Games and Scales

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The construction and use of *Suslin representations* for sets of reals lies at the heart of descriptive set theory. Indeed, virtually every paper in descriptive set theory in the Cabal Seminar volumes deals with such representations in one way or another. Most of the papers in the section to follow focus on the construction of optimally definable Suslin representations via game-theoretic methods. In this introduction, we shall attempt to put those papers in a broader historical and mathematical context. We shall also give a short synopsis of the papers themselves, and describe some of the work done later to which they are related.

1 Some definitions and history

A *tree* on a set X is a subset of $X^{<\omega}$ closed under initial segments. If T is a tree on $X \times Y$, then we regard T as a set of pairs (s,t) of sequences with $\mathrm{dom}(s) = \mathrm{dom}(t)$. If T is a tree, we use $[T]$ for the set of infinite branches of T, and if T is on $X \times Y$, we write

$$p[T] = \{x \in X^\omega \mid \exists y \in Y^\omega \forall n < \omega((x \upharpoonright n, y \upharpoonright n) \in T)\}.$$

We call $p[T]$ the *projection* of T, and say that T is a *Suslin representation* of $p[T]$, or that $p[T]$ is Y*-Suslin* via T. For $s \in X^{<\omega}$, let $T_s = \{u \mid (s,u) \in T\}$, and put $T_x = \bigcup_n T_{x\upharpoonright n}$. Then $x \in p[T]$ iff $[T_x] \neq \emptyset$ iff T_x is illfounded.

Any set $A \subseteq X^\omega$ is trivially A-Suslin. For the most part, useful Suslin representations come from trees on some $X \times Y$ such that Y is wellordered. Assuming (as we do) the Axiom of Choice (AC), this is no restriction on Y, but we can parlay it into an important and useful restriction by requiring in addition that T be *definable* in some way or other. A variant of this approach is to require that T belong to a model of AD. If T is definable, and X and Y are definably wellordered, and $p[T]$ is nonempty, then the leftmost branch (x,f) of T gives us a definable element x of $p[T]$. (Here "leftmost" can be determined by the lexicographic order on $X \times Y$.)

The simplest nontrivial X to consider are the countable ones. This is by far the most well-studied case in the Cabal volumes. In this case, one may regard $p[T]$ as a subset of the

Baire space ω^ω, that is, as a set of "logician's reals". Thus if A is a nonempty set of reals, κ is an ordinal, and A is κ-Suslin via a definable tree, then A has a definable element.

Suslin representations were first discovered in 1917 by Suslin ([60]), who isolated the class of ω-Suslin sets of reals, showed that it properly includes the Borel sets, and showed that sets in this class have various regularity properties. (For example, they are all Baire and Lebesgue measurable, and have the perfect set property.) Suslin also found a beautiful characterization of the Borel sets of reals as those which are both ω-Suslin and have ω-Suslin complements. (The ω-Suslin sets of reals are precisely the $\boldsymbol{\Sigma^1_1}$ sets of reals, almost by definition.)

Definable Suslin representations yield definable elements, and in the "boldface" setting of classical descriptive set theory, this comes out as a uniformization result. Here we say that a function f *uniformizes* a relation R iff $\mathrm{dom}(f) = \{x \mid \exists y R(x,y)\}$, and $\forall x \in \mathrm{dom}(f) R(x, f(x))$. If R is a Σ^1_1 relation, say $R = p[T]$ where T is a tree on $(\omega \times \omega) \times \omega$, then we can use leftmost branches to uniformize R: let $f(x) = y$, where (y, h) is the leftmost branch of T_x. One can calculate that for any open set U, $f^{-1}(U)$ is in the σ-algebra generated by the Σ^1_1 sets, and is therefore Lebesgue and Baire measurable. This classical uniformization result was proved by Jankov and Von Neumann around 1940 ([61]). The "lightface", effective refinement of a uniformization theorem is a *basis theorem*, where we say a pointclass Λ is a basis for a pointclass Γ just in case every nonempty set of reals in Γ has a member which is in Λ. Kleene ([20]) proved the lightface version of the Jankov-Von Neumann result. He observed that if $A \subseteq \omega^\omega$ is lightface Σ^1_1, then $A = p[T]$ for some recursive tree T, and that the leftmost branch of T is recursive in the set W of all Godel numbers of wellfounded trees on ω. Thus $\{x \mid x \leq_T W\}$ is a basis for Σ^1_1.

In 1935-38, toward the end of the classical period, Novikoff and Kondo constructed definable, ω_1-Suslin representations for arbitrary $\boldsymbol{\Sigma^1_2}$ sets, and used them to show every $\boldsymbol{\Sigma^1_2}$ relation has a $\boldsymbol{\Sigma^1_2}$ uniformization. (See [23], [21].) The effective refinement of this landmark theorem is due to Addison, who showed that the ω_1-Suslin reprentations of nonempty lightface Σ^1_2 sets constructed by Novokoff and Kondo yield, via leftmost branches, lightface Δ^1_2 elements for such sets.

Logicians often meet Suslin representations through the Shoenfield Absoluteness theorem. Shoenfield ([46]) showed that a certain tree T on $\omega \times \omega_1$ which comes from the Novikoff-Kondo construction is in L. Because wellfoundedness is absolute to transitive models of ZF, he was able to conclude that the leftmost branch of T is in L, and thus, that every nonempty Σ^1_2 set of reals has an element in L. From this it follows easily that L is Σ^1_2 correct. This method of using definable Suslin representations to obtain correctness and absoluteness results for models of set theory is very important.

In addition to definability, there is a second very useful property a Suslin representation might have. We call a tree T on $X \times Y$ *homogeneous* just in case there is a family $\langle \mu_s \mid s \in X^{<\omega} \rangle$ such that

(1) for all s, μ_s is a countably complete 2-valued measure (i.e. ultrafilter) on $\{u \mid (s,u) \in T\}$,

(2) if $s \subseteq t$, and $\mu_s(A) = 1$, then $\mu_t(\{u \mid u \upharpoonright \mathrm{dom}(s) \in A\}) = 1$, and

(3) for any $x \in p[T]$ and any $\langle A_i \mid i < \omega \rangle$ such that $\mu_{x \upharpoonright i}(A_i) = 1$ for all i, there is a $f \in Y^\omega$ such that $f \upharpoonright i \in A_i$ for all i.

We say T is *κ-homogeneous* if the measures μ_s can be taken to be κ-additive. If T is κ-homogeneous, then we also call $p[T]$ a κ-homogeneously Suslin set. We write Hom_κ for the pointclass of κ-homogeneous sets, and Hom_∞ for the pointclass $\bigcap_\kappa \mathrm{Hom}_\kappa$.

The concept of homogeneity is implicit in Martin's 1968 proof ([25]) of Π^1_1 determinacy, and was first explicitly isolated by Martin and Kechris. Martin showed that if κ is a measurable cardinal, then every Π^1_1 set of reals is κ homogeneous, via a Shoenfield tree on $\omega \times \kappa$. He also showed that every homogeneously Suslin set of reals is determined. Martin's proof became the template for all later proofs of definable determinacy from large cardinal hypotheses. Indeed, the standard characterization of descriptive set theory, as the study of the good behavior of definable sets of reals, would perhaps be more accurate if one replaced "definable" by "∞-homogeneously Suslin".

There are two natural weakenings of homogeneity. First, a tree T on $X \times (\omega \times Y)$ is *κ-weakly homogeneous* just in case it is κ- homogeneous when viewed as a tree on $(X \times \omega) \times Y$. Thus the weakly homogeneous subsets of X^ω are just the existential real quantifications of a homogeneous subsets of $X^\omega \times \omega^\omega$, and Martin's [25] shows in effect that whenever κ is measurable, all Σ^1_2 sets of reals are κ-weakly homogeneous. Second, a pair of trees S and T, on $X \times Y$ and $X \times Z$ respectively, are *κ-absolute complements* iff

$$V[G] \models p[S] = X^\omega \setminus p[T]$$

whenever G is V-generic for a poset of cardinality $< \kappa$. The fundamental Martin-Solovay construction, also from 1968 (see [28]), shows that every κ-weakly homogeneous tree has a κ-absolute complement. The projection of a κ-absolutely complemented tree is said to be *κ-universally Baire*. This concept was first explicitly isolated and studied by Feng, Magidor, and Woodin in [5]. Any universally Baire set has the Baire property and is Lebesgue measurable, but one cannot show in ZFC alone that such sets must be determined. (See [5].) On the other hand, if there are arbitrarily large Woodin cardinals, then for any set of reals A, A is κ-homogeneous for all κ iff A is κ-weakly homogeneous for all κ iff A is κ-universally Baire for all κ. (This is work of Martin, Solovay, Steel, and Woodin; see [22, Theorem 3.3.13] for one expostion, and [59] for another.)

