A KINETIC APPROACH TO BURGERS EQUATION WITH WHITE NOISE INITIAL DATA

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Abstract. In this paper, we offer a simpler proof to Groeneboom's result using recent progress made in bridging kinetic theory to scalar conservation laws with random initial data. According to Groenboom's work, if the initial data of Burgers' equation is white noise, then at a later time the solution is a piecewise linear function that is interrupted by random jumps. Groenboom uses Brownian excursion theory to find an explicit formula for the jump kernel in terms of the Airy function. In this paper, we show that if $\rho(x,t)$ is the solution of Burgers' equation with white noise initial data, then the small t limit of the process $x \mapsto \int_0^x V(\rho(y,t))\,dy$ is a Brownian motion, for any function $V : \mathbb{R} \to \mathbb{R}$, that is of of zero average with respect to the one-dimensional marginal of ρ. We also provide an explicit formula for the variance of the limiting Brownian motion. This general central limit theorem is related to Groenboom's work when $V(\rho) = \rho$.

1. Introduction and main theorem

In order to understand hydrodynamic turbulence via a simplified model, Burgers has suggested the following partial differential equation

$$\rho_t + \rho\rho_x = 0, \tag{1.1}$$

where $\rho : \mathbf{R} \times [0,\infty) \to \mathbf{R}$ is a function of two variables $(x,t) \in \mathbf{R} \times [0,\infty)$, where the variable x represents *space* and t represents *time*. Burgers has considered the Burgers equation above with Brownian white noise initial data, with an aim to understand the statistical moments and correlations of the entropy solution $\rho(x,t)$. The problem of interest is therefore to determine the law of the solution at later times $t > 0$, i.e., to determine the distribution of the stochastic process $(\rho(x,t) : x \in \mathbf{R})$ for any fixed $t > 0$. This open problem has resisted to mathematical physicists for several decades, until its resolution by Groeneboom in [Gr], who was initially interested in a completely different question related to the large scale behavior of isotonic estimators. These two questions are related via the variational formula - the *Lax-Oleinik* formula - which gives a closed form of the entropy solution of Burgers equation. Indeed, for any initial condition $\rho(\cdot,0) := \rho^0 \in L^\infty(\mathbf{R})$, the equation (1.1) admits a weak solution (in the distribution sense). If we impose more physical restrictions on the solutions, we can achieve well-posedeness, in which case we talk about an *entropy* solution. We present below briefly, the closed form of this *entropy* solution for bounded initial condition. We refer the unfamiliar reader with scalar conservation laws to Evans [E] monograph.

Let us define the potential $U_0(x) := \int_0^x \rho^0(y)dy$, and consider the corresponding Hamilton-Jacobi equation $u_t + \frac{1}{2}u_x^2 = 0$ with *unknown* $u : \mathbf{R} \times [0,\infty) \to \mathbf{R}$, and with initial condition $u(\cdot,0) = U_0$. Similarly to Burgers equation, this equation in u admits a notion of *physical* solution referred to as *viscosity* solution and is given by the *Hopf-Lax* formula below

$$u(x,t) = \min_{y\in\mathbf{R}} \left(U_0(y) + \frac{(x-y)^2}{2t} \right). \tag{1.2}$$

Formally, at least, the spatial derivative u_x of u verifies the Burgers equation. This notion is made rigorous by the fact that the *entropy* solution ρ is given by the closed formula

$$\rho(x,t) = \frac{x - y(x,t)}{t} \tag{1.3}$$

where $y(x,t)$ is the rightmost maximizer in the expression (1.2).

Coming back to our problem of interest, where the initial condition $\rho^0 := \xi$ is a Brownian white noise, one can still make sense of an *entropy* solution even though $\xi \notin L^\infty(\mathbf{R})$. Indeed, since the anti-derivative of the Brownian white noise (which is a two-sided Brownian motion) is dominated at infinity by parabolas, the minimization problem in (1.2) is achieved, and almost surely $y(x,t)$ exists and is finite.

The purpose of the present paper is to give an alternative proof of Groneboom's result, and to determine the law of $x \mapsto \rho(x,t)$ for every $t > 0$ when U_0 is a two-sided Brownian motion. This proof is based on a different approach relying on kinetic theory, that has seen several developments in the recent decade. This approach has been initiated by the work of Menon and Srinivasan in [MS], and a series of recent works have showed its power by solving several open standing problems on closure theorems of scalar conservation laws (or equivalently Hamilton-Jacobi equations) with general Hamiltonian and random initial data.

Fix $\sigma > 0$, and let ρ^σ be the unique entropy solution to

$$\begin{cases} \rho^\sigma_t + \rho^\sigma \rho^\sigma_x = 0 & \text{for } (x,t) \in \mathbf{R} \times (0,\infty), \\ \rho^\sigma(x,0) = \xi^\sigma(x) & \text{for } x \in \mathbf{R}, \end{cases}$$

where $\xi^\sigma(x)$ is a Brownian white noise with the diffusion coefficient σ^2. By this, we mean that ρ^σ is given by $\rho^\sigma(x,t) = \frac{x - y^\sigma(x,t)}{t}$, where $y^\sigma(x,t)$ is the largest y at which the process $y \mapsto \sigma B(y) + \frac{(x-y)^2}{2t}$ achieves its minimum, and where B is a standard two-sided linear Brownian motion. Using Brownian scaling, it is easy to see that

$$(\rho^\sigma(x,t), x \in \mathbf{R}) \stackrel{d}{=} \left(\sigma^{\frac{2}{3}} t^{-\frac{1}{3}} \rho^1((\sigma t)^{-\frac{2}{3}} x, 1), x \in \mathbf{R}\right). \tag{1.4}$$

Hence it suffices to evaluate the law at $t = 1$. The main theorem of this article, and that is due to Groeneboom is the following:

