Distinct iterable branches

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December 1, 2003

The basic problem of inner model theory is how to construct mice satisfying hypotheses appreciably stronger than "there is a Woodin limit of Woodin cardinals". We have a family of constructions, the K^c*-constructions*, which ought to produce such mice under the appropriate hypotheses on V. Perhaps the most important thing we lack is a proof that the countable elementary submodels of premice produced by a K^c-construction are $\omega_1 + 1$-iterable. The best partial results in this direction are those of Neeman ([4]) for K^c-constructions making use of full background extenders over V, and those of Andretta, Neeman, and Steel ([1]) for arbitrary K^c-constructions.

Let $\mathcal{M}$ be a countable premouse embedded by π into a level of the K^c-construction $\mathbb{C}$. If $\mathbb{C}$ uses only full extenders over V as its background extenders, then π and $\mathbb{C}$ enable one to lift an evolving iteration tree $\mathcal{T}$ on $\mathcal{M}$ to an iteration tree $\mathcal{T}^*$ on V. (See [3, §12].) The good behavior of $\mathcal{T}^*$ guarantees that of $\mathcal{T}$. The natural conjecture here is that V is $\omega_1 + 1$-iterable *with respect to such trees* $\mathcal{T}^*$ by the strategy of choosing the unique wellfounded branch. The open question here is uniqueness, since by [2] the uniqueness of the wellfounded branch chosen by $\mathcal{T}^*$ at limit stages strictly less than λ implies the existence of a wellfounded branch to be chosen at λ.[1] There is reasonably good evidence for the truth of this conjecture, at least in the case π is lexicographically minimal among all embeddings of $\mathcal{M}$ to a model of $\mathbb{C}$. (See [5] for a discussion.)[2]

In many situations, one must rely on K^c-constructions making use of partial background extenders. Given a countable $\mathcal{M}$ embedded by π into a level of such a construction $\mathbb{C}$, it is no longer possible to use π and $\mathbb{C}$ to lift iteration trees on $\mathcal{M}$ to trees on V, and so the conjecture of the last paragraph doesn't make sense. Instead of branches which lift to wellfounded branches of a tree on V, what we get here from the basic existence theorem of [7, §9], for countable trees $\mathcal{T}$ on $\mathcal{M}$, is a maximal branch b of $\mathcal{T}$ such that b is π*-realizable.*

[1] Here we assume the background extenders of $\mathbb{C}$ are all $2^{\aleph_0}$-closed. This can be arranged easily in applications.

[2] Woodin has constructed an iteration tree on V having distinct cofinal, wellfounded branches, however, his tree is not induced by a tree on a premouse.

[3] There may be more than one π-realizable branch (see [2, §5]), but if π is lexicographically minimal with respect to some enumeration of $\mathcal{M}$, and $\mathcal{M}$ is tame, then the π-realizable branch is unique, and so we get an ω_1+1-iteration strategy for $\mathcal{M}$. (See [1].)

One might be tempted to conjecture that for arbitrary countable $\mathcal{M}$ embedded in a lexicographically minimal way by π in a level of a K^c-construction, every countable iteration tree on $\mathcal{M}$ has at most one cofinal, π-realizable branch. This would imply that such $\mathcal{M}$ are ω_1+1-iterable. This conjecture is not actually very plausible to anyone familiar with its proof in the case that $\mathcal{M}$ is tame, but it is not easy to produce a counterexample. The main point of this paper is to do that, for a mouse $\mathcal{M}$ which is just beyond the tame mice.

In fact, we shall prove a slightly stronger result concerning the complexity of iteration strategies on mice projecting to ω. This result is easier to state, as it does not involve K^c-constructions or realizations. Suppose that $\mathcal{M}$ is an ω-sound, ω_1+1- iterable mouse such that $\rho_\omega(\mathcal{M})=\omega$; then $\mathcal{M}$ has a *unique* ω_1+1-iteration strategy Σ. If $\mathcal{T}$ is an iteration tree of countable limit length by Σ, then

$$\Sigma(\mathcal{T}) = \text{ unique cofinal } b \text{ such that } \Phi(\mathcal{T}^\frown b) \text{ is } \omega_1+1\text{-iterable},$$

where $\Phi(\mathcal{U})$ is the phalanx of models of the iteration tree $\mathcal{U}$. (It is easy to see that any iteration strategy for $\mathcal{M}$ has the weak Dodd-Jensen property with respect to all enumerations of $\mathcal{M}$, and hence [6, 4.2] applies.) It is natural to ask whether we have

$$\Sigma(\mathcal{T}) = \text{ unique cofinal } b \text{ such that } \mathcal{M}_b^\mathcal{T} \text{ is } \omega_1+1\text{-iterable}$$

in general. If $\mathcal{M}$ is tame, then indeed we do, but for arbitrary $\mathcal{M}$ it seems highly likely that the interaction of $\mathcal{M}_b^\mathcal{T}$ with the earlier models of $\mathcal{T}$ must be considered in identifying the good branch of $\mathcal{T}$. We shall show this is in fact the case.

