

DECIDABILITY OF SOME COMPLICATED STRUCTURES DEFINABLE IN $\mathbb{C}(t)$

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In memory of Thanases Pheidas, 1958 – 2023

ABSTRACT. Several properly countable unions of algebraic sets in \mathbb{C}^n are definable in $\mathbb{C}(t)$ including the set CM of j -invariants of complex elliptic curves with complex multiplication. It has been suggested that one could prove the undecidability of $\text{Th}(\mathbb{C}(t))$ by showing that the theory of the structure $\mathfrak{CM} := (\mathbb{C}, +, \cdot, 0, 1, \text{CM})$ of the field of complex numbers considered with a unary predicate picking out CM is undecidable. We show using an effective version of the André-Oort conjecture that to the contrary that $\text{Th}(\mathfrak{CM})$ is stable and decidable. We discuss some related structures on the complex numbers definable in $\mathbb{C}(t)$ and how their theories may be connected to the Zilber-Pink conjectures.

It is a long standing open problem [14] whether the first-order theory of the field $\mathbb{C}(t)$ of rational functions in the single variable t with coefficients from the field \mathbb{C} of complex numbers is decidable. In various public lectures, Pheidas described a strategy, which he attributed to an anonymous reviewer of a grant proposal¹, for showing that this theory is undecidable based on interpreting a complicated structure on the complex numbers.

Using the fact that there are no nonconstant rational maps from the projective line to curves of positive genus, it is easy to see that \mathbb{C} is definable in $\mathbb{C}(t)$. For example, we have

$$\mathbb{C} = \{a \in \mathbb{C}(t) : (\exists y)a^3 + y^3 = 1\} .$$

The harder observation is that the set CM of j -invariants of elliptic curves with complex multiplication is also definable in $\mathbb{C}(t)$. The set CM is a countably infinite set of algebraic integers and is known to have rich arithmetic. As such, one might expect that the structure $\mathfrak{CM} := (\mathbb{C}, +, \cdot, 0, 1, \text{CM})$ of the field of complex numbers considered with a unary predicate picking out CM would have a complicated and, in particular, an undecidable theory. Since \mathfrak{CM} is interpretable in $\mathbb{C}(t)$, if this expectation were verified, then one would have succeeded in showing that the theory of $\mathbb{C}(t)$ is undecidable.

In this note, we observe that it follows from a theorem of Binyamini [1] on an effective version of the André-Oort conjecture for products of modular curves that the theory of \mathfrak{CM} is stable and decidable.

¹After I talked about this topic some years ago, Clifton Ealy wrote to me saying that he had asked Lou van den Dries whether he may have been the source of this observation. Van den Dries' answer on June 30th, 2008, was that he recalled having learned about the interpretation of the endomorphism rings (as groups) of elliptic curves in $\mathbb{C}(t)$ from Pheidas and then noticed the consequence that this would give the definability of CM . This is not exactly a confirmation that he was the source of the observation, but it is also not exactly a denial.

The considerations which give the definability of \mathbf{CM} also yield the definability of several other sets which are naturally properly countable unions of algebraic sets. For example, the set

$$\mathbf{Isog} := \{(j(E), j(E/\Gamma)) \in \mathbb{C} \times \mathbb{C} : E \text{ an elliptic curve, } \Gamma < E \text{ a finite subgroup}\}$$

is definable in $\mathbb{C}(t)$ where $j(E)$ denotes the j -invariant of the elliptic curve E . I would *guess* that the Zilber-Pink conjecture would be relevant in the analysis of such structures as $\mathfrak{CMJ} := (\mathbb{C}, +, \cdot, 0, 1, \mathbf{CM}, \mathbf{Isog})$. However, I have been unsuccessful both in showing that stability of \mathfrak{CMJ} follows from the Zilber-Pink conjecture and that decidability of the theory of \mathfrak{CMJ} follows from an effective form of the Zilber-Pink conjecture.

This paper is organized as follows. In Section 1 we recall some basic results and constructions involving elliptic curves. In Section 2 we explain in detail how to define \mathbf{CM} , \mathbf{Isog} , and related sets in $\mathbb{C}(t)$. Section 3 contains the proof of the stability and decidability of $\text{Th}(\mathfrak{CM})$. Section 4 contains some questions and observations around the connections between the Zilber-Pink conjecture and the complexity of the logical theory of the structures on \mathbb{C} interpreted in $\mathbb{C}(t)$.

As noted above, I learned about the strategy for proving the undecidability of $\text{Th}(\mathbb{C}(t))$ from Thanases Pheidas. He and I talked about the relevance of the André-Oort conjecture to this problem on several occasions, initially at the 2007 ICMS workshop on Number Theory and Computability and most recently at the Summer 2022 MSRI program on Decidability, Definability and Computability in Number Theory. I had been hoping to resolve the kinds of problems described in Section 4 before releasing this paper, but it is clear that I have waited already too long. I had wanted to show the final version to Thanases, but it is too late for that. Instead, this paper is dedicated to his memory.

I have presented these ideas in various lectures, including at the 2023 Arizona Winter School. Further exposition of the number theoretic background is provided in my notes [15] for those lectures.

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1. SOME BACKGROUND ON ELLIPTIC CURVES

An elliptic curve E over a field K is an irreducible, projective, one-dimensional, (necessarily commutative) algebraic group over K . For more details on elliptic curves, consult [7] or [16]. For higher dimensional abelian varieties, see [9].

Notation 1.1. We shall take our elliptic curves in Weierstrass form. For $g_2 \in \mathbb{C}(t)$ and $g_3 \in \mathbb{C}(t)$ with $\Delta(g_2, g_3) := g_2^3 - 27g_3^2 \neq 0$ we write $E = E_{g_2, g_3}$ for the elliptic curve whose affine equation is $y^2 = 4x^3 - g_2x - g_3$ and one point at infinity. As explained in our references, there is an algebraic group structure on E in which the point at infinity is the identity and addition is given by the secant-tangent method.

In practice, we will only work with constant g_2 and g_3 .

Notation 1.2. The algebraic j -invariant of E is

$$j^{\text{alg}}(g_2, g_3) := \frac{12^3 g_2^3}{\Delta}$$

and it is known that $E_{g_2, g_3} \cong E_{g'_2, g'_3}$ as algebraic groups if and only if $j^{\text{alg}}(g_2, g_3) = j^{\text{alg}}(g'_2, g'_3)$.

Using this invariance under isomorphism, we write $j(E)$ for $j^{\text{alg}}(g_2, g_3)$ if E is an elliptic curve which is isomorphic to the elliptic curve E_{g_2, g_3} .

We may understand elliptic curves over \mathbb{C} via complex analysis. See [16, Chapter VI] and [7, Chapter 11] for some of this theory.

Notation 1.3. Let $\text{GL}_2(\mathbb{R})^+$ denote the group of 2×2 matrices with real entries and positive determinant. There is an action of $\text{GL}_2(\mathbb{R})^+$ on \mathfrak{h} via the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau := \frac{a\tau + b}{c\tau + d}.$$

If $\tau \in \mathfrak{h} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$, then the complex Lie group $T_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ is isomorphic, as a complex Lie group, to (the analytification of) an elliptic curve. This identification is achieved through the use of Eisenstein series and the Weierstrass \wp -function. The exact formulae are not really necessary for our discussion, but the sake of culture we recall them.

Notation 1.4. The Weierstrass \wp -function is a meromorphic map $\wp : \mathbb{C} \times \mathfrak{h} \rightarrow \mathbb{C}$ defined by $\wp(z, \tau) := \frac{1}{z^2} + \sum_{(n,m) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(z-n-m\tau)^2} - \frac{1}{(n+m\tau)^2}$. We write $\wp'(z, \tau)$ for the derivative of \wp with respect to z .

Notation 1.5. For $k \in \mathbb{Z}_+$ the Eisenstein function $E_{2k} : \mathfrak{h} \rightarrow \mathbb{C}$ is defined by

$$E_{2k}(\tau) := \sum_{(n,m) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(n+m\tau)^{2k}}$$

We set $g_2(\tau) := 60G_4(\tau)$, $g_3(\tau) := 140G_6(\tau)$, and define Klein's j -function by $j(\tau) := j^{\text{alg}}(g_2(\tau), g_3(\tau))$.

With Notation 1.4 and 1.5 in place, we may recall some key facts about these functions.

The complex torus T_τ is isomorphic to (the complexification of) the elliptic curve $E_{g_2(\tau), g_3(\tau)}$ via the map $z \mapsto (\wp(z, \tau), \wp'(z, \tau))$.