Although our discussion of homogeneity has focussed on its use in situations where the Axiom of Choice and the existence of large cardinals is assumed, the concept is also quite important in contexts in which full AD is assumed. AD gives us not just measures, but homogeneity measures; indeed, assuming AD, a set of reals is homogeneously Suslin iff both it and its complement are θ-Suslin. (This result of Martin from the 80's can be found in [30].) The analysis of homogeneity measures is a central theme in the work of Kunen, Martin, and Jackson ([48], [10],[11]) which located the projective ordinals among the alephs. The reader should see Jackson's surveys [9] and [8] for more on homogeneity and the projective ordinals in the AD context.

2 Construction methods

One could group the methods for producing useful Suslin representations as follows:

(1) the Martin-Solovay construction,

(2) trees to produce an elementary submodel, and

(3) scale constructions using comparison games.

We discuss these methods briefly:

2.1 The Martin-Solovay construction

The Martin-Solovay construction makes use of homogeneity. If T on $X \times Y$ is is κ- weakly homogeneous via the system of measures $\vec{\mu}$, and $|X| < \kappa$, then the construction produces a tree $\mathrm{ms}(T, \vec{\mu})$ which is a κ-absolute complement for T. The construction of $\mathrm{ms}(T, \vec{\mu})$ is effective, and its basic properties can be proved to hold in ZF+DC. Martin and Solovay ([28]) applied it with T the Shoenfield tree for Σ^1_2 and $\vec{\mu}$ its weak homogeneity measures implicit in Martin's [25]. They showed thereby that if κ is measurable, then for any Σ^1_3 formula φ, there is a tree U such that $p[U] = \{x \in \omega^\omega \mid \varphi(x)\}$ is true in every generic extension of V by a poset of size $< \kappa$.

The Martin-Solovay tree $\mathrm{ms}(T, \vec{\mu})$ is definable from T and $\vec{\mu}$. Now suppose T be on $\omega \times Y$. There is a simple variant of $\mathrm{ms}(T, \vec{\mu})$ which is definable from T and the restrictions of the measures in $\vec{\mu}$ to $\bigcup\{L[T, x] \mid x \in \omega^\omega\}$. Let us call this variant $\mathrm{ms}^*(T, \vec{\mu})$. If T is the Shoenfield tree, so that $T \in L$, then one can define these restricted weak homogeneity measures, and hence $\mathrm{ms}^*(T, \vec{\mu})$ itself, from the sharp function on the reals. Martin and Solovay showed this way that Δ^1_4 is a basis for Π^1_2, and Mansfield later improved their result by showing the class of Π^1_3 singletons is a basis for Π^1_2. (See [24].) These results are not optimal, however. We do not know whether one can get the optimal basis and uniformization results in the projective hierarchy using the Martin-Solovay construction.

Under appropriate large cardinal hypotheses, the Martin-Solovay tree is itself homogeneous. (See [32] for a precise statement.) Thus under the appropriate large cardinal hypotheses, one can show via the Martin-Solovay construction that the pointclass Hom_∞ is closed under complements and real quantification.

2.2 The tree to produce an elementary submodel

If a set A of reals admits a definition with certain condensation and generic absoluteness properties, then A is universally Baire. More precisely, let κ be a cardinal, and $\varphi(v_0, v_1)$ a formula in the language of set theory, and t any set. Let $\tau > \kappa$, $X \prec V_\tau$ be countable, and let M be the transitive collapse of X, with $\bar{\kappa}$ and $\bar{t}$ the images of κ and t under collapse. We say X is *generically* $\langle\varphi, A\rangle$*-correct* iff whenever g is M-generic for a poset of size $< \bar{\kappa}$ is M, then for all reals $y \in M[g]$,

$$y \in A \Leftrightarrow M[g] \models \varphi[y, \bar{t}].$$

If the set of generically $\langle\varphi, A\rangle$ correct X is club in $P_{\omega_1}(V_\tau)$, then A admits a κ-absolutely complemented Suslin representation T. The construction of T is relatively straightforward: if $(y, f) \in [T]$, then f will have built an X in our club of generically correct hulls, together with a proof that $M[g] \models \varphi[y, \bar{t}]$, for some g generic over the collapse M of X. (This construction is due to **Magidor? Woodin?**. See [5] or [59].)

One can use either stationary tower forcing (cf. the *Tree Production Lemma*,[22] or [59]) or iterations to make reals generic ([54, sec. 7]) to obtain, for various interesting $\langle\varphi, A\rangle$, a club of generically $\langle\varphi, A\rangle$-correct X.

If one replaces V_τ by an appropriate direct limit of mice, then the tree to produce an elementary submodel becomes definable, at a level corresponding to the definability of the iteration strategies for the mice in question. See the concluding paragraphs of [57], and [54, sec. 8]. One can use this to get optimal basis and uniformization results for various pointclasses, for example $(\Sigma^2_1)^{L(\mathbb{R})}$. It is difficult to obtain the optimal basis and uniformization results for Π^1_3 by these methods, but, building on work of Hugh Woodin, Itay Neeman has succeeded in doing so. (This work is unpublished.)

2.3 Propagation of scales using comparison games

The simplest method for obtaining optimally definable Suslin representations makes direct use of the determinacy of certain infinite games. It was discovered in 1971 by Moschovakis, who used it to extend the Novikoff-Kondo-Addison theorems to the higher levels of the projective hierarchy. (The original paper is [34]; see also [15] and [38, Chapter 6].) As part of this work, Moschovakis introduced the basic notion of a *scale*, which we now describe.

Let T be a tree on $\omega \times \lambda$, and $A = p[T]$. One can get a "small" subtree of T which still projects to A by considering only ordinals $< \lambda$ which appear in some leftmost branch. The *scale of* T does this, then records the resulting subtree as a sequence of *norms*, i.e. ordinal-valued functions, on A. More precisely, for $x \in A$ and $n < \omega$, put

$$\varphi_n(x) = |\langle l_x(0), ..., l_x(n)\rangle|_{\text{lex}},$$

where for $u \in \lambda^{n+1}$, $|u|_{\text{lex}}$ is the ordinal rank of u in the lexicographic order on λ^n. Then

$$\vec{\varphi} = \langle \varphi_n \mid n < \omega \rangle$$

is the scale of T. It has the properties:

(a) Suppose that $x_i \in A$ for all $i < \omega$, and $x_i \to x$ as $i \to \infty$, and for all n, $\varphi_n(x_i)$ is eventually constant as $n \to \infty$, then

  (i) (limit property) $x \in A$, and

  (ii) (lower semi-continuity) for all n, $\varphi_n(x) \leq$ the eventual value of $\varphi_n(x_i)$ as $i \to \infty$.

(b) (refinement property) if $x, y \in A$ and $\varphi_n(x) < \varphi_n(y)$, then $\varphi_m(x) < \varphi_m(y)$ for all $m > n$.

A sequence of norms on A with property (a) is called a *scale on A*. Any scale on A can be easily transformed into a scale on A with the refinement property. If $\vec{\varphi}$ is a scale on A, then we define the tree of $\vec{\varphi}$ to be

$$T_{\vec{\varphi}} = \{(\langle x(0), ..., x(n-1)\rangle, \langle \varphi_0(x), ..., \varphi_{n-1}(x)\rangle) \mid n < \omega \text{ and } x \in A\}.$$

It is not hard to see that $p[T_{\vec{\varphi}}] = A$. If $\vec{\varphi}$ has the refinement property, and $\vec{\psi}$ is the scale of $T_{\vec{\varphi}}$, then $\vec{\psi}$ is equivalent to $\vec{\varphi}$, in the sense that for all n, x and y, $\psi_n(x) \leq \psi_n(y)$ iff $\varphi_n(x) \leq \varphi_n(y)$. The reader should see [15, 6B] and [8, **?**] for more on the relationship between scales and Suslin representations.

