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Boltzmann–Grad Limits for Stochastic Hard Sphere Models*

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Abstract: We study a d-dimensional stochastic particle system in which the particles travel deterministically in between stochastic collisions. The collisions are elastic and occur with a probability of order ε^α when two particles are at a distance less than ε. When the number of particles N goes to infinity and $N\varepsilon^{d+\alpha-1}$ goes to a nonzero constant, we show that the particle density converges to a solution of the Boltzmann equation provided that $\alpha \geq d+1$.

1. Introduction

A long-standing open problem in statistical mechanics is the derivation of the Boltzmann-equation from the *hard sphere model.* In the hard sphere model, one starts with N spheres of diameter ε that travel according to their velocities and collide elastically. In a Boltzmann-Grad limit, we send $N \to \infty$, $\varepsilon \to 0$ in such a way that $N\varepsilon^{d-1} \to Z$, where Z is a positive finite number. If $f(x, v, t)$ denotes the density of particles of velocity v, then f satisfies the Boltzmann equation

$$f_t + v \cdot f_x = \int_{\mathbb{R}^d} \int_{\mathbb{S}} (n \cdot (v - v_*))^+ [f(x, v')f(x, v'_*) - f(x, v)f(x, v_*)] dn\, dv_*, \tag{1.1}$$

where $\mathbb{S}$ denotes the unit sphere, dn denotes the $d-1$-dimensional Hausdorff measure on $\mathbb{S}$, and

$$v' = v - (n \cdot (v - v_*))n,$$
$$v'_* = v_* + (n \cdot (v - v_*))n.$$

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The derivation of (1.1) from the hard sphere model was established by Lanford for short times in [La]. Later Illner and Pulvirenti [IP, P] replaced the smallness on time with a smallness on the initial density in a suitable norm.

The finiteness of the mean free path is the main property of the hard sphere model that is responsible for its kinetic behavior. The condition $N = O(\varepsilon^{1-d})$ implies that on average a particle experiences finitely many collisions in one unit of time. This property is still valid if we increase the number of particles to $N = O(\varepsilon^{1-\alpha-d})$ but reduce the probability of a collision to $O(\varepsilon^{\alpha})$. (Equivalently, we increase the range of interaction from $O(N^{1/(1-d)})$ to $O(N^{1/(1-\alpha-d)})$.) In this way we obtain a family of models by varying α, and it turns out that for large α many probabilistic arguments become available. To avoid some technical issues, we consider a suitable smoothing of the hard sphere. More precisely, we take a nonnegative continuous function V of compact support V and assume that a collision occurs with a stochastic rate equal to $V^{\varepsilon}(|x_i - x_j|)B(v_i - v_j, n_{ij})$, where $V^{\varepsilon}(r) = \varepsilon^{\alpha-1}V(r/\varepsilon)$, x_i and x_j are the positions of the colliding particles, v_i and v_j are the velocities of the colliding particles, and $n_{ij} = (x_i - x_j)/|x_i - x_j|$. We assume $B(0, n) = 0$ so that only particles of different velocities can collide. As a result, only for a time of order $O(\varepsilon)$ the rate $V^{\varepsilon}(|x_i - x_j|)B(v_i - v_j, n_{ij})$ is nonzero. This in particular implies that the true rate of collision is of order $O(\varepsilon) \times O(\varepsilon^{\alpha-1}) = O(\varepsilon^{\alpha})$. Indeed we show that if V is chosen so that $\int V(|x|)dx = 1$ and $\alpha \geq d+1$ then the microscopic particle densities will converge to a solution of the Boltzmann equation

$$f_t + v \cdot f_x = \int_{\mathbb{R}^d} \int_{\mathbb{S}} B(v - v_*, n)[f(x, v')f(x, v'_*) - f(x, v)f(x, v_*)]dn\,dv_* \quad (1.2)$$

as $\varepsilon \to 0$.

When $d \geq 2$, the best existence result available for (1.1) is due to DiPerna and Lions [DLi1]. This existence result is formulated for the so-called renormalized solution and the uniqueness for such solutions is an open problem. Because of this what we show in this article is that the limit points of the microscopic particle densities as $\varepsilon \to 0$ are all DiPerna-Lions solutions. Note however that if we already know a bounded strong solution exists, then there exists a unique renormalized solution [Li].

In Rezakhanlou-Tarver [RT] and Rezakhanlou [R1] we established a Boltzmann-type equation for stochastic models in dimension one. In these articles we considered discrete-velocity models in which $\alpha = 1$ and the velocities belong to a finite set. Note that when $d = 1$, Eq. (1.2) is trivial because of the elastic collision. However, we may consider more general collision rules for which the conservation of momentum is still valid but the conservation of the kinetic energy is violated. For such one dimensional models, one should be able to relax the finiteness assumption of [RT] and derive a Boltzmann-type equation for the macroscopic particle densities provided that $\alpha \geq 1$.

A variation of our model has been studied in Rezakhanlou [R2] to derive an Enskog type equation for the macroscopic particle densities. In [R2] we examined a system in which particles collide elastically with probability $O(N^{-1})$ when two particles are at distance σ. The particle density now satisfies the Enskog equation that is similar to (1.1) except that the expression in brackets is replaced with

$$\sigma^{d-1}[f(x, v')f(x - \sigma n, v'_*) - f(x, v)f(x + \sigma n, v_*)]\,.$$

The organization of the paper is as follows. In Sect. 2 the main result is stated. In Sect. 3 the proof of the main result is sketched. In Sect. 4 we establish the entropy and entropy production bounds. In Sect. 5 the velocity averaging techniques are used to prove the compactness of the collision term. This will be used in Sect. 6 to establish a variant of Stosszahlensatz (Boltzmann's molecular chaos principle) for the microscopic loss term. Sects. 7 and 8 are devoted to the Stosszahlensatz for the microscopic gain term. The proof of the kinetic limit is carried out in Sects. 9 and 10. In Sect. 11 we address an entropy production bound on the macroscopic densities.

2. Notation and Main Result

This section is devoted to the statement of the main result. We start with a description of our stochastic models.

In our models we have N particles in the d-dimensional torus $\mathbb{T}^d$. Define the state space $\mathcal{E} = (\mathbb{T}^d \times \mathbb{R}^d)^N$; $\mathbf{q} \in \mathcal{E}$ is the N-tuple,

$$\mathbf{q} = (\mathbf{x}, \mathbf{v}) = (q_1, \dots, q_N)\ , \quad \mathbf{x} = (x_1, \dots, x_N)\ , \quad \mathbf{v} = (v_1, \dots, v_N)\ ,$$

where $q_i = (x_i, v_i)$. The process $\mathbf{q}(t)$ is a Markov process with the infinitesimal generator $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_c$. We have that for any smooth $g : \mathcal{E} \to \mathbb{R}$,

$$\mathcal{A}_0 g(\mathbf{q}) = \sum_{i=1}^{N} v_i \cdot \frac{\partial g}{\partial x_i}(\mathbf{q})\ , \tag{2.1}$$

$$\mathcal{A}_c g(\mathbf{q}) = \frac{1}{2} \sum_{i,j=1}^{N} V^{\varepsilon}(|x_i - x_j|) B(v_i - v_j, n_{ij}) \left(g\left(S^{ij}\mathbf{q}\right) - g(\mathbf{q}) \right), \tag{2.2}$$

where $V^{\varepsilon}(r) = \varepsilon^{\alpha-1} V\left(\frac{r}{\varepsilon}\right)$ with $V : \mathbb{R} \to [0, \infty)$ a continuous function of compact support such that $\int_{\mathbb{R}^d} V(|x|)dx = 1$; $B : \mathbb{R}^d \times \mathbb{S} \to [0, \infty)$ is a Lipschitz continuous function such that $B(v' - v'_*, n) = B(v - v_*, n)$ and $\sup_{n,v} B(v, n)|v|^{-1} < \infty$; $n_{ij} = \frac{x_i - x_j}{|x_i - x_j|}$, and $S^{ij}\mathbf{q}$ is the configuration obtained from $\mathbf{q}$ by replacing (v_i, v_j) with (v_i^j, v_j^i), where

$$\begin{aligned} v_i^j &= v_i - ((v_i - v_j) \cdot n_{ij}) n_{ij}, \\ v_j^i &= v_j - ((v_j - v_i) \cdot n_{ij}) n_{ij} = v_j + ((v_i - v_j) \cdot n_{ij}) n_{ij}\ . \end{aligned}$$

We also assume that the function

$$A(x, v) := V(|x|) B\left(v, \frac{x}{|x|}\right)$$

is twice differentiable in x and its second x–derivatives are Lipschitz continuous in both x and v variables. Note that when B is not identically constant, even the continuity of the function A implies that V vanishes in a neighborhood of 0.

Convention 2.1. The meaning of the expression $V^{\varepsilon}(|x_i - x_j|)$ is as follows. The points x_i and x_j each have d coordinates in the circle $\mathbb{T}$. The k^{th} difference $x_i^k - x_j^k$ is defined to be the signed distance between x_i^k and x_j^k. Hence, we may regard $x_i - x_j$ as a point in $\mathbb{R}^d$. Also, for $x \in \mathbb{T}^d$, $t \in \mathbb{R}$ and $v \in \mathbb{R}^d$, the point $x + tv \in \mathbb{T}^d$ is defined *mod* 1. Interpretations of this sort will be assumed throughout the paper without mentioning.

Let $f^0 : \mathbb{T}^d \times \mathbb{R}^d \to [0, \infty)$ be a measurable function such that

$$\int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (f^0(x, v))^p \ \exp(\alpha_0 |v|^2)\, dxdv < +\infty \tag{2.3}$$

for some $p > 1$ and $\alpha_0 > 0$. We then define

$$\mu^0(d\mathbf{q}) := F^0(\mathbf{q})d\mathbf{q} := \frac{1}{Z^N} \prod_{i=1}^{N} f^0(x_i, v_i)$$

with $Z = \int f^0(x, v)dxdv$. We also define the number ε by the relationship $\varepsilon^{d+\alpha-1}N = Z$.

Given a configuration $\mathbf{q}$, define the empirical measure π by

$$\pi(t, dq; \mathbf{q}) = \pi(t, dq) := \varepsilon^{d+\alpha-1} \sum_{i=1}^{N} \delta_{(x_i(t), v_i(t))}(dq) \ .$$

The transformation $\mathbf{q}(\cdot) \mapsto \pi$ induces a probability measure $\mathcal{P}_N$ on the space $\mathcal{D} = L^\infty([0, T]; \mathcal{M})$, where $\mathcal{M}$ is the space of measures $\pi(t, dq)$ with $\pi(t, \mathbb{T}^d \times \mathbb{R}^d) = Z$ and $\mathcal{M}$ is equipped with the weak topology. Observe that by the law of large numbers for the independent random variables we have

$$\lim_{N\to\infty} \int \left| \int J(x, v)\pi(t, dx, dv) - \int J(x, v) f^0(x, v)dxdv \right| \mu^0(d\mathbf{q}) = 0 \ .$$

Define

$$\begin{aligned} Q^+(f, f)(v) &= \int_{\mathbb{R}^d} \int_{\mathbb{S}} B(v - v_*, n) f(v') f(v'_*) dn\, dv_* \ , \\ Q^-(f, f)(v) &= \int_{\mathbb{R}^d} \int_{\mathbb{S}} B(v - v_*, n) f(v) f(v_*) dn\, dv_* \ , \end{aligned} \tag{2.4}$$

and $Q = Q^+ - Q^-$. We say that f is a *renormalized solution* of (1.2) if

$$f \in L^1(\mathbb{T}^d \times \mathbb{R}^d \times [0, T]), \quad f \geq 0 \ , \quad \frac{Q^\pm(f, f)}{1 + f} \in L^1(\mathbb{T}^d \times \mathbb{R}^d \times [0, T]) \ ,$$

for every positive T and for every Lipschitz continuous $\beta : [0, \infty) \to \mathbb{R}$ that satisfies $\sup_r (1 + r)|\beta'(r)| < \infty$, we have that

$$\beta(f)_t + v \cdot \beta(f)_x = \beta'(f) Q(f, f)$$

in weak sense.