A map of complex Lie groups between two one-dimensional complex tori $\psi : T_\tau \rightarrow T_{\tau'}$ is given by multiplication by some complex number λ for which $\lambda(\mathbb{Z} + \mathbb{Z}\tau) \subseteq (\mathbb{Z} + \mathbb{Z}\tau')$. We say that the complex torus T_τ has complex multiplication if and only if there is some $\lambda \in \mathbb{C} \setminus \mathbb{Z}$ for which $\lambda(\mathbb{Z} + \mathbb{Z}\tau) \subseteq (\mathbb{Z} + \mathbb{Z}\tau)$. It is an easy computation to check that T_τ has complex multiplication if and only if $[\mathbb{Q}(\tau) : \mathbb{Q}] = 2$, that is, τ is a quadratic, imaginary number. We say that an elliptic curve E over \mathbb{C} has complex multiplication if it (or, really, its analytification) is a complex torus with complex multiplication. Alternatively, we may detect that the elliptic curve has complex multiplication through its endomorphism ring $\text{End}(E)$, the set of maps of algebraic groups $\psi : E \rightarrow E$ with addition defined by $(\psi + \phi)(x) = (\psi(x) +_E \phi(x))$ where the second addition is in the sense of the algebraic group structure on E and multiplication defined by composition of functions. For an elliptic curve E over the complex numbers, either $\text{End}(E) \cong \mathbb{Z}$ or $\text{End}(E)$ is an order in a quadratic imaginary field, and in particular has rank two as an abelian group. The elliptic curve E has complex multiplication if and only if we are in this second case.

It is known that $j : \mathfrak{h} \rightarrow \mathbb{C}$ is surjective. Moreover, it follows from the properties of the algebraic j -function that $j(\tau) = j(\tau')$ if and only if $T_\tau \cong T_{\tau'}$ if and only if

$E_{g_2(\tau), g_3(\tau)} \cong E_{g_2(\tau'), g_3(\tau')}$ where the first isomorphism is taken in the category of complex Lie groups and the second is in the category of algebraic groups.

This equivalence may be described via the action of $\mathrm{GL}_2(\mathbb{R})^+$ on \mathfrak{h} as $j(\tau) = j(\tau')$ if and only if there is some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $\tau' = \gamma \cdot \tau$.

Taking $\gamma \in \mathrm{GL}_2(\mathbb{Q})^+$ instead, we find that the complex tori T_τ and $T_{\gamma \cdot \tau}$ are all *isogenous*, even by cyclic isogenies. That is, there are maps of complex tori $T_\tau \rightarrow T_{\gamma \cdot \tau}$ whose kernels are finite cyclic groups. This relation is expressed by the modular polynomials.

Notation 1.6. For each positive integer $\ell \in \mathbb{Z}_+$, we let $\Phi_\ell(x, y) \in \mathbb{Z}[x, y]$ be the ℓ^{th} modular polynomial [8, page 335]. This polynomial Φ_ℓ is characterized by the properties that it is monic in x and for any pair of complex numbers (j_1, j_2) we have $\Phi_\ell(j_1, j_2) = 0$ if and only if there are elliptic curves E_1 and E_2 with $j(E_1) = j_1$, $j(E_2) = j_2$, and a map of algebraic groups $\psi : E_1 \rightarrow E_2$ with kernel isomorphic to $\mathbb{Z}/\ell\mathbb{Z}$. It is a general fact that if two elliptic curves E_1 and E_2 are isogenous, meaning that there is a map of algebraic groups $\psi : E_1 \rightarrow E_2$ with finite kernel, then E_1 has complex multiplication if and only if E_2 does [16, Exercise VI.6.9].

For $\gamma \in \mathrm{GL}_2(\mathbb{Q})^+$ there is a number $\ell(\gamma) \in \mathbb{Z}_+$ so that $\Phi_{\ell(\gamma)}(j(\tau), j(\gamma \cdot \tau)) \equiv 0$. That is, the elliptic curves $E_{g_2(\tau), g_3(\tau)}$ and $E_{g_2(\gamma \cdot \tau), g_3(\gamma \cdot \tau)}$ are all isogeneous by maps $\psi : E_{g_2(\tau), g_3(\tau)} \rightarrow E_{g_2(\gamma \cdot \tau), g_3(\gamma \cdot \tau)}$ having kernel cyclic of order $\ell(\gamma)$. Moreover, for each ℓ there are $\gamma \in \mathrm{GL}_2(\mathbb{Q})^+$ with $\ell(\gamma) = \ell$.

2. SOME STRUCTURES INTERPRETED IN $\mathbb{C}(t)$

In this section we explain in detail how CM , Isog , and related structures are defined in $\mathbb{C}(t)$. As noted in the introduction, these constructions are not due to me, but as they are not documented in the literature, they are included here.

We start by recording the observation (which already appears in [14]) that the field of constants is definable in $\mathbb{C}(t)$.

Lemma 2.1. *The field of complex numbers \mathbb{C} is a definable subset of $\mathbb{C}(t)$.*

Proof. See [5, Section IV.2] for the necessary results on the genus of algebraic curves.

It suffices to consider any two variable polynomial $f(x, y) \in \mathbb{Z}[x, y]$ for which the equation $f(x, y) = 0$ defines the affine part of a smooth curve X of genus at least one and for which the one variable polynomial $f(a, y)$ is nonconstant for every choice of $a \in \mathbb{C}$. Then the formula $\phi(x) := (\exists y)f(x, y) = 0$ defines \mathbb{C} . Indeed, if $a \in \mathbb{C}$, then because $f(a, y) \in \mathbb{C}[y]$ is a nonconstant polynomial, it has a root. On the other hand, if $a \in \mathbb{C}(t)$ and $b \in \mathbb{C}(t)$ witnesses that $\phi(a)$ holds, then (a, b) defines a rational map $\mathbb{P}_{\mathbb{C}}^1 \dashrightarrow X$. As the genus of X is at least one, this map is necessarily constant, which means in particular that a is constant.

For a concrete choice of f , we may take $f(x, y) = x^3 + y^3 - 1$. □

There is a standard construction to interpret a finite algebraic extension L of the field K in K . Indeed, for any fixed positive integer d the class of all extensions of K of degree d is uniformly interpretable. Let us formalize this fact with the following proposition.

Proposition 2.2. *For each positive integer d the class of field extensions of degree d is uniformly interpretable relative to the theory of fields. That is, there is a formula $\theta(\mu)$ in the d^2 variables μ , a formula ξ in d variables, and definable functions*

\otimes_μ and \oplus_μ taking $2d$ arguments returning a d tuple, depending on the parameters μ so that for any field K and any tuple $\mu \in K^{d^3}$ with $K \models \theta(\mu)$ we have that $L_\mu := (K^d, \oplus_\mu, \otimes_\mu)$ is a field and $\xi(K) \subseteq L_\mu$ picks out a subfield isomorphic to K for which $[L_\mu : K] = d$. Moreover, for every every extension L of K of degree d there is some choice of parameters μ for which $L_\mu \cong L$ over the identification of K with $\xi(K)$.

Proof. While this construction is standard, allow us to recall some of the details. For an extension L of the field K of degree d , we may fix some K -basis $\{e_1, \dots, e_d\}$ of L . Making a linear change of variables if need be, we may assume that $e_1 = 1$.

Let μ be the $d \times d \times d$ array of coefficients for which

$$e_i e_j = \sum_{k=1}^d \mu_{i,j,k} e_k$$

Note that the coefficients $\mu_{i,j,k}$ are uniquely determined as $\{e_1, \dots, e_d\}$ is a K -basis. Since $e_i e_j = e_j e_i$, we could get away with recording these valued only for $i \leq j$, but there is no harm in writing too much.

Let $\xi(x_1, \dots, x_d)$ be the formula $\bigwedge_{i=2}^d x_i = 0$. Define

$$(x_1, \dots, x_d) \oplus_\mu (y_1, \dots, y_d) := (x_1 + y_1, \dots, x_k + y_k, \dots, x_d + y_d)$$

and

$$(x_1, \dots, x_d) \otimes_\mu (y_1, \dots, y_d) := \left(\sum_{1 \leq i,j \leq d} x_i y_j \mu_{i,j,1}, \dots, \sum_{1 \leq i,j \leq d} x_i y_j \mu_{i,j,k}, \dots, \sum_{1 \leq i,j \leq d} x_i y_j \mu_{i,j,d} \right)$$

Then, $L \cong L_\mu = (K^d, \oplus_\mu, \otimes_\mu)$ and $\xi(K)$ identifies with the image of K in L under the usual embedding of K -algebras.

In general, the formula $\theta(\mu)$ asserts that L_μ is a field with multiplicative identity $(1, 0, \dots, 0)$ by relativizing the finitely many field axioms to K . \square

Let us use the construction of Proposition 2.2 to encode the function fields of the elliptic curves E_{g_2, g_3} as g_2 and g_3 range through \mathbb{C} with $\Delta \neq 0$.

