Complexity of Isomorphism for First Order Theories

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How little we know

We say a Borel equivalence relation $E$ on $X$ is *smooth* if there is a Borel measurable $f : X \to Y$ for some Polish space $Y$ such that $xEy \iff f(x) = f(y)$, i.e., $E \leq_B i(Y)$.

Let $x \in 2^\omega$. Identifying $2^\omega$ and $2^{\omega^2}$ view $x$ as coding $(x_0, x_1, \ldots) \in (2^\omega)^\omega$. Let $A_x = \{x_0, x_1, \ldots\}$. Define $x =^+ y$ if and only if $A_x = A_y$. (This was $\cong_2$ in Chris’ tutorial) $=^+$ is Borel but not smooth.

**Problem:** Find a first order theory $T$ such that $\cong_T$ is not smooth and $=^+ \not\leq_B \cong_T$.

A counterexample to Vaught’s Conjecture would do but there must be something simpler.
A Borel equivalence relation $E$ is **countable** if every $E$-class is countable.

A Borel equivalence relation $E$ is **essentially countable** if $E \sim_B E'$ for some countable Borel equivalence relation $E'$.

**Example**

- $x, y \in 2^\omega; xE_0y \iff \exists n \forall m > n \ x(m) = y(m)$;
- (Vitali relation) $x, y \in \mathbb{R}; xE_vy \iff x - y \in \mathbb{Q}$.

$E_0$ and $E_v$ are countable but not smooth and $E_0 \sim_B E_v$.

$\equiv^+$ is not essentially countable, indeed if $E$ is countable then $E \not\sim_B \equiv^+$.

**Problem** Is there a first order theory $T$ such that $\cong_T$ is essentially countable but not smooth?
Hjorth–Kečhris on $L_{\omega_1,\omega}$

**Theorem (Hjorth–Kečhris)**

If $E$ is a countable Borel equivalence relation there is $\phi \in L_{\omega_1,\omega}$ such that $\sim \subset_B E$.

**Theorem (Feldman–Moore)**

If $E$ is a countable Borel equivalence relation on $X$ there is a countable group $G$ and a Borel measurable action of $G$ on $X$ such that $xEy$ if and only if $x$ and $y$ are in the same $G$ orbit.

There is a universal orbit equivalence relation for $G$.

Let $U_G$ be the shift equivalence relation on $(2^\omega)^G$.

If $E$ is an orbit equivalence relation for a $G$ action, then $E \leq_B U_G$. 
Let $\mathcal{L} = \{\hat{g} : g \in G\} \cup \{U_n(x) : n \in \omega\}$, where the $\hat{g}$ are unary functions and $U_n$ is a unary predicate.

Let $\sigma$ assert that $\mathcal{M}$ is principle homogenous space for $G$, i.e., $G$ acts faithfully and transitively.

For $\mathcal{M} \models \sigma$ let $f_\mathcal{M} : G \to 2^\omega$

$$f_\mathcal{M}(g)(n) = 1 \iff \mathcal{M} \models U_n(\hat{g}(0)).$$

Then $\mathcal{M} \cong \mathcal{N} \iff f_\mathcal{M} U_G f_\mathcal{N}$. So $\cong_{\sigma} \leq_B U_G$.

An easy argument also shows $U_G \leq_B \cong_{\sigma}$.
We have reductions

\[ F : X \to \text{Mod}(\sigma) \quad \text{and} \quad H : \text{Mod}(\sigma) \to (2^\omega)^G \]

such that

\[ x Ey \iff F(x) \cong F(y) \]

and

\[ \mathcal{M} \cong \mathcal{N} \iff H(\mathcal{M}) U_G H(\mathcal{N}) \]

**Theorem (Luzin–Novikov Uniformization)**

Let \( F : X \to Y \) be Borel measurable such that \( F^{-1}(x) \) is countable for all \( x \). Then

i) If \( A \subseteq X \) is Borel then so is \( F(A) \);

ii) There is a Borel \( s : F(X) \to X \) such that \( F(s(x)) = x \) for all \( x \in F(X) \).
\[ F : X \rightarrow \text{Mod}(\sigma) \text{ and } H : \text{Mod}(\sigma) \rightarrow (2^\omega)^G \]

\[ xEy \Leftrightarrow F(x) \cong F(y) \text{ and } \mathcal{M} \cong \mathcal{N} \Leftrightarrow H(\mathcal{M})U_G H(\mathcal{N}) \]

- \( E \) is countable so \( H \circ F \) has countable sections. Thus \( H(F(X)) \) is Borel.
- \( U_G \) is a countable equivalence relation. Thus

\[ [H(F(X))]_{U_G} = \{ x : \exists y \in H(F(X)) \ y U_G x \} \text{ is Borel.} \]

- \( [F(X)]_{\cong_\sigma} = H^{-1}([H(F(X))]_{U_G}) \) is Borel.
- \( [F(X)]_{\cong_\sigma} \) is Borel and invariant, hence, by Lopez-Escobar it is \( \text{Mod}(\phi) \) for some \( \phi \in \mathcal{L}_{\omega_1,\omega} \) and \( F \) gives a Borel reduction of \( E \) to \( \text{Mod}(\phi) \).
• We have $E \leq_B \text{Mod}(\phi)$.

• Let $S = \{(x, y) : x \in X \text{ and } y \cong F(x)\}$. For any $y$ there are countably many $x$ such that $(x, y) \in S$, thus there is a Borel measurable $s : \text{Mod}(\phi) \to X$ such that $(s(y), y) \in S$.

