Core Models with More Woodin Cardinals

J. R. Steel

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In this paper, we shall prove two theorems involving the construction of core models with infinitely many Woodin cardinals. We assume familiarity with [11], which develops core model theory the one Woodin level, and with [12] and [6], which extend the fine structure theory of [5] to mice having many Woodin cardinals. The most important new problem of a general nature which we must face here concerns the iterability of K^c with respect to uncountable iteration trees.

Our first result is the following theorem, a slightly stronger version of which was proved independently and earlier by Woodin.[1] The theorem settles positively a conjecture of Feng, Magidor, and Woodin [2].

Theorem. *Let Ω be measurable. Then the following are equivalent*:

(a) *for all posets* $\mathbb{P}, \mathbb{Q} \in \mathbb{V}_\Omega$, $L(\mathbb{R})^{V^{\mathbb{P}}} \equiv L(\mathbb{R})^{V^{\mathbb{Q}}}$,

(b) *for every poset* $\mathbb{P} \in V_\Omega$, $V^{\mathbb{P}} \models AD^{L(\mathbb{R})}$,

(c) *for every poset* $\mathbb{P} \in \mathbb{V}_\Omega, \mathbb{V}^{\mathbb{P}} \models$ *there is no uncountable sequence of distinct reals in* $L(\mathbb{R})$,

(d) *there is an Ω-iterable premouse of height Ω which satisfies "there are infinitely many Woodin cardinals".*

It is an immediate corollary that if every set of reals in $L(\mathbb{R})$ is weakly homogeneous, then $\mathrm{AD}^{L(\mathbb{R})}$ holds.[2] We shall also indicate some extensions of the theorem to pointclasses beyond $L(\mathbb{R})$, and mice with more than ω Woodin cardinals.

Our second result is an improved lower bound on the consistency strength of the failure of the unique branches hypothesis (UBH; cf. [3]) for certain sorts of iteration trees. Recall that an iteration tree $\mathcal{T}$ is called *nonoverlapping* iff whenever E and F are extenders such that E

[1] Woodin proved the theorem under the weaker hypothesis that Ω is inaccessible in the Fall of 1991. We proved the theorem as stated in the Spring of 1992.

[2] The author proved this earlier, in the Fall of 1990, by similar but simpler methods.

is used before F along some branch of $\mathcal{T}$, then $\mathrm{lh}(E) \leq \mathrm{crit}(F)$. (See [13].) Nonoverlapping trees have special importance because the iteration trees which come up in inner model theory are all linear compositions of nonoverlapping trees.

Theorem. *If there is a nonoverlapping iteration tree $\mathcal{T}$ on V, having distinct cofinal, wellfounded branches b and c, then there is an inner model with infinitely many Woodin cardinals, and if in addition, $\delta(\mathcal{T}) \in ran(i_b) \cap ran(i_c)$, then there is an inner model with a strong cardinal which is a limit of Woodin cardinals.*

The best lower bound for this failure of UBH known previously was two Woodin cardinals (cf. [11, §7]). That argument also needed a measurable cardinal in the hypothesis, whereas here we do not. It should be noted that UBH can indeed fail: in unpublished work, Woodin has shown that the existence of a nontrivial elementary embedding of some $L(V_{\lambda+1})$ to itself implies that there is a countable, nonoverlapping iteration tree on V which has distinct cofinal, wellfounded branches. Of course, the consistency-strength gap bracketed by our new results and Woodin's result is still enormous.

The paper is organized as follows. Sections 1, 2, and 3 extend the theory of [11] to mice having many Woodin cardinals. The key problem is the iterability of K^c with respect to uncountable trees. We do have iterability with respect to countable trees just as in [11], provided that the K^c-construction does not reach a non-tame mouse. In fact, the κ-completeness of the background extenders for K^c enables us to handle trees in $V^{\mathbb{P}}$, for any poset $\mathbb{P}$ of size less κ. In §1 we use this observation to extend the fine-structural results of [5] to the levels of K^c. This tells us that the K^c construction converges, and to a model of height Ω; moreover, $(\alpha^+)^{K^c} = \alpha^+$ for almost all $\alpha < \Omega$.[3]

The iterability of K^c with respect to trees of size strictly less than the completeness of the background extenders does not suffice for all of [11], however. In order to construct "true K" from K^c, we must know that K^c is iterable with respect to trees of size Ω. We cannot quite show this, but in §2 we come close enough for our applications. We show, under a certain anti-large-cardinal hypothesis (see 2.15), that there is a premouse $\mathcal{M}$ such that $K^c(\mathcal{M})$ is fully iterable above $\mathcal{M}$, where $K^c(\mathcal{M})$ is the model obtained by doing the K^c construction with initial model $\mathcal{M}$. This enables us to construct $K(\mathcal{M})$, true K over $\mathcal{M}$, as in [11].[4] The definition of $K(\mathcal{M})$, which employs $\mathcal{M}$ as a parameter, defines the same class in all generic extensions of V by posets in V_Ω.

§3 extends the results of §6 of [11] on the inductive definition of K. From this we get that if $\mathcal{M}$ is coded by a real, then $K(\mathcal{M}) \cap HC \in L(\mathbb{R})$. It follows that unless some $K^c(\mathcal{M})$ satisfies "There are ω Woodins" then some $L(\mathbb{R})^{V^{\mathbb{P}}}$ has an uncountable sequence of distinct reals. This leads quickly to a proof of theorem 1.

We prove theorem 2 in §4.

[3] Provided all mice are tame, that is.

[4] Actually, we need a few more properties of the $\mathcal{M}$ we construct; see 2.14.

1 Existence and fine structure of K^c

Fix, for the remainder of the paper, a measurable cardinal Ω. For $\mathcal{M}$ a premouse, we set $o(\mathcal{M}) = \mathrm{OR} \cap \mathcal{M}$. All premice $\mathcal{M}$ we consider are such that $o(\mathcal{M}) \leq \Omega$; if $o(\mathcal{M}) = \Omega$ we call $\mathcal{M}$ a weasel, and if $o(\mathcal{M}) < \Omega$, we call $\mathcal{M}$ a set premouse.

We need also to consider premice built over a transitive set X, the obvious relativization to X of the standard notion. If $\mathcal{M}$ is an X-premouse, then the first level of $\mathcal{M}$ is $J_0^{\mathcal{M}}[X] =$ the rudimentary closure of $X \cup \{X\}$, and each nonempty E_α on $\mathcal{M}$-sequence is an extender over $J_\alpha^{\vec{E}}[X]$ which is sufficiently complete that its canonical embedding is the identity on $X \cup \{X\}$; otherwise the definition of "fine extender sequence" ([10, 2.4]) goes over verbatim. Projecta, standard parameters, cores, and soundness are defined as usual, but with names for arbitrary elements of $X \cup \{X\}$ allowed in all definitions, and $X \cup \{X\}$ contained in all hulls.

In fact, we shall only need this relativization in the very special case that X is itself a passive premouse $\mathcal{P}$, and $\mathcal{M}$ is an potential premouse in which $\mathcal{P}$ appears as a strong cutpoint. By sticking to that case we can simplify the notation a bit, so that is what we shall do.

Definition 1.1 *We say that η (or $\mathcal{J}_\eta^{\mathcal{M}}$) is a strong cutpoint of $\mathcal{M}$ iff $\eta \leq o(\mathcal{M})$, and for all extenders E on the $\mathcal{M}$-sequence, if $crit(E) \leq \eta$ then $lh(E) < \eta$. We write $\mathcal{N} \trianglelefteq^* \mathcal{M}$ iff $\mathcal{N}$ is a strong cutpoint of $\mathcal{M}$.*

(This strengthens the usual notion of cutpoint, in that η is not allowed as a critical point of an extender on the $\mathcal{M}$-sequence.) So for our purposes, we can take relativized premice to be pairs $(\mathcal{M}, \eta)$, where $\mathcal{M}$ is an potential premouse, and η is a strong cutpoint of $\mathcal{M}$.

Definition 1.2 *An potential r-premouse is a structure $\mathcal{M} = (\mathcal{M}_0, \eta)$ such that $\mathcal{M}_0$ is a potential premouse, and η is a strong cutpoint of $\mathcal{M}_0$. We call $\mathcal{J}_\eta^{\mathcal{M}}$ the lower part of $\mathcal{M}$ and say that $\mathcal{M}$ is an r-premouse over $\mathcal{J}_\eta^{\mathcal{M}}$; call $\eta = \eta(\mathcal{M})$ the cut of $\mathcal{M}$, and $\mathcal{M}_0$ the reduct of $\mathcal{M}$.*

For any potential premouse $\mathcal{P}$, the pair $(\mathcal{P}, o(\mathcal{P}))$ is an r-premouse; we regard it as structure by taking $o(\mathcal{P})$ as a (trivial) predicate. If $\eta(\mathcal{M}) < o(\mathcal{M})$ we treat $\eta(\mathcal{M})$ as a constant of the structure $\mathcal{M}$. If $\eta(\mathcal{M}) \leq \gamma \leq o(\mathcal{M})$, then $\mathcal{J}_\gamma^{\mathcal{M}} = ((\mathcal{J}_\gamma^{\mathcal{M}})_0, \eta(\mathcal{M}))$; these are the initial segments of $\mathcal{M}$. We do not regard proper initial segments of the reduct of $\mathcal{M}$ as initial segments of $\mathcal{M}$.

The projecta, cores, and parameters of a potential r-premouse $\mathcal{M}$ are defined as in the unrelativised case, except that when doing so we work in a language with names for each $\alpha \leq \eta(\mathcal{M})$. Thus by fiat, the anti-core embedding from $\mathfrak{C}_{k+1}(\mathcal{M})$ to $\mathfrak{C}_k(\mathcal{M})$ is the identity on $\eta(\mathcal{M}) + 1$. We say $\mathcal{M}$ is *k-sound* iff $\mathcal{M} = \mathfrak{C}_k(\mathcal{M})$.

Definition 1.3 *An r-premouse is a potential r-premouse all of whose proper initial segments are ω-sound.*

Iterability for r-premice is defined just as for ordinary premice, with the important exception that we only consider iteration trees on $\mathcal{M}$ such that *all extenders used have length, and hence critical point, strictly greater than the cut of $\mathcal{M}$.* (This is a point where it would be more natural to be working in the greater generality of lower parts which are arbitrary transitive sets.)

It is straightforward to relativize the construction of K^c in §1 of [11]. Let $\mathcal{P}$ be a set premouse. We call a sequence of r-premice $\langle \mathcal{N}_\xi \mid \xi \leq \beta \rangle$, where $\beta \leq \Omega$, a *$\mathcal{P}$-construction* just in case each $\mathcal{N}_\xi$ has lower part $\mathcal{P}$, and the following conditions are met.

First,

$$\mathcal{N}_0 = (\mathcal{P}, o(\mathcal{P})).$$

Now suppose that $\mathcal{N}_\xi$ is defined. If $\mathfrak{C}_\omega(\mathcal{N}_\xi)$ does not exist , then $\xi = \beta$. (That is, our construction stops here.) Otherwise set

$$\mathcal{M}_\xi = \mathfrak{C}_\omega(\mathcal{N}_\xi).$$

Now suppose $\mathcal{M}_\xi$ exists, and $\xi < \Omega$; then $\xi < \beta$, and $\mathcal{N}_{\xi+1}$ comes from $\mathcal{M}_\xi$ in one of the following two ways. Suppose $\mathcal{M}_\xi = (J^{\vec{E}}_\gamma, \in, \vec{E}, H, o(\mathcal{P}))$

Case 1. $H = \emptyset$ (that is, $\mathcal{M}_\xi$ is passive) and there is an extender F with critical point κ over $J^{\vec{E}}_\gamma$ such that $\mathcal{N}_{\xi+1} = (J^{\vec{E}}_\gamma, \in, \vec{E}, \tilde{F}, o(\mathcal{P}))$, and

(1) $(J^{\vec{E}}_\gamma, \in, \vec{E}, \tilde{F}, o(\mathcal{P}))$ is an r-premouse,

(2) κ is inaccessible and $\kappa > o(\mathcal{P})$,

(3) $(\kappa^+)^{\mathcal{M}_\xi} = \kappa^+ \Rightarrow$ for stationary many $\beta < \kappa$, β is inaccessible and $(\beta^+)^{\mathcal{M}_\xi} = \beta^+$, and further, letting ν be the sup of the generators of F,

(4) for any $\mathcal{A} \subset P(\kappa) \cap J^{\vec{E}}_\gamma$ such that $|A| \leq \kappa$, there is a pair (N, G) such that,

(a) N is transitive, power admissible, $V_\kappa \cup \mathcal{A} \subseteq N$, ${}^\omega N \subseteq N$, and G is an extender over N,

(b) $F \cap ([\nu]^{<\omega} \times N) = G \cap ([\nu]^{<\omega} \times N)$,

(c) $V_{\nu+1} \subseteq \mathrm{Ult}(N, G)$, and $\mathrm{Ult}(N, G)$ is closed under ω-sequences, and

(d) letting $i\colon N \to \mathrm{ult}(N, G)$ be the canonical embedding, we have that $\mathcal{M}_\xi$ agrees with $i(\mathcal{J}^{\mathcal{M}_\xi}_\kappa)$ below $o(\mathcal{M}_\xi)$.

Case 2. $\mathcal{N}_{\xi+1} = ((J^{\vec{W}}_{\gamma+1}, \in, \vec{W}, \emptyset), o(\mathcal{P}))$, where $\vec{W} = \vec{E} \frown \langle H \rangle$.

Finally, for λ a limit ordinal, $\mathcal{N}_\lambda$ is determined by $\langle \mathcal{N}_\xi \mid \xi < \lambda \rangle$ just as in [11]: $o(\mathcal{N}_\lambda) = \liminf_{\xi \to \lambda} (\rho_\omega(\mathcal{N}_\xi)^+)^{\mathcal{N}_\xi}$, and for $\omega\alpha < o(\mathcal{N}_\lambda)$, $\mathcal{J}^{\mathcal{N}_\lambda}_\alpha =$ eventual value of $\mathcal{J}^{\mathcal{N}_\xi}_\alpha$ as $\xi \to \lambda$.

The construction above is that of [11], except that:

(i) Our structures are r-premice over some fixed $\mathcal{P}$.

(ii) We do not restrict to 1-small mice, in fact we make no smallness assumption. (However, we soon shall.)

(iii) We do not require that the construction add an extender whenever possible. That is, we do not require $\mathcal{N}_{\xi+1}$ be defined by case 1 whenever there is an F as in (1)–(4). Moreover, in case 1 we do not require that F be chosen with ν minimal.

(iv) In case 1, condition (4) must hold for all $\mathcal{A}$ of cardinality κ, not just for all countable $\mathcal{A}$. This strengthening of the background certificate condition from [11] plays only one very minor role here, in our proof of 2.13.

If $\mathbb{C} = \langle \mathcal{N}_\xi \mid \xi \leq \beta \rangle$ is a $\mathcal{P}$-construction, then $\mathcal{N}^{\mathbb{C}}_\xi = \mathcal{N}_\xi$ is its ξth model. We also set $\mathcal{M}^{\mathbb{C}}_\xi = \mathfrak{C}_\omega(\mathcal{N}_\xi)$ when this exists, and

$$\mathrm{Add}(\mathbb{C}) = \inf\{\mathrm{crit}(\dot{F}^{\mathcal{N}_\xi}) \mid \xi \leq \beta\}.$$

Thus $o(\mathcal{P}) < \mathrm{Add}(\mathbb{C}) < \Omega$, and $\mathrm{Add}(\mathbb{C})$ is inaccessible. $\mathrm{Add}(\mathbb{C})$ is no greater than the smallest critical point above $o(\mathcal{P})$ of an extender on the sequence of the final model $\mathcal{N}_\beta$ which is total on $\mathcal{N}_\beta$.

We are particularly interested in $\mathcal{P}$-constructions which add as many extenders as possible, subject perhaps to a lower bound θ on $\mathrm{Add}(\mathbb{C})$.

Definition 1.4 *A $\mathcal{P}$ construction $\mathbb{C}$ is θ-maximal, where $\theta < \Omega$, iff whenever $\mathcal{M}^{\mathbb{C}}_\xi$ exists and $\mathcal{M}^{\mathbb{C}}_\xi = (J^{\vec{E}}_\gamma, \in, \vec{E}, \emptyset, o(\mathcal{P}))$, and there is an extender F over $\mathcal{M}_\xi$ such that crit$(F) > \theta$ and $(J^{\vec{E}}_\gamma, \in, \vec{E}, \tilde{F}, 0(\mathcal{P}))$ satisfies (1)–(4) of case 1 above, then $\mathcal{N}^{\mathbb{C}}_{\xi+1} = (J^{\vec{E}}_\gamma, \in, \vec{E}, \tilde{F}, o(\mathcal{P}))$ for some such F with $\nu(F)$ as small as possible.*

The bicephalus argument of §9 of [5] shows that there is exactly one θ-maximal $\mathcal{P}$ construction.

Definition 1.5 $\mathbb{C}(\mathcal{P}, \theta)$ *is the unique θ-maximal $\mathcal{P}$ construction. We set $K^c(\mathcal{P}, \theta) = \mathcal{N}^{\mathbb{C}(\mathcal{P},\theta)}_\Omega$, provided of course that $\mathcal{N}^{\mathbb{C}(\mathcal{P},\theta)}_\Omega$ exists. Finally, $K^c = K^c((V_\omega, \in), \omega)$, if this exists.*

For our purposes here, it will suffice that $\mathbb{C}(\mathcal{P}, \theta)$ is *some* θ-maximal $\mathcal{P}$ construction. We do not need to know that it is the unique one.

