The tree of a Moschovakis scale is homogeneous

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We work in the theory $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ throughout this paper.

Let T be a tree on $\omega \times \kappa$, for some ordinal κ. The *projection of* T is the set of reals

$$p[T] = \{x \in \omega^\omega \mid \exists f \in \kappa^\omega \forall n (x \restriction n, f \restriction n) \in T\}.$$

We say that $p[T]$ is *Suslin* via T. A *homogeneity system for* T is a system $\langle \mu_s \mid s \in \omega^{<\omega} \rangle$ of countably additive, 0-1 valued measures[1] such that for all $s, t \in \omega^{<\omega}$ and $x \in \omega^\omega$

$$\mu_s(T_s) = 1,$$

and

$$s \subseteq t \Rightarrow \mu_t \text{ projects to } \mu_s,$$

and

$$x \in p[T] \Rightarrow \langle \mu_{x \restriction n} \mid n < \omega \rangle \text{ is countably complete.}$$

Here, $T_s = \{u \mid (s, u) \in T\}$, and μ_t *projects to* μ_s iff whenever $\mu_s(A) = 1$, then $\mu_t(\{u \mid u \restriction \mathrm{dom}(s) \in A\}) = 1$, and the tower of measures $\langle \mu_{x \restriction n} \mid n < \omega \rangle$ is called *countably complete* just in case whenever $\langle A_n \mid n < \omega \rangle$ is a sequence of sets such that $\mu_{x \restriction n}(A_n) = 1$ for all n, then $\exists f \in \kappa^\omega \forall n (f \restriction n \in A_n)$.[2] If $\bar{\mu}$ is a homogeneity system for T, then we shall say $p[T]$ is *homogeneously Suslin* via T and $\bar{\mu}$.

Which sets of reals are homogeneously Suslin? One can use partition cardinals to propagate homogeneity, and thereby show that all projective sets are homogeneously Suslin. (This is a result of Kunen and Martin; cf. [2, §6].) This method bogs down a bit past the projective sets, however, and leaves open, for example, whether all hyperprojective sets of reals are homogeneously Suslin. In the other direction, every homogeneously Suslin set is Suslin,

[1] That is, countably complete ultrafilters.

[2] Taking $A_n = T_{x \restriction n}$, we see that the countable completeness of the tower $\langle \mu_{x \restriction n} \mid n < \omega \rangle$ implies $x \in p[T]$.

and by [4], its complement in $\mathbb{R}$ is Suslin as well. Thus if $V = L(\mathbb{R})$, every homogeneously Suslin set is $\boldsymbol{\Delta}^2_1$.

In this paper we shall show that the homogeneously Suslin sets are precisely those Suslin sets whose complements are also Suslin. Thus if $V = L(\mathbb{R})$, a set of reals is homogeneously Suslin if and only if it is $\boldsymbol{\Delta}^2_1$. The homogeneous trees in question are not generated by the partition cardinal construction; rather they are trees associated to scales produced by Moschovakis' second periodicity construction.

This result is due to the first author. We also include a result of the second author concerning the additivity of the homogeneity measures produced by this construction.

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Our main result is

Theorem 1.1 (Martin) *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$*; then for any* $A \subseteq \mathbb{R}$*, the following are equivalent:*

(1) A is homogeneously Suslin,

(2) A and $\mathbb{R} \setminus A$ are Suslin.

Proof. (1) $\Rightarrow$ (2) follows at once from [4]. Now let A be Suslin and co-Suslin. Since $\emptyset$ is homogeneously Suslin, we may assume A is nonempty.

Let $\langle u_i \mid i < \omega \rangle$ be an enumeration without repetitions of $\omega^{<\omega}$. We write $n(u)$ for the unique $n \in \omega$ such that $u = u_n$. Let us choose our enumeration so that

$$(\mathrm{dom}(u) \subseteq \mathrm{dom}(v) \wedge \forall i \in \mathrm{dom}(u)(u(i) \leq v(i))) \Rightarrow n(u) \leq n(v)).$$

This implies that proper initial segments of u are enumerated before u.[3] This enumeration lets us identify Lipschitz continuous functions on $\mathbb{R}$ (i.e. strategies for player II in games on ω) with reals. Namely, for any $\rho \in \omega^{<\omega}$, let $\rho^* : \omega^{<\omega} \to \omega^{<\omega}$ with domain $\{u \mid n(u) \in \mathrm{dom}(\rho)\}$ be given by

$$\rho^*(u)(k) = \rho(n(u \upharpoonright (k+1)),$$

whenever $k \in \mathrm{dom}(u)$. Let $\rho^*(\emptyset) = \emptyset$. For any $\sigma \in \omega^\omega$, we let

$$\sigma^* = \bigcup_{n<\omega} (\sigma \upharpoonright n)^*,$$

[3] For example, we can let $n(u) \leq n(v)$ iff $\Pi_{i \in \mathrm{dom}(u)} p_i^{u(i)+1} \leq \Pi_{i \in \mathrm{dom}(v)} p_i^{v(i)+1}$, where p_i is the i^{th} prime.

and we extend σ^* to a Lipschitz continuous function on $\mathbb{R}$, which we also call σ^*, by setting

$$\sigma^*(x) = \bigcup_{i<\omega} \sigma^*(x \restriction i).$$

So ρ^* is a finite neighborhood condition in the space of all Lipschitz continuous functions on $\mathbb{R}$, and σ^* is the set of all neighborhood conditions coded into σ, as well as the Lipschitz continuous function meeting all these neighborhood conditions. We shall tend to use Greek letters $\rho, \sigma, ..$ for finite or infinite sequences of natural numbers when we are thinking of them as codes in this fashion, and Latin letters $s, x, ..$ otherwise.

Let

$$A^* = \{\sigma \in \omega^\omega \mid \forall x \in \omega^\omega (\sigma^*(x) \in A\}$$

be the set of codes for Lipschitz functions mapping into A. It is easy to see that A is Lipschitz-reducible to A^*; simply send x to the code for the constantly x function. It will therefore suffice to show that A^* is homogeneously Suslin.

Let $\langle \varphi_n \mid n < \omega \rangle$ be a very good scale on A. Moschovakis' second periodicity construction ([5])) produces a very good scale on A^*. It is most natural to index the norms in the Moschovakis scale with finite sequences, so we shall do that. Let $u \in \omega^{<\omega}$ and let $\sigma_1, \sigma_2 \in A^*$. Consider the game $G(u, \sigma_1, \sigma_2)$: Player I plays out $x_1 \in \mathbb{R}$, and player II plays out $x_2 \in \mathbb{R}$, the players alternating moves, with I going first, as usual. Let $u^\frown x \in \mathbb{R}$ be the result of appending x to u, that is, $(u^\frown x)(n) = u(n)$ for $n \in \text{dom}(u)$, and $(u^\frown x)(n) = x(n - \text{dom}(u))$ otherwise. Since $\sigma_i \in A^*$, we have $\sigma_i^*(u^\frown x_i) \in A$ for $i = 1, 2$. We say

$$(x_1, x_2) \text{ wins } G(u, \sigma_1, \sigma_2) \text{ for II } \Leftrightarrow \varphi_{\text{dom}(u)}(\sigma_1^*(u^\frown x_1)) \leq \varphi_{\text{dom}(u)}(\sigma_2^*(u^\frown x_2)).$$