Theorem 0.1 *Suppose there is a Woodin limit of Woodin cardinals; then there is an ω-sound, ω_1+1-iterable mouse $\mathcal{M}$ such that $\rho_1(\mathcal{M})=\omega$, and an iteration tree $\mathcal{T}$ of countable limit length played according to the unique ω_1+1-iteration strategy Σ for $\mathcal{M}$, such that*

(a) $D^\mathcal{T}=\emptyset$, that is, $\mathcal{T}$ does not drop anywhere, and

(b) there is a cofinal branch c of $\mathcal{T}$ such that $c\neq\Sigma(\mathcal{T})$, but $\mathcal{M}_c^\mathcal{T}$ is ω_1+1-iterable.

The large cardinal hypothesis of 0.1 is not optimal, but we have used it so that we can quote the mouse-existence results of [4]. One does need at least a nontame $\mathcal{M}$ for the conclusion of 0.1, and in fact one needs at least two cardinals which are strong past the same Woodin cardinal, as we shall show in 1.1. This is probably enough, at least roughly speaking, but we do not know how to show that.

[3] If b does not drop, then letting $\mathcal{N}$ be the target model for π, this means that there is a $\sigma\colon\mathcal{M}_b^\mathcal{T}\to\mathcal{N}$ such that $\pi=\sigma\circ i_b^\mathcal{T}$. See [7] for the full definition.

It is worth noting that theorem 0.1 concerns mice and iteration trees in the sense of [3] and [6]. We do not know how to construct λ-*mice* and λ-*trees* of Jensen's theory (see [10]) as in the theorem. One needs significantly more than mice with two cardinals strong past the same Woodin cardinal (see 1.1). A Woodin limit of Woodin cardinals should suffice, but for the moment, we do not know how to how to construct a version of the example in 0.1 in the λ-setting under any hypotheses.

1 Strength from distinct iterable branches

It will help motivate the construction if we show that it requires nontame mice. Part (2) of the result to follow explains why one needs still more in order to obtain an example in the λ-indexing.

Theorem 1.1 *Let $\mathcal{M}$ be $k+1$-sound, and suppose $\rho_{k+1}(\mathcal{M}) = \omega$. Let Σ be the unique (k, ω_1+1)-iteration strategy for $\mathcal{M}$, and let $\mathcal{T}$ be a k-maximal tree of countable limirt length according to Σ such that $D^{\mathcal{T}} = \emptyset$ and $\deg^{\mathcal{T}}(\eta) = k$ for all $\eta < \mathrm{lh}(\mathcal{T})$. Suppose there is a cofinal branch c of $\mathcal{T}$ such that $\mathcal{M}_c^{\mathcal{T}}$ is (k, ω_1+1)-iterable, but $c \neq \Sigma(\mathcal{T})$; then*

(1) There is some $\mathcal{P} \trianglelefteq \mathcal{M}$, and $\kappa_0 < \kappa_1 < \delta < \eta_0 < \eta_1 \leq o(\mathcal{P})$, such that $\mathcal{P} \models \delta$ is Woodin, and for $i = 1, 2$, $\mathcal{P}|\eta_i$ is active, and κ_i is the critical point of its last extender;

(2) suppose also that whenever $\alpha T \beta T \gamma$, then $i^{\mathcal{T}}_{\alpha,\beta}(\mathrm{crit}(i^{\mathcal{T}}_{\alpha,\beta})) \leq \mathrm{crit}(i^{\mathcal{T}}_{\beta,\gamma})$; then there is a $\mathcal{P} \trianglelefteq \mathcal{M}$ such that $\mathcal{P}$ satisfies "there is a measurable cardinal μ such that μ is a limit of Woodin cardinals, and of cardinals which are $< \mu$-strong.

Proof. Let $b = \Sigma(\mathcal{T})$; we compare the phalanx $\Phi(\mathcal{T}^\frown b)$ with $\mathcal{M}_c^{\mathcal{T}}$ using the given iteration strategies for the two. Let $\mathcal{Q}$ and $\mathcal{R}$ be the last models on the $\Phi(\mathcal{T}^\frown b)$ and $\mathcal{M}_c^{\mathcal{T}}$-sides respectively. So $\mathcal{Q}$ is the last model of a normal tree $\mathcal{U}$ extending $\mathcal{T}^\frown b$, and $\mathcal{R}$ is the last model of a tree $\mathcal{V}$ on $\mathcal{M}_c^{\mathcal{T}}$.

Claim 1. $\mathcal{Q} = \mathcal{R}$.

Proof. Note that neither $\mathcal{Q}$ nor $\mathcal{R}$ is ω-sound. (If there is a drop on the branch leading to $\mathcal{Q}$, then clearly it is not ω-sound. If not, then as $\mathcal{M}$ projects to ω, so does $\mathcal{Q}$, and hence $\mathcal{Q}$ is not $k+1$-sound. Similarly for $\mathcal{R}$.) Hence neither can be a proper initial segment of the other. $\square$

Claim 1. Neither $\mathcal{M}$-to-$\mathcal{Q}$ nor $\mathcal{M}_c^{\mathcal{T}}$-to-$\mathcal{R}$ drops in model or degree.

Proof. Suppose one of the two branches drops in model. Then the new subset of ω definable over $\mathcal{M}$ is not definable over $\mathcal{Q} = \mathcal{R}$, and hence both branches drop in model. This leads to the usual contradiction: letting E and F be the extenders used at the sites of the last drops in $\mathcal{M}$-to-$\mathcal{Q}$ and $\mathcal{M}_c^{\mathcal{T}}$-to-$\mathcal{R}$ respectively, we have that E is compatible with F, and then the initial segment condition gives a contradiction.