Theorem 2.2. *Assume that* $\alpha \geq d + 1$. *Then the family* $\{\mathcal{P}_N : N \in \mathbb{N}\}$ *is tight. Moreover every limit point of* $\mathcal{P}_N$ *is concentrated on the set of renormalized solutions of (1.2) such that* $f(x, v, 0) = f^0(x, t)$ *and*

$$\sup_t \int f(1 + |x|^2 + |v|^2 + \log^+ f)dxdv < \infty \ . \tag{2.5}$$

Ideally we would like to prove that any limit point of the sequence $\{\mathcal{P}_N : N \in \mathbb{N}\}$ is concentrated on the space of functions f such that

$$\int_0^\infty \iint \iint_{\mathbb{S}} B(v - v_*, n)(f' f'_* - f f_*) \log\left(\frac{f' f'_*}{f f_*}\right) dn\, dx dv\, dv_* dt < \infty\,. \quad (2.6)$$

Presumably the method of this article can be used to establish (2.6) by differentiating the expression

$$E_N \int \iint f^{\delta,\varepsilon}(x, v, t) \log^+ f^{\delta,\varepsilon}(x, v, t)\, dx dv\;,$$

where E_N denotes the expectation with respect to the measure $\mathcal{P}_N$, $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, and $f^{\delta,\varepsilon}(x, v, t) := f^{\delta,\varepsilon}(x, v; \mathbf{q}(t))$ is a microscopic approximation of the density and is defined by (4.4). Instead we would rather pursue a quicker approach and only prove a consequence of (2.6) that is good enough for many known properties of the solutions. See Sect. 11 for more details.

We only prove Theorem 2.2 when $\alpha = d + 1$. The interested reader can check that the proof also works when $\alpha > d + 1$. Note that when $\alpha = d + 1$, then N and ε are related by $\varepsilon^{2d} N = Z$ and $V^\varepsilon(r) = \varepsilon^d V(r/\varepsilon)$.

3. Sketch of proofs

In this section, we sketch the proofs and explain some of the main ideas. The first general global existence result for the Boltzmann equation was established by DiPerna and Lions in the prominent article [DLi1]. An important aspect of the Boltzmann equation is the smoothing effect of its flow term $\partial_t + v \cdot \partial_x$. This is now known as the velocity averaging lemma and was quantitatively formulated and studied by Glose et al. in [GLiPS]. The velocity averaging lemma is recalled in Sect. 5 as Lemma 5.4 and has the following flavor: If both f and $\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x}$ belong to a weakly compact subset of $L^1(\mathbb{T}^d \times \mathbb{R}^d \times [0, T])$ and ψ is a bounded smooth function, then the velocity average $\int f(x, v, t)\psi(v) dv =: \rho(x, t)$ belongs to a strongly compact subset of $L^1(\mathbb{T}^d \times \mathbb{R}^d \times [0, T])$. The velocity averaging lemma and DiPerna-Lions approach play an essential role in the present article.

We used the empirical measure

$$\pi_\varepsilon(dq; \mathbf{q}) = \varepsilon^{2d} \sum_{i=1}^N \delta_{(x_i, v_i)}(dq)$$

as a candidate for the microscopic density in Sect. 2. Because of the nonlinearity of the collision term, it is necessary to replace $\pi_\varepsilon(dq; \mathbf{q})$ with a smoother candidate. One possibility is to take a smooth nonnegative function $\eta : \mathbb{R}^{2d} \to \mathbb{R}$ of compact support and total integral 1, and consider

$$(\pi_\varepsilon * \eta^\delta)(x, v) = f^{\delta,\varepsilon}(x, v; \mathbf{q}) = \varepsilon^{2d} \sum_{i=1}^N \eta^\delta\,(x_i - x, v_i - v)\,,$$

where $\eta^\delta(z) = \delta^{-2d}\eta(z/\delta)$ for a small positive δ. Needless to say that for a smooth test function J,

$$\int J\, d\pi_\varepsilon = \int J(\pi_\varepsilon * \eta^\delta)\, dx dv + \text{Error}(\delta), \quad (3.1)$$

where Error(δ) $\to$ 0 as $\delta \to 0$. In other words, if we choose a sequence $\delta = \delta(\varepsilon)$ that goes to zero as $\varepsilon \to 0$, then $f^{\delta(\varepsilon),\varepsilon}$ behaves weakly like the empirical measure π. If, however, we study $f^{\delta(\varepsilon),\varepsilon}$ as a pointwise function, the behavior of $f^{\delta,\varepsilon}$ depends critically on the way $\delta(\varepsilon)$ goes to zero. For example, if $\delta(\varepsilon) = \varepsilon$, then $f^{\varepsilon}(x, v; \mathbf{q}) := f^{\varepsilon,\varepsilon}(x, v; \mathbf{q})$ is a *Poisson-like* random variable, and is not expected to approximate the macroscopic density for small ε. To see this, observe that whenever $\eta\left(\frac{x_i - x}{\varepsilon}, \frac{v_i - v}{\varepsilon}\right) \neq 0$, then (x_i, v_i) belongs to a set of volume $O(\varepsilon^{2d})$. If particles are scattered evenly in space, only $NO(\varepsilon^{2d}) = O(1)$ particles are involved in the calculation of $f^{\varepsilon}(x, v; \mathbf{q})$. As a result, we do not have enough particles to benefit from the expected ergodic property of the model. Because of this, the random function f^{ε} does not approximate the macroscopic particle density in a strong sense. In the same way, we may argue that the function $f^{\varepsilon}(x, v; \mathbf{q})$ is rather rough as a function of (x, v). In other words, no (x, v)-regularity of the function f^{ε} should be expected. In a crucial step, we show in Sect. 5 that the velocity averages of f^{ε} are regular in (x, t)-variable. More precisely, if $\rho^{\varepsilon}(x, t) = \int f^{\varepsilon}(x, v; \mathbf{q}(t))\psi(v)dv$ for a smooth function ψ, then

$$E_N \sup_{|h|<\delta} \int_0^T \int |\rho^{\varepsilon}(x + h, t + \alpha) - \rho^{\varepsilon}(x, t)| \leq \text{const.}(\log\log|\log\delta|)^{-\alpha_0}\ , \tag{3.2}$$

where E_N denotes the expected value and $\alpha_0 = (2d+2)^{-1}(d+3)^{-1}$. The proof of (3.2) involves an *entropy bound*, an *entropy production bound*, and the aforementioned velocity averaging lemma. Section 4 is devoted to several entropy–like bounds, an entropy production bound, and their consequences. For example we show in Lemma 4.5 that

$$\sup_N E_N \sup_{t\in[0,T]} \int f^{\varepsilon}(x, v; \mathbf{q}(t)) \log f^{\varepsilon}(x, v; \mathbf{q}(t)) dxdv < \infty\ . \tag{3.3}$$

Also, a microscopic version of (2.6) is the content of Lemma 4.7.

A sketch of the proof of (3.2) is in order. We will see in Sect. 5 that weakly f^{ε} satisfies

$$f_t^{\varepsilon} + v \cdot f_x^{\varepsilon} = \Gamma^{\varepsilon} + N^{\varepsilon} + e^{\varepsilon}, \tag{3.4}$$

where Γ^{ε} is a collision-like term and N^{ε} is a martingale. The term e^{ε} has bounded L^1 norm and comes from replacing the differential operator $v_i \cdot \partial_x$ with $v \cdot \partial_x$ whenever $\eta^{\delta}(x_i - x, v_i - v) \neq 0$. As in (2.4), we write $\Gamma^{\varepsilon} = \Gamma_+^{\varepsilon} - \Gamma_-^{\varepsilon}$, where Γ_+^{ε} and Γ_-^{ε} represent the microscopic loss and gain terms respectively. Since no bound on the L^1-norm of $\Gamma_{\pm}^{\varepsilon}$ is available, we switch to a *renormalized* version of f^{ε}. The entropy bound (3.3) allows us to replace f^{ε} with $g_n^{\varepsilon} = \frac{f^{\varepsilon}}{1+n^{-1}f^{\varepsilon}}$ for a large number n. It turns out that g^{ε} satisfies an equation similar to (3.4):

$$g_t^{\varepsilon} + v \cdot g_x^{\varepsilon} = \hat{\Gamma}^{\varepsilon} + \hat{N}^{\varepsilon} + \hat{e}^{\varepsilon} = \hat{\Gamma}_+^{\varepsilon} - \hat{\Gamma}_-^{\varepsilon} + \hat{N}^{\varepsilon} + \hat{e}, \tag{3.5}$$

where $\hat{\Gamma}_{\pm}^{\varepsilon}$ is close to a term that looks like $\Gamma_{\pm}^{\varepsilon}\left(1 + n^{-1}f^{\varepsilon}\right)^{-2}$. It turns out that the entropy bound (3.3) can be used to show that $\hat{\Gamma}_-^{\varepsilon}$ belongs to a weakly compact subset of L^1. To treat $\hat{\Gamma}_+^{\varepsilon}$, we use our bound on $\hat{\Gamma}_-^{\varepsilon}$ and the microscopic analog of the entropy production bound (2.6). As a result, the renormalized collision terms $\hat{\Gamma}_+^{\varepsilon}$ and $\hat{\Gamma}_-^{\varepsilon}$ belong to a weakly compact subset of L^1. In the same fashion, we treat the martingale term $\hat{N}^{\varepsilon}$. We then directly apply the velocity averaging lemma (Lemma 5.4) to establish (3.2).

It is for the derivation of (3.2) that the condition $\alpha = d + 1$ (in general $\alpha \geq d + 1$) plays a crucial role. To have $\hat{\Gamma}^{\varepsilon}_{\pm}$ bounded in L^1, we are forced to choose $\delta(\varepsilon) = \varepsilon$ in our choice of the density $f^{\delta,\varepsilon}$. This is because we can find two positive constants c_0 and c_1 such that the term $a_{ij} := V^{\varepsilon}(|x_i - x_j|)\eta^{\varepsilon}(x_i - x, v_i - v)$ is bounded above by $c_1\varepsilon^d \mathbb{1}(|x_j - x| \leq c_0\varepsilon)\eta^{\varepsilon}(x_i - x, v_i - v)$. Such a bound would allow us to take advantage of the renormalization because a double sum of a_{ij} is bounded above by a product of density like quantities. Only if we assume $\alpha \geq d + 1$, then f^{ε} is of order one and a renormalization of f^{ε} has a chance to work. Indeed for $\alpha < d + 1$, the function f^{ε} is a large function of order $O(\varepsilon^{a-d-1})$ and has a small support of volume $O(\varepsilon^{d+1-\alpha})$. For such a function we do not expect to have (3.2), and in fact a compactness for its renormalization g^{ε}_n is not good enough to yield (3.2).

After our success in proving the regularity of ρ^{ε}, it is tempting to derive the macroscopic equation (1.1) by passing to the limit in (3.4) or its renormalized variation (3.5). Indeed the microscopic loss term can be expressed as

$$\begin{aligned}\Gamma^{\varepsilon}_{-}(x, v) &= \frac{1}{2}\sum_{i,j} V^{\varepsilon}(|x_i - x_j|)\eta\left(\frac{x_i - x}{\varepsilon}, \frac{v_i - v}{\varepsilon}\right) B(v_i - v_j, n_{ij}) \\ &= \frac{1}{2}\sum_{i} \eta\left(\frac{x_i - x}{\varepsilon}, \frac{v_i - v}{\varepsilon}\right) K(x_i, v_i)\ ,\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}K(x, v) &= \sum_{j} V^{\varepsilon}(|x - x_j|) B\left(v - v_j, \frac{x - x_j}{|x - x_j|}\right) \\ &=: \varepsilon^d \sum_{j} A\left(\frac{x - x_j}{\varepsilon}, v - v_j\right)\ .\end{aligned} \tag{3.7}$$

An important assumption of Boltzmann, known as *Stosszahlensatz*, asserts that a pair of particles before a collision behave like independent random variables. Such an assumption allows us to replace the collision term Γ^{ε}_{-} with something like the macroscopic collision term $Q_{-}(f^{\varepsilon}, f^{\varepsilon})$. For our rigorous derivation of the Boltzmann equation, we need to establish a suitable variant of Stosszahlensatz. Indeed our variant can be simply described in terms of the random function K. Roughly, if $\eta\left(\frac{x_i - x}{\varepsilon}, \frac{v_i - v}{\varepsilon}\right) \neq 0$ for some i in (3.6), then $|x_i - x|$ and $|v_i - v|$ are of order $O(\varepsilon)$. If $K(x, v)$ is sufficiently regular in (x, v)-variables, then we can replace $K(x_i, v_i)$ with $K(x, v)$. When such a replacement is performed, we can replace $\Gamma^{\varepsilon}_{-}(x, v)$ with $f^{\varepsilon}(x, v)K(x, v)$. On the other hand, the regularity bound (3.2) can be used to assert,

$$\begin{aligned}K(x, v) &\approx \varepsilon^{-2d} \iint V^{\varepsilon}(|x - y|) B\left(v - w, \frac{x - y}{|x - y|}\right) f^{\varepsilon}(y, w)\,dy\,dw \\ &\approx \iint_{\mathbb{S}} B(v - w, n) f^{\varepsilon}(x, w)\,dw\,dn.\end{aligned}$$

(See Sect. 6 for more details.) The above plausible argument explains the role of the regularity estimate (3.2) in establishing the Stosszahlensatz for the loss term. Before we move to the next step and discuss our variant of Stosszahlensatz for the gain term, let us pause here to mention that in spite of the appeal of the above argument, the choice of our microscopic density f^{ε} for the derivation of the macroscopic equation is *wrong*.