For the basic results about K^c constructions we need to know that every countable elementary submodel of an $\mathcal{N}_\xi^{\mathcal{P}}$ is $(\omega_1 + 1)$-iterable. We do not know how to prove this without some additional smallness assumption on the $\mathcal{N}_\xi^{\mathcal{P}}$. One which suffices is:

Definition 1.6 *An r-premouse $\mathcal{M}$ is tame iff whenever E is an extender on the $\mathcal{M}$-sequence, and $\eta(\mathcal{M}) < crit(E) \leq \delta < lh(E)$, then $\mathcal{J}_{lh(E)}^{\mathcal{M}} \not\models \delta$ is a Woodin cardinal.*

We shall outline here a proof that every countable elementary submodel of a tame $\mathcal{N}_\xi^{\mathbb{C}}$ is (ω_1+1)-iterable. Stronger results have been proved in [1][5], but for a different kind of mouse, and by arguments which add a layer of complexity to the one we sketch here.

Recall from [9] that $\pi\colon \mathcal{M} \to \mathcal{N}$ is called a near k-embedding just in case π has all the properties of a k-embedding in the sense of [5], except that possibly $\pi``\rho_k(\mathcal{M})$ may not be cofinal in $\rho_k(\mathcal{Q})$. Equivalently, π is a weak k-embedding in the sense of [5] which is fully $r\Sigma_{k+1}$ elementary.

Definition 1.7 *Let $\mathbb{C}$ be a $\mathcal{P}$-construction, and $\pi\colon \mathcal{M} \to \mathfrak{C}_{\mathfrak{k}}(\mathcal{N}_\xi^{\mathbb{C}})$ be a near k-embedding; then we say $\mathcal{M}$ is k-realizable in $\mathcal{N}_\xi^{\mathbb{C}}$ via π.*

A k-realization provides "background extenders" for the extenders on the $\mathcal{M}$-sequence with index above its cut.[6]

Definition 1.8 *Let $\mathcal{M}$ be k-realizable in $\mathcal{N}_\xi^{\mathbb{C}}$ via π. Let $\mathcal{T}$ be a k-maximal tree on $\mathcal{M}$. Let $\alpha < lh\mathcal{T}$, and $\sigma : \mathcal{M}_\alpha^{\mathcal{T}} \to \mathfrak{C}_j(\mathcal{N}_\gamma^{\mathbb{C}})$ be a near j-embedding. Then we say that $\mathcal{M}_\alpha^{\mathcal{T}}$ is $(\pi, \mathcal{T})$* realizable *(in $\mathcal{N}_\gamma^{\mathbb{C}}$, via σ) just in case $\pi \restriction \delta + 1 = \sigma \restriction \delta + 1$, and*

1. $D^{\mathcal{T}} \cap [0, \alpha]_T \neq \emptyset \Rightarrow \gamma < \xi$, *and*
2. $D^{\mathcal{T}} \cap [0, \alpha]_T = \emptyset \Rightarrow (\gamma = \xi \wedge \pi = \sigma \circ i_{0\alpha}^{\mathcal{T}})$.

If b is a maximal well-founded branch of $\mathcal{T}$, then we say $\mathcal{M}_b^{\mathcal{T}}$ is $(\pi, \mathcal{T})$-realizable in $(\mathcal{N}_\gamma^{\mathbb{C}})$ via σ if, letting $\mathcal{T}'$ be an extension of $\mathcal{T}$ such that $b = [0, \theta]_{T'}$, for $\theta = lh\mathcal{T}$, $\mathcal{M}_\theta^{\mathcal{T}'}$ is realizable in this way.

We can now state precisely what the iterability proof of [11] gives us. Recall that a putative iteration tree is just like an ordinary iteration tree, except that its last model (if it has one) may be ill-founded.

[5] After we proved these results.

[6] We are *not* requiring here that $o(\mathcal{P}) + 1 \subseteq \mathrm{ran}(\pi)$; typically, $\mathcal{M}$ will be countable and $\mathcal{P}$ will be uncountable.

Theorem 1.9 *Let $\mathbb{C}$ be a $\mathcal{P}$-construction. Let G be $\mathbb{P}$-generic over V, where $\mathbb{P}$ has size less than* $\mathrm{Add}(\mathbb{C})$*; then the following is true in $V[G]$: if $\mathcal{M}$ is countable and k-realizable in $(\mathcal{N}_\xi^{\mathbb{C}})^V$ via π, and $\mathcal{T}$ is a k-maximal putative iteration tree of countable length on $\mathcal{M}$, then either*

1. *$\mathcal{T}$ has a maximal branch b such that $\mathcal{M}_b^{\mathcal{T}}$ is $(\pi, \mathcal{T})$ realizable, or*
2. *$lh\mathcal{T} = \theta + 1$, for some θ, and $\mathcal{M}_\theta^{\mathcal{T}}$ is $(\pi, \mathcal{T})$ realizable.*

We note that $\mathcal{M}$ and $\mathcal{T}$ may not be in V, although the $\mathcal{P}$ construction $\mathbb{C}$ in which we realize $\mathcal{M}_b^{\mathcal{T}}$ or $\mathcal{M}_\theta^{\mathcal{T}}$ is in V. Even if $\mathcal{M}, \mathcal{T} \in V$, b and the realizing map $\sigma : \mathcal{M}_b^{\mathcal{T}} \to \mathfrak{C}_k(\mathcal{N}_\xi^{\mathbb{C}})$ may not be in V.

The proof of iterability in [11] adapts easily to yield 1.9. Note that since $\mathrm{crit}(E_\alpha^{\mathcal{T}}) > \delta$ for all $\alpha + 1 < lh\mathcal{T}$, the $\mathcal{P}$ construction $\mathbb{C}$ supplies background certificates for the images (under π and later enlargement maps) of the $E_\alpha^{\mathcal{T}}$'s. The fact that $\mathcal{T}$ is not necessarily in V is not a problem because of the completeness of the background extenders. If (N, F) is a background certificate for $\mathbb{C}$ in V, then letting $\kappa = \mathrm{crit}\ (F)$, we have that $\mathbb{P}$ has size $< \kappa$. Now let $i : N \to \mathrm{Ult}(N, F)$ be the canonical embedding. Since $V_\kappa = V_\kappa^N$, G is N-generic, and i lifts to an embedding $i^* : N[G] \to \mathrm{Ult}(N, F)[G]$ as usual: simply set $i^*(\tau^G) = i(\tau)^G$. Letting F^* be the extender induced by i^*, we have $\mathrm{Ult}(N, F)[G] = \mathrm{Ult}(N[G], F^*)$. Let ν be the sup of the generators of the extender for which (N, F) is a background certificate. Since $V_{\nu+1} \subset \mathrm{Ult}(N, F)$ in $V, V_{\nu+1}[G] = (V_{\nu+1})^{V[G]} \subset \mathrm{Ult}(N, F)[G]$. Thus $(N[G], F^*)$ has all the properties required of a background certificate in $V[G]$. We can run the iterability proof from [11] inside $V[G]$, using these fattened background certificates.

Unfortunately, the iteration strategies we get from 1.9 do not quite win the iteration game $G_k(\omega_1, \omega_1 + 1)$ used in [6] in the proof of the weak Dodd-Jensen lemma. Fortunately, they are strong enough that the proof of the weak Dodd-Jensen lemma still goes through. Let $G_k^*(\omega_1, \omega_1 + 1)$ be the game which consists of ω_1 rounds, each round being a play of the ordinary iteration game of length $\omega_1 + 1$ on a model provided by the previous rounds, just as in $G_k(\omega_1, \omega_1 + 1)$ *except that* : if round α lasts $\omega_1 + 1$ moves (i.e. player I does not exit at a countable stage), then the game is over, and there are no further rounds.[7] Thus if we play $G_k^*(\omega_1, \omega_1 + 1)$ with a countable starting model, then all models at stages strictly before the game is over are also countable. In contrast, in $G_k(\omega_1, \omega_1 + 1)$ the model at the end of round 0 might have size ω_1, and we would continue playing further rounds with it. The restriction to countable models at intermediate stages is necessary to obtain an iteration strategy from 1.9.

We say that $\mathcal{M}$ is $(k, \omega_1, \omega_1 + 1)^*$-*iterable* iff player II has a winning strategy in $G_k^*(\omega_1, \omega_1 + 1)$.

[7]The models in our game are r-premice, and we require that all extenders played by I have critical point above the (common) cut of these premice.

Corollary 1.10 *Let G be $\mathbb{P}$-generic over V, where $\mathbb{P}$ has size less than* $\mathrm{Add}(\mathbb{C})$. *The following is then true in $V[G]$: if $\mathcal{M}$ is tame, countable and k-realizable in $(\mathcal{N}_\xi^{\mathbb{C}})^V$, then $\mathcal{M}$ is $(k,\omega_1,\omega_1+1)^*$-iterable.*

Proof.(*Sketch.*) We first describe how II should play round 0 of $G_k^*(\omega_1,\omega_1+1)$.

Fix an enumeration $\vec{e}=\langle e_i \mid i<\omega\rangle$ of the universe of $\mathcal{M}$. Let π be the $(k,\vec{e})$-minimal realization of $\mathcal{M}$ in $\mathbb{C}$, in the sense of [6]. Say $\pi\colon \mathcal{M}\to\mathfrak{C}_{\mathfrak{k}}(\mathcal{N}_\xi)$. We claim the the strategy of choosing unique $(\pi,\mathcal{T})$-realizable branches does not lose for II at any countable stage during round 0 (in other words, it is a (k,ω_1)-iteration strategy for $\mathcal{M}$). If not, then by 1.9, playing according to it will lead to a tree $\mathcal{T}$ having distinct cofinal $(\pi,\mathcal{T})$-realizable branches b and c. Now $\mathcal{M}_b^{\mathcal{T}}$ and $\mathcal{M}_c^{\mathcal{T}}$ are tame, and 1.9 gives us realization strategies for them in the weak iteration game. By [12], this is enough to compare them. (We use here that if $\mathcal{T}$ is an iteration tree of length ω_1 picking unique realizable branches at each countable limit stage, then $\mathcal{T}$ has a cofinal wellfounded branch. See below.) By tameness again, the comparison is above $\delta(\mathcal{T})$.

Let $\mathcal{P}$ be the final model on the $\mathcal{M}_b$-side, and $\mathcal{Q}$ the final model on the $\mathcal{M}_c$-side, and suppose without loss of generality that $\mathcal{P}\trianglelefteq\mathcal{Q}$. If the branch $\mathcal{M}$-to$\mathcal{M}_c$-to-$\mathcal{Q}$ drops, then $\mathcal{M}$-to-$\mathcal{M}_b$-to-$\mathcal{P}$ does not, moreover $\mathcal{Q}$ realized in $\mathfrak{C}_{\mathrm{j}}(\mathcal{N}_\eta)$ for some $(j,\eta)<_{\text{lex}}(k,\xi)$. But then $\mathcal{P}$ is realized in $\mathfrak{C}_{\mathrm{j}}(\mathcal{N}_\eta)$, and since $\mathcal{M}$-to-$\mathcal{P}$ does not drop, so is $\mathcal{M}$. This contradicts the minimality of π.

Thus neither $\mathcal{M}$-to-$\mathcal{M}_b$-to-$\mathcal{P}$ nor $\mathcal{M}$-to-$\mathcal{M}_c$-to-$\mathcal{Q}$ drops (in model or degree). Let

$$j\colon \mathcal{M}_b\to\mathcal{P} \text{ and } l\colon \mathcal{M}_c\to\mathcal{Q}$$

be the iteration maps. Using the $(k,\vec{e})$-minimality of π, we get easily

$$\mathcal{P}=\mathcal{Q} \text{ and } j\circ i_b = l\circ i_c.$$

But every element of $\mathcal{M}_b$ is of the form $i_b(f)(a)$ for some finite $a\subseteq\delta(\mathcal{T})$, and similarly for $\mathcal{M}_c$, moreover i_b and i_c are the identity below $\delta(\mathcal{T})$. It follows then that j"$\mathcal{M}_b=l$"$\mathcal{M}_c$, so that $\mathcal{M}_b=\mathcal{M}_c$. Thus j and l are the identity. But then $i_b=i_c$, so $b=c$.

Now suppose $\mathcal{T}$ is a tree of length ω_1 played according to the strategy we have just described; we must show $\mathcal{T}$ has a cofinal wellfounded branch. Let H be $V[G]$-generic for collapsing ω_1. By 1.9, there is a maximal realizable branch b of $\mathcal{T}$ in $V[G][H]$. Now for any $\lambda<\omega_1$, the fact that $[0,\lambda]_T$ was the unique cofinal realizable branch of $\mathcal{T}\restriction\lambda$ in $V[G]$ is absolute to $V[G][H]$, so b must be cofinal in ω_1. If $c\neq b$ is another cofinal realizable branch of $\mathcal{T}$ in $V[G][H]$, then we can take in $V[G]$ a countable Skolem hull of some big $V_\alpha[G]$, with everything relevant in the hull, and collapse it to a transitive P, getting $\bar{\mathcal{T}}$,$\bar{\mathbb{C}}$, etc. Letting $\lambda=\omega_1^P$, we have $\bar{T}=\mathcal{T}\restriction\lambda$. Let $\bar{H}\in V[G]$ be P-generic for collapsing λ; then in $P[\bar{H}]$ we have distinct cofinal branches of $\mathcal{T}\restriction\lambda$ which are realizable in $\bar{\mathbb{C}}$, and hence, by composing with the uncollapse of P, realizable in $\mathbb{C}$. This is a contradiction. Thus b is the unique cofinal realizable branch of $\mathcal{T}$ in $V[G][H]$, so $b \in V[G]$, as desired.

If player I exits round 0 at some countable stage, then there is a last model $\mathcal{M}_\alpha^\mathcal{T}$ of the tree produced so far, and I picks $\mathcal{Q} \trianglelefteq \mathcal{M}_\alpha^\mathcal{T}$ and i; round 1 is then going to produce an i-maximal iteration tree on $\mathcal{Q}$. Player II now fixes an enumeration $\vec{t}$ of the universe of $\mathcal{Q}$ in order type ω. If $[0, \alpha]_T$ does not drop in model or degree, $\mathcal{Q} = \mathcal{M}_\alpha^\mathcal{T}$, and $i = k$, then let σ be the k-realization of $\mathcal{Q}$ in $\mathfrak{C}_k(\mathcal{N}_\xi)$ which extends π (in that $\pi = \sigma \circ i_{0,\alpha}^\mathcal{T}$), and which is $\vec{t}$-lexicographically minimal among all such extensions of π. Otherwise, let σ be the $(i, \vec{t})$-minimal realization of $\mathcal{Q}$ in $\mathbb{C}$. (Note that in the latter case, $\sigma \colon \mathcal{Q} \to \mathfrak{C}_i(\mathcal{N}_\eta)$ for $(\eta, i) <_{\text{lex}} (\xi, k)$.) We now get through round 1 by picking unique σ-realizable branches; the argument we gave for round 0 adapts quite easily.

The obvious extension of this procedure is a winning strategy for II in the full ω_1-round game $G_k^*(\omega_1, \omega_1 + 1)$. $\square$

It is a routine matter to relativise to $\mathcal{P}$ the basic fine-structural results of [5], as they appear in their most general form, for iterable mice with no smallness assumption, in 3.1 and 3.2 of [6]. (See also [7].) Notice here that the argument of [6] actually shows that every countable $(k, \omega_1, \omega_1 + 1)^*$-iterable mouse has a $(k, \omega_1 + 1)$-iteration strategy with the weak Dodd-Jensen property.

By relativising the proofs of 3.1 and 3.2 of [6], we get

Theorem 1.11 *Let $\mathbb{C}$ be a $\mathcal{P}$-construction, and let $\mathcal{N}_\xi^\mathbb{C}$ be tame; then $\mathfrak{C}_\omega(\mathcal{N}_\xi^\mathbb{C})$ exists.*

Similarly, we obtain analogues of the results of §9 and §10 on bicephali and pseudo-premice. Since it is obvious how to state and prove these, we omit further detail.

Definition 1.12 *A $\mathcal{P}$-construction $\mathbb{C}$ is tame iff whenever $\mathcal{N}_\xi^\mathbb{C}$ exists, then $\mathcal{N}_\xi^\mathbb{C}$ is tame.*

The proof of §11 in [5] as modified in §1 of [11], gives

Theorem 1.13 *Suppose that μ is a normal measure on Ω, and that $\mathbb{C}(\mathcal{P}, \theta)$ is tame; then*

$$(\alpha^+)^{K^c(\mathcal{P},\theta)} = \alpha^+,$$

for μ-a.e. α.

2 Uncountable iteration trees and true K

We have managed to make do so far with iterability with respect to iteration trees which are of size at most ω_1. However, the proofs of §3–§5 of [11] concerning the hull and definability properties, and the construction of true K, require the iterability of K^c for trees of size Ω;

more precisely, they require the $(\omega, \omega, \Omega+1)$-iterability of K^c. We shall prove an approximation to that in this section: we shall show that if there is no proper class model satisfying "there are arbitrarily large Woodin cardinals", then there is a $\mathcal{P}$ such that for all θ, $K^c(\mathcal{P}, \theta)$ exists and is $(\omega, \omega, \Omega+1)$-iterable.

Our anti-large cardinal hypothesis is much more restrictive than tameness, and it should be possible to relax it, but we do not see how to do so. On the other hand, the fact that we only get the desired result relativised to some $\mathcal{P}$ seems inevitable. If we could prove in ZFC + "Ω is measurable" + "there is no proper class model with 2 Woodin cardinals" that K^c is $(\omega, \omega, \Omega+1)$-iterable, the arguments of [11] would then show that the existence of a saturated ideal on ω_1 together with a measurable cardinal implies that there is a proper class model with 2 Woodin cardinals. This is not the case, since Shelah has shown the existence of a saturated ideal on ω_1 plus a measurable cardinal is no stronger than the existence of one Woodin cardinal with a measurable above it.

Iterabilty with respect to uncountable trees is always proved by giving an *absolute* definition of a *unique* iteration strategy, and then reflecting a failure of this strategy to a failure of it on a countable tree.[8] We shall be able to do this if we are working in a universe closed under structures of the following form.