There are least two benefits to considering the scale of a tree: first, it becomes easier to state and prove optimal definability results, and second, the construction of Suslin representations using comparison games becomes clearer. Concerning definability, we have

Definition 2.1 *Let Γ be a pointclass, and $\vec{\varphi}$ a scale on A, where $A \in \Gamma$; then we call $\vec{\varphi}$ a* Γ-scale on A *just in case the relations*

$$R(n, x, y) \Leftrightarrow x \in A \wedge (y \notin A \vee \varphi_n(x) \leq \varphi_n(y)),$$

and

$$S(n, x, y) \Leftrightarrow x \in A \wedge (y \notin A \vee \varphi_n(x) < \varphi(y))$$

are each in Γ. We say Γ has the scale property *just in case every set in Γ admits a Γ-scale, and write Scale(Γ) in this case.*

Moschovakis showed that if Γ is a pointclass which is closed under universal real quantification, has other mild closure properties, and has the scale property, then every Γ relation has a Γ uniformization, and the Γ singletons are a basis for Γ. ([15, 3A-1]). He also showed that assuming $\boldsymbol{\Delta^1_{2n}}$ determinacy, both Π^1_{2n+1} and Σ^1_{2n+2} have the scale property. ([15, 3B, 3C].) From this, one gets the natural generalization of Novikoff-Kondo-Addison to the higher levels of the projective hierachy.

Moschovakis' construction of scales goes by *propagating* them from a set A to a set B obtained from A via certain logical operations. One starts with the fact that Σ^0_1 has the scale property, and uses these propagation theorems to obtain definable scales on more complicated sets. The propagation works at the level of the individual norms in the scales.

For example, if φ is a norm of A, where $A \subseteq X \times Y^\omega$, and

$$B(y) \Leftrightarrow \exists x A(x, y),$$

then we obtain the "inf" norm on B by setting

$$\psi(y) = \inf\{\varphi(x, y) \mid A(x, y)\}.$$

If either X is an ordinal, or $X = \omega^\omega$, then inf norms can be used to transform a scale on A into a scale on B. (See [15, 3B-2].) This transformation has a simple meaning in terms of the tree of the scale; if $X = \omega^\omega$, it corresponds to regarding a tree on $(Y \times \omega) \times \kappa$ as a tree on $Y \times (\omega \times \kappa)$.

Definable scales do not propagate under negation or universal quantification over ordinals. (Otherwise, it would be possible to assign to each countable ordinal α a scale on the set of wellorders of ω of order type α, in a definable way. This would then yield a definable function picking a codes for the countable ordinals.) Moschovakis' main advance in [34] was to show that universal quantification over the reals propagates definable scales. Here it is definitely important to work with scales, rather than their associated trees. As before, the propagation takes place at the level of individual norms. Let φ be a norm on A, where $A \subseteq \mathbb{R} \times Y$, and let

$$B(y) \Leftrightarrow \forall x A(x, y).$$

To each $y \in B$, we associate $f_y\colon \mathbb{R} \to \mathrm{OR}$, where

$$f_y(x) = \varphi(x, y).$$

Our norm on B records an ordinal measure of the growth rate of f_y. Namely, given $f, g\colon \mathbb{R} \to \mathrm{OR}$, we let $G(f, g)$ be the game on ω: I plays out x_0, II plays out x_1, the players alternating moves as usual. Player II wins iff $f(x_0) \leq g(x_1)$. (Thus a winning strategy for II witnesses that g grows at least as fast as f, in an effective way.) Now put

$$f \leq^* g \Leftrightarrow \text{II has a winning strategy in } G(f, g).$$

Granted full AD, one can show $\leq^*$ is a prewellorder of all the ordinal-valued functions on $\mathbb{R}$, and granted only determinacy for sets simply definable from φ, one can show that $\leq^*$ prewellorders the f_y for $y \in B$. Our norm on B is then given by

$$\psi(y) = \text{ ordinal rank of } f_y \text{ in } \leq^* \restriction \{f_z \mid B(z)\}.$$

(See [15, 2C-1].) The norm ψ is generally called the "fake sup" norm obtained from φ; the ordinal $\psi(y)$ measures how difficult it is to verify $A(x, y)$ at arbitrary x.

The fake-sup construction was first used in [1], to propagate the prewellordering property, which involves only one norm. Granted enough determinacy, the construction can be used to transform a scale on A into a scale on B, where $B(y) \Leftrightarrow \forall x A(x, y)$. The key additional idea is to record, for each basic neighborhood N_s, the ordinal rank of $f_y \upharpoonright N_s$ in $\leq^* \upharpoonright \{f_z \upharpoonright N_s \mid B(z)\}$. See[15, 3C-1].

Using more sophisticated comparison games, one can combine the techniques for propagating scales under universal and existential real quantification, as well as existential ordinal quantification. This leads to the propagation of scales under various *game quantifiers.* We shall discuss these results in more detail in the next section.

Although the fake-sup method of propagating scales was invented in order to obtain optimally definable scales, one can show that under AD, the tree of the scale it produces is very often homogeneous. (The tree of any scale is the surjective image of $\mathbb{R}$, so it is too small to be homogeneous in V.) See [32], where it is also shown that the tree very often has the "generic codes" property of [18].

3 Individual papers

We pass to an extended table of contents for the papers in the block to follow, together with pointers to some related results and literature. We also include a number of proof sketches. Some of these sketches will only make sense to readers with significant background knowledge. We have included references to fuller explanations in the literature when possible.

Notes on the Theory of Scales ([15])

This is a survey paper, written in 1971. It is still an excellent starting point for anyone seeking basic information regarding the construction and use of scales under determinacy hypotheses. It is truly remarkable how much of the descriptive set theory that is founded on large cardinals and determinacy emerged in the early years of the subject.

The paper begins in §2 – §4 with the inf and fake-sup constructions, and their corollaries regarding the scale property and uniformization in the projective hierarchy.

Theorem 3.1 (Moschovakis 1970) *Assume all $\boldsymbol{\Delta^1_{2n}}$ games are determined; then*

(1) Π^1_{2n+1} and Σ^1_{2n+2} have the scale property, and hence

(2) every Π^1_{2n+1} (respectively Σ^1_{2n+2}) relation on $\mathbb{R}$ can be uniformized by a Π^1_{2n+1} (respectively Σ^1_{2n+2}) function.

In §6, the *projective ordinals*

$$\begin{aligned}\boldsymbol{\delta^1_n} &= \sup\{\alpha \mid \alpha \text{ is the order type of a} \\ &\quad \boldsymbol{\Delta^1_n} \text{ prewellorder of } \mathbb{R}\}\end{aligned}$$

are introduced. One can show that, assuming PD, any Π^1_{2n+1}-norm on a complete Π^1_{2n+1} set has length $\boldsymbol{\delta^1_{2n+1}}$; see [38, 4C.14]. From the scale property for Π^1_{2n+1} one then gets that all Π^1_{2n+1} sets are $\boldsymbol{\delta^1_{2n+1}}$-Suslin, and thence that all Σ^1_{2n+2} sets are $\boldsymbol{\delta^1_{2n+1}}$-Suslin. (For $n = 0$, this reduces to the classical Novikoff-Kondo result that all Σ^1_2 sets are ω_1-Suslin.) The size of the projective ordinals, both in inner models of AD, and in the full universe V, is therefore a very important topic. It is a classical result that $\boldsymbol{\delta^1_1} = \boldsymbol{\omega_1}$, while the size of the larger projective ordinals has been the subject of much later work, some of which will be collected in a block of papers in a later volume in this series.

§7 proves the Kunen-Martin theorem:

Theorem 3.2 (Kunen, Martin) *Every κ-Suslin wellfounded relation on $\mathbb{R}$ has rank $< \kappa^+$.*

This basic result has important corollaries concerning the sizes of the projective ordinals. For example, because all Σ^1_{2n+2} sets are $\boldsymbol{\delta^1_{2n+1}}$-Suslin, we have that $\boldsymbol{\delta^1_{2n+2}} \leq (\boldsymbol{\delta^1_{2n+1}})^+$, and in particular, $\boldsymbol{\delta^1_2} \leq \boldsymbol{\omega_2}$.