This is because $f^\varepsilon(x,v)$ is a Poisson-like random object and does not approximate the macroscopic density. In fact what we obtained, namely $Q^-(f^\varepsilon, f^\varepsilon)$ does not approximate the macroscopic loss term simply because $Q^-(f^\varepsilon, f^\varepsilon)$ is a nonlinear function of a Poisson-like random variable f^ε. This is also evident from (3.4) because a simple calculation reveals that the martingale term N^ε does not go away as $\varepsilon \to 0$, i.e., f^ε remains random as $\varepsilon \to 0$. However, if we consider $f^{\delta(\varepsilon),\varepsilon}$ for a choice of $\delta(\varepsilon)$ that satisfies $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$ and $\lim_{\varepsilon\to 0}\delta(\varepsilon)/\varepsilon = +\infty$, then $f^{\delta(\varepsilon)}$ should approximate the macroscopic density because now

$$f^{\delta(\varepsilon),\varepsilon}(x,v;\mathbf{q}) = \left(\frac{\delta(\varepsilon)}{\varepsilon}\right)^{2d}\sum_i \eta\left(\frac{x_i - x}{\delta(\varepsilon)}, \frac{v_i - v}{\delta(\varepsilon)}\right)$$

involves $N \times \delta(\varepsilon)^{2d} = O\left(\left(\frac{\delta(\varepsilon)}{\varepsilon}\right)^{2d}\right)$ many particles, and since $\frac{\delta(\varepsilon)}{\varepsilon} \to \infty$, we are dealing with a large number of particles. Hence we expect $f^{\delta(\varepsilon)}$ to approximate the macroscopic density for small ε by a law of large numbers. We can then derive an equation similar to (3.4) for $f^{\delta(\varepsilon),\varepsilon} =: \tilde{f}^\varepsilon$;

$$\tilde{f}^\varepsilon_t + v\cdot \tilde{f}^\varepsilon_x = \tilde{\Gamma}^\varepsilon + \tilde{N}^\varepsilon + \tilde{e}^\varepsilon,$$

where $\tilde{\Gamma}^\varepsilon$ corresponds to the collision term, $\tilde{N}^\varepsilon$ is the martingale term, and $\tilde{e}^\varepsilon$ is a small error that goes to zero as $\varepsilon \to 0$. After a renormalization, we arrive at

$$\tilde{g}^\varepsilon_t + v\cdot \tilde{g}^\varepsilon_x = \bar{\Gamma}^\varepsilon + \bar{N}^\varepsilon + \bar{e}^\varepsilon,$$

where $\tilde{g}^\varepsilon = \frac{\tilde{f}^\varepsilon}{1+n^{-1}\tilde{f}^\varepsilon}$. It turns out that $\bar{N}^\varepsilon \to 0$ as $\varepsilon \to 0$ because $\delta(\varepsilon)/\varepsilon \to +\infty$. As before, $\bar{\Gamma}^\varepsilon$ is more or less like $\tilde{\Gamma}^\varepsilon\left(1+n^{-1}\tilde{f}^\varepsilon\right)^{-2}$. Also, $\tilde{\Gamma}^\varepsilon = \tilde{\Gamma}^\varepsilon_+ - \tilde{\Gamma}^\varepsilon_-$ where, for instance,

$$\begin{aligned}\tilde{\Gamma}^\varepsilon_-(x,v) &= \frac{1}{2}\varepsilon^{2d}\sum_{i,j} V^\varepsilon(|x_i - x_j|)\eta^{\delta(\varepsilon)}(x_i - x, v_i - v)B(v_i - v_j, n_{ij})\\ &=: \frac{1}{2}\varepsilon^{2d}\sum_i \eta^{\delta(\varepsilon)}(x_i - x, v_i - v)K(x_i, v_i)\;.\end{aligned}$$

As before, the Stosszahlensatz can be achieved for the loss term if we can replace $K(x_i, v_i)$ with $K(x, v)$. Of course, we only need to make such a replacement for the renormalized loss term $\bar{\Gamma}^\varepsilon_-$ that is more or less of the form $\tilde{\Gamma}^\varepsilon_-\left(1+n^{-1}\tilde{f}^\varepsilon\right)^{-2}$. Some care is needed to carry out the replacement of $K(x_i, v_i)$ with $K(x, v)$ because K is only (x, v)-regular in L^1-sense, i.e., (3.2) holds. The renormalization factor involves $\tilde{f}^\varepsilon$ that is not so compatible with the type of expression we have for K; the function $K(x, v)$ is a velocity average of a density-like function that resembles f^ε and not $\tilde{f}^\varepsilon = f^{\delta(\varepsilon),\varepsilon}$. This creates a rather delicate situation that is handled by choosing $\delta(\varepsilon)$ in such a way that the smallness of $K(x_i, v_i) - K(x, v)$ would compensate for the incompatibility of $f^{\delta(\varepsilon),\varepsilon}$ with f^ε. The punchline is that we need to choose a sequence $\delta(\varepsilon)$ that satisfies

$$\lim_{\varepsilon\to 0}\delta(\varepsilon)\varepsilon^{-1}(\log\log\log\log|\log\varepsilon|)^{-\frac{1}{2d}} = 0. \tag{3.8}$$

We will see in Theorem 6.1 that for a choice of

$$\delta(\varepsilon) = \varepsilon(\log\log\log\log|\log\varepsilon|)^{\frac{1}{2d+1}} ,$$

the term $K(x_i, v_i)$ can be replaced with $K(x, v)$ in $\bar{\Gamma}^{\varepsilon}_{-}$. To give a partial justification for (3.8), let us assume that something stronger than (3.2) holds for the function K, namely

$$\sup_{|a|,|w|\le\delta} |K(x+a, v+w) - K(x, v)| \le const.(\log\log|\log\delta|)^{-\alpha_0} . \tag{3.9}$$

As a consequence,

$$|K(x_i, v_i) - K(x, v)| \le const.(\log\log|\log\varepsilon|)^{-\alpha_0} ,$$

whenever $\eta^{\delta(\varepsilon)}(x_i - x, v_i - v) \ne 0$. To avoid the incompatibility of $f^{\delta(\varepsilon)}$ with f^{ε}, we apply the crude inequality

$$f^{\delta(\varepsilon),\varepsilon} \ge const.\left(\frac{\delta(\varepsilon)}{\varepsilon}\right)^{-2d} f^{\varepsilon} . \tag{3.10}$$

To guarantee that the smallness of $K(x_i, v_i) - K(x, v)$ is not fully annulled by the large factor $(\delta(\varepsilon)/\varepsilon)^{2d}$, we may require

$$\lim_{\varepsilon\to 0} \delta(\varepsilon)\varepsilon^{-1}(\log\log|\log\varepsilon|)^{-\frac{\alpha_0}{2d}} = 0 . \tag{3.11}$$

It turns out that since K satisfies something like (3.2) instead of (3.9), the requirement on $\delta(\varepsilon)$ is (3.8) instead of (3.11).

As the reader will find out, the microscopic density we will use in Sects. 6-11 are of the form

$$\bar{f}^{\varepsilon}(x, v; \mathbf{q}) = \left(\frac{\delta_1(\varepsilon)}{\varepsilon}\right)^d \left(\frac{\delta_2(\varepsilon)}{\varepsilon}\right)^d \sum_i \eta\left(\frac{x_i - x}{\delta_1(\varepsilon)}, \frac{v_i - v}{\delta_2(\varepsilon)}\right) .$$

Clearly $\bar{f}^{\varepsilon} = f^{\delta(\varepsilon),\varepsilon}$ when $\delta_1 = \delta_2 = \delta$. For the presentation of this section we decided to use $f^{\delta(\varepsilon)}$. However the density $\bar{f}^{\varepsilon}$ with δ_1 and δ_2 satisfying $\lim_{\varepsilon\to 0}\delta_2/\delta_1 = 0$ will simplify some arguments in Sects 9 and 10. See for example (9.24) and (9.25).

To this end, let us assume that the function η is of the form $\eta(x, v) = \zeta(x)\zeta(v)$. Again, the gain term is approximately equal to $\tilde{\Gamma}^{\varepsilon}_{+}\left(1 + n^{-1}\bar{f}^{\varepsilon}\right)^{-2}$, where

$$\tilde{\Gamma}^{\varepsilon}_{+}(x, v) = \sum_{i,j} V^{\varepsilon}(|x_i - x_j|)\tilde{\zeta}^{\varepsilon}(x_i - x)\tilde{\zeta}^{\varepsilon}(v_i^j - v)B(v_i - v_j, n_{ij}) ,$$

with $\tilde{\zeta}^{\varepsilon}(a) = \left(\frac{\varepsilon}{\delta(\varepsilon)}\right)^d \zeta\left(\frac{a}{\delta(\varepsilon)}\right)$. In fact the Stosszahlensatz for the gain term is achieved in two steps. In our first step, which is carried out in Sect. 7, we establish a variant of Stosszahlensatz that is useful only when we show that the macroscopic density is a supersolution. This allows us to generously replace $\bar{\Gamma}^{\varepsilon}_{+}$ with a smaller quantity whenever appropriate. For example, if we define

$$u^{\varepsilon}(x) = u^{\varepsilon}(x; \mathbf{q}) := \sum_j V^{\varepsilon}(|x_j - x|))(|v_j| + 1) ,$$

and pick a smooth function J of the variable v, then we have that the expression

$$\int \bar{\Gamma}_+^\varepsilon(x,v)(1+n^{-1}\tilde{f}^\varepsilon(x,v;\mathbf{q}))^{-2}J(v)dv\ ,$$

is bounded above by

$$\int (1+n^{-1}\tilde{f}^\varepsilon(x,v;\mathbf{q}))^{-2}\sum_{i,j}V^\varepsilon(|x_i-x_j|)\tilde{\zeta}^\varepsilon(x_i-x)\tilde{\zeta}^\varepsilon(v_i^j-v)$$
$$B(v_i-v_j,n_{ij})(1+\ell^{-1}u^\varepsilon(x_i;\mathbf{q}))^{-1}\ J(v)\ dv\ , \tag{3.12}$$

for every positive ℓ. We then show that the omission of the term $(1+n^{-1}\tilde{f}^\varepsilon(x,v;\mathbf{q}))^{-2}$ from the right-hand side causes an error that is small for large n. This turns out to be useful because we would rather have a renormalization of the form $(1+\ell^{-1}u^\varepsilon(x_i;\mathbf{q}))^{-1}$ instead of $(1+n^{-1}\tilde{f}^\varepsilon(x,v;\mathbf{q}))^2$ when we are dealing with the gain term. This stems from the fact that $u^\varepsilon(x;\mathbf{q})$ is a velocity averaging for which the regularity (3.2) applies. After dropping $(1+n^{-1}\tilde{f}^\varepsilon(x,v;\mathbf{q}))^{-2}$ from (3.12), we are left with

$$\frac{1}{2}\varepsilon^d\sum_{i,j}V^\varepsilon(|x_i-x_j|)B(v_i-v_j,n_{ij})\tilde{\zeta}^\varepsilon(x_i-x)(1+\ell^{-1}u^\varepsilon(x_i))^{-1}J^\varepsilon(v_i^j), \tag{3.13}$$

where $J^\varepsilon(v)=\varepsilon^{-d}\int\tilde{\zeta}^\varepsilon(v-w)J(w)dw$. We can now express (3.13) as

$$\sum_i\tilde{\zeta}^\varepsilon(x_i-x)\tilde{K}(x_i,v_i)(1+\ell^{-1}u^\varepsilon(x_i))^{-1},$$

where

$$\tilde{K}(x,v)=\frac{1}{2}\sum_j V^\varepsilon(|x-x_j|)B\left(v-v_j,\frac{x-x_j}{|x-x_j|}\right)J^\varepsilon$$
$$\left(v-(v-v_j)\frac{x-x_j}{|x-x_j|}\frac{x-x_j}{|x-x_j|}\right).$$

It turns out that now we are in a position to repeat our treatment for the loss term where $\tilde{K}(1+\ell^{-1}u^\varepsilon)^{-1}$ plays the role of K.