Putting

$$\sigma_1 \leq_u \sigma_2 \Leftrightarrow \text{II has a winning strategy in } G(u, \sigma_1, \sigma_2),$$

Moschovakis shows that each $\leq_u$ is a prewellorder of A^*. For $\sigma \in A^*$ and $u \in \omega^{<\omega}$, we let

$$\theta_u(\sigma) = \text{ rank of } \sigma \text{ in } \leq_u ,$$

and we have by Moschovakis' arguments that $\langle \theta_u \mid u \in \omega^{<\omega} \rangle$ is a scale on A^*.[4] It is clear that $\theta_u(\sigma)$ only depends on $\sigma \restriction \{i \mid u \subseteq u_i\}$. It is also clear that

$$\theta_u(\sigma) \leq \theta_u(\tau) \Leftrightarrow \forall i \exists j (\theta_{u^\frown\langle i \rangle}(\sigma) \leq \theta_{u^\frown\langle j \rangle}(\tau).$$

[4]To see the limit property: let $\sigma_i \to \sigma$ (mod$\bar{\theta}$). Let $x \in \mathbb{R}$; we must show $\sigma^*(x) \in A$. We can thin out the σ_i's so that $\sigma_{n+1} \leq_{x \restriction n} \sigma_n$ for all n; let Σ_n be a winning strategy for II in $G(x \restriction n, \sigma_{n+1}, \sigma_n)$ witnessing this. Playing the Σ_n against each other simultaneously, we obtain z_n so that $z_n \to x$ and $\sigma_n^*(z_n) \to \sigma^*(x)$ (mod $\bar{\varphi}$). (Here z_n is defined by: $x \restriction n \subseteq z_n$, and for $k \geq n$, $z_n(k) = \Sigma_n(\langle x(n), z_{n+1}(n+1), ..., z_{n+1}(k) \rangle)$.) Since $\bar{\varphi}$ is a scale on A, $\sigma^*(x) \in A$.

The tree of the scale $\bar{\theta}$ is given by:

$$T = \{(\sigma \restriction n, \langle \theta_{u_0}(\sigma), ..., \theta_{u_{n-1}}(\sigma)\rangle) \mid \sigma \in A^*\}.$$

By the limit property of scales, $p[T] = A^*$. We shall show that T is homogeneous. The homogeneity measures μ_ρ on T_ρ come from a game in which the two players interact to produce a potential member of T_ρ.

We need a Suslin representation of the complement of A in order to define our game, so fix a tree S on some $\omega \times \kappa$ such that

$$p[S] = \mathbb{R} \setminus A.$$

From S we get a Suslin representation of $\mathbb{R} \setminus A^*$: for $\sigma \in \mathbb{R} \setminus A^*$, we define the *leftmost witness that* $\sigma \notin A^*$ to be the pair (x, f), where $x \in \mathbb{R}$ and $f \in \kappa^\omega$ are defined inductively by

$$x(n) = \text{ least } k \text{ s.t.} \exists y \exists g((x \restriction n)^\frown\langle k\rangle \subseteq y \wedge f \restriction n \subseteq g \wedge (\sigma^*(y), g) \in [S]),$$

and

$$f(n) = \text{ least } \alpha \text{ s.t. } \exists y \exists g(x \restriction (n+1) \subseteq y \wedge (f \restriction n)^\frown\langle\alpha\rangle \subseteq g \wedge (\sigma^*(y), g) \in [S].$$

Now let $\rho \in \omega^{<\omega}$ and $X \subseteq T_\rho$. We shall say that $\mu_\rho(X) = 1$ if and only if Player II has a winning strategy in the game $G_\rho(X)$, which is played as follows. Players I and II alternate playning natural numbers, with I moving first, as usual. At the end of a run I has produced $\tau_1 \in \mathbb{R}$ and II has produced τ_2. Set $\sigma_i = \rho^\frown\tau_i$. Player II wins $G_\rho(X)$ if and only if either

(1) $\sigma_1 \notin A^*$ and $\sigma_2 \in A^*$, or

(2) $\sigma_1 \notin A^*$ and $\sigma_2 \notin A^*$, and letting (x_1, f_1) and (x_2, f_2) be the leftmost witnesses respectively, and k_i least such that $x_i \restriction k_i \notin \operatorname{dom}(\rho^*)$, we have $(x_1 \restriction k_1, f_1 \restriction k_1) \leq_{\text{lex}} (x_2 \restriction k_2, f_2 \restriction k_2)$, or

(3) $\sigma_1 \in A^*$ and $\sigma_2 \in A^*$, and for all $u \in \operatorname{dom}(\rho^*)$, $\theta_u(\sigma_1) \leq \theta_u(\sigma_2)$, and $\langle \theta_{u_i}(\sigma_2) \mid i \in \operatorname{dom}(\rho)\rangle \in X$.

Here the lexicographic order $\leq_{\text{lex}}$ is defined in a way which parallels our definition of the leftmost witnesses: $(s, t) \leq_{\text{lex}} (u, v)$ iff $(s, t) = (u, v)$, or there is an $i \in \operatorname{dom}(s)$ such that $s \restriction i = u \restriction i, t \restriction i = v \restriction i$, and either $s(i) < u(i)$, or $s(i) = u(i)$ and $t(i) < v(i)$. So $\leq_{\text{lex}}$ is a partial, not not total, order on $\omega^{<\omega} \times \kappa^{<\omega}$. We also define $\leq_{\text{lex}}$ on $\omega^\omega \times \kappa^\omega$ in the obvious way: $(x, f) \leq_{\text{lex}} (y, g) \Leftrightarrow \forall k(x \restriction k, f \restriction k) \leq_{\text{lex}} (y \restriction k, g \restriction k)$. Here the order is total.

It is clear that $\mu_\rho(T_\rho) = 1$, since II can win $G_\rho(T_\rho)$ by simply copying I. It is equally clear that $\mu_\rho(\emptyset) = 0$, and that $X \subseteq Y \wedge \mu_\rho(X) = 1 \Rightarrow \mu_\rho(Y) = 1$. For the rest of the proof that the μ_ρ constitute a homogeneity system, we need some terminology concerning embeddings of one Lipschitz continuous function in another.

Let $\sigma, \tau, \pi \in \omega^\omega$. We say π *embeds* σ *in* τ iff $\sigma^* = \tau^* \circ \pi^*$. We say that π is an embedding *over* ρ, for $\rho \in \omega^{<\omega}$, iff $\pi^*(u) = u$ for all $u \in \mathrm{dom}(\rho^*)$. For any $\pi \in \omega^\omega$, we say

$$\pi \text{ is II-safe } \Leftrightarrow \forall u \in \omega^{<\omega}(u \leq_{lex} \pi^*(u)),$$

and if π is an embedding over ρ,

$$\pi \text{ is I-safe over } \rho \Leftrightarrow \forall u \in \omega^{<\omega} \setminus \mathrm{dom}(\rho^*)(u <_{lex} \pi^*(u)).$$

We say $\tau \in \omega^\omega$ is the *amalgamation of* $\langle \sigma_i \mid i \in X\rangle$ *via* $\langle \pi_i \mid i \in X\rangle$ iff each π_i embeds σ_i into τ for all i, and $\{\mathrm{ran}(\pi_i) \mid i \in X\}$ is a partition of ω^ω. We say the amalgamation is II-safe if all the π_i are II-safe, and over ρ if all the π_i are embeddings over ρ.