Notation 2.3. For $g_2 \in \mathbb{C}$ and $g_3 \in \mathbb{C}$ with $\Delta(g_2, g_3) \neq 0$, let μ be the array describing multiplication on the field $L_\mu := L^{g_2, g_3} := \mathbb{C}(t)[y]/(y^2 - 4t^3 + g_2 t + g_3)$ relative to the basis $(e_1, e_2) = (1, y)$. That is, $\mu_{1,1,1} = 1$, $\mu_{1,1,2} = 0$, $\mu_{1,2,1} = 0$, $\mu_{1,2,2} = 1$, $\mu_{2,1,1} = 0$, $\mu_{2,1,2} = 1$, $\mu_{2,2,1} = 4t^3 - g_2 t - g_3$, and $\mu_{2,2,2} = 0$.

Proposition 2.4. For $r = 0, 1$, and 2 there is formula $\rho_r(g_2, g_3, g'_2, g'_3)$ of the four variables g_2, g_3, g'_2 , and g'_3 so that $\mathbb{C}(t) \models \rho_r(g_2, g_3, g'_2, g'_3)$ if and only if

- g_2, g_3, g'_2 , and g'_3 all belong to \mathbb{C} ,
- $\Delta(g_2, g_3) \neq 0$ and $\Delta(g'_2, g'_3) \neq 0$, and
- $\text{rk Hom}(E_{g_2, g_3}, E_{g'_2, g'_3}) = r$

Proof. By Lemma 2.1 the first two conditions may be expressed with a formula. Using the fact that every rational map from one elliptic curve to another takes the form of a map of algebraic groups followed by a translation, we see that the interpretable group $E(L^{g_2, g_3}) = E_{g'_2, g'_3}(\mathbb{C}(E_{g_2, g_3}))$ fits into the following exact sequence:

$$0 \longrightarrow E_{g'_2, g'_3}(\mathbb{C}) \longrightarrow E_{g'_2, g'_3}(L^{g_2, g_3}) \longrightarrow \text{Hom}(E_{g_2, g_3}, E_{g'_2, g'_3}) \longrightarrow 0$$

That is, as a group, we may identify $\text{Hom}(E_{g_2, g_3}, E_{g'_2, g'_3})$ with the interpretable group $E_{g'_2, g'_3}(L^{g_2, g_3})/E_{g'_2, g'_3}(\mathbb{C})$. On general grounds (see [16, Theorem VI.6.1]), this group is isomorphic to one of

- 0, if E_{g_2, g_3} and $E_{g'_2, g'_3}$ are not isogenous,
- \mathbb{Z} , if E_{g_2, g_3} and $E_{g'_2, g'_3}$ are isogenous and $E_{g'_2, g'_3}$ does not have complex multiplication, or
- \mathbb{Z}^2 , if E_{g_2, g_3} and $E_{g'_2, g'_3}$ are isogenous and $E_{g'_2, g'_3}$ has complex multiplication.

We may recognize the first case that $\text{Hom}(E_{g_2, g_3}, E_{g'_2, g'_3}) = 0$, with the formula expressing that $E_{g'_2, g'_3}(L^{g_2, g_3}) = E_{g'_2, g'_3}(\mathbb{C})$, giving the formula $\rho_0(g_2, g_3, g'_2, g'_3)$. To recognize the second case we use the formula expressing that there is some $P \in E_{g'_2, g'_3}(L^{g_2, g_3}) \setminus E_{g'_2, g'_3}(\mathbb{C})$ and that for all $Q \in E_{g'_2, g'_3}(\lambda)$ there is some $R \in E_{g'_2, g'_3}(L^{g_2, g_3})$ so that either $[2]R = Q$ or $[2]R = Q + P$, where multiplication by two and addition are taken in the sense of the group law on $E_{g'_2, g'_3}$. This gives the formula $\rho_1(g_2, g_3, g'_2, g'_3)$. Finally, $\rho_2(g_2, g_3, g'_2, g'_3)$ may be taken to be the formula asserting that there are P and Q in $E_{g'_2, g'_3}(L^{g_2, g_3})$ so that for all R none of the equations $[2]R = P$, $[2]R = Q$, or $[2]R = P + Q$ hold. \square

Corollary 2.5. *Each of the sets CM and Isog is definable in $\mathbb{C}(t)$. Hence, each of the structures $\mathfrak{CM} = \mathbb{C}, +, -, \cdot, 0, 1, \text{CM}$ and $\mathfrak{CM}\mathfrak{I} = \mathbb{C}, +, -, \cdot, 0, 1, \text{CM}, \text{Isog}$ is interpretable in $\mathbb{C}(t)$.*

Proof. The set CM is defined by the formula

$$\exists u \exists v (\rho_2(u, v, u, v) \wedge x = j^{\text{alg}}(u, v))$$

and Isog is defined by

$$\exists u \exists v \exists w \exists z (x = j^{\text{alg}}(u, v) \wedge y = j^{\text{alg}}(w, z) \wedge (\rho_1(u, v, w, z) \vee \rho_2(u, v, w, z))) .$$

\square

3. STABILITY OF CM

The structure \mathfrak{CM} is not interpretable in \mathbb{C} considered as a field. Indeed, the algebraically closed field \mathbb{C} is strongly minimal but the set CM is a countably infinite set of algebraic integers, which means that it is both infinite and its complement is infinite. One might expect that some of the arithmetic of the algebraic integers could be recovered in \mathfrak{CM} . As the theory of the ring of all algebraic integers is itself decidable [17], this would not yield undecidability of \mathfrak{CM} , but it would imply instability. In this section, we observe that it follows from the André-Oort conjecture for product of modular curves [10] and a theorem of Pillay [12] on expansions of strongly minimal theories that \mathfrak{CM} is stable and then using Pillay's theorem and an effective version of the André-Oort conjecture due to Binyamini [1], that the theory of \mathfrak{CM} is decidable.

Notation 3.1. Recall that for any \mathcal{L} structure \mathfrak{M} and a nonempty subset $A \subseteq M^n$ of the n^{th} Cartesian power of the universe M of \mathfrak{M} (for some natural number n), the induced structure on $\mathfrak{A}^{\text{ind}}$ on A is the structure having universe A and an m -ary relation R_ϕ for each \mathcal{L} formula ϕ in the variables $x_{i,j}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ to be interpreted at $R_\phi^{\mathfrak{A}} = A^m \cap \phi(\mathfrak{M})$. We refer to this associated relational language as \mathcal{L}^{ind} . Strictly speaking, we should indicate the arity n in the notation for the name of this language, but we omit it.

Proposition 3.2. *The structure CM^{ind} induced on the set of j -invariants of elliptic curves with complex multiplication from the field $(\mathbb{C}, +, -, \cdot, 0, 1)$ of complex numbers considered as structure for the language of rings is stable.*

We will prove Proposition 3.2 by giving a more precise description of the induced structure on \mathbf{CM} using modular polynomials.

Let us introduce a language for expressing the relations defined by modular polynomials and then discuss two structures in this language.

Notation 3.3. Let \mathcal{L}^{mod} be the relational language having an n -ary relation symbol R_X for each component of an embedded affine algebraic variety $X \subseteq \mathbb{A}_{\mathbb{C}}^n$ defined by a system of equations of the form $\Phi_{\ell_k}(x_{i_k}, x_{j_k}) = 0$ for some sequence $(i_1, j_1, n_1), \dots, (i_m, j_m, n_m)$ where $1 \leq i_k \leq j_k \leq n$ and $\ell_k \in \mathbb{Z}_+$ for $1 \leq k \leq m$. We call such X *special varieties*. A variety for which we further allow equations of the form $x_i = c$ for $c \in \mathbb{C}$ is called *weakly special*.

Let \mathbb{C}^{mod} be the \mathcal{L}^{mod} -structure with universe \mathbb{C} in which for each such X we interpret $R_X^{\mathbb{C}^{\text{mod}}}$ as $X(\mathbb{C})$. Let \mathbf{CM}^{mod} be the substructure of \mathbb{C}^{mod} with universe \mathbf{CM} .

Remark 3.4. An equation of the form $\Phi_{\ell}(x, x) = 0$ with $\ell > 1$ defines a finite set of \mathbf{CM} -points and more generally a pair of equations $\Phi_{\ell}(x, y) = 0$ and $\Phi_{\ell'}(x, y) = 0$ with $\ell \neq \ell'$ will define a finite set of pairs or \mathbf{CM} -points. It follows from these observations that if $X \subseteq \mathbb{A}_{\mathbb{C}}^n$ is a special variety, then for each projection $\rho_j : \mathbb{A}_{\mathbb{C}}^n \rightarrow \mathbb{A}_{\mathbb{C}}^1$ given by $\rho_j(x_1, \dots, x_n) = x_j$, either the image of X under ρ_j is infinite, or $\rho_j(X) = \{\xi\}$ for some \mathbf{CM} -point ξ .

