Kinetic Description of Hamilton-Jacobi PDE III

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PDE/Probability Student Seminar

Outline

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Secondary Polytope

Minkowski-Alexandrov Problem and Optimal Transport

Hamilton-Jacobi Dynamics

Outline

Dual Tessellations/Legendre Transform

Given a finite P and a map $f : P \to \mathbb{R}$, we define two piecewise linear convex functions:

$$u(x) = f^*(x) = \sup_{\rho \in P}(x \cdot \rho - f(\rho))$$

$$u^*(\rho) = f^{**}(\rho) = \sup_x (x \cdot \rho - u(x)) = f^o(\rho) = \text{ convex hull of f.}$$

We may find f^o as follows:

1. Plot points $\{(x, f(x)) : \ x \in P\}$.
2. Take the convex hull of the set $\{(x, f(x)) : \ x \in P\}$.
3. The lower boundary of the convex hull is the graph of $f^o = u^*$.

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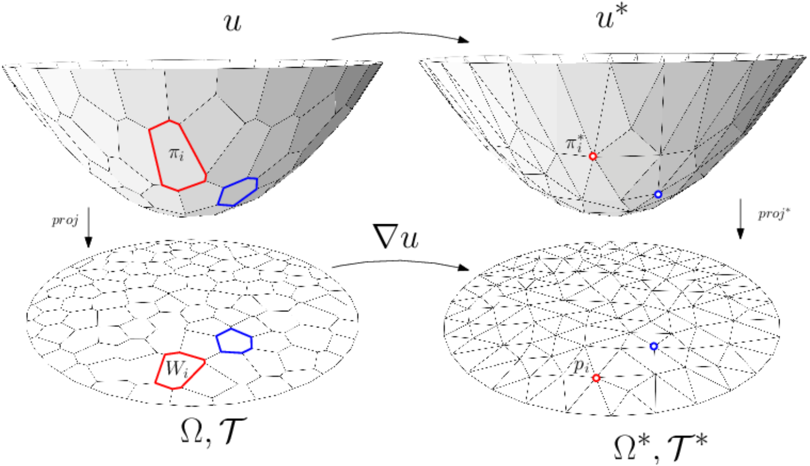
Legendre Transform

For generic *f*:



(Courtesy of N. Lei, W. Chen, Z. Luo, X. Gu 2019)

Laguerre Tessellation/Delaunay Triangulation

1. The function u is piecewise linear.
   Domains of the linearity of u yield a Laguerre tessellation:

$$\mathbf{X}(f) := \{X(\rho) : \ \rho \in \mathbb{R}^d\}, \quad X(\rho) = \partial u^*(\rho).$$

The function u^* is not differentiable at $\rho \in P$.
$\partial u^*(\rho)$ is the set of slopes of all supporting planes to the graph of u^* at ρ. For $\rho \in P$,

$$x \in X(\rho) \quad \Longrightarrow \quad u(x) = x \cdot \rho - f(\rho).$$

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Write X for the set of vertices in $\mathbf{X}(f)$.
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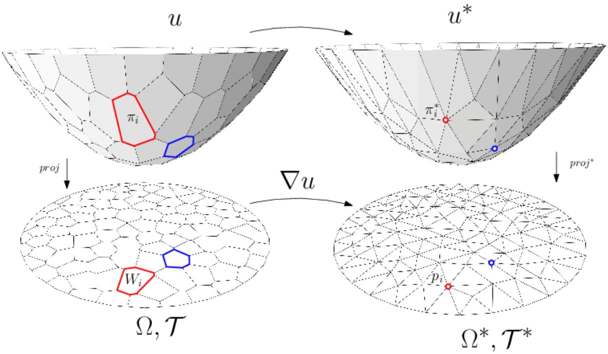
For $x \in X$,

$$\rho \in P(x) \implies u^*(\rho) = x \cdot \rho - u(x).$$

For generic f, the graph associated with **X** is of degree $d+1$.
For generic f, the tessellation **P** is a triangulation.



