text{ is wellfounded},$$

so

$$N \models \langle i(\Sigma)(x \restriction n, i(f \restriction n)) \mid n < \omega \rangle \text{ is wellfounded.}$$

But $\langle i(\Sigma)(x \restriction n, i(f \restriction n)) \mid n < \omega \rangle$ is a tower of measures from σ concentrating on $i(T_x)$. This proves the claim, and hence the theorem. $\square$

Remark 3.2 It is possible to prove the theorem using the extender algebra at δ, and iterations to make reals generic, rather than stationary tower forcing. One then uses that T and U are complementing in $V[G]$, whenever G is generic for the extender algebra at δ. Here is a sketch: let κ be T-reflecting in δ. Let $X \prec V_\theta$ be countable, with θ large, and $\kappa, \delta, T, U \in X$. Take $\sigma = \{\mu \in X \mid \mu \text{ is a } \kappa\text{-complete measure concentrating on some } T_s \cap V_\kappa\}$. We claim σ witnesses that T is κ-weakly homogeneous. To see this, let $\pi : N \cong X$ be the transitive collapse. Let $x \in p[T]$. We can iterate $N \xrightarrow{i} M$ to make x generic over M for the extender algebra at $i(\delta)$, and we have

$$\begin{array}{ccc} N & \xrightarrow{\pi} & V_\theta \\ & \searrow_{i} & \uparrow \tau \\ & & M \end{array}$$

for some realizing map τ. Letting $\bar{T} = \pi^{-1}(T)$, etc., we can arrange $\text{crit}(i) > \bar{\kappa}$. Then $x \in p[i(\bar{T})]$, as otherwise $x \in p[i(\bar{U})]$, so $x \in p[\sigma(i(\bar{U}))]$, so $x \in p[U]$. Now then if Σ is the appropriate $\Sigma_{\lambda,j}$, and $(x, f) \in p[i(\bar{T} \cap \bar{\lambda})]$, then $(\tau \circ i)(\bar{\Sigma})(x, \tau(f))$ is a wellfounded tower from σ concentrating on T_x.

This seems to require a certain amount of iterability, but by being more careful, we can make do with a form of iterability which is provable. (Use $\delta > \kappa$ which is Woodin in $L(V_\delta, T)$, and is the least such. In the extender algebra, use only identities induced by T-strong extenders with critical point above κ.)

Remark 3.3 3.1 also holds for trees T and U on $Y \times Z$, where $|Y| < \delta$. Both proofs generalize easily.

The functions $\Sigma_{\lambda,j}$ of the last theorem are useful in other ways. Here is another example:

Theorem 3.4 (Woodin) *Let δ be Woodin, and let T on $\omega \times Z$ be any tree; then there is a $\kappa < \delta$ such that*

$$V^{\text{Col}(\omega,\kappa)} \models T \textit{ is } \alpha\textit{-weakly homogeneous, for all } \alpha < \delta.$$

Proof. Let

$$S = \{\xi < \delta \mid \xi \text{ is } T\text{-reflecting in } \delta\},$$

and let κ be S-reflecting in δ. We shall show that in $V^{\mathrm{Col}(\omega,2^{2^\kappa})}$, T is α-weakly homogeneous for all $\alpha < \delta$. So let $\alpha < \delta$ be given. Let $\xi < \delta$ be such that $\kappa, \alpha < \xi$ and ξ is T-reflecting in δ. Let $\xi < \lambda < \delta$, where λ is large enough that

$$\overset{\mathrm{Col}(\omega,2^{2^\kappa})}{\Vdash} \quad p[T] = p[T \cap V_\lambda],$$

and pick $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$, $V_\lambda \subseteq M$, and $j(S) \cap V_\lambda = S \cap V_\lambda$. Thus

$$M \models \xi \text{ is } j(T)\text{-reflecting in } j(\delta),$$

and we can find an embedding $k : M \to N$ (with extender) in N such that

$$V^M_{j(\kappa)+2} \subseteq N \text{ and } k(j(T)) \cap V^N_{j(\kappa)} = j(T) \cap V^N_{j(\kappa)}.$$

Working still in V, we can use j and k to associate to each κ-complete measure μ on some $T_s \cap V_\kappa$ a ξ-complete measure $\nu = \nu(\mu)$ on $T_s \cap V_\xi$: we put, for $A \subseteq T_s \cap V_\xi$

$$A \in \nu \Leftrightarrow k(A) \cap V^N_{j(\kappa)} \in j(\mu).$$

Thus ν is an "average" of measures which knit together into $j(\mu)$ in the k-ultrapower. It is easy to check that ν is a ξ-complete measure on V_ξ. Note $\nu(T_s) = 1$, since

$$\begin{aligned} k(T_s \cap V_\xi) \cap V^N_{j(\kappa)} &= k(j(T_s) \cap V_\xi) \cap V^N_{j(\kappa)} \\ &= j(T_s) \cap V^N_{j(\kappa)} \in j(\mu), \end{aligned}$$

using the reflecting properties of j and k.

Now let G be V-generic over $\mathrm{Col}(\omega, 2^{2^\kappa})$, and for $\nu \in \mathrm{meas}_\kappa(V_\kappa)$, let ν^* be the canonical extension of ν to $V[G]$; note here $2^{2^\kappa} < \xi$ and ν is ξ-complete. Letting

$$\sigma = \{\nu^* \mid \nu \in \mathrm{meas}_\kappa(V_\kappa)\},$$

we have that in $V[G]$, σ is a countable family of ξ-complete measures. In order to see σ witnesses that T is ξ-weakly homogeneous in $V[G]$, fix $x \in p[T]$ in $V[G]$. By choice of λ, we have an f such that $(x, f) \in [T \cap V_\lambda]$. Now consider the tower

$$\langle \nu(\Sigma_{\lambda,j}(x \restriction n, f \restriction n))^* \mid n \in \omega \rangle.$$

This is a tower concentrating on T_x, and its measures are in σ. If it is illfounded, then the tree of all attempts to produce $(y, g) \in [T \cap V_\lambda]$ together with an infinite descending sequence $\overrightarrow{\alpha}$ in $\mathrm{Ult}(V, \langle \nu(\Sigma_{\lambda,j}(y \restriction n, g \restriction n)) \mid n < \omega \rangle)$ has a branch $(y, g, \bar{\alpha})$ in V. (Note here it doesn't matter whether we compute the ultrapowers in V of $V[G]$, as the measures are ξ-complete.) But for such (y, g), $\Sigma_{\lambda,j}(y, g)$ is wellfounded, so $j(\Sigma_{\lambda,j}(y, g))$ is wellfounded, which easily implies $\langle \nu(\Sigma_{\lambda,j}(y \restriction n, g \restriction n)) \mid n < \omega \rangle$ is wellfounded, a contradiction.

Thus $\langle \nu(\Sigma_{\lambda,j}(x \restriction n, f \restriction n))^* \mid n < \omega \rangle$ is wellfounded in $V[G]$, and we have shown σ witnesses that T is ξ-weakly homogeneous in $V[G]$. $\square$

In fact, if κ is merely a λ-strong cardinal, as witnessed by j, then in $V^{\mathrm{Col}(\omega, 2^{2^\kappa})}$ we get an approximation to λ-weak homogeneity for $j(T)$.

Theorem 3.5 (Woodin) *Let $\lambda = |V_\lambda|$, and $j : V \to M$ witness that κ is λ-strong, with ${}^\omega M \subseteq M$, and let T be any tree on some $\omega \times Z$. Let G be V-generic over $\mathrm{Col}(\omega, 2^{2^\kappa})$; then*

$$V[G] \models j(T) \text{ has a } \lambda\text{-absolute complement.}$$

Proof. By a Skolem hull argument, we have a tree T^* on $\omega \times \kappa$ such that $p[T] = p[T^*]$ in any $< \kappa$-generic extension of V. Notice that $\mathrm{meas}_\kappa(\kappa^{<\omega})$ is countable in $V[G]$. Let a

$$m : \omega \xrightarrow{\text{onto}} j''\mathrm{meas}_\kappa(\kappa^{<\omega})$$

be an enumeration in $V[G]$ such that each $m(e)$ concentrates on κ^n, for some $n \le e$. Of course, the measures in $j''\mathrm{meas}_\kappa(\kappa^{<\omega})$ do not extend to $V[G]$, however, they *do* extend to $M[G]$, and in fact to $M[G][H]$ whenever H is size $< \lambda$ generic over $V[G]$, and hence over $M[G]$. This will be enough for our purpose, which is to form an analog of the Martin-Solovay tree. More precisely, we put

$$\begin{aligned}(s, \langle \alpha_0, \dots, \alpha_{n-1} \rangle) \in S \quad \Leftrightarrow \quad & s \in \omega^n \wedge \alpha_0 < j(\kappa)^+ \wedge \forall i, e \\ & (i < e < n \wedge m(e)(j(T^*)_s) = 1 \wedge m(e) \text{ projects to } m(i) \\ & \Rightarrow \alpha_e < \pi_{m(i), m(e)}(\alpha_i)).\end{aligned}$$

We claim S is a λ-absolute complement for $j(T)$ in $V[G]$. For let $x \in V[G][H]$ be a real, where H is size $< \lambda$ generic over $V[G]$.

For $(s, t) \in j(T^*)$, let $\Sigma(s, t)$ be the measure on T^*_s given by

$$A \in \Sigma(s, t) \Leftrightarrow t \in j(A).$$

Then if (x, f) is a branch of $j(T^*)$ in V, we have that $\Sigma(x, f)$ is a wellfounded tower of measures concentrating on T^*_x, and hence $\langle j(\Sigma(x \restriction n, f \restriction n)) \mid n < \omega \rangle$ is a wellfounded tower in M concentrating on $j(T^*)_x$. A simple absoluteness argument shows this remains true for any branch (x, f) of $j(T^*)$ in $V[G][H]$; that is, $\mathrm{Ult}(M, \langle j(\Sigma(x \restriction n, f \restriction n)) \mid n < \omega \rangle)$ is wellfounded. So if $x \in p[j(T)]^{V[G][H]} = p[j(T)]^{M[G][H]} = p[j(T^*)]^{M[G][H]}$, then one of the towers S_x is trying to prove illfounded is actually wellfounded, so that $x \notin p[S]$.

On the other hand, suppose $x \notin p[j(T)]$. Then $x \notin p[j(T^*)]$, so we have a rank function

$$f(u) = |u|_{T^*_x},$$

and $f \in M[G][H]$ because $x \in M[G][H]$. For $m(e)$ a measure in $j''\sigma$ concentrating on some $j(T^*)_{x\restriction n}$, let

$$\alpha_e = [f]_{m(e)},$$

which makes sense because f is equal modulo $m(e)$ to a function in M. Set $\alpha_e = 0$ if $m(e)$ is not such a measure. It is easy to check that $\langle \alpha_e \mid e \in \omega \rangle$ is an infinite branch of S_x, as desired. □

Corollary 3.6 *Let κ be λ-strong, where $\lambda = |V_\lambda|$, and let T and U be λ-absolute complements. Let G be V-generic over* $\mathrm{Col}(\omega, 2^{2^k a})$*; then in $V[G]$ there are λ-absolute complements R and S such that $p[S] = \exists^{\mathbb{R}} p[T]$ in all generic extensions.*

Proof. Here T is a tree on $(\omega \times \omega) \times Z$ for some Z. Let S be T, regarded as a tree on $\omega \times (\omega \times Z)$. So $p[S] = \exists^{\mathbb{R}} p[T]$ in all generic extensions.

By the theorem, in $V[G]$ there is a λ-absolute complement R for $j(S)$, where $j : V \to M$ witnesses κ is λ-strong. It is enough then to see that $p[S] = p[j(S)]$ in $V[G][H]$, for any size $< \lambda$ generic H. For this, it is enough that $p[T] = p[j(T)]$ in $V[G][H]$. But clearly $p[T] \subseteq p[j(T)]$. If $(x, y) \notin p[T]$, then $(x, y) \in p[U]$, so $(x, y) \in p[j(U)]$, so $(x, y) \notin p[j(T)]$. Thus $p[j(T)] \subseteq p[T]$, and we are done. □

Corollary 3.7 (Woodin) *If there are n strong cardinals which are $\leq \kappa$, where $1 \leq n < \omega$, then in $V^{\mathrm{Col}(\omega, 2^{2^\kappa})}$:*

- *(a) For any η, there is a tree T_η such that in any $< \eta$-generic extension, $p[T_\eta]$ is the universal Σ^1_{n+3} set,*
- *(b) all Σ^1_{n+2} sets are ∞-universally Baire, and*
- *(c) any two set generic extensions are Σ^1_{n+3} equivalent, that is, if $x \in \mathbb{R} \cap V[G] \cap V[H]$, and φ is Σ^1_{n+3}, then $V[G] \models \varphi[x]$ iff $V[H] \models \varphi[x]$.*

The proof of this corollary is an easy induction on n, with the Martin- Solovay trees for Σ^1_3 providing the starting point in the $n = 1$ case.

Corollary 3.8 *If there are infinitely many strong cardinals below λ, then in $V^{\mathrm{Col}(\omega,\lambda)}$, projective formulae are absolute for all further set forcing.*

4 The tree production lemma

We shall show that formulae with certain generic absoluteness properties define universally Baire sets.

Let $\varphi(v_0, v_1)$ be a Σ_n formula of the language of set theory, and let a be a parameter (not necessarily a real parameter). We are interested in the κ-universal Baireness of $\{x \in \mathbb{R} | \varphi(x, a)\}$.

Let $X \prec_{\Sigma_n} V$, with X countable and $\kappa, a \in X$. Let

$$\pi : N \cong X \prec_{\Sigma_n} V$$

be the transitive collapse, with $\pi(\bar{\kappa}) = \kappa$ and $\pi(\bar{a}) = a$. We then say that X *is* (φ, a, κ)-*generically correct* iff whenever g in V is N-generic over some $\mathbb{P} \in H^N_{\bar{\kappa}}$, then for all reals $x \in N[g]$,

$$N[g] \models \varphi[x, \bar{a}] \Leftrightarrow V \models \varphi[x, a].$$

Lemma 4.1 *Let $\varphi(v_0, v_1)$ be a Σ_n formula, let a be a parameter, and let M be transitive with $H_\kappa \cup \{\kappa\} \subseteq M$, and $\sigma : M \to V$ a Σ_{n+5}-elementary embedding with $a \in \mathrm{ran}(\sigma)$ and $\mathrm{crit}(\sigma) > \kappa$. The following are equivalent:*

(1) There are club many $X \in P_{\omega_1}(M)$ such that $\sigma''X$ is (φ, a, κ)-generically correct.

(2) There are trees T and U such that whenever G is V-generic over some $\mathbb{P} \in H_\kappa$, then

$$V[G] \models p[T] = \{x \mid \varphi(x, a)\} \text{ and } p[U] = \{x \mid \neg \varphi(x, a)\}.$$

Proof. Assume (1), and let $F : M^{<\omega} \to M$ be such that whenever $X \in P_{\omega_1}((M)$ and $F''X^{<\omega} \subseteq X$, then $X \prec M$ and $\sigma''X$ is (φ, a, k)-generically correct. For $x, y \in \mathbb{R}$ let

$$\begin{aligned} A(x,y) \quad \Leftrightarrow \quad & y \text{ codes a transitive } (N, \epsilon, \bar{\kappa}, \bar{a}) \models \mathsf{ZFC}^-, \text{ and} \\ & \exists g (g \text{ is} < \bar{\kappa}\text{-generic over} N \wedge x \in N[g] \wedge N[g] \models \varphi[x, \bar{a}]. \end{aligned}$$

Here $y \in {}^\omega\omega$ codes $(N, \epsilon, \bar{\kappa}, \bar{a})$ as follows: we have $(\omega, E_y, 0, 1) \cong (N, \epsilon, \bar{\kappa}, \bar{a})$, where $\langle n, m \rangle \in E_y$ iff $y(2^n \cdot 3^m) = 0$. By ZFC^- we mean all the Σ_5 consequences of ZFC; these are of course all true in M. The set of y which are codes is Π^1_1, so A is Σ^1_2, so there is a tree S on $\omega \times \kappa$ such that $p[S] = A$ in any size $< \kappa$ generic extension. We now define T on $\omega \times \omega \times \kappa \times M$. Let $\langle u_n \mid n < \omega \rangle$ be an enumeration of $\omega^{<\omega}$, with $\mathrm{dom}\, u_n \subset n$. Put $a^* = \sigma^{-1}(a)$, and $(u, v, r, s) \in T$ iff

(a) $(u, v, r) \in S$,

(b) $0 \in \mathrm{dom}(s) \Rightarrow s(0) = \kappa$, and $1 \in \mathrm{dom}(s) \Rightarrow s(1) = a^*$,

(c) $2k+2 \in \mathrm{dom}(s) \Rightarrow s(2k+2) = F(s \circ u_k)$, and

(d) if $2^n \cdot 3^m \in \mathrm{dom}(v)$, then $v(2^n \cdot 3^m) = 0$ iff $s(n) \in s(m)$.