We can give a better description of the special varieties using Klein's j -function and the action of $\text{GL}_2(\mathbb{Q})^+$ on \mathfrak{h} .

Convention 3.5. In what follows we will use i and j as indices. We trust that this will cause no confusion with $i = \sqrt{-1}$ and the j -function.

Notation 3.6. A *pre-weakly-special datum* is given by

- $n \in \mathbb{Z}_+$,
- $\pi_0 \subseteq \{1, \dots, n\}$,
- Π a partition of $\{1, \dots, n\} \setminus \pi_0$,
- $\xi : \pi_0 \rightarrow \mathbb{C}$, and
- $\gamma = (\gamma_{\pi})_{\pi \in \Pi}$ where $\gamma_{\pi} : \pi^2 \rightarrow \text{GL}_2(\mathbb{Q})^+$ satisfies that for $\{i, j, k\} \subseteq \pi$, $\gamma(i, i) = \text{id}$ and $\gamma(i, k) = \gamma(j, k) \cdot \gamma(i, j)$.

If ξ satisfies that for all $j \in \pi_0$ we have, $[\mathbb{Q}(\xi(j)) : \mathbb{Q}] = 2$, then we call $(\pi_0, \Pi, \xi, \gamma)$ a *pre-special datum*.

We associate the complex analytic subvariety $\mathfrak{X}_{(\pi_0, \Pi, \xi, \gamma)}$ of \mathfrak{h}^n defined by the equations

- $\tau_j = \xi(j)$ for $j \in \pi_0$ and
- $\gamma(i, j) \cdot \tau_i = \tau_j$ for $\{i, j\} \subseteq \pi \in \Pi$

Proposition 3.7. *Given a pre-weakly special datum $(\pi_0, \Pi, \xi, \gamma)$, if we further choose $i_{\pi} \in \pi$ for each $\pi \in \Pi$, then the map $\nu : \mathfrak{h}^{\Pi} \rightarrow \mathfrak{h}^n$ given by $\nu((\tau_{\pi})_{\pi \in \Pi}) = (\nu_1, \dots, \nu_n)$ with*

- $\nu_j = \xi(j)$ if $j \in \pi_0$ and
- $\nu_j = \gamma(i_{\pi}, j) \cdot \tau_{\pi}$ if $j \in \pi \in \Pi$

is a bijection between \mathfrak{h}^{Π} and $\mathfrak{X}_{(\pi_0, \Pi, \xi, \gamma)}$.

Proof. We leave it to the reader to verify that the map ν simply expresses $\mathfrak{X}_{(\pi_0, \Pi, \xi, \gamma)}$ as the graph of a function. \square

Proposition 3.8. *For any pre-(weakly)-special datum $(\pi_0, \Pi, \xi, \gamma)$, the restriction of the map $j^{\times n} : \mathfrak{h}^n \rightarrow \mathbb{C}^n$ given by $(\tau_1, \dots, \tau_n) \mapsto (j(\tau_1), \dots, j(\tau_n))$ to $\mathfrak{X}_{(\pi_0, \Pi, \xi, \gamma)}$ is surjective onto the \mathbb{C} -points of a (weakly) special variety we denote as $X_{(\pi_0, \Pi, \xi, \gamma)}$.*

Proof. Using the characterization of the modular polynomials from Notation 1.6, we see that the image of $\mathfrak{X}_{(\pi_0, \Pi, \xi, \gamma)}$ under $j^{\times n}$ is contained in the algebraic variety X defined by $x_i = j(\xi(i))$ for $i \in \pi_0$ and $\Phi_{\ell(\gamma_\pi)}(x_i, x_j) = 0$ for $\{i, j\} \in \pi \in \Pi$. As presented, each component of X is weakly special. If we presume as well that $(\pi_0, \Pi, \xi, \gamma)$ is a pre-special datum, then the equations $x_i = j(\xi(i))$ set the i^{th} coordinate equal to a CM-point and may be seen as picking a component of a variety defined by $\Phi_\ell(x_i, x_i) = 0$ for suitable choice of ℓ . That is, the components of X are special.

By Proposition 3.7, $\mathfrak{X}_{(\pi_0, \Pi, \xi, \gamma)}$ is an irreducible complex analytic space. Hence, the closure of its image under $j^{\times n}$ is a component Y of X . It follows from Proposition 3.7, that if $J \subseteq \{1, \dots, n\}$ is any selector set for Π and $\rho_J : \mathbb{A}_{\mathbb{C}}^n \rightarrow \mathbb{A}_{\mathbb{C}}^{\Pi}$ is the projection defined by $(\rho_J)_\pi(x_1, \dots, x_n) = x_j$ where $\{j\} = J \cap \pi$, then $\rho_J(j^{\times n} \mathfrak{X}_{(\pi_0, \Pi, \xi, \gamma)}) = \mathbb{C}^{\Pi}$. It follows that $j^{\times n}(\mathfrak{X}_{(\pi_0, \Pi, \xi, \gamma)}) = Y(\mathbb{C})$. That is, we may take $Y = X_{(\pi_0, \Pi, \xi, \gamma)}$. \square

With the next result we see that the (weakly) special varieties are exactly those obtained as images under the j -function of the complex analytic varieties $\mathfrak{X}_{(\pi_0, \Pi, \xi, \gamma)}$ for some pre-(weakly)-special datum $(\pi_0, \Pi, \xi, \gamma)$.

Proposition 3.9. *For each $n \in \mathbb{Z}_+$ and (weakly) special variety $X \subseteq \mathbb{A}_{\mathbb{C}}^n$ there is a pre-(weakly)-special datum $(\pi_0, \Pi, \xi, \gamma)$ so that $X = X_{(\pi_0, \Pi, \xi, \gamma)}$.*

Proof. Let $\pi_0 := \{i \in \{1, \dots, n\} : \dim \rho_i(X) = 0\}$ where $\rho_i : \mathbb{A}_{\mathbb{C}}^n \rightarrow \mathbb{A}_{\mathbb{C}}^1$ is the projection $(x_1, \dots, x_n) \mapsto x_i$. As X is irreducible, if $\dim \rho_i(X) = 0$, then there is some $a_i \in \mathbb{C}$ with $\rho_i(X) = \{a_i\}$. For each $i \in \pi_0$ pick some $\tilde{a}_i \in \mathfrak{h}$ with $j(\tilde{a}_i) = a_i$. Note that if X is special, then necessarily each \tilde{a}_i is quadratic imaginary.

Define an equivalence relation \sim on $\{1, \dots, n\} \setminus \pi_0$ by $i \sim j$ if and only if there is some $\ell \in \mathbb{Z}_+$ so that $\Phi_\ell(x_i, x_j) = 0$ holds on X .

Note that if $i \sim j$, then there is exactly one $\ell := \ell(i, j)$ for which $\Phi_\ell(x_i, x_j) = 0$ holds on X for if $\Phi_k(x_i, x_j)$ were also to hold for $k \neq \ell$, then for every point in X , x_i would be CM, implying that $i \in \pi_0$ contrary to our definition of \sim .

Let Π be the partition associated to \sim . For each pair (i, j) with $i \sim j$, pick some $\delta(i, j) \in \text{GL}_2(\mathbb{Q})^+$ with $\ell(\delta(i, j)) = \ell(i, j)$. Then the preimage of X under $j^{\times n}$ is equal to

$$\bigcup_{(\epsilon_1, \dots, \epsilon_n) \in \text{SL}_2(\mathbb{Z})^n} \{(\tau_1, \dots, \tau_n) \in \mathfrak{h}^n : \tau_i = \epsilon_i \cdot a_i \text{ for } i \in \pi_0 \text{ and } \tau_j = \epsilon_j \delta(i, j) \epsilon_i \cdot \tau_i \text{ for } i \sim j\}.$$

Pick $(\epsilon_1, \dots, \epsilon_n)$ so that corresponding component is nonempty.

Define $\xi : \pi_0 \rightarrow \mathfrak{h}$ by $\xi(i) := \epsilon_i \cdot \tilde{a}_i$ and for $\pi \in \Pi$ and $\{i, j\} \in \pi$ set $\gamma_\pi(i, j) := \epsilon_j \delta(i, j) \epsilon_i$. By the consistency of the analytic equations, γ satisfies the cocycle conditions. Visibly, $X(\mathbb{C}) = j^{\times n}(\mathfrak{X}_{(\pi_0, \Pi, \xi, \gamma)})$. \square

Let us consider images of pre-(weakly)-special under projections.