Definition 2.1 *1. Let $\mathcal{Q}$ be an r-premouse with cut δ. We say that $\mathcal{Q}$ kills the Woodin property at k iff*

(a) $\mathcal{Q}$ is $k+1$-sound, and

(b) either

- *$\rho_{k+1}(\mathcal{Q}) < \delta$, or*
- *there is an $f : \delta \to \delta$ which is $r\Sigma_{k+1}^{\mathcal{Q}}$ and amenable to $\mathcal{J}_\delta^{\mathcal{Q}}$, such that for no E on the $\mathcal{Q}$-sequence do we have $i_E(f)(crit(E)) < \nu(E)$, but there is no such f which is $r\Sigma_k^{\mathcal{Q}}$.*

2. Suppose there is an r-premouse $\mathcal{Q}$ over $\mathcal{M}$ such that $\mathcal{Q}$ kills the Woodin property at k, and every countable elementary submodel of $\mathcal{Q}$ is (k, ω_1+1)-iterable; then we write $\mathcal{Q}(\mathcal{M})$ for the unique such r-premouse and $k(\mathcal{M})$ for the unique such k.

A straightforward coiteration of countable Skolem hulls shows that there is at most one pair $(\mathcal{Q}, k)$ as in the definition of $\mathcal{Q}(\mathcal{M})$. We leave $\mathcal{Q}(\mathcal{M})$ undefined if there is no r-premouse killing the Woodin property of $\mathcal{M}$ which is iterable in the way described above.

Definition 2.2 *Let $\mathcal{T}$ be a k-maximal iteration tree on $\mathcal{M}$ such that $lh(\mathcal{T})$ is a limit ordinal; then we set*

$$\delta(\mathcal{T}) = \sup\{\, lh(E_\alpha^{\mathcal{T}}) \mid \alpha < lh(\mathcal{T})\},$$

[8] This idea traces back to [3].

and

$$\mathcal{M}_\infty(\mathcal{T}) \;=\; \text{the unique passive } \mathcal{P} \text{ such that } o(\mathcal{P}) = \delta(\mathcal{T}) \text{ and } \forall\alpha < \delta(\mathcal{T})(\mathcal{P} \text{ agrees with } \mathcal{M}^{\mathcal{T}}_\alpha \text{ below } lh(E^{\mathcal{T}}_\alpha)),$$

and, if it is defined,

$$\mathcal{Q}(\mathcal{T}) = \mathcal{Q}(\mathcal{M}_\infty(\mathcal{T})).$$

Definition 2.3 *Let $\mathcal{T}$ be a k-maximal iteration tree on $\mathcal{M}$ of limit length, let b be a cofinal wellfounded branch of $\mathcal{T}$, and suppose some initial segment of $((\mathcal{M}^{\mathcal{T}}_b)_0, \delta(\mathcal{T}))$ kills the Woodin property at some k; then $\mathcal{Q}(b,\mathcal{T})$ is the unique such initial segment of $((\mathcal{M}^{\mathcal{T}}_b)_0, \delta(\mathcal{T}))$, and $k(b,\mathcal{T})$ is the unique k at which $\mathcal{Q}(b,\mathcal{T})$ kills the Woodin property.*

Notice that if $\mathcal{Q}(b,\mathcal{T})$ exists and $\delta(\mathcal{T}) \in \mathcal{Q}(b,\mathcal{T})$, then the reduct of $\mathcal{Q}(b,\mathcal{T})$ is just the longest initial segment $\mathcal{Q}$ of the reduct of $\mathcal{M}^{\mathcal{T}}_b$ such that $\mathcal{Q} \models \delta(\mathcal{T})$ is Woodin. There is a failure of $\delta(\mathcal{T})$ to be Woodin definable over $\mathcal{Q}(b,\mathcal{T})$.[9]

Definition 2.4 *An r-premouse $\mathcal{M}$ is properly small iff $\mathcal{M}$ is tame, and whenever $\eta(\mathcal{M}) < \delta < o(\mathcal{M})$, then $\mathcal{M} \models \delta$ is not Woodin, and either $\mathcal{M}$ is a weasel, or $\mathcal{M} \models \mathsf{ZF}^-$ plus "there is a largest cardinal".*

We shall restrict our attention, in the uncountable case, to iteration trees $\mathcal{T}$ whose base model is properly small. The following lemma explains the advantage in doing so.

Lemma 2.5 *Let $\mathcal{T}$ be an k-maximal tree on $\mathcal{M}$, and suppose $\mathcal{M}$ is properly small; then for every maximal well-founded branch b of $\mathcal{T}$: $Q(b,\mathcal{T})$ exists, and if $\mathcal{Q}(b,\mathcal{T}) = ((\mathcal{M}^{\mathcal{T}}_b)_0, \delta(\mathcal{T}))$, then $k(b,\mathcal{T}) \leq \deg^{\mathcal{T}}(b)$.*

Proof. If b drops in model or degree, then $\rho_{n+1}(\mathcal{M}^{\mathcal{T}}_b) < \delta(\mathcal{T})$, where $n = \deg^{\mathcal{T}}(b)$, and the lemma follows at once. If b does not drop, then since $\mathcal{M}$ has a largest cardinal, $\delta(\mathcal{T}) < o(\mathcal{M}^{\mathcal{T}}_b)$. Since $\mathcal{T}$ is above $\eta(\mathcal{M})$, and i_b is elementary, $\delta(\mathcal{T})$ is not Woodin in $\mathcal{M}^{\mathcal{T}}_b$, and once again the lemma is obvious. □

If an iteration strategy chooses a branch b of $\mathcal{T}$, then $\mathcal{M}^{\mathcal{T}}_b$ should be iterable, and hence $\mathcal{Q}(b,\mathcal{T})$ should be iterable if it is defined. But the iterability of $\mathcal{Q}(b,\mathcal{T})$ identifies b:

Theorem 2.6 ([10], Cor. 6.14) *Let $\mathcal{T}$ be k-maximal; then there is at most one cofinal, wellfounded branch b of $\mathcal{T}$ such that: $\mathcal{Q}(b,\mathcal{T})$ exists, and every countable elementary submodel of $\mathcal{Q}(b,\mathcal{T})$ is $(k(b,\mathcal{T}), \omega_1 + 1)$-iterable.*

[9] The case $\rho_{n+1}(\mathcal{Q}(b,\mathcal{T})) < \delta(\mathcal{T})$ represents a failure of $\delta(\mathcal{T})$ to be a cardinal at all.

Put another way: if $\mathcal{Q}(\mathcal{T})$ exists, then there is at most one branch b of $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$. Theorem 2.6 is just a fine-structural refinement of the fact that $\delta(\mathcal{T})$ is Woodin with respect to all functions in $\mathcal{M}_b^{\mathcal{T}} \cap \mathcal{M}_c^{\mathcal{T}}$.

Definition 2.7 *An iteration tree $\mathcal{T}$ is self-verifying iff for all limit $\lambda < lh(\mathcal{T})$, $\mathcal{Q}(\mathcal{T} \restriction \lambda)$ exists, and $\mathcal{Q}(\mathcal{T} \restriction \lambda) = \mathcal{Q}([0, \lambda]_T, \mathcal{T})$.*

So if $\mathcal{T}$ is an iteration tree on $\mathcal{M}$ of length less than Ω which is played according to an Ω-iteration strategy for $\mathcal{M}$, and $\mathcal{M}$ is properly small, then $\mathcal{T}$ is self-verifying.[10]

Theorem 2.8 *Let $\mathcal{M}$ be a k-sound, properly small r-premouse, and suppose that every countable elementary submodel of $\mathcal{M}$ is $(k, \omega_1 + 1)$-iterable. Let $\mathcal{T}$ be a self-verifying tree on $\mathcal{M}$, and suppose that $\mathcal{Q}(\mathcal{T})$ exists. Then there is a cofinal branch b of $\mathcal{T}$ such that $\mathcal{Q}(\mathcal{T}) = \mathcal{Q}(b, \mathcal{T})$.*

Proof. Suppose not, and let $\mathcal{M}$ and $\mathcal{T}$ be a counterexample. Let $\mathcal{Q} = \mathcal{Q}(\mathcal{T})$. Let $\pi\colon P \to V_\theta$ be elementary, where P is countable and transitive and $\pi(\langle \bar{\mathcal{M}}, \bar{\mathcal{T}}, \bar{\mathcal{Q}} \rangle) = \langle \mathcal{M}, \mathcal{T}, \mathcal{Q} \rangle$. Since iterability goes down to elementary submodels, $\bar{\mathcal{T}}$ is self-verifying, and $\bar{\mathcal{Q}} = \mathcal{Q}(\bar{\mathcal{T}})$. By hypothesis, $\bar{\mathcal{M}}$ is $(k, \omega_1 + 1)$-iterable, and since $\bar{\mathcal{T}}$ is self-verifying, this implies that there is a cofinal branch b of $\bar{\mathcal{T}}$ such that $\mathcal{M}_b^{\bar{\mathcal{T}}}$ is $(\deg^{\mathcal{T}}(b), \omega_1 + 1)$-iterable for trees above $\delta(\mathcal{T})$. But the coiteration of $\mathcal{M}_b^{\bar{\mathcal{T}}}$ with $\bar{\mathcal{Q}}$ is above $\delta(\mathcal{T})$, and therefore $\bar{\mathcal{Q}} = \mathcal{Q}(b, \bar{\mathcal{T}})$. A simple absoluteness argument now shows that $P \models$ "there is a cofinal branch b of $\bar{\mathcal{T}}$ such that $\bar{\mathcal{Q}} = \mathcal{Q}(b, \bar{\mathcal{T}})$". Applying π to this statement, we get an immediate contradiction. □

Our next result gives us a model $\mathcal{P}$ such that all iteration trees on $K^c(\mathcal{P})$ have $\mathcal{Q}$-structures.

Theorem 2.9 *There is no infinite sequence $\langle (\mathcal{M}_i, \mathcal{T}_i) \mid i < \omega \rangle$ such that for each $i \geq 0$:*

1. *$\mathcal{M}_i$ is a properly small r-premouse, all of whose countable elementary submodels are $(\omega, \omega_1, \omega_1)^*$-iterable,*
2. *$\mathcal{T}_i$ is a self-verifying tree on $\mathcal{M}_i$ of limit length,*
3. *$\delta(\mathcal{T}_i)$ is a strong cutpoint of $\mathcal{M}_{i+1}$, and $\delta(\mathcal{T}_i) \leq \eta(\mathcal{M}_{i+1})$; moreover $(\mathcal{M}_\infty(\mathcal{T}_i))_0 \trianglelefteq (\mathcal{M}_{i+1})_0$,*
4. *every countable elementary submodel of $((\mathcal{M}_{i+1})_0, \delta(\mathcal{T}_i))$ is $(\omega, \omega_1 + 1)$-iterable, and*

[10]In fact, this shows there is at most one $(\omega, \omega_1 + 1)$-iteration strategy for $\mathcal{M}$. However, there may be distinct $(\omega, \omega_1, \omega_1 + 1)^*$-iteration strategies for a properly small $\mathcal{M}$, since player I may drop gratuitously at the beginning of a new round to a mouse with definably-Woodin cardinals. The fact that there may be distinct such strategies is the cause of some circumlocutions below.

5. $\mathcal{M}_{i+1} \models \delta(\mathcal{T}_i)$ *is Woodin.*

Proof. Suppose $\langle (\mathcal{M}_i, \mathcal{T}_i) \mid i \in \omega \rangle$ is such a sequence. Let $\pi : H \to V_{\Omega+\omega}$ be elementary, where H is countable and transitive, with $\pi((\widehat{\mathcal{M}_i}, \widehat{\mathcal{T}_i})) = (\mathcal{M}_i, \mathcal{T}_i)$ for all i.

Since $\widehat{\mathcal{M}_i}$ is $(\omega, \omega_1, \omega_1 + 1)^*$-iterable, and $\widehat{\mathcal{T}_i}$ is self-verifying, we have a cofinal branch b_i of $\mathcal{T}_i$ such that $\mathcal{M}_{b_i}^{\mathcal{T}_i}$ is $(\deg^{\mathcal{T}_i}(b_i), \omega_1 + 1)$-iterable. Let $Q_i = Q(b_i, \widehat{\mathcal{T}_i})$; this exists because $\widehat{\mathcal{M}_i}$ is properly small. We can now compare $(Q_i)_0$ with $(\widehat{\mathcal{M}_{i+1}})_0$. Since these two premice agree through $\delta(\widehat{\mathcal{T}_i})$, which is Woodin in $\widehat{\mathcal{M}_{i+1}}$, the comparison trees $\mathcal{U}_i$ on Q_i and W_{i+1} on $\widehat{\mathcal{M}_{i+1}}$ have all critical points strictly above $\delta(\widehat{\mathcal{T}_i})$. This means that our assumptions (in particular, assumption (d)) yield enough iterability that the comparison terminates. Let $\mathcal{P}_i$ be the last model of $\mathcal{U}_i$ and $\mathcal{S}_{i+1}$ the last model of W_{i+1}. Since $\delta(\widehat{\mathcal{T}_i})$ is Woodin in $\widehat{\mathcal{M}_{i+1}}$, while $\mathcal{Q}_i$ kills the Woodin property, we have that $\mathcal{S}_{i+1}$ is a proper initial segment of $\mathcal{P}_i$. This implies that W_{i+1} does not drop along the branch leading to $\mathcal{S}_{i+1}$, so that we have a fully elementary map $\sigma_{i+1} \colon \widehat{\mathcal{M}_{i+1}} \to \mathcal{S}_{i+1}$.

Now consider the diagram

$$\begin{array}{ccccccccccccc}
\widehat{\mathcal{M}_0} & \xrightarrow{\widehat{\mathcal{T}}_0} & Q_0 & \xrightarrow{\mathcal{U}_0} & \mathcal{P}_0 & \triangleright & \mathcal{S}_1 & \xrightarrow{\sigma_1\widehat{\mathcal{T}}_1} & & \xrightarrow{\tau\mathcal{U}_1} & & \triangleright & & \bullet\ \bullet\ \bullet \\
 & & & & & & \sigma_1 \uparrow & & \tau \uparrow & & \psi \uparrow & & \uparrow \psi & \\
 & & & & & & \widehat{\mathcal{M}_1} & \xrightarrow{\bar{\mathcal{T}}_1} & Q_1 & \xrightarrow{\mathcal{U}_1} & \mathcal{P}_1 & \triangleright & \mathcal{S}_2 & \bullet\ \bullet\ \bullet \\
 & & & & & & & & & & & & \sigma_2 \uparrow & \\
 & & & & & & & & & & & & \widehat{\mathcal{M}_2} & \\
 & & & & & & & & & & & & & \bullet \\
 & & & & & & & & & & & & & \quad\bullet
\end{array}$$

Here $\sigma_1 \widehat{\mathcal{T}_1}$ is the copied version of $\widehat{\mathcal{T}_1}$ on $\mathcal{S}_1$. The map τ of Q_1 into the last model of $\sigma_1 \widehat{\mathcal{T}_1}$ comes from the copying construction. Similarly, $\tau\mathcal{U}_1$ comes from copying and ψ is the associated map of $\mathcal{P}_1$ into the last model of $\tau\mathcal{U}_1$. These copying constructions succeed, that is, copy the full trees $\widehat{\mathcal{T}_1}$ and $\mathcal{U}_1$, because $\widehat{\mathcal{T}_1}$ and $\mathcal{U}_1$ choose unique cofinal iterable branches at each limit stage. This is true for $\widehat{\mathcal{T}_1}$ because $\widehat{\mathcal{M}_1}$ is properly small, so that $\mathcal{Q}(b, \widehat{\mathcal{T}_1} \restriction \lambda)$ exists for any branch b cofinal in $\lambda < \mathrm{lh}(\widehat{\mathcal{T}_1})$, and $\widehat{\mathcal{T}_1}$ is self-verifying, and so chooses the unique b such that $\mathcal{Q}(b, \widehat{\mathcal{T}_1})$ is iterable. It is true for $\mathcal{U}_1$ because $\mathcal{Q}_1$ projects to $\delta(\widehat{\mathcal{T}_1})$ and $\mathcal{U}_1$ uses only critical points above $\delta(\widehat{\mathcal{T}_1})$, so that once again $\mathcal{Q}(b, \mathcal{U}_1 \restriction \lambda)$ exists for any branch b cofinal in $\lambda < \mathrm{lh}(\mathcal{U}_1)$.

One can fill in the remainder of the diagram similarly. (For example, because $\widehat{\mathcal{T}_2}$ is self-verifying, the copied trees $\sigma_2 \mathcal{T}_2$ and $\psi\sigma_2\mathcal{T}_2$ are according to all $(\omega, \omega, \omega_1)^*$-iteration strategies for their respective base models, and hence the copying process does not break down.) Now across the top row we have, composing the displayed trees, a play of $G^*_\omega(\omega, \omega_1 + 1)$ on $\bar{\mathcal{M}}_0$

according to a winning strategy[11] for player II in which proper drops occur, from $\mathcal{P}_i$ to $\mathcal{Q}_i$, at the beginning of each new round. This is a contradiction. $\square$

Definition 2.10 *A set premouse $\mathcal{P}$ is full iff whenever $\mathcal{M}$ is a k-sound r-premouse over $\mathcal{P}$, and every countable elementary submodel of $\mathcal{R}$ is (k, ω_1+1)-iterable, then $\rho_{k+1}(\mathcal{M}) \geq o(\mathcal{P})$.*

Corollary 2.11 *There is a full premouse $\mathcal{P}$ such that for any tame $\mathcal{P}$-construction $\mathbb{C}$, and any cutpoint η of $\mathcal{N}_\Omega^{\mathbb{C}}$ such that $\forall\delta > \eta(\mathcal{N}_\Omega^{\mathbb{C}} \not\models \delta$ is Woodin $)$, the r-premouse $((\mathcal{N}_\Omega^{\mathbb{C}})_0, \eta)$ is $(\omega, \Omega+1)$-iterable.*

Proof. Suppose not. Let $\mathcal{P}_0 = (V_\omega, \in, \emptyset, \emptyset)$, so that $\mathcal{P}_0$ is full.

