# SCALAR CONSERVATION LAWS WITH MONOTONE PURE-JUMP MARKOV INITIAL CONDITIONS

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ABSTRACT. In 2010 Menon and Srinivasan published a conjecture for the statistical structure of solutions $\rho$ to scalar conservation laws with certain Markov initial conditions, proposing a kinetic equation that should suffice to describe $\rho(x,t)$ as a stochastic process in $x$ with $t$ fixed. In this article we verify an analogue of the conjecture for initial conditions which are bounded, monotone, and piecewise constant. Our argument uses a particle system representation of $\rho(x,t)$ over $0 \le x \le L$ for $L > 0$, with a suitable random boundary condition at $x = L$.

## 1. INTRODUCTION

We are interested in the statistical properties of solutions to the Cauchy problem for the scalar conservation law

$$\begin{cases} \rho_t = H(\rho)_x & \text{in } \mathbb{R} \times (0,\infty) \\ \rho = \xi & \text{on } \mathbb{R} \times \{t = 0\} \end{cases} \tag{1.1}$$

where $\xi = \xi(x)$ is a stochastic process. In the special case

$$H(p) = -\frac{p^2}{2} \tag{1.2}$$

(1.1) is the well-known inviscid Burgers' equation, which has often been considered with random initial data. Burgers himself, in his investigation of turbulence [8], was already moving in this direction, though it would be some time before problems in this area were addressed rigorously.

Several papers, including [4, 5, 9–11, 26], have developed a theory showing a kind of integrability for the evolution of the *law* of $\rho(\cdot,t)$ for certain initial data. This article is concerned with pushing stochastic integrability results beyond the Burgers setting to general scalar conservation laws, an effort begun in earnest by Menon and Srinivasan [21, 24]. Before stating our main result, we first survey the major developments in this subject.

This work is adapted from [19].

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1.1. **Background.** The Burgers case $H(p) = -p^2/2$ has seen extensive interest. Recall that if

$$u(x,t) = \int_{-\infty}^{x} \rho(x',t)\,dx' \tag{1.3}$$

then $u(x,t)$ formally satisfies

$$u_t + \frac{1}{2}(u_x)^2 = 0 \tag{1.4}$$

and is determined by the Hopf-Lax formula [17]:

$$u(x,t) = \inf_y \left\{ \int_{-\infty}^{y} \xi(x')\,dx' + \frac{(x-y)^2}{2t} \right\}. \tag{1.5}$$

The backward Lagrangian $y(x,t)$ is the rightmost minimizer in (1.5), and it is standard [13] that this determines both the viscosity solution $u(x,t)$ to (1.4) and the entropy solution to $\rho_t + \rho\rho_x = 0$,

$$\rho(x,t) = \frac{x - y(x,t)}{t}. \tag{1.6}$$

Groeneboom, in a paper [15] concerned not with PDE but concave majorants[1] and isotonic estimators, determined the statistics of $y(x,t)$ for the case where $\xi(x)$ is white noise, which we understand to mean that the initial condition for $u(x,t)$ in (1.4) is a standard Brownian motion $B(x)$. Among the results of [15] is a density for

$$\sup\{x \in \mathbb{R} : B(x) - cx^2 \text{ is maximal}\} \tag{1.7}$$

expressed in terms of the Airy function. It has been observed (see e.g. [20]) that the Airy function seems to arise in a surprising number of seemingly unrelated stochastic problems, and it is notable that Burgers' equation falls in this class.

Several key papers appeared in the 1990s; we rely on the introduction of Chabanol and Duchon's 2004 paper [11], which recounts some of this history, and discuss those portions most relevant for our present purposes. In 1992, Sinai [26] explicitly connected Burgers' equation with white noise initial data to convex minorants of Brownian montion. Three years later Avellaneda and E [3] showed for the same initial data that the solution $\rho(x,t)$ is a Markov process in $x$ for each fixed $t > 0$.

Carraro and Duchon's 1994 paper [10] defined a notion of *statistical solution* to Burgers' equation. This is a time-indexed family of probability measures $\nu(t,\cdot)$ on a function space, defined in terms of the characteristic functional

$$\hat{\nu}(t,\phi) = \int \exp\left( i \int \rho(x)\phi(x)\,dx \right) \nu(t,d\rho) \tag{1.8}$$

for test functions $\phi$. Namely, $\nu(t,\phi)$ must satisfy a differential equation obtained by formally differentiating $\hat{\nu}(t,\phi)$ under the assumption that $\rho$ distributed according to $\nu(t,d\rho)$ solves Burgers' equation. This statistical solution approach was further developed in 1998 by the same authors [9] and by Chabanol and Duchon [11]. It does have a drawback: given a (random) entropy solution $\rho(x,t)$ to the inviscid Burgers' equation, the law of $\rho(\cdot,t)$ is a statistical solution, but it is not clear that a statistical solution yields a entropy solution, and at least one example is known [5] when these notions differ. Nonetheless, [9, 10] realized that it was natural to consider Lévy process initial data, which set the stage for the next development.

[1]Straightforward manipulations of (1.5) relate this to a Legendre transform.

In 1998, Bertoin [5] proved a remarkable closure theorem for Lévy initial data. We quote this, with adjustments to match our notation.

**Theorem 1.1** ([5, Theorem 2])**.** *Consider Burgers' equation with initial data $\xi(x)$ which is a Lévy process without positive jumps for $x \geq 0$, and $\xi(x) = 0$ for $x < 0$. Assume that the expected value of $\xi(1)$ is positive, $\mathbb{E}\xi(1) \geq 0$. Then, for each fixed $t > 0$, the backward Lagrangian $y(x,t)$ has the property that*

$$y(x,t) - y(0,t) \tag{1.9}$$

*is independent of $y(0,t)$ and is in the parameter $x$ a subordinator, i.e. a nondecreasing Lévy process. Its distribution is the same as that of the* first *passage process*

$$x \mapsto \inf\{z \geq 0 : t\xi(z) + z > x\}. \tag{1.10}$$

*Further, denoting by $\psi(s)$ and $\Theta(t,s)$ $(s \geq 0)$ the Laplace exponents of $\xi(x)$ and $y(x,t) - y(0,t)$,*

$$\mathbb{E}\exp(s\xi(x)) = \exp(x\psi(s)) \tag{1.11}$$

$$\mathbb{E}\exp[s(y(x,t) - y(0,t))] = \exp(x\Theta(t,s)), \tag{1.12}$$

*we have the functional identity*

$$\psi(t\Theta(t,s)) + \Theta(t,s) = s. \tag{1.13}$$

*Remark.* The requirement $\mathbb{E}\xi(1) \geq 0$ can be relaxed, with minor modifications to the theorem, in light of the following elementary fact. Suppose that $\xi^1(x)$ and $\xi^2(x)$ are two different initial conditions for Burgers' equation, which are related by $\xi^2(x) = \xi^1(x) + cx$. It is easily checked using (1.5) that the corresponding solutions $\rho^1(x,t)$ and $\rho^2(x,t)$ are related for $t > 0$ by

$$\rho^2(x,t) = \frac{1}{1+ct}\left[\rho^1\left(\frac{x}{1+ct}, \frac{t}{1+ct}\right) + cx\right]. \tag{1.14}$$

This observation is found in a paper of Menon and Pego [23], but (as this is elementary) it may have been known previously. Using (1.14) we can adjust a statistical description for a case where $\mathbb{E}\xi(1) \geq 0$ to cover the case of a Lévy process with general mean drift.

We find Theorem 1.1 remarkable for several reasons. First, in light of (1.6), it follows immediately that the solution $\rho(x,t) - \rho(0,t)$ is for each fixed $t$ a Lévy process in the parameter $x$, and we have an example of an infinite-dimensional, nonlinear dynamical system (the PDE, Burgers' equation) which preserves the independence and homogeneity properties of its random initial configuration. Second, the distributional characterization of $y(x,t)$ is that of a *first* passage process, where the definition of $y(x,t)$ following (1.5) is that of a *last* passage process. Third, (1.13) can be used to show [22] that if $\psi(t,q)$ is the Laplace exponent of $\rho(x,t) - \rho(0,t)$, then

$$\psi_t + \psi\psi_q = 0 \tag{1.15}$$

for $t > 0$ and $q \in \mathbb{C}$ with nonnegative real part. This shows for entropy solutions what had previously been observed by Carraro and Duchon for statistical solutions [9], namely that the Laplace exponent (1.12) evolves according to Burgers' equation!

In 2007 Menon and Pego [23] used the Lévy-Khintchine representation for the Laplace exponent (1.12) and observed that the evolution according to Burgers' equation in (1.15) corresponds to a Smoluchowski coagulation equation [2, 27],

with additive collision kernel, for the jump measure of the Lévy process $y(\cdot, t)$. The jumps of $y(\cdot, t)$ correspond to shocks in the solution $\rho(\cdot, t)$. The relative velocity of successive shocks can be written as a sum of two functions, one depending on the positions of the shocks and the other proportional to the sum of the sizes of the jumps in $y(\cdot, t)$. Regarding the sizes of the jumps as the usual masses in the Smoluchowski equation, it is plausible that Smoluchowski equation with additive kernel should be relevant, and [23] provides the details that verify this.

It is natural to wonder whether this evolution through Markov processes with simple statistical descriptions is a *miracle* [22] confined to the Burgers-Lévy case, or an instance of a more general phenomenon. However, extending the results of Bertoin [5] beyond the Burgers case $H(p) = -\frac{1}{2}p^2$ remains a challenge. A different particular case, corresponding to $L(q) = (-H)^*(q) = |q|$, is a problem of determining Lipschitz minorants, and has been investigated by Abramson and Evans [1]. From the PDE perspective this is not as natural, since $(-H)^*(q) = |q|$ corresponds to

$$-H(p) = +\infty\, \mathbf{1}(|p| > 1), \tag{1.16}$$

i.e. $H(p)$ takes the value 0 on $[-1, +1]$ and is equal to $+\infty$ elsewhere. So [1], while very interesting from a stochastic processes perspective, has a specialized structure which is rather different from those cases we will consider.

The biggest step toward understanding the problem for a wide class of $H$ is found in a 2010 paper of Menon and Srinivasan [24]. Here it is shown that when the initial condition $\xi$ is a *spectrally negative*[2] strong Markov process, the backward Lagrangian process $y(\cdot, t)$ and the solution $\rho(\cdot, t)$ remain Markov for fixed $t > 0$, the latter again being spectrally negative. The argument is adapted from that of [5] and both [5, 24] use the notion of splitting times (due to Getoor [14]) to verify the Markov property according to its bare definition. In the Burgers-Lévy case, the independence and homogeneity of the increments can be shown to survive, from which additional regularity is immediate using standard results about Lévy processes [18]. As [24] points out, without these properties it is not clear whether a Feller process initial condition leads to a Feller process in $x$ at later times. Nonetheless, [24] presents a very interesting conjecture for the evolution of the generator of $\rho(\cdot, t)$, which has a remarkably nice form and follows from multiple (nonrigorous, but persuasive) calculations.

We now give a partial statement of this conjecture. The generator $\mathcal{A}$ of a stationary, spectrally negative Feller process acts on test functions $J \in C_c^\infty(\mathbb{R})$ by

$$(\mathcal{A}J)(y) = b(y)J'(y) + \int_{-\infty}^{y} (J(z) - J(y))\, f(y, dz) \tag{1.17}$$

where $b(y)$ characterizes the drift and $f(y, \cdot)$ describes the law of the jumps. If we allow $b$ and $f$ to depend on $t$, we have a family of generators. The conjecture of [24] is that the evolution of the generator $\mathcal{A}$ for $\rho(\cdot, t)$ is given by the Lax equation

$$\dot{\mathcal{A}} = [\mathcal{A}, \mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A} \tag{1.18}$$

for $\mathcal{B}$ which acts on test functions $J$ by

$$(\mathcal{B}J)(y) = -H'(y)b(t, y)J'(y) - \int_{-\infty}^{y} \frac{H(y) - H(z)}{y - z}(J(z) - J(y))\, f(t, y, dz). \tag{1.19}$$

[2] i.e. without positive jumps

An equivalent form of the conjecture (1.18) involves a kinetic equation for $f$. The key result in the present article verifies that this kinetic equation holds in the special case we will consider. Before we state this, let us establish our working notation.