Theorem 1 ([Gr])**.** *The process $(\rho^{\frac{1}{\sqrt{2}}}(x,1), x \in \mathbf{R})$ is a stationary piecewise-linear Markov process with the generator $\mathcal{A}$, given by its action on test functions $\varphi \in C_c^\infty(\mathbf{R})$,*

$$\mathcal{A}\varphi(y) = \varphi'(y) + \int_{-\infty}^{y} (\varphi(z) - \varphi(y)) n(y,z) dz.$$

The jump density n is given by

$$n(y,z) = \frac{J(z)}{J(y)} K(y - z) \ , \quad y > z \tag{1.5}$$

where J and K are positive functions defined on the line and the positive half-line respectively, whose Laplace transforms

$$j(q) := \int_{-\infty}^{\infty} e^{-qy} J(y) dy, \qquad k(q) := \int_0^{\infty} e^{-qy} K(y) dy, \tag{1.6}$$

are meromorphic functions on $\mathbf{C}$ *given by*

$$j(q) = \frac{1}{\mathrm{Ai}(q)}, \qquad k(q) = -2\frac{d^2}{dq^2}\log\,\mathrm{Ai}(q),$$

where Ai *denotes the first Airy function defined by*

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt.$$

Groeneboom's original proof of the previous theorem relied extensively on Ito's excursion theory and results on path decompositions of Markov process at the time they achieve their ultimate maximum. In this paper, we will give a different approach relying on kinetic theory. Indeed, the following result due to Kaspar-Rezakhanlou in [KR2] gives a complete description of the law at fixed times of solutions to scalar conservation laws with piecewise-deterministic Markov initial data. Kaspar and Rezakhanlou's theorem holds for any general scalar conservation laws with C^2 convex flux H, but as we are only interested in Burgers equation, we will only state it in the case when H is quadratic, and omit some technical details in it, as we will be using this theorem as an *ansatz* (indeed, the assumptions of the theorem do not hold in our case due to some technical difficulties in proving it, however it will serve to make an educated guess on a non-trivial solution to a kinetic equation).

Theorem 2 ([KR2])**.** *Consider an initial data* ρ^0 *that is a Markov process of bounded variation, and whose generator is given by its actions on test functions* $\varphi \in C_c^\infty(\mathbf{R})$

$$\mathcal{A}^0\varphi(y) = b^0(y)\varphi'(y) + \int_{-\infty}^{y} (\varphi(z) - \varphi(y))n^0(y,z)dz.$$

Let ρ *be the unique entropy solution to Burgers equation with initial condition* $\rho(x,0) = \rho^0(x), x \in \mathbf{R}$*, then for any* $t > 0$ *the process* $x \mapsto \rho(x,t)$ *is again Markovian with bounded variation and with generator* $\mathcal{A}^t$ *given by*

$$\mathcal{A}^t\varphi(y) = b(y,t)\varphi'(y) + \int_{-\infty}^{y} (\varphi(z) - \varphi(y))n(y,z,t)\ dz. \tag{1.7}$$

The drift $(y,t) \mapsto b(y,t)$ *verifies the ODE with parameter* $b_t(y,t) = -b(y,t)^2$*, and the jump kernel* n *verifies the following kinetic equation*

$$\partial_t n(y,z,t) - \frac{1}{2t}(y-z)(\partial_y n - \partial_z n)(y,z,t) = Q(n,n)(y,z,t), \tag{1.8}$$

where $Q = Q^+ - Q^-$ *is a quadratic operator with* $Q^-(n,n) = nL(n)$*, and*

$$\begin{aligned} Q^+(n,n)(y,z,t) =& \frac{y-z}{2}\int_z^y n(y,w,t)n(w,z,t)\ dw, \\ L(n)(y,z,t) =& \int_{-\infty}^{z} \frac{y-w}{2}n(z,w,t)\ dw + \int_{-\infty}^{y} \frac{w-z}{2}n(y,w,t)\ dw. \end{aligned} \tag{1.9}$$

Remark 1. (i) *The process* $x \mapsto \rho(x,t)$ *is a concatenation of smooth pieces corresponding to the flow of the ODE with drift* $b(\cdot,t)$ *(i.e the ODE* $\dot{x}(t) = b(x(t),t)$*) interrupted by Markovian jumps at rate* $(y,z) \mapsto n(y,z,t)$*.*

(ii) *In [KR2] original statement, it is assumed that the Markov process* ρ^0 *starts at the origin at* $x = 0$*. However, in our case, we will be using it for stationary initial data. The fact that Burgers equation is translation invariant, implies that the solution remains stationary in* space *at later times.*

Let us give here an outline of our proof. From a theorem of Menon and Srinivasan in [MS] using last exit times, we know that for any fixed $t > 0$ that the process $x \mapsto \rho^\sigma(x,t)$ is a Markov process. Moreover, it is a deterministic fact that *entropy* solutions are of bounded variation at later times. Therefore, if $x \mapsto \rho^\sigma(x,t)$ were to be a Feller process, then it must have a generator of the form (1.7). As such, to determine the law of $x \mapsto \rho^\sigma(x,t)$, one only need to find the expression of the corresponding drift $b(y,t)$ and jump kernel $n(y,z,t)$. Our proof contains three steps, listed as follows