Now suppose that we have defined $\mathcal{P}_i$, and that $\mathcal{P}_i$ is full. Let $\mathbb{C}_i$ be a $\mathcal{P}_i$-construction which is a counterexample to the claim, and η_i the relevant cutpoint of $\mathcal{N}_\Omega^{\mathbb{C}_i}$. Let

$$\mathcal{M}_i = ((\mathcal{N}_\Omega^{\mathbb{C}_i})_0, \eta_i).$$

The strategy of iterating $\mathcal{M}_i$ by choosing the unique cofinal branch b of $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$ must fail. Since $\mathcal{M}_i$ is properly small, we then have by 2.5, 2.6, and 2.8 that there is an ω-maximal tree $\mathcal{T}_i$ on $\mathcal{M}_i$ such that

$$\mathcal{T}_i \text{ is self-verifying, and } \mathcal{Q}(\mathcal{T}_i) \text{ does not exist.}$$

We then set

$$\mathcal{P}_{i+1} = \mathcal{M}_\infty(\mathcal{T}_i)_0.$$

We must check that $\mathcal{P}_{i+1}$ is full. Letting $\mathbb{C}$ be a $\mathcal{P}_{i+1}$-construction, and k, ξ such that $\mathfrak{C}_k(\mathcal{N}_\xi)$ exists, we have $o(\mathcal{P}_{i+1}) \leq \rho_{k+1}(\mathcal{N}_\xi)$ because otherwise $\mathcal{Q}(\mathcal{T}_i)$ exists; in fact, $\mathcal{Q}(\mathcal{T}_i) = \mathfrak{C}_k(\mathcal{N}_\xi)$. For the same reason, $\delta(\mathcal{T}_i) = o(\mathcal{P}_{i+1})$ is Woodin in $\mathfrak{C}_{\mathfrak{k}}(\mathcal{N}_\xi)$.

This completes the inductive definition of $\mathcal{P}_i$, $\mathcal{M}_i$, and $\mathcal{T}_i$. It is easy to see that $\langle(\mathcal{M}_i, \mathcal{T}_i) \mid i < \omega\rangle$ is a counterexample to 2.9. This is a contradiction. $\square$

The following lemmas will be useful in showing certain premice are full.

Lemma 2.12 *Suppose W is an $(\omega, \Omega+1)$-iterable r-weasel with cut $\mathcal{P}$ such that W has no Woodin cardinals strictly greater than $o(\mathcal{P})$, and $(\alpha^+)^W = \alpha^+$ for stationarily many $\alpha < \Omega$. Let $\mathcal{R}$ be a k-sound r-premouse with cut $\mathcal{P}$ such that $\mathcal{R} = \mathcal{H}_{k+1}^{\mathcal{R}}(o(\mathcal{P}) \cup \{q\})$ for some q, and every countable elementary submodel of $\mathcal{R}$ is (k, ω_1+1)-iterable. Then $\mathcal{R} \in W$.*

[11] In fact, according to all winning strategies, as we just in effect checked.

Proof. We compare $\mathcal{R}$ with W. On the $\mathcal{R}$ side, our iteration strategy is simply to insure that the tree $\mathcal{T}$ on $\mathcal{R}$ we are building is self-verifying. If this iteration strategy doesn't fail us, then the comparison terminates, and standard arguments give the desired conclusion. So suppose $\mathcal{T}$ is a self-verifying tree of limit length on $\mathcal{R}$ produced by following this strategy in its comparison with W, and that the comparison has not yet terminated. Let S be the current model in the tree on W. Notice that $\mathcal{M}_\infty(\mathcal{T})$ is an initial segment of S, and either $\delta(\mathcal{T})$ is not Woodin in S, or the branch W-to-S has dropped. It follows that for some $\mathcal{I} \trianglelefteq (S)_0, (\mathcal{I}, \delta(\mathcal{T}))$ kills the Woodin property. But then $(\mathcal{I}, \delta(\mathcal{T})) = \mathcal{Q}(\mathcal{T})$, so $\mathcal{Q}(\mathcal{T})$ exists, and the proof of 2.8 gives a cofinal branch b of $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$.

We must also show that no self-verifying tree on $\mathcal{R}$ can go bad at a successor step, by giving rise to an illfounded ultrapower. This is done by a straightforward Lowenheim-Skolem argument; we leave the details to the reader. We have shown then that the $\mathcal{R}$ versus W comparison does not break down, and this proves the lemma. □

Lemma 2.13 *Let $\mathbb{C}$ be a tame $\mathcal{P}$-construction which is θ-maximal, for some θ. Suppose that there are arbitrarily large $\delta < \Omega$ such that $\mathcal{N}_\Omega^{\mathbb{C}} \models \delta$ is Woodin. Then there are arbitrarily large $\delta < \Omega$ such that $\mathcal{N}_\Omega^{\mathbb{C}} \models \delta$ is Woodin, and $\mathcal{J}_\gamma^{\mathcal{N}_\Omega^{\mathbb{C}}}$ is full, where $\gamma = (\delta^+)^{\mathcal{N}_\Omega^{\mathbb{C}}}$.*

Proof. Let $\delta > o(\mathcal{P})$ be Woodin in $\mathcal{N} = \mathcal{N}_\Omega^{\mathbb{C}}$. It will be enough to find an $\eta \geq \delta$ which is Woodin in $\mathcal{N}$, and such that $\mathcal{J}_\gamma^{\mathcal{N}}$ is full, where $\gamma = (\eta^+)^{\mathcal{N}}$. We are done if we can take $\eta = \delta$, so assume not. This means that, setting $\gamma_0 = (\delta^+)^{\mathcal{N}}$, we have a $k+1$-sound r-premouse $\mathcal{R}$ over $\mathcal{J}_{\gamma_0}^{\mathcal{N}}$ such that $\rho_{k+1}(\mathcal{R}) < \gamma_0$ and all countable elementary submodels of $\mathcal{R}$ are $(k, \omega_1 + 1)$-iterable. Since $\mathcal{R}$ collapses γ_0, and $\mathcal{N}$ is universal (by cheapo covering), the attempt to compare the two must fail. We now consider how that can happen.

Let us attempt to compare $\mathcal{R}$ with $(\mathcal{N}, \gamma_0)$, choosing branches b of the tree $\mathcal{T}$ on $\mathcal{R}$ we are building so that $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$ for as long as possible.

Claim. $(\mathcal{N}, \gamma_0)$ never moves in this comparison with $\mathcal{R}$.

Proof. Suppose otherwise, and letting ξ be such that $\mathcal{N}_\xi^{\mathbb{C}} = \mathcal{J}_{\gamma_0}^{\mathcal{N}}$, let (τ, i) be lexicographically least such that $(\mathfrak{C}_i(\mathcal{N}_\tau^{\mathbb{C}}), \gamma_0)$ moves in its comparison with $\mathcal{R}$, when $\mathcal{R}$ is iterated by choosing the unique cofinal b such that $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$, before that prescription for iterating $\mathcal{R}$ breaks down.

Note that $i = 0$, as otherwise $\mathfrak{C}_{i-1}(\mathcal{N}_\tau)$ is an iterate of $\mathcal{R}$ via a k-maximal tree $\mathcal{U}$. ($\mathcal{R}$ cannot come out shorter because it defines a subset of γ_0 not in $\mathcal{N}$). But then $\mathfrak{C}_i(\mathcal{N}_\tau)$ occurs along the branch $\mathcal{R}$-to-$\mathfrak{C}_{i-1}(\mathcal{N}_\tau)$ of $\mathcal{U}$, either as its last model, or at the site of its last drop. Thus some initial segment of $\mathcal{U}$ constitutes its comparison with $\mathfrak{C}_i(\mathcal{N}_\tau)$, a contradiction.

Let $\mathcal{T}$ be the iteration tree on $\mathcal{R}$ produced by comparing it with $\mathcal{N}_\tau$, up to the stage at which the first extender is used on the $\mathcal{N}_\tau$ side. Since τ is minimal, this extender is the last extender F of $\mathcal{N}_\tau$. Let F be a (κ, λ) extender. Since $|\mathcal{R}| < \kappa$, we have

$$\mathrm{lh}(\mathcal{T}) := \alpha + 1 > \kappa,$$

and setting
$$\mathcal{A} = V_{\kappa+1} \cap \mathcal{M}_\alpha^{\mathcal{T}} = V_{\kappa+1} \cap \mathcal{M}_\kappa^{\mathcal{T}} = V_{\kappa+1} \cap \mathcal{N}_\tau,$$
we have that
$$|\mathcal{A}| \leq \kappa.$$
Let (N, G) be an $\mathcal{A}$-certificate for F, that is, let (N, G) satisfy clauses (1)-(4) of case 1 in our description of K^c-constructions. Let
$$j \colon N \to \mathrm{ult}(N, G) := P$$
be the canonical embedding. Since $V_\kappa \cup \{\mathcal{J}_\kappa^{\mathcal{N}_\tau}\} \subseteq N$ and the prescription for choosing branches of $\mathcal{T}$ is local, $\mathcal{T} \restriction \kappa \in N$. Since $\mathcal{Q}(\mathcal{T} \restriction \kappa) \in \mathcal{A} \subseteq N$, we have $\mathcal{T} \restriction \kappa + 1 \in N$. Set
$$\mathcal{S} = j(\mathcal{T} \restriction \kappa + 1).$$
Thus in P, the tree on $\mathcal{R}$ of length $j(\kappa+1)$ which comes from comparing $\mathcal{R}$ with $j(\mathcal{J}_\kappa^{\mathcal{N}_\tau})$ and choosing branches with iterable $\mathcal{Q}$-structures is $\mathcal{S}$. Because $j(\mathcal{J}_\kappa^{\mathcal{N}_\tau})$ agrees with $\mathcal{N}_\tau$ below $\mathrm{lh}(F)$ and P is correct about the iterability of all countable elementary submodels of what it thinks are $\mathcal{Q}$-structures, $\mathcal{T}$ is an initial segment of $\mathcal{S}$.

Note that $[0, \kappa)_T \subseteq [0, j(\kappa)]_S$, so that $\kappa \in [0, j(\kappa)]_S$. If $x \in \mathcal{M}_\kappa^{\mathcal{T}}$, say
$$x = i_{\beta,\kappa}^{\mathcal{T} \restriction \kappa+1}(y),$$
then
$$j(x) = i_{\beta, j(\kappa)}^{\mathcal{S}}(y) = i_{\kappa, j(\kappa)}^{\mathcal{S}}(x).$$
It follows F is an initial segment of the extender derived from $i_{\kappa,j(\kappa)}^{\mathcal{S}}$. Letting $\beta + 1$ be the S-successor of κ on $[0, j(\kappa)]_S$, and $E = E_\beta^{\mathcal{S}}$, we then have that E and F are compatible. We must have $\beta \geq \alpha$, as otherwise E is used in $\mathcal{T}$, but at the same time is on the $\mathcal{N}_\tau$-sequence or an ultrapower away from it by the initial segment condition on F. Thus $E \restriction \nu = F \restriction \nu$, where $\nu = \nu(F)$, so that F is the trivial completion of $E \restriction \nu$. But F must be on the $\mathcal{M}_\beta^{\mathcal{S}}$-sequence, since otherwise it is on the sequence of $\mathrm{ult}(\mathcal{J}_\nu^{\mathcal{M}_\beta^{\mathcal{S}}}, H)$, where $H = E_\nu^{\mathcal{M}_\beta^{\mathcal{S}}}$. Since $\mathcal{M}_\beta^{\mathcal{S}}$, $\mathcal{N}_\tau$, and $\mathrm{ult}(N_\tau, F)$ agree below $\mathrm{lh}(F)$, we get that $F \in \mathrm{ult}(\mathcal{N}_\tau, F)$, a contradiction. Since F is on the $\mathcal{M}_\beta^{\mathcal{S}}$-sequence, and $\mathrm{lh}(E_\gamma^{\mathcal{S}}) \geq \mathrm{lh}(F)$ for all $\gamma \geq \alpha$, we have that F is on the $\mathcal{M}_\alpha^{\mathcal{S}}$-sequence. But $\mathcal{M}_\alpha^{\mathcal{S}} = \mathcal{M}_\alpha^{\mathcal{T}}$, so F was not part of a disagreement at stage α, contradiction. □

Thus the comparison of $\mathcal{R}$ with $(\mathcal{N}, \gamma_0)$ breaks down because we produce a self-verifying tree $\mathcal{T}$ on $\mathcal{R}$ of limit length which has no cofinal branch b such that $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$. By 2.8, we have that $\mathcal{Q}(\mathcal{T})$ does not exist. Now since $(\mathcal{N}, \gamma_0)$ has not moved in this comparison, we have that $\mathcal{M}_\infty(\mathcal{T}) = (\mathcal{J}_\delta^{\mathcal{N}}, \gamma_0)$, where $\delta = \delta(\mathcal{T})$. Clearly δ is Woodin in $\mathcal{N}$, as otherwise some initial segment of $\mathcal{N}$ would serve as (the reduct of) $\mathcal{Q}(\mathcal{T})$. Let γ_1 be the cardinal successor of δ in $\mathcal{N}$. If $\mathcal{J}_{\gamma_1}^{\mathcal{N}}$ is full, we are done, so assume otherwise. This gives us an r-premouse $\mathcal{R}_1$

over $\mathcal{J}_{\gamma_1}$. Once again, $\mathcal{R}_1$ and $\mathcal{N}$ cannot be compared. By attempting to do so, we produce a tree $\mathcal{T}_1$ on $\mathcal{R}_1$, etc.

We now proceed to a contradiction as in the proof of 2.9. Let S be the transitive collapse of a countable elementary submodel of some large V_σ. Let $\bar{\mathcal{R}}_i$ be the image under collapse of $\mathcal{R}_i$, etc. It is not hard to see that the $\bar{\mathcal{R}}_i$ constitute an infinite descending chain in the mouse order. The reason is just that $\mathcal{Q}(b_i, \bar{\mathcal{T}}_i) = \mathcal{Q}(\bar{\mathcal{T}}_i)$ does exist for the unique cofinal, realizable branch of $\bar{\mathcal{T}}_i$ by our iterability hypothesis on $\mathcal{R}_i$. We have that $\mathcal{Q}(b_i, \bar{\mathcal{T}}_i)$ is less than or equal to $\bar{\mathcal{R}}_i$ in the mouse order. But no initial segment of $\mathcal{R}_{i+1}$ kills the Woodin property of $\delta(\mathcal{T}_i)$, so $\bar{\mathcal{R}}_{i+1}$ is a proper initial segment of $\mathcal{Q}(b_i, \bar{\mathcal{T}}_i)$.

This contradiction completes the proof. $\square$

Definition 2.14 *A set premouse $\mathcal{P}$ is* excellent *iff for all sufficiently large $\theta < \Omega$:*

- *$\mathcal{P}$ is full,*
- *$K^c(\mathcal{P}, \theta)$ exists, and is $(\omega, \omega, \Omega + 1)$-iterable, and*
- *$K^c(\mathcal{P}, \theta) \models$ there are no Woodin cardinals above $o(\mathcal{P})$.*

In order to prove that there are excellent premice, we shall need an anti-large-cardinal hypothesis.

Definition 2.15 ($\dagger$) *is the assertion: for any $\theta < \Omega$, there is a set premouse $\mathcal{P}$ and a $\mathcal{P}$-construction $\mathbb{C}$ of additivity at least θ, such that either $\mathbb{C}$ is not tame, or for arbitrarily large $\delta < \Omega$, $\mathcal{N}_\Omega^{\mathbb{C}} \models \delta$ is Woodin , and $\mathcal{J}_\gamma^{\mathcal{N}_\Omega^{\mathbb{C}}}$ is full, where $\gamma = (\delta^+)^{\mathcal{N}_\Omega^{\mathbb{C}}}$.*

Theorem 2.16 *If* ($\dagger$) *is false, then there is an excellent set premouse.*

Proof. We first show that there is a $\mathcal{P}$ which is *pretty good*, in that it has all the properties of an excellent mouse, but with the iterability condition weakened to $(\omega, \Omega + 1)$-iterability. We then show that any pretty good mouse is excellent.

Let θ_0 be such that ($\dagger$) fails for $\theta = \theta_0$. For any $\mathcal{P}$-construction $\mathbb{C}$ we set

$$\kappa(\mathbb{C}) = \sup \{ \delta \mid \mathcal{N}_\Omega^{\mathbb{C}} \models \delta \text{ is Woodin } \},$$

where we take the sup to be $o(\mathcal{P})$ if there are no such δ. Our anti-large cardinal hypothesis implies $\kappa(\mathbb{C}(\mathcal{P}, \theta)) < \Omega$, for all $\mathbb{C}$ with additivity at least θ_0.

By 2.11, we can fix a full premouse $\mathcal{P}_0$ such that for any $\mathcal{P}_0$-construction $\mathbb{C}$ such that $\kappa(\mathbb{C}) < \Omega$, the r-premouse $((\mathcal{N}_\Omega^{\mathbb{C}})_0, \kappa(\mathbb{C}))$ is $(\omega, \Omega + 1)$-iterable.

Claim 1. There is a pretty good premouse.

Proof. Assume not. We define by induction on α set premice $\mathcal{P}_\alpha$ and initial segments of θ_0-additive $\mathcal{P}_0$-constructions $\mathbb{C}_\alpha$ such that for all $\alpha > 0$:

- $\mathcal{P}_\alpha$ is full, and $\mathcal{P}_\alpha \models$ "there is a largest cardinal",
- $\beta < \alpha \to \mathcal{P}_\beta$ is a strong cutpoint of $\mathcal{P}_\alpha$,
- $\mathcal{P}_\alpha$ is the last model of $\mathbb{C}_\alpha$, and if $\beta < \alpha$, then $\mathbb{C}_\beta = \mathbb{C}_\alpha \restriction \mathrm{lh}(\mathbb{C}_\beta)$.