A similar argument shows neither branch drops in degree. The main thing to notice is that the base model $\mathcal{M}_c^{\mathcal{T}}$ of $\mathcal{V}$ is k-sound. $\square$

Claim 2. Let $i \colon \mathcal{M}_c^{\mathcal{T}} \to \mathcal{R}$ be the canonical embedding; then i is not the identity, and $\mathrm{crit}(i) < \delta(\mathcal{T})$.

Proof. If $\mathcal{M}_c^{\mathcal{T}} = \mathcal{R}$, then $\mathcal{Q} = \mathcal{R}$ is $(k+1, \delta(\mathcal{T}))$-sound. This implies $\mathcal{Q} = \mathcal{M}_b^{\mathcal{T}}$. But $\mathcal{M}_b^{\mathcal{T}} = \mathcal{M}_c^{\mathcal{T}}$ implies $b = c$ (see [6]), so we have a contradiction. Thus $\mathcal{M}_c^{\mathcal{T}} \neq \mathcal{R}$, and i is not the identity.

If $\mathrm{crit}(i) \geq \delta(\mathcal{T})$, then as $\mathcal{M}_c^{\mathcal{T}}$ is $(k+1, \delta(\mathcal{T}))$-sound, we get that the first extender used in the branch $\mathcal{M}_c^{\mathcal{T}}$-to-$\mathcal{R}$ is compatible with an extender used in the branch $\mathcal{M}$-to-$\mathcal{Q}$, a contradiction. Thus $\mathrm{crit}(i) < \delta(\mathcal{T})$. $\square$

Claim 3. Let F be the first extender used in the branch $\mathcal{M}_c^{\mathcal{T}}$-to-$\mathcal{R}$; then for some extender E used in c, we have $\mathrm{crit}(E) < \mathrm{crit}(F) < \nu(E)$.

Proof. Let $j = i \circ i_c$. Since $\mathcal{M}$ is pointwise definable, we have that j is the canonical embedding from $\mathcal{M}$ to $\mathcal{Q}$ given by $\mathcal{U}$. Let $\kappa = \mathrm{crit}(F)$, and suppose toward contradiction that the claim fails. It follows that $\mathcal{M}_c^{\mathcal{T}}$ has the $\mathcal{M}$-hull property at κ, in the sense that for any $A \subseteq \kappa$ such that $A \in M_c^{\mathcal{T}}$,

$$A = i_c(f)(a) \cap \kappa, \text{ for some } a \in [\kappa]^{<\omega}.$$

But then $\mathcal{R}$ has the $\mathcal{M}$-hull property at κ, since

$$A = i(A) \cap \kappa = i(i_c(f)(a)) \cap \kappa = j(f)(a) \cap \kappa,$$

for $A = i_c(f)(a) \cap \kappa$, with $a \in [\kappa]^{<\omega}$. Further, $\mathcal{R}$ does not have the $\mathcal{M}$-definability property at κ, since if $\kappa = j(f)(a)$ for $a \in [\kappa]^{<\omega}$, then $\kappa = i(i_c(f))(a) = i(i_c(f)(a))$, so $\kappa \in \mathrm{ran}(i)$. But $\mathcal{Q} = \mathcal{R}$, and j is the canonical embedding of $\mathcal{U}$. It follows that κ is the critical point of an extender G used in $\mathcal{M}$-to-$\mathcal{Q}$. It is now easy to use the $\mathcal{M}$-hull property at κ in $\mathcal{Q}$ to see that G is compatible with F. This is a contradiction. $\square$

It is now easy to prove (1). Let E and F be as in claim 3, and let $\kappa = \mathrm{crit}(F)$. Let $\mathcal{P}^*$ be the active initial segment of the model in $\mathcal{V}$ from which F is taken whose last extender is F. Note that $\delta(\mathcal{T})$ is Woodin in $\mathcal{P}^*$. Now $E \restriction \kappa \in \mathcal{P}^*$, so on the sequence of $\mathrm{Ult}(\mathcal{P}^*, F)$ there is an extender H with $\mathrm{crit}(E) = \mathrm{crit}(H)$ and $\delta(\mathcal{T}) < \mathrm{lh}(H) < \mathrm{lh}(F)$. (This uses the initial segment condition.) By coherence, H is on the $\mathcal{P}^*$-sequence. So with $\kappa_0 = \mathrm{crit}(H), \kappa_1 = \kappa$, and $\delta = \delta(\mathcal{T})$, we have the pattern of (1) realized in $\mathcal{P}^*$. By elementarity, we can realize this pattern in some $\mathcal{P} \trianglelefteq \mathcal{M}$.

For (2), let E, F, etc., be as in the last paragraph. The additional hypothesis of (2) guarantees that $i_E(\kappa_0) < \delta$. This implies κ_1 is $< i_E(\kappa_0)$-strong in $\mathcal{P}^*$, and thus κ_0 is a limit of cardinals which are $< \kappa_0$-strong in $\mathcal{P}^*$. Since δ is Woodin in $\mathrm{Ult}(\mathcal{P}^*, H)$, κ_0 is a limit of Woodins in $\mathcal{P}^*$. We can now pull back this situation in $\mathcal{P}^*$ to some $\mathcal{P} \trianglelefteq \mathcal{M}$, and thereby obtain (2). □

2 The construction

We now prove theorem 0.1. Our construction realizes the situation we showed must occur in the proof of 1.1, part (1), in pretty much the simplest possible way.