- *(Step 1)* We make the assumption that Theorem 2 hold for the more general situation where the initial condition $\rho(\cdot,0)$ is stationary. Therefore, should the Feller property hold in our case (Burgers equation with white noise initial data), the drift b and jump kernel n must satisfy the above equations. Our goal in this first step will be to find one non-trivial solution (b,n) to this system of equations by making some *ansatzes* on the form of these solutions based on the homogeneous properties of the Brownian motion initial potential.
- *(Step 2)* Once we have a candidate (b,n) solving the above equations, we construct a family of measures $(\mu_\epsilon)_{\epsilon>0}$ on the space $\mathcal{C}([0,\infty),\mathcal{D}(\mathbf{R},\mathbf{R}))$ (recall that $\mathcal{D}(\mathbf{R},\mathbf{R})$ is the set of real-valued càdlàg functions defined on the real-line) as follows: for any $\epsilon > 0$, the measure μ_ϵ is the law of the unique *entropy* solution started from initial condition given by the stationary Markov process with drift $b(\cdot,\epsilon)$ and jump kernel $n(\cdot,\cdot,\epsilon)$.
- *(Step 3)* We show that the family of measures $(\mu_\epsilon)_{\epsilon>0}$ is tight, and thus admit a limit along a certain subsequence $\epsilon_n \downarrow 0$. We denote this limit $\bar{\mu}$.
- *(Step 4)* We show that under the law of the limit $\bar{\mu}$, the law of the marginal

$$x \in \mathcal{C}([0,\infty),\mathcal{D}(\mathbf{R},\mathbf{R})) \mapsto x(0) \in \mathcal{D}(\mathbf{R},\mathbf{R})$$

  is a two-sided linear Brownian motion. Moreover, $\bar{\mu}$-almost surely, x is an entropy solution to Burgers equation. By uniqueness of the entropy solution, we deduce the main theorem.

2. The form of the Markov Process

The purpose of this section is twofold: we partially justify the form of our kernel n as in (1.5), and use the kinetic equation (1.8) to determine J and K. As we will see in this section that modulo some translation and scaling, K and J must satisfy (1.6).

We start by making some *ansatzes* on the form of the generator of $x \mapsto \rho^\sigma(x,t)$. As we mentioned in the previous section, we assume that the solution at later times is a Feller process with an infinitesimal generator $\mathcal{A}^t$ as in (1.7). Due to the scaling property (1.4), we restrict ourselves to the case $\sigma = \frac{1}{\sqrt{2}}$ in the statement of the main Theorem, and consequently we don't make any reference to the diffusion factor σ (we always assume that we start with a Brownian white noise with variance $\frac{1}{2}$). Again from (1.4), it is not hard to show that the drift b and kernel n must verify the two identities

$$b(y,t) = t^{-1}b(t^{\frac{1}{3}}y,1) =: t^{-1}b(t^{\frac{1}{3}}y), \qquad n(y,z,t) = t^{-\frac{1}{3}}n(t^{\frac{1}{3}}y,t^{\frac{1}{3}}z,1) =: t^{-\frac{1}{3}}n(t^{\frac{1}{3}}y,t^{\frac{1}{3}}z). \tag{2.1}$$

Let us start by finding a candidate solution to b. As b solves the ODE with parameter $b_t(y,t) = -b(y,t)^2$, we must have that $-t^{-2}b(t^{\frac{1}{3}}y) + \frac{1}{3}t^{-\frac{5}{3}}yb'(t^{\frac{1}{3}}y) = -t^{-2}b^2(t^{\frac{1}{3}}y)$. This equation is equivalent to $b(r) - \frac{1}{3}rb'(r) = b(r)^2$ for all $r \in \mathbf{R}$. Let $f(r) = b(r^{\frac{1}{3}})$, then $f'(r) = \frac{1}{3}r^{-\frac{2}{3}}b'(r^{\frac{1}{3}}) = r^{-1}(f(r) - f(r)^2)$ which is equivalent to $\left(\frac{r}{f(r)}\right)' = 1$, which implies that $f = r/(c+r)$ for a constant c. The solution $f = 1$ is the only non-singular solution. Hence we take $b(r) = 1$ and thus $b(y,t) = t^{-1}$ is our non-singular candidate solution to the ODE that verifies the self-similarity condition.

Let us move on now to finding a non-trivial candidate for $n(y,z,t)$. Recall that the kinetic equation in the case of Burgers equation takes the form (1.8), with Q as in (1.9).

2.1. **The Kernel n.** We first partially justify the form of the kernel n that appeared in (1.5). We first claim that in our setting, it is quite natural to express n as

$$n(y,z) = \frac{J(z)}{J(y)} K(y,z), \tag{2.2}$$

for a suitable pair functions J and K. We can derive some of the properties of this pair by using the fact that the process $\rho(x) = \rho(x,1)$ is a stationary Markov process, and a symmetry of our PDE. It is worth emphasizing that the argument we are using is general and works even when the Burgers equation is replaced the general scalar conservation law

$$\rho_t + H(\rho)_x = 0,$$

for a convex function H. Equivalently, we may write $\rho = u_x$, where u satisfies the Hamilton-Jacobi equation,

$$u_t + H(u_x) = 0, \qquad u(x,0) = \sigma B(x). \tag{2.3}$$

To simplify our presentation, we additionally assume that H is an even function. Since the processes $x \mapsto B(x)$ and $x \mapsto B(-x)$ have the same law, we deduce that $(u(x,t) : x \in \mathbb{R}) \overset{d}{=} (u(-x,t) : x \in \mathbb{R})$. As a result

$$(\rho(x,t) : x \in \mathbb{R}) \overset{d}{=} (-\rho(-x,t) : x \in \mathbb{R}). \tag{2.4}$$

Observe that if the process $\rho(\cdot)$ is a stationary Markov process, and $a > 0$, then the sequence $(\rho_j = \rho(ja) : \ j \in \mathbb{Z})$ is a stationary Markov chain. Conditioned on $\rho(0) = \rho_0$ and $\rho(na) = \rho_n$, the law of $(\rho_1, \dots, \rho_{n-1})$ has a Gibbsian representation of the form

$$Z_n(\rho_0,\rho_n)^{-1} \prod_{j=1}^{n} g(\rho_{j-1},\rho_j) \ \prod_{j=1}^{n-1} d\rho_j,$$

for a normalizing constant $Z_n(\rho_0,\rho_n)$. Let us define the operators

$$\mathcal{L}\varphi(\rho) = \int g(\rho,\rho_*)\varphi(\rho_*) \ d\rho_*, \qquad \mathcal{L}^*\varphi(\rho_*) = \int g(\rho,\rho_*)\varphi(\rho) \ d\rho.$$