Remark 1.2 The following two facts help explain our interest in safe amalgamations:

(a) If τ is an amalgamation of $\langle \sigma_i \mid i \in X\rangle$ and each $\sigma_i \in A^*$, then $\tau \in A^*$. If the amalgamation is over ρ, then $\theta_u(\sigma_i) \leq \theta_u(\tau)$ for all i and all $u \in \mathrm{dom}(\rho^*)$.

(b) If the amalgamation is via II-safe embeddings π_i, and $\tau \notin A^*$, then letting (x, f) be the leftmost witness that $\tau \notin A^*$, and $x = \pi_i(y)$, then $\sigma_i^*(y) = \tau^*(x)$, so (y, f) is a witness that $\sigma_i \notin A^*$ such that $(y, f) \leq_{lex} (x, f)$.

Claim 1. For any $\rho \in \omega^{<\omega}$ and $X \subseteq T_\rho$, either $\mu_\rho(X) = 1$ or $\mu_\rho(T_\rho \setminus X) = 1$.

Proof. Suppose $\mu_\rho(X) \neq 1$, and fix a winning strategy Σ for I in $G_\rho(X)$. Player II cannot quite afford to simply use Σ as his own strategy in $G_\rho(T_\rho \setminus X)$, because of the demand that $\theta_u(\sigma_1) \leq \theta_u(\sigma_2)$ for all $u \in \mathrm{dom}(\rho^*)$ in II's payoff condition (3). So II will also embed I's play into his own as he proceeds. In order not to lose because of payoff condition (2), he will use a II-safe embedding. More precisely, II's winning strategy in $G_\rho(T_\rho \setminus X)$ is to insure that if I plays out $x \in \mathbb{R}$, then II's response y is such that $\tau = \rho^\frown y$ is a II-safe amalgamation over ρ of $\sigma = \rho^\frown x$ and $\psi = \rho^\frown \Sigma(y)$.

We show first that such a play (x, y) is a win for II in $G_\rho(T_\rho \setminus X)$. Suppose first that $\tau \in A^*$. Then both σ and ψ are in A^*, since they embed in τ. Since the embedding is over ρ,

$$\theta_u(\sigma) \leq \theta_u(\tau) \text{ and } \theta_u(\psi) \leq \theta_u(\tau)$$

for all $u \in \mathrm{dom}(\rho^*)$. Now (ψ, τ) comes from a play of $G_\rho(X)$ according to I's winning strategy Σ, and the second set of inequalities displayed therefore imply

$$\langle \theta_{u_0}(\tau), ..., \theta_{u_{\mathrm{dom}(\rho)-1}}(\tau)\rangle \in T_\rho \setminus X.$$

But then the first set of inequalities show that (x, y), which gives rise to (σ, τ), is a win for II under payoff clause (3).

Suppose next that $\tau \notin A^*$, and let (z, f) be the leftmost witness of this fact. Since the amalgamation yielding τ is II-safe, we can find $(w, g) \leq_{lex} (z, f)$ such that either (w, g) is the leftmost witness that $\sigma \notin A^*$ or (w, g) is the leftmost witness that $\psi \notin A^*$. (See remark 1.2.) The latter is impossible, since it implies that $(\Sigma(y), y))$ is a defeat for Σ under payoff condition (2) of $G_\rho(X)$. The former implies that (x, y) is a win for II in $G_\rho(T_\rho \setminus X)$ under payoff condition (2), as desired.

Finally, we must see that II can play his y in response to x so that $\rho^\frown y$ is the desired amalgamation. For each $u \in \mathrm{dom}(\rho^*)$, let

$$I_u^\rho = \{i \mid u^\frown\langle i\rangle \notin \mathrm{dom}(\rho^*)\},$$

and

$$h_u^1, h_u^2 \colon I_u^\rho \to I_u^\rho$$

be order-preserving functions such that $\mathrm{ran}(h_u^1)$ and $\mathrm{ran}(h_u^2)$ partition I_u^ρ, and

$$k \leq h_u^i(k)$$

for all $k \in I_u^\rho$ and all $i \in \{0, 1\}$. We define our embeddings π_1 and π_2 by

$$\pi_i^*(v)(k) = h_{v\restriction k}^i(v(k))$$

if $v \restriction k \in \mathrm{dom}(\rho^*)$ but $v \restriction k+1 \notin \mathrm{dom}(\rho^*)$, and

$$\pi_i^*(v)(k) = v(k)$$

otherwise. Notice that $\mathrm{dom}(\pi_i^*(v)) = \mathrm{dom}(\pi_i(v))$, and that $\pi_i^*(v)$ differs from v at at most one k, and for that k, $v(k) \leq \pi_i^*(v)(k)$. It follows that

$$\forall v(n(v) \leq n(\pi_i^*(v))).$$

Player II can now determine $y(k)$, granted $x \restriction (k+1)$, as follows: let u be such that

$$n(u) = \mathrm{dom}(\rho) + k.$$

Then

$$u = \pi_i^*(v)$$

for exactly one pair i, v; moreover,

$$n(v) = \mathrm{dom}(\rho) + j$$

for some $j \leq k$. We let

$$y(k) = x(j) \text{ if } i = 0$$

and

$$y(k) = \Sigma(y \restriction k)(j) \text{ if } i = 1.$$

We leave any further detail to the reader. $\square$

Claim 2. If $\mu_\rho(X_i) = 1$ for all $i < \omega$, then $\mu_\rho(\bigcap_i X_i) = 1$.

Proof. Let Σ_i be a winning strategy for II in $G_\rho(X_i)$. For each $u \in \mathrm{dom}(\rho^*)$, let

$$h^1_u, h^2_u \colon I^\rho_u \to I^\rho_u$$

be as in the proof of claim 1, but with

$$k < h^2_u(k)$$

for all $k \in I^\rho_u$ in addition. Let π_1 and π_2 be the II-safe embeddings over ρ determined from h^1_u and h^2_u as in claim 1. Our additional property of h^2_u guarantees that π_2 is in fact I-safe over ρ.

Our winning strategy Σ for II in $G_\rho(X)$ will behave as follows: suppose I plays out $x \in \omega^\omega$, and let $\sigma = \rho^\frown x$. Let y be II's response as dictated by Σ, and $\tau = \rho^\frown y$. Let φ be the amalgamation of σ and τ via π_1 and π_2. (Here π_2 is used to embed τ into φ.) Since σ and τ extend ρ and the amalgamation is over ρ, φ extends ρ. Let $\varphi = \rho^\frown z$, let $w_i = \Sigma_i(z)$, and let $\psi_i = \rho^\frown w_i$. Player II's strategy Σ will be to arrange that τ is a II-safe amalgamation over ρ of $\langle \psi_i \mid i < \omega \rangle$.

