Kinetic Description of Hamilton-Jacobi PDE II

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PDE/Probability Student Seminar

Outline

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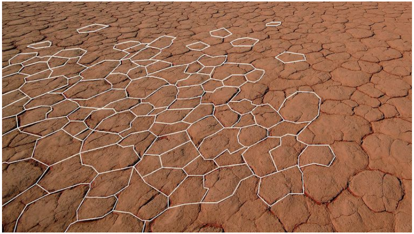
Voronoi Tessellation

Voronoi tessellations are used to model/study various phenomena in nature:



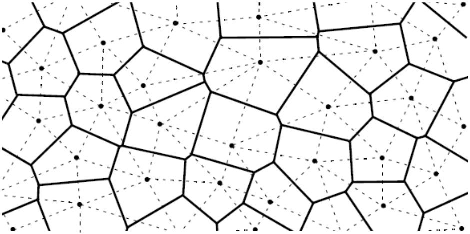
Voronoi Tessellation

Given n distinct points $\rho_1, \ldots, \rho_n$ (in general position), consider the optimization problem

$$w(x) = \min_i |x - \rho_i|.$$

For each i, set

$$X(\rho_i) = \{x : w(x) = |x - \rho_i|\}.$$



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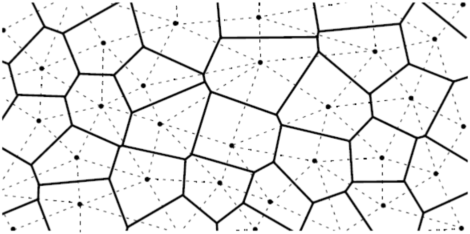

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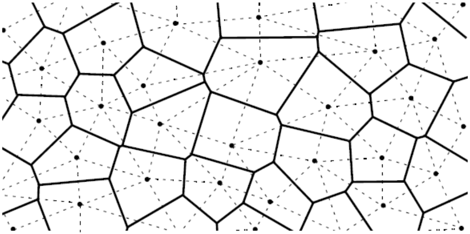

Voronoi Tessellation (alternative formulation)

For $f(\rho) = |\rho|^2/2$, consider

$$u(x) = \sup_i (x \cdot \rho_i - f(\rho_i)) = \frac{1}{2}|x|^2 - \frac{1}{2}w(x)^2.$$

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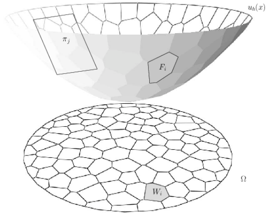
Set $P = \{\rho_1, \ldots, \rho_n\}$, $h(\rho) = f(\rho) + \infty \mathbb{1}(\rho \notin P)$, then $u = h^*$.



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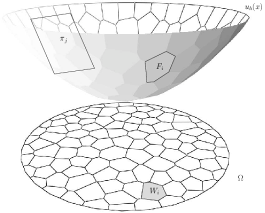

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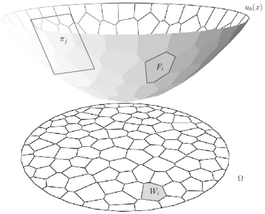

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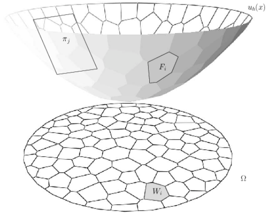

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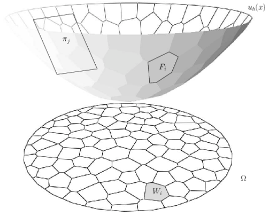

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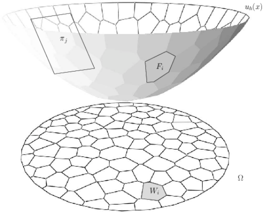

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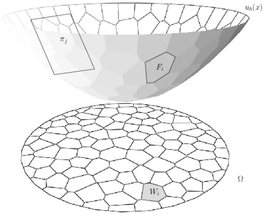

Voronoi Tessellation (Some Remarks)

Write $\hat{P}$ for the convex hull of P.