Similarly, replacing φ by $\neg\varphi$, we can define a Σ^1_2 relation $B(x,y)$, and from the Shoenfield tree for B, a tree U on $\omega \times \omega \times \kappa \times M$.

Let $(x,y,f,\pi) \in [T]$. Then $(x,y,f) \in [S]$, so $A(x,y)$. By (b) and (d), π is an isomorphism between $(\omega, E_y, 0, 1)$ and $(X, \epsilon, \kappa, a^*)$, where $X = \mathrm{ran}(\pi)$. By (c), $F``X^{<\omega} \subseteq X$. Thus $X \prec M$ and $\sigma``X$ is (φ, a, κ)-generically correct. It follows then that $\varphi(x,a)$ is true in V. Thus $p[T] \subseteq \{x \mid \varphi(x,a)\}$ in V, and similarly, $p[U] \subseteq \{x \mid \neg\varphi(x,a)\}$ in V.

Now let G be size $< \kappa$ generic over V, and suppose $V[G] \models \varphi[x,a]$. Then $M[G] \models \varphi[x,a^*]$, as σ is sufficiently elementary. We can find a countable $Z \prec M[G]$ with $G, a^*, x, \kappa \in Z$ such that setting $X = Z \cap M$, $F''X^{<\omega} \subseteq X$. Letting $X = \mathrm{ran}(\pi)$, we can find y, f such that $(x,y,f,\pi) \in [T]$. Thus $x \in p[T]^{V[G]}$. Similarly, if $V[G] \models \neg\varphi[x,a]$, then $x \in p[U]^{V[G]}$, and hence $x \notin p[T]^{V[G]}$, since $p[T]$ and $p[U]$ are disjoint in V, hence in $V[G]$. Thus $p[T] = \{x \mid \varphi(x,a)\}$ in $V[G]$, and similarly for U, as desired.

For the (2) $\Rightarrow$ (1) direction, just note that σ is sufficiently elementary that there must be trees T and U as in (2) (with a^* replacing a) such that $T, U \in M$. But then any countable $X \prec M$ such that $\kappa, a^*, T, U \in X$ is such that $\sigma``X$ is (φ, a, κ) generically correct. $\square$

Recall that $\mathbb{Q}_{<\delta}$ is Woodin's "countable" stationary tower forcing (see appendix). Conditions in $\mathbb{Q}_{<\delta}$ are stationary sets $b \subseteq P_{\omega_1}(Z)$, for some $Z \in V_\delta$.

Theorem 4.2 (Tree production lemma, Woodin) *Let $\varphi(v_0, v_1)$ be a formula, let a be a parameter, and let δ be a Woodin cardinal. Suppose*

(1) (Generic absoluteness) If G is $< \delta$-generic over V, and H is $< \delta^+$-generic over $V[G]$, then for all $x \in \mathbb{R} \cap V[G]$,

$$V[G] \models \varphi[x,a] \text{ iff } V[G][H] \models \varphi[x,a],$$

and

(2) (Stationary tower correctness) If G is $\mathbb{Q}_{<\delta}$-generic, and $j : V \to M \subseteq V[G]$ is the generic elementary embedding, then for all $x \in \mathbb{R} \cap V[G]$

$$V[G] \models \varphi[x,a] \text{ iff } M \models \varphi[x, j(a)].$$

Then there are trees T and U such that whenever g is $< \delta$- generic over V, then

$$V[G] \models (p[T] = \{x \mid \varphi(x,a)\} \wedge p[U] = \{x \mid \neg\varphi(x,a)\}).$$

In particular, $\{x \mid \varphi(x,a)\}$ is δ-universally Baire.

Proof. It is enough to find for each $\kappa < \delta$ trees T_κ and U_κ which work for all$< \kappa$-generic g, since then we can take $T = \bigoplus_{\kappa<\delta} T_\kappa$ and $U = \bigoplus_{\kappa<\delta} U_\kappa$. For if g is $< \kappa$-generic over V, where $\kappa < \delta$, then if $V[g] \models \varphi[x, a]$, then $x \in p[T_\kappa]$, so $x \in p[T]$. On the other hand, if $V[g] \models \neg\, \varphi[x, a]$, then $x \in p[U_\kappa]$, so $x \notin p[T]$. Note here that if $p[U_\kappa] \cap p[T_\alpha] \neq \emptyset$ in some $< \delta$ generic $V[H]$, then $p[U_\kappa] \cap p[T_\alpha] \neq \emptyset$ in V, and this easily contradicts condition (1).

So fix $\kappa < \delta$. Let φ be Σ_n. Let M be transitive, $H_{\kappa^+} \subseteq M$, and $|M| < \delta$, and $\sigma : M \to V$ be Σ_{n+5} elementary, with $a = \sigma(a^*)$ and $\sigma \restriction \kappa^+ =$ identity. Let

$$\begin{aligned} b \;=\; & \{X \in P_{\omega_1}(M) \mid X \prec M \text{ and} \sigma'' X \\ & \text{is } (\varphi, a, k)\text{-generically correct} \end{aligned}$$

It is enough to show b contains a club in $P_{\omega_1}(M)$. If not, $P_{\omega_1}(M) \setminus b$ is a condition in $\mathbb{Q}_{<\delta}$, so we can find a $\mathbb{Q}_{<\delta}$ generic G such that $P_{\omega_1}(M) \setminus b \in G$. Let

$$j : V \to N \subseteq V[G]$$

be the generic embedding. Then $j''M \in j(P_{\omega_1}(M) \setminus b)$. Since $j''M \prec j(M)$, we have that $j(\sigma)''j''M$ is not $(\varphi, j(a), j(\kappa))$ correct in N. Since $j(\sigma)(j(z)) = j(\sigma(z))$, we see that $j(\sigma)''j''M$ collapses to M, and the image of $j(\kappa)$ under the collapse is κ, while the image of $j(a) = j(\sigma(a^*))$ is just a^*. But then for any $g \in N$ which is M-generic over poset of size $< \kappa$ in M, and any $x \in \mathbb{R} \cap M[g]$, we have

$$\begin{aligned} M[g] \models \varphi[x, a^*] \quad & \Leftrightarrow V[g] \models \varphi[x, a] \\ & \Leftrightarrow V[G] \models \varphi[x, a] \\ & \Leftrightarrow \qquad\qquad N \models \varphi[x, j(a)]. \end{aligned}$$

The first equivalence holds because σ lifts, the second by generic absoluteness, and the third by stationary tower correctness. Thus $j(\sigma)''j''M$ is $(\varphi, j(a), j(\kappa))$-generically correct in N, a contradiction. $\square$

The Tree Production Lemma was first used by Woodin, although he did not formally state it, in the case that the parameter $a \in \mathbb{R}$, so that $j(a) = a$. The author made the trivial adaptation to the case $a \notin \mathbb{R}$ as part of the proof of the following theorem. Woodin then formally isolated the Tree Production Lemma as we have stated it.

Theorem 4.3 (Steel) *Let λ be a limit of Woodin cardinals; then every* $\mathrm{Hom}_{<\lambda}$ *set has a* $\mathrm{Hom}_{<\lambda}$ *scale.*

For the proof, we need some elementary lemmas. The first is well-known. Let μ be a κ-complete ultrafilter on I, and g be $< \kappa$- generic over V. In $V[G]$, for $A \subseteq I$ put

$$A \in \mu^* \Leftrightarrow \exists\, B \in \mu (B \subseteq A).$$

Then μ^* is a κ-complete ultrafilter on I in $V[g]$. Moreover

Proposition 4.4 *If $I \in V$, and g is $< \kappa$ generic over V, and $\nu \in V[g]$ is a κ-complete ultrafilter over I in $V[g]$, then $\nu = \mu^*$ for some $\mu \in V$.*

Proof. Note first that if $A \in \nu$, then there is a set $B \subseteq A$ such that $B \in \nu$ and $B \in V$. (Work in $V[g]$, and let $A = \dot{A}_g$. By κ-completeness of ν, we can fix $p \in g$ so that p decides "$\check{i} \in \dot{A}$" for μ-a.e. $i \in I$. Then take $B = \{i \mid p \Vdash \check{i} \in \dot{A}\}$.)

Let $\nu = \dot{\nu}_g$. We claim there is a set $B \in \nu \cap V$ such that for all $C \subseteq B$ such that $C \in V$

$$\|\check{C} \in \dot{\nu}\| = \|\check{B} \in \dot{\nu}\| \text{ or } \|\check{C} \in \dot{\nu}\| = 0.$$

We can then define in V

$$\mu = \{C \subseteq I \mid \|\check{B} \in \dot{\nu}\| \leq \|\check{C} \in \dot{\nu}\|\},$$

and it is easy to see that $\mu = \nu \cap V$, so that $\nu = \mu^*$.

If there is no B as desired, then working in $V[g]$, we define a κ-sequence of sets $B_\alpha \in \nu \cap V$ such that

$$\alpha < \beta \Rightarrow \|\check{B}_\alpha \in \dot{\nu}\| > \|\check{B}_\beta \in \dot{\nu}\|.$$

We get $B_{\alpha+1}$ from the fact that B_α is not as desired. At limit $\lambda < \kappa$, let $A = \bigcap_{\alpha<\lambda} B_\alpha$. Since $A \in \nu$, we can find $B_\lambda \in \nu \cap V$ so that $B_\lambda \subseteq A$, and continue. But now g was generic for a poset of size $< \kappa$, so there cannot be a strictly decreasing κ-sequence of Boolean values, even in $V[g]$. □

The second lemma we need is a minor variation on the well-known fact that if μ and ν are normal ultrafilters on κ and λ, with $\kappa < \lambda$, and $j : V \to M = \text{Ult}(V, \mu)$ is the canonical embedding, then $j(\nu) = \nu \cap M$, and $\text{Ult}(M, j(\nu))$, which is the ultrapower computed using functions in M, is the same as $\text{Ult}^*(M, \nu)$, where the $*$ indicates that the ultrapower is computed using functions in V. The variation comes from letting j be a generic embedding.

Lemma 4.5 *Let δ be Woodin, and G be $\mathbb{Q}_{<\delta}$ generic over V, with*

$$j : V \to M \subseteq \mathit{Ult}(V, G)$$

the canonical embedding. Let μ be a δ^+-complete ultrafilter on some I, with $\mu \in V$. Then

(1) For any $A \subseteq j(I)$ in M,

$$A \in j(\mu) \Leftrightarrow \exists\ B \in \mu(j(B) \subseteq A),$$

(2) $\mathit{Ult}(M, j(\mu)) \cong \mathit{Ult}^*(M, \mu^*)$, where the first ultrapower is computed using all $f : j(I) \to M$ such that $f \in M$, and the second ultrapower is computed using all $f : I \to M$ such that $f \in V[G]$, or equivalently, all $f : I \to M$ such that $f \in V$.

(3) *Let* $\nu \leq_{RK} \mu$ *via* $p : I \to J$*, that is, let* μ *be the measure given by* $\nu(A) = \mu(p^{-1}(A))$ *for* $A \subseteq J$*. Let*

$$i : Ult(V, \nu) \to Ult(V, \mu)$$

be the canonical embedding given by $i([f]_\nu) = [f \circ p]_\mu$ *for all* $f \in V$*. Let*

$$i^* : Ult(V[G], \nu^*) \to Ult(V[G], \mu^*)$$

be its lift to $V[G]$*, given by* $i^*([f]_{\nu^*}) = [f \circ p]_{\mu^*}$ *for all* $f \in V[G]$*. (Thus* $i \subseteq i^*$*.) Then*

$$j(i) = i^* \restriction Ult(M, j(\nu)),$$

and in particular, $j(i)$ *agrees with* i *on the ordinals.*

Proof. The $\Leftarrow$ direction of (1) is trivial. For the $\Rightarrow$ direction, let $A \in j(\mu)$, and $A = [f]_G$ where $f \in V$. We may assume $f : P_{\omega_1}(Z) \to \mu$ for some $Z \in V_\delta$. It is easy to see that

$$B = \bigcap \{f(X) \mid X \in P_{\omega_1}(Z)\}$$

is as desired.

For (2), note first

Claim. If $f : I \to M$ and $f \in V[G]$, then there is an $h \in M$ and $B \in \mu$ such that

$$f(u) = h(j(u))$$

for all $u \in I$.

Proof. Work in $V[G]$. Let $\dot{f}_G = f$. For each $u \in I$, pick $a_u \in G$ such that for some $g \in V$ with domain $P_{\omega_1}(Z_u)$

$$a_u \overset{\mathbb{Q}_{<\delta}}{\Vdash} \dot{f}(\check{u}) = [\check{g}]_{\dot{G}}.$$

We can fix $a_u = a$ and $Z_u = Z$ for μ^*-a.e. u. We then have a set $B \in \mu$ such that $a_u = a$ and $Z_u = Z$ for all $u \in B$. Going back to V, we can find for each $u \in B$ a function g_u with domain $P_{\omega_1}(Z)$ such that

$$a \overset{Q_{<\delta}}{\Vdash} \dot{f}(\check{u}) = [\check{g}_u].$$

Now, for $X \in P_{\omega_1}(Z)$ in V, set

$$h_X(u) = g_u(X)$$

for all $u \in B$. Then for all $u \in B$,

$$[\lambda X.h_X]_G(j(u)) = [g_u]_G = f(u),$$

by Los' theorem and the fact that $a \in G$. Thus $h = [\lambda X.h_X]_G$ is as desired.

Now for each $f : I \to M$ in $V[G]$, pick an $h_f \in M$ such that $h_f(j(u)) = f(u)$ for μ^*-a.e. u. Notice that if $[h_f]_{j(\mu)} \neq [h_g]_{j(\mu)}$, then by (1) we can find $B \in \mu$ s.t. $h_f(v) \neq h_g(v)$ for all $v \in j(B)$, and hence $h_f(j(u)) \neq h_g(j(u))$ for all $u \in B$, so that $f(u) \neq g(u)$ for all $u \in B$; that is, $[f]_{\mu^*} \neq [g]_{\mu^*}$. Thus the map

$$\pi([f]_{\mu^*}) = [h_f]_{j(\mu)}$$

is well-defined on equivalence classes. A similar argument shows

$$[f]_{\mu^*} \in [g]_{\mu^*} \text{ iff } [h_f]_{j(\mu)} \in [h_g]_{j(\mu)},$$

so π is an $\in$-isomorphism with its range. But if $h\colon j(I) \to M$ and $h \in M$, then letting $f(u) = h(j(u))$ for all $u \in I$, we have $f \in V[G]$ and $[h]_{j(\mu)} = [h_f]_{j(\mu)}$. Thus π is onto, and hence an isomorphism of the ultrapowers in question.