We first show that there is a strategy Σ such that plays (x, y) according to Σ have the property that there are z and w_i as above. Let $g^i_u \colon I^\rho_u \to I^\rho_u$ for $i \in \omega$ be nondecreasing functions whose ranges partition I^ρ_u, for each $u \in \mathrm{dom}(\rho^*)$. Let $\langle \eta_i \mid i \in \omega \rangle$ be the II-safe amalgamating functions these g^i_u determine (as in claim 1). It is easy to see that there is a Lipschitz function which, given x, will produce y, z, and w_i so that the conditions above hold. Note first that $z \restriction k+1$, and hence $w_i \restriction k+1$, are determined by $x \restriction k$ and $y \restriction k$. For to determine $z(k)$, let $n(v) = \mathrm{dom}(\rho) + k$. If $v = \pi^*_1(r)$, then set $z(k) = x(n(r) - \mathrm{dom}(\rho))$, noting that $n(r) \leq n(v)$, so that $n(r) - \mathrm{dom}(\rho) \leq k$. If $v = \pi^*_2(r)$, then $n(r) < n(v)$ because of our additional requirement guaranteeing the I-safety of π_2. Thus it makes sense to set $z(k) = y(n(r) - \mathrm{dom}(\rho))$. It is enough, then, to determine $y(k)$ from $x \restriction k+1$ together with $z \restriction k+1$ and $w_i \restriction k+1$. Let $n(v) = \mathrm{dom}(\rho) + k$, and let $v = \eta^*_i(r)$, and note that $n(v) \leq n(r)$. We then set $y(k) = w_i(n(r) - \mathrm{dom}(\rho))$.

We now verify that Σ wins $G_\rho(\bigcap_{i<\omega} X_i)$ for II. Let (x, y) be a play by Σ, with z and w_i for $i \in \omega$ associated as above. Let $\sigma = \rho^\frown x$, $\tau = \rho^\frown y$, $\varphi = \rho^\frown z$, and $\psi_i = \rho^\frown w_i$ for all i.

Suppose first that $\tau \notin A^*$, and let (p, f) be the leftmost witness to this fact. Let $p = \pi^*_i(q)$; then (q, f) is a witness that $\psi_i \notin A^*$, and $(q, f) \leq_{\mathrm{lex}} (p, f)$ because π_i is II-safe.

Now (z, w_i) is a play by Σ_i, which is winning for II, and hence we have a witness (r, g) that $\varphi \notin A^*$ such that $(r \restriction k_1, g \restriction k_1) \leq_{\text{lex}} (q \restriction k_2, f \restriction k_2)$, where k_1, k_2 are least such that $r \restriction k_1 \notin \text{dom}(\rho^*)$ and $q \restriction k_2 \notin \text{dom}(\rho^*)$. We cannot have $r \in \text{ran}(\pi_2^*)$, for if $r = \pi_2^*(s)$, then because π_2 is I-safe, we have $s \restriction k_1 <_{\text{lex}} r \restriction k_1$, so (s, g) is a witness that $\tau \notin A^*$, and $(s \restriction k_1, g \restriction k_1) <_{\text{lex}} (q \restriction k_2, f \restriction k_2)$, so $(s, g) <_{\text{lex}} (q, f) \leq_{\text{lex}} (p, f)$, contrary to our choice of (p, f). Thus $r = \pi_1^*(s)$ for some s. But then (s, g) is a witness that $\sigma \notin A^*$, and

$$(s \restriction k_1, g \restriction k_1) \leq_{\text{lex}} (r \restriction k_1, g \restriction k_1) \leq_{\text{lex}} (p \restriction k_2, f \restriction k_2),$$

and since all embeddings above are over ρ, we have k_1 is least such that $s \restriction k_1 \notin \text{dom}(\rho^*)$ and k_2 is least such that $p \restriction k_2 \notin \text{dom}(\rho^*)$. It follows that (x, y) is a win for II in $G_\rho(\bigcap X_i)$ under payoff condition (2).

Next, suppose $\tau \in A^*$. We may assume that $\sigma \in A^*$, as otherwise we are done in virtue of payoff condition (1). But then $\varphi \in A^*$, as it is an amalgamation of σ and τ, and $\psi_i \in A^*$ for all i, as τ is an amalgamation of $\langle \psi_i \mid i \in \omega \rangle$. Further, for $u \in \text{dom}(\rho^*)$ we have

$$\theta_u(\sigma) \leq \theta_u(\varphi) \leq \theta_u(\psi_i) \leq \theta_u(\tau)$$

for all i. (Here the first and last inequalities come from the properties of amalgamations, and the middle one holds because Σ_i wins for II.) This shows II has met his norm-inequality obligations. We also have, for $u \in \text{dom}(\rho^*)$,

$$\theta_u(\tau) \leq \theta_u(\varphi \leq \theta_u(\psi_i) \leq \theta_u(\tau),$$

for similar reasons. Thus

$$\langle \theta_{u_n}(\tau) \mid n \in \text{dom}(\rho) \rangle = \langle \theta_{u_n}(\psi_i) \mid n \in \text{dom}(\rho) \rangle \in X_i,$$

for all i. This implies that (x, y) is a win for II in $G_\rho(\bigcap_{i<\omega} X_i)$ under payoff condition (3). $\square$

So μ_ρ is a countably complete ultrafilter concentrating on T_ρ. We now verify compatibility.

Claim 3. If $\rho, \varphi \in \omega^{<\omega}$, $\rho \subseteq \varphi$, $\mu_\rho(X) = 1$, and $Y = \{w \mid w \restriction \text{dom}(\rho) \in X\}$, then $\mu_\varphi(Y) = 1$.

Proof. Fix a winning strategy Σ for II in $G_\rho(X)$. Our strategy for II in $G_\varphi(Y)$ is roughly the following. If I plays σ extending φ, and ψ is the extension of ρ which Σ would play when σ is regarded as I's play in $G_\rho(X)$, then II will play a safe amalgamation τ of ψ and the parts of σ below u's in $\text{dom}(\varphi^*) \setminus \text{dom}(\rho^*)$.

More precisely, for $u \in \text{dom}(\rho^*)$, let

$$h_u : I_u^\rho \to I_u^\varphi$$

be an order-preserving bijection, where $I_u^\rho = \{i \mid u^\frown i \notin \mathrm{dom}(\rho^*)\}$, and I_u^φ is defined similarly. Note that $i \leq h_u(i)$ for all i, as $I_u^\varphi \subseteq I_u^\rho$. We define a II-safe embedding over ρ by:

$$\pi^*(u^\frown\langle i\rangle^\frown v) = u^\frown\langle h_u(i)\rangle^\frown v$$

whenever $u \in \mathrm{dom}(\rho^*)$ and $i \in I_u^\rho$, and of course $\pi^*(u) = u$ for all $u \in \mathrm{dom}(\rho^*)$. We can now define τ from σ and ψ by letting

$$\tau^*(\pi^*(u)) = \psi^*(u),$$

and

$$\tau^*(v) = \sigma^*(v) \text{ for } v \notin \mathrm{ran}(\pi^*).$$

We leave it to the reader to check that $\tau(n)$ is determined by $\sigma \restriction n$ (via $\psi \restriction n$), for all n, so that we have legitimately described a strategy for II in $G_\varphi(Y)$. (Notice here that τ does extend φ, because $\mathrm{ran}(\pi^*)$ is disjoint from $\mathrm{dom}(\varphi^*) \setminus \mathrm{dom}(\rho^*)$.) In order to see that this strategy wins for II, suppose σ, τ, and ψ are as above.