1. If ρ is an extreme point of $\hat{P}$, then $X(\rho)$ is unbounded.
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3. Each $X(\rho)$ is a polyhedron/polytope.
4. We say P is generic (points in P are in general position) if no k points of P lie on a $k-1$ affine set (for $k \in \{2, \ldots, d+1\}$), and no set of $d+2$ points in P lie on the boundary of a ball whose interior does not intersect P.
5. For generic P, we have a graph of degree $d+1$;
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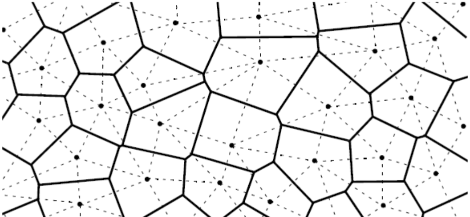

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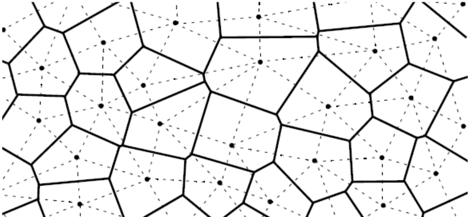

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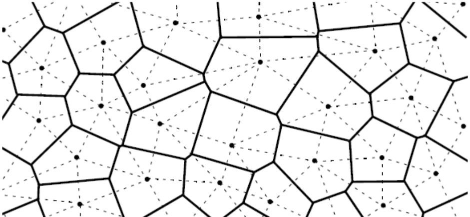

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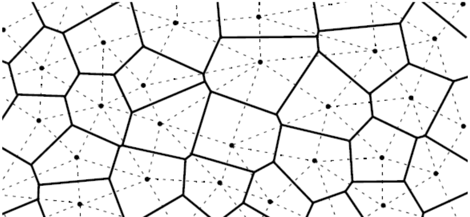

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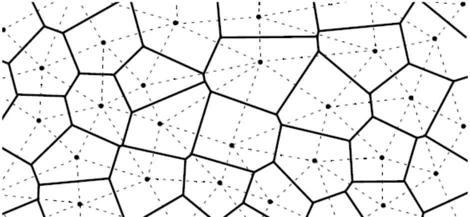

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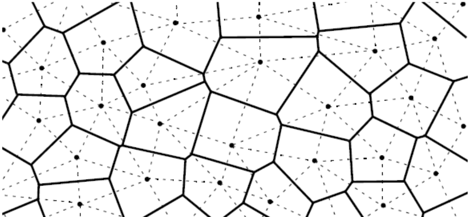

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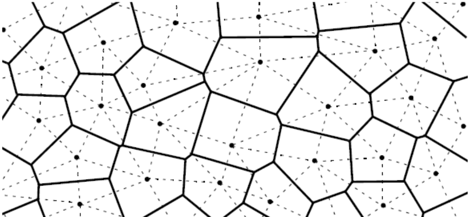

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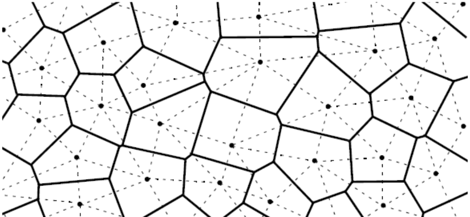