1.2. **Notation.** Here we collect some of the various notation used later in the article.

Write $H[p_1, \dots, p_n]$ for the $n^{\text{th}}$ divided difference of $H$ through $p_1, \dots, p_n$. For each $n$ this function is symmetric in its arguments, and given by the following formulas in the cases $n = 2$ and $n = 3$ where $p_1, p_2, p_3$ are distinct:

$$H[p_1, p_2] = \frac{H(p_2) - H(p_1)}{p_2 - p_1} \quad \text{and} \quad H[p_1, p_2, p_3] = \frac{H[p_2, p_3] - H[p_1, p_2]}{p_3 - p_1}. \tag{1.20}$$

The definition for general $n$ and standard properties can be found in several numerical analysis texts, including [16].

We write $\delta_x$ for the usual point mass assigning unit mass to $x$ and zero elsewhere.

For $C > 0$ and $n$ a positive integer, write

$$\Delta_n^C = \{(a_1, \dots, a_n) \in \mathbb{R}^n : 0 < a_1 < \cdots < a_n < C\} \tag{1.21}$$

and $\overline{\Delta_n^C}$ for the closure of this set in $\mathbb{R}^n$. We write $\partial\Delta_n^C$ for the boundary of the simplex, and $\partial_i\Delta_n^C$ for its various faces:

$$\begin{aligned} \partial_0\Delta_n^C &= \{(a_1, \dots, a_n) \in \partial\Delta_n^C : a_1 = 0\} \\ \partial_i\Delta_n^C &= \{(a_1, \dots, a_n) \in \partial\Delta_n^C : a_i = a_{i+1}\}, \qquad i = 1, \dots, n-1 \\ \partial_n\Delta_n^C &= \{(a_1, \dots, a_n) \in \partial\Delta_n^C : a_n = C\}. \end{aligned} \tag{1.22}$$

Write $\mathcal{M}$ for the set of finite, regular (signed) measures on $[0, P]$, which is a Banach space when equipped with the total variation norm $\|\cdot\|_{\text{TV}}$. Call its nonnegative subset $\mathcal{M}_+$.

Let $\mathcal{K}$ denote the set of bounded signed kernels from $[0, P]$ to $[0, P]$, i.e. bounded Borel-measurable mappings $[0, P] \to \mathcal{M}$. $\mathcal{K}$ is a Banach space when equipped with the norm on $k = k(\rho_-, d\rho_+)$ given by

$$\|k\| = \sup\{\|k(\rho_-, \cdot)\|_{\text{TV}} : \rho_- \in [0, P]\}. \tag{1.23}$$

Write $\mathcal{K}_+$ for the subset of $\mathcal{K}$ with range contained in $\mathcal{M}_+$.

1.3. **Main result.** In this section we provide a statistical description of solutions to the scalar conservation law when the initial condition is an increasing, pure-jump Markov process given by a rate kernel $g$. For this we require some assumptions on the rate kernel $g$ and the Hamiltonian $H$.

**Assumption 1.2.** The initial condition $\xi = \xi(x)$ is a pure-jump Markov process starting at $\xi(0) = 0$ and evolving for $x > 0$ according to a rate kernel $g(\rho_-, d\rho_+)$. We assume that for some constant $P > 0$ the kernel $g$ is supported on

$$\{(\rho_-, \rho_+) : 0 \le \rho_- \le \rho_+ \le P\} \tag{1.24}$$

and has total rate which is constant in $\rho_-$:

$$\lambda = \int g(\rho_-, d\rho_+) \tag{1.25}$$

for all $0 \le \rho_- \le P$.

**Assumption 1.3.** The Hamiltonian function $H : [0, P] \to \mathbb{R}$ is smooth, convex, has nonnegative right-derivative at $p = 0$ and noninfinite left-derivative at $p = P$.

Our statistical description consists of a one-dimensional marginal and a rate kernel generating the rest of the path. The evolution of the rate kernel is given by the following kinetic equation, and the evolution of the marginal will be described in terms of the solution to the kinetic equation.

**Theorem 1.4.** *Suppose Assumptions 1.2 and 1.3 hold. Then there are unique positive kernels*

$$f : [0, \infty) \to \mathcal{K}_+ \tag{1.26}$$

$$\ell : [0, \infty) \to \mathcal{M}_+ \tag{1.27}$$

*satisfying the initial conditions*

$$f(0, \rho_-, d\rho_+) = g(\rho_-, d\rho_+) \tag{1.28}$$

$$\ell(t, d\rho_0) = \delta_0(d\rho_0) \tag{1.29}$$

*and solving, in an integral sense, the equations*

$$f_t = \mathcal{L}^\kappa f \qquad \text{and} \qquad \ell_t = \mathcal{L}^0 \ell, \tag{1.30}$$

*where the operators $\mathcal{L}^\kappa$ and $\mathcal{L}^0$ are as follows, with integration over $\rho_*$ only:*

$$\begin{aligned} &\mathcal{L}^\kappa f(t, \rho_-, d\rho_+) \\ &= \int (H[\rho_*, \rho_+] - H[\rho_-, \rho_*]) f(t, \rho_-, d\rho_*) f(t, \rho_*, d\rho_+) \\ &\quad - \left[ \int H[\rho_+, \rho_*] f(t, \rho_+, d\rho_*) - \int H[\rho_-, \rho_*] f(t, \rho_-, d\rho_*) \right] f(t, \rho_-, d\rho_+) \end{aligned} \tag{1.31}$$

*and*

$$\begin{aligned} \mathcal{L}^0 \ell(t, d\rho_0) = \int H[\rho_*, \rho_0] \ell(t, d\rho_*) f(t, \rho_*, d\rho_0) \\ - \left[ \int H[\rho_0, \rho_*] f(t, \rho_0, d\rho_*) \right] \ell(t, d\rho_0). \end{aligned} \tag{1.32}$$

*Furthermore, the total integrals are conserved:*

$$\lambda = \int f(t, \rho_-, d\rho_+) \tag{1.33}$$

$$1 = \int \ell(t, d\rho_0) \tag{1.34}$$

*for all $t \geq 0$ and all $0 \leq \rho_- \leq P$. Finally, if*

$$g(\rho_-, [0, \rho_-] \cup \{P\}) = 0 \tag{1.35}$$

*for all $0 \leq \rho_- < P$, then $f(t, \cdot, \cdot)$ has the same property for $t > 0$.*

The kernels described by the Theorem 1.4 are precisely what we need to describe the statistics of the solution $\rho$, which brings us to our main result:

**Theorem 1.5.** *When Assumptions 1.2 and 1.3 hold, the solution $\rho$ to*

$$\begin{cases} \rho_t = H(\rho)_x & (x, t) \in \mathbb{R} \times (0, \infty) \\ \rho = \xi & (x, t) \in \mathbb{R} \times \{t = 0\} \end{cases} \tag{1.36}$$

*for each fixed $t > 0$ has $x = 0$ marginal given by $\ell(t, d\rho_0)$ and for $0 < x < \infty$ evolves according to rate kernel $f(t, \rho_-, d\rho_+)$.*

1.4. **Organization.** The remainder of this article is organized as follows. In Section 2 we show that Theorem 1.5 will follow from a similar statistical characterization for the solution to the PDE over $x \in [0, L]$ with a time-dependent *random* boundary condition at $x = L$. The latter we can study using a sticky particle system whose dimension is random and unbounded, but almost surely finite. Elementary arguments are used to check that our candidate for the law matches the evolution of the random initial condition according to the dynamics. Next in Section 3 we show existence and uniqueness of the solutions to marginal and kinetic equations of Theorem 1.4, which are needed to construct the candidate law. The concluding Section 4 indicates some desired extensions and similar questions for future work. To keep the main development concise, proofs of selected lemmas have been deferred to Appendix A.

## 2. A random particle system

As functions of $x$, the solutions $\rho(x, t)$ we consider all have the form depicted in Figure 1. From a PDE perspective this situation is standard—a concatenation of Riemann problems for the scalar conservation law—and we can describe the solution completely in terms of a particle system. Each shock consists of some number of particles stuck together, and the particles move at constant velocities according to the Rankine-Hugoniot condition except when they collide. The collisions are totally inelastic.

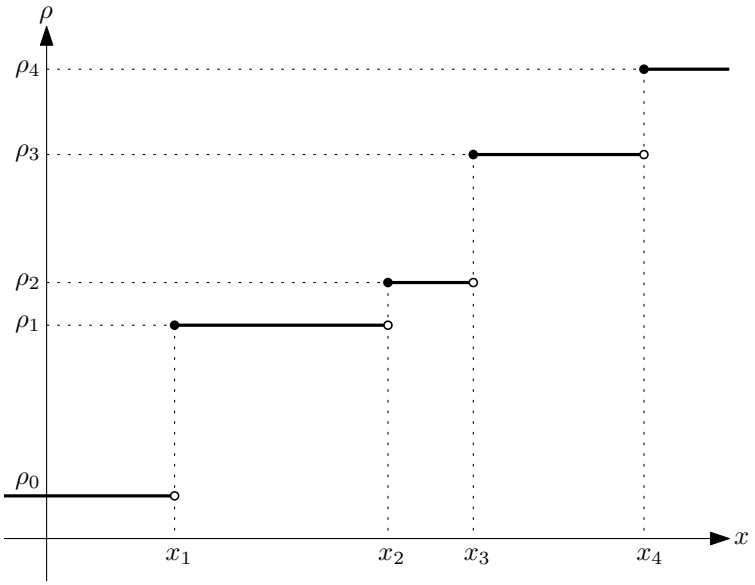


FIGURE 1. For each $t > 0$, the solution $\rho(x, t)$ is a nondecreasing, pure-jump process in $x$. We will see that for any fixed $L > 0$, we have a.s. finitely many jumps for $x \in [0, L]$ and that $\rho(\cdot, t)$ on this interval can be described by two (finite) nondecreasing sequences $(x_1, \ldots, x_N; \rho_0, \ldots, \rho_N)$.

The utility of this particle description is lessened, however, by the fact that the dynamics are quite infinite dimensional for our pure-jump initial condition $\xi(x)$, $x \in [0, +\infty)$. To have a simple description as motion at constant velocities, punctuated by *occasional* collisions, we might argue that on each fixed bounded interval of space we have finitely many collisions in a bounded interval of time, and piece together whatever statistical descriptions we might obtain for the solution on these various intervals. Inspired by those situations in statistical mechanics where boundary conditions become irrelevant in an infinite-volume limit, we pursue a different approach. We construct solutions to a problem on a bounded space interval $[0, L]$, and choose a random boundary condition at $x = L$ to obtain an exact match with the kinetic equations. The involved analysis will all pertain to the following result.

**Theorem 2.1.** *Suppose Assumptions 1.2 and 1.3 hold. For any fixed $L > 0$, consider the scalar conservation law*

$$\begin{cases} \rho_t = H(\rho)_x & (x,t) \in (0,L) \times (0,\infty) \\ \rho = \xi & x \in [0,L] \times \{t = 0\} \\ \rho = \zeta & (x,t) \in \{x = L\} \times (0,\infty) \end{cases} \tag{2.1}$$

*with initial condition $\xi$ (restricted to $[0, L]$), open boundary[3] at $x = 0$, and random boundary $\zeta$ at $x = L$. Suppose the process $\zeta$ has $\zeta(0) = \xi(L)$ and evolves according to the time-dependent rate kernel $H[\rho_,\rho_+]f(t, \rho_, d\rho_+)$ independently of $\xi$ given $\xi(L)$. Then for all $t > 0$ the law of $\rho(\cdot, t)$ is as follows:*

*(i) the $x = 0$ marginal is $\ell(t, d\rho_0)$, and*
*(ii) the rest of the path is a pure-jump process with rate kernel $f(t, \rho_-, d\rho_+)$.*

To prove our main result we can send $L \to \infty$, applying Theorem 2.1 on each $[0, L]$, and use bounded speed of propagation to limit the respective influences of far away particles (unbounded system) or truncation with random boundary (bounded system). The argument is quite short.