Proposition 3.10. *We are given a pre-(weakly)-special datum $(\pi_0, \Pi, \xi, \gamma)$ with $n > 1$. Write $\varpi : \mathfrak{h}^n \rightarrow \mathfrak{h}^{n-1}$ for the projection map given by $(\tau_1, \dots, \tau_n) \mapsto$*

$(\tau_1, \dots, \tau_{n-1})$. Then there is a pre-(weakly)-special datum $(\pi'_0, \Pi', \xi', \gamma')$ so that the restriction of ϖ to $\mathfrak{X}_{(\pi_0, \Pi, \xi, \gamma)}$ is surjective onto $\mathfrak{X}_{(\pi'_0, \Pi', \xi', \gamma')}$

Proof. There are three cases to consider:

- $n \in \pi_0$
- $\{n\} \in \Pi$
- There is some $\eta \in \Pi$ with $|\eta| > 1$ and $n \in \eta$

In the first case we set $\pi'_0 := \pi_0 \setminus \{n\}$, $\Pi' := \Pi$, $\xi' := \xi \upharpoonright \pi'_0$, and $\gamma' := \gamma$.

In the second case we set $\pi'_0 := \pi_0$, $\Pi' := \Pi \setminus \{n\}$, $\xi' := \xi$, and $\gamma' := ((\gamma_\pi)_{\pi \in \Pi'})$.

In the last case we set $\pi'_0 := \pi_0$, $\Pi' := (\Pi \setminus \{\eta\}) \cup (\eta \setminus \{n\})$, $\xi' := \xi$, and $\gamma' := ((\gamma'_\pi)_{\pi \in \Pi'})$ where $\gamma'_\pi = \gamma_\pi$ for $\pi \in \Pi \setminus \{\eta\}$ and $\gamma'_{\eta \setminus \{n\}} = \gamma_\eta \upharpoonright (\eta \setminus \{n\})^2$.

We leave it to the reader to verify that these choices work. \square

Combining Proposition 3.8, Proposition 3.9, and Proposition 3.10, we may compute images of (weakly) special varieties under projections.

Corollary 3.11. *If $n > 1$, $X \subseteq \mathbb{A}_{\mathbb{C}}^n$ is a (weakly) special variety, and $\rho : \mathbb{A}_{\mathbb{C}}^n \rightarrow \mathbb{A}_{\mathbb{C}}^{n-1}$ is the projection to the first $n-1$ coordinates, then there is a (weakly) special variety $Y \subseteq \mathbb{A}_{\mathbb{C}}^{n-1}$ so that $\rho(X(\mathbb{C})) = Y(\mathbb{C})$.*

Proof. By Proposition 3.9, we may find a pre-(weakly)-special datum $(\pi_0, \Pi, \xi, \gamma)$ so that we may express $X = X_{(\pi_0, \Pi, \xi, \gamma)}$ with $X(\mathbb{C}) = j^{\times n}(\mathfrak{X}_{(\pi_0, \Pi, \xi, \gamma)})$. By Proposition 3.10, there is a pre-(weakly)-special datum $(\pi'_0, \Pi', \xi', \gamma')$ so that the projection $\varpi : \mathfrak{h}^n \rightarrow \mathfrak{h}^{n-1}$ onto the first $n-1$ coordinates takes $\mathfrak{X}_{(\pi_0, \Pi, \xi, \gamma)}$ onto $\mathfrak{X}_{(\pi'_0, \Pi', \xi', \gamma')}$. By Proposition 3.8, the image of $\mathfrak{X}_{(\pi'_0, \Pi', \xi', \gamma')}$ under $j^{\times(n-1)}$ is the set of \mathbb{C} -points of a (weakly) special variety $Y = X_{(\pi'_0, \Pi', \xi', \gamma')} \subseteq \mathbb{A}_{\mathbb{C}}^{n-1}$. From the commutativity of the diagram

$$\begin{array}{ccccc}
 & & \mathfrak{X}_{(\pi'_0, \Pi', \xi', \gamma')} & \xrightarrow{\quad} & \mathfrak{h}^{n-1} \\
 & \nearrow & \downarrow & & \downarrow \varpi \\
 \mathfrak{X}_{(\pi_0, \Pi, \xi, \gamma)} & \xrightarrow{\quad} & \mathfrak{h}^n & \xrightarrow{\quad} & \mathfrak{h}^n \\
 \downarrow & & \downarrow & & \downarrow j^{\times(n-1)} \\
 & \nearrow & Y(\mathbb{C}) & \xrightarrow{j^{\times n}} & \mathbb{C}^{n-1} \\
 X(\mathbb{C}) & \xrightarrow{\quad} & \mathbb{C}^n & \xrightarrow{\rho} & \mathbb{C}^{n-1}
 \end{array}$$

we conclude that the projection $X(\mathbb{C}) \rightarrow Y(\mathbb{C})$ is onto. \square

Proposition 3.12. *The set \mathbb{C} given together with the finite unions of weakly special subvarieties of each Cartesian power as its closed sets forms a complete Zariski geometry.*

Proof. We need to verify Axioms (Z0), (Z1), (Z2), and (Z3) of [6].

The implicit hypothesis of closure under intersection and finite union is attained by taking unions of the equations (for closure under intersection) and then we allow for finite unions of weakly special varieties to get the closure under finite unions.

The condition (Z0) is achieved observing that the diagonal in \mathbb{C}^n given by $x_i = x_j$ is defined by $\Phi_1(x_i, x_j) = 0$ and a condition $x_i = c$ for a fixed $c \in \mathbb{C}$ is permitted because we are working with *weakly* special varieties. Condition (Z1) (and even the stronger condition of completeness) is given by Corollary 3.11. Conditions (Z2) and (Z3) are inherited from algebraic geometry in that $\mathbb{A}_{\mathbb{C}}^1$ is a smooth curve. \square

Corollary 3.13. *The theory of $\mathbb{C}_{\mathbb{C}}^{mod}$, the expansion of the structure \mathbb{C}^{mod} by constants naming all elements, has quantifier elimination and is strongly minimal.*

Proof. Apply [6, Proposition 2.1 and Corollary 2.7]. \square

We need the result of Corollary 3.13 without naming constants. The proof of [6, Proposition 2.1] goes through under a weakening (Z0)'. Let us make this precise.

Proposition 3.14. *Let D be a set and for each $n \in \mathbb{N}$ let \mathcal{T}_n be a topology on D . Let $S \subseteq D$ be an infinite set. We presume that with these topologies, D satisfies conditions (Z1), (Z2) and (Z3) of the definition of a Zariski geometry and the weakened condition (Z0)':*

- for each pair $(m, n) \in \mathbb{N}^2$ of natural numbers and sequence $(i_1, \dots, i_m) \in \{1, \dots, n\}^m$ the map $D^n \rightarrow D^m$ given by $(x_1, \dots, x_n) \mapsto (x_{i_1}, \dots, x_{i_m})$ is continuous,
- each polydiagonal defined by $x_i = x_j$ in D^n is closed,
- for $c \in S$ and $n \in \mathbb{N}$ each constant map $D^n \rightarrow D$ given by $(x_1, \dots, x_n) \mapsto c$ is continuous and $\{c\} \in \mathcal{T}_1$, and
- for each finite $Y \in \mathcal{T}_1$, $Y \subseteq S$.

Then the structure \mathfrak{D} with universe D and predicates for each closed set admits quantifier elimination and is strongly minimal.

Proof. Let us work through the proof of [6, Proposition 2.1] noting where the apparent use of the extra constant symbols may be replaced by working just with S .

The proof of [6, Lemma 2.2] that for each k the space D^k is irreducible seems to make use of the condition that $\{x\}$ is closed for each $x \in D$ to show that if $F \subseteq D^{k+1}$ is closed then $F^* := \{a \in D^k : (a, x) \in F \text{ for all } x \in D\}$ is also closed in that $F^* = \bigcap_{x \in D} F_x$ where $F_x = \{a \in D^k : (a, x) \in F\}$ would be an intersection of closed sets if we were to assume (Z0). Let us note that $\bigcap_{x \in D} F_x = \bigcap_{x \in S} F_x$. The inclusion \subseteq is immediate from the inclusion $S \subseteq D$. For the other inclusion, if $a \in D^k \setminus \bigcap_{x \in D} F_x$, then by (Z3), there are only finitely many $x \in D$ with $a \in F_x$. In particular, there is some $x \in S$ with $a \notin F_x$. That is, $a \notin \bigcap_{x \in S} F_x$.

The proof of [6, Lemma 2.3] that the closure of the image of an irreducible set under a projection remains irreducible and that each component of the closure of an image of a closed set comes from a single component does not use (Z0).