In Sect. 8 we establish a variant of Stosszahlensatz that is needed when we treat the macroscopic densities as subsolutions. This time we study

$$\left(1+n^{-1}\tilde{u}^\varepsilon(x)\right)^{-1}\int\bar{\Gamma}_+^\varepsilon(x,v)J(v)dv,$$

where

$$\tilde{u}^\varepsilon(x)=\varepsilon^d\sum_j\tilde{\zeta}^\varepsilon(x_j-x)\left(|v_j|^{3/2}+1\right).$$

Our Stosszahlensatz for the collision term allows us to replace the microscopic collision terms with suitable nonlinear functionals of densities that enjoy some stabilities with respect to the weak topology. This will be used in Sects. 9 and 10 to pass to the limit and derive the macroscopic equation (1.2).

4. Entropy and Entropy Production Bound

We start with the entropy bound. Define

$$\nu_\beta(d\mathbf{q}) = \left(\frac{\beta}{\pi}\right)^{\frac{d}{2}} \exp(\beta \sum_i |v_i|^2) d\mathbf{q} \; .$$

Using the property $B(v' - v'_*, n) = B(v - v_*, n)$, it is not hard to deduce that the collision operator $\mathcal{A}_c$ is reversible with respect to the measure ν_β. That is, for every bounded continuous functions η_1 and η_2,

$$\int \eta_2 \mathcal{A}_c \eta_1 \, d\nu_\beta = \int \eta_1 \mathcal{A}_c \eta_2 \, d\nu_\beta \; .$$

From this we can readily deduce that ν_β is an invariant measure and that the adjoint $\mathcal{A}^*$ of $\mathcal{A}$ with respect to ν_β equals to $-\mathcal{A}_0 + \mathcal{A}_c$.

If $\mathbf{q}(0)$ is distributed according to $\mu^0(d\mathbf{q}) = F^0(\mathbf{q})d\mathbf{q} =: G^0(\mathbf{q})\nu_\beta(d\mathbf{q})$, then at later times $\mathbf{q}(t)$ is distributed according to

$$\mu(t, d\mathbf{q}) = F(t, \mathbf{q})d\mathbf{q} =: G(t, \mathbf{q})\nu_\beta(d\mathbf{q}) \; , \tag{4.1}$$

where G is a solution to the *forward equation*

$$G_t = \mathcal{A}^* G \; . \tag{4.2}$$

As in Lemma 2.2 of [RT] we can easily show

Lemma 4.1. *Choose $\beta = \alpha_0/(p-1)$ for α_0 and p as in (2.3). Then there exists a constant $\bar{c}$ such that*

$$\sup_t \int (G(t, \mathbf{q}))^p \, \nu_\beta(d\mathbf{q}) \leq \int \left(G^0(\mathbf{q})\right)^p \nu_\beta(d\mathbf{q}) \leq \exp(\bar{c}N) \; .$$

Regard $\mathbb{T}^d$ as the box $[0, 1]^d$ with opposite faces identified, and partition $\mathbb{T}^d \times \mathbb{R}^d$ into sets of the form,

$$\prod_{r=1}^{d} [a_r, b_r) \times \prod_{r=1}^{d} [a'_r, b'_r) \; ,$$

of side length δ. Let us write $\mathcal{J}^\delta$ for such a partition . We then define

$$\mathcal{N}(\mathbf{q}; K) = \mathcal{N}(x_1, v_1, \ldots, x_N, v_N; K) = \sum_{i=1}^{N} 1\!\!1((x_i, v_i) \in K)$$

for every set $K \subseteq \mathbb{T}^d \times \mathbb{R}^d$ and

$$\Phi^\varepsilon(\mathbf{q}) := \sum_{I \in \mathcal{J}^\varepsilon} \phi(\mathcal{N}(\mathbf{q}; I)) \; ,$$

where $\phi(z) = z \log z$. Similarly, we partition $\mathbb{T}^d$ into sets of the form

$$\prod_{r=1}^{d} [a_r, b_r)$$

of side length δ and write $\tilde{\mathcal{J}}^{\delta}$ for the resulting partition. We then define

$$\tilde{\Phi}^{\varepsilon}(\mathbf{q}) := \sum_{I\in\tilde{\mathcal{J}}^{\varepsilon^2}} \phi(\mathcal{N}(\mathbf{q};\, I\times\mathbb{R}^d))\ ,$$
$$\hat{\Phi}^{\varepsilon}(\mathbf{q}) := \sum_{I\in\tilde{\mathcal{J}}^{\varepsilon}} \phi(\varepsilon^d\mathcal{N}(\mathbf{q};\, I\times\mathbb{R}^d)).$$

Note that each $I\in\tilde{\mathcal{J}}^{\varepsilon}$ can be written as a union of $O(\varepsilon^{-d})$ sets in $\tilde{\mathcal{J}}^{\varepsilon^2}$. From this and convexity of ϕ, it is not hard to show that there exists a constant c such that

$$\hat{\Phi}^{\varepsilon}(\mathbf{q}) \le c\tilde{\Phi}^{\varepsilon}(\mathbf{q})\ . \tag{4.3}$$

Using Lemma 4.1, we can repeat the proof of Theorem 4.1 of [R2] to deduce

Proposition 4.2. *There exists a constant $C_0(T)$ such that*

$$E_N \sup_{0\le s\le T} \exp\left[\frac{p-1}{2p}\left(\Phi^{\varepsilon}(\mathbf{q}(s)) + \tilde{\Phi}^{\varepsilon}(\mathbf{q}(s))\right)\right] \le \exp(C_0(T)N)\ .$$

Define

$$h(\delta) = \begin{cases} |1+\log\delta|^{-1} & \text{if } \delta<1 \\ 1 & \text{otherwise.}\end{cases}$$

Fix a continuous function $\eta : \mathbb{R}^d\times\mathbb{R}^d \to [0,\infty)$ of compact support with $\iint \eta\, dx\, dv = 1$ and define

$$\alpha^{\varepsilon}(x,v) = \int \eta^{\varepsilon}(x-z, v-w)\alpha(z,w)\, dz\, dw,$$

where $\eta^{\varepsilon}(x,v) = \varepsilon^{-2d}\eta\left(\frac{x}{\varepsilon},\frac{v}{\varepsilon}\right)$. As in [RT] and [R2], we have the following consequences of Proposition 4.2:

Proposition 4.3. (i) *There exists a constant $C_1(T,r)$ such that*

$$E_N \sup_{0\le s\le T}\left[N^{-1}\Phi^{\varepsilon}(\mathbf{q}(s)) + N^{-1}\tilde{\Phi}^{\varepsilon}(\mathbf{q}(s))\right]^r \le C_1(T,r)\ ,$$

for every positive integer r.

(ii) *There exists a constant $C_1(\eta)$ such that for every nonnegative α,*

$$\sum_{i=1}^{N}\alpha^{\varepsilon}(x_i,v_i) \le C_1(\eta)\|\alpha\|_{L^\infty}\ h(\|\alpha\|_{L^1})\ (N+\Phi^{\varepsilon}(\mathbf{q})).$$

(iii) *There exists a constant $\hat{C}_1$ such that for every measurable set $K\subseteq\mathbb{T}^d\times\mathbb{R}^d$ and $\tilde{K}\subseteq\mathbb{T}^d$,*

$$\mathcal{N}(\mathbf{q};\, K) \le \hat{C}_1 h(|B_\varepsilon K|)(N+\Phi^{\varepsilon}(\mathbf{q}))\ ,$$
$$\mathcal{N}(\mathbf{q};\, \tilde{K}\times\mathbb{R}^d) \le \hat{C}_1 h(|\tilde{B}_\varepsilon\tilde{K}|)(N+\tilde{\Phi}^{\varepsilon}(\mathbf{q}))\ ,$$

where $B_\varepsilon K = K+\varepsilon[0,1]^{2d}$ and $\tilde{B}_\varepsilon\tilde{K} = \tilde{K}+\varepsilon[0,1]^d$.

Let $\eta : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ be a continuous function such that $\int \eta dxdv = 1$, and define $\zeta_1, \zeta_2 : \mathbb{R}^d \to [0, \infty)$ by $\zeta_1(x) = \int \eta(x, v)dv$ and $\zeta_2(v) = \int \eta(x, v)dx$. Define

$$f^{\delta,\varepsilon}(x, v; \mathbf{q}) = \left(\frac{\varepsilon}{\delta}\right)^{2d} \sum_{i=1}^{N} \eta\left(\frac{x_i - x}{\delta}, \frac{v_i - v}{\delta}\right) , \tag{4.4}$$

$$u_a^{\varepsilon}(x; \mathbf{q}) = \varepsilon^d \sum_{i=1}^{N} \zeta_1\left(\frac{x_i - x}{\varepsilon}\right)(|v_i|^a + 1) . \tag{4.5}$$

We simply write f^ε for $f^{\delta,\varepsilon}$ when $\delta = \varepsilon$ and u^ε for u_a^ε when $a = 1$.

Lemma 4.4. *There exist constants C_2, $C_2(T)$ and $C_2(T, a)$ such that if η satisfies $\eta(x, v) = 0$ for (x, v) with $|x| + |v| \geq r$, $\|\eta\|_{L^\infty} \leq 1$ and $a \in [1, 2)$, then*

$$E_N \sup_t \int (|v|^2 + 1) f^\varepsilon(x, v; \mathbf{q}(t))dxdv \leq C_2(1 + \varepsilon^2 r^2), \tag{4.6}$$

$$E_N \sup_{0\leq t\leq T} \int f^\varepsilon(x, v; \mathbf{q}(t)) \log^+ f^\varepsilon(x, v; \mathbf{q}(t))\, dxdv \leq C_2(T)(1 + r^{2d} \log r), \tag{4.7}$$

$$\begin{aligned} E_N \sup_{0\leq t\leq T} \int u_a^\varepsilon(x; \mathbf{q}(t)) \quad &\left[\log^+ u_a^\varepsilon(x; \mathbf{q}(t))\right]^{1-a/2} dx \\ &\leq C_2(T, a)\left(1 + r^{2d} \log r + \varepsilon^a r^3\right) . \end{aligned} \tag{4.8}$$

Proof. The bound (4.6) is a consequence of the conservation of the kinetic energy; one can readily show

$$\begin{aligned} &\int \sum_i \eta\left(\frac{x_i(t) - x}{\varepsilon}, \frac{v_i(t) - v}{\varepsilon}\right)(|v|^2 + 1)dvdx \\ &= \varepsilon^d \int \sum_i \zeta_2\left(\frac{v_i(t) - v}{\varepsilon}\right)(|v|^2 + 1)dv \\ &= \varepsilon^{2d} \int \sum_i \zeta_2(v)(|v_i(t) - \varepsilon v|^2 + 1)dv \\ &\leq 2Z\varepsilon^2 r^2 + 2Z\varepsilon^{2d} \sum_i |v_i(t)|^2 + Z \\ &= 2Z\varepsilon^2 r^2 + 2Z\varepsilon^{2d} \sum_i |v_i(0)|^2 + Z . \end{aligned}$$

(Recall $Z = \varepsilon^{2d} N$.)

The proof of (4.7) is an immediate consequence of Proposition 4.3(i) and the fact that there exists a constant c_0 such that,

$$f^\varepsilon(x, v; \mathbf{q}) \leq c_0 \mathcal{N}(\mathbf{q}; I_{c_0\varepsilon r}(x, v)) \leq c_0 \sum \left\{\mathcal{N}(\mathbf{q}; I) : I \in \mathcal{J}^\varepsilon,\ I \cap I_{c_0\varepsilon r}(x, v) \neq \emptyset\right\} ,$$

where $I_\alpha(x, v)$ is a cube with center (x, v) and side length α.