For (3), let $p : I \to J$ and $\nu(A) = 1$ iff $\mu(p^{-1}(A)) = 1$. We need to see that the diagram

$$\begin{array}{ccc} \mathrm{Ult}^*(M,\nu^*) & \xrightarrow{i^*} & \mathrm{Ult}^*(M,\mu^*) \\ \sigma \updownarrow & & \updownarrow \tau \\ \mathrm{Ult}(M,j(\nu)) & \xrightarrow{j(i)} & \mathrm{Ult}(M,j(\mu)) \end{array}$$

commutes, where σ and τ are the isomorphisms of part (2). This then means $j(i) = i^* \restriction \mathrm{Ult}^*(M,\nu^*)$ after the ultrapowers have been transitivised. So let $X = [f]_{\nu^*} \in \mathrm{Ult}^*(M,\nu^*)$. Then

$$\begin{aligned} j(i)(\sigma(x)) &= j(i)([h_f]_{j(\nu)}) \\ &= [h_f \circ j(p)]_{j(\nu)}, \end{aligned}$$

and

$$\begin{aligned} \tau(i^*(x)) &= \tau([f \circ p]_{\mu^*}) \\ &= [h_{f\circ p}]_{j(\mu)}, \end{aligned}$$

where we have adopted the notation from the proof of (2) in analyzing σ and τ. From the proof of (2), we see that it suffices to show that $h_f \circ j(p)$ and $h_{f\circ p}$ agree at all points in $j''B$, for some $B \in \mu$. But now

$$h_{f\circ p}(j(u)) = (f \circ p)(u) = f(p(u))$$

for all $u \in B_0$, where $B_0 \in \mu$. Also

$$\begin{aligned} (h_f \circ j(p))(j(u)) &= h_f(j(p)(j(u))) \\ &= h_f(j(p(u))) \\ &= f(p(u)) \end{aligned}$$

for all $u \in B_1$, where $B_1 \in \mu$. Thus $h_{f \circ p}(j(u)) = (h_f \circ j(p))(j(u))$ for all $u \in B_0 \cap B_1$, as desired. □

From the last lemma we get a stationary tower correctness result for homogeneity systems.

Corollary 4.6 *Let δ be Woodin, G be V-generic over $\mathbb{Q}_{<\delta}$, and*

$$j : V \to M \subseteq Ult(V, G)$$

be the generic embedding.

(1) Let $\langle \mu_s^* \mid s \in \omega^{<\omega} \rangle$ be a homogeneity system in $V[G]$, where each μ_s is a δ^+-complete ultrafilter in V (although the system *$\langle \mu_s \mid s \in \omega^{<\omega} \rangle$ may not be in V); then for any γ*

$$ms(\langle \mu_s^* \mid s \in \omega^{<\omega} \rangle, \gamma)^{V[G]} = ms(\langle j(\mu_s) \mid s \in \omega^{<\omega} \rangle, \gamma)^M,$$

and

$$(S_{\langle \mu_s^* | s \in \omega^{<\omega} \rangle})^{V[G]} = (S_{\langle j(\mu_s) | s \in \omega^{<\omega} \rangle})^M.$$

(2) If $\lambda > \delta$ is a limit of Woodin cardinals, then $\mathrm{Hom}_{<\lambda}^{V[G]}$ is an initial segment of $j(\mathrm{Hom}_{<\lambda})$ under Wadge reducibility.

Proof. This follows at once from the last lemma. □

The corollary also holds for weak homogeneity systems, but we shall not need this fact. The corollary is due independently to Woodin and the author.

Proof of Theorem 4.3. Fix $\gamma_0 < \lambda$ such that

$$\mathrm{Hom}_{\gamma_0} = \mathrm{Hom}_{<\lambda}.$$

Let $\gamma_0 < \delta_0 < \delta_1 < \delta_2 < \lambda$, where the δ_i's are Woodin. Let $B \in \mathrm{Hom}_{<\lambda}$. It will be enough to find a scale $\{\theta_n\}$ on B such that the relation

$$S(n, x, y) \Leftrightarrow \theta_n(x) \leq \theta_n(y)$$

is δ_1^+-universally Baire. For then, S is δ_0^+-weakly homogeneous by 3.1, and hence γ_0-homogeneous by 2.1, and hence in $\mathrm{Hom}_{<\lambda}$.

Let $\langle \mu_s \mid s \in \omega^{<\omega} \rangle$ be a homogeneity system consisting of δ_2^+-complete measures such that

$$\mathbb{R} \setminus B = S_{\langle \mu_s | s \in \omega^{<\omega} \rangle}.$$

Let γ be a strong limit cardinal of cofinality $> \delta_2$, and let

$$T = ms(\langle \mu_s \mid s \in \omega^{<\omega} \rangle, \gamma).$$

Let $\{\theta_n\}$ be the leftmost-branch scale of T; that is, for $x \in p[T]$ define $\ell_x : \omega \to OR$ by

$$\ell_x(n) = \text{least } \alpha \text{ such that } \exists\, g(g \restriction n = \ell_x \restriction n \wedge g(n) = \alpha \wedge g \in [T_x]),$$

and put

$$\theta_n(x) \leq \theta_n(y) \text{ iff } \ell_x \restriction n \leq_{\text{lex}} \ell_y \restriction n,$$

where $\leq_{\text{lex}}$ is the lexicographic order. It is well-known folklore that $\{\theta_n\}$ is a scale on $p[T] = B$.

To see that the relation $S(n, x, y)$ is δ_1^+-universally Baire, we apply the tree production lemma at δ_2. Let $\varphi(v_0, v_1)$ be the natural formula defining S from the parameter T.

For generic absoluteness, let G be size $< \delta_2$ generic over V, and H size δ_2 generic over $V[G]$, and $(n, x, y) \in V[G]$. Clearly $x \in p[T]^{V[G]}$ iff $x \in p[T]^{V[G][H]}$, and similarly for y; also, $\ell_x^{V[G]} = \ell_x^{V[G][H]}$ and similarly for y. Thus

$$V[G] \models \ell_x \restriction n \leq_{\text{lex}} \ell_y \restriction n \text{ iff } V[G][H] \models \ell_x \restriction n \leq_{\text{lex}} \ell_y \restriction n,$$

as desired.

For stationary tower correctness, let $j : V \to M \subseteq V[G]$ where G is V-generic over $\mathbb{Q}_{<\delta_2}$. Clearly $j(\gamma) = \gamma$. But then

$$\begin{aligned} j(T) &= j(ms(\langle \mu_s \mid s \in \omega^{<\omega} \rangle, \gamma)) \\ &= ms(\langle j(\mu_s) \mid s \in \omega^{<\omega} \rangle, \gamma)^M \\ &= ms(\langle \mu_s^* \mid s \in \omega^{<\omega} \rangle, \gamma)^{V[G]} \\ &= T, \end{aligned}$$

using our previous lemmas. The absoluteness of wellfoundedness then tells us

$$V[G] \models \varphi[(n, x, y), T] \text{ iff } M \models \varphi[(n, x, y), j(T)],$$

as desired. □

5 $(\Sigma_1^2)^{\text{Hom}_{<\lambda}}$ absoluteness

The observation that $\text{Hom}_{<\lambda}^{V[G]}$ is a Wadge-initial segment of $j(\text{Hom}_{<\lambda})$, recorded in 4.6, can be used to strengthen our projective generic absoluteness result. What we get is "$(\Sigma_1^2)^{\text{Hom}_{<\lambda}}$ generic absoluteness". More precisely

Theorem 5.1 (Woodin) *Let $A \in \mathrm{Hom}_{<\lambda}$, where λ be a limit of Woodins, and let G be $< \lambda$-generic over V; then for any sentence φ in the language of set theory expanded by adding two new unary predicate symbols,*

$$\exists B \in \mathrm{Hom}^V_{<\lambda}(HC^V, \in, A, B) \models \varphi \Leftrightarrow \exists B \in \mathrm{Hom}^{V[G]}_{<\lambda}(HC^{V[G]}, \in, A^{V[G]}, B) \models \varphi.$$

Proof. The left-to-right direction is an immediate consequence of our projective generic absoluteness result, 2.6. For the right-to- left direction, let H be $\mathbb{Q}_{<\delta}$-generic over V for some Woodin cardinal $\delta < \lambda$, with $G \in V[H]$. (Since $\mathbb{Q}_{<\delta}$ collapses all $\eta < \delta$, general forcing theory tells us there is such an H.) By the upward absoluteness of $(\Sigma^2_1)^{\mathrm{Hom}_{<\lambda}}$ from $V[G]$ to $V[H]$, we have a $B \in \mathrm{Hom}^{V[H]}_{<\lambda}$ such that

$$(HC^{V[H]}, \in, A^{V[H]}, B) \models \varphi.$$

Letting $j\colon V \to M = \mathrm{Ult}(V, H)$ be the generic embedding, we see from 4.6 that $j(A) = A^{V[H]}$ and $B \in j(\mathrm{Hom}^V_{<\lambda})$. Of course, $HC^{V[H]} = HC^M$ as well. Thus

$$M \models [\exists B \in j(\mathrm{Hom}^V_{<\lambda})(HC, \in, j(A), B) \models \varphi].$$

The elementarity of j now yields the desired conclusion. $\square$

Remark 5.2 Let M_ω be the minimal iterable proper class mouse satisfying "there are infinitely many Woodin cardinals", and let λ be the supremum of the Woodin cardinals of M_ω. If $\alpha < \omega_1^{M_\omega}$, then the canonical iteration strategy for $M_\omega|\alpha$ is $\mathrm{Hom}^{M_\omega}_{<\lambda}$, in the sense that there is a λ-absolutely complemented tree $T \in M_\omega$ which projects to this iteration strategy in all $< \lambda$-generic extensions of M_ω. It follows then that in M_ω, every real is $(\Sigma^2_1)^{\mathrm{Hom}_{<\lambda}}$ in a countable ordinal. This statement, and the statement

$$\forall x \in \mathbb{R} \exists \Gamma \in \mathrm{Hom}_{<\lambda}(\Gamma \text{ is an } \omega_1\text{- iteration strategy for the mouse } N \wedge x \in N)$$

have the form $\forall x \in \mathbb{R}\psi$, where ψ "is" $(\Sigma^2_1)^{\mathrm{Hom}_{<\lambda}}$. The statements are false in $M_\omega[x]$, where x is a Cohen real over M_ω. This puts a limit on what statements about $\mathrm{Hom}_{<\lambda}$ are provably $< \lambda$- generically absolute which lies just beyond the positive result of 5.1.

Remark 5.3 It follows from the last remark that if

$$j\colon M_\omega \to N = \mathrm{Ult}(M_\omega, G)$$

the the generic embedding, where G is $\mathbb{Q}_{<\delta}$-generic over M_ω, then

$$\mathrm{Hom}^{M_\omega[G]}_{<\lambda} \neq j(\mathrm{Hom}^{M_\omega}_{<\lambda}).$$

The reason is that N satisfies both of the statements $\forall x \in \mathbb{R}\psi$ referred to in 5.2, while $M_\omega[G]$ satisfies neither.[5] Since N and $M_\omega[G]$ have the same reals, it must be that $\mathrm{Hom}^N_{<\lambda} \neq \mathrm{Hom}^{M_\omega[G]}_{<\lambda}$.

Remark 5.4 A closely related fact is that M_ω satisfies the statement "there is a $(\Sigma^2_1)^{\mathrm{Hom}_{<\lambda}}$ wellorder of the reals". The wellorder is not a $\mathrm{Hom}_{<\lambda}$ set from the point of view of M_ω, since, for example, it does not have the Baire property. This shows that it is possible (consistent with the existence of infinitely many Woodins) that there are $(\Sigma^2_1)^{\mathrm{Hom}_{<\lambda}}$ sets of reals which are not $\mathrm{Hom}_{<\lambda}$. This in turn shows that the stationary tower correctness hypothesis of the tree production lemma is necessary, since the formula $(\Sigma^2_1)^{\mathrm{Hom}_{<\lambda}}$ formula defining a wellorder of the reals in M_ω *is* generically absolute over M_ω.

Remark 5.5 The last three remarks probably generalize from M_ω to any other mouse M and ordinal λ such that $M \models \lambda$ is a limit of Woodin cardinals. This is already known for many of the more natural M. We would guess, for example that the assertion that every real is $(\Sigma^2_1)^{\mathrm{Hom}_{<\lambda}}$- definable from a countable ordinal, for all λ which are limits of Woodin cardinals, is consistent with the existence of arbitrarily large superstrong cardinals. This is because we would guess that current inner model theory, which is based on the existence of homogeneously Suslin iteration strategies, goes at least this far.

Remark 5.6 Finally, the last remark should not be taken to mean that superstrong cardinals yield no generic absoluteness beyond that given by 5.1. Large cardinal hypotheses stronger than "there are infinitely many Woodin cardinals" will imply there are more $\mathrm{Hom}_{<\lambda}$ sets of reals, and hence that more statements can be expressed in $(\Sigma^2_1)^{\mathrm{Hom}_{<\lambda}}$ form. For example, if we add that there is a measurable cardinal above λ, then we have that $\mathbb{R}^\sharp$ is in $\mathrm{Hom}_{<\lambda}$, which implies that the first order theory of $L(\mathbb{R})$ is $< \lambda$-generically absolute. (This much generic absoluteness fails in M_ω.) One actually gets a many-one reduction of the theory of $L(\mathbb{R})$ to the set Σ of all $(\Sigma^2_1)^{\mathrm{Hom}_{<\lambda}}$ truths. Still stronger large cardinal hypotheses provide explicit provable many-one reductions to Σ of the truth sets for more powerful languages, and hence yield still stronger generic absoluteness theorems.

Woodin's Ω-conjecture implies that, granted there are arbitrarily large Woodin cardinals, all generic absoluteness theorems come through many-one reductions to $(\Sigma^2_1)^{\mathrm{Hom}_\infty}$.

6 The derived model theorem

Let λ be a limit of Woodin cardinals. As explained in the last section, we cannot hope to show that $L(\mathbb{R}, \mathrm{Hom}_{<\lambda}) \models \mathsf{AD}$, since if V is a canonical inner model, then $L(\mathbb{R}, \mathrm{Hom}_{<\lambda})$ has a

[5]Let x be a real in $M_\omega[G] \setminus M_\omega$. If x were $(\Sigma^2_1)^{\mathrm{Hom}_{<\lambda}}$ in $M_\omega[G]$, then by 5.1 that would remain true in $M_\omega[H]$ for some $\mathrm{Col}(\omega, \eta)$-generic H, with $\eta < \lambda$ sufficiently large. So x would be OD in $M_\omega[H]$, and hence x would be in M_ω, a contradiction.

wellorder of $\mathbb{R}$ in it. (At least this is true if $V = M_\omega$, and in many other cases.) Nevertheless, one can find a model very close to $L(\mathbb{R}, \mathrm{Hom}_{<\lambda})$ which satisfies AD. This so-called *derived model* is obtained by collapsing everything below λ to be countable.

More precisely, let λ be a limit of Woodin cardinals, and let G be V-generic over $\mathrm{Col}(\omega, <\lambda)$. Let us write $G \restriction \alpha$ for $G \cap \mathrm{Col}(\omega, < \alpha)$. We set

$$\mathbb{R}^* = \mathbb{R}^*_G = \bigcup_{\alpha<\lambda} \mathbb{R} \cap V[G \restriction \alpha)],$$

and

$$\begin{aligned}\mathrm{Hom}^* = \mathrm{Hom}^*_G &= \{p[T] \cap \mathbb{R}^* \mid \exists\alpha < \lambda(T \in V[G \restriction \alpha) \wedge \\ &\quad V[G \restriction \alpha] \models T \text{ is } \lambda\text{-absolutely complemented})\}.\end{aligned}$$

Put another way, for any $\alpha < \lambda$ and $A \in \mathrm{Hom}^{V[G\restriction\alpha]}_{<\lambda}$, we set

$$A^* = \bigcup_{\alpha<\beta<\lambda} A^{V[G\restriction\beta]},$$

and we have

$$\mathrm{Hom}^* = \{A^* \mid \exists\alpha < \lambda \mid A \in \mathrm{Hom}^{V[G\restriction\alpha]}_{<\lambda}\}.$$

Then $L(\mathbb{R}^*, \mathrm{Hom}^*)$ is called a *derived model* of V at λ. Of course, it is not literally accurate to speak of *the* derived model, since the model depends not just on V and λ, but on $\mathbb{R}^*$, which can be realized in different ways with different G. However, the forcing is sufficiently homogeneous that the first order theory of $L(\mathbb{R}^*, \mathrm{Hom}^*)$ is independent of G, so there is no ambiguity if we say that "the" derived model at λ satisfies φ.

Theorem 6.1 (Derived model theorem, Woodin) *Let λ be a limit of Woodin cardinals, and $L(\mathbb{R}^*, \mathrm{Hom}^*)$ be a derived model at λ; then*

(1) $L(\mathbb{R}^*, \mathrm{Hom}^*) \models \mathsf{AD}^+$,

(2) $\mathrm{Hom}^* = \{A \subseteq \mathbb{R}^* \mid A$ *is Suslin and co-Suslin in* $L(\mathbb{R}^*, \mathrm{Hom}^*)\}$.