§8 investigates the way in which Suslin representations yield ∞-Borel representations. It is shown that κ-Suslin sets are κ^{++}-Borel (i.e. can be built up from open sets using complementation and wellordered unions of length $< \kappa^{++}$). Of course, if CH holds, then *every* set of reals is a union of ω_1 singletons; the true content of the result of §8 lies in the fact that the κ^{++}-Borel representation is definable from the κ-Suslin representation. §8 also shows that, assuming PD, every $\boldsymbol{\Delta^1_{2n+1}}$ set is $\boldsymbol{\delta^1_{2n+1}}$-Borel. If $n = 0$, this is just Suslin's original theorem. In order to obtain a converse when $n > 0$, we must impose a definability restriction on our $\boldsymbol{\delta^1_{2n+1}}$-Borel representation, since again, it could be that every set of reals is $\omega_1 + 1$-Borel. One way to do that is to assume full AD, and Martin showed that indeed, assuming AD, every $\boldsymbol{\Delta^1_{2n+1}}$ set is $\boldsymbol{\delta^1_{2n+1}}$-Borel. So we have

Theorem 3.3 (Martin, Moschovakis) *Assume AD; then the $\boldsymbol{\Delta^1_{2n+1}}$ sets of reals are precisely the $\boldsymbol{\delta^1_{2n+1}}$-Borel sets.*

See [38, 7D.9]. This fully generalizes Suslin's 1917 theorem to the higher levels of the projective hierarchy.

§5 and §9 introduce inner models, obtained from Suslin representations, which have certain degrees of correctness. In §5, it is shown that for $n \geq 2$, there is a unique, minimal Σ^1_n-correct inner model M^*_n containing all the ordinals; the model is obtained by closing under constructibility and an optimally definable Skolem function for Σ^1_n. (Kechris and Moschovakis call this model M_n, but "M_n" is now generally used to denote a different model; see below.) §9 considers the model $L[T]$, where T is the tree of a Π^1_{2n+1} scale on a complete Π^1_{2n+1} set. These models have proved more important in later work than the M^*_n. It is shown that if $n = 0$, then $L[T] = L$; in particular, $L[T]$ is independent of the Π^1_{2n+1} scale and complete set chosen. Moschovakis conjectured that $L[T]$ is independent of these choices if $n > 0$ as well, and more vaguely, that it is a "correct higher level analog of L".

Becker's paper [2] contains an excellent summary of what was known in 1977 about the models of §5 and §9. The independence conjecture, which inspired a great deal of work, became Victoria Delfino problem 3. Harrington and Kechris ([7]) made a significant advance by showing that the reals of $L[T]$, where T is the tree of any Π^1_{2n+1} scale on a complete Π^1_{2n+1} set, are the largest countable Σ^1_{2n+2} set of reals, and hence independent of the choice of T. Building on this work, Moschovakis made a step forward in the late 70's with the introduction of the model H_Γ, for Γ a pointclass which *resembles* Π^1_1 in a certain technical sense, and has the scale property. (See [38, 8G.17 ff.].) Assuming $\boldsymbol{\Delta^1_{2n}}$-determinacy, the pointclass Π^1_{2n+1} is an example of such a Γ, but there are many more examples. The model H_Γ is of the form $L[U]$, where U is a universal $\exists^{\mathbb{R}}\Gamma$ (in the codes) subset of the prewellordering ordinal of Γ, and one can think of it as a fragment of HOD corresponding to Γ-definability. Using the Harrington-Kechris work, Moschovakis showed that H_Γ is independent of the universal set and Γ-norm used to define U, that it includes $L[T]$, for tree T of a Γ scale on a complete Γ set, and that $\mathbb{R} \cap H_\Gamma$ is the largest countable $\exists^{\mathbb{R}}\Gamma$ set of reals. (See [38, 8G.17 ff.].) Moschovakis' results require a bit more than $\boldsymbol{\Gamma}$-determinacy.

The independence of $L[T]$ was finally proved by Becker and Kechris ([3]), who showed

Theorem 3.4 (Becker, Kechris 1984) *Let Γ be a pointclass which resembles Π^1_1 and has the scale property, and suppose AD holds in $L(\boldsymbol{\Gamma}, \mathbb{R})$. Let T be the tree of any Γ-scale on a complete Γ set; then $L[T] = H_\Gamma$.*

The Becker-Kechris proof makes heavy use of a class of games introduced by Martin in order to obtain an approximation to 3.4.

Not long after the last of the Cabal Seminar volumes appeared, our understanding of the large cardinal side of the "equivalence" between large cardinals and determinacy caught up with our understanding of the determinacy side. This equivalence is mediated by the canonical inner models for large cardinal hypotheses, which are sometimes called *extender models*. We can now identify each of the models of §5 and §9 as an extender model, and thereby understand it much more deeply than we could using only pure descriptive set theory. For example, most nontrivial facts in the first order theory of $L[T]$ (e.g., that the GCH, and Jensen's diamond and square principles, hold in $L[T]$) seem to require its identification as an extender model for proof. The identifications are as follows: Here and in the rest of the paper, for $0 \leq n \leq \omega$, we let M_n be the minimal iterable proper class extender model with n Woodin cardinals. If $n \geq 2$ is even, then M^*_n is $L[M_{n-2}|\gamma]$, where γ is least such that $\gamma = \omega_1^{L[M_{n-2}|\gamma]}$ and $L[M_{n-2}|\gamma]$ is Σ^1_n-correct. (For $n > 2$, we have that $\gamma < \omega_1^{M_{n-2}}$.) If n is odd, then M^*_n is the minimal proper class extender model Q such that if S is an initial segment of Q projecting to ω, then $M_{n-2}(S)^\sharp$ is an initial segment of Q. These identifications are implicit in [55]. Finally, if $n \geq 3$ is odd, and T is the tree of a Π^1_n scale on a complete Π^1_n set, then there is an iterate Q of M_{n-1} such that $L[T] = L[Q|\boldsymbol{\delta}^1_{\mathbf{n}}]$. This identification is implicit in [57], where the parallel fact with the pointclass Π_n replaced by $\Sigma_1^{L(\mathbb{R})}$, and M_{n-1} replaced by M_ω, is proved. So we have

Theorem 3.5 (Steel 1994) *Assume there are ω Woodin cardinals with a measurable above them all, and let $\Gamma = \Pi^1_{2n+1}$ or $\Gamma = \Sigma_1^{L(\mathbb{R})}$; then H_Γ is an iterable extender model.*

In a similar vein, the prewellordering and scale theorems of §2 - §4 can now be proved using extender models. In the prewellordering case, the proof is due to Woodin, and in the scale case, to Neeman; in neither case is the proof published, but see [55]. These proofs require significantly more theory than the comparsion game approach, but in some ways they give deeper insight into the meaning of the norms being constructed.

Finally, Suslin and ∞-Borel representations are related to Lebesgue measurability, the Baire property, and the perfect set property in §10 and §11. Solovay's breakthrough results from 1966 on the regularity of Σ^1_2 sets under large cardinal hypotheses ([47]) are thereby extended to other pointclasses. A basic result on the existence of largest countable sets is proved (in effect):

Theorem 3.6 (Kechris, Moschovakis) *Suppose Γ is adequate, ω-parametrized, has the scale property, and is closed under $\exists^{\mathbb{R}}$, and suppose all $\boldsymbol{\Gamma}$ games are determined; then there is a largest countable Γ set of reals.*

When it exists, the largest countable Γ set is called C_Γ. The theorem is implicit in the proof of Theorem 11B-2, which proves the existence of C_Γ for $\Gamma = \Sigma^1_{2n}$. Kechris' paper [13] contains further basic information in this area. The sets C_Γ are quite important, partly because many of them show up naturally as the set of reals in some canoncal inner model. For example, Solovay showed that $C_{\Sigma^1_2} = \mathbb{R} \cap L$ ([15, 11B-1]), and we now know that for any n, $C_{\Sigma^1_{2n+2}} = \mathbb{R} \cap M_{2n}$. (See [55]. Note that $M_0 = L$.) In general, under the hypotheses of 3.4, we have $C_{\exists^{\mathbb{R}}\Gamma} = \mathbb{R} \cap H_\Gamma$. (See [38, 8G.29].)

Kechris [13] shows that assuming $\boldsymbol{\Pi}^1_{\mathbf{2n+1}}$- determinacy, there is a largest countable Π^1_{2n+1} set of reals $C_{\Pi^1_{2n+1}}$. This result is due to Guaspari, Kechris, and Sacks for n=0, in which case $C_{\Pi^1_{2n+1}}$ has an inner-model-theoretic meaning as the set of reals Δ^1_{2n+1}-equivalent to the first order theory of some level of M_{2n} projecting to ω. It is open whether this characterization of $C_{\Pi^1_{2n+1}}$ holds also for $n > 0$.