Suppose $\mathcal{P}_\alpha$ and $\mathbb{C}_\alpha$ are as above; we shall define $\mathcal{P}_{\alpha+1}$ and $\mathbb{C}_{\alpha+1}$. We claim first that if $\mathbb{C}$ is any θ_0-additive $\mathcal{P}_\alpha$-construction such that $\kappa(\mathbb{C}) < \Omega$, then $((\mathcal{N}_\Omega^{\mathbb{C}})_0, \kappa(\mathbb{C}))$ is $(\omega, \Omega+1)$-iterable. To see this, fix such a $\mathbb{C}$, and let $\mathbb{D} = \mathbb{C}_\alpha$ if $\alpha > 0$, and $\mathbb{D} = \emptyset$ otherwise. The concatenation $\mathbb{D}*\mathbb{C}$ is equivalent to a $\mathcal{P}_0$-construction with last model $((\mathcal{N}_\Omega^{\mathbb{C}})_0, o(\mathcal{P}_0))$, because the relativised cores taken in $\mathbb{C}$ correspond exactly to the cores which would be taken in a $\mathcal{P}_0$-construction extending $\mathbb{D}$. (The reason is that $\mathcal{P}_\alpha$ is full, so all projecta of $\mathbb{C}$ are at least $o(\mathcal{P}_\alpha)$; moreover $\mathcal{P}_\alpha$ has a largest cardinal, so that $o(\mathcal{P}_\alpha) \in \mathcal{H}$, whenever $\mathcal{H}$ is one of the hulls used to form a core in the $\mathcal{P}_0$-construction equivalent to $\mathbb{D} * \mathbb{C}$.) It follows from our choice of $\mathcal{P}_0$ that $\mathcal{N}_\Omega^{\mathbb{C}}$ is $(\omega, \Omega+1)$-iterable.

Since $\mathcal{P}_\alpha$ is not pretty good, there must then be some $\theta > \theta_0$ such that $K^c(\mathcal{P}_\alpha, \theta)$ has Woodin cardinals above $o(\mathcal{P}_\alpha)$. Let ν be the cardinal successor of the sup of the Woodin cardinals in $K^c(\mathcal{P}_\alpha, \theta)$, and set

$$\mathcal{P}_{\alpha+1} = ((\mathcal{J}_\nu^{K^c(\mathcal{P}_\alpha,\theta)})_0, o(\mathcal{P}_0)).$$

Let $\mathbb{C}_{\alpha+1}$ be the initial segment of a $\mathcal{P}_0$-construction equivalent to $\mathbb{D} * \mathbb{C}(\mathcal{P}_\alpha, \theta)$ whose last model is $\mathcal{P}_{\alpha+1}$.

It is easy to verify our induction hypotheses. To see that $\mathcal{P}_{\alpha+1}$ is full, we use 2.12 with $W = (K^c(\mathcal{P}_\alpha, \theta)_0, o(\mathcal{P}_{\alpha+1}))$, which is $(\omega, \Omega+1)$-iterable by the argument given two paragraphs ago.

Suppose now λ is a limit, and we have defined $\mathcal{P}_\alpha$ and $\mathbb{C}_\alpha$ for $\alpha < \lambda$. Let $\mathcal{R}$ be the unique r-premouse whose proper initial segments are just the proper initial segments of the $\mathcal{P}_\alpha$ for $\alpha < \lambda$. Let $\mathbb{C}$ be a θ_0-additive $\mathcal{P}_0$-construction extending the union of the $\mathbb{C}_\beta$ for $\beta < \alpha$ which is maximal (i.e. adds extenders whenever possible) subject to this constraint. Letting $\gamma = \bigcup_{\alpha<\lambda} \mathrm{lh}(\mathbb{C}_\alpha)$, we have that $\mathcal{R} = \mathcal{N}_\gamma^{\mathbb{C}}$. We can now show by induction on ξ that $o(\mathcal{R})$ is a cutpoint of $\mathcal{N}_\xi^{\mathbb{C}}$ and $\rho_\omega(\mathcal{N}_\xi) \geq o(\mathcal{R})$, for all $\xi > \gamma$. The former implies the latter because $\mathcal{P}_\alpha$ is full for all $\alpha < \lambda$. But if the latter holds at all $\eta < \xi$, then $\mathcal{R}$ is an initial segment of $\mathcal{N}_\xi$, and $o(\mathcal{R})$ is a cardinal of $\mathcal{N}_\xi^{\mathbb{C}}$. It follows from our construction that $\mathcal{R}$, and hence $\mathcal{N}_\xi$, satisfies "$o(\mathcal{R})$ is a limit of Woodin cardinals". Since $\mathcal{N}_\xi^{\mathbb{C}}$ is tame, $o(\mathcal{R})$ is a cutpoint of $\mathcal{N}_\xi^{\mathbb{C}}$.[12]

We now proceed as in the successor case. Let ν be the cardinal successor of $\kappa(\mathbb{C})$ in $\mathcal{N}_\Omega^{\mathbb{C}}$. (Notice that $o(\mathcal{R}) \leq \kappa(\mathbb{C})$.) We let $\mathcal{P}_\lambda$ be $\mathcal{J}_\nu^{\mathbb{C}}$, and we let $\mathbb{C}_\lambda$ be the initial segment of $\mathbb{C}$

[12] We don't know that all the anti-core embeddings of $\mathcal{N}_\xi^{\mathbb{C}}$ fix $o(\mathcal{R})$, so we cannot interpret the part of $\mathbb{C}$ beyond γ as an $\mathcal{R}$-construction.

whose last model is $\mathcal{P}_\lambda$. We have that $\mathcal{P}_\lambda$ is full for the same reason as in the successor case, and the remaining induction hypotheses are trivial.

But now the union $\mathbb{C}$ of the $\mathbb{C}_\alpha$, for $\alpha < \Omega$, is a θ_0-additive $\mathcal{P}_0$-construction such that for arbitrarily large $\delta < \Omega$, $\mathcal{N}_\Omega^\mathbb{C}$ satisfies "δ is Woodin ", and $\mathcal{J}_\gamma^{\mathcal{N}_\Omega^\mathbb{C}}$ is full, where $\gamma = (\delta^+)^{\mathcal{N}_\Omega^\mathbb{C}}$. This contradicts our anti-large cardinal hypothesis. □

We shall now show that any pretty good premouse is excellent. The argument contains information on the generic absoluteness of iterability which we shall need later, so we state it with that in mind.

Claim 2. Let $\mathcal{P}$ be pretty good, and let W be a properly small, $(\omega, \Omega+1)$-iterable r-weasel over $\mathcal{P}$. Let G be generic over V for a partial order of size strictly less than Ω; then W is $(\omega, \omega_1, \Omega+1)$-iterable in $V[G]$.

Proof. Let $\mathcal{P}$, W, and G be as in the hypotheses. We shall actually only show that W has a slightly weakened form of $(\omega, \omega_1, \Omega+1)$-iterability in $V[G]$; namely, we shall only consider plays of the iteration game in which whenever R is the base model on which one of the component rounds is to be played, then R is a weasel, and there is an iteration map $i \colon W \to R$ provided by the previous rounds. The advantage here is that R is then properly small, so we can cite 2.5 for trees on R. The same basic ideas, combined with significantly more fine-structural detail, will yield the full claim 3.[13] Let us call a winning strategy for II in this weaker game a *quasi-$(\omega, \omega_1, \Omega+1)$-iteration strategy* . We shall also need to consider the same restriction on the $(\omega, \omega, \kappa^+ +1)$ iteration game as played on a properly small initial segment $\mathcal{J}_\kappa^W$ of W. There again, the restriction on I is that there be iteration maps from $\mathcal{J}_\kappa^W$ to the base models of each later round.

Subclaim A. Let κ be a sucessor cardinal of W above $o(\mathcal{P})$, and let $\theta < (\kappa^+)^{V[G]}$. Let Σ and Λ be quasi-$(\omega, \omega_1, \kappa^+ + 1)$ iteration strategies for $\mathcal{J}_\kappa^W$ in $V[G]$. Then Σ and Λ agree on positions of length $< (\kappa^+)^{V[G]}$.

Proof. Work in $V[G]$. If the claim fails, then Σ and Λ must diverge by reaching a common R as the base model for some round, and a common ω-maximal iteration tree $\mathcal{T}$ on R of limit length $\lambda < \kappa^+$, and choosing distinct cofinal branches b and c of $\mathcal{T}$. But then $\mathcal{Q}(b, \mathcal{T})$ and $\mathcal{Q}(c, \mathcal{T})$ exist by 2.5, and $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(c, \mathcal{T})$ because they are iterable enough to be compared (see 2.6). So $b = c$, contradiction. □

Subclaim B. For any successor cardinal $\kappa \geq o(\mathcal{P})$ of W, and any $\mu < \Omega$, $\mathcal{J}_\kappa^W$ is quasi-(ω, ω_1, μ)- iterable in $V[G]$.

Proof. Fix κ, and let $\theta \geq \max(\kappa, \mu)$ be large enough that $K^c(\mathcal{P}, \theta)$ is $(\omega, \Omega+1)$-iterable. Working in V, let us compare $\mathcal{J}_\kappa^W$ with $K^c(\mathcal{P}, \theta)$. Since both structures are $(\omega, \Omega+1)$-iterable in V, we can do this. Since $K^c(\mathcal{P}, \theta)$ computes successor cardinals correctly stationarily often

[13]The weaker iterability we are considering now is all that is used in §3 - §5 of [11] anyway.

below Ω, the $K^c(\mathcal{P},\theta)$ side comes out longer. That is, we have an ω-maximal iteration tree $\mathcal{T}$ on $K^c(\mathcal{P},\theta)$, having length $\gamma+1$, and an elementary embedding σ of $\mathcal{J}^W_\kappa$ into an initial segment of $\mathcal{M}^{\mathcal{T}}_\gamma$.

Still working in V, let $\pi : N \to V_{\Omega+\omega}$ be elementary, where N is transitive and of cardinality κ, and $\kappa+1 \subseteq N$ and $\pi \restriction (\kappa+1) =$ identity, and all objects mentioned in the last paragraph are in ran π. Let $\pi(\bar{\mathcal{T}}) = \mathcal{T}$, $\pi(\bar{\sigma}) = \sigma$, etc. Let $\pi(\mathcal{R}) = K^c(\mathcal{P},\theta)$. Thus $\bar{\mathcal{T}}$ is an ω-maximal iteration tree on $\mathcal{R}$, which is properly small. Moreover, $\bar{\sigma}$ is an elementary embedding from $\mathcal{J}^W_\kappa$ into an initial segment of the last model of $\bar{\mathcal{T}}$.

Let H be $V[G]$-generic over $\mathrm{Col}(\omega,\theta)$. Since $\bar{\mathcal{T}}$ and $\mathcal{R}$ are countable in $V[G][H]$, by 1.9 either the last model of $\bar{\mathcal{T}}$ is realizable in a model of the θ-maximal $\mathcal{P}$-construction $\mathbb{C}$ which produced $K^c(\mathcal{P},\theta)$, or there is a maximal branch b of $\bar{\mathcal{T}}$ which is so realizable. We claim the latter cannot happen. For otherwise, working back in V we can find a $\psi\colon S \to V_\eta$, where S is countable and transitive, and everything of interest is in the range of ψ. Let $\psi(\mathbb{C}^*) = \mathbb{C}, \psi(\mathcal{T}^*) = \mathcal{T}$, etc. Let G^* and H^* be generic over S for the appropriate forcings, and so that in $S[G^*][H^*]$ we have a maximal branch b^* of $\mathcal{T}^*$ which is realizable in some model of $\mathbb{C}^*$ via some τ. Let $\lambda^* = \sup(b^*)$, and $c^* = [0,\lambda^*]_{T^*}$. The existence of $\psi\colon \mathcal{Q}(c^*,\mathcal{T}^* \restriction \lambda^*) \to \mathcal{Q}(c,\mathcal{T} \restriction \lambda)$ and the $(\deg(c^*),\Omega+1)$-iterabilty of $\mathcal{Q}(c,\mathcal{T} \restriction \lambda)$[14] imply that $\mathcal{Q}(c^*,\mathcal{T}^* \restriction \lambda^*)$ is $(\deg(c^*),\omega_1+1)$- iterable. But $\mathcal{Q}(b^*,\mathcal{T}^* \restriction \lambda^*)$ is $(\deg(b^*),\omega_1+1)$-iterable, since it is realized in a model of $\mathbb{C}$ by $\psi \circ \tau$. We then have $\mathcal{Q}(c^*,\mathcal{T}^* \restriction \lambda^*) = \mathcal{Q}(b^*,\mathcal{T}^* \restriction \lambda^*)$ by comparison, so $b^* = c^*$, a contradiction.

Returning to $V[G][H]$, we have a map τ realizing $\mathcal{M}^{\bar{\mathcal{T}}}_\gamma$ in a model of $\mathbb{C}$. But then $\tau \circ \bar{\sigma}$ realizes $\mathcal{J}^W_\kappa$ in $\mathbb{C}$. Thus, by 1.10, $\mathcal{J}^W_\kappa$ is $(\omega,\omega_1,\omega_1+1)$-iterable in $V[G][H]$. The restriction of any iteration strategy witnessing this to plays of the "quasi" game in $V[G]$ is itself in $V[G]$, and witnesses the truth of subclaim B. $\square$

It is easy now to piece together the strategies given by subclaim B, in a way which is coherent by subclaim A, and thereby obtain a quasi- $(\omega,\omega_1,\Omega+1)$ iteration strategy for W in $V[G]$. $\square$

Definition 2.17 *Let $\mathcal{M}$ be an r-premouse; then we say $\mathcal{M}$ is very sound iff there is an r-weasel W such that $\mathcal{M} \trianglelefteq W$ and*

(a) W is properly small and $(\omega,\omega_1,\Omega+1)$-iterable,

(b) Ω is thick in W, and

(c) whenever Γ is thick in W, we have $\mathcal{M} \subseteq H^W(\eta(W) \cup \Gamma)$.

Here "$H^W(\eta(W) \cup \Gamma)$" denotes the uncollapsed hull. The notion of a thick class is defined in [11], §3, and it goes over to r-weasels verbatim.

[14]Since trees on $\mathcal{Q}(c,\mathcal{T} \restriction \lambda)$ must be above $\delta(\mathcal{T} \restriction \lambda)$, they can be interpreted as continuations of $\mathcal{T} \restriction \lambda$ in the one-round iteration game.

Theorem 2.18 *For any set premouse $\mathcal{P}$, there is at most one r-weasel S with lower part $\mathcal{P}$ such that $\mathcal{J}_\alpha^S$ is very sound for all $\alpha < \Omega$. If $\mathcal{P}$ is excellent, then there is at least one such W.*

Proof. As in §5 of [11]. □

Definition 2.19 *If there is an r-weasel S with lower part $\mathcal{P}$ such that $\mathcal{J}_\alpha^S$ is very sound for all $\alpha < \Omega$, then we let $K(\mathcal{P})$ be the unique such weasel.*

Theorem 2.20 *Let $\mathcal{P}$ be excellent; then*

(a) $K(P) \preceq K^c(\mathcal{P}, \theta)$ *for all* θ,

(b) $(\alpha^+)^{K(\mathcal{P})} = \alpha^+$ *for almost every (in any normal measure)* $\alpha < \Omega$.

Proof. As in §5 of [11]. □

By 2.20, if $\mathcal{P}$ is excellent then $K(\mathcal{P})$ is $(\omega, \omega_1, \Omega + 1)$-iterable.

We do not know whether excellence is preserved by forcing of size $< \Omega$, although we suspect so.[15] For our purposes, however, the following is enough.

Theorem 2.21 *Let $\mathcal{P}$ be excellent in V, and let G be V-generic over a partial order of size strictly less than Ω. Then $V[G] \vDash K(\mathcal{P})$ exists, and $K(\mathcal{P})^V = K(\mathcal{P})^{V[G]}$.*

Proof. Let $\mathcal{M}$ be a proper initial segment of $K(\mathcal{P})^V$ It will be enough to show $\mathcal{M}$ is very sound in $V[G]$. Let $W \in V$ be an r-weasel which witnesses in V that $\mathcal{M}$ is very sound. It is clear that W continues to witness this in $V[G]$, provided that W is $(\omega, \omega_1, \Omega + 1)$-iterable in $V[G]$. But this follows at once from claim 3 in the proof of 2.16. □

3 The strength of $L(\mathbb{R})$ generic absoluteness

In this section we prove

Theorem 3.1 *Let Ω be measurable. Then the following are equivalent:*

(a) for all posets $\mathbb{P}, \mathbb{Q} \in \mathbb{V}_\Omega$, $L(\mathbb{R})^{V^{\mathbb{P}}} \equiv L(\mathbb{R})^{V^{\mathbb{Q}}}$,

[15]Ralf Schindler has shown that indeed it is. For let $\mathcal{P}$ be excellent, and work in $V[G]$, where G is $< \Omega$-generic over V. By 2.21, $K(\mathcal{P})$ still exists, and is $\Omega + 1$-iterable. This implies $\mathcal{P}$ is still full. Moreover, any $K^c(\mathcal{P}, \theta)$ is an iterate of $K(\mathcal{P})$ by the argument of [8]. This implies that $K^c(\mathcal{P}, \theta)$ is $(\omega, \omega, \Omega + 1)$-iterable and has no Woodin cardinals above its cut, as desired.