AD^+ is the theory $\mathsf{AD} + \mathsf{DC}_\mathbb{R}$ + Ordinal Determinacy + "all sets of reals are ∞-Borel". These are local consequences of scales [6]: if every set in M is Suslin in some perhaps bigger model N of AD having the same reals as M, then $M \models \mathsf{AD}^+$.[7] Many of the consequences of being Suslin in a larger model of AD are theorems of AD^+. The following converse to the derived model theorem is further evidence of the significance of AD^+.[8]

[6] AD^+ used to be called "Within Scales".

[7] In particular, if every set of reals in M is $\mathrm{Hom}_{<\lambda}$, then $M \models \mathsf{AD}^+$.

[8] AD^+ was first isolated by Woodin. It is open whether any or all of the additional axioms of AD^+ are provable in $\mathsf{ZF} + \mathsf{AD}$.

Theorem 6.2 (Woodin) *Let $M \models \mathsf{AD}^+$, and let Γ be the pointclass consisting of all sets of reals which are Suslin and co-Suslin in M; then $L(\mathbb{R}^M, \Gamma)$ is a derived model of some N at some λ.*

The model N referred to in 6.2 exists in a generic extension of M. Its λ is ω_1^M, as it must be if $\mathbb{R}^M$ is to be the set of reals of a derived model at λ. We shall not prove 6.2 in these notes.

According to 6.1 and 6.2, being the pointclass of all Suslin and co-Suslin sets in a model of AD^+ is equivalent to being the pointclass of all Suslin and co-Suslin sets of a derived model (and this is equivalent to being the Hom^* of a derived model). Woodin has found a generalization of the derived model construction, and shown that the generalized derived models it produces are precisely the models of AD^+. We shall not prove this strengthening of the derived model theorem here. [9]

We proceed toward the proof of the derived model theorem. The following little lemma will be useful.

Lemma 6.3 *Let G be $\text{Col}(\omega, < \lambda)$-generic over V, where λ is a limit of Woodin cardinals. For any $\alpha < \lambda$ and $A \in \text{Hom}_{<\lambda}^{V[G\restriction\alpha]}$, $(HC^{V[G\restriction\alpha]}, \in, A) \prec (HC_G^*, \in, A^*)$.*

Proof. From our projective absoluteness result 2.6, we have that whenever $\alpha < \beta < \gamma < \lambda$, then $(\text{HC}^{V[G\restriction\beta]}, \in, A^{V[G\restriction\beta]}) \prec (\text{HC}^{V[G\restriction\gamma]}, \in, A^{V[G\restriction\gamma]})$. The lemma now follows by the Tarski-Vaught theorem on unions of elementary chains. □

The heart of the matter is the following reflection result.

Lemma 6.4 *Let G be $\text{Col}(\omega, < \lambda)$-generic over V, where λ is a limit of Woodin cardinals. Let $A \in \text{Hom}_{<\lambda}^{V[G\restriction\alpha]}$, where $\alpha < \lambda$. Let φ be a sentence in the language of set theory with two additional unary predicate symbols, and suppose that*

$$\exists B \subseteq \mathbb{R}^*(B \in L(\mathbb{R}^*, \text{Hom}^*) \wedge (HC^*, \in, A^*, B) \models \varphi);$$

then

$$\exists B(B \in \text{Hom}_{<\lambda}^{V[G\restriction\alpha]} \wedge (HC^{V[G\restriction\alpha]}, \in, A, B) \models \varphi).$$

Before proving 6.4, let us use it to complete the proof of the derived model theorem. So let G be $\text{Col}(\omega, < \lambda)$-generic over V, where λ is a limit of Woodins, and $\mathbb{R}^* = \mathbb{R}_G^*$ and $\text{Hom}^* = \text{Hom}_G^*$. We show first that $L(\mathbb{R}^*, \text{Hom}^*) \models \mathsf{AD}$. For if not, there is a $B \in L(\mathbb{R}^*, \text{Hom}^*)$ such that

$$(\text{HC}^*, \in, B) \models \text{ the game with payoff } B \text{ is not determined.}$$

[9] Let λ be a limit of Woodin cardinals, and $\mathbb{R}^* = \mathbb{R}_G^*$ where G is $\text{Col}(\omega, < \lambda)$-generic over V. Let M be the union of all models $P \in V(\mathbb{R}^*)$ such that $P \models \mathsf{AD}^+$ (plus $V = L(P(\mathbb{R}))$?); then M is a generalized derived model of V at λ.

By 6.4, we can find $B \in \mathrm{Hom}^V_{<\lambda}$ such that

$$(\mathrm{HC}, \in, B) \models \text{ the game with payoff } B \text{ is not determined.}$$

This contradicts Martin's theorem 1.7.

The remaining axioms of AD^+ are true in $L(\mathbb{R}^*, \mathrm{Hom}^*)$ for similar reasons. In each case the axiom can be expressed in the form "$\forall B \subseteq \mathbb{R}(\mathrm{HC}, \in, B) \models \varphi$", and there are no $\mathrm{Hom}_{<\lambda}$ sets B such that $(\mathrm{HC}^V, \in, B) \models \varphi$. For the axiom $\mathsf{DC}_{\mathbb{R}}$ both parts are obvious. The other two axioms have the form $\forall B \subseteq \mathbb{R} \exists C \subseteq \mathrm{OR}$...., but using the Coding Lemma the quantifier on C can be reduced to a real quantifier over the field of a prewellorder which is projective in B. For Ordinal Determinacy, this is obvious, but for the assertion that B has an infinity-Borel code C, we need a preliminary argument which bounds the least size of such a code by some ordinal projective in B. This can be done.[10] Finally, the fact that there are no $\mathrm{Hom}_{<\lambda}$ counterexamples B to Ordinal Determinacy or the assertion that every set of reals is ∞-Borel follows from the fact that every $\mathrm{Hom}_{<\lambda}$ set has a $\mathrm{Hom}_{<\lambda}$ scale, together with $\mathrm{Hom}_{<\lambda}$- determinacy.[11]

To see that all Hom^* sets are Suslin in $L(\mathbb{R}^*, \mathrm{Hom}^*)$, fix C in Hom^*. We then have $A \in \mathrm{Hom}^{V[G \restriction \alpha]}_{<\lambda}$, for some $\alpha < \lambda$, such that $C = A^*$. By 4.3 there is $B \in \mathrm{Hom}^{V[G \restriction \alpha]}_{<\lambda}$ which codes a scale on A. This fact can be expressed using only real quantifiers, and thus by 6.3, B^* codes a scale on A^* in $L(\mathbb{R}^*, \mathrm{Hom}^*)$, so C is Suslin in $L(\mathbb{R}^*, \mathrm{Hom}^*)$, as desired. Since Hom^* is closed under complement, all Hom^* sets are co-Suslin in $L(\mathbb{R}^*, \mathrm{Hom}^*)$.

Conversely, suppose A is Suslin and co-Suslin in $L(\mathbb{R}^*, \mathrm{Hom}^*)$, and let T and U be the trees which witness this. We can fix a set $C \in \mathrm{Hom}^*$ such that T and U are ordinal definable over $L(\mathbb{R}^*, \mathrm{Hom}^*)$ from C. (Every set in $L(\mathbb{R}^*, \mathrm{Hom}^*)$ has this form.) We then have $W \in V[G \restriction \alpha]$, where $\alpha < \lambda$, such that $C = p[W] \cap \mathbb{R}^*$. It follows that T and U are definable in $V[G]$ from the parameter $\mathbb{R}^*$ and parameters in $V[G \restriction \alpha]$. But $V[G] = V[G \restriction \alpha][H]$ where H is generic for $\mathrm{Col}(\omega, < \lambda)$, and there is a term τ such that $\tau_H = \mathbb{R}^*$ and $\mathrm{Col}(\omega, < \lambda)$ is homogeneous with respect to τ, in that $\forall p, q \exists \pi (\pi$ is an automorphism of $\mathrm{Col}(\omega, < \lambda)$ and $\pi(p)$ is compatible with q and $\pi\tau = \tau)$. Since T and U are subsets of $V[G \restriction \alpha]$, we have that $T, U \in V[G \restriction \alpha]$. But now T and U project to complements over $\mathbb{R}^*$, and hence in any $V[G \restriction \beta]$ for $\beta < \lambda$. Since the collapse forcing is universal, this implies that T and U are $< \lambda$-absolute complements in $V[G \restriction \alpha]$. Thus $p[T] \in \mathrm{Hom}^*$, as desired. This completes the proof of the derived model theorem, modulo 6.4. □

One key step toward the proof of 6.4 is to show that the reals of a symmetric collapse below λ can be realized as the reals of a stationary tower ultrapower. For this we use the following elementary lemma. For G $\mathrm{Col}(\omega, < \lambda)$ generic, let HC^*_G be the collection of hereditarily countable sets having codes in $\mathbb{R}^*_G$.

[10] The locality of ∞-Borel codes is due to Woodin.

[11] For Ordinal Determinacy, this is due independently to Moschovakis ([9]) and Woodin. It is folklore that all Suslin sets are ∞-Borel; see e.g. [2].

Lemma 6.5 *Let N be a countable transitive model of* ZFC*, and let λ be a strong limit cardinals of N. Let X be countable. The following are then equivalent:*

(1) $X = HC^*_G$ for some G which is $\mathrm{Col}(\omega, <\lambda)$-generic over N,

(2) $\forall y \in X(y$ is $<\lambda$ generic over N and $V_\lambda^{N[y]} \subseteq X)$, and $\forall y \in X \exists f \in X(f : \omega \overset{onto}{\to} y)$.

Proof. It is clear that (1) implies (2). For the converse, let $X = \{y_n \mid n < \omega\}$. We construct the desired G by defining $G \restriction \alpha_n$ by induction on n, where α_n is an increasing sequence with limit λ determined by the construction. We maintain that $G \restriction \alpha_n$ is coded by a real in X as part of the induction. Let $\langle D_n \mid n < \omega \rangle$ enumerate the dense subsets of $\mathrm{Col}(\omega, <\lambda)$ lying in N. Given such $G \restriction \alpha_n$, we have by hypothesis that y_n is $<\lambda$-generic over N, and hence over $N[G \restriction \alpha_n]$. By general forcing theory, the complete Boolean algebra for adding y is a complete subalgebra of the collapse algebra at some $\beta < \lambda$ such that $\alpha_n < \beta$. Thus $y_n \in N[G \restriction \alpha_n][H]$ for some $\mathrm{Col}(\omega, \beta)$-generic H. We can take $H \in X$, because $V_\lambda^{N[G, y_n} \subseteq X$ and every set in X has a counting in X. It is now easy to find α_{n+1} and $G \restriction \alpha_{n+1} \in X$ extending $G \restriction \alpha_n$ such that $H \in N[G \restriction \alpha_{n+1}]$ and $G \restriction \alpha_{n+1} \cap D_n \neq \emptyset$.

This completes the construction. It is clear that G is $\mathrm{Col}(\omega, <\lambda)$- generic over N, and $\mathrm{HC}^*_G = X$. □

We now look at stationary tower forcing up to λ. Since λ may not itself be Woodin, $\mathbb{Q}_{<\lambda}$-generic G may be such that $\mathrm{Ult}(V, G)$ is illfounded. However, because λ is a limit of Woodins, we can find $G \subseteq \mathbb{Q}_{<\lambda}$ such that $\mathrm{Ult}(V, G)$ has wellfounded part as long as desired, and such that $\mathbb{R} \cap \mathrm{Ult}(V, G)$ is the set of reals in a symmetric collapse.

Our G will not actually be $\mathbb{Q}_{<\lambda}$-generic. However, $G \cap \mathbb{Q}_{<\delta}$ will be $\mathbb{Q}_{<\delta}$-generic for cofinally many Woodin cardinals $\delta < \lambda$. This is enough to make sense of $\mathrm{Ult}(V, G)$, since the functions used in computing this ultrapower all have domain of the form $P_{\omega_1}(V_\xi)$ for some $\xi < \lambda$ (and are in V). If $\xi < \delta$ and $G \cap \mathbb{Q}_{<\delta}$ is V-generic, then $G \cap \mathbb{Q}_{<\delta}$ measures all subsets of $P_{\omega_1}(V_\xi)$ which lie in V. Thus $\mathrm{Ult}(V, G)$ makes sense.

Let us call δ a *successor Woodin cardinal* if δ is a Woodin Cardinal which is not a limit of Woodins.

Lemma 6.6 (Woodin) *Let λ be a limit of Woodin cardinals, let H be $\mathrm{Col}(\omega, <\lambda)$-generic over V, and let $\alpha \in OR$; then for any $b \in \mathbb{Q}_{<\lambda}$ there is a $\mathbb{Q}_{<\lambda}$-generic G over V such that $b \in G$ and*

(a) for any successor Woodin cardinal $\delta < \lambda$ such that $\bigcup b \in V_\delta$, $G \cap \mathbb{Q}_{<\delta}$ is $\mathbb{Q}_{<\delta}$-generic over V,

(b) α is in the wellfounded part of Ult(V, G), and

(c) $\mathbb{R} \cap Ult(V,G) = \mathbb{R}^*_H$, *and* Hom^*_H *is a Wadge initial segment of* $i_G(Hom_{<\lambda})$, *where* $i_G \colon V \to Ult(V,G)$ *is the canonical embedding.*

Proof. There is such a G if and only if there is such a G in $V[H]^{\mathrm{Col}(\omega,\alpha)}$, because this universe is Σ^1_1-correct. Thus the existence of such a G is a first order question about $\mathbb{R}^*_H, b, V_\lambda$ and α inside $V[H]$. Because $\mathbb{R}^*_H$ is the denotation of a symmetric term, this question is decided by the empty condition in $\mathrm{Col}(\omega, <\lambda)$. So it is enough just to find *some* H and G related as in the statement of 6.6.

For this, we need the following sublemma

Sublemma 6.7 *There are stationarily many* $X \in P_{\omega_1}(V_\lambda)$ *such that* $(X \cap \bigcup b) \in b$, *and whenever* $\delta \in X$ *is a successor Woodin cardinal such that* $\bigcup b \in V_\delta$, *and* $A \in X$ *is a maximal antichain in* $\mathbb{Q}_{<\delta}$, *then there is an* $a \in A$ *such that* $(X \cap \bigcup a) \in a$.

Proof sketch. If there is an $a \in X \cap A$ such that $(X \cap \bigcup a) \in a$, then one says that X *captures* A. In order to see that there are stationarily many X capturing all their maximal antichains at successor Woodins below λ but above $\bigcup b$, and such that $(X \cap \bigcup b) \in b$, it is enough to find one such $X \prec V_{\lambda+\omega}$ with $\lambda \in X$.

We construct X as the union of a countable elementary chain. Let X_0 be any countable elementary submodel of $V_{\lambda+\omega}$ such that $\lambda \in X_0$ and $(X_0 \cap \bigcup b) \in b$. We can find such an X_0 because b is stationary. Given $X_\alpha \prec V_{\lambda+\omega}$, let δ be the least successor Woodin in X_α not yet considered, and such that $\bigcup b \in V_\delta$. We form an elementary chain $Y_i \prec V_{\lambda+\omega}$ for $i < \omega$, setting $Y_0 = X_\alpha$ and $\gamma_0 = \bigcup\{\eta \mid \eta < \delta \text{ and } \eta \text{ is Woodin }\}$. Given Y_n and γ_n, let A be the "next" maximal antichain of $\mathbb{Q}_{<\delta}$, and let $\gamma_{n+1} \in Y_n$ be such that $\gamma_n < \gamma_{n+1} < \delta$ and $A \cap \mathbb{Q}_{<\gamma_{n+1}}$ is semiproper. We can find such a γ_{n+1} since δ is Woodin. Now we get Y_{n+1} which captures A and such that $Y_n \prec Y_{n+1} \prec V_{\lambda+\omega}$ and $Y_{n+1} \cap V_{\gamma_n} = Y_n \cap V_{\gamma_n}$, as a consequence of semiproperness. The end-extension below γ_{n+1} relationship guarantees that all antichains captured at earlier stages are still captured by Y_{n+1}, and that $Y_{n+1} \cap \bigcup b = Y_n \cap \bigcup b \in b$. Let $X_{\alpha+1} = \bigcup_n Y_n$. With a little care as to the meaning of "next antichain", we shall have that $X_{\alpha+1}$ captures all maximal antichains of $\mathbb{Q}_{<\delta}$ such that $A \in X$.