The proof of [6, Lemma 2.4] uses (Z0) in a way similar to what appears in [6, Lemma 2.2] in that one needs to check that for a closed set $C \subseteq D^n \times D$ that the set $\{a \in D^n : C_a = D\}$ is closed, which there is achieved by realizing this set as $\bigcap_{x \in D} C_x$ but we could see it as $\bigcap_{x \in S} C_x$.

The dimension theorem in [6, Lemma 2.5] implicitly uses the same argument as in [6, Lemma 2.2] in showing that the product of two irreducible sets is irreducible. For our version of this result we need the last condition in (Z0)' to see that each set in \mathcal{T}_n contains a dense set of points with coordinates from S .

The proof of [6, Lemma 2.6] seems to use $(Z0)$ in that various choices of points in D are made. It is enough to note that these may be taken from S . Finally, when the proof of [6, Proposition 2.1] is completed on page 8, $(Z0)$ is again used as it was for [6, Lemma 2.2] and we may replace that use with an intersection over S . \square

It follows from Proposition 3.14 that the theory of \mathbb{C}^{mod} eliminates quantifiers.

Corollary 3.15. *The theory of \mathbb{C}^{mod} eliminates quantifiers.*

The proof of quantifier elimination for \mathbb{C}^{mod} is effective.

Proposition 3.16. *There is a computable function from \mathcal{L}^{mod} to itself, $\phi \mapsto \widehat{\phi}$, so that for every ϕ we have that $\widehat{\phi}$ is quantifier-free and $\mathbb{C}^{\text{mod}} \models \phi \leftrightarrow \widehat{\phi}$.*

Proof. Working by recursion on the complexity of ϕ we see that it suffices to consider that case that $\phi = \exists x_{n+1}\psi$ where ψ is quantifier-free in the free variables x_1, \dots, x_{n+1} . With the usual operations in Boolean algebras, we may computably convert ψ to $\check{\psi}$ expressed as

$$\check{\psi} = \bigvee_{X \in \mathcal{X}} \left(R_X \wedge \neg \left(\bigvee_{Y \in \mathcal{Y}_X} R_Y \right) \right)$$

where \mathcal{X} is a finite set of special varieties and for each $X \in \mathcal{X}$, \mathcal{Y}_X is a finite set of proper special subvarieties of X . Corollary 3.11 gives a way to compute the projection of each such X and Y . The argument for Corollary 3.15 constructs $\widehat{\phi}$ from the predicates for each of these projections. \square

In fact, the equivalence given by Proposition 3.16 also works for CM^{mod} .

Proposition 3.17. *With the function of Proposition 3.16, for every formula ϕ of \mathcal{L}^{mod} we have $\text{CM}^{\text{mod}} \models \phi \leftrightarrow \widehat{\phi}$*

Proof. Working by induction on ϕ , it suffices to consider the case that $\phi = \exists x_{n+1}\psi$ where ψ is quantifier-free in the free variables x_1, \dots, x_{n+1} . As in the proof of Proposition 3.17, we may put ψ into disjunctive normal form and then moving \bigvee across the existential quantifier, we may assume that $\psi = R_X \wedge \neg(\bigvee_{Y \subseteq \mathcal{Y}} R_Y)$ where $X \subseteq \mathbb{A}_{\mathbb{C}}^{n+1}$ is a special subvariety and \mathcal{Y} is a finite set of proper special subvarieties of X . If $a \in \text{CM}^n$ and $\text{CM}^{\text{mod}} \models \exists x_{n+1}\psi(a, x_{n+1})$, then $\mathbb{C}^{\text{mod}} \models \exists x_{n+1}\psi(a, x_{n+1})$, which implies by Proposition 3.16 that $\mathbb{C}^{\text{mod}} \models \widehat{\phi}(a)$. Since $\widehat{\phi}$ is quantifier-free, it follows that $\text{CM}^{\text{mod}} \models \widehat{\phi}(a)$. In the other direction, if $\text{CM}^{\text{mod}} \models \widehat{\phi}(a)$, then again because $\widehat{\phi}$ is quantifier-free, $\mathbb{C}^{\text{mod}} \models \widehat{\phi}(a)$. This yields that $\mathbb{C}^{\text{mod}} \models \exists x_{n+1}\psi(a, x_{n+1})$. That is, $X_a(\mathbb{C}) \setminus (\bigcup_{Y \in \mathcal{Y}} Y_a(\mathbb{C})) \neq \emptyset$. If $\dim X_a = 0$, then either some equation of the form $\Phi_\ell(x_i, x_{n+1}) = 0$ with $1 \leq i \leq n$ and $\ell \in \mathbb{Z}_+$ or $x_{n+1} = c$ for some $c \in \text{CM}$ holds on X . Thus, either every point in X_a satisfies $\Phi_\ell(a_i, x) = 0$ or $x = c$. Either way, $X_a \subseteq \text{CM}$, where in the first case we use the fact that an elliptic curve isogenous to a CM-elliptic curve also has CM. Thus, in this case that $\dim X_a = 0$ and $\text{CM}^{\text{mod}} \models \widehat{\phi}(a)$, we have that $\text{CM}^{\text{mod}} \models \exists x_{n+1}\psi(a, x_{n+1})$. The other possibility is that $\dim X_a = 1$. Nonemptiness of $X_a(\mathbb{C}) \setminus (\bigcup_{Y \in \mathcal{Y}} Y_a(\mathbb{C}))$ implies that each Y_a is finite. Thus, any $c \in \text{CM}$ outside of the finite set $\bigcup_{Y \in \mathcal{Y}} Y_a(\mathbb{C})$ witnesses that $\text{CM}^{\text{mod}} \models \exists x_{n+1}\psi(a, x_{n+1})$. \square

It follows from Proposition 3.17 that CM^{mod} is an elementary substructure of \mathbb{C}^{mod} .

Corollary 3.18. $\mathbf{CM}^{\text{mod}} \preceq \mathbb{C}^{\text{mod}}$

Proof. Apply the Tarski-Vaught test: for any \mathcal{L}^{mod} -formula ϕ in the free variables x_1, \dots, x_{n+1} and tuple $a \in \mathbf{CM}^n$ we have by Proposition 3.17 and then Proposition 3.16 that

$$\begin{aligned} \mathbf{CM}^{\text{mod}} \models \exists x_{n+1} \phi(a, x_{n+1}) &\iff \mathbf{CM}^{\text{mod}} \models \widehat{\phi}(a) \\ &\iff \mathbb{C}^{\text{mod}} \models \widehat{\phi}(a) \\ &\iff \mathbb{C}^{\text{mod}} \models \exists x_{n+1} \phi(a, x_{n+1}) \end{aligned}$$

□

Corollary 3.19. *The theory of \mathbf{CM}^{mod} is strongly minimal, and, a fortiori, stable.*

Proof. The structure \mathbb{C}^{mod} is a reduct of the strongly minimal structure of \mathbb{C} in the language of rings. Hence, $\text{Th}(\mathbb{C}^{\text{mod}})$ is strongly minimal. As $\mathbf{CM}^{\text{mod}} \preceq \mathbb{C}^{\text{mod}}$, we have that $\text{Th}(\mathbf{CM}^{\text{mod}}) = \text{Th}(\mathbb{C}^{\text{mod}})$. Hence, $\text{Th}(\mathbf{CM}^{\text{mod}})$ is also strongly minimal. □

With the next result we see that \mathbf{CM}^{mod} is the structure induced on \mathbf{CM} from the language of rings.

Proposition 3.20. *The structures \mathbf{CM}^{mod} and \mathbf{CM}^{ind} , where here we induce the structure from \mathbb{C} in the language of rings, are bi-definable.*

Proof. This is the content of the André-Oort conjecture for products of modular curves [10, Theorem 1.1*].

Indeed, by quantifier elimination for algebraically closed fields, each formula ϕ in the language of rings with free variables amongst $\{x_1, \dots, x_n\}$ we may write

$$\phi(\mathbb{C}) = \bigcup_{X \in \mathcal{X}} \left(X(\mathbb{C}) \setminus \bigcup_{Y \in \mathcal{Y}_X} Y(\mathbb{C}) \right)$$

where \mathcal{X} is a finite set of irreducible varieties and for each $X \in \mathcal{X}$ the set \mathcal{Y}_X is a finite set of proper closed irreducible subvarieties of X .

For $X \in \mathcal{X}$ and $Y \in \mathcal{Y}_X$ define $\widetilde{X} := \overline{X(\mathbb{C}) \cap \mathbf{CM}^n}^{\text{Zariski}}$ and $\widetilde{Y} := \overline{Y(\mathbb{C}) \cap \mathbf{CM}^n}^{\text{Zariski}}$.