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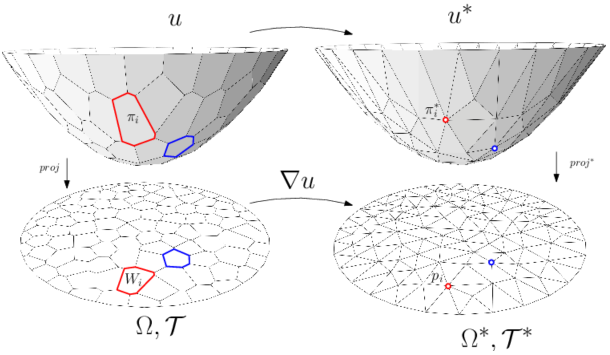

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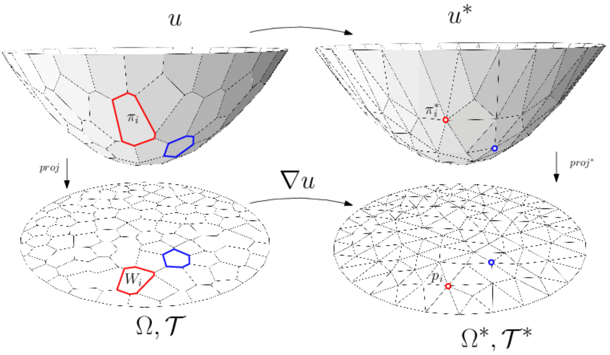

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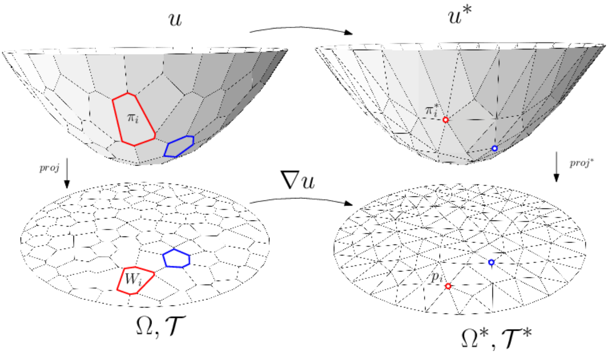
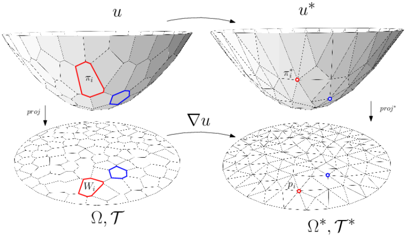


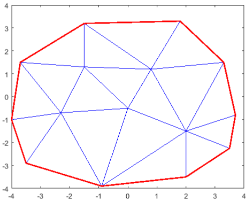
Triangulations

We first focus on $u^* = f^o$. We wish to develop a better understanding of the operation $f \mapsto f^o$. We fix a finite set P and very f. The set of $f : P \to \mathbb{R}$ is identified as $\mathbb{R}^n$ if $\sharp P = n$. (Remember $P \subset \mathbb{R}^d$.) The function $f^o : \hat{P} \to \mathbb{R}$, where

$$\hat{P} = Conv(P).$$

Without loss of generality we may assume that $\dim \hat{P} = d$. $\hat{P}$ is a polytope in $\mathbb{R}^d$ and serves as our primary polytope. Note that as we go from f to f^o the main challenge comes from the tessellation $\mathbf{P}(f)$ which is a triangulation for generic f.





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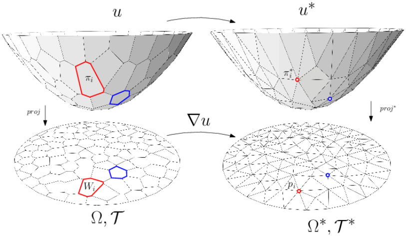

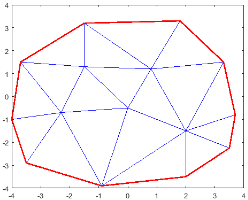

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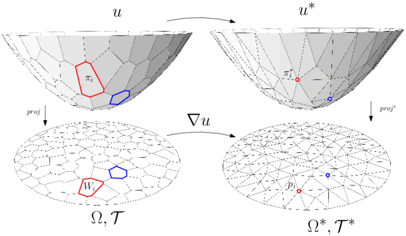

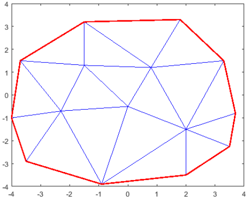

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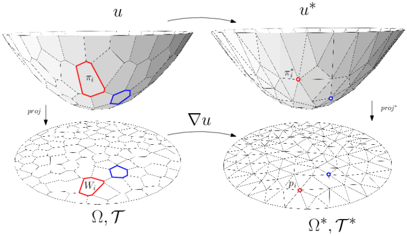