At limit stages τ, set $X_\tau = \bigcup_{\alpha<\tau} X_\alpha$. It is not hard to show that there is some countable α such that X_α captures all maximal antichains at successor Woodins which it knows about, so that $X = X_\alpha$ is as desired. □

We proceed to the proof of 6.6.[12]

Fix an α and b; we may as well assume $\alpha > \lambda$. We claim there are G and H as desired in $V^{\mathrm{Col}(\omega,\alpha)}$.

[12]The argument to follow is due to the author. Woodin had a somewhat different way of using the sublemma. The observation that Hom^* is a Wadge initial segment of $i_G(\mathrm{Hom}_{<\lambda})$ (6.6(c)) is due independently to the author.

Let $\theta = \alpha + \omega$, and let

$$\alpha, \lambda \in X \prec V_\theta,$$

where X is countable and in the stationary set given by sublemma 6.7. Let

$$\pi \colon N \cong X$$

be the transitive collapse, and let $\pi(\langle \bar{\alpha}, \bar{\lambda}, \bar{b}\rangle) = \langle \alpha, \lambda, b\rangle$. We define $G \subseteq \mathbb{Q}^N_{<\bar{\lambda}}$ by using initial segments of X as our typical objects. More precisely, for $a \in \mathbb{Q}^N_{<\bar{\lambda}}$, let

$$a \in G \Leftrightarrow \pi``\bigcup a \in \pi(a).$$

It will be enough to show that for some H, G and H have the properties (a), (b), and (c) of 6.6 vis-a-vis N and $\bar{\alpha}, \bar{\lambda}, \bar{b}$. For then by Σ^1_1 absoluteness, $N^{\mathrm{Col}(\omega,\alpha)}$ satisfies that there are G and H with these properties (at $\bar{\alpha}, \bar{\lambda}$, and $\bar{b}$), and since π is elementary, we are done.

It is clear that $\bar{b} \in G$. For (a), let δ be a successor Woodin cardinal of N below $\bar{\lambda}$, and let A be a maximal antichain in $\mathbb{Q}^N_{<\delta}$. Then $\pi(A)$ is a maximal antichain in $\mathbb{Q}_{<\pi(\delta)}$ and $\pi(A) \in X$, so X captures $\pi(A)$, say via $\pi(a)$. This means

$$\pi``\bigcup a = X \cap \bigcup \pi(a) \in \pi(a) \wedge \pi(a) \in \pi(A),$$

so that $a \in G \cap A$. Thus G meets all the necessary maximal antichains.

For (b), we can embed $\mathrm{Ult}(N, G)$ into V_θ as follows: let $f \in N$ be a function, and $\mathrm{dom}(f) = P_{\omega_1}(V_\gamma)^N$ where $\gamma < \bar{\lambda}$. We set

$$\sigma([f]_G) = \pi(f)(\pi``\bigcup \mathrm{dom}(f)).$$

It is easy to check that σ is well-defined and elementary (and extends π, in that $\pi = \sigma \circ \infty_G$, where i_G is the generic embedding). Thus $\mathrm{Ult}(V, G)$ is in fact fully wellfounded, and so $\bar{\alpha}$ is in its wellfounded part.

It follows immediately from (a) and lemma 6.5 that there is a $\mathrm{Col}(\omega, < \bar{\lambda})$-generic H over N such that $\mathbb{R}^*_H = \mathbb{R} \cap \mathrm{Ult}(N, G)$. To see that $\mathrm{Hom}^*_H \subseteq i_G(\mathrm{Hom}^N_{<\bar{\lambda}})$, fix $\eta < \lambda$ and $A \in \mathrm{Hom}^{N[H \restriction \eta]}_{<\lambda}$; we must see that $A^* \in i_G(\mathrm{Hom}^N_{<\bar{\lambda}})$. Let $\gamma > \eta$ be a successor Woodin cardinal such that $H \restriction \eta \in N[G \cap \mathbb{Q}_{<\gamma}]$. Clearly, $A^* = (A^{N[G \cap \mathbb{Q}_{<\gamma}]})^*$, so to save notation, let us re-name $A = A^{N[G\cap \mathbb{Q}_{<\gamma}]}$. It follows from 4.6 that $A \in i_\gamma(\mathrm{Hom}^N_{<\lambda})$, where

$$i_\gamma \colon N \to \mathrm{Ult}(N, G \cap \mathbb{Q}^N_{<\gamma})$$

is the canonical embedding. Let

$$\sigma \colon \mathrm{Ult}(N, G \cap \mathbb{Q}^N_{<\gamma}) \to \mathrm{Ult}(N, G)$$

be the natural embedding. (That is, $\sigma([f]_{G\cap\mathbb{Q}_{<\gamma}}) = [f]_G$.) It is enough to see that $\sigma(A) = A^*$. For that, it is enough to see that

$$i_{\gamma,\delta}(A) = A^{N[G\cap\mathbb{Q}_{<\delta}]}$$

whenever $\delta > \gamma$ is a successor Woodin cardinal and $i_{\gamma,\delta}\colon \mathrm{Ult}(N, G\cap\mathbb{Q}_{<\gamma}) \to \mathrm{Ult}(N, \mathbb{Q}_{<\delta})$ is the canonical embedding. But pick any δ^+-complete homogeneity system $\bar{\nu}$ in $N[G\cap\mathbb{Q}_{<\gamma}]$ such that

$$A = S_{\bar{\nu}}^{N[G\cap\mathbb{Q}_{<\gamma}]},$$

so that $\nu_s = \mu_s^*$ for some $\mu_s \in N$, and

$$A = S_{\langle i_\gamma(\mu_s) | s\in\omega^{<\omega}\rangle}$$

in $\mathrm{Ult}(N, G\cap\mathbb{Q}_{<\gamma})$. Then

$$i_{\gamma,\delta}(A) = S_{\langle i_\delta(\mu_s) | s\in\omega^{<\omega}\rangle}$$

in $\mathrm{Ult}(N, G\cap\mathbb{Q}_{<\delta})$, and

$$A^{N[G\cap\mathbb{Q}_{<\delta}]} = S_{\langle \mu_s^{**} | s\in\omega^{<\omega}\rangle}$$

in $N[G\cap\mathbb{Q}_{<\delta}]$ as a consequence of the Martin-Solovay construction. Now 4.6, applied at δ, gives the desired conclusion. □

Proof of 6.4 Let H be $\mathrm{Col}(\omega, <\lambda)$-generic over V, and let $A \in \mathrm{Hom}_{<\lambda}^{V[H\restriction\alpha]}$, and assume there is a $B \in L(\mathbb{R}^*, \mathrm{Hom}^*)$ such that $(\mathrm{HC}^*, \in, A^*, B) \models \varphi$. Let us call such a B a *φ-witness for A^**. What we are looking for is a φ-witness for A in $\mathrm{Hom}_{<\lambda}^{V[H\restriction\alpha]}$. By 5.1, it will suffice to find a φ-witness for $A^{V[H\restriction\beta]}$ in $\mathrm{Hom}_{<\lambda}^{V[H\restriction\beta]}$, for some $\beta < \lambda$. We consider two cases:

Case 1. There is an $C^* \in \mathrm{Hom}^*$ such that some $B \in L(C^*, \mathbb{R}^*)$ is a φ-witness for A^*.

Proof. By increasing α, we may as well assume that $C \in \mathrm{Hom}_{<\lambda}^{V[H\restriction\alpha]}$. We can also easily arrange that $A \leq_w C$.

Let γ_0 be least such that there is some φ-witness B for A^* with $B, A^* \in L_{\gamma_0}(C^*, \mathbb{R}^*)$.[13] Fix $x_0 \in \mathbb{R}^*$ such that some such B is ordinal definable over $L_{\gamma_0}(C^*, \mathbb{R}^*)$ from x_0 and A^*, C^*. We may as well assume $x_0 \in V[H\restriction\alpha]$. The least sequence of ordinals from which one can define a φ-witness B from $\langle x_0, A^*, C^*\rangle$ over $L_{\gamma_0}(C^*, \mathbb{R}^*)$ is definable from $\langle x_0, A^*, C^*\rangle$ over $L_{\gamma_0}(C^*, \mathbb{R}^*)$, and so we may assume that B is definable without ordinal parameters. Say

$$u \in B \Leftrightarrow L_{\gamma_0}(C^*, \mathbb{R}^*) \models \psi[\langle x_0, A^*, C^*\rangle, u].$$

Let

$$\bar{\varphi}(v_0, v_1) = \text{“}v_0 \text{ is a } \varphi\text{-witness for } v_1\text{”},$$

[13]We set $L_0(Z, \mathbb{R}^*) = \mathbb{R}^* \cup \{Z\}$, then iterate first order definability as usual.

and let $\theta(v,u)$ be the natural formula defining B from $\langle x_0, A^*, C^*\rangle$:

$$\begin{aligned}\theta(v,u) \quad = \quad & \text{“}v \text{ is } \langle v_0, v_1, v_2\rangle \text{ where } L(v_2,\mathbb{R}) \models \exists B\bar{\varphi}(B, v_1) \\ & \text{and if } \gamma_0 = \text{ is the least } \gamma \text{ s.t. } L_\gamma(v_2,\mathbb{R}) \models \exists B\bar{\varphi}(B,v_1), \\ & \text{then } L_{\gamma_0}(v_2,\mathbb{R}) \models \psi[v,u]\text{”}\end{aligned}$$

The key is that θ gives us an absolute definition of B. More precisely, letting $N = V[H \restriction \alpha]$ and $g \in \mathrm{HC}^*$,

Claim. For all $u \in \mathbb{R} \cap N[g]$

$$N[g] \models \theta[\langle x_0, A^{N[g]}, C^{N[g]}\rangle, u] \Leftrightarrow u \in B.$$

Proof. Of course, $\mathbb{R}^*_H = \mathbb{R}^*_K$ for some $\mathrm{Col}(\omega, <\lambda)$- generic K over $N[g]$. (This follows from 6.5.) Let $G \subseteq \mathbb{Q}_{<\lambda}$ be as given by 6.6, with $N[g]$ playing the role of V, K the role of H, and γ_0 the role of α. Let

$$i\colon N[g] \to \mathrm{Ult}(N[g], G)$$

be the canonical embedding; then

$$i(A^{N[g]}) = A^* \text{ and } i(C^{N[g]}) = C^*$$

by the proof of 6.6(c). Since γ_0 is in the wellfounded part of $\mathrm{Ult}(N[g],G)$ and $\mathbb{R}^* = \mathbb{R} \cap \mathrm{Ult}(N[g],G)$, we get

$$\mathrm{Ult}(N[g],G) \models (L(i(C^{N[g]}),\mathbb{R}) \models \exists B\bar{\varphi}[B, i(A^{N[g]}]$$

and

$$\mathrm{Ult}(N[g],G) \models \theta[\langle x_0, i(A^{N[g]}), i(C^{N[g]}\rangle, u] \Leftrightarrow u \in B.$$

Since i is elementary and the identity on reals, we have proved our claim. □

Taking $g = \emptyset$ in the above, it follows from the meaning of θ that $B \cap \mathbb{R}^N$ is a φ-witness for A in the sense of N. We will be done with case 1 if we show that $B \cap \mathbb{R}^N \in \mathrm{Hom}^N_{<\lambda}$. This follows easily from the tree production lemma. Let (T,U) and (R,S) be $<\lambda$-absolutely complementing pairs in N such that

$$p[T] = A \text{ and } p[R] = C.$$

Let

$$\tau(\langle x_0, T, R\rangle, u) = \theta(\langle x_0, p[T], p[R]\rangle, u).$$

(The author trusts the reader will untangle the confusion of language and metalanguage here.) We apply the tree production lemma to τ, with $\langle x_0, T, R\rangle$ playing the role of the parameter a. The generic absoluteness hypothesis of the lemma is an immediate consequence of our claim and the universality of the symmetric collapse. For stationary tower correctness, let $\delta < \lambda$ be Woodin in N, and

$$i\colon N \to M = \mathrm{Ult}(N, G)$$

the canonical embedding associated to a $\mathbb{Q}_{<\delta}$-generic G over N. Then

$$N[G] \models p[T] = p[i(T)] \text{ and } p[R] = p[i(R)],$$

as the reader who is still with us can easily show. Thus for $u \in \mathbb{R} \cap N[G]$,

$$\begin{aligned} M \models \tau[\langle x_0, i(T), i(R)\rangle, u] &\Leftrightarrow M \models \theta[\langle x_0, p[i(T)]^M, p[i(R)]^M\rangle, u] \\ &\Leftrightarrow N[G] \models \theta[\langle x_0, p[T]^{N[G]}, p[R]^{N[G]}\rangle, u] \\ &\Leftrightarrow N[G] \models \tau[\langle x_0, T, R\rangle, u] \end{aligned}$$

This completes the proof of 6.4 case 1. □

Case 2. Otherwise.

Proof. From case 1 and our proof of the derived model theorem modulo 6.4, we get

$$\forall C \in \mathrm{Hom}^*(L(C, \mathbb{R}^*) \models \mathsf{AD}^+).$$

For $C \in \mathrm{Hom}^*$, let $C^\sharp$ be the type of a club class of indiscernibles for $L(C, \mathbb{R}^*)$, in the language of set theory expanded by names for each $x \in \mathbb{R}^*$. We regard $C^\sharp$, if it exists, as a subset of $\mathbb{R}^*$ under some natural coding.

Claim 1. $\forall C \in \mathrm{Hom}^*(C^\sharp$ exists and $C^\sharp \in \mathrm{Hom}^*)$.

Proof. Fix C, and let $D \in \mathrm{Hom}^*$ be such that $C \notin L(D, \mathbb{R}^*)$. Now Hom^* is semi-linearly ordered by $\leq_w$ by 6.3[14], and clearly D is not Wadge-reducible to either C or $\mathbb{R}^* \setminus C$. Hence $C \leq_w D$. But now

$$\mathsf{AD}^+ \models \forall C \subseteq \mathbb{R}(\exists D \subseteq \mathbb{R}(D \notin L(C, \mathbb{R}) \Rightarrow C^\sharp \text{ exists }).$$

(See [13]; the result is due independently to the authors of that paper and to Kechris and Woodin.[15]) Since $L(D, \mathbb{R}^*) \models \mathsf{AD}^+$, we get $L(D, \mathbb{R}^*) \models C^\sharp$ exists. But $L(D, \mathbb{R}^*)$ is correct about sharps because it has all the ordinals, so $C^\sharp$ exists. Finally, $C^\sharp \equiv_w \oplus_{n<\omega} B_n$, where B_n

[14]The continuous reductions in question here are coded by reals in $\mathbb{R}^*$.

[15]Here is a short sketch. Work in $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$. By Wadge, every set of reals in $L(C, \mathbb{R})$ is $\leq_w D$. Thus $\theta^{L(C,\mathbb{R})} < \theta$, so we can find a measurable cardinal κ such that $\theta^{L(C,\mathbb{R})} < \kappa$. Let U be a κ-complete normal ultrafilter on κ. One can use U to get the dersired indiscernibles in the usual way. The key here is that U is $\mathbb{R}$-complete over $L(C, \mathbb{R})$: is $A_x \in U$ for all $x \in \mathbb{R}$, and the function $x \mapsto A_x$ is in $L(C, \mathbb{R})$, then $\bigcap_x A_x \in U$.

is the type of the first n indiscernibles. $B_n \in L(C, \mathbb{R}^*)$, so $B_n \leq_w D$, for all n. This implies that $\oplus_{n<\omega} B_n \leq_w D$. Since Hom^* under Wadge reducibility, $C^\sharp \in \mathrm{Hom}^*$. $\square$

Claim 2. For any $g \in \mathrm{HC}^*$, $\mathrm{Hom}_{<\lambda}^{V[g]}$ is closed under sharps.