$s$ is a reduction of $\cong_\phi$ to $E$ so $E \sim_B \cong_\phi$. 
I know one simple general fact about $\cong_T$ for first order theories.

**Theorem**

Let $T$ be a complete first order theory such that $S_1(T)$ is uncountable. Then $=^+ \leq_B \cong_T$.

- Need to look at small $T$ to answer earlier problems;
- This is not so surprising. The set of types realized has to be part of the invariants of a structure.

*When all you have a hammer, everything looks like a nail.*
Scott sets and $S$-saturated models

**Definition**

$S \subseteq 2^\omega$ is a Scott set if

i) If $x, y \in S$ and $z \leq_T x \oplus y$, then $z \in S$.

ii) If $\mathcal{T} \subseteq 2^{<\omega}$ is an infinite tree recursive in some element of $S$, then there is $f \in [\mathcal{T}] \cap S$.

**Definition**

Let $\mathcal{M} \models T$ and let $S$ be a Scott set containing $T$. Then $\mathcal{M}$ is $S$-saturated if

i) $\text{tp}(\bar{a}) \in S$ for all $\bar{a} \in \mathcal{M}$;

ii) If $p(v, \bar{w}) \in S$, $\bar{a} \in \mathcal{M}$ and $p(v, \bar{a})$ is finitely satisfiable, then $p(v, \bar{a})$ is realized in $\mathcal{M}$. 
S-saturated models

S-saturated models were studied by Knight–Nadel, Wilmers, Macintyre–Marker.

Theorem

Let $S$ be a countable Scott set with $T \in S$.

i) There is a countable $S$-saturated model of $T$.

ii) If $M$ and $N$ are countable $S$-saturated models of $T$, then $M \cong N$.

Theorem

Suppose $T$ is not small and $r$ codes a perfect tree of types. If $S_1$ and $S_2$ are countable Scott sets containing $r$ and $M_i$ is a countable $S_i$-saturated models of $T$ then

$$M_1 \cong M_2 \iff S_1 = S_2.$$
Reducing $\equiv^+$ to $\cong_T$

Let $r$ code a perfect tree of types.

- There is a countable family of Borel measurable functions $\mathcal{F}$ such that if $A \subseteq 2^\omega$, then $\text{cl}_\mathcal{F}(A)$ is a Scott set containing $A \cup \{r\}$;
- There is a perfect set $P \subseteq 2^\omega$ that is $\mathcal{F}$-independent, i.e. if $A \subseteq P$ then $\text{cl}_\mathcal{F}(A) \cap P = A$. Let $f : 2^\omega \to P$ be a continuous bijection.
- For $A \subset 2^\omega$ countable, let $\mathcal{M}_A$ be a countable $\text{cl}_\mathcal{F}(f(A))$-saturated model of $T$.
  Then $x \equiv^+ y \iff \mathcal{M}_{Ax} \cong \mathcal{M}_{Ay}$.

**Corollary**

If $T$ is not small and $h\text{Mod}(T)$ is the set of $\omega$-homogeneous models of $T$, then $\equiv^+ \sim_B \cong |h\text{Mod}(T)|$
Smooth-by-Smooth

One can ask many questions of the form: If $\cong_{T'}$ behaves well for all completions of $T$ does $\cong_T$ behave well?

**Problem** Suppose $\cong_{T'}$ is smooth for all completions $T'$ of $T$. Is $\cong_T$-smooth?

**Definition**

We call an equivalence relation $E$ **smooth-by-smooth** if there there is a smooth equivalence relation $E^* \supseteq E$ such that $E|[x]_{E^*}$ is smooth for all $x$.

Elementary equivalence is smooth, so $\cong_T$ above would be smooth-by-smooth.

Does smooth-by-smooth $\Rightarrow$ smooth?
There is $\Sigma^1_1$ smooth-by-smooth equivalence relation with all equivalence classes Borel that is not Borel. Let $(x, y) E (x', y')$ if and only if $x = x'$ and at least one of the following:

i) $y = y'$;

ii) $x \notin WO$;

iii) $x, y, y'$ code linear orders, $y$ and $y'$ embed into $x$ and $y \equiv y'$.

$(x, y) E^*(x', y')$ if and only if $x = x'$

- If $x \notin WO$, the $[(x, y)]_{E^*}$ is the single $E$-class $\{x\} \times 2^\omega$;

- If $x \in WO$ has order type $\alpha$ then $[(x, y)]_{E^*}$ is has one $E$-class for all $\beta \leq \alpha$ and many singleton classes

- $E$ has classes of arbitrary Borel complexity.
A Borel smooth-by-smooth equivalence relation is smooth.

Let $E$ be a Borel equivalence relation. Then $E$ is non-smooth if and only if there is a Borel probability measure $\mu$ that is
i) non $E$-atomic; i.e., $\mu([x]_E) = 0$ for all $x$;
ii) $E$-ergodic, i.e., $\mu(A) = 0$ or 1 for all $E$-invariant Borel $A$.

For $E_0$ Lebesgue measure works.
Borel smooth-by-smooth

Suppose $E^* \supset E$ is smooth but $E$ is not smooth.

There is a non-$E$-atomic, $E$-ergodic probability measure $\mu$ on $X$.

$\mu$ is $E^*$-ergodic. Thus since $E^*$ is smooth there must be some $x$ with $\mu([x]_{E^*}) = 1$.

But then $\mu|[x]_{E^*}$ is $E$-ergodic and non $E$-atomic, thus $E|[x]_{E^*}$ is not smooth.

Thus if $\cong_T$ is Borel and $\cong_{T'}$ is smooth for all completions of $T$, then $\cong_T$ is smooth.