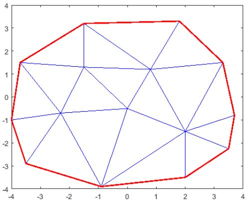

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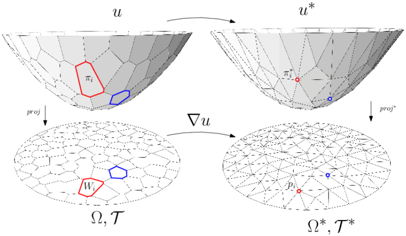

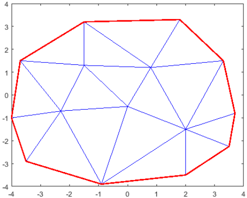

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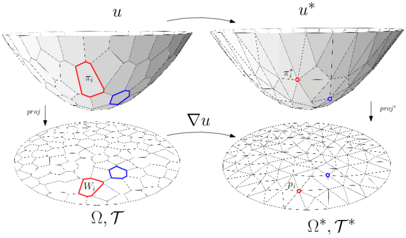

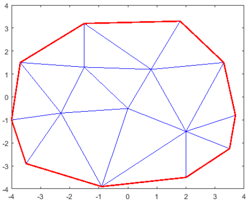

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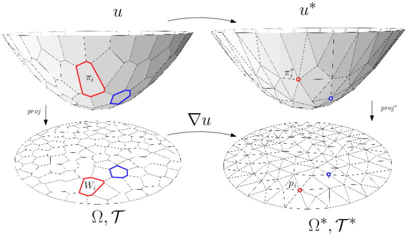

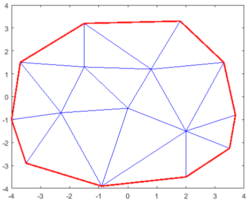

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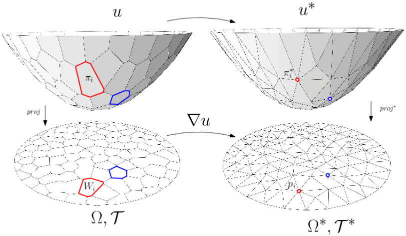

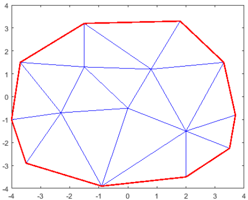

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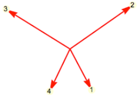

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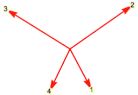

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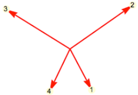

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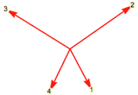

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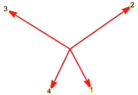

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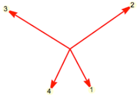

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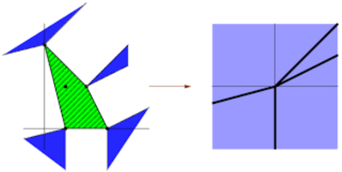

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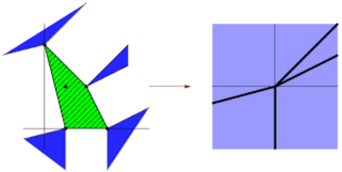

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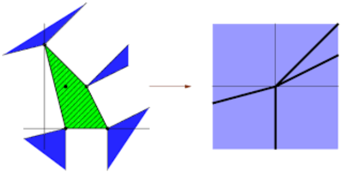

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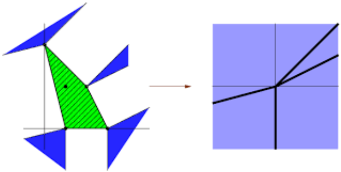

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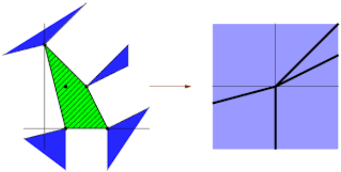

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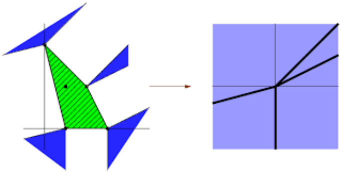