Proof. Let $C \in \mathrm{Hom}_{<\lambda}^{V[g]}$. Let $B \in \mathrm{Hom}_{<\lambda}^{V[g][h]}$, where $h \in \mathrm{HC}^*$, be such that $B^* = (C^*)^\sharp$. Such a B exists by our first claim. The relation $X = Y^\sharp$ on sets of reals is uniformly Π_1^1 in X, Y, so it follows at once from 6.3 that $B = (C^{V[g][h]})^\sharp$ holds in $V[g][h]$. But then by $(\Sigma_1^2)^{\mathrm{Hom}_{<\lambda}}$-absoluteness (cf. 5.1), $V[g]$ has a sharp for C in its $\mathrm{Hom}_{<\lambda}$. $\square$

We let $L_0(\mathbb{R}^*, \mathrm{Hom}^*) = \mathbb{R}^* \cup \mathrm{Hom}^*$, and obtain $L_\gamma(\mathbb{R}^*, \mathrm{Hom}^*)$ for $\gamma > 0$ by iterating first order definability, as usual. Let

$$\gamma_0 = \text{ least } \gamma \text{ s.t. } \exists B(L_\gamma(\mathbb{R}^*, \mathrm{Hom}^*) \models \bar{\varphi}[B, A^*]) \wedge \forall C \in \mathrm{Hom}^*(|C|_w < \gamma)).$$

Again, let $N = V[H \restriction \alpha]$.

Claim 3. Let $g \in \mathrm{HC}^*$, and let $G \subseteq \mathbb{Q}_{<\lambda}^{N[g]}$ be such that $G \cap \mathbb{Q}_{<\delta}^{N[g]}$ is $N[g]$-generic, for arbitrarily large Woodin cardinals $\delta < \lambda$ of $N[g]$. Suppose $\mathbb{R}^* = \mathbb{R} \cap \mathrm{Ult}(N[g], G)$ and γ_0 is in the wellfounded part of $\mathrm{Ult}(N, G)$; then letting

$$i_G \colon N \to \mathrm{Ult}(N[g], G)$$

be the canonical embedding, we have

$$i_G(\mathrm{Hom}_{<\lambda}^{N[g]}) = \mathrm{Hom}^* .$$

Proof. If not, let $C \in i_G(\mathrm{Hom}_{<\lambda}^{N[g]}) \setminus \mathrm{Hom}^*$ be Wadge minimal, so that

$$\mathrm{Ult}(N[g], G) \models \mathrm{Hom}^* = \{A \mid A <_w C,$$

and

$$L_{\gamma_0}(\mathbb{R}^*, \mathrm{Hom}^*) \subseteq L(C, \mathbb{R})^{\mathrm{Ult}(N[g],G)}.$$

Note $(C^\sharp)^{\mathrm{Ult}(N[g],G)}$ exists and is in $i_G(\mathrm{Hom}_{<\lambda}^{N[g]}$ by claim 2; moreover every set Wadge reducible to it in the sense of $\mathrm{Ult}(N[g], G)$ is in $i_G(\mathrm{Hom}_{<\lambda}^{N[g]})$. It follows that

$$\mathrm{Ult}(N[g], G) \models \exists B \in i_G(\mathrm{Hom}_{<\lambda}^{N[g]}(B \text{ is a } \varphi\text{-witness for } A^*).$$

Noting that $i_G(A^{N[g]}) = A^*$, we see that there is a $B \in \mathrm{Hom}_{<\lambda}^{N[g]}$ such that B is a φ-witness for $A^{N[g]}$. By 6.3, B^* is a φ-witness for A^*, and of course, $B^* \in \mathrm{Hom}^*$. This contradicts our case hypothesis. $\square$

We can now complete the proof of 6.4 just as we did in case 1. We take a minimal-in-$L(\mathbb{R}^*, \mathrm{Hom}^*)$ φ-witness B for A^*, and show that that B has a sufficiently absolute definition that it yields a $\mathrm{Hom}^N_{<\lambda}$ witness for A. Where we used in case 1 that $i_G(C) = C^*$ for generic embeddings induced by $\subseteq \mathbb{Q}_{<\lambda}$, we use here that $i_G(\mathrm{Hom}^{N[g]}_{<\lambda}) = \mathrm{Hom}^*$ for such embeddings. As in case 1, this gives us the generic absoluteness of the definition of B needed in the tree production lemma. For stationary tower correctness, we use

Claim 4. Let $\delta < \lambda$ be a successor Woodin of N, and K be $\mathbb{Q}_{<\delta}$-generic over N; then

$$i_K(\mathrm{Hom}^N_{<\lambda}) = \mathrm{Hom}^{N[K]}_{<\lambda},$$

where i_K is the generic embedding.

Proof. There is a φ-witness for $A^{N[K]}$ in $L(\mathbb{R}, \mathrm{Hom}_{<\lambda})^{N[K]}$ by 6.6. If claim 4 fails, that just as in the proof of claim 3, we get a φ-witness for A in $\mathrm{Hom}^N_{<\lambda}$, which gives us a φ-witness for A^* in Hom^*, a contradiction. □

This completes the proof of 6.4. □

There is a corollary worth pointing out:

Corollary 6.8 *Let $L(\mathbb{R}^*, \mathrm{Hom}^*)$ be a derived model, and suppose there is a φ-witness for A^* in $L(\mathbb{R}^*, \mathrm{Hom}^*)$; then there is a φ-witness for A^* in Hom^*.*

That is, in the derived model, every Σ^2_1 fact has a Suslin-co-Suslin witness. The result is an easy corollary of 6.4 and 6.3.

7 Scale(Σ^2_1) in derived models

Woodin has shown

Theorem 7.1 (Woodin) *Assume AD^+; then*

(1) The pointclass Σ^2_1 has the Scale Property, and

(2) Every lightface Σ^2_1 collection of sets of reals has a lightface Δ^2_1 member.

In this section we shall prove part of this theorem, namely, we shall show that (1) and (2) of 7.1 hold in any derived model. Woodin's original proof of 7.1 used this fact, together with his result that all models of AD^+ are derived models in a certain sense. (See 6.2.)[16]

In fact what we show is

[16]Woodin later found a proof of 7.1 which avoids the derived model theorem.

Lemma 7.2 (Woodin) *Assume* AD^+*, and suppose also that whenever* $A \subseteq \mathbb{R}$ *is Suslin and co-Suslin, and there is a* φ*-witness for* A*, then there is a* φ*-witness* B *for* A *such that* B *is Suslin and co-Suslin. Then for any Suslin-co-Suslin set* A*:*

(1) The pointclass $\Sigma^2_1(A)$ *has the Scale Property, and*

(2) Every $\Sigma^2_1(A)$ *collection of sets of reals has a* $\Delta^2_1(A)$ *member.*

Of course, it follows at once that if $L(\mathbb{R}^*, \mathrm{Hom}^*)$ is a derived model, and $A \in \mathrm{Hom}^*$, then in $L(\mathbb{R}^*, \mathrm{Hom}^*)$, $\Sigma^2_1(A)$ has the Scale Property, and every $\Sigma^2_1(A)$ collection of sets of reals has a $\Delta^2_1(A)$ member. These facts will be useful in the further theory of derived models which we shall develop in later sections.

The proof of 7.2 which we shall give involves techniques unlike those we have been using, and the reader without some experience with AD will no doubt find it impenetrable.

Our first lemma shows how coarse the definabilty requirement on a Σ^2_1 scale is.

Lemma 7.3 (Woodin) *Assume* $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$*. The following are equivalent:*

(1) Σ^2_1 *has the Scale Property,*

(2) if U *is a* Σ^2_1 *set of reals, then for any* $x \in U$ *there is a tree* T *on some* $\omega \times \kappa$ *such that*

- *(a)* $x \in p[T]$ *and* $p[T] \subseteq U$*, and*
- *(b) for some* $A \subseteq \mathbb{R}$*,* T *is ordinal definable in* $L(A, \mathbb{R})$*.*[17]

Proof. To see (1) $\Rightarrow$ (2), simply take T to be the tree of a Σ^2_1 scale on U. Clearly, T works simultaneously for all $x \in U$. Now assume (2). We define a scale $\{\psi_i\}$ on U as follows: for $x \in U$, let

$$\psi_0(x) = \langle \alpha, \beta, \gamma, \varphi \rangle,$$

where

$$\begin{aligned} \alpha &= |A|_w, \text{ for } A \text{ Wadge-minimal such that} \\ &\quad\ \exists T \in \mathrm{OD}^{L(A,\mathbb{R})}(x \in p[T] \wedge p[T] \subseteq U) \end{aligned}$$

and $\langle \beta, \gamma, \varphi \rangle$ is the lexicographically minimal tuple such that for some (equivalently all) A such that $|A|_w = \alpha$, φ defines over $L_\beta(A, \mathbb{R})$ from parameter γ a tree T on some $\omega \times \kappa$ such that $x \in p[T] \subseteq U$. We identify the range of ψ_0 with an ordinal by ordering tuples lexicographically. It is then easy to check that ψ_0 is a Σ^2_1-norm. Let us write T^x for the tree T which arises in the definition of $\psi_0(x)$. For $x \in U$, set

$$l_x = \text{ leftmost branch of } (T^x)_x,$$

[17] No parameters other than ordinals are allowed. In particular, then definition cannot mention A.

and for $i > 0$ let

$$\psi_i(x) = \langle \psi_0(x), l_x \upharpoonright i \rangle.$$

Again, we use the lexicographic order to identify $\mathrm{ran}(\psi_i)$ with an ordinal. It is easy to see that the ψ_i constitute a Σ^2_1 scale on U. □

Remark 7.4 Let us write

$$P_\alpha(\mathbb{R}) = \{A \subseteq \mathbb{R} \mid |A|_w < \alpha\}.$$

Woodin has shown that AD^+ implies that for any Σ_1 formula of the language of set theory $\varphi(v_0, v_1)$ and any $A \subseteq \mathbb{R}$,

$$L(P\mathbb{R}) \models \varphi[A, P(\mathbb{R})] \Rightarrow \exists \alpha, \beta < \Theta(L_\alpha(P_\beta(\mathbb{R})) \models \varphi[A, P_\beta(\mathbb{R})].$$

It follows at once that for any $A \subseteq \mathbb{R}$,

$$A \in \mathrm{OD} \Leftrightarrow \exists B \subseteq \mathbb{R}(A \in \mathrm{OD}^{L(B,\mathbb{R})}).$$

We have stated 7.3 in the somewhat more complicated way we have in order to avoid using this equivalence, whose proof we do not know at the moment. In what follows, we shall often write "OD in some $L(B, \mathbb{R})$" when it might seem more natural to simply write "OD". We usually do so because the former notion is clearly Σ^2_1, and we want to avoid quoting Woodin's result that the two notions are equivalent.

We shall need to use homogeneity representations in the choiceless world of AD. The following basic theorem of Martin characterizes the sets of reals which are homogeneously Suslin via trees on some $\omega \times \kappa$. Although the proof involves some very pretty constructions of measures from games, we shall omit it, since such techniques are rather far from the other techniques we are using. See [8] for a proof.

Let Θ be the least ordinal which is not the surjective image of $\mathbb{R}$.

Theorem 7.5 (Martin) *Assume $\mathsf{AD} + \mathsf{DC}_\mathbb{R}$; then for any $A \subseteq \mathbb{R}$, the following are equivalent:*

(1) $A = p[T]$, for some homogeneous tree T on some $\omega \times \kappa$, where $\kappa < \Theta$;

(2) A is Suslin and co-Suslin.

Proof. Assuming (1), it is clear that A is Suslin. But the Martin-Solovay construction requires only $\mathsf{DC}_\mathbb{R}$. (One uses the Coding Lemma to code functions from κ^n to κ^+ by reals, and then $\mathsf{DC}_\mathbb{R}$ to show the appropriate ultrapowers are wellfounded.) Thus A is co-Suslin.

The author will fill in the rest later. □

Part of the reason homogeneity systems yield the ordinal definable trees required in 7.3 is Kunen's theorem that all measures are ordinal definable.

Theorem 7.6 (Kunen) *Assume* $\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$*, and let* μ *be a measure on some ordinal* $\kappa < \Theta$*; then* μ *is ordinal definable.*

Proof. By the Coding Lemma, there is a surjective map $x \mapsto C_x$ from $\mathbb{R}$ onto $P(\kappa)$. Let D be the set of Turing degrees, and for $d \in D$, let

$$f(d) = \text{ least } \alpha \text{ such that } \alpha \in \bigcap\{C_x \mid \exists y \in d(x \leq_T y) \wedge \mu(C_x) = 1\}.$$

Since μ is countably complete, $f(d)$ exists for all d. Clearly, if $C \subseteq \kappa$, and $f(d) = g(d)$ on a Turing cone, then

$$\mu(C) = 1 \Leftrightarrow \text{ for a cone of } d, g(d) \in C.$$

This gives us a definition of μ from $[f]_\nu$, where ν is Martin's cone measure on D. But since f maps into κ, $[f]_\nu$ "is" an ordinal. □

The proof of 7.2 which we shall give differs a bit from Woodin's original one. It makes use of certain observations concerning the continuous propagation of homogeneity representations which, so far as the author knows, are due to him. The first lemma in this direction elaborates on a basic construction due to Martin.

Lemma 7.7 (Steel) *Assume* AD $+\mathsf{DC}_{\mathbb{R}}$*; then for any* $\kappa < \Theta$ *there is an ordinal definable function* $F\colon meas(\kappa) \to \bigcup_{\beta<\Theta} meas(\beta)$ *such that*

(a) for all $\mu \in meas(\kappa)$, $\dim(\mu) = \dim(F(\mu))$,

(b) for all μ, ν *in* $meas(\kappa)$*,* μ *projects to* ν *iff* $F(\mu)$ *projects to* $F(\nu)$*, and*

(c) for all towers of measures $\langle \mu_n \mid n < \omega \rangle \in meas(\kappa)^\omega$,

$$\langle \mu_n \mid n < \omega \rangle \textit{ is wellfounded } \Leftrightarrow \langle F(\mu_n) \mid n < \omega \rangle \textit{ is illfounded.}$$

Proof. Let λ have the strong partition property $\lambda \to (\lambda)^\lambda$, and $\kappa < \lambda < \Theta$. We get the measures $F(\mu)$ we need from the strong partition property in a standard way.[18] For any $X \subseteq \lambda$ and set W equipped with a wellorder $<_W$ of order type $\leq \lambda$, let

$$[X]^W = \{f\colon W \to X \mid \forall a, b(a <_W b \Rightarrow f(a) < f(b))\}.$$

For any unbounded $X \subseteq \lambda$, let $\pi_X\colon \lambda \to X$ be the increasing enumeration of X, and set

$$X^* = \{\sup(\{\pi_X(\omega\xi + n) \mid n < \omega\}) \mid \xi < \lambda\}.$$

[18]The construction to follow is due to Martin. The new observation here is just that Martin's construction does not require a *tree* for which $\bar{\mu}$ is a homogeneity system.

The strong partition property gives us a measure σ_W on $[\lambda]^W$: for $\mathcal{A} \subseteq [\lambda]^W$, we put

$$\sigma(\mathcal{A}) = 1 \Leftrightarrow \exists C(C \text{ is club in } \lambda \text{ and } [C^*]^W \subseteq \mathcal{A}).$$

It is not hard to see that σ_W is a countably complete measure on $[\lambda]^W$.[19]

For any $n < \omega$, let

$$W_n = (\bigcup_{i \leq n} \kappa^i, \leq_{\text{bk}}),$$

where $\leq_{\text{bk}}$ is the Brouwer-Kleene order: $s \leq_{\text{bk}} t$ iff $t \subseteq s$ or $\exists k \in \text{dom}(s) \cap \text{dom}(t)(s \restriction k = t \restriction k \wedge s(k) < t(k))$. (So W_0 consists of one point, and W_n is a suborder of W_{n+1}, for all n.) Given a measure $\mu \in \text{meas}(\kappa)$ of dimension $n > 0$, with projections μ_i to measures of dimension i for each $i \leq n$, we define a measure $F(\mu)$ on $i_\mu(\lambda)^n$ by

$$A \in F(\mu) \Leftrightarrow \text{ for } \sigma_{W_n} \text{ a.e. } f, \langle [f \restriction \kappa^1]_{\mu_1}, [f \restriction \kappa^2]_{\mu_2}, ..., [f \restriction \kappa^n]_{\mu_n} \rangle \in A.$$

If μ concentrates on $\kappa^0 = \{\emptyset\}$, so is principal, we let $F(\mu) = \mu$. Clearly $F(\mu) \in \text{meas}(i_\mu(\lambda))$, $i_\mu(\lambda) < \Theta$, and $\dim(\mu) = \dim(F(\mu))$.