*Proof of Theorem 1.5.* Fix any $t > 0$. We will write $\hat{\rho}(x, t)$ for the solution to (1.36) over the semi-infinite $x$-interval $[0, +\infty)$ with initial data $\hat{\xi}(x)$. We take the right-continuous version of the solution. For $L$ to be specified shortly, write $\rho(x, t)$ for the solution and $\xi(x)$ for the initial data corresponding to (2.1).

Fix any $x_1, \dots, x_k \in [0, +\infty)$, and let $X = \max\{x_1, \dots, x_k\}$. Choose $L > X + tH'(P)$. We couple the bounded and unbounded systems by requiring

$$\hat{\xi}|_{[0,L]} = \xi, \tag{2.2}$$

allowing the random boundary $\zeta$ to evolve independently of $\hat{\xi}$ given $\hat{\xi}(L)$

Recall [13] that the scalar conservation law has finite speed of propagation. Our solutions are bounded in $[0, P]$, so the speed is bounded by $H'(P)$. Since $\hat{\rho}(x, 0)$ and $\rho(x, 0)$ are a.s. equal on $[0, L]$, we have also $\hat{\rho}(x, t) = \rho(x, t)$ a.s. for $x \in [0, L - tH'(P)] \supset [0, X]$. From this we deduce the distributional equality

$$(\hat{\rho}(x_1, t), \dots, \hat{\rho}(x_k, t)) \stackrel{d}{=} (\rho(x_1, t), \dots, \rho(x_k, t)). \tag{2.3}$$

[3]Using Assumption 1.3 and $\xi, \zeta \geq 0$, the shocks and characteristics only flow outward across $x = 0$. Any boundary condition we would assign, unless it involved negative values, would thus be irrelevant.

By Theorem 2.1, the latter distribution is exactly that of a process started according to $\ell(t, d\rho_0)$, evolving according to rate kernel $f(t, \rho_-, d\rho_+)$. This process is killed deterministically at $x = L$, which does not alter finite dimensional distributions prior to $x = L$. Since $\rho(x, t)$ is right-continuous and has the correct finite dimensional distributions, the result follows. □

2.1. **The dynamics.** Our work in the remainder of this section is to prove Theorem 2.1. We begin by describing precisely those particle dynamics which determine the solution to the PDE.

Figure 1 illustrates a parametrization of a nondecreasing pure-jump process on $[0, +\infty)$ with heights $\rho_0, \rho_1, \rho_2, \ldots$ and jump locations $x_1, x_2, \ldots$. Our sign restriction on the jumps excludes rarefaction waves, and we have only constant values separated by shocks. Going forward in time, the shocks move according to the Rankine-Hugoniot condition

$$\dot{x}_i = -H[\rho_{i-1}, \rho_i] \tag{2.4}$$

until they collide. We say that each shock consists initially of one *particle* moving at the velocity indicated above. Then result of a collision can be characterized in two equivalent ways:

(i) At the first instant when $x_i = x_{i+1}$, the $i^{\text{th}}$ particle is annihilated, and the velocity of the $(i+1)^{\text{th}}$ particle changes from $-H[\rho_i, \rho_{i+1}]$ to

$$\dot{x}_{i+1} = -H[\rho_{i-1}, \rho_{i+1}]. \tag{2.5}$$

In the case[4] where several consecutive particles collide with each other at the same instant, all but the rightmost particle is annihilated. Since we only seek a statistical description for $x \geq 0$, we annihilate the first particle when $x_1 = 0$, replace $\rho_0$ with $\rho_1$, and relabel the other particles accordingly.

(ii) When $x_{i-1} < x_i = x_{i+1} = \cdots = x_{j-1} = x_j < x_{j+1}$, the particles $i$ through $j$ all move with common velocity $-H[\rho_{i-1}, \rho_j]$. Following Brenier et al [6, 7] we call these a *sticky particle dynamics*. We can additionally take the position $x = 0$ to be absorbing: $\dot{x}_i = 0$ whenever $x_i = 0$.

We adopt the first viewpoint for convenience, as this is compatible with [12] (see the text following Definition 2.3 below), but a suitable argument could be given for the second alternative as well.

**Definition 2.2.** For $L$ as in Theorem 2.1, the configuration space $Q$ for the sticky particle dynamics is

$$Q = \bigsqcup_{n=0}^{\infty} Q_n, \qquad Q_n = \Delta_n^L \times \overline{\Delta_{n+1}^P}. \tag{2.6}$$

A typical configuration is $q = (x_1, \ldots, x_n; \rho_0, \ldots, \rho_n) \in Q_n$ when $n > 0$, or $q = (\rho_0) \in Q_0 = \{\rho_0 : \ 0 \leq \rho_0 \leq P\}$ when $n = 0$.

**Definition 2.3.** Our notation for the particle dynamics is as follows:

(i) For $0 \leq s \leq t$ and $q \in Q$, write

$$\phi_s^t q = \phi_0^{t-s} q \tag{2.7}$$

[4] which will almost surely not occur for the randomness we consider

for the deterministic evolution from time $s$ to $t$ of the configuration $q$ according to the annihilating particle dynamics for the PDE, *without* random entry dynamics at $x = L$.

(ii) Given a configuration $q = (x_1, \dots, x_n, \rho_0, \dots, \rho_n)$ and $\rho_+ > \rho_n$, write $\epsilon_{\rho_+} q$ for the configuration $(x_1, \dots, x_n, L, \rho_0, \dots, \rho_n, \rho_+)$.

(iii) Write $\Phi_s^t q$ for the *random* evolution of the configuration according to deterministic particle dynamics interrupted with random entries at $x = L$ according to the boundary process $\zeta$ of (2.1), where the latter has been started at time $s$ with value $\rho_n$. In particular, if the jumps of $\zeta$ between times $s$ and $t$ occur at times $s < \tau_1 < \cdots < \tau_k < t$ with values $\rho_{n+1} < \cdots < \rho_{n+k}$, then

$$\Phi_s^t q = \phi_{\tau_k}^t \epsilon_{\rho_{n+k}} \phi_{\tau_{k-1}}^{\tau_k} \epsilon_{\rho_{n+k-1}} \cdots \phi_{\tau_1}^{\tau_2} \epsilon_{\rho_{n+1}} \phi_s^{\tau_1} q. \tag{2.8}$$

**Proposition 2.4.** *For any $s, q$, the process $\Phi_s^t q$ is strong Markov.*

This assertion follows after recognizing $\Phi_s^t q$ as a piecewise-determisitic Markov process described in some generality by Davis [12]. Namely, we augment the configuration space $Q$ to add the time parameter to each component $Q_n$, and then have a deterministic flow according to the vector field

$$(1; -H[\rho_0, \rho_1], \dots, -H[\rho_{n-1}, \rho_n]; 0, \dots, 0). \tag{2.9}$$

With rate $\int H[\rho_n, \rho_+] f(t, \rho_n, d\rho_+)$ we jump to the indicated point in $Q_{n+1}$, and upon hitting a boundary we transition to $Q_k$ for suitable $k < n$, annihilating particles in the manner described above.

2.2. **Checking the candidate measure.** Our goal is to take $q$ distributed according to the initial condition and exactly describe the law of $\Phi_0^T q$ for each $T > 0$. Using the kinetic (1.31) and marginal equations (1.32), we construct for each time $t \geq 0$ a candidate law $\mu(t, dq)$ on $Q$ as follows. Take $N$ to be Poisson with rate $\lambda L$, $x_1, \dots, x_N$ uniform on $\Delta_N^L$, and $\rho_0, \dots, \rho_N$ distributed on $\overline{\Delta_{N+1}^P}$ according to the marginal $\ell$ and transitions $f$ independently of the $x_i$:

$$e^{-\lambda L} \sum_{n=0}^{\infty} \delta_n(dN) \mu_n(t, dq), \tag{2.10}$$

where $\mu_0(t, dq) = \mathbf{1}_{\rho \in [0,P]}\ \ell(t, d\rho_0)$, and

$$\mu_n(t, dq) = \mathbf{1}_{\Delta_n^L}(x_1, \dots, x_n)\, dx_1 \cdots dx_n\, \ell(t, d\rho_0) \prod_{j=1}^{n} f(t, \rho_{j-1}, d\rho_j), \tag{2.11}$$

for $n > 0$. When we have verified (1.33), it will be immediate that the total mass of this is one.

We decompose the mapping from configurations $q$ to solutions (as functions of $x$ over $[0, L]$) into the map from $q$ to the measure

$$\pi(q, dx) = \rho_0 \delta_0 + \sum_{i=1}^{n} (\rho_i - \rho_{i-1}) \delta_{x_i} \qquad (q \in Q_n) \tag{2.12}$$

and integration over $[0, x]$. Our claim is that when $q$ is distributed according to $\mu(0, dq)$, the law of the measure $\pi(\Phi_0^t q, \cdot)$ is identical to that of $\pi(q', \cdot)$ where $q'$ is distributed according to $\mu(t, dq')$.

We now describe the structure of the proof of Theorem 2.1. Fix some time $T > 0$ and consider $F(t,q) = \mathbb{E}G(\Phi_t^T q)$ where $G$ takes the form of a Laplace functional:

$$G(q) = \exp\left(-\int J(x)\,\pi(q,dx)\right) = \exp\left(-\rho_0 J(0) - \sum_{i=1}^{n}(\rho_i - \rho_{i-1})J(x_i)\right) \quad (2.13)$$

for $J \geq 0$ a continuous function on $[0,L]$. We aim to show that

$$\frac{d}{dt}\int \mathbb{E}G(\Phi_t^T q)\mu(t,dq) = 0 \quad (2.14)$$

for $0 < t < T$, from which it will follow that

$$\int \mathbb{E}G(\Phi_0^T q)\,\mu(0,dq) = \int G(q)\,\mu(T,dq) \quad (2.15)$$

for all $G$ of the form in (2.13). Using the standard fact that Laplace functionals completely determine the law of any random measure [18], this will suffice to show that law of $\pi(\Phi_0^T q, dx)$ for $q$ distributed as $\mu(0,dq)$ is precisely the pushforward through $\pi$ of $\mu(T,dq)$, and obtain the result.

We might verify (2.14) by establishing regularity for $F(t,q) = \mathbb{E}G(\Phi_t^T q)$ in $t$ and the $x$-components of $q$. We speculate that it should be possible to do so if $J$ is smooth with $J'(0) = 0$ and we content ourselves to divide the configuration space into finitely many regions, each of which corresponds to a definite order of deterministic collisions in $\phi$. On the other hand, the measures $\mu(t,dq)$ enjoy considerable regularity (uniformity, in fact) in $x$, and we pursue instead an argument along these lines. Here continuity of $F(t,q)$ in $t$ will suffice.

**Lemma 2.5.** *Let $G$ take the form of (2.13). Then $F(t,q) = \mathbb{E}G(\Phi_t^T q)$ is a bounded function which is uniformly continuous in $t$ uniformly in $q$:*

$$w(\delta) = \sup\{|F(t,q) - F(s,q)| \,:\, s,t \in [0,T],\, |t-s| < \delta,\, q \in Q\} \to 0 \quad (2.16)$$

*as $\delta \to 0+$.*

To differentiate in (2.14) we will need to compare $\mathbb{E}G(\Phi_s^T q)$ and $\mathbb{E}G(\Phi_t^T q)$ for $0 < s < t < T$. Our next observation is that when the $t - s$ is small, we can separate the deterministic and stochastic portions of the dynamics over the time interval $[s,t]$. The idea is that in an interval of time $[s,t]$, the probability of multiple particle entries is $O((t-s)^2)$, and a single particle entry has probability $O(t-s)$ which permits additional $o(1)$ errors.