For (4.8), observe

$$\begin{aligned}\int f^\varepsilon(x,v;\mathbf{q})(|v|^a+1)dv &= \sum_{i=1}^N \int \eta\left(\frac{x_i-x}{\varepsilon},\frac{v_i-v}{\varepsilon}\right)(|v|^a+1)dv\\ &= \sum_{i=1}^N \varepsilon^d \int \eta\left(\frac{x_i-x}{\varepsilon},v\right)(|v_i-\varepsilon v|^a+1)dv\\ &\ge \varepsilon^d \sum_{i=1}^N \zeta_1\left(\frac{x_i-x}{\varepsilon}\right)(2^{1-a}|v_i|^a+1)\\ &\quad -\varepsilon^a r\varepsilon^d \sum_{i=1}^N \zeta_1\left(\frac{x_i-x}{\varepsilon}\right).\end{aligned}$$

Since,

$$\int f^\varepsilon(x,v;\mathbf{q})dv =: \rho^\varepsilon(x;\mathbf{q}) = \varepsilon^d \sum_{i=1}^N \zeta_1\left(\frac{x_i-x}{\varepsilon}\right),$$

it suffices to bound,

$$\sup_N E_N \sup_{0\le t\le T} \int \tilde{u}^\varepsilon(x;\mathbf{q}(t)) \log^+ \tilde{u}^\varepsilon(x;\mathbf{q}(t))dx, \tag{4.9}$$

where $\tilde{u}^\varepsilon(x;\mathbf{q}) = \int (|v|^a+1)f^\varepsilon(x,v;\mathbf{q})dv$.

Observe that we may write

$$\tilde{u}^\varepsilon(x;\mathbf{q}) = \int \frac{|v|^a+1}{\gamma(v)}\gamma(v)f^\varepsilon(x,v;\mathbf{q})dv,$$

where $\gamma(v) = (2\pi)^{-d/2}\exp(-|v|^2/2)$. If $\tilde{\phi}(z) = z(\log^+ z)^{1-a/2}$, then by Jensen's inequality and the elementary inequalities

$$\begin{aligned}\log^+ AB &\le \log^+ A + \log^+ B,\\ (A+B)^{1-a/2} &\le A^{1-a/2} + B^{1-a/2},\\ AB^{1-a/2} &\le \frac{a}{2}A^{2/a} + \left(1-\frac{a}{2}\right)B,\end{aligned}$$

we deduce,

$$\begin{aligned}\tilde{\phi}(\tilde{u}^\varepsilon(x;\mathbf{q})) &\le \int \tilde{\phi}\left(\frac{|v|^a+1}{\gamma(v)} f^\varepsilon(x,v;\mathbf{q})\right)\gamma(v)dv\\ &\le \int (|v|^a+1)f^\varepsilon(x,v;\mathbf{q})[\log^+ f^\varepsilon(x,v;\mathbf{q})]^{1-a/2}dv\\ &\quad + \int (|v|^a+1)\left[\log(|v|^a+1)+\frac{d}{2}\log 2\pi + \frac{1}{2}|v|^2\right]^{1-a/2} f^\varepsilon(x,v;\mathbf{q})dv\\ &\le \int (|v|^a+1)f^\varepsilon(x,v;\mathbf{q})[\log^+ f^\varepsilon(x,v;\mathbf{q})]^{1-a/2}dv\\ &\quad + c_1\int (|v|^2+1)f^\varepsilon(x,v;\mathbf{q})dv\\ &\le c_2\int (|v|^2+1)f^\varepsilon(x,v;\mathbf{q})dv + c_2\int f^\varepsilon(x,v;\mathbf{q})\log^+ f^\varepsilon(x,v;\mathbf{q})dv,\end{aligned}$$

for some constants c_1 and c_2. This, (4.7) and (4.6) imply (4.8). □

For our purposes, we also need Lemma 4.4 for a function η that is not necessarily of compact support. To this end, let us write $\mathcal{L}$ for the set of continuous functions $\eta : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ such that for some positive k, the function

$$L_k(\eta)(x, v) := \sup_{|z|,|w|\le k} \eta(x + z, v + w) , \tag{4.10}$$

belongs to $L^1(\mathbb{R}^{2d})$.

Lemma 4.5. *Suppose that $\eta \in \mathcal{L}$. Then there exist constants $\hat{C}_2(T, \eta)$ and $\hat{C}_2(T, \eta, a)$ such that for every $a \in [1, 2)$,*

$$E_N \sup_{0\le t\le T} \int f^\varepsilon(x, v; \mathbf{q}(t)) \log^+ f^\varepsilon(x, v; \mathbf{q}(t))\, dxdv \le \hat{C}_2(T, \eta) , \tag{4.11}$$

$$E_N \sup_{0\le t\le T} \int u_a^\varepsilon(x; \mathbf{q}(t)) \left[\log^+ u_a^\varepsilon(x; \mathbf{q}(t))\right]^{1-a/2} dx \le \hat{C}_2(T, \eta, a) . \tag{4.12}$$

Proof. Take a continuous function $\beta : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ such that $\iint \beta dxdv = 1$ and its support is contained in the ball with center at the origin and radius k. We certainly have

$$\begin{aligned} f^\varepsilon(x, v; \mathbf{q}) &= \iint \sum_{i=1}^N \eta\left(\frac{x_i - x}{\varepsilon}, \frac{v_i - v}{\varepsilon}\right) \varepsilon^{-2d} \beta\left(\frac{y - x_i}{\varepsilon}, \frac{w - v_i}{\varepsilon}\right) dydw \qquad (4.13)\\ &\le \iint \sum_{i=1}^N L_k(\eta)\left(\frac{y - x}{\varepsilon}, \frac{w - v}{\varepsilon}\right) \varepsilon^{-2d} \beta\left(\frac{y - x_i}{\varepsilon}, \frac{w - v_i}{\varepsilon}\right) dydw . \end{aligned}$$

Put

$$\begin{aligned} L_k(\eta)_\varepsilon(x, v) &= \varepsilon^{-2d} L_k(\eta)\left(-\frac{x}{\varepsilon}, -\frac{v}{\varepsilon}\right) ,\\ \hat{f}^\varepsilon(x, v; \mathbf{q}) &= \sum_{i=1}^N \beta\left(\frac{x - x_i}{\varepsilon}, \frac{v - v_i}{\varepsilon}\right) . \end{aligned}$$

We can now rewrite (4.13) as

$$f^\varepsilon(x, v; \mathbf{q}) \le (L_k(\eta)_\varepsilon * \hat{f}^\varepsilon)(x, v) , \tag{4.14}$$

where $*$ denotes the convolution. The bound (4.11) is now an immediate consequence of (4.14), Jensen's inequality and (4.7).

For the proof of (4.12), set

$$\beta_1(x) = \int \beta(x, v)dv , \quad \gamma(x) = \int L_k(x, v)dv , \quad \gamma_\varepsilon(x) = \varepsilon^{-d}\gamma\left(-\frac{x}{\varepsilon}\right) .$$

We then have

$$u_a^\varepsilon(x;\mathbf{q}) = \iint \varepsilon^d \sum_{i=1}^N \eta\left(\frac{x_i - x}{\varepsilon}, v\right) \varepsilon^{-d}\beta_1\left(\frac{y - x_i}{\varepsilon}\right)(|v_i|^a + 1)dydv \quad (4.15)$$

$$\le \iint \sum_{i=1}^N \gamma\left(\frac{y-x}{\varepsilon}\right)\beta_1\left(\frac{y-x_i}{\varepsilon}\right)(|v_i|^a+1)dy$$

$$= \gamma_\varepsilon * \hat{u}^\varepsilon(x)\ , \quad (4.16)$$

where

$$\hat{u}^\varepsilon(x) = \varepsilon^d \sum_i \beta_1\left(\frac{x - x_i}{\varepsilon}\right)(|v_i|^a + 1)\ .$$

The bound (4.12) is now an immediate consequence of (4.15), Jensen's inequality and (4.8). □

Using a similar idea, we can also allow a function η that merely belongs to $\mathcal{L}$ in Proposition 4.3(ii). More precisely,

Lemma 4.6. *There exists a constant $\tilde{C}_1(k)$ such that if $\eta \in \mathcal{L}$ with $L_k(\eta) \in L^1$, then*

$$\sum_{i=1}^N \alpha^\varepsilon(x_i, v_i) \le \tilde{C}_1(k)\|L_k(\eta)\|_{L^1}\|\alpha\|_{L^\infty}\ h(\|\alpha\|_{L^1})\ (N + \Phi^\varepsilon(\mathbf{q})),$$

where $\phi(z) = z\log^+ z$ and α^ε is as in Proposition 4.3.

Proof. Without loss of generality, we may assume that $\|L_k(\eta)\|_{L^1} \le 1$. Take a continuous function $\beta : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ such that $\iint \beta dxdv = 1$ and its support is contained in the ball with center at the origin and radius k. We use Proposition 4.3(ii) to assert that for some constant $\tilde{C}_1$,

$$\sum_{i=1}^N \alpha^\varepsilon(x_i, v_i) = \iint\iint \sum_{i=1}^N \eta^\varepsilon(x_i - z, v_i - w)\alpha(z,w)\varepsilon^{-2d}\beta \left(\frac{x_i - y}{\varepsilon}, \frac{v_i - v}{\varepsilon}\right) dzdwdydv$$

$$\le \iint\iint \sum_{i=1}^N L_k(\eta)^\varepsilon(y - z, v - w)\alpha(z,w)\varepsilon^{-2d}\beta \left(\frac{x_i - y}{\varepsilon}, \frac{v_i - v}{\varepsilon}\right) dzdwdydv$$

$$= \iint \sum_{i=1}^N (L_k(\eta)^\varepsilon * \alpha)(y,v)\varepsilon^{-2d}\beta\left(\frac{x_i - y}{\varepsilon}, \frac{v_i - v}{\varepsilon}\right) dydv$$

$$\le C_1(\beta)\|L_k(\eta)^\varepsilon * \alpha\|_{L^\infty}\ h\left(\|L_k(\eta)^\varepsilon * \alpha\|_{L^1}\right)(N + \Phi^\varepsilon(\mathbf{q}))$$

$$\le \tilde{C}_1(k)\|\alpha\|_{L^\infty}\ h(\|\alpha\|_{L^1})(N + \Phi^\varepsilon(\mathbf{q}))\ ,$$

where

$$L_k(\eta)^\varepsilon(x,v) = \varepsilon^{-2d}L_k(\eta)\left(\frac{x}{\varepsilon}, \frac{v}{\varepsilon}\right)\ .$$

This completes the proof of lemma. □

We now turn to the entropy production bound.

Lemma 4.7. *There exists a constant C_3 such that*

$$\int_0^\infty \sum_{i,j} V^\varepsilon(|x_i - x_j|) B(v_i - v_j, n_{ij}) \psi\left(\frac{G(t, S^{ij}\mathbf{q})}{G(t,\mathbf{q})}\right) G(t,\mathbf{q}) \nu_\beta(d\mathbf{q}) \leq C_3 N ,$$

where $\psi(z) = z \log z - z + 1$.