For (b) let μ project to ν, and suppose ν has dimension i, where $i > 0$. (If $i = 0$, (b) is trivial.) Let $A \in F(\nu)$; then we can find a club C in λ such that for any $f \in [C^*]^{W_i}$, $\langle [f \restriction \kappa^0]_{\mu_0}, ..., [f \restriction \kappa^i]_{\mu_i} \rangle \in A$. But then for any $f \in [C^*]^{W_n}$, $f \restriction \kappa^i \in [C^*]^{W_i}$, so C witnesses that for $F(\mu)$ a.e. $\langle \alpha_1, ..., \alpha_n \rangle$, $\langle \alpha_1, ..., \alpha_i \rangle \in A$. Since A was arbitrary, we have that $F(\mu)$ projects to $F(\nu)$, as desired.

For (c), let $\langle \mu_n \mid n < \omega \rangle \in \text{meas}(\kappa)^\omega$ be a tower of measures. Notice that if $n > 0$, then $F(\mu_n)$ concentrates on tuples $\langle \alpha_1, ..., \alpha_n \rangle$ such that whenever $1 \leq i < n$, then $i_{\mu_i, \mu_{i+1}}(\alpha_i) > \alpha_{i+1}$. (This comes down to the fact that whenever $f \in [\lambda]^{W_{i+1}}$, then for all $s \in \kappa^{i+1}$, $f(s) < f(s \restriction i)$, because $s \leq_{\text{bk}} s \restriction i$.) Thus

$$\langle F(\mu_n) \mid n < \omega \rangle \text{ is wellfounded } \Rightarrow \langle \mu_n \mid n < \omega \rangle \text{ is illfounded },$$

since by meeting countably many measure one sets in the $F(\bar{\mu})$ tower, we produce an infinite descending chain in $\text{Ult}(V, \bar{\mu})$.

For the converse, suppose $\bar{\mu}$ is illfounded. We can then find a tree T on κ such that $\mu_n(T \cap \kappa^n) = 1$ for all n, but T is wellfounded. In order to see that $\langle F(\mu_n) \mid n < \omega \rangle$ is countably complete, fix sets $A_n \in F(\mu_n)$, for each $n \geq 1$. We seek a "fiber" for the A_n's. Let C_n be club in λ and such that

$$f \in [C_n^*]^{W_n} \Rightarrow \langle [f \restriction \kappa^1]_{\mu_1}, ..., [f \restriction \kappa^n]_{\mu_n} \rangle \in A_n.$$

[19]For example, given $\mathcal{A} \subseteq [\lambda]^W$, partition $[\lambda]^\lambda$ by letting $F(X) = 0$ iff $g \in \mathcal{A}$ where $g \in [\lim(X)^*]^W$ is such that $\text{ran}(g)$ is an initial segment of $\lim(X)^*$. Let $H \in [\lambda]^\lambda$ be homogeneous for this partition, and $C = \lim(H)$. Then either $[C^*]^W \subseteq \mathcal{A}$ or $[C^*]^W \cap \mathcal{A} = \emptyset$. So either $\mathcal{A}$ or its complement gets measure one.

Let

$$C = \bigcap_{n<\omega} C_n,$$

and let $f\colon T \to C^*$ preserve the Brouwer-Kleene order (which is a wellorder of order type ¡ κ^+ when restricted to T), and be such that if $u, v \in T$ and $u <_{\text{bk}} v$, then $C^* \cap (f(u), f(v))$ has order type at least κ^+. This spacing in C^* of the points in $\operatorname{ran}(f)$ guarantees that for any n, we can find a $g \in [C^*]^{W_n}$ such that $g \restriction (T \cap W_n) = f$. It follows that $[g \restriction \kappa^i]_{\mu_i} = [f \restriction \kappa^i]_{\mu_i}$ for $i = 1, .., n$, and therefore

$$\langle [f \restriction \kappa^1]_{\mu_1}, ..., [f \restriction \kappa^n]_{\mu_n} \rangle \in A_n,$$

for all n. This is the desired fiber for the A's. That finishes the proof of (c). □

The set of homogeneity systems over ω^k with support Z is a closed set in the topological space $\operatorname{meas}(Z)^{\omega^{k^{<\omega}}}$, where this space is given the Baire topology induced by any and all enumerations of $\omega^{k^{<\omega}}$. For any set $Y \subseteq \operatorname{meas}(Z)$, let

$$\begin{aligned} \mathcal{H}_Y^k &= \{\bar{\mu} \mid \bar{\mu} \text{ is a homogeneity system over } \omega^k \text{ and} \\ &\quad \forall s \in \omega^{k^{<\omega}} (\mu_s \in Y) \end{aligned}$$

$\mathcal{H}_Y^k$ is again a closed set in the space $\operatorname{meas}(Z)^{\omega^{k^{<\omega}}}$. The topology of $\mathcal{H}_Y^k$ is generated by *finite partial homogeneity systems* (from Y, of dimension k), that is, functions $h\colon T \to Y$, where T is a finite tree on ω^k, such that whenever $s \in T$, then $\langle h(s \restriction i) \mid i \leq \operatorname{dom}(s) \rangle$ is a (finite) tower of measures. Given such a finite partial homogeneity system h, the set

$$N_h = \{\bar{\mu} \in \mathcal{H}_Y^k \mid h = \bar{\mu} \restriction T\}$$

is clopen in $\mathcal{H}_Y^k$, and the N_h's generate its topology. Let

$$h_Y^k = \{h \mid h \text{ is a finite partial homogeneity system from } Y \text{ of dimension } k\}.$$

For any $\pi\colon h_Y^k \to h_Z^n$ we let π^* be the function on $\mathcal{H}_Y^k$ given by

$$\pi^*(\bar{\mu}) = \bigcup \{\pi(h) \mid h \in h_Y^k \wedge h \subseteq \bar{\mu}\}.$$

Let us call π *good* if

$$\pi^*\colon \mathcal{H}_Y^k \to \mathcal{H}_Z^n$$

is a total, continuous function. (This reduces to some elementary, concrete properties of π.)

We wish to capture formulae with real quantifiers by continuous transformations π^* on homogeneity systems. More precisely, let $\mathcal{L}^*$ be the language with a unary predicate symbol

$\dot{A}$, together with one k-ary relation symbol $\dot{T}$ for each k-ary recursive relation $T \subseteq (\omega^\omega)^k$. For any formula $\varphi(v_0, ..., v_{n-1})$ of $\mathcal{L}^*$, and any $A \subseteq \mathbb{R}$, let

$$\varphi^A = \{\bar{x} \in \mathbb{R}^n \mid (\mathbb{R}, T, A)_{T \text{ recursive}} \models \varphi[\bar{x}],$$

where of course A interprets $\dot{A}$ and T interprets $\dot{T}$.[20] We then have

Lemma 7.8 (Steel) *Assume* $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$. *Let* $\varphi(v_0, ..., v_{n-1})$ *be a formula of* $\mathcal{L}^*$, *and let* $Y \subseteq$ *meas*(κ) *for some* $\kappa < \Theta$ *be such that* $|Y| < \Theta$, *and* $\mathcal{H}^1_Y \neq \emptyset$. *Then there is a* $\beta < \Theta$ *and* $Z \subseteq$ *meas*(β) *such that* $|Z| < \Theta$, *together with a good*

$$\pi \colon h^1_Y \to h^n_Z$$

such that

$$\forall \bar{\mu} \in \mathcal{H}^1_Y (\varphi^{S_{\bar{\mu}}} = S_{\pi^*(\bar{\mu})}.$$

Moreover, if Y *is ordinal definable, then so are* Z,π, *and* π^*.

Proof. The proof is by induction on φ.

If $\varphi = \dot{T}(v_{i_1}, ..., v_{i_k})$, then φ^A is recursive, hence homogeneous, and independent of A, so we let π be the appropriate constant function. If $\varphi = \dot{A}(v_i)$, then $\varphi^A = \{\bar{x} \in \mathbb{R}^n \mid x_i \in A\}$, and the desired π is a minor perturbation of π designed to accomodate the change of arity.[21] This finishes the atomic case.

If $\varphi = \neg\psi$, we can use 7.7. For let $\pi \colon h^1_Y \to h^n_Z$ witness the lemma for ψ, and let $F \colon Z \to \text{meas}(\beta)$ be an OD tower-flipping function as in 7.7. For any $h \in h^1_Y$, we let $\sigma(h)$ have the same domain as $\pi(h)$, and

$$\sigma(h)(s) = F(\pi(h)(s)).$$

It is clear that σ works for φ.

If $\varphi = \psi \wedge \rho$, and π and σ witness the lemma for φ and ρ, then it is not hard to construct a good τ with domain h^1_Y such that

$$\text{Ult}(V, \tau^*(\bar{\mu})) \cong \text{Ult}(\text{Ult}(V, \pi^*(\bar{\mu})), j(\sigma^*(\bar{\mu})),$$

where $j \colon V \to \text{Ult}(V, \pi^*(\bar{\mu}))$ is the canonical embedding.[22] We leave the details to the reader, since the case $\varphi = \psi \wedge \rho$ can anyway be subsumed under the case $\varphi = \forall v \psi$ to follow.[23]

[20]The notation $\varphi(v_1, ..., v_{n-1})$ does not presume that all v_i for $i < n$ actually occur in φ. We should therefore write $(n, \varphi)^A$, but we will allow n to be understood from context.

[21]Our induction is really on pairs n, φ such that all free variables of φ are among $v_0, ..., v_{n-1}$.

[22]Since we have assumed AD, j will not be elementary. However, for any $\nu \in \text{meas}(\gamma)$, $j(\nu) \in \text{meas}(j(\gamma))$, as the reader can easily check, and this is enough to make sense of the iteration.

[23]This argument in the $\wedge$ case works without AD, however, while the $\forall$ argument does not.

Finally, let $\varphi = \varphi(v_1) = \forall v_2 \psi(v_1, v_2)$, where we have taken $n = 1$ for notational simplicty. Let $\pi\colon h^1_Y \to h^2_Z$ witness the lemma for ψ. It will be enough to find a $\beta < \Theta$ and a good $\sigma\colon h^2_Z \to h^1_{\mathrm{meas}(\beta)}$ such that for all $\bar{\nu} \in \mathcal{H}^2_Z$ and $x \in \mathbb{R}$,

$$\sigma^*(\bar{\nu})_x \text{ is wellfounded } \Leftrightarrow \forall y (\bar{\nu})_{(x,y)} \text{ is wellfounded.}$$

For then, it is easy to find a good τ such that $\tau^* = \sigma^* \circ \pi^*$, and τ witnesses the lemma for φ.

We get σ from the standard construction, due to Martin, which obtains homogeneity from weak homogeneity, using partition cardinals. We need a little care, however, because we are not given a homogeneous *tree*.

Fix $F\colon Z \to \mathrm{meas}(\gamma)$ be a tower-flipping function as in 7.7. Given $\bar{\nu} \in \mathcal{H}^2_Z$, we shall define $\bar{\mu} = \sigma^*(\bar{\nu})$. It will be clear from the construction that $\sigma*$ is continuous. To begin with, set

$$\nu^*_{(s,t)} = F(\nu_{(s,t)})$$

for all $(s,t) \in \mathrm{dom}(\bar{\nu})$. Inspecting the construction of 7.7, we see that for any $(x,y) \in \mathbb{R}^2$, $\bar{\nu}^*_{(x,y)}$ concentrates on descending chains in $\mathrm{Ult}(V, \bar{\nu}_{(x,y)})$. Our tower $\bar{\mu}_x$ will concentrate on attempts to prove continuously that all $\bar{\nu}^*_{x,y)}$ are illfounded. The construction generalizes that in 7.7.

Let $\langle u_i \mid i < \omega \rangle$ enumerate $\omega^{<\omega}$ so that $\forall i \forall k \exists j \le i (u_i \restriction k = u_j)$, and let $n_i = \mathrm{dom}(u_i)$. Let

$$W_n = (\bigcup_{i \le n} (\omega^i \times \gamma^i), \le_{\mathrm{bk}}),$$

let λ be a strong partition cardinal such that $\gamma < \lambda < \Theta$, and for any ordered set W let σ_W be the strong partition measure on $[\lambda]^W$ defined in 7.7. For $f \in [\lambda]^{W_n}$ and $u \in \omega^k$, where $k \le n$, let

$$f_u(t) = f(u,t)$$

for all $t \in \gamma^k$. Let

$$\beta = \sup\{ i_{\nu_{(s,t)}}(\lambda) \mid s,t \in \omega^{<\omega} \}.$$

For $s \in \omega^k$, where $k > 0$, we define a measure μ_s concentrating on $[\beta]^k$ by

$$\mu_s(A) = 1 \Leftrightarrow \text{ for } \sigma_{W_k}\text{-a.e. } f \ \langle [f_{u_1}]_{\nu^*_{s\restriction n_1, u_1}}, ..., [f_{u_n}]_{\nu^*_{s\restriction n_k, u_k}} \rangle \in A.$$

Let μ_0 be the principal measure on $\{\emptyset\}$.

$\bar{\mu}_x$ concentrates on attempts to continuously illfound all $\bar{\nu}^*_{(x,y)}$ below the image of λ. Thus if $\bar{\mu}_x$ is wellfounded, any fiber meeting the appropriate measure one sets witnesses that $\forall y (\bar{\nu}^*_{(x,y)}$ is illfounded. Conversely, let x be such that all $\bar{\nu}^*_{(x,y)}$ are illfounded. Let $\mu_{x\restriction n}(A_n) = 1$ for all n. Let C be club in λ, and contain all the clubs witnessing the A_n contain projections of measure one sets with respect to σ_{W_n}. Since all $\bar{\nu}_{(x,y)}$ are wellfounded, the tree T on $\omega \times \beta$ of all attempts to build a (y, g) such that g is an infinite descending chain in $\mathrm{Ult}(V, \bar{\nu}_{(x,y)})$ below the image of λ is wellfounded. Let

$$f \colon T \to C^*$$

preserve $\leq_{\mathrm{bk}}$ on T. Then f determines a fiber for the A_n's, as in the proof of 7.7, and we are done.

It is easy to check that the continuous homogeneity transformations we have *defined* are ordinal definable. □

Proof of 7.2. We shall prove the result for Σ^2_1, and leave it to the reader to provide the easy generalization to $\Sigma^2_1(A)$, where A is Suslin and co-Suslin.

We begin by showing Σ^2_1 has the Scale Property. For this, we shall show that (2) of 7.3 holds. So let U be Σ^2_1; say

$$x \in U \Leftrightarrow \exists B \subseteq \mathbb{R} x \in \neg\varphi^B,$$

where $\varphi = \varphi(v_1)$ is an $\mathcal{L}^*$ formula. Fix $x \in U$. By hypothesis, there is a Suslin, co-Suslin B such that $x \in \neg\varphi^B$. By the theorems of Martin and Kunen, we can fix $A \subseteq \mathbb{R}$ such that for some $\kappa < \Theta^{L(A,\mathbb{R})}$, there is a homogeneity system $\bar{\mu}$ over ω, with support κ, such that $\bar{\mu} \in L(A, \mathbb{R})$ and

$$x \in (\neg\varphi)^{S_{\bar{\mu}}},$$

and

$$\forall s \in \omega^{<\omega}(\mu_s \in \mathrm{OD}^{L(A,\mathbb{R})})$$

. We now work in $L(A, \mathbb{R})$. Fix $\alpha_0 < \Theta$ such that

$$\forall s(\mu_s \in L_{\alpha_0}(A, \mathbb{R})),$$

and let

$$Y = \mathrm{meas}(\kappa) \cap L_{\alpha_0}(A, \mathbb{R}).$$

Y and each of its elements are OD. Let

$$\pi \colon h^1_Y \to h^1_Z$$

be the OD good function for $\neg\varphi$ given by 7.8, where $Z \subseteq \mathrm{meas}(\lambda)$. Thus

$$\exists \bar{\mu} \in \mathcal{H}^1_Y(\pi^*(\bar{\mu})_x \text{ is illfounded }),$$

and for any $z \in \mathbb{R}$ and $\bar{\mu} \in \mathcal{H}^1_Y$,

$$\pi^*(\bar{\mu})_z \text{ is illfounded } \Rightarrow z \in U.$$

Let $\beta = \sup\{i_\nu(\lambda) \mid \nu \in Z\}$, and let T be the tree on $\omega \times (Y \times \beta)$ which attempts to build a triple $(z, \bar{\mu}, g)$ such that $\bar{\mu} \in \mathcal{H}^1_Y$ and g is an infinite descending chain in $\mathrm{Ult}(V, \pi^*(\bar{\mu}))$ below the image of λ. It follows from the statements just displayed that $x \in p[T]$, and $p[T] \subseteq U$. Since Y has an OD wellorder of length $< \Theta$, we may regard T as a tree $\omega \times \gamma$ for some $\gamma < \Theta$. Thus T witnesses (2) of 7.3, as desired.