The mouse $\mathcal{M}$ supporting an iteration tree with distinct iterable branches will be the sharp of the minimal model of the $\mathsf{AD}_{\mathbb{R}}$-hypothesis. More precisely:

Definition 2.1 *For any set X, $M^\sharp_{adr}(X)$ is the minimal active ω_1+1-iterable X-premouse $\mathcal{M}$ such that for some strictly increasing sequence $\langle \kappa_i \mid i < \omega \rangle$ with $\sup_i \kappa_i = \lambda < o(\mathcal{M})$, we have*

$$\begin{aligned} \mathcal{M} \models\ & \lambda \textit{ is a limit of Woodin cardinals, and} \\ & \forall i \ (\kappa_i \textit{ is } < \lambda\textit{- strong}). \end{aligned}$$

Of course, we mean that Woodinness and strongness must be witnessed by extenders on the sequence of $\mathcal{M}$. It is easy to see that $\rho_1(M^\sharp_{\mathrm{adr}}(X)) = \sup(X))$. It is a result of Neeman that if δ is a Woodin limit of Woodin cardinals, then for all $X \in V_\delta$, $M^\sharp_{\mathrm{adr}}(X)$ exists, and moreover it has a unique δ-iteration strategy in V and in all its generic extensions by posets of size $< \delta$. (See [4].) The existence of $M^\sharp_{\mathrm{adr}}(\emptyset)$ suffices for our construction.[4] The author believes he can derive this from the existence in V of a cardinal λ which is a limit of Woodins and $< \lambda$-strongs, together with a measurable cardinal above λ, but this has not been checked carefully. (See [9].)

The assertion that there is a λ which is a limit of Woodins and of $< \lambda$-strongs is called the $\mathsf{AD}_{\mathbb{R}}$-*hypothesis.* This is because Woodin has shown the assertion to be equiconsistent with $\mathsf{AD}_{\mathbb{R}}$. One direction of this is Woodin's theorem that the *derived model* associated to the symmetric collapse below any λ witnessing the $\mathsf{AD}_{\mathbb{R}}$-hypothesis satisfies $\mathsf{AD}_{\mathbb{R}}$. This theorem, which helped motivate our construction, is exposited in [8, see theorem 8.3].

If $\mathcal{N} \equiv M^\sharp_{\mathrm{adr}}(X)$, then $\lambda^{\mathcal{N}}$ and $\kappa_i^{\mathcal{N}}$ for $0 \le i < \omega$ are the unique ordinals satisfying in $\mathcal{N}$ the definitions satisfied in $M^\sharp_{\mathrm{adr}}(X)$ by the λ and κ_i of 2.1. Let also

$$\delta_i^{\mathcal{N}} = \text{ least Woodin cardinal of } \mathcal{N} \text{ above } \kappa_i^{\mathcal{N}}.$$

Let $\mathcal{M} = M^\sharp_{\mathrm{adr}}(\emptyset)$, and let Σ be the unique ω_1+1-iteration strategy for $\mathcal{M}$. We shall need at one point that Σ can be extended to a (ω, ω_1+1) iteration strategy, which we also

[4]Implicit in its existence is its ω_1+1-iterability in V.

call Σ.[5] Since $\mathcal{M}$ projects to ω, we can regard it as a real. Let $\mathcal{E}_0$ be Woodin's extender algebra at $\delta_0^{\mathcal{M}}$, where the extenders used to determine the identities are restricted to have critical points $> \kappa_0^{\mathcal{M}}$. By Woodin's basic result (cf. [6, §7]), there is a nondropping iteration tree $\mathcal{U}$ on $\mathcal{M}$ which is played according to Σ, with last model $\mathcal{N}$ and canonical embedding

$$i^{\mathcal{U}}: \mathcal{M} \to \mathcal{N}$$

such that

$$\mathcal{M} \text{ is } i^{\mathcal{U}}(\mathcal{E}_0)\text{-generic over } \mathcal{N}.$$

We also have that $\mathcal{U}$ only involves extenders coming from the sequence of $\mathcal{M}|\delta_0^{\mathcal{M}}$ or its images, and having critical point $> \kappa_0^{\mathcal{M}}$. In particular, $\mathrm{crit}(i^{\mathcal{U}}) > \kappa_0^{\mathcal{M}}$. We write

$$\kappa = \kappa_0^{\mathcal{M}} = \kappa_0^{\mathcal{N}}.$$

Let us also fix g to be $\mathcal{N}$-generic over $i^{\mathcal{U}}(\mathcal{E}_0)$, and such that

$$\mathcal{M} \in \mathcal{N}[g].$$

The following lemma has some independent interest.

Lemma 2.1.1 *Let $\mathcal{T}$ be a normal iteration tree by Σ extending $\mathcal{U}$, with a branch giving rise to an embedding $i: \mathcal{N} \to \mathcal{S}$ with $\mathrm{crit}(i) > \delta_0^{\mathcal{N}}$. Let H be $\mathcal{S}[g]$-generic for a poset of size $< \lambda^{\mathcal{S}}$ in $\mathcal{S}$; then the function*

$$X \mapsto M^{\sharp}_{adr}(X),$$

defined on all $X \in V_{\lambda^{\mathcal{S}}}^{\mathcal{S}[g][H]}$, is a member of $\mathcal{S}[g][H]$.