Propagation of the Scale Property by Game Quantifiers ([39])
Scales on $\boldsymbol{\Sigma}^1_1$ Sets ([49])

Moschovakis unified his results on scale propagation under the real quantifiers into a single theorem on the propagation of scales under the *game quantifier* on ω. Letting $A \subseteq \mathbb{R} \times \mathbb{R}$, we put

$$\Game y A(x,y) \Leftrightarrow \exists n_0 \forall n_1 \exists n_2 \forall n_3 ... A(x, \langle n_i \mid i < \omega \rangle),$$

where we interpret the right hand side as meaning its quantifier string has a Skolem function, that is, that player I wins the game on ω with payoff $A_x = \{y \mid A(x,y)$. We write $\Game A$ for $\{x \mid \Game y A(x,y)\}$, and if Γ is a pointclass, we set $\Game\Gamma = \{\Game A \mid A \in \Gamma\}$. The following is often called the *third periodicity* theorem. It dates from approximately 1973; see [35] or [38, 6E].

Theorem 3.7 (Moschovakis) *Let Γ be an adequate, ω-parameterized pointclass closed under quantification over ω, and suppose $\Gamma(x)$-determinacy holds for all reals x. Suppose Γ has the scale property; then*

(a) $\Game\Gamma$ has the scale property, and

(b) if G is a game on ω with payoff set in Γ, and the player whose payoff is Γ has a winning strategy in G, then that player has a $\Game\Gamma$ winning strategy.

The proof involves a more sophisticated comparison game: given a norm φ on A, one gets a norm on $\Game A$ using comparison games in which the two players play out the games with payoff A_{x_1} and A_{x_2} simultaneously, in different roles on the two boards, each trying to win in his role as player I with lower φ-norm than the other. The first paper in the present pair gives a thorough exposition of the proof of this theorem. (See also [38, **Section?**].)

It is easy to see that $\Game\Pi^1_n = \Sigma^1_{n+1}$, and assuming $\boldsymbol{\Sigma^1_n}$-determinacy, that $\Game\Sigma^1_n = \Pi^1_{n+1}$. Setting $\Sigma^1_0 = \Sigma^0_1$, this is true for $n = 0$ as well. Thus 3.7 subsumes 3.1. Part (b) of 3.7, on the existence of canonical winning strategies, is very useful. In the special case of projective sets, we get

Corollary 3.8 (Moschovakis) *Assume $\boldsymbol{\Delta^1_{2n}}$-determinacy, and let G be a game with Σ^1_{2n} payoff, and suppose the player with Σ^1_{2n} payoff has a winning strategy; then he has a Δ^1_{2n+1} winning strategy.*

Moschovakis' proof used $\boldsymbol{\Sigma^1_{2n}}$-determinacy, but Martin later showed this follows from $\boldsymbol{\Delta^1_{2n}}$-determinacy, so we have stated the theorem in its sharper form. Of course, we also get Δ^1_{2n+2} strategies for games won by a player with Π^1_{2n+1} payoff from 3.8, but this already follows easily from the basis theorem for Π^1_{2n+1}. It is easy to see that these definability bounds on winning strategies are optimal.

It is natural to ask what are the optimally definable scales and winning strategies for the projective pointclasses which zig when they should have zagged, that is, for Σ^1_{2n+1} and

Π^1_{2n+2}. The second paper in this pair gives part of the answer. Let $\alpha - \Pi^1_1$ be the α^{th} level of the difference hierarchy over Π^1_1 (see [49]). and let

$$\Lambda_0 = \bigcup_{k<\omega} \omega k - \Pi^1_1.$$

Steel gives a simple proof in [49] that every Σ^1_1 set admits a very good scale whose associated prewellorders are each in Λ_0, and in fact, each set in Λ_0 admits a very good scale whose associated prewellorders are all in Λ_0. (We are *not* demanding that the sequence of prewellorders be in Λ_0.) Now let $\Game^{(n)} = \Game....\Game$ be the n-fold composition of the game quantifier on ω; then the proof of third periodicity theorem easily gives

Theorem 3.9 (Steel 1980) *Let $n \geq 1$, and suppose all $\Game^{(n-1)}\Lambda_0(x)$ games are determined, for all reals x; then*

- *(a) every $\Game^{(n)}\Lambda_0$ set admits a very good scale, all of whose norms are $\Game^{(n)}\Lambda_0$, and*
- *(b) if G is a game with payoff in $\Game^{(n-1)}\Lambda_0$, then there is a winning strategy σ for G such that for any k, $\sigma \restriction \{p \mid \mathrm{lh}(p) \leq k\}$ is in $\Game^{(n)}\Lambda_0$.*

It is easy to see that for $n \geq 1$,

$$(\Sigma^1_n \cup \Pi^1_n) \subseteq \Game^{(n-1)} \bigcup_{k<\omega} \omega k - \Pi^1_1 \subseteq \Delta^1_{n+1}.$$

The best bounds on the definability of very good scales and winning strategies for Σ^1_n sets (n odd) and Π^1_n sets (n even) are just those given by Theorem 3.9 and this inclusion. That the bounds cannot be improved follows from Martin [26]; see below.

(We should note here that Busch [4] showed that every Σ^1_1 set admits a scale all of whose prewellorders are $(\omega+3) - \Pi^1_1$. However, the Busch scale is not very good, and transforming it to a very good scale involves taking intersections, which drives us up to Λ_0. The third periodicity propagation technique requires, in effect, that the input scale be very good.)

The progress of inner model theory has shed some light on these results. Neeman [40] gives an inner-model-theoretic proof that every Σ^1_{2n} game won by the player with Σ^1_{2n} payoff has as Δ^1_{2n+1} winning strategy, as a byproduct of his proof of Σ^1_{2n} determinacy from the existence and iterability of M_{2n-1}. Neeman's work also gives an insight into the pointclasses $\Game^{(n)}\Lambda_0$. For $n \geq 0$, let T^n_k be the theory in M_n of its first k indiscernibles. Thus the reals in M_n are just those reals which are recursive in some T^n_k. One can show that every $\Game^{n+1}\omega k - \Pi^1_1$ real is recursive in T^n_k, and that T^n_k itself is $\Game^{(n+1)}\omega(k+1) - \Pi^1_1$. The proof is an induction on n, with the base case $n = 0$ being due to Martin, as part of his proof of $\omega k - \Pi^1_1$-determinacy from the existence of the sharp of $M_0 = L$. (Here is a proof sketch of the $n > 0$ case for experts: To reduce a $\Game^{n+1}\omega k - \Pi^1_1$ real to T^n_k, we ask questions about what is forced in collapse of the bottom Woodin of M_n about its first k indiscernibles. The answer we get will reflect $\Game^{n+1}\omega k - \Pi^1_1$ truth because every real, and in particular a winning strategy witnessing or refuting the outer $\Game$ quantifier, is generic over an iterate of M_n for this collapse. To show that T^n_k itself is $\Game^{(n+1)}\omega(k+1) - \Pi^1_1$, we use a game in which the players play a putative $M^\sharp_n$'s, say P and Q respectively, and then the two are coiterated inside $M_{n-1}(\langle P, Q\rangle)$.)

It follows that $M^\sharp_n$ is Turing equivalent to the set of true $\Game^{(n+1)}\Lambda_0$ sentences. From this we see that any game with $\Game^{(n)}\Lambda_0$ payoff has a winning strategy which is recursive in $M^\sharp_n$. (By 3.14(b) below, no better definability bound is possible.) In particular, every nonempty Σ^1_{2n+1} set has a member recursive in $M^\sharp_{2n}$, using the trivial game in which I must play a member of the set and a witness to the Π^1_{2n} matrix, and II does nothing. This gives us an inner-model-theoretic proof of Martin and Solovay's generalization of the Kleene Basis Theorem for Σ^1_1 ([14, 5.6]).

Inductive Scales on Inductive Sets ([36])
Scales on Coinductive Sets ([37])
The Extent of Scales in $L(\mathbb{R})$ ([29])

It is natural to try to extend the civilizing influence of definable scales to more complicated sets. The remaining papers in this block use the comparison game construction to do that, while showing that, most of the time, the scales produced are definable in the simplest possible logical form.