Decompose these as $\widetilde{X} = \bigcup_{Z \in \mathcal{Z}_X} Z$ and then for each $Z \in \mathcal{Z}_X$ write $Z \cap \bigcup_{Y \in \mathcal{Y}_X} \widetilde{Y}$ as $\bigcup_{W \in \mathcal{W}_Z} W$ where each W is irreducible.

By the André-Oort conjecture, each Z and W is a special variety. That is, R_Z and R_W are predicates in \mathcal{L}^{mod} .

Using the fact that $\widetilde{X} \subseteq X$ and $\widetilde{Y} \subseteq Y$, we see that $\widetilde{X}(\mathbb{C}) \cap \mathbf{CM}^n = X(\mathbb{C}) \cap \mathbf{CM}^n$, and likewise for \widetilde{Y} and Y in place of \widetilde{X} and X .

Putting this all together we have

$$\begin{aligned}
R_\phi(\mathfrak{CM}) &= \phi(\mathbb{C}) \cap \mathbb{C}\mathbb{M}^n \\
&= \bigcup_{X \in \mathcal{X}} \left(X(\mathbb{C}) \setminus \bigcup_{Y \in \mathcal{Y}_X} Y(\mathbb{C}) \right) \cap \mathbb{C}\mathbb{M}^n \\
&= \bigcup_{X \in \mathcal{X}} \left(\tilde{X}(\mathbb{C}) \setminus \bigcup_{Y \in \mathcal{Y}_X} \tilde{Y}(\mathbb{C}) \right) \cap \mathbb{C}\mathbb{M}^n \\
&= \bigcup_{X \in \mathcal{X}} \bigcup_{Z \in \mathcal{Z}_X} \left(Z(\mathbb{C}) \setminus \bigcup_{W \in \mathcal{W}_Z} W(\mathbb{C}) \right) \cap \mathbb{C}\mathbb{M}^n \\
&= \bigcup_{X \in \mathcal{X}} \bigcup_{Z \in \mathcal{Z}_X} \left(Z(\mathbb{C}) \cap \mathbb{C}\mathbb{M}^n \setminus \bigcup_{W \in \mathcal{W}_Z} (W(\mathbb{C}) \cap \mathbb{C}\mathbb{M}^n) \right) \\
&= \hat{\phi}(\mathfrak{CM})
\end{aligned}$$

where

$$\hat{\phi} = \bigvee_{X \in \mathcal{X}} \bigvee_{Z \in \mathcal{Z}_X} \left(R_Z \wedge \neg \bigwedge_{W \in \mathcal{W}_Z} R_W \right)$$

is a quantifier-free formula in \mathcal{L}^{mod} .

Thus, every basic predicate in \mathcal{L}^{ind} is interpreted in \mathfrak{CM} by a quantifier-free \mathcal{L}^{mod} formula. Hence, the every \mathcal{L}^{ind} -definable set in $\mathbb{C}\mathbb{M}^{\text{ind}}$ is already \mathcal{L}^{mod} -definable. Since $\mathbb{C}\mathbb{M}^{\text{mod}}$ is a reduct of $\mathbb{C}\mathbb{M}^{\text{ind}}$, we see that $\mathbb{C}\mathbb{M}^{\text{mod}}$ and $\mathbb{C}\mathbb{M}^{\text{ind}}$ are bidefinable. \square

Stability of $\mathbb{C}\mathbb{M}^{\text{ind}}$ follows from Corollary 3.19 and Proposition 3.20.

Corollary 3.21. *The theory of $\mathbb{C}\mathbb{M}^{\text{ind}}$ is stable.*

In fact, by the effective André-Oort theorem of Binyamini, the bidefinability of $\mathbb{C}\mathbb{M}^{\text{mod}}$ and $\mathbb{C}\mathbb{M}^{\text{ind}}$ is itself effective.

Proposition 3.22. *There is a computable function $\phi \mapsto \tilde{\phi}$ from the language of rings to \mathcal{L}^{mod} so that for every formula ϕ of the language of rings, we have $\mathbb{C}\mathbb{M}^{\text{ind}} \models R_\phi \leftrightarrow \tilde{\phi}$*

Proof. The theory of algebraically closed fields itself has an effective quantifier elimination algorithm. So to compute $\tilde{\phi}$ we may take ϕ to be quantifier-free. Our proof of Proposition 3.20 shows that to compute $\tilde{\phi}$ we need only have a way to compute from an algebraic variety $X \subseteq \mathbb{A}_{\mathbb{Q}^{\text{alg}}}^n$ the (possibly reducible) variety $\overline{X(\mathbb{C}) \cap \mathbb{C}\mathbb{M}^n}^{\text{Zariski}}$. Using the degree bounds of [1, Theorem 1] as described in the footnote to that theorem, we may carry out such a computation. \square

Using a theorem of Casanovas and Ziegler on expansions of stable structures, we conclude that the theory of \mathfrak{CM} is tame.

Theorem 3.23. *The theory of \mathfrak{CM} is stable and decidable.*

Proof. The theory of the complex numbers in the language of rings is strongly minimal, and, hence, does not have the finite cover property. The countable set $\mathbb{C}\mathbb{M}$ of $\mathbb{C}\mathbb{M}$ -points is small in \mathbb{C} and by Corollary 3.21 has stable induced structure. Hence, by [2, Theorem A], $\mathfrak{CM} = (\mathbb{C}, +, \cdot, 0, 1, \mathbb{C}\mathbb{M})$ has a stable theory. From [2,

Corollary 2.2] we see that $\text{Th}(\mathfrak{CM})$ is axiomatized by $\text{Th}(\mathbb{C}, +, \cdot, 0, 1) = \text{ACF}_0$ and $\text{Th}(\text{CM}^{\text{ind}})$. Combining Proposition 3.22 and Proposition 3.17, we see that $\text{Th}(\text{CM}^{\text{ind}})$ is decidable. Hence, $\text{Th}(\mathfrak{CM})$ is also decidable. \square

Remark 3.24. While the results of [12] are stated in the special case where the predicate by which the algebraically closed field is expanded names a finite rank subgroup of a semiabelian variety, the proofs would suffice for our application in Theorem 3.23.

4. QUESTIONS ABOUT DECIDABILITY OF OTHER STRUCTURES INTERPRETABLE IN $\mathbb{C}(t)$

While Theorem 3.23 shows that the approach to proving the undecidability by observing the the structure \mathfrak{CM} is interpretable in $\mathbb{C}(t)$ will not succeed, it does not resolve the question of whether the other structures interpreted using similar methods are decidable. In this section, we highlight some of those questions and discuss what we would need to know in order to resolve them.

Let us recall that $\text{Isog} := \{(x, y) \in \mathbb{C} : (\exists E)(\exists E')(\exists \psi : E \rightarrow E' \text{ isogeny })j(E) = x \wedge j(E') = y\}$ is definable in $\mathbb{C}(t)$. Using $a \in \mathbb{C}$ as a parameter, Isog_a , the set of j -invariants of elliptic curves isogenous to an elliptic curve with j -invariant a , is thus also definable in $\mathbb{C}(t)$. Using the Andr e-Pink-Zannier Conjecture as proven by Rodolphe and Yafaev [13, Theorem 1.3] in place of the Andr e-Oort conjecture, we obtain an analogue of Proposition 3.20.

Proposition 4.1. *For any complex number $a \in \mathbb{C}$ the structures $(\text{Isog}_a^{\text{ind}})_{\text{Isog}_a}$ and $(\text{Isog}_a^{\text{mod}})_{\text{Isog}_a}$ are bidefinable.*

Remark 4.2. Note that in Proposition 4.1 we need to allow for parameters from Isog_a . Implicitly, we were already doing this for CM^{mod} in Proposition 3.20 as every element of CM is already definable by a predicate there.

Unlike for CM^{mod} , $\text{Isog}_a^{\text{mod}}$ is not an elementary substructure of \mathbb{C}^{mod} . However, we may still deduce quantifier elimination for $(\text{Isog}_a^{\text{mod}})_{\text{Isog}_a}$ from what we know for \mathbb{C}^{mod} .