If e^{θ_0} is the largest eigenvalue of $\mathcal{L}$, then we can find functions J and J^* such that

$$\mathcal{L}J = e^{\theta_0} J, \qquad \mathcal{L}J^* = e^{\theta_0} J^*.$$

With the aid of (J, J^*) we can turn our Gibbsian description to a Markovian description. More precisely, the sequence $(\rho_j : \ j \in \mathbb{Z})$ is a stationary Markov chain with the kernel

$$h(\rho,\rho_*) = e^{-\theta_0} \frac{J(\rho_*)}{J(\rho)} g(\rho,\rho_*),$$

and an invariant measure $d\pi = JJ^* \ d\rho$. This in turn suggests writing the kernel as in (2.2). The property (2.4) means that $g(\rho,\rho_*) = g(-\rho_*,-\rho)$. This in turn implies that $J^*(\rho) = J(-\rho)$, and as a consequence,

$$\pi(d\rho) =: \ell(\rho) \ d\rho = J(\rho)J(-\rho) \ d\rho. \tag{2.5}$$

Furthermore, the reversed process $x \mapsto \rho(-x)$ is a Markov process with the generator $\mathcal{A}^*$ (the adjoint of $\mathcal{A}$ with respect to the inner product of $L^2(\pi)$), and the jump rate density

$$n^*(y,z) = \frac{\ell(z)}{\ell(y)} n(z,y) = \frac{J(-z)}{J(-y)} K(z,y).$$

Hence the process $x \mapsto \rho(-x)$ is a Markov process with the jump rate density

$$\frac{J(z)}{J(y)} K(-z,-y).$$

From (2.4) we deduce $K(y,z) = K(-z,-y)$. This is the case if $K(y,z)$ depends on $y - z$.

Remark 2. *A deeper explanation of this fact would rely on using the exact form of the law of the post-maximum Brownian motion with parabolic drift (recall the definition of the pure-jump process $x \mapsto y(x,t)$ that is the inverse Lagrangian). In this case, the value of $K(y,z)$ is linked to a functional of the law of a Brownian motion conditioned to a certain event and to take the values y and z at its extremities. One can show by some little work that the value $K(y+c, z+c)$ for some constant c is equivalent to the same functional but now with adding a drift c to the Brownian motion, but it is a common fact that the law of the Brownian motion with drift on an interval conditioned to take fixed values on the end-intervals is independent of this drift. We omit the details of this heuristic.*

We next use the self-similarity relation (1.4) to express our kinetic equation (1.8) as an equation for the function $(y,z) \mapsto n(y,z)$. From

$$\partial_t n(y,z,t)|_{t=1} = 3^{-1}(-n + y\partial_y n + z\partial_z n)(y,z),$$

we learn that the left-hand side of (1.8) at $t = 1$ can be written as

$$\frac{1}{3}(-n + y\partial_y n + z\partial_z n) - \frac{1}{2}(y-z)(\partial_y n - \partial_z n) = -\frac{1}{3}n + \left(\frac{z}{2} - \frac{y}{6}\right)\partial_y n + \left(\frac{y}{2} - \frac{z}{6}\right)\partial_z n.$$

As a result, the kinetic equation (1.8) is equivalent to

$$-2n + (3z - y)\partial_y n + (3y - z)\partial_z n = 6Q(n,n). \tag{2.6}$$

Proposition 1. *Assume that the kernel n is of the form (1.5) for a pair of C^1 functions J and K. Then n satisfies (2.6) if and only if $K(0) = 0$, and there are constants c_0, c_1, and c_2 such that the following equations hold:*

$$3(K \star \hat{J})(y) - y(K \star J)(y) = c_0 J(y), \tag{2.7}$$

$$J'(y) + (K \star J)(y) = (c_2 y^2 + c_1) J(y), \tag{2.8}$$

$$2K(s) + 3s(K \star K)(s) + 4sK'(s) = (c_2 s^3 + c_1 s)K(s), \tag{2.9}$$

where $\hat{J}(y) = yJ(y)$.

Proof Let us set

$$\lambda(y) = \int n(y,z)\, dz, \qquad \eta(y) = \int z n(y,z)\, dz.$$

It is not hard to check that

$$\int Q(n,n)(y,z)\, dz = 0.$$

From this, the the kinetic equation (2.6), and an integration by parts, we deduce

$$0 = -2\lambda(y) + 3\eta'(y) - y\lambda'(y) + \lambda(y) + 2ya = -\lambda(y) - y\lambda'(y) + 3\eta'(y) + 2ya,$$

where $a = K(0)$. This leads to $3\eta(y) - y\lambda(y) + ay^2 = c_0$, for a constant c_0. In terms of J and K, we have

$$3(K \star \hat{J})(y) - y(K \star J)(y) = (c_0 - ay^2)J(y). \tag{2.10}$$

Observe

$$\begin{aligned}6L(n)(y,z) =&3\eta(y) - 3\eta(z) + 3y\lambda(z) - 3z\lambda(y) = y\lambda(y) - z\lambda(z) + 3y\lambda(z) - 3z\lambda(y)\\ =&(3y - z)\lambda(z) + (y - 3z)\lambda(y) + a(y^2 - z^2).\end{aligned}$$

We now go back to the kinetic equation (2.6) and rewrite it as $X^1 = X^2$, where

$$\begin{aligned}X_1 =& -2 - (3z - y)\frac{J'(y)}{J(y)} + (3y - z)\frac{J'(z)}{J(z)} - 4s\frac{K'(s)}{K(s)},\\ X_2 =&3s\frac{(K \star K)(s)}{K(s)} - (3y - z)\lambda(z) - (y - 3z)\lambda(y) + a(y^2 - z^2),\end{aligned}$$

where $s = y - z$. We first send $s \to 0$ in the both of the equation $X_1 = X_2$ to deduce that if $a \neq 0$, then we must have $-2 = 0$, which is absurd. As a result $a = 0$, and (2.10) becomes (2.7).