Proof. Note that we can extend i to act on $\mathcal{N}[g]$. Fix X; we show informally how to compute $M^{\sharp}_{\mathrm{adr}}(X)$ inside $\mathcal{S}[g][H]$. First, let $\alpha < \lambda^{\mathcal{S}}$ be an inaccessible cardinal of $\mathcal{S}$ large enough that $g * H$ is generic over $\mathcal{S}|\alpha$, and $X \in (\mathcal{S}|\alpha)[g][H]$. Let E be an extender on the $\mathcal{S}$ sequence such that $\kappa = \mathrm{crit}(E)$ and $\alpha \leq \mathrm{lh}(E)$. In $\mathcal{S}[g]$ we have $\mathcal{M}$ *as a set*, and hence we can form $\mathrm{Ult}(\mathcal{M}, E)$ *as a set*. Since $\mathcal{M}|\kappa^{+\mathcal{M}} = \mathcal{S}|\kappa^{+\mathcal{S}}$, we have that

$$\mathrm{Ult}(\mathcal{M}, E)|i_E^{\mathcal{M}}(\kappa) = \mathrm{Ult}(\mathcal{S}, E)|i_E^{\mathcal{S}}(\kappa),$$

and thus

$$\mathrm{Ult}(\mathcal{M}, E)|\alpha) = \mathcal{S}|\alpha$$

[5] This follows from [4]. One does not have uniqueness for the extended strategy; indeed, our main result 0.1 is just a proof that the extended strategy is not unique.

by coherence. Thus X is generic over $\mathrm{Ult}(\mathcal{M},E)$ for a partial order of size $< \lambda^{\mathrm{Ult}(\mathcal{M},E)}$. We can now use the modified background-extender construction of [9] to re-build $M^\sharp_{\mathrm{adr}}(X)$ from $\mathrm{Ult}(\mathcal{M},E)[X]$. Once again, the key is that we get $M^\sharp_{\mathrm{adr}}(X)$ *as a set* in $\mathcal{S}[g][H]$. $\square$

Remark. One can go further, and show that under the hypotheses of lemma 2.1.1, the function $x \mapsto M^\sharp_{\mathrm{adr}}(x)$, defined for $x \in \mathbb{R}$, is in the derived model of $\mathcal{S}[g]$, and in fact has Wadge rank Θ_0 there. These facts are due to Woodin.

Now let $\mu = \kappa_1^{\mathcal{N}}$, and let E^* and F^* be such that

- F^* is an extender on the $\mathcal{N}$-sequence, $\mathrm{crit}(F^*) = \mu$, and $\delta^{+\mathcal{N}} < \mathrm{lh}(F^*)$, and
- E^* is an extender on the $\mathcal{N}$-sequence, $\mathrm{crit}(E^*) = \kappa$, and $\mathrm{lh}(F^*) < \mathrm{lh}(E^*)$.

Let E be the trivial completion of $E^* \restriction \mu^{+\mathcal{N}}$. There is a canonical embedding from $\mathrm{Ult}(\mathcal{N},E)$ to $\mathrm{Ult}(\mathcal{N},E^*)$ which is the identity on $\mu^{+\mathcal{N}}$, and hence letting

$$\delta = \text{ least Woodin of } \mathrm{Ult}(\mathcal{N},E) \text{ greater than } \mu,$$

we have an extender F such that

- F is an extender on the $\mathrm{Ult}(\mathcal{N},E)$-sequence, $\mathrm{crit}(F) = \mu$, and $\delta^{+\mathrm{Ult}(\mathcal{N},E)} < \mathrm{lh}(F) < i_E^{\mathcal{N}}(\kappa)$, and
- E is an extender on the $\mathcal{N}$-sequence, and $\mu^{+\mathcal{N}} < \mathrm{lh}(E)$.

(We have moved from E^*, F^* to E, F so that the iteration tree we construct will be normal, and in particular, so that the lengths of the extenders it uses will be increasing.)

Let $\mathcal{V}$ be the normal iteration tree extending $\mathcal{U}$ obtained by applying E to the appropriate model, namely $\mathcal{M}$, and let

$$\mathcal{P} = \mathrm{Ult}(\mathcal{M},E).$$

Note $\mathcal{P}$ agrees with $\mathrm{Ult}(\mathcal{N},E)$ to $i_E^{\mathcal{N}}(\kappa) = i_E^{\mathcal{M}}(\kappa)$, so $F \in \mathcal{P}$, δ is Woodin in $\mathcal{P}$, and g is $i^{\mathcal{U}}(\mathcal{E}_0)$-generic over $\mathcal{P}$. Let $\mathcal{E}_1$ be the extender algebra of $\mathcal{P}[g]$ on $\mathrm{lh}(E)$-many generators, defined using only extenders G on the $\mathcal{P}$-sequence such that $\mathrm{lh}(E) < \mathrm{crit}(G)$ and $\nu(G)$ is a cardinal of $\mathcal{P}$.[6] We can identify E with a subset of $\mathrm{lh}(E)$, and hence there is an iteration tree $\mathcal{W}^*$ on $\mathcal{P}$, with last model $\mathcal{Q}$ and canonical embedding $k\colon \mathcal{P} \to \mathcal{Q}$ such that

(1) $\mathcal{W}^*$ is by Σ, and $\mathcal{W}^*$ lifts to a tree on $\mathcal{P}[g]$ (which we also call $\mathcal{W}^*$), so that k extends to an embedding (which we also call k) from $\mathcal{P}[g]$ to $\mathcal{Q}[g]$, and

(2) h is $k(\mathcal{E}_1)$-generic over $\mathcal{Q}[g]$, and $E \in \mathcal{Q}[g][h]$.