Proof. Recall that $\mathbf{q}(t)$ is distributed according to

$$\mu(t, d\mathbf{q}) = G(t,\mathbf{q}) \nu_\beta(d\mathbf{q}) ,$$

with G solving (4.2). Define

$$H(t) = \int \log G(t,\mathbf{q}) \mu(t, d\mathbf{q}) = \int G(t,\mathbf{q}) \log G(t,\mathbf{q}) \nu_\beta(d\mathbf{q}) ,$$

with β as in Lemma 4.1. Recall that ν_β is invariant for both $\mathcal{A}_0$ and $\mathcal{A}_c$. A straightforward calculation yields

$$\begin{aligned} \frac{d}{dt} H(t) &= \int (\mathcal{A} \log G)(t,\mathbf{q}) G(t,\mathbf{q}) \nu_\beta(d\mathbf{q}) \\ &= \int (\mathcal{A}_c \log G) G d\nu_\beta + \int \mathcal{A}_0 G \, d\nu_\beta \\ &= \int (\mathcal{A}_c \log G) G d\nu_\beta + \int \mathcal{A}_c G \, d\nu_\beta . \end{aligned}$$

We now use

$$\begin{aligned} \int (\mathcal{A}_c \log G) G d\nu_\beta &= \int \sum_{i,j} V^\varepsilon(|x_i - x_j|) B(v_i - v_j, n_{ij}) \log \frac{G(S^{ij}\mathbf{q})}{G(\mathbf{q})} G(\mathbf{q}) \nu_\beta(d\mathbf{q}) \\ &= \int \sum_{i,j} V^\varepsilon(|x_i - x_j|) B(v_i - v_j, n_{ij}) \log \frac{G(\mathbf{q})}{G(S^{ij}\mathbf{q})} G(S^{ij}\mathbf{q}) \nu_\beta(d\mathbf{q}), \end{aligned}$$

to deduce

$$\frac{d}{dt} H(t) = -\int \sum_{i,j} V^\varepsilon(|x_i - x_j|) B(v_i - v_j, n_{ij}) \psi\left(\frac{G(t, S^{ij}\mathbf{q})}{G(t,\mathbf{q})}\right) G(t,\mathbf{q}) \nu_\beta(d\mathbf{q}).$$

This completes the proof because $H(0) \leq const \cdot N$. $\square$

5. Compactness of Averaged Densities

Recall that the microscopic density f^ε is defined by $f^\varepsilon(x,v;\mathbf{q}) = \sum_i \eta\left(\frac{x_i - x}{\varepsilon}, \frac{v_i - v}{\varepsilon}\right)$ for a nonnegative continuous function η of compact support such that $\int \eta dx dv = 1$. On account of the collision term, we would like to study

$$K^\varepsilon(x,v;\mathbf{q}) = \sum_i \varepsilon^d V\left(\frac{|x-x_i|}{\varepsilon}\right) B\left(v - v_i, \frac{x - x_i}{|x-x_i|}\right) .$$

More generally, we may take two continuous functions $\xi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $\zeta : \mathbb{R}^d \to [0,\infty)$, and define

$$\hat{f}^\varepsilon(x,v,w;\mathbf{q}) = \sum_i \xi\left(\frac{x-x_i}{\varepsilon}, v - w\right)\zeta\left(\frac{v_i - w}{\varepsilon}\right) .$$

We assume $\int \zeta dx = 1$ and that ζ is of compact support. The function $\hat{f}^\varepsilon$ is a density-like function and we would like to study its *average*

$$\rho^\varepsilon(x,v,t) = \varepsilon^d \sum_i \xi\left(\frac{x - x_i(t)}{\varepsilon}, v - v_i(t)\right) .$$

The main objective of this section is a strong compactness result for the averaged density ρ^ε. For this we will need some conditions on ξ. To state these conditions, we fix a constant $b \in [0,1)$ and define several seminorms:

$$\begin{aligned}
\mathcal{R}_0(\xi) &= \sup_x |\xi(x,v)|(1+|v|)^{-b-1}, \\
\mathcal{R}_1(\xi) &= \int (1+|x|)|\xi(x,0)|dx, \\
\mathcal{R}_2(\xi) &= \int (1+|x|) \sup_{v\neq w} |\xi(x,v) - \xi(x,w)||v-w|^{-1}(1+|v|+|w|)^{-b}\, dx, \qquad (5.1)\\
\mathcal{R}_3(\xi) &= \sup_v \sup_{x\neq y} |\xi(x,v) - \xi(y,v)||x-y|^{-1}(1+|v|)^{-b-1}, \\
\mathcal{R}_4(\xi) &= \sup_v \sup_{x\neq y} |\xi(x,v) - \xi(y,v)||x-y|^{-1} .
\end{aligned}$$

Theorem 5.1. *There exists a constant $C_4(T)$ such that if*

$$\mathcal{R}(\xi) := \mathcal{R}_0(\xi) + \mathcal{R}_1(\xi) + \mathcal{R}_2(\xi) + \mathcal{R}_3(\xi) \le 1, \qquad (5.2)$$

then

$$\begin{aligned}
E_N \sup_{|h|<\delta} \sup_{\alpha\in[0,\delta]} \int_0^T \int |\rho^\varepsilon(x+h,v,t+\alpha) - \rho^\varepsilon(x,v,t)|dxdt \\
\le C_4(T)(1+|v|^{(b+3)/2})\left[(\log\log|\log\delta|)^{-\alpha_b} + \varepsilon\right] .
\end{aligned}$$

for every v, where $\alpha_b = (2d+b+2)^{-1}(d+3)^{-1}$.

To prepare for the proof of Theorem 5.1, let us define

$$\hat\rho^\varepsilon(x,v,t)=\int \hat f^\varepsilon(x,v,w;\mathbf{q}(t))dw\ .$$

We now state a lemma that is equivalent to Theorem 5.1.

Lemma 5.2. *There exists a constant $\tilde C_4(T)$ such that if ξ satisfies (5.2), then*

$$E_N\sup_{|h|<\delta}\sup_{\alpha\in[0,\delta]}\int_0^T\int|\hat\rho^\varepsilon(x+h,v,t+\alpha)-\hat\rho^\varepsilon(x,v,t)|dxdt \le \tilde C_4(T)(1+|v|^{(b+3)/2})(\log\log|\log\delta|)^{-\alpha_b}\ ,$$

for every v.

We first demonstrate how Lemma 5.2 implies Theorem 5.1.

Proof of Theorem 5.1. Since ξ satisfies (5.2), there exists an integrable function γ such that

$$|\xi(x,v)-\xi(x,w)|\le\gamma(x)(1+|v|+|w|)^b|v-w|\ ,\quad \int\gamma(x)dx\le 1\ . \tag{5.3}$$

Note that we can find a constant c_1 such that if $\zeta(z)\neq 0$, then $|z|\le c_1\varepsilon$. From this and (5.3) we deduce,

$$\begin{aligned}|\rho^\varepsilon(x,v,t)&-\hat\rho^\varepsilon(x,v,t)|\\ &=\left|\int\sum_i\left(\xi\left(\frac{x-x_i}{\varepsilon},v-v_i\right)-\xi\left(\frac{x-x_i}{\varepsilon},v-w\right)\right)\zeta\left(\frac{v_i-w}{\varepsilon}\right)dw\right|\\ &\le c_2\varepsilon\int\sum_i\gamma\left(\frac{x-x_i}{\varepsilon}\right)\zeta\left(\frac{v_i-w}{\varepsilon}\right)(1+|v-v_i|^b)dw\end{aligned}$$

for some constant c_2. As a result of this, the elementary inequality $|v-v_i|^b\le 2+2|v|^b+2|v_i|^2$, and the conservation of the kinetic energy we have,

$$\int_0^T\int|\rho^\varepsilon(x,v,t)-\hat\rho^\varepsilon(x,v,t)|dxdt\le c_3\varepsilon(1+|v|^b)\ ,$$

for a constant c_3. This and Lemma 5.2 imply Theorem 5.1. □

For the proof of Lemma 5.2, we first replace the density $\hat f^\varepsilon$ with the *renormalized density*

$$g_n^\varepsilon(x,v,w;\mathbf{q}(t))=\frac{n\hat f^\varepsilon(x,v,w;\mathbf{q}(t))}{n+\hat f^\varepsilon(x,v,w;\mathbf{q}(t))}\ ,$$

where n is a positive integer. Define $m^\varepsilon(x,v,t)=m^\varepsilon_{n,\ell_0}(x,v,t)=\int g_n^\varepsilon(x,v,w;\mathbf{q}(t))\chi_{\ell_0}(w)dw$, where $\chi_{\ell_0}(w)=\mathbb{1}(|w|\le\ell_0)$. The next lemma is the main ingredient for the proof of Lemma 5.2.

Let $\mathcal{C}_r$ denote the set of continuous $\xi:\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty)$ such that $\xi(x,v)=0$ for $|x|+|v|\ge r$ and $\|\xi\|_{L^\infty}+\mathcal{R}_4(\xi)\le 1$.

Lemma 5.3. *There exists a constant $C_5(T)$ such that if $\xi \in \mathcal{C}_r$ and $r \geq 1$, then*

$$E_N \sup_{|h|<\delta} \sup_{\alpha\in[0,\delta]} \int_0^T \int |m^\varepsilon(x+h,v,t+\alpha) - m^\varepsilon(x,v,t)|dxdt$$
$$\leq C_5(T) r^{2d} \log rn\ell_0^{d+1} \left(\log^+(n\ell_0)\right)^{1/2} (\log|\log\delta|)^{-1/2} ,$$

for every v.

An important tool to be used for the proof of Lemma 5.4 is the celebrated *averaging lemma*:

Lemma 5.4. *There exists a constant $C_6(T)$ such that if $\hat{m}(x,t) = \int g^0(x-wt,w)\chi_{\ell_0}(w)dw$ and $\tilde{m}(x,t) = \int m(x,w,t)\chi_{\ell_0}(w)dw$ with $m(x,w,t) = \int_0^t \gamma(x-w(t-s),w,s)ds$, then*

$$\int_0^T \int_0^T \iint \frac{(\tilde{m}(x,t)-\tilde{m}(y,s))^2}{|(x,t)-(y,s)|^{d+2}} dxdydsdt \leq C_6(T)\ell_0^d \|\gamma\|_{L^2_{\ell_0}} \|m\|_{L^2_{\ell_0}} ,$$
$$\int_0^T \int_0^T \iint \frac{(\hat{m}(x,t)-\hat{m}(y,s))^2}{|(x,t)-(y,s)|^{d+2}} dxdydsdt \leq C_6(T)\ell_0^d \|g^0\|^2_{L^2_{\ell_0}} ,$$

where

$$\|\gamma\|^2_{L^2_{\ell_0}} = \int_0^T \iint \gamma^2(x,w,t)\chi_{\ell_0}(w)dxdwdt,$$
$$\|g^0\|_{L^2_{\ell_0}} = \iint (g^0)^2(x,w)\chi_{\ell_0}(w)dxdw .$$

See for example [GLiPS] for a proof.

Proof of Lemma 5.3. Step 1. First observe that it suffices to establish the lemma for a ξ that is continuously differentiable. This is because if ξ is merely x-Lipschitz, then we may approximate it by continuously differentiable functions and pass to the limit. From now on we assume that ξ is continuously differentiable.

We define $\beta(r) = \frac{nr}{r+n}$ and $F(x,v,w;\mathbf{q}) = \beta(\hat{f}^\varepsilon(x,v,w;\mathbf{q})) = g_n^\varepsilon(x,v,w;\mathbf{q})$. Write $g(x,v,w,t)$ for $g_n^\varepsilon(x,v,w;\mathbf{q}(t))$, and set $\hat{g}(x,v,w,t) := g(x+wt,v,w,t)$. Evidently $\hat{g}(x,v,w,t) = F(x+wt,v,w;\mathbf{q}(t))$. It is well known that the process

$$M(x,v,w,t) = F(x+wt,v,w;\mathbf{q}(t)) - F(x,v,w;\mathbf{q}(0))$$
$$- \int_0^t \left(\frac{\partial}{\partial s} + \mathcal{A}\right) F(x+ws,v,w;\mathbf{q}(s))ds ,$$

is a martingale and that its quadratic variation $E_N(M(x,v,w,t) - M(x,v,w,s))^2$ is given by

$$E_N \int_s^t (\mathcal{A}F^2 - 2F\mathcal{A}F)(x+w\theta,v,w;\mathbf{q}(\theta))d\theta \tag{5.4}$$
$$= E_N \int_s^t (\mathcal{A}_c F^2 - 2F\mathcal{A}_c F)(x+w\theta,v,w;\mathbf{q}(\theta))d\theta .$$

As a result, we may write

$$\hat{g}(x,v,w,t) = \hat{g}(x,v,w,0) + \int_0^t A(x+ws,v,w,s)ds + \int_0^t D(x+ws,v,w,s)ds + M(x,v,w,t) \ ,$$

where

$$A(x,v,w,s) = \left(w \cdot \frac{\partial}{\partial x} + \mathcal{A}_0\right) F(x,v,w;\mathbf{q}(s))$$
$$D(x,v,w,s) = \mathcal{A}_c F(x,v,w;\mathbf{q}(s)) \ .$$

Hence

$$g(x,v,w,t) = g(x-wt,v,w,0) + \int_0^t A(x-w(t-s),v,w,s)ds + \int_0^t D(x-w(t-s),v,w,s)ds + M(x-wt,v,w,t) \ . \quad (5.5)$$

Our goal is to show that the w-average of each term on the right-hand side of (5.5) is strongly compact with respect to the L^1 topology . In view of the averaging lemma, we would like to show that the functions A and D are weakly compact in L^1.