Let us set

$$\xi := \frac{J'}{J} + \lambda = \frac{J' + K \star J}{J},$$

so that we can rewrite our kinetic equation in a more compact form:

$$-(3z - y)\xi(y) + (3y - z)\xi(z) = 2 + 3s\frac{(K \star K)(s)}{K(s)} + 4s\frac{K'(s)}{K(s)}. \tag{2.11}$$

Note that the right-hand side depends on s only. On the other hand, if we set $\zeta(y) = \xi(y) - \xi(0)$, then the left-hand side of (2.11) can be rewritten as

$$\begin{aligned}X(s,y) :=& -(2y - 3s)\xi(y) + (2y + s)\xi(y - s) = s(3\xi(y) + \xi(y + s)) + 2y(\xi(y - s) - \xi(y))\\ =&s(3\zeta(y) + \zeta(y + s)) + 2y(\zeta(y - s) - \zeta(y)) + 4s\xi(0),\end{aligned}$$

which is independent of y. From $X(s,y) = X(s,0)$ we deduce

$$s(3\zeta(y) + \zeta(y + s)) + 2y(\zeta(y - s) - \zeta(y)) = s\zeta(s).$$

By dividing both sides by s and sending $s \to 0$ we learn that $4\zeta(y) - 2y\zeta'(y) = 0$. This means that $\zeta(y) = c_2y^2$ for a constant c_2. Hence $\xi(y) = c_2y^2 + c_1/4$ for constants c_1 and c_2. This and the definition of ξ yield (2.8). Moreover, for such ξ,

$$-(3z - y)\xi(y) + (3y - z)\xi(z) = c_2s^3 + c_1s.$$

From this and (2.11) we deduce (2.9). □

We now focus on solving the equations (2.7)-(2.9) for K and J. Observe that if $c \in \mathbb{R}$, and $\bar{J}(y) = J(cy)$, $\bar{K}(y) = c^2K(cy)$, then

$$\begin{aligned}\frac{\bar{J}'(y) + (\bar{K} \star \bar{J})(y)}{\bar{J}(y)} &= c\ \frac{J'(cy) + (K \star J)(cy)}{J(cy)},\\ \frac{3(\bar{K} \star \hat{\bar{J}})(y) - y(\bar{K} \star \bar{J})(y)}{J(y)} &= \frac{3(K \star \hat{J})(cy) - y(K \star J)(cy)}{J(cy)},\\ \frac{2\bar{K}(s) + 3s(\bar{K} \star \bar{K})(s) + 4s\bar{K}'(s)}{\bar{K}(s)} &= \frac{2K(cs) + 3(cs)(K \star K)(cs) + 4(cs)K'(cs)}{K(cs)}.\end{aligned}$$

From this we learn that if (J,K) solves (2.7)-(2.9), for the constant (c_1,c_2,c_3), then $(\bar J,\bar K)$ solves (2.7)-(2.9),for constant (c_1,c^3c_2,cc_3). From this we learn that without loss of generality, we may assume that $c_2=1$.

From now on, we assume that $c_2=1$. We apply the Laplace transform to the equations (2.7) and (2.8), to obtain

$$3(kj')(q)-(jk)'(q)=c_0j(q),\qquad (jk)(q)+qj(q)=j''(q)+c_3j(q). \tag{2.12}$$

Note that if (j,k) is a solution, and $\hat j(q)=j(q+c_3)$, then $(\hat j,k)$ is solves the same system of equations for $c_3=0$. Hence, without loss of generality, we may assume that $c_3=0$. In this, (2.12) is equivalent to equations $k'j+c_0j=2j'k$, and $j(q)(k(q)+q)=j''(q)$. These can be rewritten as

$$l'(q)=k(q)+q-l^2(q),\qquad k'(q)=2l(q)k(q)-c_0, \tag{2.13}$$

where $l=j'/j$. To solve (2.13), we first derive an equation for l:

$$l''=k'-2ll'+1=2lk-2ll'+1-c_0=2l(k-l')+1-c_0=2l(l^2-q)+1-c_0.$$

Let us assume that $c_0=2$. Define $h:=l'-l^2+q$. Then

$$h'=l''-2ll'+1=l''-2l(h+l^2-q)+1=-2lh.$$

This means that $h'/h=-2l=-2j'/j$, which in turn implies that $h=c_4j^{-2}$, for a constant c_4. In summary, once we find j, such that

$$l'-l^2+q=h,\qquad h=c_4j^{-2},\quad l=j'/j, \tag{2.14}$$

then we set $k=h+2(l^2-q)$, which satisfies the second equation of (2.13). It remains to solve (2.14) for a given constant c_4. To achieve this, we set $A=j^{-1}$, so that $A'=-j'/j^2=-l/j$, and

$$\frac{A''(q)}{A(q)}=(l^2-l')(q)=q-h=q-c_4A^2(q).$$

In summary

$$A''(q)=qA(q)-c_4A^3(q).$$

When $c_4=0$, we obtain a special solution which correspond to Groeneboom's calculation.