**Lemma 2.6.** *Let $0 < s < t \leq T$ and $q \in Q_n$. There exist a random variable $\tau \in (s,t)$ a.s., with law depending on $s,t$, and $q$ only through $\rho_n$, and a constant $C_{\lambda,H'(P)}$ independent of $q,s,t$ so that*

$$\left|F(s,q) - F(t,\phi_s^t q) - (t-s)\int [\mathbb{E}F(t,\epsilon_{\rho_+}\phi_s^\tau q) - F(t,q)]H[\rho_n,\rho_+]f(t,\rho_n,d\rho_+)\right| \leq C_{\lambda,H'(P)}[(t-s)^2 + (t-s)w(t-s)]. \quad (2.17)$$

We proceed to an analysis of the deterministic portion of the flow, $\phi_s^t q$, where $q$ is distributed according to $\mu(t,dq)$. For this we introduce some notation. Given $q \in Q_n$, we write

$$\sigma(q) = \min\{r \geq s \,:\, \phi_s^r q \in (\partial\Delta_n^L) \times \overline{\Delta_{n+1}^P}\}. \quad (2.18)$$

For each fixed $n$, we consider several subsets of the set of $n$-particle configurations $Q_n$. In particular we need to separate those configurations which experience an exit at $x = 0$ or a collision in a time interval shorter than $t-s$. For $i = 0, \dots, n-1$, write:

$$\begin{aligned} A_i &= \{q \in Q_n : \sigma(q) \le t \text{ and } \phi_s^{\sigma(q)} q \in (\partial_i \Delta_n^L) \times \overline{\Delta_{n+1}^P}\} \\ U &= Q_n \setminus \bigcup_{i=0}^{n-1} A_i \\ B &= \{q \in Q_n : x_n \ge L - (t-s)H[\rho_{n-1}, \rho_n]\} \\ V &= Q_n \setminus B. \end{aligned} \tag{2.19}$$

Figure 2 illustrates these sets in the case $n = 2$. We observe in particular that $\phi_s^t U$ and $V$ differ by a $\mu_n(t,\cdot)$-null set, and that the terms associated with the boundary faces are related to configurations with one fewer particle.

**Lemma 2.7.** *For each positive integer $n$ and with errors bounded by*

$$C_{L,n,\lambda,H'(P)}[(t-s) + w(t-s)] \tag{2.20}$$

*we have the following approximations:*

$$\begin{aligned} &(t-s)^{-1} \int F(t, \phi_s^t q) \mathbf{1}_{A_0}(q) \mu_n(t, dq) \\ &\quad \approx \int F(t,q) \frac{H[\rho_*, \rho_0] \ell(t, d\rho_*) f(t, \rho_*, d\rho_0)}{\ell(t, d\rho_0)} \mu_{n-1}(t, dq), \end{aligned} \tag{2.21}$$

$$\begin{aligned} &(t-s)^{-1} \int F(t, \phi_s^t q) \mathbf{1}_{A_i}(q) \mu_n(t, dq) \\ &\quad \approx \int F(t,q) \frac{(H[\rho_*, \rho_i] - H[\rho_{i-1}, \rho_*]) f(t, \rho_{i-1}, d\rho_*) f(t, \rho_*, d\rho_i)}{f(t, \rho_{i-1}, d\rho_i)} \mu_{n-1}(t, dq) \end{aligned} \tag{2.22}$$

*for $i = 1, \dots, n-1$, and*

$$\begin{aligned} &(t-s)^{-1} \int F(t, \phi_s^t q) \mathbf{1}_{B}(q) \mu_n(t, dq) \\ &\quad \approx \int F(t, \epsilon_{\rho_+} q) H[\rho_{n-1}, \rho_+] f(t, \rho_{n-1}, d\rho_+) \mu_{n-1}(t, dq). \end{aligned} \tag{2.23}$$

*Remark.* It is *essential* that the integral over $B$ can be approximated by an integral over $(x_1, \dots, x_{n-1}, L; \rho_0, \dots, \rho_{n-1}, \rho_+)$. The measure $\mathcal{L}^* \mu$ below does not have any singular factors like $\delta(x_n = L)$. In particular, the result of integrating over $B$ will partially cancel with the term arising from random particle entries.

The final ingredient for the proof of Theorem 2.1 is the time derivative of $\mu(t, dq)$, which we now record.

**Lemma 2.8.** *For any $n \ge 0$ and any $0 \le s < t$ we have*

$$\|\mu_n(t,\cdot) - \mu_n(s,\cdot) - (t-s)(\mathcal{L}^* \mu_n)(t,\cdot)\|_{\mathrm{TV}} = o(t-s) \tag{2.24}$$

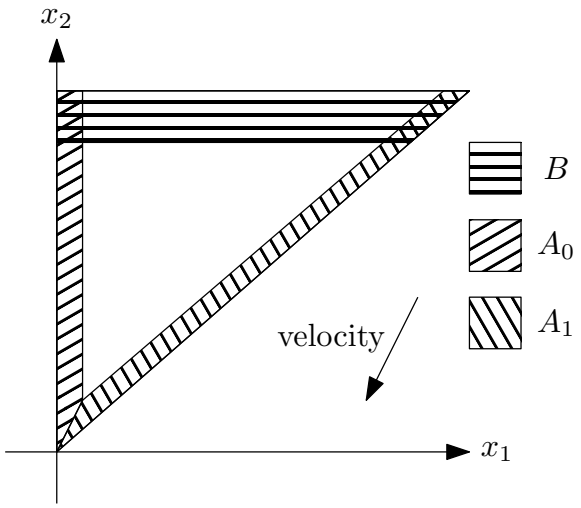


FIGURE 2. The deterministic flow $\phi$ on the $x$-simplex is translation at constant velocity unless this translation crosses a boundary of $\Delta_n^L$. In the case $n = 2$ pictured above, points in $A_0$ and $A_1$ hit the boundary faces $x_1 = 0$ and $x_1 = x_2$, respectively. The remaining portion of the simplex is mapped to the simplex minus the set $B$.

*where the norm is total variation and* $(\mathcal{L}^*\mu_n)(t, dq)$ *is defined to be the signed kernel*

$$\left[\frac{(\mathcal{L}^0\ell)(t,d\rho_0)}{\ell(t,d\rho_0)} + \sum_{i=1}^{n} \frac{(\mathcal{L}^\kappa f)(t,\rho_{i-1},d\rho_i)}{f(t,\rho_{i-1},d\rho_i)}\right] \times e^{-\lambda L}\mathbf{1}_{\Delta_n^L}(x_1,\dots,x_n)\,dx_1\cdots dx_n\,\ell(t,d\rho_0)\prod_{j=1}^{n} f(t,\rho_{j-1},d\rho_j). \quad (2.25)$$

The expression for the measure above is to be understood formally; the correct interpretation involves replacement, not division. All of the "divisors" above are present as factors of $\ell(t, d\rho_0)\prod_{j=1}^n f(t, \rho_{j-1}, d\rho_j)$, and the fractions indicate that the appearance of the denominator in this portion is to be replaced with the indicated numerator.

So that we know what to expect, before proceeding we note that when we sum over $i$ in (2.25), some of the terms arising from $\mathcal{L}^0$ and $\mathcal{L}^\kappa$ cancel. Namely, the bracketed portion of (2.25) expands as

$$\begin{aligned}&\frac{\int H[\rho_*,\rho_0]\ell(t,d\rho_*)f(t,\rho_*,d\rho_0)}{\ell(t,d\rho_0)} - \int H[\rho_0,\rho_*]f(t,\rho_0,d\rho_*)\\ &+\sum_{i=1}^{n}\left[\frac{\int(H[\rho_*,\rho_i]-H[\rho_{i-1},\rho_*])f(t,\rho_{i-1},d\rho_*)f(t,\rho_*,d\rho_i)}{f(t,\rho_{i-1},d\rho_i)}\right.\\ &\qquad\left.-\int(H[\rho_i,\rho_*]f(t,\rho_i,d\rho_*)-H[\rho_{i-1},\rho_*]f(t,\rho_{i-1},d\rho_*))\right]\end{aligned} \quad (2.26)$$

The gain terms associated with the kinetic equations we leave as they are, but note that the "loss" terms telescope, and the above may be shortened to

$$\frac{\int H[\rho_*,\rho_0]\ell(t,d\rho_*)f(t,\rho_*,d\rho_0)}{\ell(t,d\rho_0)} - \int H[\rho_n,\rho_*]f(t,\rho_n,d\rho_*) + \sum_{i=1}^{n}\frac{\int (H[\rho_*,\rho_i]-H[\rho_{i-1},\rho_*])f(t,\rho_{i-1},d\rho_*)f(t,\rho_*,d\rho_i)}{f(t,\rho_{i-1},d\rho_i)}. \quad (2.27)$$

We are ready to prove our statistical characterization of the bounded system.

*Proof of Theorem 2.1.* Consider $\int F(t,q)\,\mu(t,dq) - \int F(s,q)\,\mu(s,dq)$ for times $s$ and $t$ with $0 < s < t \le T$:

$$\int [F(t,q)-F(s,q)]\,\mu(t,dq) + \int F(t,q)[\mu(t,dq)-\mu(s,dq)] - \int [F(t,q)-F(s,q)][\mu(t,dq)-\mu(s,dq)] = \text{(I)} + \text{(II)} + \text{(III)} \quad (2.28)$$

Since $F(t,q)$ is uniformly continuous in $t$ uniformly in $q$ by Lemma 2.5 and $\mu$ is a probability measure, (I) $\to 0$ as $t-s \to 0$. Using Lemma 2.8 and the fact that $|F| \le 1$, (II) $\to 0$ as $t-s \to 0$, and in fact

$$\int F(t,q)\frac{\mu(t,dq)-\mu(s,dq)}{t-s} \to \int F(t,q)(\mathcal{L}^*\mu)(t,dq) \quad (2.29)$$

as $s \to t-$ with $t$ fixed. Using both Lemmas 2.5 and 2.8, we see (III) is $o(t-s)$. Thus $\int F(t,q)\,\mu(t,dq)$ is continuous in $t$; we will show additionally that it is differentiable from below in $t$ with one-sided derivative equal to 0 for all $0 < t < T$. In light of (2.29), our task is to show that $-\int F(t,q)\,(\mathcal{L}^*\mu)(t,dq)$ approximates (I) up to an $o(t-s)$ error.

Using Lemma 2.6 we have the following approximation of the portion of (I) involving $\mu_n$, with error bounded by $C_{\lambda,H'(P)}[(t-s)^2 + (t-s)w(t-s)]$:

$$\int \Bigg[F(t,q)(\mathbf{1}_B(q)+\mathbf{1}_V(q)) - F(t,\phi_s^t q)\left(\mathbf{1}_U(q) + \sum_{i=0}^{n-1}\mathbf{1}_{A_i}(q)\right) + (t-s)F(t,q)\int H[\rho_n,\rho_+]f(t,\rho_n,d\rho_+) - (t-s)\int \mathbb{E}F(t,\epsilon_{\rho_+}\phi_s^\tau q)H[\rho_n,\rho_+]f(t,\rho_n,d\rho_+)\Bigg]\mu_n(t,dq). \quad (2.30)$$

We have $\int [F(t,q)\mathbf{1}_V(q) - F(t,\phi_s^t q)\mathbf{1}_U(q)]\,\mu_n(t,dq) = 0$ because the deterministic flow $\phi_s^t$ maps $U$ to $V$ (modulo lower-dimensional sets) and preserve the Lebesgue measure in spatial coordinates. Making replacements using Lemma 2.7 and reordering the terms, we find with an error bounded by $C_{L,n,\lambda,H'(P)}[(t-s)+w(t-s)]$

that

$$\begin{aligned}
&\int \frac{F(t,q)-F(s,q)}{t-s}\mu_n(t,dq)\\
&\approx \int F(t,\epsilon_{\rho_+}q)H[\rho_{n-1},\rho_+]f(t,\rho_{n-1},d\rho_+)\mu_{n-1}(t,dq)\\
&\quad - \int \mathbb{E}F(t,\epsilon_{\rho_+}\phi_s^\tau q)H[\rho_n,\rho_+]f(t,\rho_n,d\rho_+)\mu_n(t,dq)\\
&\quad - \Bigg[\int F(t,q)\left(\sum_{i=1}^{n-1}\frac{(H[\rho_*,\rho_i]-H[\rho_{i-1},\rho_*])f(t,\rho_{i-1},d\rho_*)f(t,\rho_*,d\rho_i)}{f(t,\rho_{i-1},d\rho_i)}\right.\\
&\qquad\qquad \left.+\frac{H[\rho_*,\rho_0]\ell(t,d\rho_*)f(t,\rho_*,d\rho_0)}{\ell(t,d\rho_0)}\right)\mu_{n-1}(t,dq)\\
&\qquad\qquad - \int F(t,q)\int H[\rho_n,\rho_+]f(t,\rho_n,d\rho_+)\mu_n(t,dq)\Bigg]
\end{aligned} \tag{2.31}$$

for positive integers $n$. In the case $n=0$, we have $\phi_s^t q = q$, and the approximation is

$$\int \frac{F(t,q)-F(s,q)}{t-s}\mu_0(t,dq) \approx \int [F(t,\epsilon_{\rho_*}q)-F(t,q)]H[\rho_0,\rho_*]\mu_0(t,dq). \tag{2.32}$$

We observe that the bracketed portion of (2.31) nearly matches (2.27), except that part involves $\mu_{n-1}$ and part involves $\mu_n$. Furthermore, we have

$$\begin{aligned}
&\left|\int \mathbb{E}F(t,\epsilon_{\rho_+}\phi_s^\tau q)H[\rho_n,\rho_+]f(t,\rho_n,d\rho_+)\mu_n(t,dq)\right.\\
&\qquad \left.- \int F(t,\epsilon_{\rho_+}q)H[\rho_n,\rho_+]f(t,\rho_n,d\rho_+)\mu_n(t,dq)\right| \le C_{H'(P),n,L}(t-s).
\end{aligned} \tag{2.33}$$

This follows since $\tau$ is conditionally independent of $q$ given $\rho_n$, and $\tau \in (s,t)$ a.s. so that all but $O(t-s)$ volume of the $x$-simplex is simply translated by $\phi_s^\tau$ to a region of identical volume. We make the replacement indicated by (2.33) in (2.31) without changing the form of the error.