Suppose first $\tau \notin A^*$, and let (x, f) be the leftmost witness to this fact. If $x \notin \mathrm{ran}(\pi^*)$, then (x, f) is a witness that $\sigma \notin A^*$, so II has won under payoff condition (2). Now suppose $x = \pi^*(y)$, and let u be the longest initial segment of x, or equivalently y, such that $u \in \mathrm{dom}(\rho^*)$. Since (y, f) is a witness that $\psi \notin A^*$, and (σ, ψ) is a play by Σ, we have a leftmost witness (z, g) that $\sigma \notin A^*$ such that

$$(z \restriction (j+1), g \restriction (j+1)) \leq_{\mathrm{lex}} (y \restriction (k+1), f \restriction (k+1)),$$

where j is largest such that $z \restriction j \in \mathrm{dom}(\rho^*)$ and k is largest such that $y \restriction k \in \mathrm{dom}(\rho^*)$. Note that $k = \mathrm{dom}(u)$ since $x \restriction (k+1) \notin \mathrm{dom}(\rho^*)$ and $\pi^*(y) = x$. If the inequality just displayed is strict, then we have by the II-safety of π^*

$$(z \restriction l, g \restriction l) <_{\mathrm{lex}} (y \restriction i), f \restriction i)) \leq_{\mathrm{lex}} (x \restriction i, f \restriction i),$$

where $l \geq j+1$ is least such that $z \restriction l \notin \mathrm{dom}(\varphi^*)$, and $i \geq k+1$ is least such that $x \restriction i \notin \mathrm{dom}(\varphi^*)$. Thus II has won $G_\varphi(Y)$ under clause (2). So suppose that

$$(z \restriction (j+1), g \restriction (j+1)) = (y \restriction (k+1), f \restriction (k+1)).$$

If $I_u^\rho = I_u^\varphi$, then we see from the definition of π^* that $x = y$, and also $j = k$ is largest so that $x \restriction k = z \restriction k$ is in $\mathrm{dom}(\varphi^*)$, so that again II wins $G_\varphi(Y)$ under clause (2). Finally, if $I_u^\rho \neq I_u^\varphi$, then $y(k) < h_u(y(k)) = x(k)$, so $y \restriction (k+1) <_{\mathrm{lex}} x \restriction (k+1)$, which leads to

$$(z \restriction l, g \restriction l) <_{\mathrm{lex}} (x \restriction i, f \restriction i)$$

for the φ-critical l and i, both of which are $\geq k+1$. So again, II wins $G_\varphi(Y)$ under clause (2).

Next, suppose $\tau \in A^*$. If $\sigma \notin A^*$, we are done, so assume $\sigma \in A^*$. Thus (σ, ψ) is a win for II in $G_\rho(X)$ under clause (3). We shall show that $\theta_u(\tau) = \theta_u(\psi)$ for all $u \in \mathrm{dom}(\rho^*)$ so that (σ, τ) is a win for II in $G_\varphi(Y)$ under clause (3). We proceed by reverse induction on $\mathrm{dom}(u)$. Suppose that $\theta_v(\tau) = \theta_v(\psi)$ whenever $v \in \mathrm{dom}(\rho^*)$ and v is a proper extension of u. To see that $\theta_u(\psi) \leq \theta_u(\tau)$, it is enough to show that whenever v is any one-term extension of u, then there are one-term extensions r of u such that $\theta_v(\psi) \leq \theta_r(\tau)$. We can take $r = v$ if $v \in \mathrm{dom}(\rho^*)$, by induction. Otherwise, we take $r = \pi^*(v)$, which works because π^* embeds ψ into τ. To see that $\theta_u(\tau) \leq \theta_u(\psi)$, again let v be any one-term extension of u; we must find a one-term extension r of u such that $\theta_v(\tau) \leq \theta_r(\psi)$. Again, $r = v$ works if $v \in \mathrm{dom}(\rho^*)$. If $v \notin \mathrm{dom}(\varphi^*)$, then $r = (\pi^*)^{-1}(v)$ works, because π^* is an "isomorphism" between ψ^* below r and τ^* below v. Finally, if $v \in \mathrm{dom}(\varphi^*) \setminus \mathrm{dom}(\rho^*)$, then τ^* agrees with σ^* below v. But $\theta_u(\sigma) \leq \theta_u(\psi)$ because Σ is winning, so there is a one-term extension r of u such that $\theta_v(\tau) = \theta_v(\sigma) \leq \theta_r(\psi)$.

This completes the proof of claim 3. □

Finally, we must show the countable completeness of towers associated to $\sigma \in A^*$.

Claim 4. Let $\sigma \in A^*$, and suppose $\mu_{\sigma \restriction n}(X_n) = 1$ for all n; then there is an infinite sequence f such that $f \restriction n \in X_n$ for all n.

Proof. By claim 3, we may as well assume that X_{n+1} projects into X_n, for all n. Let Σ_n be a winning strategy for II in $G_{\sigma \restriction n}(X_n)$. By playing all the Σ_n together in the usual way, we obtain τ_n for $n < \omega$ such that for each n,

$$\sigma \restriction n \subseteq \tau_n$$

and

$$(\tau_{n+1}, \tau_n) \text{ arises from a play by } \Sigma_n$$

in which I is responsible for τ_{n+1} and II is responsible for τ_n.[5]

We claim that $\tau_n \in A^*$ for all n. For if $\tau_n \notin A^*$, then since the Σ's win for II, $\tau_k \notin A^*$ for all $k \geq n$, by induction on k. Letting (x_k, f_k) be the leftmost witness that $\tau_k \notin A^*$, payoff condition (2) implies that (x_k, f_k) converges to some (x, f). It is clear that (x, f) is a witness that $\sigma \notin A^*$, a contradiction.

Thus (τ_{n+1}, τ_n) falls under II's payoff condition (3), for all n. But this implies that for each $u \in \omega^{<\omega}$, $\theta_u(\tau_k) \geq \theta_u(\tau_{k+1})$ for all $k > n(u)$, so that we can let

$$f(n) = \text{ eventual value of } \theta_{u_n}(\tau_k)$$

as $k \to \omega$. It is easy to see that $f \restriction n \in X_n$ for all n. □

[5] Formally, $\tau_n = (\sigma \restriction n)^\frown x_n$, where $x_n(k) = \Sigma_n(\langle \sigma(n) \rangle^\frown (x_{n+1} \restriction k))$.

These claims complete the proof of Theorem 1.1. $\square$

We conclude with two further observations.

First, the tree T constructed in the proof of 1.1 has the generic-codes property isolated by Kechris and Woodin ([3]).

Let T be the tree of a scale $\vec{\theta}$, and let $(r, v) \in T$. Consider the Banach-Mazur game $G^{**}(r, v)$ on T in which at round n, I plays $(r_{2n}, v_{2n}) \in T$, then II plays $(r_{2n+1}, v_{2n+1}) \in T$, and we require $(r, v) \subseteq (r_0, v_0)$ and $(r_n, v_n) \subseteq (r_{n+1}, v_{n+1})$ for all n. Letting $x = \bigcup_n r_n$ and $f = \bigcup_n v_n$, we say II wins $G_{**}(r, v)$ iff for all $k \in \mathrm{dom}(r)$, $\theta_{u_k}(x) = v(k)$. (By the lower semi-continuity of the scale $\vec{\theta}$, $\theta_{u_k}(x) \leq v(k)$.) Kechris and Woodin say that T has the generic codes property if for all $(r, v) \in T$, player II has a winning strategy in $G^{**}(r, v)$. For any Suslin set A, they construct a T with the generic codes property projecting to A.