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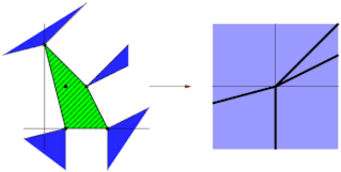

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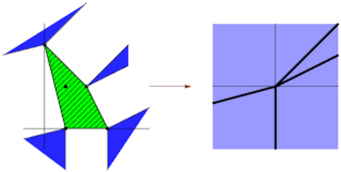

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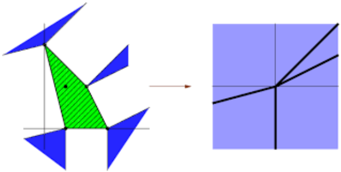

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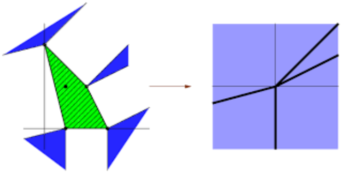

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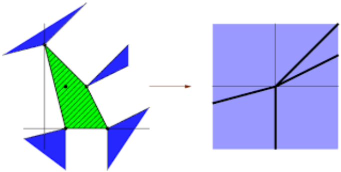

A Recipe for Secondary Polytope

Recall that a piecewise linear convex function yields a tessellation with convex cells.

The fan **C** is a tessellation with convex cones $\mathcal{C}(\mathbf{T})$ for cells.

Natural Question: Is there a convex (concave) U function that would yield **C**?

1. We want U to be linear on each $\mathcal{C}(\mathbf{T})$ but of different slopes on different cells.

2. The set of slopes would generate the secondary polytope $\Sigma(P)$.

3. Equivalently U^* is 0 in $\Sigma(P)$, and ∞ outside $\Sigma(P)$.

Recall that $f^o = u^*$ is the convex hull of f:

$$f^o = \inf\{\hat{f}_{\mathbf{T}} : \ \mathbf{T} \in \mathcal{T}\}.$$

Claim: $U(f) = \int_{\hat{P}} f^o(\rho)\, d\rho$ is concave and does the job!

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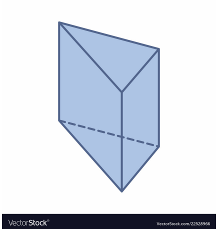

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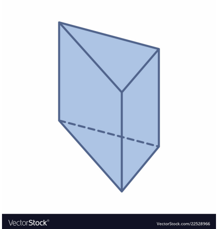

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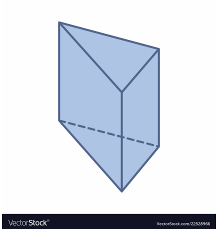

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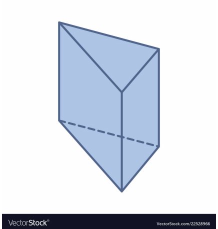

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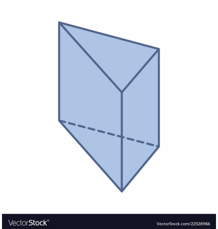

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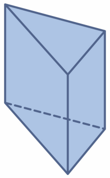

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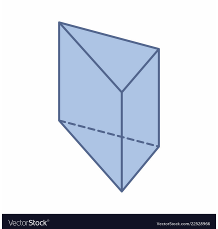

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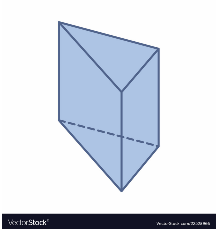

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$$\Sigma(P) = \text{ Convex hull of } \{\sigma_{\mathbf{T}} :\ \mathbf{T} \in \mathcal{T}\}.$$

1. If $\sharp P = d + 1$, $\dim \hat{P} = d$ (points in P are affinely independent), then $\Sigma(P)$ is a single point.
2. If $\sharp P = n$, $\dim \hat{P} = d$, then $\dim \Sigma(P) = n - d - 1$.
3. If $\sharp P = d + 2$, $\dim \hat{P} = d$, and any proper subset of P affinely independent. then $\Sigma(P)$ is a line segment. Such a P is called a circuit. Two cases to consider:
3(i). Let P be as in 3, no point of P is in the interior of $\hat{P}$.
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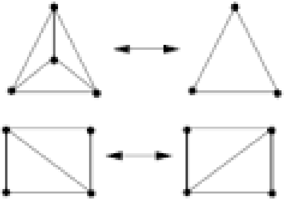