For any positive integer $N$ we define

$$\Gamma_N(t) = \int F(t,q)\sum_{n=0}^{N}\mu_n(t,dq). \tag{2.34}$$

Summing (2.31), (2.32), and our approximation for $\mu_n(t,\cdot)-\mu_n(s,\cdot)$ from (2.27) gives

$$\begin{aligned}
&\frac{\Gamma_N(t)-\Gamma_N(s)}{t-s}\\
&\approx \int F(t,q)\left(\sum_{i=1}^{N}\frac{(H[\rho_*,\rho_i]-H[\rho_{i-1},\rho_*])f(t,\rho_{i-1},d\rho_*)f(t,\rho_*,d\rho_i)}{f(t,\rho_{i-1},d\rho_i)}\right.\\
&\qquad\qquad \left.+\frac{H[\rho_*,\rho_0]\ell(t,d\rho_*)f(t,\rho_*,d\rho_0)}{\ell(t,d\rho_0)}\right)\mu_N(t,dq)\\
&\quad - \int F(t,\epsilon_{\rho_+}q)H[\rho_N,\rho_+]f(t,\rho_N,d\rho_+)\mu_N(t,dq)
\end{aligned} \tag{2.35}$$

with an $o(t-s)$ error. Call the right side $\gamma_N(t)$, so that the left derivative of $\Gamma_N$ is $\gamma_N$. Since $\Gamma_N$ is a Lipschitz function, we deduce

$$\Gamma_N(T) - \Gamma_N(0) = \int_0^T \gamma_N(t)\, dt. \tag{2.36}$$

We have $\gamma_N(t)$ bounded uniformly in $t$ by

$$|\gamma_N(t)| \leq \frac{3H'(P)L^N\lambda^{N+1}}{(N-1)!}, \tag{2.37}$$

and thus $\gamma_N \to 0$ uniformly in $t$. It follows that

$$\Gamma_N(T) - \Gamma_N(0) \to 0 \tag{2.38}$$

as $N \to \infty$. Recognizing the limits of $\Gamma_N(T)$ and $\Gamma_0(T)$ as $\int G(q)\mu(T, dq)$ and $\int \mathbb{E}G(\Phi_0^T q)\mu(0, dq)$, respectively, we have verified (2.14) and completed the proof. □

Having shown the candidate measure $\mu(t, dq)$ constructed using the solutions $\ell(t, d\rho_0)$ and $f(t, \rho_-, d\rho_+)$ given by Theorem 1.4, our next task is to verify the latter, which we undertake in the next section.

## 3. The kinetic and marginal equation

The primary goal of the present section is to prove Theorem 1.4 concerning the kernel $f(t, \rho_-, d\rho_+)$ and marginal $\ell(t, d\rho_0)$, so that the candidate measure $\mu(t, dq)$ in the previous section is well-defined and has the properties required for the argument there.

While we introduce our notation, we also discuss the intuitive meaning of the terms of our kinetic equation, comparing with that of Menon and Srinivasan [24]. Let us write $\mathcal{L}^\kappa$ for the operator on $\mathcal{K}$ given by the right-hand side of (1.31),

$$\begin{aligned}(\mathcal{L}^\kappa k)(\rho_-, d\rho_+) = &\int (H[\rho_*, \rho_+] - H[\rho_-, \rho_*])k(\rho_-, d\rho_*)k(\rho_*, d\rho_+) \\ &- \left[\int H[\rho_+, \rho_*]k(\rho_+, d\rho_*) - \int H[\rho_-, \rho_*]k(\rho_-, d\rho_*)\right] k(\rho_-, d\rho_+),\end{aligned} \tag{3.1}$$

so that the kinetic equation is $f_t = \mathcal{L}^\kappa f$.

The first term, which we call the *gain* term, corresponds to the production of a shock connecting states $\rho_-$ and $\rho_+$ by means of collision of shocks connecting states $\rho_-, \rho_*$ and $\rho_*, \rho_+$. Such shocks have relative velocity given by

$$H[\rho_*, \rho_+] - H[\rho_-, \rho_*] = H[\rho_-, \rho_*, \rho_+](\rho_+ - \rho_-). \tag{3.2}$$

In the Burgers case, the second divided difference $H[\cdot, \cdot, \cdot]$ is constant, and the above is proportional to the sum of the increment $\rho_+ - \rho_-$, analogous to mass in the Smoluchowski equation.

The second line of (3.1) we call the "loss" term, though this need not be of definite sign. To better understand this, note that when we have proven Theorem 1.4, we will know that $f(t, \rho_-, d\rho_+)$ has total mass which is constant in $(t, \rho_-)$. In this case

the loss term may be rewritten equivalently as

$$\left[\int (H[\rho_+,\rho_*]-H[\rho_-,\rho_+])f(t,\rho_+,d\rho_*) \right. \\ \left. -\int (H[\rho_-,\rho_*]-H[\rho_-,\rho_+])f(t,\rho_-,d\rho_*)\right] f(t,\rho_-,d\rho_+), \quad (3.3)$$

which corresponds precisely with the kinetic equation of [24]. The meaning of the first line of (3.3) is clear: we lose a shock connecting states $\rho_-,\rho_+$ when a shock connecting $\rho_+,\rho_*$ collides with this, and the relative velocity is precisely $H[\rho_+,\rho_*]-H[\rho_-,\rho_+]$.

The second line of (3.3) is less easily understood. One would expect to find here a loss related to a shock connecting $\rho_-,\rho_+$ colliding with $\rho_*,\rho_-$ for $\rho_*<\rho_-$, particularly if we were viewing $f$ as a *jump density* as [24] views its corresponding $n(t,y,dz)$. Viewed as a *rate kernel*, it is not clear that $f$ should suffer such a loss: in some sense we must condition on being in state $\rho_-$ for $f(t,\rho_-,\cdot)$ to be relevant at all. At this point we cannot offer an intuitive kinetic reason for the second line of (3.3), but in light of the rigorous results of this article we can be assured that this is correct for our model.

We return to the task at hand, showing existence and uniqueness of $f(t,\rho_-,d\rho_+)$. The argument proceeds through an approximation scheme for $\exp(\lambda H'(P)t)f(t,\cdot,\cdot)$, where we can easily maintain positivity. To avoid endowing $\mathcal{K}$ with a weak topology, we show directly the approximations are Cauchy, rather than appealing to Arzela-Ascoli.

*Proof of Theorem 1.4, part I.* Write $c=\lambda H'(P)$ and consider for each positive integer $n$ the continuous paths $h^n:[0,\infty)\to\mathcal{K}$ defined by $h^n(0)=g$ for all $n$ and

$$\dot{h}^n(t)=e^{-cj/n}(\mathcal{L}^\kappa h^n)(j/n)+ch^n(j/n), \qquad t\in(j/n,(j+1)/n). \quad (3.4)$$

We claim the following properties of $h^n$:

(a) $h^n(t)\geq 0$ for all $t\geq 0$,
(b) for each $t\geq 0$ the total integral $\int h^n(t,\rho_-,d\rho_+)$ is constant in $\rho_-\in[0,P)$,
(c) for all $t\geq 0$

$$\|h^n(t)\|\leq\lambda(1+c/n)^{\lceil nt\rceil}\leq\lambda e^{c(t+1)}=:M(t), \quad (3.5)$$

and
(d) $h^n(t,\rho_-,[0,\rho_-]\cup\{P\})=0$ for all $t$ and all $\rho_-\in[0,P)$.

Since $h^n$ is piecewise-linear in $t$ and the properties above are preserved by convex combinations, it suffices to verify this at $t=j/n$ for all integers $j\geq 0$.

We proceed by induction. Abbreviate $h^n_j=h^n(j/n)$. The $j=0$ case holds by the hypotheses on $g$. Assume now that the claim holds for $j$, and consider $j+1$. We have

$$\begin{aligned} n(h^n_{j+1}-h^n_j)=e^{-cj/n}&\left[\int (H[\rho_*,\rho_+]-H[\rho_-,\rho_*])h^n_j(\rho_-,d\rho_*)h^n_j(\rho_*,d\rho_+)\right. \\ &+\left(\int H[\rho_-,\rho_*]h^n_j(\rho_-,d\rho_*)\right)h^n_j(\rho_-,d\rho_+) \\ &\left.-\left(\int H[\rho_+,\rho_*]h^n_j(\rho_+,d\rho_*)\right)h^n_j(\rho_-,d\rho_+)\right] \\ &+ch^n_j(\rho_-,d\rho_+). \end{aligned} \quad (3.6)$$

In particular, for any $\rho_+$ we have

$$\begin{aligned} c - e^{-cj/n} \int H[\rho_+, \rho_*] h_j^n(\rho_+, d\rho_*) &\geq c - e^{-cj/n} H'(P) \|h_j^n\| \\ &\geq c - e^{-cj/n} H'(P) \lambda (1 + c/n)^j \\ &= c \left[ 1 - e^{-cj/n} (1 + c/n)^j \right] \geq 0, \end{aligned} \tag{3.7}$$

using property (c) for case $j$; thus $n(h_{j+1}^n - h_j^n) \geq 0$. This and (a) $h_j^n \geq 0$ give $h_{j+1}^n \geq 0$, property (a) for case $j+1$. Furthermore, the total integral of the right-hand side of (3.6) is exactly

$$c \int h_j^n(\rho_-, d\rho_+) = c \|h_j^n\|, \tag{3.8}$$

since (b) implies the bracketed portion integrates to 0. Also using (b) for case $j$, the change in the total integral from $h_j^n$ to $h_{j+1}^n$ is independent of $\rho_-$, verifying (b) for case $j+1$. Using (c) for case $j$ we find

$$\|h_{j+1}^n\| \leq \|h_j^n\| + cn^{-1} \|h_j^n\| = (1 + c/n) \|h_j^n\| \leq \lambda (1 + c/n)^j, \tag{3.9}$$

so that (c) holds for case $j+1$. Finally, we observe that property (d) for $h_j^n$ implies

$$\mathcal{L}^\kappa h_j^n(\rho_-, [0, \rho_-] \cup \{P\}) = 0, \tag{3.10}$$

and thus (d) holds for case $j+1$.