We now argue that the equations (2.7) and (2.8) imply (2.9) when $c_2=1$ and $c_3=0$. Indeed applying Laplace transform to both sides (2.9) yields

$$k'''=3(k^2)'+4qk'+2k \tag{2.15}$$

We differentiate the second equation and use the first two identities above in (2.15)

$$\begin{aligned}k''&=2lk'+2l'k=4l^2k-2c_0l+2k(k+q-l^2)\\&=2k^2+2kq+2kl^2-2c_0l\end{aligned}$$

We differentiate a second time to get

$$\begin{aligned}k'''&=2(k^2)'+2k'q+2k+2k'l^2+4kll'-2c_0l'\\&=2(k^2)'+2k'q+2k+2k'(k+q-l')+2l'(k'+c_0)-2c_0l'\\&=3(k^2)'+4k'q+2k.\end{aligned}$$

This is exactly (2.15), and thus follows from (2.14).

3. Central Limit Theorem

As mentioned above, the construction of the measure μ_ϵ is straightforward. Let

$$\begin{aligned}\Phi : \mathcal{D}(\mathbf{R}) &\to \mathcal{C}([0,\infty), \mathcal{D}(\mathbf{R},\mathbf{R}))\\ g &\mapsto \rho\end{aligned}$$

the map that maps any initial condition g to ρ the unique entropy solution to Burgers equation with initial condition $\rho(\cdot, 0) = g$. For $\epsilon > 0$, we define the measure μ_ϵ on $\mathcal{C}([0,\infty), \mathcal{D}(\mathbf{R})))$ as the push-forward measure of Φ of the stationary Markov process ρ^ϵ with generator

$$\mathcal{A}^\epsilon \varphi(\rho^-) = \frac{1}{\epsilon}\varphi'(\rho^-) + \int_{-\infty}^{\rho^-} (\varphi(\rho^+) - \varphi(\rho^-)) n^\epsilon(\rho^-, \rho^+) d\rho^+$$

and stationary marginal density given by $\rho \mapsto \frac{1}{2}J^\epsilon(\rho)J^\epsilon(-\rho)$, where $n^\epsilon(y,z) = \frac{J^\epsilon(z)}{J^\epsilon(y)}K^\epsilon(y-z)$, with $J^\epsilon(x) = J(\epsilon^{\frac{1}{3}}x)$ and $K^\epsilon(x) = \epsilon^{-\frac{1}{3}}K(\epsilon^{\frac{1}{3}}x)$, where J and K are given above. The fact that $\rho \mapsto J^\epsilon(\rho)J^\epsilon(-\rho)$ is a stationary density for this Markov process follows from the spectral considerations that we mentioned above when considering the integral operator with kernel n. We give a quick proof of this stationarity for the sake of completeness and that follows for the equations verified by J and K above, We restrict ourselves to $\epsilon = 1$ by self-similiarity, and denote the corresponding generator $\mathcal{A}^1$ by $\mathcal{L}$. To prove that the marginal with density $\ell(\rho) = \frac{1}{2}J(\rho)J(-\rho)$ with respect to Lebesgue measure is stationary for this Markov process, it suffices to prove that for any test function φ, we have that $\int_\mathbf{R} \mathcal{L}(\varphi)(u)\ell(u)du = 0$. First, let us give a more friendly expression of the generator $\mathcal{L}$ in terms of convolutions using the formulas above verified by the functionals J and K. For a test function φ

$$\mathcal{L}\varphi = \varphi' + \frac{(\varphi J) \star K - \varphi\,(J \star K)}{J}$$

Using the relation $(J \star K)(s) + J'(s) = s^2 J(s)$, we have then

$$\begin{aligned}\mathcal{L}\varphi &= \varphi' + \frac{(\varphi J) \star K - \varphi\left(x^2 J - J'\right)}{J}\\ &= \frac{\varphi' J + \varphi J' + (\varphi J) \star K - x^2(\varphi J)}{J}\\ \mathcal{L}\varphi &= \frac{(\varphi J)' + (\varphi J) \star K - x^2(\varphi J)}{J}\end{aligned}$$

Thus

$$\int_\mathbf{R} \mathcal{L}(\varphi)(u)\ell(u)du = \frac{1}{2}\int_\mathbf{R} (h'(u) + (h \star K)(u) - u^2 h(u))J(-u)du$$

where $h = \varphi J$. Now

$$\begin{aligned}\int_\mathbf{R} (h \star K)(u)J(-u)du &= \int_\mathbf{R}\int_\mathbf{R} h(z)K(u-z)J(-u)dzdu\\ &= \int_{\mathbf{R}\times\mathbf{R}} h(-z)K(z-u)J(u)dudz\\ &= \int_\mathbf{R} h(-z)(J \star K)(z)dz\end{aligned}$$

Hence

$$\begin{aligned}\int_{\mathbf{R}} \mathcal{L}(f)(u)\ell(u)du &= \frac{1}{2}\int_{\mathbf{R}} (h'(-u)J(u) + h(-u)(J \star K)(u) - u^2 h(-u)J(u))du \\ &= \frac{1}{2}\int_{\mathbf{R}} (h'(-u)J(u) + h(-u)(u^2 J(u) - J'(u)) - u^2 h(-u)J(u))du \\ &= -\frac{1}{2}\int_{\mathbf{R}} (h(-\cdot)J)'(u)du = 0\end{aligned}$$

confirming that ℓ is indeed a density of a stationary distribution for the Markov process with generator $\mathcal{L}$. The fact that ℓ is a density (its integral is equal to 1) is a fact from the analysis of contour integrals on Airy functions that is discussed in detail in Groeneboom's work.

3.1. **Step 3.** To prove the tightness of the family of measures $(\mu_\epsilon)_{\epsilon>0}$, we can use the following theorem due to Prohorov for the criterion on the tightness of a family of probability measures defined on the space of Banach-valued continuous functions.