The papers in this group, which represent work done in late 1979, exploit the uniformities in the comparison game method of propagating scales. Let us use $\exists^{\mathrm{or}}$, $\exists^{\mathbb{R}}$, and $\forall^{\mathbb{R}}$ to stand for existential quantification over the ordinals, over the reals, and universal quantification over the reals, respectively. Because the propagation of scales under these operations is uniform in the scales, one gets inductive scales on inductive sets; this is done in [36]. (A set is inductive iff it is $\Sigma_1^{J_{\kappa_{\mathbb{R}}}(\mathbb{R})}$, where $\kappa_{\mathbb{R}}$ is least κ such that $J_\kappa(\mathbb{R}) \models \mathsf{KP}$.) Since AD implies that the pointclass of inductive sets is closed under real quantification and wellordered unions, it seemed at first that one needed a radically new idea to go further. (One cannot hope to show that the class of scaled sets is closed under complement!) The existence of definable scales for coinductive sets became Victoria Delfino problem 2. However, it turned out that what was missing was more in the nature of a subtle observation: the comparison game propagation of scales under $\exists^{\mathrm{or}}$, $\exists^{\mathbb{R}}$, and $\forall^{\mathbb{R}}$ acts at the level of individual norms–it corresponds, in each case, to a *continuous* operation on the input scale. Moschovakis realized this, and realized that it could be used to define scales on any set A definable in the form

$$A(x) \Leftrightarrow \exists x_0 \exists \alpha_0 \forall x_1 \exists x_2 \exists \alpha_1 \forall x_3 ... \forall n R(\langle x_0|n, ..., x_n|n\rangle, \langle \alpha_0, ..., \alpha_n\rangle),$$

where R is definable, the α's are ordinals, and the x's are reals. (This is done by simultaneously defining scales on each of the ω-many sets defined by the formula on the right with some initial segment of its quantifiers removed. The scale on any such set is obtained from the scale on the set corresponding to removing one more quantifier by the continuous operation corresponding to that quantifier.)

The expression displayed on the right hand side above gives what is called a *closed game representation* of A: it asserts that player I wins the infinite game in which he plays the even x's and the α's, while II plays the odd x's, and the payoff indicated by the matrix is closed in the product of the Baire topology on $\mathbb{R}$ and the discrete topology on the ordinals. What [37] shows, in effect, is that granted sufficient determinacy, any set with a closed game representation admits a definable scale. (The converse is trivial.) In the special case that the game involves no ordinal moves, one gets

Theorem 3.10 (Moschovakis 1979) *Suppose all games in $J_{\kappa_{\mathbb{R}}+1}(\mathbb{R})$ are determined, and let A be coinductive; then A admits a scale whose associated prewellorders are each in $J_{\kappa_{\mathbb{R}}+1}(\mathbb{R})$.*

Martin and Steel showed in [29] that in fact, every $\Sigma_1^{L(\mathbb{R})}$ set admits a closed game representation in $L(\mathbb{R})$, which together with Moschovakis' work and some simple definability calculations implies

Theorem 3.11 (Martin, Steel 1979) *Assume $\mathsf{AD}^{L(\mathbb{R})}$; then the pointclass $\Sigma_1^{L(\mathbb{R})}$ has the scale property.*

Kechris and Solovay had observed earlier that, assuming AD, the relation "$x, y \in \mathbb{R}$ and y is not ordinal definable from x" is ordinal definable, but admits no uniformization, and hence no scale, which is ordinal definable from a real. (Let f be a uniformizing function, and suppose f is ordinal definable from x; then $f(x)$ is ordinal definable from x, a contradiction.) If $V = L(\mathbb{R})$, then this relation is Π_1, while every set whatsoever is ordinal definable from a real, so we have a Π_1 set which admits no scale at all. A simple Wadge argument then shows that assuming $\mathsf{AD}^{L(\mathbb{R})}$, the sets admitting scales in $L(\mathbb{R})$ are precisely the $\Sigma_1^{L(\mathbb{R})}$ sets.

Under suitable large cardinal assumptions, one can construct natural models of AD properly larger than $L(\mathbb{R})$. These models, and $L(\mathbb{R})$ itself, all satisfy a certain strengthening of AD called AD^+. The theory AD^+ was isolated by Woodin, and part of his work is the following far-reaching generalization of 3.11:

Theorem 3.12 (Woodin, mid 90's) *Assume AD^+; then the pointclass Σ_1^2 has the scale property.*

Note here that $\Sigma_1^{L(\mathbb{R})} = (\Sigma_1^2)^{L(\mathbb{R})}$, so that Woodin's theorem reduces to that of Martin and Steel if our model of AD^+ is $L(\mathbb{R})$. No proof of 3.12 has been published as yet, but the theory AD^+ is described in [62, Section 9.1], where 3.12 and related results are stated as Theorem 9.7. A proof that Σ_1^2 has the scale property in those models of AD^+ obtained from models with large cardinals via the standard means, i.e. the derived model construction, is exposited in [59, §7].

The Largest Countable This, That, and the Other ([26])

Moschovakis [37] shows that the norms of the scale on a coinductive set it constructs are each first order definable over $J_{\kappa_{\mathbb{R}}}(\mathbb{R})$. It is natural to ask whether one can do better: does every coinductive set admit a scale such that for some fixed $n < \omega$, all the norms are $\mathbf{\Sigma_n^{J_{\kappa_{\mathbb{R}}}(\mathbb{R})}}$? In [26], Martin proves an important reflection result which implies that the answer is "no". Let us write $y \in \mathrm{OD}^{\alpha}(x)$ to mean that y is ordinal definable from x over $J_\alpha(\mathbb{R})$.

Theorem 3.13 (Martin 1980) *Assume inductive determinacy, and suppose $x, y \in \mathbb{R}$ and $y \in OD^{\kappa_{\mathbb{R}}}(x)$; then $y \in OD^{\alpha}(x)$ for some $\alpha < \kappa_{\mathbb{R}}$.*

(Though not literally stated in [26], this is a fairly direct consequence of Lemma 4.1.) Now the relation "$x, y \in \mathbb{R}$ and $\forall \alpha < \kappa_{\mathbb{R}}(y \notin \mathrm{OD}^{\alpha}(x))$" is coinductive, and by the Kechris-Solovay argument, it cannot be uniformized by a function in $J_{\kappa_{\mathbb{R}}+1}(\mathbb{R})$, and hence it admits no scale whose sequence of associated prewellorders is in $J_{\kappa_{\mathbb{R}}}(\mathbb{R})$. Thus one cannot improve Moschovakis' definability bound.

Martin's reflection result is part of a characterization of C_Γ, of the largest countable Γ set of reals, for various pointclasses Γ. Letting $\mathrm{IND} = \Sigma_1^{J_{\kappa_{\mathbb{R}}}}(\mathbb{R})$ be the pointclass of (lightface) inductive sets, it is easy to see that assuming inductive determinacy, $C_{\mathrm{IND}} = \{y \mid y \in \mathrm{OD}^{\kappa_{\mathbb{R}}}(\emptyset)\}$. (See [50, 2.11].) This characterizes the members of C_{IND} in terms of definability from ordinals, in a way which parallels Kechris' characterization of $C_{\Sigma^1_{2n}}$ as $\{y \mid y$ is Δ^1_{2n} in a countable ordinal$\}$. Martin found characterizations of C_{IND} and $C_{\Sigma^1_{2n}}$ in terms of definability wihout parameters:

Theorem 3.14 (Martin 1980) *Assume all games in $J_{\kappa_{\mathbb{R}}+1}(\mathbb{R})$ are determined; then for any real y*

(a) $y \in C_{IND}$ iff y is definable over $J_{\kappa_{\mathbb{R}}}(\mathbb{R})$ from no parameters, and

(b) $y \in C_{\Sigma^1_{2n}}$ iff y is $\Game^{(2n-1)} \bigcup_{k<\omega} \omega k - \Pi^1_1$.

The left-to-right directions make use of the existence of scales on coinductive sets in case (a), and on Π^1_{2n} sets in case (b), which are definable in the appropriate way: each norm being definable over $J_{\kappa_{\mathbb{R}}}(\mathbb{R})$ in case (a), each norm being $\Game^{(2n-1)} \bigcup_{k<\omega} \omega k - \Pi^1_1$ in case (b). The right-to-left directions are reflection arguments, and they show that the definability bounds for scales used in the other direction are best possible.

Soon after Martin proved 3.14, Kechris and Woodin used his technique, among other ideas, to prove a vaguely similar reflection result: if there is a non-determined game in

$J_{\kappa_{\mathbb{R}}+1}(\mathbb{R})$, then there is a non-determined game in $J_{\kappa_{\mathbb{R}}}(\mathbb{R})$. It follows that the hypothesis of Moschovakis' scale existence result 3.10 can be reduced to $J_{\kappa_{\mathbb{R}}}(\mathbb{R})$-determinacy. See [17], which is reprinted in Part II of this volume. Somewhat more general results along the same lines were a key ingredient in the proof of

Theorem 3.15 (Kechris, Woodin 1980 date?) *If there is a non-determined game in $L(\mathbb{R})$, then there is a non-determined game whose payoff is Suslin in $L(\mathbb{R})$.*

The structure of this proof has played an important role in later proofs of $\mathsf{AD}^{L(\mathbb{R})}$ under various hypotheses. See below.