We now consider the matter of convergence. Pairing $\mathcal{L}^\kappa$ with any $k_1, k_2 \in \mathcal{K}$ and measurable $J(\rho_+)$ with $|J| \leq 1$, we find that

$$\|\mathcal{L}^\kappa k_1 - \mathcal{L}^\kappa k_2\| \leq 3H'(P)(\|k_1\| + \|k_2\|) \|k_1 - k_2\|. \tag{3.11}$$

In particular $\mathcal{L}^\kappa$ is Lipschitz on bounded sets in $\mathcal{K}$. For brevity write

$$\mathcal{L}^h k(s, \rho_-, d\rho_+) = e^{-cs} \mathcal{L}^\kappa k(\rho_-, d\rho_+) + ck(\rho_-, d\rho_+). \tag{3.12}$$

We compute for any $m < n$ and $t \geq 0$

$$\begin{aligned} \|h^m(t) - h^n(t)\| \leq \int_0^t \|\dot{h}^m(s) - \mathcal{L}^h h^m(s)\| + \|\mathcal{L}^h h^m(s) - \mathcal{L}^h h^n(s)\| \\ + \|\mathcal{L}^h h^n(s) - \dot{h}^n(s)\| \, ds. \end{aligned} \tag{3.13}$$

Observe that

$$\begin{aligned} \|\dot{h}^n(s) - \mathcal{L}^h h^n(s)\| &= \|\mathcal{L}^h h^n(n^{-1} \lfloor ns \rfloor) - \mathcal{L}^h h^n(s)\| \\ &\leq \|e^{-cn^{-1} \lfloor ns \rfloor} \mathcal{L}^\kappa h^n(n^{-1} \lfloor ns \rfloor) - e^{-cs} \mathcal{L}^\kappa h^n(s)\| \\ &\quad + c \|h^n(n^{-1} \lfloor ns \rfloor) - h^n(s)\| \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} &\|e^{-cn^{-1} \lfloor ns \rfloor} \mathcal{L}^\kappa h^n(n^{-1} \lfloor ns \rfloor) - e^{-cs} \mathcal{L}^\kappa h^n(s)\| \\ &\quad \leq e^{-cs} \|\mathcal{L}^\kappa h^n(n^{-1} \lfloor ns \rfloor) - \mathcal{L}^\kappa h^n(s)\| \\ &\qquad + (e^{-cn^{-1} \lfloor ns \rfloor} - e^{-cs}) \|\mathcal{L}^\kappa h^n(n^{-1} \lfloor ns \rfloor)\| \\ &\quad \leq e^{-cs} 6H'(P) M(s) \|h^n(n^{-1} \lfloor ns \rfloor) - h^n(s)\| \\ &\qquad + e^{-cs} e^{cn^{-1}} n^{-1} 3H'(P) M(s)^2 \\ &\quad \leq 6ce^c \|h^n(n^{-1} \lfloor ns \rfloor) - h^n(s)\| + 3c\lambda e^{cs+3c} n^{-1}, \end{aligned} \tag{3.15}$$

We have also

$$\begin{aligned}\|h^n(n^{-1}\lfloor ns\rfloor) - h^n(s)\| &\le n^{-1}\|\mathcal{L}^h h^n(n^{-1}\lfloor ns\rfloor)\| \\ &\le n^{-1}(e^{-cs}3H'(P)M(s)^2 + M(s)) \\ &= n^{-1}(3\lambda c e^{cs+2c} + \lambda e^{c(s+1)}).\end{aligned} \tag{3.16}$$

Combining these, there is a constant $C_1$ so that

$$\|\dot{h}^n(s) - \mathcal{L}^h h^n(s)\| \le C_1 n^{-1} e^{cs} \tag{3.17}$$

for all positive integers $n$ and times $s \ge 0$.

The remaining term in the integrand of (3.13) is

$$\begin{aligned}\|\mathcal{L}^h h^m(s) - \mathcal{L}^h h^n(s)\| &\le e^{-cs}\|\mathcal{L}^\kappa h^m(s) - \mathcal{L}^\kappa h^n(s)\| + c\|h^m(s) - h^n(s)\| \\ &\le (e^{-cs}6H'(P)M(s) + c)\|h^m(s) - h^n(s)\| \\ &= (6e^c + c)\|h^m(s) - h^n(s)\| = C_2\|h^m(s) - h^n(s)\|.\end{aligned} \tag{3.18}$$

All together,

$$\|h^m(t) - h^n(t)\| \le \int_0^t C_2\|h^m(s) - h^n(s)\| + C_1(n^{-1} + m^{-1})e^{cs}\,ds, \tag{3.19}$$

and by Gronwall

$$\|h^m(t) - h^n(t)\| \le C_1(n^{-1} + m^{-1})e^{C_2 t}\frac{e^{ct} - 1}{c}. \tag{3.20}$$

From this we see that $h^n$ is Cauchy, hence convergent to a continuous $h : [0, \infty) \to \mathcal{K}$ satisfying

$$h(t) = g + \int_0^t [e^{-cs}(\mathcal{L}^\kappa h)(s) + ch(s)]\,ds \tag{3.21}$$

for all $t \ge 0$, having properties (a,b,d) above and $\|h(t)\| \le M(t)$.

We define $f(t) = e^{-ct}h(t)$, and find using (3.21) that $f$ solves $\dot{f} = \mathcal{L}^\kappa f$ with $f(0) = g$. Using the local Lipschitz property of $\mathcal{L}^\kappa$, the solution is unique. Properties (a,b,d) hold for $f$, and (b) in particular gives for each fixed $\rho_-$

$$\frac{d}{dt}\int f(t, \rho_-, d\rho_+) = \int (\mathcal{L}^\kappa f)(t, \rho_-, d\rho_+) = 0. \tag{3.22}$$

Thus $\int f(t, \rho_-, d\rho_+) = \lambda$ for all $t \ge 0$, $0 \le \rho_- < P$. □

We now turn to $\ell(t, d\rho_0)$, which is intended to serve as the marginal at $x = 0$ for our solution $\rho(x, t)$ for fixed $t > 0$. We define a time-dependent family of operators $\mathcal{L}^0$ acting on measures $\nu(d\rho_0)$ in $\mathcal{M}$ by

$$\begin{aligned}(\mathcal{L}^0\nu)(t, d\rho_0) = &\int H[\rho_*, \rho_0]\nu(d\rho_*)f(t, \rho_*, d\rho_0) \\ &- \left[\int H[\rho_0, \rho_*]f(t, \rho_0, d\rho_*)\right]\nu(d\rho_0).\end{aligned} \tag{3.23}$$

Again the integration is over $\rho_*$ only, and for each $t$ we have $(\mathcal{L}^0\nu)(t, \cdot) \in \mathcal{M}$. Our evolution equation for $\ell(t, d\rho_0)$ is the linear, time-inhomogeneous $\ell_t = \mathcal{L}^0\ell$.

*Proof of Theorem 1.4, part II.* Note that (3.23) is exactly the forward equation for a pure-jump Markov process evolving according to a time-varying rate kernel $H[\rho_0, \rho_+]f(t, \rho_0, d\rho_+)$, so we could obtain existence and uniqueness of solutions along these lines. We outline an argument similar to that employed for $f$, for the sake of completeness.

Define for each $n$ the continuous path $h^n : [0, \infty) \to \mathcal{M}$ with $h^n(0) = \delta_0$ and

$$\dot{h}^n(t) = e^{-cj/n}\mathcal{L}^0 h^n(j/n) + ch^n(j/n), \qquad t \in (j/n, (j+1)/n). \tag{3.24}$$

Proceeding as in the proof for $f$ we verify that for each $j$ the $h^n_j = h^n(j/n)$ are nonnegative and, since $\mathcal{L}^0 h^n(j/n)$ has total integral zero, $\|h^n_j\| \leq (1 + c/n)^{\lceil nt \rceil}$. Observing that $\mathcal{L}^0$ is linear and bounded uniformly over $t$,

$$\sup_t \|(\mathcal{L}^0\nu)(t, \cdot)\|_{\text{TV}} \leq 2\lambda H'(P)\|\nu\|_{\text{TV}}, \tag{3.25}$$

we easily show $h^n$ is Cauchy on bounded time intervals, with limit $h : [0, \infty) \to \mathcal{M}_+$ satisfying

$$h(t) = \delta_0 + \int_0^t [e^{-cs}(\mathcal{L}^0 h)(s) + ch(s)]\, ds. \tag{3.26}$$

Take $\ell(t) = e^{-ct}h(t)$ to find that $\ell(t) \geq 0$ solves

$$\ell(t) = \delta_0 + \int_0^t (\mathcal{L}^0\ell)(s)\, ds. \tag{3.27}$$

Uniqueness follows using (3.25). Finally, since $\mathcal{L}^0\nu$ as zero total integral for any $\nu$, we find that $\int \ell(t, d\rho_0) = 1$ for all $t$. □

We close this section with a remark concerning the kinetic equation $f_t = \mathcal{L}^\kappa f$ without the assumption that the initial kernel $g$ has bounded support $[0, P]$. When the initial condition $\xi(x)$ is unbounded, growing for example linearly in the case of quadratic $H$, the solution to the scalar conservation law $\rho_t = H(\rho_x)$ on the semi-infinite domain should blow up in finite time. We likewise expect that the solution $f$ to the kinetic equation will blow up, but control of certain moments prior to this blow up may allow us to run the argument of Theorem 2.1 with only superficial changes. At the moment we lack the sort of estimates one obtains in the Smoluchowski case (e.g. coming from a closed equation for the second moment), but we hope to revisit this in future work.

## 4. Conclusion

We review what has been accomplished: using an exact propagation of chaos calculation for a bounded system with a suitably selected random boundary condition, we have derived a complete description of the law of the solution $\rho(x, t)$ to the scalar conservation law $\rho_t = H(\rho)_x$ for $x, t > 0$ when $\rho(x, 0) = \xi(x)$ is a bounded, monotone, pure-jump Markov process with constant jump rate. Notably, we have recovered the Markov property of the solution and a statistical description of the shocks *simultaneously*. This may be regarded as both a strength of our present analysis and a shortcoming of our present understanding (a soft argument for the preservation of the Markov property in the particle system would be illuminating).

We emphasize how our approach using random dynamics on a bounded interval has made things considerably easier: by constructing a bounded system which has

*exactly* the right law, we are relieved of the burden of determining precisely how wrong the law would be with deterministic boundary.

Our result lends additional support to the conjecture of Menon and Srinivasan [24] and we hope that the sticky particle methods described herein will provide another approach, quite different from that of Bertoin [5], which might be adapted to resolve the full conjecture. One of the authors is attempting a similar particle approach in the non-monotone setting, and hopes to report on this in future work.

## Appendix A. Proofs of lemmas

*Proof of Lemma 2.5.* Boundedness of $G$ is obvious from its definition as a Laplace functional. For continuity, we begin by considering $G(\phi_0^t q)$ with $q$ fixed. As $t$ varies, $\phi_0^t q$ varies continuously, with $\rho$-values unchanging and $x$-values changing with rate bounded by $H'(P)$, *except* for finitely many times when collisions occur. Since $G(\phi_0^t q)$ depends on $\phi_0^t q$ via $\pi(\phi_0^t q, \cdot)$, the former varies continuously across these collision times. More concretely, because $J \in C([0, L])$, $J$ is uniformly continuous, and so

$$w_J(\delta) = \sup\{|J(x) - J(y)| : x, y \in [0, L], |x - y| \le \delta\} \tag{A.1}$$

has the property that $w_J(\delta) \to 0$ as $\delta \to 0+$. Using the above,

$$|\log G(\phi_0^t q) - \log G(q)| \le \sum_{i=1}^{n} (\rho_i - \rho_{i-1}) w_J(tH'(P)) \le P w(tH'(P)). \tag{A.2}$$

Since the exponential is uniformly continuous for nonpositive arguments, we find that $G(\phi_0^t q)$ is continuous at $t = 0$ uniformly in $q$. Write

$$w_G(\delta) = \sup\{|G(\phi_0^t q) - G(q)| : q \in Q, 0 \le t \le \delta\}, \tag{A.3}$$

and observe that $w_G(\delta) \to 0$ and $\delta \to 0+$.