Theorem 3 (Prohorov)**.** *For a function X, define its modulus of continuity by $\omega_X(\delta) = \sup_{||t-s||<\delta} ||X_t - X_s||$ for any $\delta > 0$. A family Γ of probability measures on a space of Banach-valued continuous functions is tight if and only if the following conditions holds*

(1) *For each positive η, there exists an a such that*

$$\mathbf{P}\Big\{x : ||x(0)|| > a\Big\} \le \eta, \quad \textit{for all } \mathbf{P} \in \Gamma$$

(2) *For each positive ϵ and η, there exists a $\delta \in (0,1)$ such that*

$$\mathbf{P}\Big\{x : \omega_x(\delta) \ge \epsilon\Big\} \le \eta, \quad \textit{for all } \mathbf{P} \in \Gamma$$

We can apply this theorem in our context and use the self-similarity to show that there is no blow-up when $\epsilon \downarrow 0$.

3.2. **Step 4.** To finish our proof, we just need to prove that any limit in law of the process $x \mapsto \int_0^x \rho^\epsilon(y)dy$ along a certain subsequence has the law of a two-sided linear standard Brownian motion with variance $\frac{1}{2}$. By self-similarity, this process has the same law as the process

$$x \mapsto \epsilon^{\frac{1}{3}} \int_0^{\frac{x}{\epsilon^{\frac{2}{3}}}} \rho(y,1)dy$$

We denote $\rho := \rho(\cdot, 1)$. We will use the fact that the following process

$$M^\epsilon(x) := \exp\left(g(\rho^\epsilon(x)) - g(\rho^\epsilon(0)) - \int_0^x e^{-g}\mathcal{A}^\epsilon e^g(\rho^\epsilon(y))dy \right)$$

is a martingale for any nice function g. This martingale has the same law as the martingale (we put $\delta = \epsilon^{\frac{2}{3}}$)

$$N_\delta(x) = \exp\left(g\left(\sqrt{\delta}\rho\left(\frac{x}{\delta}\right)\right) - g(\sqrt{\delta}\rho(0)) - \int_0^{\frac{x}{\delta}} e^{-\sqrt{\delta}g}\mathcal{L}e^{\sqrt{\delta}g}(y)dy \right), \ x \in \mathbf{R}$$

where $\mathcal{L}$ is the generator of the Markov process ρ. Recall that the generator $\mathcal{L}$ is given by its action on test functions $\varphi \in \mathcal{C}_c^\infty(\mathbf{R})$:

$$\mathcal{L}\varphi(y) = \varphi'(y) + \int_{-\infty}^y (\varphi(z) - \varphi(y))\frac{J(z)}{J(y)}K(y-z)dz$$

where the Laplace transform $j(q) = \int_{\mathbf{R}} e^{-qy} J(y)dy$ of J is given by $j(q) = \frac{1}{\mathrm{Ai}(q)}$ and the Laplace transform $k(q) = \int_0^\infty e^{-qy} K(y)dy$ of K is given by $k(q) = -2\frac{d^2}{dq^2} \log \mathrm{Ai}(q)$. We can rewrite the expression of the martingale N_δ in this following form in order to exhibit the expression of the process of interest $x \mapsto \sqrt{\delta} \int_0^{\frac{x}{\delta}} \rho(y)dy$

$$N_\delta(x) = \exp\left(g\left(\sqrt{\delta}\rho\left(\frac{x}{\delta}\right)\right) - g(\sqrt{\delta}\rho(0)) - \sqrt{\delta}\int_0^{\frac{x}{\delta}} \mathcal{L}g(\rho(y))dy \right.$$
$$\left. - \int_0^{\frac{x}{\delta}} (e^{-\sqrt{\delta}g}\mathcal{L}e^{\sqrt{\delta}g} - \sqrt{\delta}\mathcal{L}g)(\rho(y))dy \right)$$

Let $x_1 < x_2 < \cdots < x_n$ be real numbers, by the martingale property, we know that for any bounded continuous functions $f_1, \cdots, f_{n-1}$, we have

$$\mathbf{E}\left[\prod_{i=1}^{n-1} f_i(N_\delta(x_i))N_\delta(x_n)\right] = \mathbf{E}\left[\prod_{i=1}^{n-1} f_i(N_\delta(x_i))N_\delta(x_{n-1})\right]$$

We fix a number $\lambda \in \mathbf{R}$. As a first step, we wish to find a function g such that $\mathcal{L}g = \lambda\mathrm{Id}$. In other words, we wish to find $\mathcal{L}^{-1}(\mathrm{Id})$ the image of the identity by the inverse of the operator $\mathcal{L}$. This will allow us to plug-in this function g in the expression of the martingale N_δ and recover partly the Laplace transform of the process of interest. Put $h = Jg$ and solve the equation $\mathcal{L}g = \lambda\mathrm{Id}$, we obtain the equality

$$\lambda\mathrm{Id} = \frac{h' + h \star K - x^2 h}{J}$$

which translates to $\lambda x J = h' + h \star K - x^2 h$. Consider now the Laplace transform of h to be $\hat{h}(q) = \int_{\mathbf{R}} e^{-qy} h(y)dy$, by taking the Laplace transform of the previous equality, one obtains

$$\begin{aligned} -\lambda j'(q) &= q\hat{h}(q) + \hat{h}(q)k(q) - \hat{h}''(q) \\ &= (q + k(q))\hat{h}(q) - \hat{h}''(q) \end{aligned}$$