Step 2. We have that the expression $\left(w \cdot \frac{\partial}{\partial x} + \mathcal{A}_0\right) F(x,v,w;\mathbf{q})$ equals to

$$\beta'(\hat{f}^\varepsilon(x,v,w;\mathbf{q})) \sum_i \left(\frac{w-v_i}{\varepsilon}\right) \cdot \hat{\xi}\left(\frac{x-x_i}{\varepsilon}, v-w\right) \zeta \left(\frac{v_i - w}{\varepsilon}\right) \ ,$$

where $\hat{\xi}(z,v)$ denotes the z-gradient of $\xi(z,v)$. As a result,

$$|A(x,v,w,s)| \leq \beta'(\hat{f}^\varepsilon(x,v,w;\mathbf{q}))\tilde{f}^\varepsilon(x,v,w;\mathbf{q}(s)) \ ,$$

where $\tilde{f}^\varepsilon(x,v,w;\mathbf{q}) = \sum_i \tilde{\eta}\left(\frac{x_i-x}{\varepsilon}, \frac{v_i-w}{\varepsilon}, v-w\right)$ for $\tilde{\eta}(z,v,w) = |v \cdot \hat{\xi}(-z,w)|\zeta(v)$. Since β' is bounded by 1, we deduce

$$|A(x,v,w,s)| \leq \tilde{f}^\varepsilon(x,v,w;\mathbf{q}(s)) \ . \quad (5.6)$$

This and (4.7) imply that there exists a constant c_0 such that

$$\sup_N E_N \sup_{s\in[0,T]} \iint \phi(|A(x,v,w,s)|)\mathbb{1}(|w| \leq \ell_0) dx dw \leq c_0 r^{2d} \log r \quad (5.7)$$

where $\phi(z) = z \log^+ z$.

Step 3. We now concentrate on the collision term. We have that $2\mathcal{A}_c F(x, v, w; \mathbf{q})$ equals to

$$
\begin{aligned}
&\sum_{i,j} V^\varepsilon\left(|x_i - x_j|\right) B(v_i - v_j, n_{ij}) \left(1 + n^{-1}\hat{f}^\varepsilon(x, v, w; \mathbf{q})\right)^{-1} \\
&\left(1 + n^{-1}\hat{f}^\varepsilon(x, v, w; S^{ij}\mathbf{q})\right)^{-1} \\
&\left[\xi\left(\frac{x - x_i}{\varepsilon}, v - w\right)\zeta\left(\frac{v_i^j - w}{\varepsilon}\right) + \xi\left(\frac{x - x_j}{\varepsilon}, v - w\right)\zeta\left(\frac{v_j^i - w}{\varepsilon}\right)\right. \\
&\left.-\xi\left(\frac{x - x_i}{\varepsilon}, v - w\right)\zeta\left(\frac{v_i - w}{\varepsilon}\right) - \xi\left(\frac{x - x_j}{\varepsilon}, v - w\right)\zeta\left(\frac{v_j - w}{\varepsilon}\right)\right] \\
=: {}& \Omega_1 + \Omega_2 - \Omega_3 - \Omega_4,
\end{aligned}
$$

where, for example, $\Omega_1 = \Omega_1(x, v, w, \mathbf{q})$ equals to

$$
\begin{aligned}
&\varepsilon^d \sum_{i,j} V\left(\frac{|x_i - x_j|}{\varepsilon}\right) B(v_i - v_j, n_{ij}) \left(1 + n^{-1}\hat{f}^\varepsilon(x, v, w; \mathbf{q})\right)^{-1} \\
&\left(1 + n^{-1}\hat{f}^\varepsilon(x, v, w; S^{ij}\mathbf{q})\right)^{-1} \xi\left(\frac{x - x_i}{\varepsilon}, v - w\right)\zeta\left(\frac{v_i^j - w}{\varepsilon}\right) .
\end{aligned}
$$

Using this decomposition we can write

$$
D(x, v, w, s) = \frac{1}{2} D^+(x, v, w, s) - \frac{1}{2} D^-(x, v, w, s) ,
$$

where

$$
\begin{aligned}
D^+(x, v, w, s) &= \Omega_1(x, v, w, \mathbf{q}(s)) + \Omega_2(x, v, w, \mathbf{q}(s)) , \\
D^-(x, v, w, s) &= \Omega_3(x, v, w, \mathbf{q}(s)) + \Omega_4(x, v, w, \mathbf{q}(s)) .
\end{aligned}
$$

The term Ω_3 is bounded above by $\hat{D}^-$ which is equal

$$
\begin{aligned}
&\varepsilon^d \sum_{i,j} V\left(\frac{|x_i - x_j|}{\varepsilon}\right) B(v_i - v_j, n_{ij})\, \xi\left(\frac{x - x_i}{\varepsilon}, v - w\right)\zeta\left(\frac{v_i - w}{\varepsilon}\right) \\
&\qquad\qquad \left(1 + n^{-1}\hat{f}^\varepsilon(x, v, w; \mathbf{q})\right)^{-1} \\
&\leq c_1 \varepsilon^d \sum_{i,j} 1\!\!1(|x_j - x| \leq c_1 r\varepsilon)|v_i - v_j|\, \xi\left(\frac{x - x_i}{\varepsilon}, v - w\right)\zeta\left(\frac{v_i - w}{\varepsilon}\right) \\
&\qquad\qquad \left(1 + n^{-1}\hat{f}^\varepsilon(x, v, w; \mathbf{q})\right)^{-1} \\
&\leq c_1 \varepsilon^d \sum_{i,j} 1\!\!1(|x_j - x| \leq c_1 r\varepsilon)(|v_j - w| + c_2\varepsilon)\xi\left(\frac{x - x_i}{\varepsilon}, v - w\right)\zeta\left(\frac{v_i - w}{\varepsilon}\right) \\
&\qquad\qquad \left(1 + n^{-1}\hat{f}^\varepsilon(x, v, w; \mathbf{q})\right)^{-1} \\
&\leq c_1 n \varepsilon^d \sum_{j} 1\!\!1(|x_j - x| \leq c_1 r\varepsilon)(|v_j - w| + c_2\varepsilon) ,
\end{aligned}
$$

for some constants c_1 and c_2. In the same fashion, we can treat Ω_4. As a result

$$\begin{aligned} D^-(x,v,w,s) &\le 2\hat{D}^-(x,v,w,s) \\ &\le 2c_1 n\varepsilon^d \sum_j \mathbb{1}(|x_j(s)-x| \le c_1 r\varepsilon)(|v_j(s)-w| + c_2\varepsilon) . \end{aligned} \tag{5.8}$$

From this and (4.8) we deduce that there exists a constant c_3 such that,

$$E_N \sup_{s\in[0,T]} \iint \tilde{\phi}(D^-(x,v,w,s))\mathbb{1}(|w| \le \ell_0) dx dw \le c_3 \tilde{\phi}(n\ell_0)(r^{2d}\log r + \varepsilon r^3) , \tag{5.9}$$

where $\tilde{\phi}(z) = z\sqrt{\log^+ z}$.

Step 4. In this step we study the function D^+. From $B(v_i^j - v_j^i, n_{ij}) = B(v_i - v_j, n_{ij})$ we deduce that

$$\begin{aligned} &E_N \Omega_1(x,v,w,\mathbf{q}(s))\ \mathbb{1}(\Omega_1(x,v,w,\mathbf{q}(s)) \ge \ell) \\ &= \int \Omega_1(x,v,w,\mathbf{q})\ \mathbb{1}(\Omega_1(x,v,w,\mathbf{q}) \ge \ell) G(s,\mathbf{q})\ \nu_\beta(d\mathbf{q}) , \end{aligned}$$

is bounded above by,

$$\begin{aligned} &\int \sum_{i,j} V^\varepsilon(|x_i - x_j|) B(v_i - v_j, n_{ij})\, \xi\left(\frac{x-x_i}{\varepsilon}, v-w\right) \zeta\left(\frac{v_i^j - w}{\varepsilon}\right) \\ &\quad \left(1 + n^{-1}\hat{f}^\varepsilon(x,v,w;S^{ij}\mathbf{q})\right)^{-1} \mathbb{1}(\Omega_1(x,v,w,\mathbf{q}) \ge \ell) G(s,\mathbf{q}) \nu_\beta(d\mathbf{q}) \\ &= \int \sum_{i,j} V^\varepsilon(|x_i - x_j|) B(v_i - v_j, n_{ij})\, \xi\left(\frac{x-x_i}{\varepsilon}, v-w\right) \zeta\left(\frac{v_i - w}{\varepsilon}\right) \\ &\quad \left(1 + n^{-1}\hat{f}^\varepsilon(x,v,w;\mathbf{q})\right)^{-1} \mathbb{1}(\Omega_1(x,v,w,S^{ij}\mathbf{q}) \ge \ell) G(s,S^{ij}\mathbf{q}) \nu_\beta(d\mathbf{q}) . \end{aligned}$$

Fix $k > 1$. We now use the elementary inequality

$$\begin{aligned} a &= \frac{a}{b} b = \frac{a}{b} b \mathbb{1}\left(\frac{a}{b} \le k\right) + \frac{a}{b} b \mathbb{1}\left(\frac{a}{b} > k\right) \\ &\le kb + \left[\frac{a}{b}\log\frac{a}{b} - \frac{a}{b} + 1\right] \frac{b}{\log k - 1} \end{aligned} \tag{5.10}$$

to deduce that $E_N \Omega_1(x,v,w,\mathbf{q}(s))\ \mathbb{1}(\Omega_1(x,v,w,\mathbf{q}(s)) \ge \ell)$ is bounded above by

$$\begin{aligned} &k \int \sum_{i,j} V^\varepsilon(|x_i - x_j|) B(v_i - v_j, n_{ij}) \xi\left(\frac{x-x_i}{\varepsilon}, v-w\right) \zeta\left(\frac{v_i - w}{\varepsilon}\right) \\ &\quad \left(1 + n^{-1}\hat{f}^\varepsilon(x,v,w;\mathbf{q})\right)^{-1} \mathbb{1}(\Omega_1(x,v,w,S^{ij}\mathbf{q}) \ge \ell) G(s,\mathbf{q}) \nu_\beta(d\mathbf{q}) \\ &+ \frac{1}{\log k - 1} \int \sum_{i,j} V^\varepsilon(|x_i - x_j|) B(v_i - v_j, n_{ij}) \xi\left(\frac{x-x_i}{\varepsilon}, v-w\right) \zeta\left(\frac{v_i - w}{\varepsilon}\right) \\ &\quad \left(1 + n^{-1}\hat{f}^\varepsilon(x,v,w;\mathbf{q})\right)^{-1} \psi\left(\frac{G(s,S^{ij}\mathbf{q})}{G(s,\mathbf{q})}\right) G(s,\mathbf{q}) \nu_\beta(d\mathbf{q}) \\ &=: \Omega_{11} + \Omega_{12} , \end{aligned}$$

for every $k > 1$. We certainly have that the expression $\int_0^T \iint \Omega_{12} dxdwds$ is bounded above by

$$\frac{1}{\log k - 1}\int_0^T \iint \int \sum_{i,j} V^\varepsilon(|x_i - x_j|)B(v_i - v_j, n_{ij})$$
$$\mathbb{1}(|x_i - x| \leq r\varepsilon)\zeta\left(\frac{v_i - w}{\varepsilon}\right)\psi\left(\frac{G(s, S^{ij}\mathbf{q})}{G(s,\mathbf{q})}\right)\nu_\beta(d\mathbf{q})dxdwds \qquad (5.11)$$
$$= \varepsilon^{2d} r^d \frac{1}{\log k - 1}\int_0^T \int \sum_{i,j} V^\varepsilon(|x_i - x_j|)B(v_i - v_j, n_{ij})$$
$$\psi\left(\frac{G(s, S^{ij}\mathbf{q})}{G(s,\mathbf{q})}\right)\nu_\beta(d\mathbf{q})ds \leq \frac{C_3 r^d}{\log k - 1} ,$$

where for the last inequality we have used Lemma 4.7.