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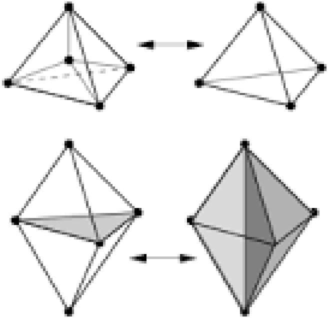

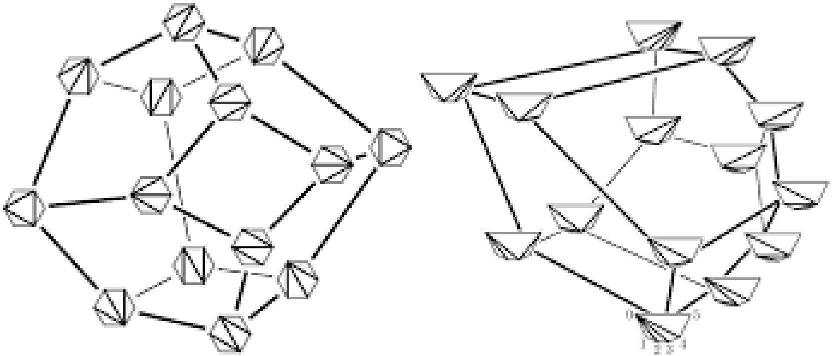
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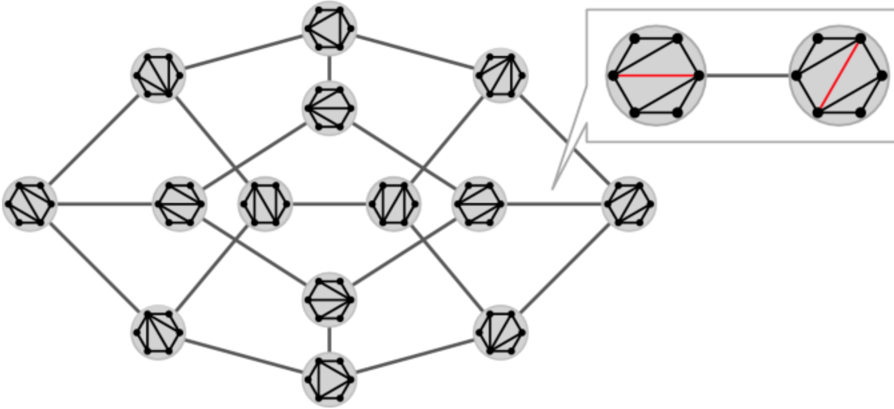

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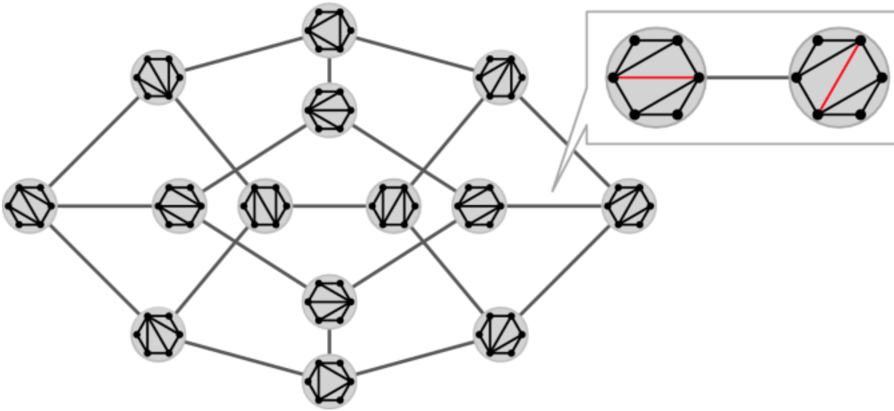