(b) *for every poset* $\mathbb{P} \in V_\Omega$, $V^{\mathbb{P}} \models AD^{L(\mathbb{R})}$,

(c) *for every poset* $\mathbb{P} \in V_\Omega$, $V^{\mathbb{P}} \models$ *all sets of reals in* $L(\mathbb{R})$ *are Lebesgue measurable,*

(d) *for every poset* $\mathbb{P} \in \mathbb{V}_\Omega$, $\mathbb{V}^{\mathbb{P}} \models$ *there is no uncountable sequence of distinct reals in* $L(\mathbb{R})$,

(e) *there is an* $(\omega, \Omega+1)$*-iterable r-weasel which satisfies "there are infinitely many Woodin cardinals".*

One could lengthen this list of equivalents, for example by replacing Lebesgue measurability with the Baire or Ramsey or Perfect Set properties. It is easy to see, using 3.2, that 3.1 implies the first theorem we stated in the introduction to this paper.

Proof. Clearly $(b) \Rightarrow (c) \Rightarrow (d)$. It is also easy to see that $(a) \Rightarrow (c)$, since if we take $\mathbb{P}$ to be the Levy collapse below the least inaccessible, then in $V^{\mathbb{P}}$, all sets in $L(\mathbb{R})$ are Lebesgue measurable. By (a) we then have that in all $V^{\mathbb{P}}$, for $\mathbb{P} \in \mathbb{V}_\Omega$, it is true that all sets of reals in $L(\mathbb{R})$ are Lebesgue measurable.

That (e) implies (a) and (b) will follow at once from work of Woodin and the author (see [12]), once we have the following lemma.

Lemma 3.2 *If (e) holds, then* $M_\omega^\sharp$ *exists; moreover, setting* $\mathcal{P} = M_\omega^\sharp$, *and letting* G *be generic over* V *for a poset in* V_Ω, *we have* $V[G] \models \mathcal{P}$ *is* $(\omega, \Omega+1)$*-iterable. In particular,* $(M_\omega^\sharp)^V = (M_\omega^\sharp)^{V[G]}$.

Proof. Let N be the r-weasel given by (e).

If $x \in V_\Omega$ is transitive, then we can iterate N so as to make x generic over the iterate for the extender algebra at the bottom Woodin above its cut. This gives a wellfounded model of height Ω which believes that $x^\sharp$ exists. Therefore $x^\sharp$ does exist.[16]

The unrelativised K^c construction done in N, using full background extenders over N which are on the sequence of N above its cut, yields a weasel M which is ω-small and satisfies "there are ω Woodin cardinals". (See [12].) Because we used full background extenders, iterations of M can be lifted to iterations of N, so that M is $(\omega, \Omega+1)$-iterable. But $M = L[\vec{E}]$, where $\vec{E} \in V_\Omega$ and hence $\vec{E}^\sharp$ exists. If $i\colon M \to P$ comes from an iteration tree on M of length strictly less than Ω, then i has a stationary class of fixed points, and hence $i(\vec{E}^\sharp) = i(\vec{E})^\sharp$. So if we let $\alpha = (c^{++})^M$ where c is an $L[\vec{E}]$-indiscernible, and $\mathcal{P} = (\mathcal{J}_\alpha^M, F)$ where F is the (c, α) extender over M coming from $\vec{E}^\sharp$, then $M_\omega^\sharp$ is just the Σ_1 Skolem hull of P, and $M_\omega^\sharp$ is $(\omega, \Omega+1)$-iterable. Since $M_\omega^\sharp$ has projectum ω, it must then have exactly one $(\omega, \Omega+1)$ iteration strategy (namely, choosing the unique b such that $\mathcal{Q}(b, \mathcal{T})$ exists and is $(\deg(b), \Omega+1)$-iterable). We call this strategy Σ.

[16]That $x^\sharp$ exists follows from the measurability of Ω, but it is of some interest that we can prove 3.2 without using the measurability of Ω.

The idea of our absoluteness-of-iterability argument is just that $\Sigma \restriction \bar{V}[\bar{G}]$ is definable over $\bar{V}[\bar{G}]$ whenever $\bar{V}$ is a countable elementary submodel of V and $\bar{G}$ is generic over it. Because the definition is uniform over all $\bar{V}, \bar{G}$, we can use it in any $V[G]$ as well.

More precisely, let G be generic over V for a poset in V_Ω; we claim that the following putative (ω, Ω) iteration strategy for $M_\omega^\sharp$ works in $V[G]$. Suppose $\mathcal{T}$ is a tree of limit length which has been produced so far. Let $\mathcal{T} \in V_\alpha[G]$, where $\alpha < \Omega$. Going back to V, iterate $M_\omega^\sharp$ so as to make V_α generic over the image of the extender algebra at the bottom Woodin cardinal. Let $i \colon M_\omega^\sharp \to S$ be the iteration map. We have $V_\alpha \subseteq S[H]$, where H is the generic object on the extender algebra, so that G is generic over $S[H]$, and $V_\alpha^{S[H][G]} = V_\alpha[G]$. Now, working in $S[H][G]$, do the full background extender version of the $M_\infty(\mathcal{T})$-construction, using background extenders from the S-sequence above the bottom Woodin cardinal of S. (Note that these extenders have canonical completions measuring all sets in $S[H][G]$.) This gives us an r-weasel W over $M_\infty(\mathcal{T})$ which is ω-small and satisfies "there are ω Woodin cardinals above $\delta(\mathcal{T})$". Let $\mathcal{R}$ be the Σ_1 Skolem hull of $(\mathcal{J}_\gamma^W, F)$, where $\gamma = (c^{++})^W$ for some W-indiscernible c, and F is the (c, γ) extender over W coding its sharp. Our iteration strategy now is: choose the unique cofinal branch b of $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T})$ exists, and is an initial segment of $\mathcal{R}$.

We must show that our b exists, and is independent of all the choices which led to it. What we must show is a first order assertion about V, via the forcing relation associated to G, so it is enough to show that if $\pi \colon \bar{V} \to V$ is elementary, where $\bar{V}$ is countable and transitive, then our assertion of existence and uniqueness of b (for $\bar{\mathcal{T}}$ now) holds in $\bar{V}$. But it is easy to see that $\bar{\Sigma}$ chooses branches with iterable $\mathcal{Q}$-structures, the iterability being witnessed by π, and hence $\bar{\Sigma} = \Sigma \restriction \bar{V}$. Let $\bar{G} \in V$ be any generic over $\bar{V}$. It is easy see that the putative iteration strategy for $M_\omega^\sharp$ in $\bar{V}[\bar{G}]$ which we have described always chooses branches with $\mathcal{Q}$-structures which are iterable out in V. Thus our putative strategy is just $\Sigma \restriction \bar{V}[\bar{G}]$, no matter what choices are made. This completes the proof. $\square$

Using 3.2 and the arguments of [12] (see also [10, §7]), we easily get that (e) implies (a) and (b).

We are left with proving $(d) \Rightarrow (e)$. This is where the core model theory of this paper comes in. We shall use this theory to extend the argument of [11, §7], which showed that (d) implies that there is a weasel with one Woodin cardinal. First, we note that (†) implies (e).

Lemma 3.3 *If* (†) *holds, then there is an* $(\omega, \Omega + 1)$*-iterable r-weasel satisfying "there are infinitely many Woodin cardinals".*

Proof. Suppose (†) holds. Fix a $\theta < \Omega$. We shall show that there is a (ω, θ)-iterable premouse $\mathcal{R}_\theta$ which has the first order properties of $M_\omega^\sharp$. Since the $\mathcal{R}_\theta$ are countable, they are iterable enough to be compared, and hence $\mathcal{R}_\theta = \mathcal{R}_\gamma$ for all θ, γ. Thus $M_\omega^\sharp$ exists and is $(\omega, \Omega + 1)$-iterable.

To see that such an $\mathcal{R}_\theta$ exists, note that since (†) holds, we have a θ-additive $\mathcal{P}$-construction which reaches an $\mathcal{M}_\xi$ which is not ω-small above $\mathcal{P}$. By the results of §1, $\mathcal{M}_\xi$ is (ω,θ)-iterable. But then we can use $\mathcal{M}_\xi$ as a background universe for a $(V_\omega, \in, \emptyset, \emptyset)$-construction, using full background extenders. The premouse $\mathcal{R}_\theta$ we get inherits the large cardinal and iterability properties of $\mathcal{M}_\xi$. $\square$

We may therefore assume toward contradiction that (†) fails, so that the results of the last section apply, and in particular, there is an excellent premouse. What is left in [11] to be extended is the material of §6 on the inductive definition of K. The following lemma will do that job.

Lemma 3.4 *Assume there is no $(\omega, \Omega+1)$- iterable r-weasel satisfying "there are infinitely many Woodin cardinals above my cut". Let $\mathcal{P}$ be excellent, and let G be generic over V for a poset in V_Ω. Suppose that $o(\mathcal{P})$ is countable in $V[G]$; then letting $\tau = \omega_1^{V[G]}$, we have that $\mathcal{J}_\tau^{K(\mathcal{P})}$ is definable over $L(\mathbb{R})^{V[G]}$ from any real coding $\mathcal{P}$.*

Proof. The notion of an α-strong mouse from §6 of [11] relativises to $\mathcal{P}$ in an obvious way, so we shall use the relativised notion without further comment. As in [11], we shall define α-strength over $L(\mathbb{R})^{V[G]}$ by induction on α. We need here to be using a notion of iterability which is definable over $L(\mathbb{R})^{V[G]}$, and for this reason we shall restrict ourselves to winning strategies in the weak iteration game of length ω. Since $K(\mathcal{P})$ is ω-small and has no Woodin cardinals above $o(\mathcal{P})$, this will be enough.

Recall that $\mathcal{M}$ is $\Game^{\mathbb{R}}\Pi^1_1$-iterable iff player II has a winning strategy in the weak iteration game of length ω on $\mathcal{M}$. (See [10, §7].) This game generalizes in an obvious way to r-premice, and to phalanxes $(W, \mathcal{M}, \alpha)$ where W and $\mathcal{M}$ are r-premice with the same lower part agreeing below α, so we shall speak of such r-premice or phalanxes as being $\Game^{\mathbb{R}}\Pi^1_1$-iterable without further comment.

Let us work in $V[G]$. The following claim is the heart of the matter.

Claim. Let $\alpha \geq o(\mathcal{P})$ be a cardinal of $K(\mathcal{P})$, with $\alpha < \omega_1$. Let $\mathcal{M}$ be a countable, properly small premouse over $\mathcal{P}$ such that $\mathcal{J}_\alpha^{\mathcal{M}} = \mathcal{J}_\alpha^{K(\mathcal{P})}$; then the following are equivalent:

(1) $\mathcal{M}$ is α-strong,

(2) whenever $\mathcal{N}$ is a countable, properly small premouse such that $\mathcal{J}_\alpha^{\mathcal{N}} = \mathcal{J}_\alpha^{K(\mathcal{P})}$, and $\mathcal{N}$ is β-strong for all $\beta < \alpha$, then $(\mathcal{N}, \mathcal{M}, \alpha)$ is $\Game^{\mathbb{R}}\Pi^1_1$-iterable.

Proof. The proof in [11, §6] of the analog of (1) ⇒ (2) generalizes routinely to our situation. The proof that (2) ⇒ (1) requires one new argument beyond those of [11, §6], so we sketch it here.

Assume (2), and let W be a weasel witnessing that $\mathcal{J}_\alpha^{K(\mathcal{P})}$ is very sound. We compare W with $(W, \mathcal{M}, \alpha)$. If we can do so in such a way that the comparison terminates, then the argument of [11] yields (1) immediately. So we need only see we have enough iterability that the comparison will terminate.

Now we have a unique $(\omega, \Omega+1)$-iteration strategy Σ for W, so we need only describe how to iterate $(W, \mathcal{M}, \alpha)$. Our method is the only reasonable one: choose the unique cofinal branch with an iterable $\mathcal{Q}$-structure. More precisely, suppose we have constructed an initial segment $\mathcal{U}$ of the comparison tree on $(W, \mathcal{M}, \alpha)$ we are building, with $\mathrm{lh}(\mathcal{U}) < \Omega$ and $\mathrm{lh}(\mathcal{U})$ a limit ordinal. We claim that $\mathcal{Q}(\mathcal{U})$ exists. For let $\mathcal{T}$ be the tree on W built so far in the comparison. Let $\mathcal{R}$ be the last model of $\mathcal{T}$ if it has one, and let $\mathcal{R} = \mathcal{M}_b^{\mathcal{T}}$ where $b = \Sigma(\mathcal{T})$ otherwise. Then $\mathcal{M}_\infty(\mathcal{U})$ is an initial segment of $\mathcal{R}$, and since W is properly small, either $\delta(\mathcal{U})$ is not Woodin in $\mathcal{R}$, or the branch W-to-$\mathcal{R}$ of $\mathcal{T}$ drops, so that $\rho_\omega(\mathcal{R}) < \delta(\mathcal{U})$. In either case, we get an η such that $((\mathcal{J}_\eta^{\mathcal{R}})_0, \delta(\mathcal{U}))$ kills the Woodin property. Since this structure is appropriately iterable, it is $\mathcal{Q}(\mathcal{U})$. We now try to extend $\mathcal{U}$ by choosing the unique cofinal branch c of $\mathcal{U}$ such that $\mathcal{Q}(c, \mathcal{U}) = \mathcal{Q}(\mathcal{U})$. If we can always do so, then the comparison terminates and the argument of [11] yields (1), as desired. So suppose otherwise.

By 2.6, it is the existence of a cofinal c such that $\mathcal{Q}(c, \mathcal{U}) = \mathcal{Q}(\mathcal{U})$ which fails, and not the uniqueness. Now let $\pi\colon N \to V_\theta$, with N countable transitive and everything relevant in the range of π. We let $\pi(\bar{W}) = W, \pi(\bar{\mathcal{U}}) = \mathcal{U}$, etc. We have $\bar{\mathcal{M}} = \mathcal{M}$ and $\bar{\alpha} = \alpha$ since $\mathcal{M}$ and α are countable, so $\bar{\mathcal{U}}$ is a tree on $(\bar{W}, \mathcal{M}, \alpha)$. Since $\pi \restriction \alpha$ is the identity map, $\bar{W}$ is β-strong for all $\beta < \alpha$, and hence $(\bar{W}, \mathcal{M}, \alpha)$ is $\Game^{\mathbb{R}}\Pi^1_1$-iterable. Note also that since $\mathcal{U}$ is self-verifying, so is $\bar{\mathcal{U}}$. Similarly, $\mathcal{Q}(\mathcal{U})$ collapses to $\mathcal{Q}(\bar{\mathcal{U}})$. A familiar absoluteness argument shows that $\bar{\mathcal{U}}$ has no cofinal branch c such that $\mathcal{Q}(c, \bar{\mathcal{U}}) = \mathcal{Q}(\bar{\mathcal{U}})$.

Since $(\bar{W}, \mathcal{M}, \alpha)$ is $\Game^{\mathbb{R}}\Pi^1_1$-iterable, we have a maximal branch c of $\bar{\mathcal{U}}$ such that $\mathcal{M}_c^{\mathcal{U}}$ is $\Game^{\mathbb{R}}\Pi^1_1$-iterable. Letting $\lambda = \sup(c)$, we then have $\mathcal{Q}(c_0, \bar{\mathcal{U}} \restriction \lambda) \neq \mathcal{Q}(\mathcal{U} \restriction \lambda)$. (If $\lambda < \mathrm{lh}(\bar{\mathcal{U}})$, this is true because $\mathcal{Q}(\bar{\mathcal{U}} \restriction \lambda) = \mathcal{Q}([0, \lambda]_{\bar{U}}, \bar{\mathcal{U}})$ and $c_0 \neq [0, \lambda]_U$; if $\lambda = \mathrm{lh}(\mathcal{U})$, then it is the last line of the last paragraph.) Since both $\mathcal{Q}(c, \bar{\mathcal{U}} \restriction \lambda)$ and $\mathcal{Q}(\bar{\mathcal{U}} \restriction \lambda)$ kill the Woodin property minimally, neither is an initial segment of the other. The following subclaim then gives the desired contradiction.

Subclaim. Let $\mathcal{R}$ and $\mathcal{S}$ be countable, ω-small r-premice with the same lower part, and suppose

(1) $\mathcal{R}$ is k-sound, $\rho_k(\mathcal{R}) \leq \eta(\mathcal{R})$, and $\Game^{\mathbb{R}}\Pi^1_1$-iterable (in that player II wins the weak (k, ω) iteration game on $\mathcal{R}$), and

(2) $\mathcal{S}$ is j-sound, $\rho_j(\mathcal{S}) \leq \eta(\mathcal{S})$, and $\mathcal{S}$ is (j, ω_1+1)-iterable.

Then $\mathcal{R}$ is an initial segment of $\mathcal{S}$, or vice-versa.

Proof. Suppose not. Fix an iteration strategy Γ witnessing that $\mathcal{R}$ is $\Game^{\mathbb{R}}\Pi^1_1$-iterable. Let Σ be the unique (j, ω_1+1)-iteration strategy for $\mathcal{S}$. We get uniqueness because Σ has to choose the unique branch with an appropriately iterable $\mathcal{Q}$-structure; any branch has a $\mathcal{Q}$-structure because we are building a maximal tree on $\mathcal{S}$, and $\mathcal{S}$ projects to $\eta(\mathcal{S})$.

We now construct a play $\langle (\mathcal{U}_i, b_i) \mid 0 < i < \omega\rangle$ according to Γ, together with a j-maximal iteration tree $\mathcal{T}$ on $\mathcal{S}$ played according to Σ, by coiterating $\mathcal{R}$ with $\mathcal{S}$. At stage i in the construction we have an initial segment $\mathcal{T}_i$ of $\mathcal{T}$ having a last model $\mathcal{S}_i$, with $\mathcal{T}_0$ being the one-model tree consisting of $\mathcal{S}_0 = \mathcal{S}$. Let $\mathcal{R}_0 = \mathcal{R}$, and let $\mathcal{R}_i = \mathcal{M}^{\mathcal{U}_i}_{b_i}$ if $i > 0$. We shall also define a strictly increasing sequence of ordinals $\langle \delta_i \mid i < \omega\rangle$, with $\delta_0 = \eta(\mathcal{R}) = \eta(\mathcal{S})$. We maintain by induction:

- $\mathcal{R}_i$ agrees with $\mathcal{S}_i$ below δ_i, and
- if $i > 0$, then there are initial segments $\mathcal{P}$ of $\mathcal{R}_i$ and $\mathcal{Q}$ of $\mathcal{S}_i$ such that $(\mathcal{P}, \delta_i)$ and $(\mathcal{Q}, \delta_i)$ kill the Woodin property, but $\mathcal{P} \neq \mathcal{Q}$.