As $j''(q) = j(q)(q + k(q))$, we get after replacing

$$\lambda j'(q)j(q) = j''(q)\hat{h}(q) - j(q)\hat{h}''(q) = (j'\hat{h})' - (j\hat{h}')'$$

which is equivalent to $(\frac{\lambda j^2}{2} + j\hat{h}')' = (j'\hat{h})'$, i.e $\left(\frac{\hat{h}}{j}\right)' = -\frac{\lambda}{2}$, or equivalently that $\hat{h}(q) = -\frac{\lambda}{2} q j(q) = -\frac{\lambda}{2}\widehat{J'}(q)$. Taking the inverse Laplace transform, we find the expression of g to be

$$g = \mathcal{L}^{-1}(\lambda\mathrm{Id}) = -\frac{\lambda}{2}\frac{J'}{J}$$

As a second step, we explicitly compute the last term of the martingale N_δ

$$\begin{aligned} e^f\mathcal{L}e^{-f} - \mathcal{L}f &= e^{-f}\left((e^f)' + \frac{(e^f J) \star K - e^f(J \star K)}{J}\right) - \left(f' + \frac{(fJ) \star K - f(J \star K)}{J}\right) \\ &= \frac{e^{-f}(e^f J) \star K - (J \star K) - (fJ) \star K + f(J \star K)}{J} \end{aligned}$$

Writing this down pointwise leads to

$$(e^f\mathcal{L}e^{-f} - \mathcal{L}f)(y) = \int_{\mathbf{R}} \frac{\left(e^{f(z)-f(y)} - 1 - (f(z) - f(y))\right) J(z)K(y - z)}{J(y)} dz$$

We substitute now $f=\sqrt{\delta}g$. We have then by Taylor expansion that

$$\left(e^{\sqrt{\delta}g}\mathcal{L}e^{-\sqrt{\delta}g}-\sqrt{\delta}\mathcal{L}g\right)(y)=\delta\int_{\mathbf{R}}(g(z)-g(y))^2n(y,z)dz+r(\delta,y)$$

where $\sup_y|r(\delta,y)|=o(\delta)$ as $\delta\downarrow 0$. This is straightforward from the expression above and the regularity of the kernel n. Now, we can write

$$N_\delta(x)=\exp\left(g\Big(\sqrt{\delta}\rho\Big(\frac{x}{\delta}\Big)\Big)-g\Big(\sqrt{\delta}\rho(0)\Big)-\lambda\sqrt{\delta}\int_0^{\frac{x}{\delta}}\rho(y)dy\right.$$
$$\left.-\frac{\lambda^2}{2}\delta\int_0^{\frac{x}{\delta}}\int_{\mathbf{R}}\Big(g(z)-g(\rho(y))\Big)^2n(\rho(y),z)dzdy+o(\delta)\right)$$

Let $B^\infty(x):=\lim_{\delta_n\downarrow 0}\sqrt{\delta_n}\int_0^{\frac{x}{\delta_n}}\rho(y)dy$ be the almost sure limit along a subsequence to our process, then by the ergodic theorem we get that $N_{\delta_n}(x)$ converges almost surely to

$$\lim_{\delta_n\downarrow 0}N_{\delta_n}(x)=\exp\left(-\lambda B^\infty(x)-\frac{\lambda^2}{2}x\int_{\mathbf{R}}\int_{\mathbf{R}}(g(z)-g(u))^2n(u,z)\ell(u)dudz\right)$$

where $\ell(u)=\frac{1}{2}J(u)J(-u)$. Now, to finish we just need to compute the integral

$$\int_{\mathbf{R}}\int_{\mathbf{R}}(g(z)-g(u))^2n(u,z)\ell(u)dudz$$

For any nice function f, let us compute $\mathcal{L}(f^2)-2f\mathcal{L}(f)$. We have

$$\begin{aligned}\mathcal{L}(f^2)-2f\mathcal{L}(f)&=\frac{(f^2J)'+(f^2J)\star K-x^2(f^2J)}{J}-2\frac{f(fJ)'+f(fJ\star K)-x^2fJ}{J}\\&=\frac{2ff'J+f^2J'+(f^2J)\star K-x^2f^2J-2ff'J-2f^2J'-2f(fJ\star K)+2x^2f^2J}{J}\\&=\frac{-f^2J'+x^2f^2J+(f^2J\star K)-2f(fJ\star K)}{J}\\&=\frac{f^2(J\star K)+(f^2J\star K)-2f(fJ\star K)}{J}\end{aligned}$$

This identity pointwise translates to

$$\begin{aligned}(\mathcal{L}(f^2)-2f\mathcal{L}(f))(y)&=\int_{\mathbf{R}}\frac{f(y)^2J(z)K(y-z)+f(z)^2J(z)K(y-z)-2f(y)f(z)J(z)K(y-z)}{J(y)}dz\\&=\int_{\mathbf{R}}(f(y)-f(z))^2n(y,z)dz\end{aligned}$$

Hence

$$\begin{aligned}\int_{\mathbf{R}}\int_{\mathbf{R}}(g(z)-g(u))^2n(u,z)\ell(u)dudz&=\int_{\mathbf{R}}(\mathcal{L}(g^2)-2g\mathcal{L}(g))(u)\ell(u)du\\&=-2\int_{\mathbf{R}}g(u)\mathcal{L}(g)(u)\ell(u)du\\&=\frac{1}{2}\int_{\mathbf{R}}\frac{J'(u)}{J(u)}uJ(u)J(-u)du\\&=\frac{1}{2}\int_{\mathbf{R}}uJ'(u)J(-u)du=c\end{aligned}$$

By doing integration by parts, we have that

$$c = -\frac{1}{2}\int_{\mathbf{R}} J(u)(J(-u) - uJ(-u))du = \frac{1}{2}\int_{\mathbf{R}} J(u)J(-u)du - c$$

Hence

$$c = \frac{1}{4}\int_{\mathbf{R}} J(u)J(-u)du = \frac{1}{2}$$

$$\mathbf{E}\left[\prod_{i=1}^{n-1} M(x_i))M(x_n)\right] = \mathbf{E}\left[\prod_{i=1}^{n-1} f_i(M(x_i))M(x_{n-1})\right]$$

where

$$M(x) := \exp\left(-\lambda B^{\infty}(x) - \frac{\lambda^2}{4}x\right)$$

this shows that M is a martingale and that B^{∞} is a standard Brownian motion with variance equal to $\frac{1}{2}$.

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