Suppose now that $0 \le s < t \le T$; we compare $\mathbb{E}G(\Phi_s^T q)$ and $\mathbb{E}G(\phi_t^T q)$. Write $\theta = t - s$. We have $G(\Phi_s^T q) = G(\Phi_{T-\theta}^T \Phi_s^{T-\theta} q)$ a.s., coupling the random entry processes on matching intervals. Then

$$\mathbb{E}|G(\Phi_{T-\theta}^T \Phi_s^{T-\theta} q) - G(\phi_{T-\theta}^T \Phi_s^{T-\theta} q)| \le C_{\lambda, H'(P)} \theta, \tag{A.4}$$

because the random entries occur at a rate bounded by $\lambda H'(P)$ and $|G| \le 1$. Using the time homogeneity and continuity for the deterministic flow,

$$\mathbb{E}|G(\phi_{T-\theta}^T \Phi_s^{T-\theta} q) - G(\Phi_s^{T-\theta} q)| \le w_G(\theta). \tag{A.5}$$

So, at the cost of an error bounded by $C_{\lambda, H'(P)}\theta + w_G(\theta)$, we compare $\mathbb{E}G(\Phi_t^T q)$ with $\mathbb{E}G(\Phi_s^{T-\theta} q)$ instead. Now the configuration $q$ is flowed on time intervals of equal length, but with random entry rates that have been shifted in time.

The random entry rate is given by $H[\rho_n, \rho_+] f(r, \rho_n, d\rho_+)$ and $f$ is TV-continuous in $r$; we have

$$\begin{aligned} \|f(r, \rho_n, d\rho_+) - f(r - \theta, \rho_n, d\rho_+)\| &= \left\| \int_{r-\theta}^{r} (\mathcal{L}^\kappa f)(\tau, \rho_n, d\rho_+)\, d\tau \right\| \\ &\le C_{\lambda, H'(P)} \theta. \end{aligned} \tag{A.6}$$

Let us define three kernels for $r \in [0, T-t]$ according to

$$\begin{aligned} \hat{f}(r, \rho_-, d\rho_+) &= f(s+r, \rho_-, d\rho_+) \wedge f(t+r, \rho_-, d\rho_+) \\ f^s(r, \rho_-, d\rho_+) &= f(s+r, \rho_-, d\rho_+) - \hat{f}(r, \rho_-, d\rho_+) \\ f^t(r, \rho_-, d\rho_+) &= f(t+r, \rho_-, d\rho_+) - \hat{f}(r, \rho_-, d\rho_+), \end{aligned} \tag{A.7}$$

where the minimum ($\wedge$) of two measures is defined as usual by choosing a third measure with which the former two are absolutely continuous, and taking the pointwise minimum of their Radon-Nikodym derivatives. That this extends measurably to the parametric case (i.e. involving kernels) is immediate from a parametric version of the Radon-Nikodym theorem [25, Theorem 2.3].

Using the above we construct a coupled random entry process for times $r \in [0, T-t]$, namely let $(\zeta^s, \zeta^t)(r)$ be the pure-jump Markov process started at $(\rho_n, \rho_n)$ from $q$ and evolving according to the generator $\mathcal{L}^z$ acting on bounded measurable functions $J(y, z)$ defined on $[0, P]^2$ according to

$$\begin{aligned} (\mathcal{L}^z J)(r, y, z) = \quad & \mathbf{1}(y = z) \int \{J(\rho_+, \rho_+) - J(y, z)\} H[y, \rho_+] \hat{f}(r, y, d\rho_+) \\ & + \mathbf{1}(y = z) \int \{J(\rho_+, z) - J(y, z)\} H[y, \rho_+] f^s(r, y, d\rho_+) \\ & + \mathbf{1}(y = z) \int \{J(y, \rho_+) - J(y, z)\} H[z, \rho_+] f^t(r, z, d\rho_+) \\ & + \mathbf{1}(y \neq z) \int \{J(\rho_+, z) - J(y, z)\} H[y, \rho_+] (\hat{f} + f^s)(r, y, d\rho_+) \\ & + \mathbf{1}(y \neq z) \int \{J(y, \rho_+) - J(y, z)\} H[z, \rho_+] (\hat{f} + f^t)(r, z, d\rho_+). \end{aligned} \tag{A.8}$$

Taking $J(y, z)$ which does not depend on $z$ we find that $\zeta^s(r)$ for $r \in [0, T-t]$ has the same law as the random boundary for $\Phi_s^{T-\theta} q$, and likewise taking $J$ which does not depend on $y$ we find the $\zeta^t(r)$ has the same law as the random boundary for $\Phi_t^T q$, verifying the coupling.

On the diagonal $y = z$, the rate at which $\mathcal{L}^z$ causes jumps that leave the diagonal (the second and third lines of (A.8)) is bounded by

$$C_{\lambda, H'(P)} \theta, \tag{A.9}$$

and the probability that such a transition never occurs in a time interval of length $T - t$ is bounded below by

$$\exp[-C_{\lambda, H'(P)}(T-t)\theta] \geq \exp[-C_{\lambda, H'(P)} T\theta]. \tag{A.10}$$

So, coupling the random entry dynamics, we find that $\Phi_s^{T-\theta} q = \Phi_t^T q$ with probability at least $\exp[-C_{\lambda, H'(P)} T\theta]$. Putting the above pieces together, we find

$$\begin{aligned} & |\mathbb{E} G(\Phi_s^T q) - \mathbb{E} G(\Phi_t^T q)| \\ & \qquad \leq C_{\lambda, H'(P)}(t-s) + w_G(t-s) + (1 - \exp(C_{\lambda, H'(P)} T(t-s))), \end{aligned} \tag{A.11}$$

and the proof is complete. □

*Proof of Lemma 2.6.* Using the Markov property of the random flow $\Phi$, we have a functional identity:

$$F(s, q) = \mathbb{E} G(\Phi_t^T \Phi_s^t q) = \mathbb{E} F(t, \Phi_s^t q). \tag{A.12}$$

Let $q$ be fixed, and consider the following events, whose union is of full measure for computing the final expectation above:

$$\begin{aligned}\mathcal{E}_0 &= \{\text{no entry at } x = L \text{ in } (s,t)\}\\ \mathcal{E}_1 &= \{\text{at least one entry at } x = L \text{ in } (s,t)\}\end{aligned} \tag{A.13}$$

Observe that on $\mathcal{E}_0$ we see only the deterministic flow $\phi$ over the time interval $(s,t)$:

$$F(t, \Phi_s^t q)\mathbf{1}_{\mathcal{E}_0} = F(t, \phi_s^t q)\mathbf{1}_{\mathcal{E}_0}, \tag{A.14}$$

and this occurs with probability

$$\mathbb{P}(\mathcal{E}_0) = \exp\left(-\int_s^t \int H[\rho_n, \rho_+] f(r, \rho_n; d\rho_+)\, dr\right). \tag{A.15}$$

We prefer an expression evaluating $f$ at only a single time, and Taylor expand the exponential around zero.

$$\begin{aligned}&\left|\int_s^t \int H[\rho_n, \rho_+] f(r, \rho_n, d\rho_+)\, dr - (t-s)\int H[\rho_n, \rho_+] f(t, \rho_n, d\rho_+)\right|\\ &\quad = \left|\int_s^t \int_t^r \int H[\rho_n, \rho_+](\mathcal{L}^\kappa f)(r', \rho_n, d\rho_+)\, dr'\, dr\right| \le C_{\lambda, H'(P)}(t-s)^2,\end{aligned} \tag{A.16}$$

so that

$$\mathbb{P}(\mathcal{E}_0) = 1 - (t-s)\int H[\rho_n, \rho_+] f(t, \rho_n, d\rho_+) + O((t-s)^2) \tag{A.17}$$

and by exhaustion

$$\mathbb{P}(\mathcal{E}_1) = (t-s)\int H[\rho_n, \rho_+] f(t, \rho_n, d\rho_+) + O((t-s)^2), \tag{A.18}$$

with both errors bounded uniformly over $Q$. On $\mathcal{E}_1$ write $\tau$ for the first time a random entry occurs for $\Phi_s^t$, and $\rho_+$ for the new boundary value, noting that the distribution of $\tau$ depends only on $q$ through $\rho_n$ and that the law of $\rho_+$ is determined by $f(\tau, \rho_n, \cdot)$. We have $\tau \in (s,t)$ so that

$$F(t, \Phi_s^t q)\mathbf{1}_{\mathcal{E}_1} = F(t, \Phi_\tau^t \epsilon_{\rho_+} \phi_s^\tau q)\mathbf{1}_{\mathcal{E}_1}. \tag{A.19}$$

Using the strong Markov property for the random boundary at the stopping time $\tau$,

$$\mathbb{E}F(t, \Phi_\tau^t \epsilon_{\rho_+} \phi_s^\tau q)\mathbf{1}_{\mathcal{E}_1} = \mathbb{E}F(\tau, \epsilon_{\rho_+} \phi_s^\tau q)\mathbf{1}_{\mathcal{E}_1}. \tag{A.20}$$

Since $\mathbb{P}(\mathcal{E}_1) = O(t-s)$, we can afford to make $o(1)$ modifications to this. Write $w(\delta)$ for the modulus of continuity of $F(t,q)$ in time, according to Lemma 2.5. Now $\tau \in (s,t)$ a.s. on $\mathcal{E}_1$, so

$$|\mathbb{E}F(\tau, \epsilon_{\rho_+} \phi_s^\tau q)\mathbf{1}_{\mathcal{E}_1} - \mathbb{E}F(t, \epsilon_{\rho_+} \phi_s^\tau q)\mathbf{1}_{\mathcal{E}_1}| \le \mathbb{P}(\mathcal{E}_1) w(t-s). \tag{A.21}$$

Next we modify the distribution from which $\rho_+$ is selected; at present, $\rho_+$ is selected according to the random measure

$$\frac{H[\rho_n, \rho_+] f(\tau, \rho_n, d\rho_+)}{\int H[\rho_n, \rho_+] f(\tau, \rho_n, d\rho_+)}. \tag{A.22}$$

Since $\tau \in (s,t)$ a.s. on $\mathcal{E}_1$, the total variation difference

$$\|f(\tau, \rho_n, d\rho_+) - f(t, \rho_n, d\rho_+)\| \le C_{\lambda, H'(P)}(t-s) \quad \text{a.s.} \tag{A.23}$$

Let us write $\hat{\rho}_+$ for an independent random variable distributed as

$$H[\rho_n, \rho_+] f(t, \rho_n, d\rho_+), \tag{A.24}$$

(note $t$ instead of $\tau$), normalized to have unit mass. Then

$$\left|\mathbb{E}F(t,\epsilon_{\rho_+}\phi_s^\tau q)\mathbf{1}_{\mathcal{E}_1} - \mathbb{E}F(t,\epsilon_{\hat{\rho}_+}\phi_s^\tau q)\mathbf{1}_{\mathcal{E}_1}\right| \le C_{\lambda,H'(P)}(t-s)\mathbb{P}(\mathcal{E}_1). \tag{A.25}$$

Note that $\mathbb{P}(\mathcal{E}_1)$ cancels with the normalization in (A.22), up to an $O(t-s)$ error, and that $\hat{\rho}_+$ is independent of $\mathcal{E}_1$. With error bounded uniformly over $q \in Q$ by $C_{\lambda,H'(P)}[(t-s)^2 + (t-s)w(t-s)]$, we have

$$\begin{aligned}\mathbb{E}F(t,\Phi_s^t q) \approx F(t,\phi_s^t q)\left(1-(t-s)\int H[\rho_n,\rho_+]f(t,\rho_n,d\rho_+)\right)\\ + (t-s)\int \mathbb{E}[F(t,\epsilon_{\rho_+}\phi_s^\tau q)\mid \mathcal{E}_1]H[\rho_n,\rho_+]f(t,\rho_n,d\rho_+),\end{aligned} \tag{A.26}$$

where the remaining expectation is now taken only over $\tau$ conditioned on $\mathcal{E}_1$, which is therefore distributed over $[s,t]$. □

*Proof of Lemma 2.7.* On any of $A_i$, $i = 0,\dots,n-1$, we have $\sigma = \sigma(q) \le t$. For such $q$ we have

$$\begin{aligned}|F(t,\phi_s^t q) - F(t,\phi_s^\sigma q)|\\ \le |F(t,\phi_\sigma^t\phi_s^\sigma q) - F(\sigma,\phi_s^\sigma q)| + |F(\sigma,\phi_s^\sigma q) - F(t,\phi_s^\sigma q)|.\end{aligned} \tag{A.27}$$

The first absolute difference is the error induced by conditioning on zero particle entries on the interval $[s,\sigma]$, which costs less than $\lambda H'(P)(t-s)$. Lemma 2.5 bounds the second of these by $w(t-s)$. Using (A.27) we can replace the integrand $F(t,\phi_s^t q)$ over $A_i$ with $F(t,\phi_s^\sigma q)$, i.e. the projection of $q$ onto $(\partial_i\Delta_n^L)\times\overline{\Delta_{n+1}^P}$ in the direction of the velocity.

