Methods of Mathematics

Kenneth A. Ribet

UC Berkeley

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Permutations and Combinations

Our first aim is to answer questions like these:

1. In how many ways can we select five representatives from a class of 287 students?
2. In how many ways can we select a President, Vice President, Secretary, Treasurer and Sommelier from a class of 287 students?

These questions are similar but genuinely different. In the first problem, the representatives all have the same status. Their roles are indistinguishable. In the second problem, the officers have roles that are distinguished by their titles (and possibly job descriptions).

The number in the first problem is the "number of 5-combinations of a set with 287 elements"; it's denoted $C(287, 5)$ or $\binom{287}{5}$.

The number in the first problem is the "number of 5-permutations of a set with 287 elements"; it's denoted $P(287, 5)$.

This morning's pop-up breakfast. Tomorrow we'll have around 20 people.

It is easy to compute $P(287, 5)$. (The second problem is the easier one and so perhaps should have been the first problem.) There are 287 ways to choose the first officer (the President, let's say) and then 286 ways to choose the second officer from among the remaining students. There are 285 ways to choose the third officer. And so on.

Thus $P(287, 5) = 287 \cdot 286 \cdot 285 \cdot 284 \cdot 283$.

In general,

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}.$$

Note that $k!$ is the product of the integers from 1 to k for $k \geq 1$. The number $0!$ is defined to be 1. (In math, the empty product is deemed to be 1.)

For combinations, you need to divide because there are r ways of ordering the r elements that you're choosing but don't want to distinguish. There are $(287 \cdot 286)/2$ ways of choosing two people from this class.

There are $(287 \cdot 286 \cdot 285)/6$ ways of choosing three people from this class. (Note $6 = 3!$.)

The number of ways of choosing five people from our class is $(287 \cdot 286 \cdot 285 \cdot 284 \cdot 283)/5!$ and, in general:

$$C(n,r) = P(n,r)/r! = \frac{n!}{(n-r)!r!}.$$

Note that $C(n,r) = C(n,n-r)$! (The last exclamation point is not a factorial. It's an exclamation point.)

Binomial theorem

If x and y are variables, then $(x+y)^0 = 1$, $(x+y)^1 = x+y$, $(x+y)^2 = x^2+2xy+y^2$, $(x+y)^3 = x^3+3x^2y+3xy^2+y^3$,

$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4, \quad \text{etc., etc.}$$

The general formula is

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

This is the binomial theorem.

The binomial coefficients form *Pascal's triangle*, where each coefficient is the sum of the two coefficients above it. (See board.) For example, $\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$; $6 = 3+3$.

In general,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

We can prove this by writing all terms as quotients of factorials and doing an ugly calculation. The suave way to see this identity is to give it a combinatorial interpretation and to verify it by combinatorial reasoning.

Here's the reasoning: We have $n+1$ things and want to choose k of them. Focus on the first "thing." Is it one of the k objects that we choose? If it is, then we complete our choice by selecting $k-1$ objects from the remaining n things. If it isn't, we have to choose all k objects from the remaining n things. The two numbers on the right count the number of ways of performing these two activities.

A weird problem

Let S be a set with 100 elements. The number of subsets of size k of S is $C(100, k)$. The number of all subsets of S is 2^{100}; this shows that $2^{100} = \displaystyle\sum_{k=0}^{100} \binom{100}{k}$.

The problem: show that

$$\sum_{k=41}^{59} \binom{100}{k} \geq \frac{3}{4} \cdot 2^{100}.$$

Using `Sage`, I found that the sum is bigger than 0.943 times 2^{100}, so the problem is asking for a coarse estimate.

Come to office hours!

Some questions (Rosen)

- How many bit strings of length n contain exactly r 1s
- Show that $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.
- Show that $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$.
- How many ways are there to select five bills from a cash box containing \$1 bills, \$2 bills, \$5 bills, \$10 bills, \$20 bills, \$50 bills, and \$100 bills? Assume that the order in which the bills are chosen does not matter, that the bills of each denomination are indistinguishable, and that there are at least five bills of each type. [This is a *bagel problem*.]
- How many different strings can be made by reordering the letters of the word SUCCESS? Same question for MISSISSIPPI.

Lunch today after SLC office hours. My proposal is that we have lunch in a DC every Tuesday. Good idea? Bad idea?

How many ways are there to distribute 39 indistinguishable donuts to 287 students?

Same question, with the previso that no student can get more than one donut.

How many ways are there to distribute 39 distinguishable prepaid $20 credit cards to 287 students?

Same question, with the previso that no student can get two or more credit cards.