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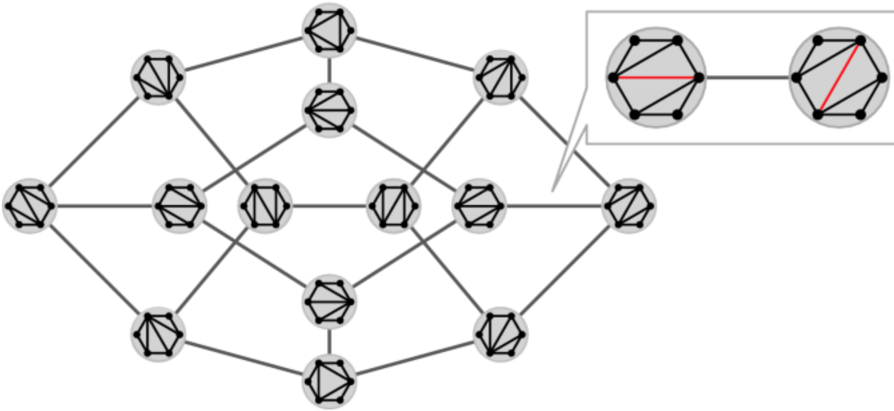

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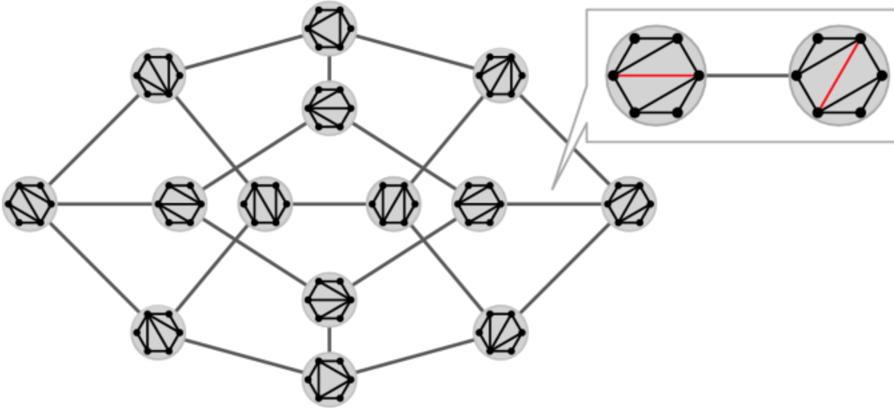

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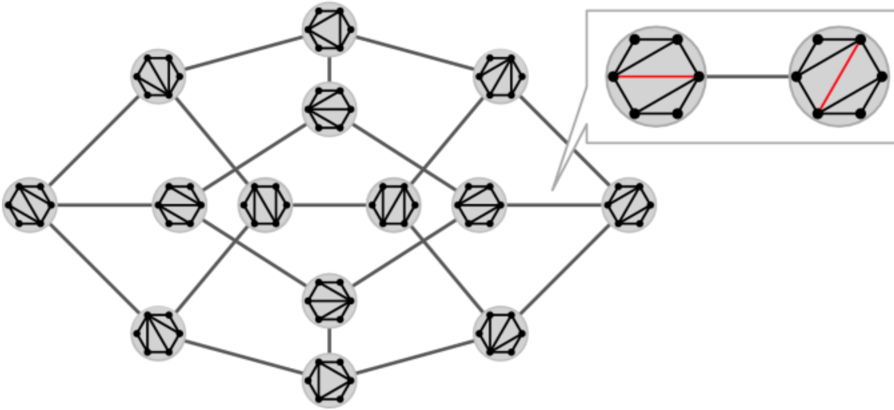

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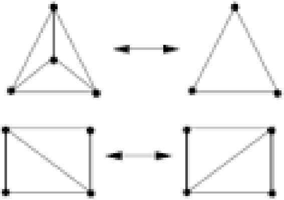

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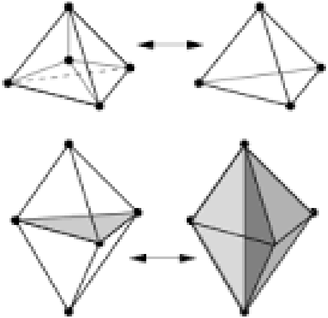

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(i) Either diagonals are swapped,

(ii) or three triangles are replaced with one triangle.

In the context of Hamilton-Jacobi equation (i) means the occurrence of a collision between two vertices of the corresponding Laguerre tessellation.