Write $v_i = -H[\rho_{i-1},\rho_i]$ for the velocities. We can integrate over $A_0$ using as a parametrization for the $x$-variables

$$(\theta, x_2,\dots,x_n) \mapsto (-\theta v_1, x_2-\theta v_2,\dots,x_n-\theta v_n), \tag{A.28}$$

which has absolute Jacobian determinant $-v_1 = H[\rho_0,\rho_1]$. Namely,

$$\begin{aligned}\int_{A_0} F(t,\phi_s^\sigma q)\mu_n(t,dq)\\ = \int_{\Delta_{n-1}^L\times\overline{\Delta_{n+1}^P}}\int_0^{(t-s)\wedge|(L-x_n)/v_n|} F(t,(0,x_2,\dots,x_n;\rho_0,\dots,\rho_n))\\ H[\rho_0,\rho_1]\; d\theta\, dx_2\cdots dx_n\,\ell(t,d\rho_0)\prod_{i=1}^n f(t,\rho_{i-1},d\rho_i).\end{aligned} \tag{A.29}$$

On all but a portion of $A_0$ with $x$-volume bounded by $C_{n,L,H'(P)}(t-s)^2$ we have $t-s < |(L-x_n)/v_n|$, and we can extend the upper limit of integration to $\theta = t-s$ with and error bounded by this. Noting that the configurations

$$(0,x_2,\dots,x_n;\rho_0,\dots,\rho_n) \text{ and } (x_2,\dots,x_n;\rho_1,\dots,\rho_n) \tag{A.30}$$

are equivalent under our sticky particle dynamics and relabeling, we obtain the claimed approximation for the integral over $A_0$.

The argument for $A_i$, $i = 1,\dots,n-1$, is similar. We use the mapping

$$(x_1,\dots,x_i,x_{i+2},\dots,x_n,\theta) \mapsto (x_1-\theta v_1,\dots,x_i-\theta v_i, x_i-\theta v_{i+1},\dots,x_n-\theta v_n), \tag{A.31}$$

with absolute Jacobian determinant $v_i - v_{i+1} = H[\rho_i, \rho_{i+1}] - H[\rho_{i-1}, \rho_i]$, and replace the upper limit of integration for $\theta$ with $t-s$ at the cost of an $O((t-s)^2)$ error. Relabeling gives the indicated approximation.

The range of

$$\phi_s^{s+\theta}(x_1, \dots, x_{n-1}, L; \rho_0, \dots, \rho_n) \tag{A.32}$$

where $(x_1, \dots, x_{n-1}) \in \Delta_{n-1}^L$, $(\rho_0, \dots, \rho_n) \in \overline{\Delta_{n+1}^P}$, and

$$0 \le \theta \le (t-s) \wedge \frac{x_1}{|v_1|} \wedge \frac{x_2 - x_1}{|v_2 - v_1|} \wedge \cdots \wedge \frac{x_n - x_{n-1}}{|v_n - v_{n-1}|} \tag{A.33}$$

is $B$, modulo sets of lower $x$-dimension. Using Lemma 2.5 we can replace $F(t, q)$ by

$$F(t, (x_1, \dots, x_{n-1}, L; \rho_0, \dots, \rho_n)) \tag{A.34}$$

with pointwise error over $B$ bounded by $C_{\lambda, H'(P)}[w(t-s) + (t-s)]$. For all but an $O((t-s)^2)$ portion of the $x$-volume, the value $t-s$ is the minimum above, and as before we obtain the indicated approximation of the integral over $B$. □

*Proof of Lemma 2.8.* Essentially we are verifying the Leibniz rule, but we are unable to find a version of this to cite for kernels. We first obtain quantitative control over our linear approximations of $f$ and $\ell$. Namely, fix any measurable $|J| \le 1$. We have

$$\begin{aligned}\int J(\rho_+)[f(t, \rho_-, d\rho_+) - f(s, \rho_-, d\rho_+) - (t-s)(\mathcal{L}^\kappa f)(t, \rho_-, d\rho_+)] \\ = \int_s^t J(\rho_+)[(\mathcal{L}^\kappa f)(\tau, \rho_-, d\rho_+) - (\mathcal{L}^\kappa f)(t, \rho_-, d\rho_+)] d\tau.\end{aligned} \tag{A.35}$$

The difference of $\mathcal{L}^\kappa f$ at different times can be expressed in terms of $f$ and $H$ again,

$$\begin{aligned}&(\mathcal{L}^\kappa f)(\tau, \rho_-, d\rho_+) - (\mathcal{L}^\kappa f)(t, \rho_-, d\rho_+) \\ &= \int (H[\rho_*, \rho_+] - H[\rho_-, \rho_*]) \left( \int_t^\tau (\mathcal{L}^\kappa f)(\tau', \rho_-, d\rho_*) d\tau' \right) f(\tau, \rho_*, d\rho_+) \\ &\quad + \int (H[\rho_*, \rho_+] - H[\rho_-, \rho_*]) f(t, \rho_-, d\rho_*) \left( \int_t^\tau (\mathcal{L}^\kappa f)(\tau', \rho_*, d\rho_+)\, d\tau' \right) \\ &\quad - \left[ \int_t^\tau \int (H[\rho_+, \rho_*](\mathcal{L}^\kappa)(\tau', \rho_+, d\rho_*) \right. \\ &\qquad \left. - H[\rho_-, \rho_*](\mathcal{L}^\kappa)(\tau', \rho_-, d\rho_*) \right) d\tau' \Big] f(\tau, \rho_-, d\rho_*) \\ &\quad - \left[ \int H[\rho_+, \rho_*] f(\tau, \rho_+, d\rho_*) \right. \\ &\qquad \left. - H[\rho_-, \rho_*] f(\tau, \rho_-, d\rho_*) \right] \int_t^\tau (\mathcal{L}^\kappa f)(\tau', \rho_-, d\rho_+)\end{aligned} \tag{A.36}$$

Noting that $\|\mathcal{L}^\kappa f\| \le 3\lambda^2 H'(P)$, we integrate over $\rho_+$, then $\rho_*$, and find that (A.35) is bounded by

$$9H'(P)^2 \lambda^3 (t-s)^2. \tag{A.37}$$

Next, for any $|J| \leq 1$,

$$\int J(\rho_0)[\ell(t,d\rho_0) - \ell(s,d\rho_0) - (t-s)(\mathcal{L}^0\ell)(t,d\rho_0)] \\ = \int_s^t J(\rho_0)[(\mathcal{L}^0\ell)(\tau,d\rho_0) - (\mathcal{L}^0\ell)(t,d\rho_0)]d\tau, \quad \text{(A.38)}$$

and

$$\begin{aligned}(\mathcal{L}^0\ell)(\tau,d\rho_0) &- (\mathcal{L}^0\ell)(t,d\rho_0) \\ &= \int H[\rho_*,\rho_0]\left(\int_t^\tau (\mathcal{L}^0\ell)(\tau',d\rho_*)\,d\tau'\right) f(\tau,\rho_*,d\rho_0) \\ &\quad + \int H[\rho_*,\rho_0]\ell(t,d\rho_*)\left(\int_t^\tau (\mathcal{L}^\kappa f)(\tau',\rho_*,d\rho_0)\right) \\ &\quad - \left[\int_t^\tau \left(\int H[\rho_0,\rho_*](\mathcal{L}^\kappa f)(\tau',\rho_0,d\rho_*)\right)d\tau'\right]\ell(\tau,d\rho_0) \\ &\quad - \left[\int H[\rho_0,\rho_*]f(t,\rho_0,d\rho_*)\right]\int_t^\tau (\mathcal{L}^0\ell)(\tau',d\rho_0)\,d\tau'.\end{aligned} \quad \text{(A.39)}$$

We have $\|\mathcal{L}^0\ell\| \leq 2\lambda H'(P)$, so by integrating over $\rho_0$ and then $\rho_*$ we find (A.38) is bounded by

$$5H'(P)^2\lambda^2(t-s)^2. \quad \text{(A.40)}$$

Returning to the problem of establishing our Leibniz rule, note that $e^{-\lambda L}\mathbf{1}_{\Delta_n^L}(dx)$ factors from both $\mu_n$ and $\mathcal{L}^*\mu_n$. It will therefore suffice to obtain a bound on the $(\rho_0,\dots,\rho_n)$ portion.

We argue by induction. In the case $n=0$, we have only the difference in (A.38), and the result holds. Now suppose that the result holds for case $n$, and consider $n+1$. Choose as a test function $J(\rho_0,\dots,\rho_{n+1})$, which is measurable and has $|J| \leq 1$, and integrate it against

$$\begin{aligned}&\ell(t,d\rho_0)\prod_{j=1}^{n+1} f(t,\rho_{j-1},d\rho_j) - \ell(s,d\rho_0)\prod_{i=1}^{n+1} f(s,\rho_{j-1},d\rho_j) \\ &= \int \ell(t,d\rho_0)\prod_{j=1}^{n} f(t,\rho_{j-1},d\rho_j)[f(t,\rho_n,d\rho_{n+1}) - f(s,\rho_n,d\rho_{n+1})] \\ &\quad + \left[\ell(t,d\rho_0)\prod_{j=1}^{n} f(t,\rho_{j-1},d\rho_j) - \ell(s,d\rho_0)\prod_{j=1}^{n} f(s,\rho_{j-1},d\rho_j)\right] f(s,\rho_n,d\rho_{n+1}).\end{aligned} \quad \text{(A.41)}$$

For each $\rho_0,\dots,\rho_n$, the function $J(\rho_0,\dots,\rho_{n+1})$ is bounded and measurable in $\rho_{n+1}$, so $f(t,\rho_n,d\rho_{n+1}) - f(s,\rho_n,d\rho_{n+1})$ can be replaced with

$$(t-s)\int J(\rho_0,\dots,\rho_{n+1})(\mathcal{L}^\kappa f)(t,\rho_n,d\rho_{n+1}) \quad \text{(A.42)}$$

plus an error no larger than $9H'(P)^2\lambda^3(t-s)^2$, which when integrated over the remaining variables grows by a factor $\lambda^n$.

In the third line of (A.41), noting $\int J(\rho_0,\dots,\rho_{n+1})f(s,\rho_n,d\rho_{n+1})$ is bounded and measurable in $(\rho_0,\dots,\rho_n)$, we apply the inductive hypothesis, replacing the

bracketed difference by

$$(t-s)\left[\frac{(\mathcal{L}^0\ell)(t,d\rho_0)}{\ell(t,d\rho_0)}+\sum_{i=1}^{n}\frac{(\mathcal{L}^\kappa f)(t,\rho_{i-1},d\rho_i)}{f(t,\rho_{i-1},d\rho_i)}\right]\ell(t,d\rho_0)\prod_{j=1}^{n}f(t,\rho_{j-1},d\rho_j) \quad \text{(A.43)}$$

plus an $o(t-s)$ error. After doing this, again noting $J(\rho_0,\dots,\rho_{n+1})$ is measurable in $\rho_{n+1}$ for each fixed $\rho_0,\dots,\rho_n$, we replace $f(s,\rho_n,d\rho_{n+1})$ with $f(t,\rho_n,d\rho_{n+1})$ at a cost of $3\lambda^2H'(P)(t-s)$, which gets multiplied by the other factor $(t-s)$.

Adding the modified versions of these two lines of (A.41), we find exactly the $\rho_0,\dots,\rho_{n+1}$ portion of $\mathcal{L}^*\mu_{n+1}$ plus an $o(t-s)$ error. □

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