It is quite easy to show that the T and $\vec{\theta}$ constructed in the proof of 1.1 have the generic codes property. To win $G^{**}(r, v)$, player II simply picks at the outset some $\sigma \in A^*$ such that $r \subseteq \sigma$ and $\theta_{u_k}(\sigma) = v(k)$ for all $k \in \mathrm{dom}(r)$. He then plays so that if (τ, f) is the output of the game, then σ is embedded into τ. Since II can only make finite extensions, he can never prevent II from further extending the embedding he is building.

Our second observation is that in certain natural situations, the homogeneity measures we have constructed are more than countably complete.

Theorem 1.3 (Steel) *Let Γ be an ω-parametrized pointclass which is closed under recursive substitution, number quantification, and universal real quantification, and suppose that every set in Γ is Suslin and co-Suslin. Let δ be the supremum of the order types of prewellorders in $\Gamma \cap \check{\Gamma}$; then every set in Γ is δ-homogeneously Suslin.*

Proof. We illustrate the proof in the special case $\Gamma = \Pi^1_3$. Let $\delta = \boldsymbol{\delta}^1_3$. We shall define a complete Π^1_3 set A^* such that the homogeneity measures we get from the proof of 1.1 are δ-complete. In this connection, it will help if we are more careful in our choice of the ingredients A (which will be Σ^1_2), the scale $\vec{\varphi}$ on A, and the Suslin representation S of $\mathbb{R} \setminus A$.

Let $U \subseteq \omega \times \mathbb{R}$ be a universal Σ^1_2 set. Fix a map

$$H : \omega \overset{\text{onto}}{\to} \mathrm{HF},$$

where HF is the set of all hereditarily finite sets. For $x \in \omega^\omega$, put $x \in \bar{A}$ iff

(a) for all n,
$$H(x(n)) = \langle r_n, v_n \rangle \in (\omega^{<\omega} \times \omega^{<\omega}),$$
where $\mathrm{dom}(r_n) = \mathrm{dom}(v_n) > 0$,

(b) $r_n \subseteq r_{n+1}$, and $r_n \neq r_{n+1}$, and

(c) letting e^x_i be the common value of $r_n(i)$ for all $i > n$, and defining $(x)_i \in \omega^\omega$ by

$$(x)_i(n) = v_{p+n}(i), \text{ where } p = \text{ least } k \text{ s.t. } i \in \mathrm{dom}(v_k),$$

we have

$$\exists i U(e^x_i, (x)_i).$$

Note that the set of $x \in \omega^\omega$ satisfying (a) and (b) in the definition of A is closed; let W_0 be the tree of this closed set. Put $x \in A$ iff $x \in \bar{A}$, or $x \notin [W_0]$.

Let $\langle \varphi^*_i \mid i < \omega \rangle$ be a very good Σ^1_2-scale on U. We get a Σ^1_2-scale on A in a routine way.[6] Let $x \in A$. If $x \notin [W_0]$, then set

$$\varphi_n(x) = \langle 0, k \rangle, \text{ where } k \text{ is least s.t. } x \restriction k \notin W_0.$$

Otherwise, $x \in \bar{A}$, and we have $U(e^x_i, (x)_i)$ for some i. Let i_0 be such that $\varphi^*_0(e^x_{i_0}, (x)_{i_0}) = \alpha$ is as small as possible, and i_0 is the least i such that $\varphi^*_0(e^x_i, (x)_i) = \alpha$, and set

$$\varphi_n(x) = \langle 1, \varphi^*_0(i_0, (x)_{i_0}), i_0, \varphi^*_1(i_0, (x)_{i_0}), ..., \varphi^*_n(i_0, (x)_{i_0}) \rangle$$

for all n. Here we use the lexicographic order to identify the range of φ_n with an ordinal. It is clear that $\vec{\varphi}$ is a Σ^1_2-scale on A. For a technical reason, we have not tried to make $\vec{\varphi}$ very good; it could happen that $\varphi_n(x_i)$ is eventually constant for each n, but the x_i do not converge. However, it *is* true that $n \leq m$ and $\varphi_n(x) < \varphi_n(y)$ imply $\varphi_m(x) < \varphi_m(y)$. This refinement property is enough to guarantee that the Moschovakis norms $\vec{\theta}$ on A^*, defined as in the proof of 1.1, constitute a scale on A^*. It is easy to check that if $\sigma \in A^*$ and $\sigma^*(u) \notin W_0$, then since $\varphi_{\mathrm{dom}(u)}$ is constant on the neighborhood determined by $\sigma^*(u)$, $\theta_u(\sigma^*) = 0$. This property, which might have been lost if we had tried to make $\vec{\varphi}$ very good, will be of use later.

Let S_0 be a tree on $\omega \times \kappa$ such that $p[S_0] = (\omega \times \mathbb{R}) \setminus U$. We construct a tree S projecting to $\mathbb{R} \setminus A$. For $s \in W_0$, let $\langle r^s_n, v^s_n \rangle = H(s(n))$, and letting n be largest in $\mathrm{dom}(s)$, set

$$k(s) = \mathrm{dom}(r^s_n) \text{ and } e^s_i = r^s_n(i) \text{ for all } i < k(s),$$

and

$$(s)_i(j) = v^s_{p+j}(i), \text{ where } p = \text{ least } l \text{ s.t. } i \in \mathrm{dom}(v^s_l).$$

(The domain of $(s)_i$ is $(n - p) + 1$.) That is, e^s_i and $(s)_i$, for $i < k(s)$, are the common initial segments of all e^x_i and $(x)_i$ for $s \subseteq x$ determined by s. Now fix a bijection

$$c \colon \lambda \to \kappa^{<\omega},$$

for some λ, with the property that $\mathrm{dom}(s) < \mathrm{dom}(t) \Rightarrow c^{-1}(s) < c^{-1}(t)$. For $s \in \omega^{<\omega}$ and $t \in \lambda^{<\omega}$, put (s, t) in S iff

[6]The definability of the scale $\vec{\varphi}$, and of the Moschovakis scale $\vec{\theta}$ on A^* it induces, are actually not relevant to 1.1 or 1.3.

(i) $s \in W_0$, $\mathrm{dom}(s) = \mathrm{dom}(t)$, and $\forall n \in \mathrm{dom}(s)$, $\mathrm{dom}(c(t(n)) = \mathrm{dom}(v_n^s)$, and

(ii) letting

$$(t)_i(j) = c(t(p+j))(i), \text{ where } p = \text{ least } l \text{ s.t. } i \in \mathrm{dom}(v_l^s),$$

(where j is such that $p + j < \mathrm{dom}(s)$, we have

$$(e_i^s, (s)_i, (t)_i) \in S_0$$

for all $i < k(s)$.

That is, S is the tree of attempts to build an (x, f) such that x satisfies (a) and (b) of the definition of $\bar{A}$, and for each $i < \omega$, $(f)_i$ (defined as above) witnesses that $\neg U(e_i^x, (x)_i)$.