Let $\mathcal{T}_i, \mathcal{U}_i, b_i, \delta_i, \mathcal{R}_i$, and $\mathcal{S}_i$ be given as above. Our hypotheses imply that $\mathcal{R}_i$ and $\mathcal{S}_i$ cannot be compared. We attempt to compare them by iterating the least disagreement, using Σ to extend $\mathcal{T}_i$ on the $\mathcal{S}_i$-side, and on the $\mathcal{R}_i$-side picking the unique branch b of the tree $\mathcal{U}$ we have built so far such that $\mathcal{Q}(b,\mathcal{U}) = \mathcal{Q}(\mathcal{U})$. Note here that, as before, $\mathcal{Q}(\mathcal{U})$ exists, since in fact it is an initial segment of the current last model in $\mathcal{T}$. There must be a limit stage $\lambda \leq \omega_1$ such that our prescription for choosing a branch b of $\mathcal{U}$ fails, and again, it is existence rather than uniqueness which fails. As before, a simple Skolem hull plus absoluteness argument shows that $\lambda < \omega_1$.

Let $\mathcal{U}_{i+1}$ be the tree on $\mathcal{R}_i$ we have at this stage, and let $b_{i+1} = \Gamma(\langle \mathcal{U}_0, \ldots, \mathcal{U}_{i+1}\rangle)$. Let $\gamma = \sup(b)$. We have that both $\mathcal{Q}(b_{i+1}, \mathcal{U}_{i+1} \restriction \gamma)$ and $\mathcal{Q}(\mathcal{U}_{i+1} \restriction \gamma)$ exist, and they are different. (If $\gamma < \mathrm{lh}(\mathcal{U}_{i+1})$, this is true because $b_{i+1} \neq [0,\gamma]_{U_{i+1}}$ and $[0,\gamma]_{U_{i+1}}$ was chosen so as to have an iterable $\mathcal{Q}$-structure. If $\gamma = \mathrm{lh}(U_{i+1})$, it is true because no cofinal branch of U_{i+1} has an iterable $\mathcal{Q}$-structure.) We set $\delta_{i+1} = \delta(\mathcal{U}_{i+1} \restriction \gamma)$. Let $\mathcal{T}_{i+1}$ be the initial segment of $\mathcal{T}$ using extenders of length strictly less than δ_{i+1}, extended by one model via Σ if this initial segment has limit length. Let $\mathcal{S}_{i+1}$ be the last model of $\mathcal{T}_{i+1}$. It is easy to see that our induction hypotheses remain true.

Let $\mathcal{R}_\infty$ and $\mathcal{S}_\infty$ be the direct limits of the $\mathcal{R}_i$ and $\mathcal{S}_i$, for i sufficiently large, respectively. (The $\mathcal{S}_i$ lie along a branch of $\bigcup_i \mathcal{T}_i$ because each δ_i is Woodin in the lined up part of the models being compared, so that no $\mathcal{T}_j$ for $j > i$ uses an extender E such that $\mathrm{crit}(E) \leq \delta_i$.) Then $\mathcal{R}_\infty$ agrees with $\mathcal{S}_\infty$ below the supremum of the δ_i; moreover each δ_i is Woodin with respect to all functions in $\mathcal{R}_\infty \cap \mathcal{S}_\infty$ via extenders on their common sequence below δ_i. Since $\mathcal{R}_\infty$ and $\mathcal{S}_\infty$ are ω-small, this implies that one is an initial segment of the other. But then, letting i be large enough that neither $\mathcal{R}_i$-to-$\mathcal{R}_\infty$ nor $\mathcal{S}_i$-to-$\mathcal{S}_\infty$ has dropped, we get that $\mathcal{Q}(b_i, \mathcal{U}_i \restriction \sup(b_i))$ is an initial segment of $\mathcal{Q}(\mathcal{U}_i \restriction \sup(b_i))$, or vice versa. This is a contradiction, so the subclaim has been proved. $\square$

The subclaim completes our proof that (2) $\Rightarrow$ (1) of the claim holds. This completes our proof of the claim. The claim yields 3.4 immediately; see [11, §6]. $\square$

We can now complete the proof of $(d) \Rightarrow (e)$ in theorem 3.1. Assume (e) fails. Let $\mathcal{P}$ be excellent, and let $\alpha < \Omega$ be such that $(\alpha^+)^{K(\mathcal{P})} = \alpha^+$. Let G be V-generic over $\operatorname{col}(\omega, \alpha)$. Then $\mathcal{J}_{\alpha^+}^{K(\mathcal{P})}$ is a member of $L(\mathbb{R})^{V[G]}$, and from it we can easily construct a sequence of distinct reals of length $\alpha^+ = \omega_1^{V[G]}$. Therefore (d) fails.

This completes the proof of 3.1. □

4 The strength of ¬UBH.

In this section we prove[17]

Theorem 4.1 *Suppose that $\mathcal{T}$ is a nonoverlapping iteration tree on V which has distinct cofinal, wellfounded branches b and c, and let $\delta = \delta(\mathcal{T})$; then*

- *$M_\omega^\sharp$ exists, and is μ-iterable, where μ is least such that $i_b(\mu) \geq \delta$ or $i_c(\mu) \geq \delta$, and*
- *if $\delta \in ran(i_b)$, where b is such that $(\delta^+)^{M_b} \leq (\delta^+)^{M_c}$, then there is an inner model with a strong cardinal which is a limit of Woodin cardinals.*

We believe that any failure of UBH implies the existence of nontame mice, and probably much more, but the result above is all we have been able to prove. The assumption that $\mathcal{T}$ is nonoverlapping is a natural restriction, but the assumption under which we get a model with a strong limit of Woodin cardinals is not. We do get the following more easily stated corollary.

Corollary 4.2 *If there is a nonoverlapping iteration tree $\mathcal{T}$ on V with distinct cofinal wellfounded branches b and c such that $\delta(\mathcal{T}) \in ran(i_b) \cap ran(i_c)$, then there is an inner model with a strong cardinal which is a limit of Woodin cardinals.*

The following lemma, which is due to Hugh Woodin, plays an important role in motivating our proof.

Lemma 4.3 (Woodin) *Let $\mathcal{T}$ be an iteration tree on V having distinct cofinal, wellfounded branches b and c such that, for $\delta = \delta(\mathcal{T})$, we have $i_b(\delta) = \delta = i_c(\delta)$; then $P(\delta) \cap M_b \cap M_c$ has cardinality $\leq \delta$.*

Proof. Let $A \subseteq \delta$, and suppose

$$A \in \operatorname{ran}(i_{\alpha,b}) \cap \operatorname{ran}(i_{\beta,c}),$$

[17]The author wishes to thank Itay Neeman for an observation which was crucial to completing the proof of Theorem 4.1.

where $\alpha \in b \setminus c$ and β is largest in $c \cap \alpha$. We claim that A is determined by α, β, and $A \cap \mathrm{crit}(i_{\beta,c})$, in the sense that whenever we have

$$i_{\alpha,b}(B) = i_{\beta,c}(C) \text{ and } C \cap \mathrm{crit}(i_{\beta,c}) = A \cap \mathrm{crit}(i_{\beta,c}(C)),$$

then

$$A = i_{\beta,c}(C).$$

Since there only δ-many possibilities for $(\alpha, \beta, A \cap \mathrm{crit}(i_{\beta,c}))$, it is enough to show this.

But now, following the uniqueness proof of [3], we can find extenders E_k used on b and F_k used on c, for $k < \omega$, such that

$$\mathrm{crit}(E_k) < \mathrm{lh}(F_k) < \mathrm{crit}(F_{k+1}) < \mathrm{lh}(E_k)$$

for all k, and

$$\mathrm{crit}(i_{\beta,c}) = \mathrm{crit}(F_0) < \mathrm{crit}(E_0).$$

(Take F_0 to be the first extender used on c after stage α, E_k to be the first extender used on b after F_k was used on c, and F_{k+1} to be the first extender used on c after E_k was used on b. Note here that if G is used in $\mathcal{T}$ to get to $\mathcal{M}_{\gamma+1}$, that is, $G = E_\gamma^{\mathcal{T}}$, and H is used after G but applied to a model M_η with $\eta \leq \gamma$, then $\mathrm{crit}(H) < \mathrm{lh}(G)$.)

We now show by induction on k that

$$i_{\beta,c}(C) \cap \mathrm{crit}(F_k) = A \cap \mathrm{crit}(F_k).$$

This is true if $k = 0$. Assume it is true for k. Let B^* and C^* be such that $A = i_{\alpha,b}(B^*) = i_{\beta,c}(C^*)$. Let $i_{\xi,\sigma}$ be the embedding along c given by the F_k-ultrapower, and $i_{\eta,\tau}$ be the embedding along b given by the E_k-ultrapower. Then $A = i_{\beta,c}(C^*)$ and $i_{\beta,c}(C)$ agree with $i_{\beta,\xi}(C^*)$ and $i_{\xi,c}(C)$ below $\mathrm{crit}(i_{\xi,c}) = \mathrm{crit}(F_k)$.The agreement is stretched by $i_{\xi,\sigma}$, so A agrees with $i_{\beta,c}(C)$ below $\mathrm{lh}(F_k)$. But $i_{\alpha,b}(B) = i_{\beta,c}(C)$, and $i_{\alpha,\eta}(B)$ agrees with $i_{\alpha,b}(B)$ below $\mathrm{crit}(i_{\eta,b}) = \mathrm{crit}(E_k)$, so A agrees with $i_{\alpha,\eta}(B)$ below $\mathrm{crit}(i_{\eta,b})$. As A also agrees with $i_{\alpha,\eta}(B^*)$ below $\mathrm{crit}(i_{\eta,b})$, and $i_{\eta,b}$ stretches this agreement, we have that A agrees with $i_{\alpha,b}(B)$ below $\mathrm{lh}(E_k)$. Since $i_{\alpha,b}(B) = i_{\beta,c}(C)$ and $\mathrm{crit}(F_{k+1}) < \mathrm{lh}(E_k)$, we have that A agrees with $i_{\beta,c}(C)$ below $\mathrm{crit}(F_{k+1})$, as desired.

Since the critical points of the F_k are cofinal in δ, we have $A = i_{\beta,c}(C)$. This completes the proof. □

In order to see how 4.3 motivates our proof of 4.1, let us consider the special case that $\delta = \delta(\mathcal{T})$ is a fixed point of both i_b and i_c. Notice first that $\mathrm{lh}(\mathcal{T})$ has cofinality ω, since b and c are both club in $\mathrm{lh}(\mathcal{T})$, while $b \cap c$ is bounded in $\mathrm{lh}(\mathcal{T})$. But now we can map $\mathrm{lh}(\mathcal{T})$ cofinally into δ by sending α to $\mathrm{lh}(E_\alpha^{\mathcal{T}})$, and therefore δ has cofinality ω.[18]

[18] Woodin has shown that if UBH fails, then it fails for some countable iteration tree. For let $\pi\colon M \to V_\theta$ be elementary, where M is countable and transitive, and let M satisfy that b and c are distinct cofinal wellfounded branches of the iteration tree $\mathcal{U}$. It is then easy to see that $\pi\mathcal{U}$ is a countable iteration tree on V, and that b and c are wellfounded branches of $\pi\mathcal{U}$, regarded as a tree on V. (Note that $\mathcal{M}_b^{\pi\mathcal{U}}$ embeds into $\mathcal{M}_{\pi(b)}^{\pi(\mathcal{U})}$.)

Now suppose that we could build K in $V_\delta^{M_b} = V_\delta^{M_c}$. We could then try to extend this K to $(\delta^+)^{M_b} = (\delta^+)^{M_c} = \delta^+$ by stacking all r-premice over this K with cut δ which project to δ and are such that every countable elementary submodel is ω_1+1-iterable. Since $i_b(\delta) = i_c(\delta) = \delta$, δ is singular in M_b and in M_c. It follows then from the proof of the weak covering theorem for K that in M_b, there are δ^+-many such r-premice, and similarly for M_c. But a simple absoluteness argument show that the countable elementary submodels of the r-premice in question are $\omega_1 + 1$-iterable in V, not just in M_b or M_c. This is enough iterability to show they are comparable, and hence the same. So $P(\delta) \cap M_b \cap M_c$ has cardinality at least δ^+, contrary to 4.3.

In order to make this argument work without the hypothesis that δ is fixed by i_b and i_c, we need first to show that δ is either singular or measurable in M_b and M_c. This is where we use the hypothesis that $\mathcal{T}$ is nonoverlapping.

Lemma 4.4 *Let $\mathcal{T}$ be a nonoverlapping iteration tree on V, and let $\alpha > 0$ and $X \subseteq M_\alpha^\mathcal{T}$ be such that $|X| \le crit(E)$ for all extenders E used in $\mathcal{T}$. Let $\rho = \sup\{\, lh(E_\beta^\mathcal{T}) \mid \beta+1 \in [0,\alpha]_T\}$; then there is a set $Y \in M_\alpha^\mathcal{T}$ such that $X \subseteq Y$ and $M_\alpha^\mathcal{T} \models |Y| \le \rho$.*

Proof. Because $\mathcal{T}$ is nonoverlapping, every element of $M_\alpha^\mathcal{T}$ is of the form $i_{0,\alpha}(f)(a)$ for some $f \in V$ and finite $a \subseteq \rho$. (See [13].) Let Z be such that $|Z| = |X|$ and every element of X is of the form $i_{0,\alpha}(f)(a)$ for some $f \in Z$ and finite $a \subseteq \rho$. Let $Z^* = i_{0,\alpha}``Z$; then $Z^* \in M_\alpha^\mathcal{T}$ and $|Z^*| \le \rho$ in $M_\alpha^\mathcal{T}$. Therefore we can take $Y = \{\, g(a) \mid g \in Z^* \text{ and } a \in [\rho]^{<\omega}\}$. □

Corollary 4.5 *Let $\mathcal{T}$ be a nonoverlapping iteration tree on V such that $lh(\mathcal{T})$ is a limit ordinal of cofinality ω. Let b be a cofinal, wellfounded branch of $\mathcal{T}$; then $\delta(\mathcal{T})$ is either singular or measurable in $M_b^\mathcal{T}$.*

Proof. Suppose $\delta = \delta(\mathcal{T})$ is regular, but not measurable, in M_b. Let $\alpha \in b$ and $\bar\delta$ be such that $i_{\alpha,b}(\bar\delta) = \delta$. Then $\bar\delta$ is regular, but not measurable, in M_α. Also, if $\beta + 1 \in b \setminus (\alpha + 1)$ and $\gamma = \mathrm{pred}_T(\beta + 1)$, then $\mathrm{crit}(E_\beta) \le i_{\alpha,\gamma}(\bar\delta)$ because $\mathcal{T}$ is nonoverlapping. Since $\bar\delta$ is not measurable in M_α, $\mathrm{crit}(E_\beta) < i_{\alpha,\gamma}(\bar\delta)$. It follows that $i_{\alpha,b}$ is continuous at $\bar\delta$. Since δ has cofinality ω in V, so does $\bar\delta$. But then the lemma implies that $M_\alpha \models \mathrm{cof}\,(\bar\delta) \le \rho$, where ρ is as in the statement of the lemma. Since $\mathcal{T}$ is nonoverlapping, $\rho < \bar\delta$, so $\bar\delta$ is singular in M_α, a contradiction. □

The next lemma connects the large cardinal hypothesis in the second conclusion of 4.1 with the hypothesis (†), whose negation we used in the general theory of K.

Lemma 4.6 *Let κ be λ-strong, where $\lambda > \kappa$ is inaccessible. Suppose that (†) holds when we take $\Omega = \kappa$; then there is an inner model with a strong cardinal which is a limit of Woodin cardinals.*

Proof. If any $\mathcal{P}$-construction reaches a nontame mouse, then we are done, so assume otherwise. Let $\mathbb{C}$ be a $\mathcal{P}$-construction of length κ (with all background certificates in V_κ) witnessing that (†) holds at κ. Let $\mathbb{D}$ be a $(\mathcal{N}_\kappa^\mathbb{C})_0$-construction of length λ which is κ-maximal. The important point here is that any bounded subset of κ which belongs to $\mathcal{N}_\lambda^\mathbb{D}$ is in $\mathcal{N}_\kappa^\mathbb{C}$, because there are arbitrarily large cutpoints γ of $\mathcal{N}_\kappa^\mathbb{C}$ such that $\mathcal{J}_\gamma^{\mathcal{N}_\kappa^\mathbb{C}}$ is full. It follows that κ is a limit of Woodin cardinals in $\mathcal{N}_\lambda^\mathbb{D}$. The maximality of $\mathbb{D}$ implies that κ is a strong cardinal in $\mathcal{N}_\lambda^\mathbb{D}$. □

Proof of 4.1. Let $\mathcal{T}$, b, and c be as in the hypotheses, and let $\delta = \delta(\mathcal{T})$. Since branches of an iteration tree are closed in the order topology on the ordinals, $\mathrm{lh}(\mathcal{T})$ has cofinality ω. We assume without loss of generality that

$$(\delta^+)^{M_b} \leq (\delta^+)^{M_c}.$$

Let

$$\mu := \text{ least } \nu \text{ such that } i_b(\nu) \geq \delta \ .$$

Since δ is a limit of measurable cardinals in M_b, μ is a limit of measurable cardinals in V.