[6]This last condition insures that the genericity iteration does not drop.

Claim. $\mathcal{W}^* = \mathcal{W}^\frown b$, where b is a cofinal branch of $\mathcal{W}$, so that $\mathcal{M}_b^{\mathcal{W}} = \mathcal{Q}$ and $i_b^{\mathcal{W}} = k$; moreover

(a) $\delta(\mathcal{W}) = k(\delta)$,

(b) $\mathcal{W} \in \mathcal{Q}[g][h]$, and

(c) $b \notin \mathcal{Q}[g][h]$.

Proof. Let $\lambda \leq \mathrm{lh}(\mathcal{W}^*)$ be a limit ordinal, and suppose $c = \Sigma(\mathcal{W}^* \restriction \lambda)$.[7] Suppose also that for all limit $\eta < \lambda$, $i_{0,\eta}^{\mathcal{W}^*}(\delta) > \delta(\mathcal{W}^* \restriction \eta)$, while $i_c^{\mathcal{W}^*}(\delta) = \delta(\mathcal{W}^* \restriction \lambda)$. Let $\mathcal{W} = \mathcal{W}^* \restriction \lambda$, and $b = c$; we show that they witness the claim.

Since $i_b(\delta) = \delta(\mathcal{W})$, all bad identities have been removed and the genericity iteration is finished. Thus $\mathcal{W}^* = \mathcal{W}^\frown b$, $\mathcal{Q} = \mathcal{M}_b^{\mathcal{W}}$, and $k = i_b^{\mathcal{W}}$. This gives part (a). To see (b), we show how $\mathcal{Q}[g][h]$ can recover $\mathcal{W}$. Notice first that $\mathcal{M}, E \in \mathcal{Q}[g][h]$, and hence $\mathrm{Ult}(\mathcal{M}, E) \in \mathcal{Q}[g][h]$. Thus the structure being iterated, $\mathcal{P}$ (and thereby $\mathcal{P}[g]$), is in $\mathcal{Q}[g][h]$. We can recover the successor steps of the iteration $\mathcal{W}$ of $\mathcal{P}$ by looking at the least disagreement between the current model in $\mathcal{W}$ and $\mathcal{Q}$. To recover the cofinal branch $[0,\eta]_W$ for $\eta < \lambda$ a limit ordinal, note that since

$$\delta(\mathcal{W} \restriction \eta) < i_{0,\eta}^{\mathcal{W}}(\delta) \leq i_b^{\mathcal{W}}(\delta),$$

and δ was the *least* Woodin of $\mathcal{P}$ above μ, we have

$$\mathcal{Q} \models \delta(\mathcal{W} \restriction \eta) \text{ is not Woodin.}$$

But $\mathcal{Q}$ agrees with $\mathcal{M}_\eta^{\mathcal{W}}$ up to their common value for $\delta(\mathcal{W} \restriction \eta)^+$, and hence the $\mathcal{Q}$-structure $\mathcal{Q}([0,\eta]_W, \mathcal{W} \restriction \eta)$ is just the first initial segment of $\mathcal{Q}$ which satisfies $\delta(\mathcal{W} \restriction \eta)$ is not Woodin. Hence the model $\mathcal{Q}[g][h]$ can find this $\mathcal{Q}$-structure, and hence can find $[0,\eta]_W$. This proves part (b).

To see (c), note that $\delta < i_E(\kappa)$, since κ is a limit of Woodins in $\mathcal{M}$, so that $i_E(\kappa)$ is a limit of Woodins in $\mathcal{P}$. Thus the composite embedding

$$j = k \circ i_E$$

is given by an extender E_j of length $k(\delta)$ having all its component measures on κ. If $b \in \mathcal{Q}[g][h]$, then as $E \in \mathcal{Q}[g][h]$, we get $E_j \in \mathcal{Q}[g][h]$. This implies that there is a surjection of $k(\delta)$ onto $j(\kappa)$ in $\mathcal{Q}[g][h]$. However, $j(\kappa)$ is inaccessible in $\mathcal{Q}$, and $g * h$ is generic over $\mathcal{Q}$ for a poset of size $k(\delta)$ in $\mathcal{Q}$, so $j(\kappa)$ is inaccessible in $\mathcal{Q}[g][h]$, a contradiction.

Finally, we claim there is a λ with the properties we described at the beginning of the proof. For if not, the proof of (b) we have just given shows that the full iteration $\mathcal{W}^* \in \mathcal{Q}[g][h]$. Our proof of (c) then yields a contradiction. $\square$

[7] Literally, Σ applies to $\mathcal{V}^\frown \mathcal{W}^* \restriction \lambda$.