Having defined the ingredients A, $\vec{\varphi}$, and S in the proof of 1.1, we let $\vec{\theta}$ be the Moschovakis scale on A^*, T be the tree of $\vec{\theta}$, and $\langle \mu_\rho \mid \rho \in \omega^{<\omega} \rangle$ be the homogeneity system for T defined there. We also carry over all the notation regarding Lipschitz maps from the proof of 1.1. Fix $\rho \in \omega^{<\omega}$, and suppose that $\mu_\rho(X_\alpha) = 1$ for all $\alpha < \gamma_0$, where $\gamma_0 < \delta$. We must show that $\mu_\rho(\bigcap_{\alpha<\gamma_0} X_\alpha) = 1$, that is, we must describe a winning strategy for II in $G_\rho(\bigcap_{\alpha<\gamma_0}(X_\alpha)$. This strategy will be, roughly, to amalgamate plays according to strategies for II in the games $G_\rho(X_\alpha)$, as was done in the proof that μ_ρ is countably complete. The new problem, that there are uncountably many such strategies, makes the amalgamation somewhat more delicate, but we can use a rather direct transcription of an argument of Kunen.[7] That argument rests on the Coding Lemma.

Strategies for II in games of the form $G_\rho(Y)$ are Lipschitz functions on ω^ω, and so can be coded by reals $\Sigma \in \omega^\omega$ as we have done above. By the Coding Lemma, there is a Σ_3^1 set B such that

$$\forall \Sigma (B(\Sigma) \Rightarrow \exists \alpha < \gamma_0 (\Sigma \text{ is a wiining strategy in } G_\rho(X_\alpha)),$$

and

$$\forall \alpha < \gamma_0 \exists \Sigma (B(\Sigma) \wedge \Sigma \text{ is a winning strategy in } G_\rho(X_\alpha)).$$

Since B is $\mathbf{\Sigma_3^1}$, we can fix a real z_0 and an $a \in \omega$ such that for all Σ

$$B(\Sigma) \Leftrightarrow \exists y \neg U(a, [\Sigma, y, z_0]).$$

For use below, we spell out our coding of tuples here: if $s_0, ..., s_{k-1} \in \omega^{<\omega}$ are sequences of the same length n, then $[s_0, ..., s_{k-1}]$ is the sequence of length n such that for $i < n$, $[s_0, ..., s_{k-1}](i) = n(\langle s_0(i), ..., s_{k-1}(i) \rangle)$.) If $x_0, ..., x_{k-1} \in \omega^\omega$, then $[x_0, ..., x_{k-1}] = \bigcup_n [x_0 \restriction n, ..., x_{k-1} \restriction n]$. Finally, if $t = [s_0, ..., s_{k-1}]$, then we call t a k-code, and write $(t)^i = s_i$. (The more common $(t)_i$ notation has already been dedicated to something slightly different.)

Let π_1, π_2 be the II-safe embeddings over ρ defined in the proof of claim 1 (induced by picking for each $u \in \mathrm{dom}(\rho^*)$ increasing maps h_u^1, h_u^2 from I_u^ρ to itself, whose ranges partition

[7] From his proof that all $\delta_{\mathbf{n}}^{\mathbf{1}}$ are measurable; see [1].

I^ρ_u). If $\sigma, \tau \in \omega^\omega$ extend ρ, then we write $\sigma \oplus \tau$ for their amalgamation via (π_1, π_2). Our strategy for II in $G_\rho(\bigcap_\alpha X_\alpha)$ will behave as follows: as I produces σ extending ρ, we produce a τ extending ρ whose membership in A^* would code the truth of the statement

$$\forall \Sigma (B(\Sigma) \Rightarrow \Sigma(\sigma \oplus \tau) \in A^*).$$

Notice that this statement is Π^1_3, and A^* is complete Π^1_3, so it is reasonable to try to construct such a τ. Of course, the statement coded by τ is self-referential. It is possible to bring in the recursion theorem at this point, but instead we shall simply give a direct construction of the desired τ.

More precisely, for each $u \in \mathrm{dom}(\rho^*)$, we want the restriction of τ^* to sequences extending u to code the truth of the statement

$$\forall \Sigma \forall y \forall x \supseteq u (U(a, [\Sigma, y, z_0]) \vee \Sigma(\sigma \oplus \tau)(x) \in A).$$

This is achieved by having τ^* regard the s in $u^\frown s$ as a 3-code for an initial segment of a possible Σ, y, and x, then compute an appropriate fragment of $\Sigma(\sigma \oplus \tau)(x)$, then give an output whose membership in A codes up the disjunction in the matrix of the formula just displayed. To make this work, we have to let $(s)^0$ give information about Σ quickly enough, so we make the convention: for any $s \in \omega^{<\omega}$, $\widetilde{s} = u_{\widehat{s(0)}} \cdots {}^\frown u_{s(\mathrm{dom}(s)-1)}$, and for $x \in \omega^\omega$, $\widetilde{x} = \bigcup_n \widetilde{x \restriction n}$. It will be $\widetilde{s^0}$ which is considered by τ^* as a possible initial segment of some Σ in B.

We turn to the details of the construction of τ from σ. Suppose we are given $\tau \restriction n$ and $\sigma \restriction n+1$; we must define $\tau(n)$. We may assume $\mathrm{dom}(\rho) \leq n$, as otherwise $\tau(n) = \rho(n)$. Our amalgamation process yields $(\sigma \oplus \tau) \restriction n+1$. Letting $n = n(v)$, our job is to define $\tau^*(v)$. Let $\bar{v} = v \restriction (\mathrm{dom}(v) - 1)$. If $\tau^*(\bar{v}) \notin W_0$, then we just let $\tau^*(v)$ be any one-term extension of $\tau^*(\bar{v})$. Assume now that $\tau^*(\bar{v}) \in W_0$. Let us write

$$v = u^\frown w,$$

where u is the longest initial segment of v in $\mathrm{dom}(\rho^*)$. If w is not a 3-code, then again we just let $\tau^*(v)$ be any one-term extension of $\tau^*(\bar{v})$ which is not in W_0; note we can always leave W_0 in one step. So assume w is a 3-code, and let

$$\Gamma = \widetilde{(w)^0}.$$

This is the approximation to Σ determined by w, the approximations to y and x being $(w)^1$ and $u^\frown (w)^2$ respectively . If $(\sigma \oplus \tau) \restriction n+1 \notin \mathrm{dom}(\Gamma^*)$ or $\rho \not\subseteq \Gamma^*((\sigma \oplus \tau) \restriction n+1)$, then once again we let $\tau^*(v)$ be any one-term extension of $\tau^*(\bar{v})$ which is not in W_0. Assume then that $(\sigma \oplus \tau) \restriction n+1 \in \mathrm{dom}(\Gamma^*)$, and that $\Gamma^*((\sigma \oplus \tau) \restriction n+1)$ extends ρ. Clearly, $n(u^\frown (w)^2) \leq n(v)$, so we can set

$$s = (\Gamma^*((\sigma \oplus \tau) \restriction n+1))^*(u^\frown (w)^2),$$

and we have $\rho^*(u) \subset s$ and $\mathrm{dom}(s) = \mathrm{dom}(v)$. If $s \notin W_0$, then again we can let $\tau^*(v)$ be any one-term extension of $\tau^*(\bar{v})$ which is not in W_0. We are finally in what we shall call the *interesting case* in the definition of $\tau^*(v)$.