We now show that Δ^2_1 is a basis for Σ^2_1. It will be enough to show every non-empty projective collection of sets of reals has a Δ^2_1 member.[24] Fix then $\varphi(v_1)$ an $\mathcal{L}^*$ formula, and put

$$\mathcal{S}(B) \Leftrightarrow 0 \in \varphi^B.$$

Then $\mathcal{S}$ is a typical projective collection of sets of reals. Assume $\mathcal{S} \neq \emptyset$. It will be enough to show

$$\exists A, B \subseteq \mathbb{R}(\mathcal{S}(B) \wedge B \in \mathrm{OD}^{L(A,R)}).$$

For the we can let A be Wadge-minimal as above, and let B be the first $\mathrm{OD}^{L(A,\mathbb{R})}$ set in $\mathcal{S}$ in some natural wellorder of $\mathrm{OD}^{L(A,\mathbb{R})}$, and it is easy to see that B is Δ^2_1.

But now let's look at the proof of $\mathrm{Scale}(\Sigma^2_1)$, in the case our real $x = 0$ and our formula is $\neg\varphi$. Let T be the tree on $\omega \times (Y \times \beta)$ produced there, and $A \subseteq \mathbb{R}$ such that T is $\mathrm{OD}^{L(A,\mathbb{R})}$. Then $0 \in p[T]$, so we can set

$$(\bar{\mu}, g) = \text{ leftmost branch of } T_0 \ ,$$

using the $\mathrm{OD}^{L(A,\mathbb{R})}$ wellorder of Y to help make sense of "leftmost". Then $\bar{\mu} \in \mathcal{H}^1_Y$, moreover $\bar{\mu}$, and hence

$$B = S_{\bar{\mu}},$$

are $\mathrm{OD}^{L(A,\mathbb{R})}$. But g witnesses that $\pi^*(\bar{\mu})_0$ is illfounded, which in turn means that $0 \notin (\neg\varphi)^{S_{\bar{\mu}}}$, that is, $0 \in \varphi^B$, as desired. □

8 Derived models of $\mathsf{AD}_\mathbb{R}$.

It is not too hard to see that if V is the minimal fully iterable canonical inner model with ω Woodin cardinals (i.e., $V = M_\omega$), then the derived model at the unique limit of Woodin cardinals has the form $L(\mathbb{R}^*)$.[25] In this case, $\mathrm{Hom}^* = (\Delta^2_1)^{L(\mathbb{R}^*)}$. That is, if we start with the weakest ground model which has a derived model, we get the weakest model of AD. It is natural to ask whether stronger large cardinal properties in V yield stronger forms of determinacy in its derived models. In fact, there seems to be a systematic, detailed correspondence, much of which has yet to be mapped out. In this section, we consider one

[24] If $\mathcal{S}(B) \Leftrightarrow \exists A \mathcal{R}(A, B)$, where $\mathcal{R}$ is projective, then a Δ^2_1 set $A \oplus B$ such that $\mathcal{R}(A, B)$ yields a Δ^2_1 set B such that $\mathcal{S}(B)$.

[25] This observation is probably due to Woodin and the author.

very natural strengthening of AD, namely $\mathsf{AD}_{\mathbb{R}}$. We shall show that if λ is a limit of Woodin cardinals and of cardinals which are $< \lambda$-strong, then the derived model at λ satisfies $\mathsf{AD}_{\mathbb{R}}$. This result is due to Hugh Woodin. As a corollary, one has that the consistency of ZFC together with the existence of such a cardinal λ implies the consistency of $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$. The author has recently proven the converse relative consistency theorem, and thus the existence of such a λ is in fact equiconsistent with $\mathsf{AD}_{\mathbb{R}}$.

$\mathsf{AD}_{\mathbb{R}}$ is a bit of a red herring here, as explained by the following unpublished results (from the early 80's?).

Theorem 8.1 (Martin, Woodin) *Assume* AD. *If every set of reals is Suslin, then* $\mathsf{AD}_{\mathbb{R}}$ *holds.*

Theorem 8.2 (Woodin) *If* $\mathsf{AD}_{\mathbb{R}}$ *holds, then all sets of reals are Suslin.*

So in the presence of AD, $\mathsf{AD}_{\mathbb{R}}$ is equivalent to the assertion that every set of reals is Suslin. A derived model $L(\mathbb{R}^*, \mathrm{Hom}^*)$ will therefore satisfy $\mathsf{AD}_{\mathbb{R}}$ if and only if $P(\mathbb{R}^*) \cap L(\mathbb{R}^*, \mathrm{Hom}^*) \subseteq \mathrm{Hom}^*$. Our main goal in this section is to prove

Theorem 8.3 (Woodin) *Let* λ *be a limit of Woodin cardinals and of cardinals which are* $< \lambda$*-strong, and let* $L(\mathbb{R}^*, \mathrm{Hom}^*)$ *be a derived model at* λ*; then*

- $P(\mathbb{R}^*) \cap L(\mathbb{R}^*, \mathrm{Hom}^*) = \mathrm{Hom}^*$,*so*
- $L(\mathbb{R}^*, \mathrm{Hom}^*) \models \mathsf{AD}_{\mathbb{R}}$

First, an well-known basic lemma:

Lemma 8.4 (Kechris,Solovay) *Assume* $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$*, and let* $A \subseteq \mathbb{R}$*. For* $x, y \in \mathbb{R}$*, put*

$$R(x, y) \Leftrightarrow \forall B \subseteq \mathbb{R}(y \notin OD^{L(B,\mathbb{R})}(A, x));$$

then

(a) R *is a* $\Pi^2_1(A)$ *relation,*

(b) $\forall x \exists y R(x, y)$,

(c) $\neg(\exists f \colon \mathbb{R} \to \mathbb{R} \exists B \subseteq \mathbb{R} \exists x_0 \in \mathbb{R}(f \in OD^{L(B,\mathbb{R})}(A, x_0) \wedge \forall x \in \mathbb{R} R(x, f(x)))$.

Proof. (a) is obvious. (b) holds because $\{y \mid \neg R(x, y)\}$ is wellordered (all its members being $\mathrm{OD}(A, x)$, hence countable. For (c), suppose f, B, x_0 were a counterexample. Then $f(x_0)$ is $\mathrm{OD}^{L(B,\mathbb{R})}(A, x_0)$, so $\neg R(x_0, f(x_0))$, a contradiction. □

So $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ implies that for any $A \subseteq \mathbb{R}$, there is a $\Pi^2_1(A)$ relation with no uniformization, and hence no scale, which is ordinal definable from A and a real in some $L(B, \mathbb{R})$. This leads at once to

Corollary 8.5 *Let $L(\mathbb{R}^*, \mathrm{Hom}^*)$ be a derived model; then the following are equivalent:*

(1) $P(\mathbb{R}^*) \cap L(\mathbb{R}^*, \mathrm{Hom}^*) = \mathrm{Hom}^*$,

(2) $\forall A \in \mathrm{Hom}^*$, *every* $\Sigma^2_1(A)^{L(\mathbb{R}^*,\mathrm{Hom}^*)}$ *set of reals is in* Hom^*.

Proof. (1) $\Rightarrow$ (2) is trivial. Now suppose toward a contradiction that (2) holds and (1) fails, and let $B \subseteq \mathbb{R}^*$ be in $L(\mathbb{R}^*, \mathrm{Hom}^*) \setminus \mathrm{Hom}^*$. Note that every set in Hom^* is Wadge reducible (in the sense of $L(\mathbb{R}^*, \mathrm{Hom}^*)$) to B, so that $L(\mathbb{R}^*, \mathrm{Hom}^*) = L(B, \mathbb{R}^*)$. Let us work now in this universe. Since $B \in L(\mathbb{R}^*, \mathrm{Hom}^*)$ we can fix $A \in \mathrm{Hom}^*$ such that $B \in \mathrm{OD}(A, \mathrm{Hom}^*)$. But $\mathrm{Hom}^* = P_\alpha(\mathbb{R}^*)$ for some α, so $\mathrm{Hom}^* \in \mathrm{OD}$, and $B \in \mathrm{OD}(A)$. Letting R be the $\Pi^2_1(A)$ relation of 8.4, we have by (2) that $R \in \mathrm{Hom}^*$, so that R has a scale in Hom^*, and hence a uniformizing function $f \in \mathrm{Hom}^*$. But the $f \in L(B, \mathbb{R}^*)$, so f is ordinal definable in $L(B, \mathbb{R}^*)$ from B and some $x \in \mathbb{R}^*$, so f is ordinal definable in $L(B, \mathbb{R}^*)$ from A and some $x \in \mathbb{R}^*$, which contradicts property (c) of 8.4. □

Proof of 8.3. Let λ be a limit of Woodins, and of cardinals which are $< \lambda$-strong, and let $L(\mathbb{R}^*, \mathrm{Hom}^*)$ be a derived model at λ. By 8.5 and 8.1, it will suffice to show that whenever $A \in \mathrm{Hom}^*$, then every $\Sigma^2_1(A)^{L(\mathbb{R}^*,\mathrm{Hom}^*)}$ set of reals is in Hom^*. We shall give the proof for $A = \emptyset$; the proof in general is a simple relativization of the one we give. (One must replace V by some intermediate extension having a λ-absolutely complemented tree projecting to A.)

So let $U \subseteq \mathbb{R}^*$ be Σ^2_1 in $L(\mathbb{R}^*, \mathrm{Hom}^*)$, say

$$U(x) \Leftrightarrow \exists B \subseteq \mathbb{R}^*(B \in L(\mathbb{R}^*, \mathrm{Hom}^*)) \wedge (\mathrm{HC}^*, \in, B) \models \varphi[x]).$$

By 7.2, U has a Σ^2_1 scale in $L(\mathbb{R}^*, \mathrm{Hom}^*)$. Let T be the tree of such a scale. Since T is OD in $L(\mathbb{R}^*, \mathrm{Hom}^*)$, $T \in V$. We must find a λ-absolute complement for T in some $V[g]$, for $g \in \mathrm{HC}^*$. But let $\kappa < \lambda$ be η-strong for all $\eta < \lambda$, and let $g \in \mathrm{HC}^*$ be $\mathrm{Col}(\omega, |\mathrm{meas}(\kappa)|)$-generic over V. We shall show that T has such an absolute complement in $V[g]$. The key to this is 3.5.

It will be enough to find, for each $\eta < \lambda$, an η-absolute complement S_η for T in $V[g]$, for then a simple amalgamation $\oplus_\eta S_\eta$ is a λ-absolute complement for T. So fix $\eta < \lambda$. We may as well assume $\eta = V_\eta$, and $\kappa < \eta$. Let γ be such that $\eta < \gamma < \lambda$, and

$$V[g] \models \overset{\mathrm{Col}(\omega,\eta)}{\Vdash} \mathrm{Hom}_\gamma = \mathrm{Hom}_{<\lambda}\,.$$

Let δ be the 5th Woodin cardinal above γ. Now we apply 3.5: going back to V, let

$$j \colon V \to M \wedge \mathrm{crit}(j) = \kappa \wedge V_\delta \subseteq M$$

be such that in $V[g]$ we have a tree S such that

$$V[g] \models S \text{ is a } \delta\text{-absolute complement for } j(T).$$

We claim that S is the desired η-absolute complement for T in $V[g]$.

Since $p[T] \subseteq p[j(T)]$, it is clear that $p[S] \cap p[T] = \emptyset$. Now let $x \in \mathbb{R}^*$ be $< \eta$-generic over $V[g]$. We can find a $\mathrm{Col}(\omega, \eta)$-generic h over $V[g]$ such that $x \in V[g][h]$. We must see $x \in p[S] \cup p[T]$. Suppose $x \notin p[S]$. Since x is $< \delta$-generic over $V[g]$, we have $x \in p[j(T)]$. Now let's look at what this means.

Note $x \in M[g][h]$, where $j(T)$ has its meaning. Moreover, $M[g][h]$ satisfies the statement that

$$p[j(T)] \cap \mathbb{R}^* = \{x \mid \exists B \in L(\mathbb{R}^*, \mathrm{Hom}^*)(\mathrm{HC}^*, \in, B) \models \varphi[x]\},$$

where this statement is phrased as a statement about the collapse up to λ over $M[g][h]$. Thus for our particular x,

$$M[g][h] \models \exists B \in L(\mathbb{R}^*, \mathrm{Hom}^*)(\mathrm{HC}^*, \in, B) \models \varphi[x],$$

where again this is a statement about the collapse up to λ. But now, applying 6.4 inside $M[g][h]$, we get a B such that

$$M[g][h] \models B \in \mathrm{Hom}_{<\lambda} \wedge (\mathrm{HC}, \in, B) \models \varphi[x].$$

Moving back to $V[g][h]$, which agrees up to δ with $M[g][h]$, we see B is δ-absolutely complemented in $V[g][h]$. But there are enough Woodin cardinals between γ and δ that this implies B is Hom_γ in $V[g][h]$, and by our choice of γ, that B is $\mathrm{Hom}_{<\lambda}$ in $V[g][h]$. But then, using 6.3, we can push up the Σ^2_1 fact that B is witnessing in $V[g][h]$ to $L(\mathbb{R}^*, \mathrm{Hom}^*)$, and we conclude that $U(x)$. That is, $x \in p[T]$, as desired. □

References

[1] Q. Feng, M. Magidor, and W.H.Woodin, Universally Baire sets of reals, in *Set theory of the Continuum*, H. Judah, W. Just, and W.H. Woodin eds., MSRI publications vol. 26 (1992), Springer-Verlag.

[2] A.S. Kechris and Y.N. Moscovakis, Notes on the theory of scales, in *Cabal Seminar 76-77* Lecture notes in Mathematics **689**, Springer-Verlag 1978 1–54 (A.S. Kechris and Y.N. Moschovakis eds.).

[3] P. Larson, The stationary tower, University Lecture Series, Amer. Math. Soc., 2004.

[4] D.A. Martin, Measurable cardinals and analytic games, *Fund. Math.* **66** (1970), 287–291.

[5] , D.A. Martin, forthcoming book on proofs of determinacy.

[6] D.A. Martin and R.M. Solovay, A basis theorem for Σ^1_3 sets of reals, em Annals of Mathematics **89** (1969), 138–159.

[7] D.A. Martin and J.R. Steel, A Proof of Projective Determinacy, *Journal of the American Mathematical Society* bf 2 (1989), 71–125.

[8] D.A. Martin and J.R. Steel, The tree of a Moschovakis scale is homogeneous, to appear (and available at http://www.math.berkeley.edu/ steel).

[9] Y.N. Moschovakis, Ordinal games and playful models, *Cabal Seminar 77-79* , Lecture notes in Mathematics **839**, A.S. Kechris,D.A. Martin, and Y.N. Moschovakis eds., Springer-Verlag 1981 169–202.

[10] I. Neeman, Determinacy in $L(\mathbb{R})$, to appear in the Handbook of Set Theory, M. Foreman and A. Kanamori eds.

[11] J. R. Steel, A stationary-tower-free proof of the derived model theorem, Advances in Logic: the North Texas Logic Conference, S. Gao, S. Jackson, and Y. Zhang eds., Contemporary Mathematics 425, Amer. Math. Soc. 2007, 1–8.

[12] J.R. Steel, unpublished lecture notes, UCLA 1994.

[13] J.R. Steel and R. Van Wesep, Two consequences of determinacy consistent with choice, *Trans. of AMS* **272** (1982) 67–85.

[14] W.H. Woodin, unpublished lecture notes, Berkeley 1993-94.

[15] W.H. Woodin, Supercompact cardinals, sets of reals, and weakly homogeneous trees,*Proc. Nat. Acad. Sci. USA* **85**, 6587–6591.