Kechris-Woodin [17] also proves something along the lines of 3.14(b): $\boldsymbol{\Delta^1_{2n}}$-determinacy implies $\Game^{2n-1}\bigcup_{k<\omega}\omega k-\Pi^1_1$-determinacy. (Martin had proved $\boldsymbol{\Delta^1_{2n}}$-determinacy implies $\boldsymbol{\Sigma^1_{2n}}$-determinacy earlier, by a different method. See [16].)

Once again, the progress of inner model theory has given us deeper insight into these results of Martin and Kechris-Woodin. Itay Neeman found an inner-model-theoretic proof of 3.14(b). Let M_n be the minimal iterable proper class extender model with n Woodin cardinals, and T^n_k be the theory in M_n of its first k indiscernibles. By [55], the reals in $C_{\Sigma^1_{2n}}$ are precisely the reals in M_{2n}, and hence are just those reals which are recursive in some T^{2n}_k. But by [40], every $\Game^{2n-1}\omega k - \Pi^1_1$ real is recursive in T_k, and that T_k itself is $\Game^{(2n-1)}\omega(k+1) - \Pi^1_1$. (We sketched this proof above.) 3.14(b) now follows easily. Mitch Rudominer found an inner-model-theoretic proof of 3.14(a); in this case, the role of M_{2n} is played by the minimal iterable extender model P having ω Woodin cardinals, which by minimality are cofinal in the ordinals of P, and the role of T^{2n}_k is played by the Σ_0 theory in P of its first k Woodin cardinals. See [45].

It is worth noting that the set of reals in M_{2n+1} is also well known from descriptive set theory; it is the set Q_{2n+1}. See [14] for the many characterizations of this set. In general, the reals of M_n, for any n, can be characterized in terms of ordinal definability as those reals which are Δ^1_{n+2} in a countable ordinal.

Neeman and Woodin have proved the Kechris-Woodin theorem within the projective hierarchy by the methods of inner model theory, and at the same time generalized it to the odd levels as well. Woodin (unpublished) showed that for any $n \geq 1$, $\boldsymbol{\Pi^1_n}$-determinacy implies that for all reals x, $P_n(x)^\sharp$ exists, and Neeman [40] showed that the existence of these mice implies $\Game^{n-1}\bigcup_{k<\omega}\omega k - \Pi^1_1$-determinacy. For $n = 1$, these are results of Harrington [6] Martin (unpublished) respectively. It is known from work of Kechris and Solovay ([16]) that these "transfer results" for determinacy in the projective hierarchy cannot be improved.

No one has as yet proved that $J_{\kappa_{\mathbb{R}}}(\mathbb{R})$-determinacy implies $J_{\kappa_{\mathbb{R}}+1}(\mathbb{R})$-determinacy by purely inner-model-theoretic methods.

Scales in $L(\mathbb{R})$ ([50])
Scales in $K(\mathbb{R})$([51])

Given that a set of reals admits a scale in $L(\mathbb{R})$, it is natural to ask what is the least level of the Levy hierarchy for $L(\mathbb{R})$ at which such a scale appears. The papers in the last group answered this question in some important cases. The paper [50], work of Steel from 1980, knits the arguments of those papers together into a complete answer.

It turns out that the (lexicographically) least $\langle\gamma, n\rangle$ such that A admits a $\mathbf{\Sigma_n^{J_\gamma(\mathbb{R})}}$ scale is the lexicographically least $\langle\gamma, n\rangle$ such that $A \in \mathbf{\Sigma_n^{J_\gamma(\mathbb{R})}}$ and $\mathbf{\Sigma_n^{J_\gamma(\mathbb{R})}}$ has the scale property. [50] characterizes those pointclasses $\mathbf{\Sigma_n^{J_\gamma(\mathbb{R})}}$ which have the scale property in terms of reflection properties. The key concept is that of a Σ_1-*gap*. Let us say $\mathcal{M} \prec_1^{\mathbb{R}} \mathcal{N}$ iff $\mathcal{M}$ is an elementary submodel of $\mathcal{N}$ with respect to Σ_1 formulae about parameters from $\mathbb{R} \cup \{\mathbb{R}\}$.

Definition 3.16 *The interval* $[\alpha, \beta]$ *is a* Σ_1-gap *iff*

1. $J_\alpha(\mathbb{R}) \prec_1^{\mathbb{R}} J_\beta(\mathbb{R})$,
2. $\forall\gamma < \alpha(J_\gamma(\mathbb{R}) \not\prec_1^{\mathbb{R}} J_\alpha(\mathbb{R}))$, *and*
3. $J_\beta(\mathbb{R}) \not\prec_1^{\mathbb{R}} J_{\beta+1}(\mathbb{R})$.

With the convention that $[(\delta_1^2)^{L(\mathbb{R})}, \infty]$ is also a Σ_1 gap, we have that Σ_1 gaps partition OR. In order to determine whether $\mathbf{\Sigma_n^{J_\gamma(\mathbb{R})}}$ has the scale property, we consider the unique Σ_1 gap $[\alpha, \beta]$ to which γ belongs. [50] shows that, assuming enough determinacy,

(1) if $\alpha < \gamma < \beta$ then $\mathbf{\Sigma_n^{J_\gamma(\mathbb{R})}}$ does not have the scale property (Kechris, Solovay),

(2) if $\gamma = \alpha$ and $n = 1$, then $\mathbf{\Sigma_n^{J_\gamma(\mathbb{R})}}$ has the scale property,

(3) if $\gamma = \alpha, n > 1$, and α is admissible, or if $\gamma = \beta$ and $[\alpha, \beta]$ is a strong gap, then $\mathbf{\Sigma_n^{J_\gamma(\mathbb{R})}}$ does not have the scale property (Martin),

(4) if $\gamma = \alpha, n > 1$, and α is inadmissible, or if $\gamma = \beta$ and $[\alpha, \beta]$ is a weak gap, and $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$, then $\mathbf{\Sigma_n^{J_\gamma(\mathbb{R})}}$ or its dual class has the scale property, according to the "zig-zag" pattern.

The weak/strong distinction for gaps is motivated by Martin's proof of 3.14; strong gaps have a reflection property which is used in this argument.

The most important applications of this analysis occur in inductive proofs of $\mathsf{AD}^{L(\mathbb{R})}$. The first of these, which set the pattern, is the Kechris-Woodin proof of 3.15, that Suslin determinacy implies determinacy in $L(\mathbb{R})$. Their argument goes roughly as follows: let β be least such that there is a non-determined game in $J_{\beta+1}(\mathbb{R})$. The failure of determinacy is a Σ_1 fact, so β ends a gap of the form $[\alpha, \beta]$. Setting $\gamma = \beta$, and letting n be least such that there is a non-determined $\mathbf{\Sigma_n}^{\mathbf{J}_\beta(\mathbb{R})}$ game, we have enough determinacy to prove (1)-(4) above. (This comes from inspecting the proof of [50], and using an observation of Kechris.) The

Kechris-Woodin result that $J_{\kappa_{\mathbb{R}}}(\mathbb{R})$-determinacy implies $J_{\kappa_{\mathbb{R}}+1}(\mathbb{R})$-determinacy generalizes routinely so as to show that we cannot be in case (3). In all other cases, we have enough determinacy to show that either $\mathbf{\Sigma_n}^{\mathbf{J}_\beta(\mathbb{R})}$ or $\mathbf{\Sigma_{n+1}}^{\mathbf{J}_\beta(\mathbb{R})}$ has the scale property. Thus the payoff of our non-determined game is Suslin in $L(\mathbb{R})$, a contradiction.

Kechris and Woodin used 3.15 to show that if $V = L(\mathbb{R})$ and there are arbitrarily large strong partition cardinals below Θ, then AD holds. With the proofs of determinacy from Woodin cardinals in the mid-80's ([30]), and the advances in techniques for constructing correct inner models with Woodin cardinals in the late 80's and 1990 ([31], [33], [56]), it became possible to use this pattern of argument to prove $\mathsf{AD}^{L(\mathbb{R})}$ under many different hypotheses. Woodin pioneered this *core model induction* method, in his 1990 proof that the Proper Forcing Axiom together with the existence of an inaccessible cardinal implies $\mathsf{AD}^{L(\mathbb{R})}$. In such an argument, one proves not just determinacy, but the existence of correct mice "certifying" the determinacy in question, by an induction. The scale analysis is used to get a definable scale on a set coding truth at the next level of correctness, and then core model theory is used to construct mice correct at that level. The method has been used a number of times since 1990, by Woodin and others. (See [19], [63].) Indeed, if the large cardinal strength of a theory T is not close to the surface, it is highly likely that any proof that T implies $\mathsf{AD}^{L(\mathbb{R})}$ will use the core model induction method. Only very recently has a paper describing and using the method been published; see [58].