Let
$$k = \mathrm{dom}(u), l = \mathrm{dom}(\bar{v}), \text{ and } m = \mathrm{dom}(r^s_{k-1}).$$

We must define $\tau^*(v)(l)$, the earlier entries having been given by $\tau^*(\bar{v})$. We do this by setting

$$H(\tau^*(v)(l)) = \langle r, v\rangle,$$

where
$$r = \langle r^s_l(0), ..., r^s_l(m-1), a, r^s_l(m), ..., r^s_l(\mathrm{dom}(r^s_l - 1))\rangle,$$

and
$$v = \langle v^s_l(0), ..., v^s_l(m-1), [\Gamma^*, (w)^1, z_0](l-k), v^s_l(m), ..., v^s_l(\mathrm{dom}(v^s_l - 1))\rangle.$$

To be pedantic, we should have written $[\Gamma^* \restriction (l+1-k), (w)^1 \restriction (l+1-k), z_0 \restriction (l+1-k)]$ above. We leave it to the reader to check that $\tau^*(v) \in W_0$. This completes the definition of $\tau^*(v)$, and thereby our description of II's putative winning strategy in $G_\rho(\bigcap_\alpha X_\alpha)$.

Let (σ, τ) be a run of $G_\rho(\bigcap_\alpha X_\alpha)$ according to this strategy. We must see II wins.

Suppose first that $\tau \notin A^*$, and let (p, f) be the leftmost witness to this fact. Note that $\tau^*(p \restriction i)$ is defined using the interesting case, for all i. Let k be largest such that $p \restriction k \in \mathrm{dom}(\rho^*)$, and
$$p = u^\frown z,$$
where $u = p \restriction k$. Then
$$z = [\Sigma, y, \bar{x}]$$
must be a 3-code. Inspecting the tree S and our definition of $\tau^*(p)$, we see that

$$(a, [\Sigma, y, z_0], (f)_m) \in [S_0], \text{ where } m = \mathrm{dom}(r^{\tau^*(p)}_{k-1},$$

which implies that $B(\Sigma)$ holds. Let $g \in \lambda^\omega$ be given as follows: if

$$c(f(j)) = \langle \alpha_0, ..., \alpha_n\rangle,$$

then $g(j) = f(j)$ if $n < m$, and otherwise

$$c(g(j)) = \langle \alpha_0, ..., \alpha_{m-1}, \alpha_{m+1}, ..., \alpha_n\rangle.$$

Letting $x = u^\frown \bar{x}$ it is easy to see that our construction guarantees that (x, g) is a witness that $\Sigma^*(\sigma \oplus \tau) \notin A^*$, and moreover, $(x, g) \leq_{\mathrm{lex}} (p, f)$. (Note here that $g(j) = f(j)$ for $j < k$, and $g(j) < f(j)$ for $j \geq k$ because $c(g(j))$ is a sequence of length one less than that of $c(f(j))$.) The remainder of the argument goes as in the countable additivity proof: since Σ is winning, there is a witness (q, h) that $\sigma \oplus \tau \notin A^*$ such that $(q \restriction k_1, h \restriction k_1) \leq_{\text{lex}} (x \restriction k_2, g \restriction k_2)$ for the appropriate k_1, k_2. The nature of the amalgamation and the fact that (p, f) is leftmost implies that (q, h) is a witness that $\sigma \notin A^*$, and that therefore (σ, τ) is a win for II under payoff clause (2).

Next, suppose $\tau \in A^*$. If $\sigma \notin A^*$, we are done, so suppose $\sigma \in A^*$. Notice that if $u \in \mathrm{dom}(\rho^*)$ and $B(\Sigma)$, then

$$\sup(\theta_u(\sigma), \theta_u(\tau)) \leq \theta_u(\sigma \oplus \tau) \leq \theta_u(\Sigma^*(\sigma \oplus \tau)),$$

because Σ is winning for II.

Claim. For $u \in \mathrm{dom}(\rho^*)$ and Σ such that $B(\Sigma)$, $\theta_u(\Sigma^*(\sigma \oplus \tau)) \leq \theta_u(\tau)$.

Proof. We describe a winning strategy Γ for II in the Moschovakis comparison game $G(u, \Sigma^*(\sigma \oplus \tau), \tau)$. Fix a witness y to $B(\Sigma)$, so that

$$\neg U(a, [\Sigma, y, z_0]).$$

Suppose that I plays out x extending u in this game (his first move being $x(\mathrm{dom}(u))$). Let $\hat{u}$ be the longest initial segment of x which is in $\mathrm{dom}(\rho^*)$, and

$$x = \hat{u}^\frown \bar{x}.$$

Then II will respond with

$$\Gamma(x) = \hat{u}^\frown [w, y, z_0], \text{ where } \Sigma = \widetilde{w},$$

and w codes Σ sufficiently rapidly.[8] It is clear that this defines a strategy Γ for II. To see that it wins, suppose I has played x extending u. If $\Sigma^*(\sigma \oplus \tau)(x)$ is not a branch of W_0, then by our construction, neither is $\tau^*(\Gamma(x))$, and moreover both reals leave W_0 at the same time, so that $\varphi_n(\Sigma^*(\sigma \oplus \tau)(x)) = \varphi_n(\tau^*(\Gamma(x))) = \langle 0, k \rangle$ for all n, where k is this common departure time. Thus II has won the Moschovakis comparison game. If $p = \Sigma^*(\sigma \oplus \tau)(x)$ is a branch of W_0, then by construction so is $q = \tau^*(\Gamma(x))$. By construction, setting $k = \mathrm{dom}(\hat{u})$ and $m = \mathrm{dom}(r^p_{k-1})$, we have

$$e^q_i = \begin{cases} e^p_i & \text{when } i < m \\ a & \text{when } i = m \\ e^p_{i-1} & \text{when } i > m \end{cases}$$

and

$$(q)_i = \begin{cases} (p)_i & \text{when } i < m \\ [\Sigma, y, z_0] & \text{when } i = m \\ (p)_{i-1} & \text{when } i > m. \end{cases}$$

[8]Namely, $\Sigma \restriction n(\sigma \oplus \tau \restriction k + 5) \subseteq w \restriction k - \mathrm{dom}(\hat{u})$ for all $k \geq \mathrm{dom}(\hat{u})$.

Using these facts, and inspecting the definition of the φ_n, it is easy to see that $\varphi_n(p) \leq \varphi_n(q)$ for all n. The key point here is just that $U(a,(q)_m)$ is false, so it cannot lower the norm values. $\square$

We can now complete the proof of 1.3. We have that whenever $B(\Sigma)$ and $u \in \mathrm{dom}(\rho^*)$, then $\theta_u(\tau) = \theta_u(\Sigma^*(\sigma \oplus \tau))$. Fixing $\alpha < \gamma_0$ and Σ a winning strategy for II in $G_\rho(X_\alpha)$ such that $B(\Sigma)$, this implies

$$\langle \theta_{u_0}(\tau), ..., \theta_{u_k}(\tau) \rangle \in X_\alpha,$$

where $k = \mathrm{dom}(\rho) - 1$. Since this is true for all $\alpha < \gamma_0$, (σ, τ) is a win for II under clause (3) in $G_\rho(\bigcap_{\alpha<\gamma_0} X_\alpha)$, as desired. $\square$

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