Proposition 4.3. *For any complex number $a \in \mathbb{C}$ the theory of $(\text{Isog}_a^{\text{mod}})_{\text{Isog}_a}$ admits quantifier elimination and is therefore strongly minimal.*

Proof. Working by induction and the usual reductions using irredundant representations of constructible sets, it suffices to show that if ϕ is a quantifier-free formula in the free variables x_1, \dots, x_{n+1} of the form $\phi = R_X \wedge \neg \bigwedge_{Y \in \mathcal{Y}} R_Y$ where X is a weakly special variety whose constant coordinates come from Isog_a and each Y is a proper weakly special variety of X , then $\exists x_{n+1} \psi$ is equivalent to a quantifier-free formula. By quantifier elimination for \mathbb{C}^{mod} there is a quantifier-free formula θ so that $\mathbb{C}^{\text{mod}} \models \theta \leftrightarrow \exists x_{n+1} \psi$. Express θ in an irredundant disjunctive normal form as $\theta = \bigvee_{Z \in \mathcal{Z}} (R_Z \wedge \neg \bigvee_{W \in \mathcal{W}_Z} R_W)$ where \mathcal{Z} is a finite set of weakly special varieties and for each $Z \in \mathcal{Z}$, \mathcal{W}_Z is a finite set of proper weakly special subvarieties of Z . Let $\mathcal{Z}' := \{Z \in \mathcal{Z} : \text{no constant coordinate of } Z \text{ lies outside of } \text{Isog}_a\}$. Likewise, let for $Z \in \mathcal{Z}'$ let \mathcal{W}'_Z consist of those $W \in \mathcal{W}_Z$ not having any constant coordinates outside of Isog_a . Let $\theta' := \bigvee_{Z \in \mathcal{Z}'} (R_Z \wedge \neg \bigvee_{W \in \mathcal{W}'_Z} R_W)$. Noting that if $V \subseteq \mathbb{A}_{\mathbb{C}}^n$ has a constant coordinate outside of Isog_a , then $V(\mathbb{C}) \cap \text{Isog}_a^n = \emptyset$, we see that $\text{Isog}_a \models \theta \leftrightarrow \theta'$. We finish the proof as with the proof of Proposition 3.17.

Suppose that for some $b \in \mathbf{Isog}_a^n$ we have $\mathbf{Isog}_a^{\text{mod}} \models \exists x_{n+1} \psi(b, x_{n+1})$. Then because \mathbf{Isog}_a is a substructure of \mathbb{C}^{mod} , we have $\mathbb{C}^{\text{mod}} \models \exists x_{n+1} \psi(b, x_{n+1})$ as well. From the characteristic property of θ , we then have that $\mathbb{C}^{\text{mod}} \models \theta(b)$, which again gives because \mathbf{Isog}_a is a substructure that $\mathbf{Isog}_a^{\text{mod}} \models \theta(b)$, which means that $\mathbf{Isog}_a \models \theta'(b)$ as θ and θ' are equivalent in \mathbf{Isog}_a . In the other directions, if $\mathbf{Isog}_a \models \theta'(b)$, then $\mathbb{C}^{\text{mod}} \models \exists x_{n+1} \psi(b, x_{n+1})$. From the equations defining X we see that either $X_b = \mathbb{A}_{\mathbb{C}}^1$ (in which case the non-emptiness of $X_b(\mathbb{C}) \setminus \bigcup_{Y \in \mathcal{Y}} Y_b(\mathbb{C})$ implies that all but finitely many elements of \mathbf{Isog}_a could serve as witnesses of $\exists x_{n+1} \psi$) or $\Phi_\ell(b_i, x)$ holds on X_b for some $1 \leq i \leq n$ and $\ell \in \mathbb{Z}_+$ so that $X_b(\mathbb{C})$ is contained in the isogeny class of b_i , which is \mathbf{Isog}_a . Either way, we see that $\mathbf{Isog}_a \models \exists x_{n+1} \psi$. \square

Corollary 4.4. *For any $a \in \mathbb{C}$ the theory of $(\mathbb{C}, +, \cdot, 0, 1, \mathbf{Isog}_a)$ is stable.*

Proof. Apply [2, Theorem A]. \square

To obtain decidability for the theory of $(\mathbb{C}, +, \cdot, 0, 1, \mathbf{Isog}_a)$ we would need to know the the function which takes a variety $X \subseteq \mathbb{A}_{\mathbb{Q}(a)^{\text{alg}}}^n$ to $\overline{\mathbf{Isog}_a^n \cap X(\mathbb{C})}$ is computable. It seems plausible that the methods of [1] could be adapted to produce such an algorithm. In the case that a is transcendental, this can be deduced from [4].

The argument for Proposition 4.3 can be applied to finitely many isogeny orbits and the set of CM j-invariants at one time. From this it follows that the structure $(\mathbb{C}, +, \cdot, 0, 1, \text{CM}, (\mathbf{Isog}_a)_{a \in \mathbb{C}})$ is stable and then if the expected analogues of Proposition 3.22 hold for \mathbf{Isog}_a , then for any computable algebraically closed subfield $K_0 \subseteq \mathbb{C}$, the structure $(\mathbb{C}, +, \cdot, 0, 1, \text{CM}, (\mathbf{Isog}_a)_{a \in K_0})$ would have a decidable theory.

We have see that the relation \mathbf{Isog} itself is definable in $\mathbb{C}(t)$. The set \mathbf{Isog} is not small relative to \mathbb{C} . Thus, the main theorems of [2] do not immediately apply. However, because the ambient theory of the complex numbers as a field is strongly minimal, the implications from stability without the finite cover property of the theory of the induced structure on \mathbf{Isog} to stability of $(\mathbb{C}, +, \cdot, 0, 1, \mathbf{Isog})$ and likewise from decidability of the induced theory to decidability of the theory with a predicate still hold. Thus, we reduce to the following questions.

Question 4.5. *Is the theory of $\mathbf{Isog}^{\text{ind}}$ stable? Is its theory decidable?*

It would seem that the Zilber-Pink conjecture should be relevant for resolving Question 4.5. However, even assuming that conjecture, the description of even the quantifier-free induced structure on \mathbf{Isog} is much more complicated than $\mathbf{Isog}^{\text{mod}}$. In [11] a differential algebraic analogue Ξ of \mathbf{Isog} is considered, namely the Kolchin closure of this set with the constant points removed. The set Ξ has decidable, stable induced structure because it is a reduct of something definable relative to the theory of differentially closed fields of characteristic zero. Moreover, Ξ satisfies a version of the Zilber-Pink conjecture. While $\mathbf{Isog}^{\text{ind}}$ is not an elementary substructure of Ξ^{ind} (because Ξ is missing algebraic points), one might hope that their theories are close enough to permit a description of the sets defined with quantifiers in $\mathbf{Isog}^{\text{ind}}$ through a comparison to the corresponding sets in Ξ^{mod} .

There are some other sets definable in $\mathbb{C}(t)$ using variants of the method used to prove Corollary 2.5. For general abelian varieties A and B defined over \mathbb{C} , it is still the case that $\text{Hom}(A, B) = B(\mathbb{C}(A))/B(\mathbb{C})$. Thus, if we have a uniformly definable family $B \rightarrow S$ of abelian varieties and uniformly definable family $\{CC(A)\}_{A \in F}$ of function fields of abelian varieties, we can describe with formulas conditions on the ranks of $\text{Hom}(A, B)$.

We do not know how to interpret function fields of higher dimensional varieties in $\mathbb{C}(t)$. If we could, then by a theorem of Eisenträger [3], we would already know that $\mathbb{C}(t)$ is undecidable. Instead, if C is a smooth projective curve of genus $g \geq 1$ over \mathbb{C} and we fix some point $P \in C(\mathbb{C})$, then there map of algebraic varieties $f : C \rightarrow J_C$ from the curve C to its Jacobian, an abelian variety of dimension g , having the property that for any rational map $h : C \dashrightarrow A$ from C to an abelian variety A with $h(P) = 0_A$, there is a unique map $\bar{h} : J_C \rightarrow A$ with $\bar{h} \circ f = h$. That is, $\text{Hom}(J_C, A) = A(\mathbb{C}(C))/A(\mathbb{C})$ so that we can uniformly interpret the groups $\text{Hom}(J_C, A)$ of maps from Jacobians to abelian varieties. I do not have useful conclusions about what we might do with these groups to shed light on the (un)decidability of $\mathbb{C}(t)$. Some natural questions about them follow.

Question 4.6. *Let $A \rightarrow S$ be a family of abelian varieties over the complex numbers. Is the set $\{a \in S(\mathbb{C}) : A_a \text{ has complex multiplication}\}$ definable in $\mathbb{C}(t)$.*

I would guess that the answer to Question 4.6 is positive and that the definition could be obtained by working with a suitable resolution of the abelian varieties in this family by Jacobians, but to my knowledge, it remains an open problem whether such resolutions always exist.

Question 4.7. *Let $A \rightarrow S$ be a family of abelian varieties over the complex numbers and let $X \rightarrow T$ be a family of smooth projective curves of some fixed genus over \mathbb{C} . Fix a natural number r . Let $H_r := \{(t, s) \in (T \times S)(\mathbb{C}) : \text{rk Hom}(J_{C_t}, A_s) = r\}$. Is the induced structure on H_r stable? It is decidable?*

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