Let $\mathcal{W}, b$ be as in the claim. Let

$$\mathcal{T} = \mathcal{V}^\frown\mathcal{W}.$$

In order to see that $\mathcal{T}$ witnesses the truth of 0.1, and thereby finish the proof, it will suffice to show that there is a cofinal branch c of $\mathcal{W}$ such that $c \neq b$ and $\mathcal{M}_c^{\mathcal{W}}$ is $\omega_1 + 1$-iterable.

Notice that $k(F)$ is on the $\mathcal{Q}$ sequence, and has length $> k(\delta)$. Let us use it now to extend $\mathcal{V}^\frown\mathcal{W}^*$ according to the rules of normal iteration trees. Since $\mathrm{crit}(k(F)) = \mu$, our new tree $\mathcal{X}$ extending $\mathcal{V}^\frown\mathcal{W}^*$ comes by adding

$$i_{k(F)}\colon \mathcal{N} \to \mathcal{R} = \mathrm{Ult}(\mathcal{N}, k(F)).$$

Notice that $i_{k(F)}$ lifts to an embedding, which we also call $i_{k(F)}$, from $\mathcal{N}[g]$ to $\mathcal{R}[g]$. We have that $\mathcal{R}$ agrees with $\mathcal{Q}$ to their common value of $k(\delta)^+$, so $\mathcal{R}[g]$ agrees with $\mathcal{Q}[g]$ to their common value of $k(\delta)^+$. It follows that

- h is $k(\mathcal{E}_1)$-generic over $\mathcal{R}[g]$, and $E, \mathcal{M}, \mathcal{P}, \mathcal{W} \in \mathcal{R}[g][h]$, and
- $b \notin \mathcal{R}[g][h]$.

Let θ be a Woodin cardinal of $\mathcal{R}$ such that $\theta > \mathrm{lh}(k(F))$, and let

$$l\colon \mathcal{R}[g][h] \to \mathcal{S}[g][h]$$

come from a genericity iteration according to Σ of $\mathcal{R}$ with all critical points above $\mathrm{lh}(k(F))$, and such that for some G which is $\mathrm{Col}(\omega, l(\theta))$-generic over $\mathcal{S}[g][h]$ we have

$$b \in \mathcal{S}[g][h][G].$$

We shall show that if b is the unique cofinal branch c of $\mathcal{W}$ such that $\mathcal{M}_c^{\mathcal{W}}$ is $\omega_1 + 1$-iterable, then b is ordinal definable in $\mathcal{S}[g][h][G]$ from parameters in $\mathcal{S}[g][h]$, so that $b \in \mathcal{S}[g][h]$, so that $b \in \mathcal{R}[g][h]$, a contradiction. This will finish the proof.

Working in $\mathcal{S}$, let $\mathcal{S}_0$ be the last (from the point of view of $\mathcal{S}$, definable proper class) model of the maximal certified array described in [9], where the certifying extenders must come from the $\mathcal{S}$-sequence and have critical points $> l(\theta)$. It is shown in [9] that $\lambda^{\mathcal{S}}$ is a limit of Woodins and of $< \lambda^{\mathcal{S}}$-strong cardinals in $\mathcal{S}_0$, so that $\mathcal{S}_0$ satisfies "I am the minimal premouse satisfying the $\mathsf{AD}_{\mathbb{R}}$-hypothesis".

Claim 2. Let τ be the least $< \lambda^{\mathcal{S}_0}$-strong cardinal of $\mathcal{S}_0$; then

$$\mathcal{S}[g][h][G] \models \exists\pi\ (\pi\colon \mathcal{M}_b^{\mathcal{W}} \to M^\sharp_{\mathrm{adr}}(\mathcal{S}_0|\tau) \text{ elementarily}.$$

Proof. Let $\mathcal{S}_0^+$ be the active premouse $(\mathcal{S}_0, L)$, where L has the last extender of $\mathcal{S}$ as a background extender. It is shown in [9] how to lift iterations of $\mathcal{S}_0^+$ to iterations of $\mathcal{S}$, and since $\mathcal{S}$ is iterable, we get that $\mathcal{S}_0$ is iterable. Let us compare $\mathcal{M}_b^{\mathcal{W}}$ with $\mathcal{S}_0^+$, using our two iteration strategies.[8] Since the two are φ-minimal for the same sentence φ, we get a common iterate $\mathcal{L}$, and iteration maps $i\colon \mathcal{M}_b^{\mathcal{W}} \to \mathcal{L}$ and $j\colon \mathcal{S}_0^+ \to \mathcal{L}$. The lifting (or resurrection) process of [9] also gives rise to $j^*\colon \mathcal{S}[g][h] \to \mathcal{Y}[g][h]$, which we can lift further to

$$j^*\colon \mathcal{S}[g][h][G] \to \mathcal{Y}[g][h][G]$$

in such a way that there is a final lifting map

$$\psi\colon \mathcal{L} \to j^*(\mathcal{S}_0^+).$$

(Here we can apply j^* to the last extender of $\mathcal{S}_0^+$ by applying it to the fragments of that extender, all of which are in $\mathcal{S}_0$.)