Claim 1. If one of the two conclusions of 4.1 fails, then there is a premouse $\mathcal{P} \in V_\mu$ and an unbounded set $S \subseteq \mu$ such that $i_b(S)$ is unbounded in δ such that for all $\kappa \in S$, κ is measurable and $\mathcal{P}$ is κ-excellent.

Proof. Of course, by "κ-excellent" we simply mean excellent in the sense of definition 2.14, with κ playing the role of the measurable cardinal Ω.

Note first that the proof of 2.11 easily shows that there is a full premouse $\mathcal{P}_0 \in V_\mu$ such that for any tame $\mathcal{P}_0$-construction $\mathbb{C}$ of any length $\kappa < \mu$, and any cutpoint η of $\mathcal{N}_\kappa^\mathbb{C}$ such that $\forall \delta > \eta(\mathcal{N}_\kappa^\mathbb{C} \not\models \delta$ is Woodin), the r-premouse $((\mathcal{N}_\kappa^\mathbb{C})_0, \eta)$ is $(\omega, \kappa+1)$-iterable.

Suppose first that $i_b(\mu) > \delta$, so that it must be the first of the two conclusions of 4.1 which fails. Fix a $\theta < \mu$ such that there is no θ-iterable $M_\omega^\sharp$. We can now repeat the proof of 2.16 so as to produce a premouse $\mathcal{P}$ extending $\mathcal{P}_0$ which is κ-excellent for all sufficiently large measurable $\kappa < \mu$. (If there is no such $\mathcal{P}$, then the proof of 2.16 yields a sequence $\langle \mathbb{C}_i \mid i < \omega + 1 \rangle$ whose concatenation is a $\mathcal{P}_0$-construction which is θ-additive and reaches a level with ω Woodins and a measurable above $o(\mathcal{P}_0)$. Note here that $\mathrm{cof}(\mu) > \omega$ because i_b is discontinuous at μ.) We can take $S = \{ \kappa \mid \nu < \kappa < \mu \text{ and } \kappa \text{ is measurable } \}$, for some ν, as our witness to claim 1.

Now suppose $i_b(\mu) = \delta$, and therefore μ is a Woodin cardinal in $L(V_\mu)$ by the results of [3]. By 4.6, (†) fails at κ for any cardinal $\kappa < \mu$ which is $< \mu$-strong, as otherwise the conclusions of 4.1 hold. (A little additional work, which we shall omit, is needed to get the $< \mu$ iterability clause of the first conclusion.) By 2.16, for any such κ there is a κ-excellent premouse $\mathcal{P} \in V_\kappa$. We claim that there is a fixed $\mathcal{P}$ which is κ-excellent for unboundedly many $\kappa < \mu$. For let S^* be the set of $\kappa < \mu$ which are $< \mu$-strong, and for $\kappa \in S^*$, let $f(\kappa)$ be the least α such that some $\mathcal{P} \in V_\alpha^{M_b}$ is κ-excellent. Notice that $f, S^* \in L(V_\mu)$, and S^*

is stationary in $L(V_\mu)$ because μ is Woodin in $L(V_\mu)$. We can find a $T \subseteq S^*$ on which f is constant such that T is stationary in $L(V_\mu)$, say with value α. Since V_α has a wellorder in $L(V_\mu)$, we can now fix $\mathcal{P}$ on an unbounded subset S of T. Since $i_b(\mu) = \delta$, $i_b(S)$ is unbounded in δ. □

Now fix $\mathcal{P}$ and S as in the conclusion of claim 1. We shall derive a contradiction from the existence of such a $\mathcal{P}$ and S, thereby completing the proof of 4.1.

For $\tau \in S$, we have an analogue K_τ of the 1-small K of [11], with $o(K_\tau) = \tau$. In the notation of section 2, K_τ is $K(\mathcal{P})$; since our measurable cardinal was fixed throughout section 2, we did not incorporate a name for it in the notation, as we are doing now. The results of [11, §6] on the inductive definition of K show that there is a fixed formula that defines $K_\tau \cap V_\nu$ over $V_{\nu+1}$, independently of τ. Thus K_τ is an initial segment of K_ξ whenever $\tau, \xi \in S$ and $\tau < \xi$. Let

$$K_\mu = \text{ unique } \mathcal{R} \text{ such that } o(\mathcal{R}) = \mu \text{ and } \forall \tau \in S(K_\tau \trianglelefteq \mathcal{R}).$$

Moving over to M_b, for $\xi \leq i_b(\mu)$ we set

$$K^b_\xi = \mathcal{J}^{i_b(K_\mu)}_\xi.$$

So for $\xi \in i_b(S)$, K^b_ξ is a reasonable version of K from the point of view of M_b. Notice that

$$K^b_\delta \in M_c,$$

because $V_\delta^{M_b} = V_\delta^{M_c}$, and the inductive definition of K is sufficiently local.

Working in V, let us call $\mathcal{R}$ a δ^+*-collapsing mouse* if and only if $\mathcal{R}$ is an r-premouse with cut δ over $(K^b_\delta)_0$, and for some $k < \omega$, $\mathcal{R}$ is $k+1$-sound, $\rho_{k+1}(\mathcal{R}) = \delta$, and every countable elementary submodel of $\mathcal{R}$ is $(k, \omega_1 + 1)$-iterable.

Claim 2. If $\mathcal{R}$ and $\mathcal{S}$ are δ^+-collapsing mice, then one of the two is an initial segment of the other.

Proof. If not, we can take a countable Skolem hull of the whole situation. The images under collapse of $\mathcal{R}$ and $\mathcal{S}$ are sufficiently iterable that they can be compared, and this leads to contradiction by a well-known argument. □

Claim 3. Let $\mathcal{R} \in M_b$; then $\mathcal{R}$ is a δ^+-collapsing mouse iff $\mathcal{M}_b \models \mathcal{R}$ is a δ^+-collapsing mouse. Similarly for $\mathcal{M}_c$.

Proof. Since the base model of $\mathcal{T}$ is V, we have $V_{\omega+2} \subseteq \mathcal{M}_b$. Thus $\mathcal{M}_b$ has all the countable (transitive) elementary submodels of $\mathcal{R}$ in it, and is correct about their $\omega_1 + 1$-iterability. A simple absoluteness argument shows that $\mathcal{M}_b$ is correct about whether a given countable transitive model is elementarily embeddable in $\mathcal{R}$. □

These two claims imply that either the δ^+-collapsing mice of M_b form an initial segment of the δ^+-collapsing mice of M_c, or vice-versa.

Claim 3. If $\mathcal{Q}(K^b_\delta)$ exists, then $\mathcal{Q}(K^b_\delta) \notin M_b \cap M_c$.

Proof. Assume otherwise. Let us work in $L(V_\delta^{M_b} \cup \{\mathcal{Q}(K^b_\delta)\})$, where δ is a Woodin cardinal by [3].

Suppose that $\mathcal{Q}(K^b_\delta)$ kills the Woodin property in the first of the two possible ways it may do so, and let

$$\rho = \rho_{k+1}(\mathcal{Q}(K^b_\delta) < \delta,$$

where $\mathcal{Q}(K^b_\delta)$ is $(k+1)$-sound and all its countable elementary submodels are (k, ω_1+1)-iterable. Sice δ is inaccessible (in the universe we are working in now), we can find

$$\pi \colon \mathcal{R} \to \mathcal{Q}(K^b_\delta)$$

such that

$$\rho < \bar{\delta} < \delta \text{ and } \pi \restriction \bar{\delta} = \text{ id },$$

where $\bar{\delta} = \pi^{-1}(\delta)$. Pick any $\tau \in i_b(S) \cap \delta$ such that $\tau > o(\mathcal{R})$. If $\bar{\delta}$, which is the cut of $\mathcal{R}$, is not a strong cutpoint of K^b_τ, then let E on the K^b_τ-sequence be such that $\text{crit}(E) \le \bar{\delta} \le \text{lh}(E)$ and $\text{lh}(E)$ is as small as possible. Let W be the r-weasel with the same lower part as $\mathcal{R}$ whose reduct is the same as the reduct of $\text{ult}(K^b_\tau, E)$ if E exists, and is the reduct of K^b_τ otherwise. We can now simply apply 2.12, and conclude that $\mathcal{R} \in W$. This gives

$$\mathcal{R} \in K^b_\tau,$$

which is a contradiction because $\mathcal{R}$ defines a subset of ρ which is not in K^b_δ.

Now suppose that $\mathcal{Q}(K^b_\delta)$ kills the Woodin property in the second of the two possible ways, and let $f \colon \delta \to \delta$ be the relevant function. Since δ is Woodin in the universe we are working in, there is an extender $E \in V_\delta$ which witnesses the Woodin property with respect to f, and is such that its restriction to K^b_δ could be added to the K^b_δ-sequence at some stage $\alpha > i_E(f)(\text{crit}(E))$. The results of [8] on the maximality of the 1-Woodin K extend routinely to the K^b_τ for $\tau \in i_b(S)$, and hence to K^b_δ because $i_b(S)$ is cofinal in δ. Thus the restriction of E is in K^b_δ, so that f was not a counterexample to the Woodin property for the sequence of K^b_δ after all, a contradiction. $\square$

Let us set

$$\lambda = \sup\{\, o(\mathcal{R}) \mid \mathcal{R} \in M_b \cap M_c \text{ and } \mathcal{R} \text{ is a } \delta^+\text{- collapsing mouse }\}.$$

It is not hard to see that $\lambda = (\delta^+)^{L[\vec{E}]}$, where $\vec{E}$ is the union of the extender sequences of the reducts of (δ^+)-collapsing mice in $M_b \cap M_c$. We write K^+_δ for the r-premouse with lower part $i_b(\mathcal{P})$ whose reduct is $\mathcal{J}^{\vec{E}}_\lambda$.

Claim 4. $\lambda = (\delta^+)^{\mathcal{M}_b}$.

Proof. Since the δ^+-collapsing mice of M_b are an initial segment of those of M_c, or vice-versa, we have $a \in \{b, c\}$ be such that

$$\lambda = \sup\{\, o(\mathcal{R}) \mid \mathcal{R} \in M_a \text{ and } \mathcal{R} \text{ is a } \delta^+\text{- collapsing mouse }\}.$$

We shall show that

$$\lambda = (\delta^+)^{M_a},$$

so that

$$\lambda \le (\delta^+)^{M_b} \le (\delta^+)^{M_a} \le \lambda,$$

because we chose b with $(\delta^+)^{M_b} \le (\delta^+)^{M_c}$. This will complete the proof of claim 4.

We work in M_a, using the proof of weak covering at δ. (See [4].) We have by 4.5 that δ is either singular or measurable. We shall assume that δ is singular; if δ is measurable, one can simply go to the Prikry extension $M_a[G]$ where δ has cofinality ω, and carry out the argument we are about to give there.[19] Suppose toward contradiction that $\lambda < \delta^+$. Since δ is a singular strong limit cardinal, there is a transitive set N such that $|N| < \delta$ and a

$$\pi \colon N \to V_\theta,$$

where $\theta >> \lambda$, such that

$$\operatorname{ran}(\pi) \text{ is cofinal in } \delta \text{ and in } \lambda,$$

and N is closed under ω-sequences. We let

$$\pi(\langle \bar{\delta}, \bar{\lambda} \rangle) = \langle \delta, \lambda \rangle \text{ and } \pi(\bar{K}) = K^+_\delta.$$

Following the proof of weak covering, we compare $\bar{K}$ with K^b_δ. The arguments of [4] show that only K^b_δ moves in this process, so that we have a self-verifying iteration tree $\mathcal{T}$ on K^b_δ with last model $\mathcal{R}$ such that $\bar{K} \trianglelefteq \mathcal{R}$, and $\mathcal{T}$ uses only extenders with length strictly less than $\bar{\delta}$. (In order to make the arguments of [4] apply, note that for arbitrarily large $\bar{\tau} < \bar{\delta}$, we have $\pi(\bar{\tau}) \in i_b(S)$, so $\pi(\mathcal{J}^{\bar{K}}_{\bar{\tau}}) = K^b_\tau$. Thus the initial segment of our comparison using extenders of length $< \bar{\tau}$ can be considered a comparison between K^b_τ and its preimage under π. Since we have extended the basic theory of the 1-Woodin K to K^b_τ, we have that only K^b_τ moves in this comparison. It doesn't matter that at this point we may be working in M_c, since $V^{M_c}_\delta = V^{M_b}_\delta$. Thus only K^b_δ moves in the initial segment of its comparison with $\bar{K}$ in which extenders of length strictly less than $\bar{\delta}$ are being used. But this initial segment is all there is to the comparison of $\bar{K}$ with K^b_δ, for we can regard it as a comparison of $\bar{K}$ with K^b_τ for some $\tau \in i_b(S) \cap \delta$ such that $\tau > \bar{\delta}$, and letting W be the last model of the tree on K^b_τ

[19] Alternatively, one can apply Theorem 3.1 of [8] in the ultrapower of M_a by a normal ultrafilter on δ.

in the initial segment of the comparison using only extenders of length strictly less than $\bar{\delta}$, show that $\bar{K} \trianglelefteq W$ as an immediate consequence of 2.12.)

Since K^b_δ is properly small, we have by 2.5 that $\mathcal{Q}(\bar{K})$ exists, and is (essentially) an initial segment of $\mathcal{R}$. Thus there is a shortest initial segment $\mathcal{S}$ of $\mathcal{R}$ such that

$$\bar{K} \trianglelefteq \mathcal{S} \text{ and } \rho_{k+1}(\mathcal{S}) \leq \bar{\delta},$$

for some k. Let k be minimal with this property. Since $\bar{\delta}$ is Woodin in $\mathcal{S}$, it is a cutpoint of $\mathcal{S}$. Let

$$\mathcal{S}^+ = \text{ult}_k(((\mathcal{S})_0, \bar{\delta}), E_\pi \restriction \delta),$$

where E_π is the extender derived from π. Standard arguments (using the countable completeness of E_π) show that $\mathcal{S}^+$ is a δ^+-collapsing mouse, and $\lambda \leq o(\mathcal{S})$, contrary to the definition of λ. □

We can now complete the proof of 4.1. Suppose first that $i_b(\mu) > \delta$. We can then fix $\tau \in i_b(S) \setminus \delta$, and we have that K^b_τ is a reasonable version of K in M_b. The proof of 2.12 shows that K^+_δ is an initial segment of K^b_τ, and since $o(K^+_\delta) = (\delta^+)^{M_b}$ and δ is Woodin in K^+_δ, we have that δ is Woodin in K^b_τ. On the other hand, all K_ξ for $\xi \in S$ were properly small, so K^b_τ is properly small, a contradiction.

Suppose next that $i_b(\mu) = \delta$. Our proof then breaks into cases according to whether i_b is continuous at δ. Assume first that it is not. We have then that $\text{cof}(\mu)$ is measurable (with $\mu = \text{cof}(\mu)$ a possibility). Let

$$j \colon V \to N = \text{ult}(V, U)$$

be the canonical embedding, where U is a normal ultrafilter on $\text{cof}(\mu)$. We have then that

$$\mu = \sup(j\text{"}\mu) < j(\mu).$$

(The first equality holds because μ is Woodin in $L(V_\mu)$.) Let K^+_μ be the preimage of K^+_δ under i_b, let

$$E = E_j \cap (K^+_\mu \times [\mu]^{<\omega}),$$

and let

$$k \colon \text{ult}(K^+_\mu, E) \to j(K^+_\mu)$$

be the natural embedding, where $\text{ult}(K^+_\mu, E)$ is the ultrapower formed using only functions in K^+_μ. (If $\text{cof}(\mu) = \mu$, then E contains only principal measures, so $\text{ult}(K^+_\mu, E) = K^+_\mu$ and $k = j$.) Since μ is regular (in fact, Woodin) in K^+_μ, the natural embedding i of K^+_μ into $\text{ult}(K^+_\mu, E)$ is continuous at μ, so that $i(\mu) = \mu$. It follows that k has critical point μ, and $k(\mu) = j(\mu)$. Note also that $o(\text{ult}(K^+_\mu, E)) = \mu^+$. One can now show easily, by a comparison argument, that $j(K^+_\mu)$ agrees with $\text{ult}(K^+_\mu, E)$ below μ^+. (The two structures agree below μ by construction, and thus below μ^+ they are each unions of sufficiently iterable μ^+-collapsing mice with the same cut.) Since μ is Woodin in K_μ^+, it is Woodin in $\mathrm{ult}(K_\mu^+, E)$, and hence in $j(K_\mu^+)$. This implies that μ is a limit of Woodins in K_μ^+, a contradiction.

Suppose finally that $i_b(\mu) = \delta$ and i_b is continuous at μ. Here we use Woodin's argument. Since

$$|P(\mu) \cap i_b^{-1}(K_\delta^+)| = \mu^+$$

and

$$K_\delta^+ \subseteq M_c,$$

and $\mathrm{lh}(\mathcal{T})$ has cofinality ω, we can find an $\alpha \in c \setminus b$ such that for some $X \subseteq P(\mu) \cap i_b^{-1}(K_\delta^+)$ of cardinality μ^+,

$$i_b'' X \subseteq \mathrm{ran}(i_{\alpha,c}).$$

Let β be least in $b \setminus \alpha$. We showed in the proof of 4.3 that if $A, B \in \mathrm{ran}(i_{\beta,b}) \cap \mathrm{ran}(i_{\alpha,c})$ and $A \cap \mathrm{crit}(i_{\beta,b}) = B \cap \mathrm{crit}(i_{\alpha,c})$, then $A = B$. By our special assumption, i_b is continuous at μ, so we have $\kappa < \mu$ such that

$$i_b(\kappa) > \mathrm{crit}(i_{\beta,b}).$$

It follows then that if $\bar{A}, \bar{B} \in X$ and $\bar{A} \cap \kappa = \bar{B} \cap \kappa$, then $i_b(\bar{A}) = i_b(\bar{B})$, and hence $\bar{A} = \bar{B}$. We have then that

$$|X| \leq 2^\kappa,$$

a contradiction. □

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