In the context of Hamilton-Jacobi equation (ii) means that the corresponding Laguerre tessellation has a triangular cell, and this cell collapses to a vertex. When this happens, we say that a coagulation has occurred. (The vertices of the cell coagulate to form a single vertex/particle.)

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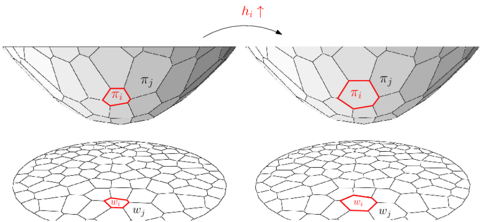
Alexandrov Problem

We now focus on u. Fix finite P, and vary $f : P \to \mathbb{R}$. is finite and fixed. We wish to understand the operation $f \mapsto u = f^*$.

$$u(x) = f^*(x) = \sup_{\rho \in P}(x \cdot \rho - f(\rho))$$

The function u is piecewise linear.
Domains of the linearity of u yield a Laguerre tessellation:

$$\mathbf{X}(f) := \{X(\rho) : \ \rho \in \mathbb{R}^d\}, \quad X(\rho) = \partial u^*(\rho).$$



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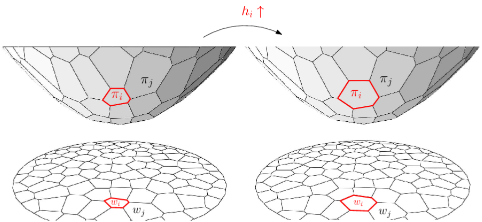

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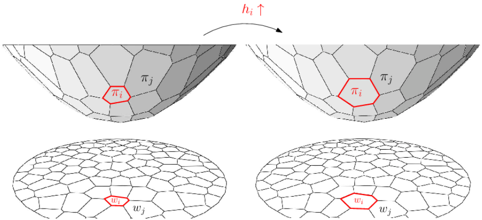

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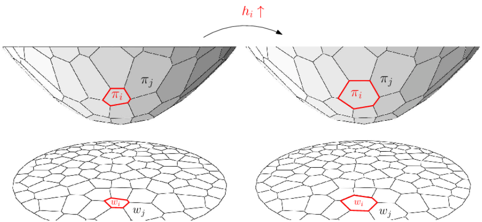

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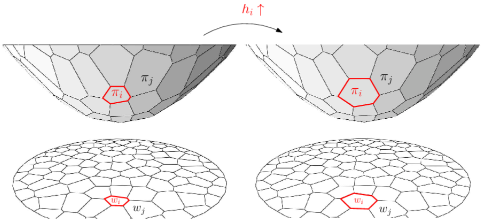

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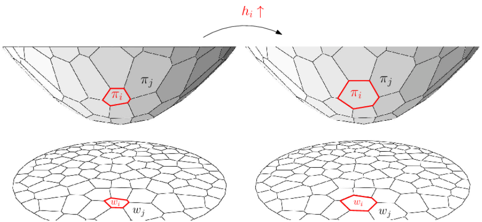

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Recall that P is fixed and we only vary f. Fix a domain Ω and define $\nu : P \to [0, \infty)$, by

$$\nu(\rho) = |X(\rho) \cap \Omega|.$$

Alexandrov: The map $f \mapsto \nu$ is a local diffeomorphism.
If ν is known, then we can recover f (and hence u) from it.
Alexandrov Problem: How to build $\nu \mapsto u$?
We wish to formulate an optimization problem for this problem.
Solution via Optimal Transport techniques: Observe that if $\rho(x) = \nabla u(x)$ (which coincides with $\partial u(x)$ almost everywhere), then $\rho : \Omega \to \mathbb{R}^d$ pushes forward Lebesgue measure to the measure

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Brenier: Given two measures λ and μ, there exists a unique (modulo a constant) convex function $u : \Omega \to \mathbb{R}$ such that $\rho = \nabla u$ pushes forward λ to μ.

Moreover ρ minimizes

$$\frac{1}{2}\int_\Omega |x - \rho(x)|^2 \, \lambda(dx).$$

Alternative formulation As in the case of u^*, examine the functional

$$E(f) = \int_\Omega f^*(x) \, \lambda(dx).$$

The map $f \mapsto E(f)$ is convex.
Claim: $f \mapsto -\nabla E(f)$ is $f \mapsto \nu$.
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