[51] extends the scale analysis of [50] to $K(\mathbb{R})$. This is useful in constructing mice with more than infinitely many Woodin cardinals by the core model induction method.

The Real Game Quantifier Propagates Scales ([27])
Long Games ([52])
The length-ω_1 open game quantifier propagates scales ([53])

These papers push the comparison game construction of scales as far as it has been pushed to date.

It is natural to ask whether there is a propagation theorem behind Moschovakis' theorem 3.10. Martin obtains a positive answer in [27], showing that the game quantifier corresponding to games of length ω on the reals propagates definable scales. In fact, he shows that the game quantifier corresponding to games of any fixed countable length propagates definable scales. (Note here that a game of length α on $\mathbb{R}$ can be simulated by a game of length $\omega\alpha$ on ω.) He also proves the other half of third periodicity, that there are definable winning strategies for these games. Both theorems require the determinacy of the games in question as a hypothesis.

Steel extends these results to the game quantifiers corresponding to clopen games of length ω_1 in [52], and then finally to the game quantifier corresponding to open games of length ω_1 in [53]. These latter results represent the limit of what has been done in this direction, so let us state them more precisely.

Let $A \subseteq \mathbb{R} \times \omega^{<\omega_1}$. For $x \in \mathbb{R}$, consider the game of length ω_1 in which I and II play natural numbers, alternating as usual, with I going first at limit ordinals. We fix some natural way of representing runs of such games by $p \in \omega^{\omega_1}$, and then declare that p is a winning run for I in G_{A_x} iff $\exists \alpha < \omega_1 A(x, p \restriction \alpha)$. Thus G_{A_x} is open for I in the topology on ω^{ω_1} whose basis consists of sets of the form $\{p \in \omega^{\omega_1} \mid q \subset p\}$, where $q \in \omega^{<\omega_1}$. We then put

$$x \in \Game^{\omega_1} A \Leftrightarrow \text{ I has a winning strategy in } G_{A_x}.$$

In order to calculate definability, we code countable sequences q of natural numbers by reals q^* in some simple way. Putting $A^*(x,y) \Leftrightarrow \exists q(y = q^* \wedge A(x,q))$, we then define, for any pointclass Γ,

$$\Game^{\omega_1} \text{ open} - \Gamma = \{\Game^{\omega_1} A \mid A^* \in \Gamma\}.$$

The main result of [53] is

Theorem 3.17 (Steel) *Let Γ be an adequate pointclass with the scale property, and suppose that all (boldface) length ω_1 open-Γ games are determined; then*

(a) $\Game^{\omega_1}$ open $- \Gamma$ has the scale property, and

(b) if the player with open payoff has a winning strategy in a length ω_1 open-Γ game, then he has a $\Game^{\omega_1}$ open Γ winning strategy.

Thus the determinacy of length ω_1 open-Δ^1_2 games implies that $\Game^{\omega_1}$ open $- \Delta^1_2$ has the scale property. It is not known how to construct definable scales on the complements of $\Game^{\omega_1}$ open $- \Delta^1_2$ sets, or show that if the closed player wins such a game, he has a definable winning strategy, under any definable determinacy or large cardinal assumption. We should note here that assuming the determinacy of the long games involved, the complements of $\Game^{\omega_1}$ open $- \Delta^1_2$ sets are just the $\Game^{\omega_1}$ closed $- \Delta^1_2$ sets. in the natural meaning of this term. All $\Game^{\omega_1}$ closed $- \Delta^1_2$ sets are Σ^2_1, and assuming CH, all Σ^2_1 sets are $\Game^{\omega_1}$ closed $- \Delta^1_2$. Woodin's Σ^2_1 absoluteness theorem ([22, Theorem 3.2.1]) can be formulated without referring to CH as follows: if there are arbitrarily large measurable Woodin cardinals, then $\Game^{\omega_1}$ closed $- \Delta^1_2$ sentences are absolute between set-generic extensions of V.

[52, §3] uses a weaker form of this theorem to show that the determinacy of games ending at the first Σ_2 admissible relative to the play, with Δ^1_2-in-the-codes payoff, implies that there is an inner model of $\mathsf{AD}_{\mathbb{R}}$ containing all reals and ordinals. This was the first proof that some form of definable determinacy implies that there is an inner model of $\mathsf{AD}_{\mathbb{R}}$. In the early 90's, Woodin showed how to obtain such a model directly, under the optimal large cardinal hypothesis. This work is unpublished, but is exposited in [59].

[52, §4] also proves the determinacy of games ending at the first Σ_1 admissible relative to the play in the theory $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}+$ "every set of reals admits a scale" + "ω_1 is $P(\mathbb{R})$-supercompact". The latter had been proved to have a model containing all reals and ordinals assuming $\mathsf{ZFC}+$ "there is a supercompact cardinal" by Woodin (unpublished, but see [59]), so [52, §4] completed the proof of a small fragment of clopen determinacy beyond fixed countable length from large cardinals. Neeman ([42]) has since proved this fragment of clopen determinacy directly, from essentially the optimal large cardinal hypotheses.

Although it is not known how to construct definable scales on $\Game^{\omega_1}\text{closed} - \Delta^1_2$ sets under any definable determinacy or large cardinal assumption, one can do so under the assumption that there are sufficiently strong countable iterable structures. What one needs is essentially a countable, ω_1+1-iterable model ZF^-+ "there is a measurable Woodin cardnal". (A slightly weaker theory suffices.) Woodin, and later independently Steel, showed that $\Game^{\omega_1}\text{closed} - \Delta^1_2$ truth can be reduced to truth in any such model, by asking what is forced in the extender algebra at its measurable Woodin cardinal. This yields a recursive function t such that if φ is a $\Game^{\omega_1}\text{closed} - \Delta^1_2$ formula, and x is a real, then

$$\varphi(x) \Leftrightarrow M \models t(\varphi)[x],$$

where M is any iterable model as above such that $x \in M$. (Sadly, this work is still unpublished. The arguments of [54, §7] are a prototype lower down.) We can now use the tree-to-produce-an-elementary-submodel method to get a definable scale. For let M_0 be the minimal extender model satisfying ZF^-+ "there is a measurable Woodin cardinal". M_0 exists and is iterable by our assumption. Because it is minimal, M_0 is pointwise definable, and hence by [54, 4.10] it has a *unique* ω_1+1 iteration strategy with the Dodd-Jensen property. We can then define the direct limit M_∞ of all countable iterates of M_0 under the iteration maps given by this strategy, as in [57]. Our Suslin representation T of the universal $\Game^{\omega_1}\text{closed} - \Delta^1_2$ set is definable from M_∞, and hence definable. Roughly speaking, T builds along each branch a putative $\Game^{\omega_1}\text{closed} - \Delta^1_2$ truth $\varphi(x)$, a model N and an embedding of N into M_∞, and a proof that x is N-generic for the extender algebra at the bottom Woodin cardinal of N, and that $N[x] \models t(\varphi)[x]$.

It is much harder to construct definable winning strategies for $\Game^{\omega_1}\text{closed} - \Delta^1_2$ games granted the existence of an iterable model of ZF^-+ "there is a measurable Woodin cardnal". One must show that there are any strategies at all, in the first place! Itay Neeman ([44], [43]) has made great progress on the questions of the existence and definability of winning strategies in length ω_1 open-Hom_∞ games. He shows that these strategies can be obtained by a logically simple transformation from iteration strategies for appropriate mice. In the case of length ω_1 closed-Δ^1_2, the appropriate mouse is essentially the minimal iterable mouse with a measurable Woodin cardinal. Letting M_0 be as above, Neeman's work shows that any length ω_1 closed-Δ^1_2 game has a winning strategy which is easily definable from the unique iteration strategy for M_0, which by its uniqueness is itself definable.

The problem of constructing iteration strategies for mice at the level of M_0 and beyond has been one of the fundamental open problems in pure set theory since the mid 1980's. Carthago delenda est!

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