Since $\mathcal{M}_b^{\mathcal{W}}$ is hereditarily countable in $\mathcal{S}[g][h][G]$,

$$j^*(\mathcal{M}_b^{\mathcal{W}}) = \mathcal{M}_b^{\mathcal{W}},$$

and thus it is enough by elementarity to show there is an embedding of $\mathcal{M}_b^{\mathcal{W}}$ into $j^*(M^\sharp_{\text{adr}}(\mathcal{S}_0|\tau))$. (By absoluteness, if there is any such embedding, there is one in $\mathcal{Y}[g][h][G]$.) Tracing through our proof of lemma 2.1.1, one can see that $j^*(M^\sharp_{\text{adr}}(\mathcal{S}_0|\tau)) = M^\sharp_{\text{adr}}(j^*(\mathcal{S}_0|\tau))$.[9]

Let U be the order zero normal measure on $j^*(\tau)$ from the $j^*(\mathcal{S}_0^+)$-sequence. It is easy to see then that

$$M^\sharp_{\text{adr}}(j^*(\mathcal{S}_0|\tau)) = \mathcal{H}_1^{\text{Ult}(j^*(\mathcal{S}_0^+),U)}(\tau+1),$$

since both mice are $\tau+1$-sound, and have τ as a cutpoint.[10] Let

$$\phi\colon j^*(\mathcal{S}_0^+) \to \text{Ult}(j^*(S_0^+), U)$$

and

$$\sigma\colon M^\sharp_{\text{adr}}(j^*(\mathcal{S}_0|\tau)) \to \text{Ult}(j^*(S_0^+), U)$$

be the canonical embeddings given by the ultrapower and the transitive collapse respectively. We then have that

$$\sigma^{-1} \circ \phi \circ \psi \circ i\colon \mathcal{M}_b^{\mathcal{W}} \to M^\sharp_{\text{adr}}(j^*(\mathcal{S}_0|\tau))$$

[8] At this point, it is important that we are *not* forming a normal extension of $\mathcal{V}^\frown\mathcal{W}^*$, but rather a tree on $\mathcal{M}_b^{\mathcal{W}}$. This tree will definitely use extenders which ought to be applied to some model in $\mathcal{V}^\frown\mathcal{W}^*$ in a normal extension of that tree. This is where we need that Σ can be extended to act on compositions of normal trees.

[9] Our method of computing the $M^\sharp_{\text{adr}}$-operator in $\mathcal{S}_0[g][h][G]$ is ultimately justified by the iterability of $\mathcal{S}_0$, and $j^*(\mathcal{S}_0) = \mathcal{Y}$ is iterable.

[10] Here $\mathcal{H}_1^{\mathcal{P}}(X)$ is the transitive collapse of the Σ_1 hull of X inside $\mathcal{P}$. There is a small fudge here, in that one must identify an unrelativised mouse $\mathcal{M}$ with cutpoint η with a mouse relativised to $\mathcal{M}|\eta$, in the obvious way.

is elementary. The main point to notice here is that $\mathcal{M}_b^{\mathcal{W}} = \mathcal{H}_1^{\mathcal{M}_b^{\mathcal{W}}}(k \circ i_E(\kappa))$, simply because $\mathcal{T}$ is a normal iteration tree on an ω-sound mouse. This implies that $\mathrm{ran}(\phi \circ \psi \circ i) \subseteq \mathrm{ran}(\sigma)$.

Thus $\sigma^{-1} \circ \phi \circ \psi \circ i$ is the desired embedding. $\square$

Since b is not ordinal definable in $\mathcal{S}[g][h][G]$ from $M^\sharp_{\text{adr}}(\mathcal{S}_0|\tau)$, we get from claim 2 that there must be a cofinal branch c of $\mathcal{W}$ such that $b \neq c$ and $\mathcal{M}_c^{\mathcal{W}}$ is embeddable into $M^\sharp_{\text{adr}}(\mathcal{S}_0|\tau)$. This embedding guarantees that $\mathcal{M}_c^{\mathcal{W}}$ is iterable, and thereby completes the proof of 0.1. $\square$

Finally, we show that the proof of 0.1 gives an example of distinct π-realizable branches, where π is a minimal embedding. For by [4], there is a fully backgrounded K^c-construction $\mathbb{C}$ and an embedding $\pi\colon M^\sharp_{\text{adr}} \to \mathcal{N}_\eta^{\mathbb{C}}$ for some η. Since $M^\sharp_{\text{adr}}$ is pointwise Σ_1 definable, there can be at most one such η and π for the given $\mathbb{C}$, and hence π is $\vec{e}$-minimal in the sense of [5] for *all* enumerations $\vec{e}$ of $M^\sharp_{\text{adr}}$. The iteration strategy Σ for $M^\sharp_{\text{adr}}$ constructed in [4] involves lifting trees to V and choosing wellfounded branches there; a standard absoluteness argument shows that such branches are π-realizable. Thus the branch $a = \langle 0\rangle^\frown b$ of the tree $\mathcal{T}$ we constructed in the proof of 0.1 is π-realizable. But also, the branch $d = \langle 0\rangle^\frown c$ we constructed was such that for some Σ-iterate $\mathcal{Z}$ of $M^\sharp_{\text{adr}}$, $\mathcal{M}_d^{\mathcal{T}}$ embeds into $\mathcal{Z}$. Since $\mathcal{Z}$ is realizable in $\mathbb{C}$, d is realizable, and since π is determined by the pointwise definability of $M^\sharp_{\text{adr}}$, any realization of d is a π-realization. So a and d are our distinct π-realizable branches of $\mathcal{T}$.

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