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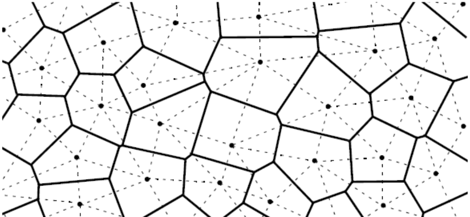
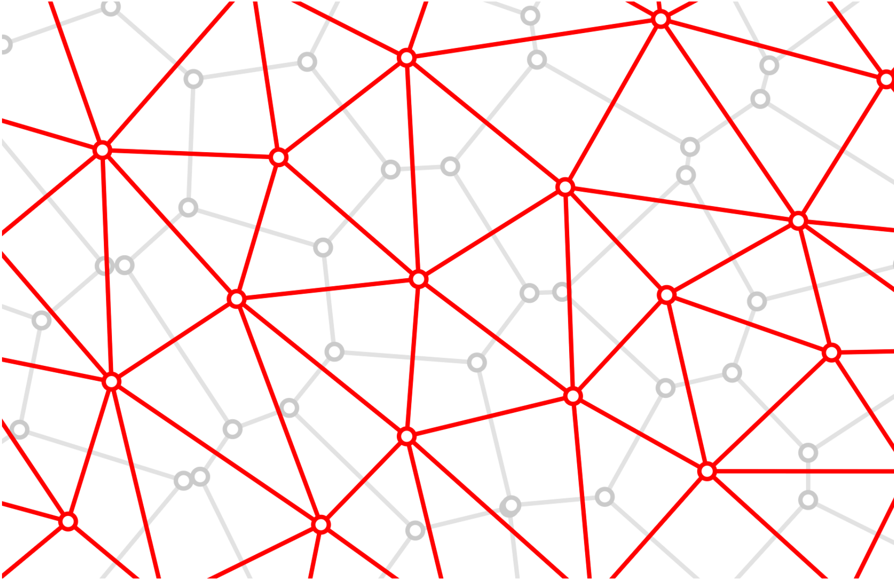


Delaunay triangulation



Laguerre Tessellation

Given a set n distinct points $P = \{\rho_1, \dots, \rho_n\}$, and $c : P \to \mathbb{R}$, consider the optimization problem

$$w(x) = \min_{\rho \in P} \{|x - \rho|^2/2 - c(\rho)\}.$$

(When $c = 0$, we are back to Voronoi scenario) For each ρ, set

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Dual Tessellation=Legendre Transform

Given a set P and a map $f : P \to \mathbb{R}$, we define a (marked) tessellation

$$\{(\rho, X(\rho)) : \ \rho \in P\}.$$

This is nothing other than

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In fact $u^* = f^o$ is the convex hull of f. On $X(\rho)$, we have $u(x) = x \cdot \rho - f(\rho) = x \cdot \rho - u^*(\rho)$. It is more convenient to consider

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Summary:

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for $P(x) = \partial u(x)$? What we have is simply the Laguerre tessellation associated with $u^* = f^o$. This is the dual tessellation. If f is generic, then cells of this dual tessellation are simplices (triangles when $d = 2$). They are also dual in graph theoretical sense. Write X for the set of vertices in the original tessellation:

$$X = \{X(\rho) : \rho \in \mathbb{R}^d, \sharp X(\rho) = 1\}.$$

Then

$$u^*(\rho) = \sup_x (x \cdot \rho - u(x)) = \sup_{x \in X} (x \cdot \rho - u(x)).$$

Summary:

1. Start from a discrete P, and $f : P \to \mathbb{R}$.
   Use $u = f^*$ to define a tessellation $\{X(\rho) : \rho \in P\}$.
2. The map $u : X \to \mathbb{R}$ and u^* in the same way yields the dual tessellation.

Dual Tessellation=Legendre Transform

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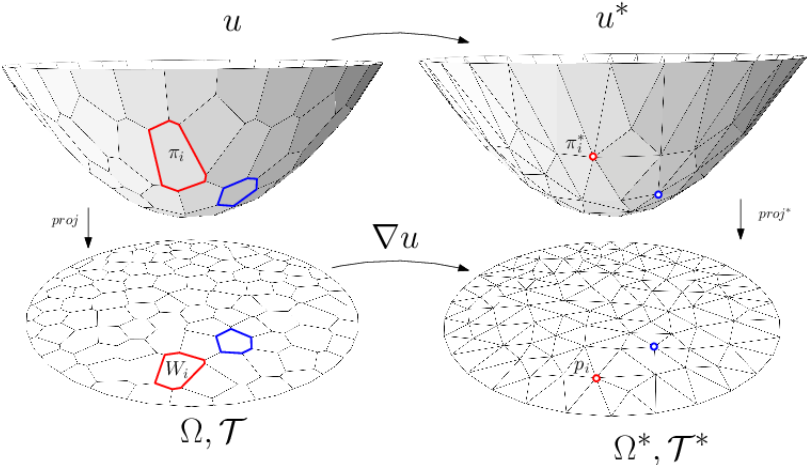
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