Step 5. We now turn to Ω_{11}. Fix $p \geq 1$. We can certainly write

$$\Omega_{11} = \Omega_{111} + \Omega_{112} ,$$

where Ω_{111} equals to

$$k\int \sum_{i,j} V^\varepsilon(|x_i - x_j|)B(v_i - v_j, n_{ij})\mathbb{1}(|v_i - v_j| \leq p)\xi\left(\frac{x - x_i}{\varepsilon}, v - w\right)\zeta\left(\frac{v_i - w}{\varepsilon}\right)$$
$$\left(1 + n^{-1}\hat{f}^\varepsilon(x, v, w; \mathbf{q})\right)^{-1}\mathbb{1}(\Omega_1(x, v, w, S^{ij}\mathbf{q}) \geq \ell)G(s,\mathbf{q})\nu_\beta(d\mathbf{q}) .$$

The term Ω_{112} is obtained from Ω_{111} by replacing $\mathbb{1}(|v_i - v_j| \leq p)$ with $\mathbb{1}(|v_i - v_j| > p)$. One can readily show that for some constant c_4,

$$\Omega_{112} \leq kc_4 \int \varepsilon^d \sum_{i,j}\mathbb{1}(|x_j - x| \leq c_4 r\varepsilon,\ |v_j - w| \geq p - c_4\varepsilon)(|v_j - w| + \varepsilon)$$
$$\xi\left(\frac{x - x_i}{\varepsilon}, v - w\right)\zeta\left(\frac{v_i - w}{\varepsilon}\right)\left(1 + n^{-1}\hat{f}^\varepsilon(x, v, w; \mathbf{q})\right)^{-1} G(s,\mathbf{q})\nu_\beta(d\mathbf{q})$$
$$\leq c_4 nk \int \varepsilon^d \sum_j \mathbb{1}(|x_j - x| \leq c_4 r\varepsilon,\ |v_j - w|$$
$$\geq p - c_4\varepsilon)(|v_j - w| + \varepsilon)G(s,\mathbf{q})\nu_\beta(d\mathbf{q}) .$$

From this we deduce that if $p \geq c_4/2$ and $\varepsilon \leq 1$, then the expression

$$\int_0^T \iint \Omega_{112}\mathbb{1}(|w| \leq \ell_0)dxdwds ,$$

is bounded above by

$$c_5 nkr^d E_N \int_0^T \int \varepsilon^{2d} \sum_j (|v_j(s) - w| + 1) \mathbb{1}(|w| \le \ell_0) \mathbb{1}(|v_j(s) - w| \ge p/2) dw ds$$
$$\le 2c_5 nkr^d p^{-1} E_N \int_0^T \int \varepsilon^{2d} \sum_j (|v_j(s) - w|^2 + |v_j(s) - w|) \mathbb{1}(|w| \le \ell_0) dw ds$$
$$\le 4c_5 nkr^d p^{-1} E_N \int_0^T \varepsilon^{2d} \sum_j (|v_j(s) - w|^2 + 1) \mathbb{1}(|w| \le \ell_0) dw ds$$
$$\le c_6 nkTr^d \ell_0^d p^{-1} E_N \varepsilon^{2d} \sum_j (|v_j(0)|^2 + \ell_0^2 + 1) ,$$

for some constants c_5 and c_6. As a result, there exists a constant c_7 such that if $p \ge c_4/2$, then

$$\int_0^T \iint \Omega_{112} \mathbb{1}(|w| \le \ell_0) dx dw ds \le c_7 nkr^d \ell_0^{d+2} p^{-1} . \tag{5.12}$$

Step 6. To treat the term Ω_{111}, we first replace $\mathbb{1}(\Omega_1(x, v, w, \mathbf{q}(s)) \ge \ell)$ with a more tractable expression. To ease the notation, let us write $\Omega_1(\mathbf{q})$ for $\Omega_1(x, v, w, \mathbf{q})$. It is not hard to show

$$\Omega_1(S^{ij}\mathbf{q}) - \Omega_1(\mathbf{q}) \le X_1(i, j, \mathbf{q}) + X_2(i, j, \mathbf{q}) + Y(i, j, \mathbf{q}) ,$$

where

$$Y(i, j, \mathbf{q}) = V^\varepsilon \left(|x_i - x_j|\right) B(v_i - v_j, n_{ij}) \xi \left(\frac{x - x_i}{\varepsilon}, v - w\right) \zeta \left(\frac{v_i - w}{\varepsilon}\right) ,$$

$$X_1(i, j, \mathbf{q}) = \sum_{k \ne j} V^\varepsilon (|x_i - x_k|) B(v_i - v_k, n_{ik}) \zeta \left(\frac{v_i^j - (v_i^j - v_k) \cdot n_{ik} n_{ik} - w}{\varepsilon}\right) \xi \left(\frac{x - x_i}{\varepsilon}, v - w\right) ,$$

$$X_2(i, j, \mathbf{q}) = \sum_{k \ne i} V^\varepsilon \left(|x_j - x_k|\right) B(v_j - v_k, n_{jk}) \zeta \left(\frac{v_k + (v_j^i - v_k) \cdot n_{jk} n_{jk} - w}{\varepsilon}\right) \xi \left(\frac{x - x_k}{\varepsilon}, v - w\right) .$$

Observe that if

$$B(v_i - v_j, n_{ij}) \mathbb{1}(|v_i - v_j| \le p) V \left(\frac{|x_i - x_j|}{\varepsilon}\right) \xi \left(\frac{x - x_i}{\varepsilon}, v - w\right) \zeta \left(\frac{v_i - w}{\varepsilon}\right) \ne 0 ,$$

then

$$|x_i - x| \le c_8 r\varepsilon , \quad |x_j - x| \le c_8 r\varepsilon , \quad |v_i - w| \le c_8 \varepsilon , \quad |v_i - v_j| \le p ,$$

for some constant c_8. As a result, the expression $\Omega_1(S^{ij}\mathbf{q}) - \Omega_1(\mathbf{q})$ is bounded above by

$$c_9\varepsilon^d \sum_k 1\!\!1(|x_k - x| \le c_9 r\varepsilon)(|v_k - w| + p) =: R(x, w, \mathbf{q}) ,$$

for some constant c_9. From this we deduce that Ω_{111} is bounded above by

$$\begin{aligned}
&k \int \sum_{i,j} V^\varepsilon(|x_i - x_j|)B(v_i - v_j, n_{ij})\xi\left(\frac{x - x_i}{\varepsilon}, v - w\right)\zeta\left(\frac{v_i - w}{\varepsilon}\right) \\
&\quad \left(1 + n^{-1}\hat{f}^\varepsilon(x, v, w; \mathbf{q})\right)^{-1} 1\!\!1(\Omega_1(x, v, w, \mathbf{q}) + R(x, w, \mathbf{q}) \ge \ell)G(s, \mathbf{q})\nu_\beta(d\mathbf{q}) \\
&=: kE_N\hat{D}^-(x, v, w, s)1\!\!1(\Omega_1(x, v, w, \mathbf{q}(s)) + R(x, w, \mathbf{q}(s)) \ge \ell) .
\end{aligned}$$

As in (5.9) we have

$$E_N \sup_{t\in[0,T]} \iint \tilde{\phi}(\hat{D}^-(x, v, w, s))1\!\!1(|w| \le \ell_0)dxdw \le c_3\tilde{\phi}(n\ell_0)(r^{2d}\log r + \varepsilon r^3) ,$$

where $\tilde{\phi}(z) = z\sqrt{\log^+ z}$. As a result, the expression

$$E_N \int_0^T \iint \Omega_{111}\ 1\!\!1(|w| \le \ell_0)dxdwds \tag{5.13}$$

is bounded above by

$$\begin{aligned}
&kE_N \int_0^T \iint \hat{D}^-(x, v, w, s)1\!\!1(|w| \le \ell_0)1\!\!1(\Omega_1(x, v, w, \mathbf{q}(s)) + R(x, w, \mathbf{q}(s)) \ge \ell) \\
&\qquad 1\!\!1(\hat{D}^-(x, v, w, s) \ge \ell_1)dxdwds \\
&+kE_N \int_0^T \iint \hat{D}^-(x, v, w, s)1\!\!1(|w| \le \ell_0)1\!\!1(\Omega_1(x, v, w, \mathbf{q}(s)) + R(x, w, \mathbf{q}(s)) \ge \ell) \\
&\qquad 1\!\!1(\hat{D}^-(x, v, w, s) < \ell_1)dxdwds \\
&\le k(\log^+ \ell_1)^{-1/2}E_N \int_0^T \iint \tilde{\phi}(\hat{D}^-(x, v, w, s))1\!\!1(|w| \le \ell_0)dxdwds \\
&\quad +k\ell_1 E_N \int_0^T |\{(x, w) : \Omega_1(x, v, w, \mathbf{q}(s))+R(x, w, \mathbf{q}(s)) \ge \ell\ ,\ |w| \le \ell_0\}|\,ds \\
&\le c_3T\tilde{\phi}(n\ell_0)(r^{2d}\log r + \varepsilon r^3)k(\log^+ \ell_1)^{-1/2} \\
&\quad +\frac{k\ell_1}{\ell} \int_0^T \iint E_N(\Omega_1(x, v, w, \mathbf{q}(s)) + R(x, w, \mathbf{q}(s)))1\!\!1(|w| \le \ell_0)dxdwds ,
\end{aligned}$$

for every $\ell_1 > 1$. (Here and below $|A|$ denotes the Lebesgue measure of a set A.) Evidently if $\ell_0 \ge 1$, then

$$E_N \iint R(x, w, \mathbf{q}(s)))1\!\!1(|w| \le \ell_0)dxdw = c_{10}r^d\ell_0^d(\ell_0 + p) , \tag{5.14}$$

for some constant c_{10}. On the other hand,

$$E_N\Omega_1(x,v,w,\mathbf{q}(s)) \le \int \sum_{i,j} V^\varepsilon(|x_i-x_j|)B(v_i-v_j,n_{ij})\,\xi\left(\frac{x-x_i}{\varepsilon},v-w\right)$$
$$\zeta\left(\frac{v_i^j-w}{\varepsilon}\right)\left(1+n^{-1}\hat f^\varepsilon(x,v,w;S^{ij}\mathbf{q})\right)^{-1} G(s,\mathbf{q})\nu_\beta(d\mathbf{q})$$
$$= \int \sum_{i,j} V^\varepsilon(|x_i-x_j|)B(v_i-v_j,n_{ij})\,\xi\left(\frac{x-x_i}{\varepsilon},v-w\right)$$
$$\zeta\left(\frac{v_i-w}{\varepsilon}\right)\left(1+n^{-1}\hat f^\varepsilon(x,v,w;\mathbf{q})\right)^{-1} G(s,S^{ij}\mathbf{q})\nu_\beta(d\mathbf{q})\ .$$

Again the elementary inequality (5.10) yields

$$E_N\Omega_1(x,v,w,\mathbf{q}(s))$$
$$\le p_1\int \sum_{i,j} V^\varepsilon(|x_i-x_j|)B(v_i-v_j,n_{ij})\xi\left(\frac{x-x_i}{\varepsilon},v-w\right)\zeta\left(\frac{v_i-w}{\varepsilon}\right)$$
$$\left(1+n^{-1}\hat f^\varepsilon(x,v,w;\mathbf{q})\right)^{-1} G(s,\mathbf{q})\nu_\beta(d\mathbf{q})$$
$$+\frac{1}{\log p_1-1}\int \sum_{i,j} V^\varepsilon(|x_i-x_j|)B(v_i-v_j,n_{ij})\xi\left(\frac{x-x_i}{\varepsilon},v-w\right)$$
$$\zeta\left(\frac{v_i-w}{\varepsilon}\right)\psi\left(\frac{G(s,S^{ij}\mathbf{q})}{G(s,\mathbf{q})}\right)G(s,\mathbf{q})\nu_\beta(d\mathbf{q})\ ,$$

for every $p_1>1$. Hence, we can repeat (5.11) to assert that for some constant c_{11},

$$E_N\int_0^T\iint \Omega_1(x,v,w,\mathbf{q}(s))\mathbb{1}(|w|\le\ell_0)dxdwds \tag{5.15}$$
$$\le p_1E_N\int_0^T\iint \hat D^-(x,v,w,s)\mathbb{1}(|w|\le\ell_0)dxdwds+\frac{c_{11}r^d}{\log p_1}\ .$$

From this, (5.14) and (5.8) we learn that (5.13) is bounded above by

$$c_{12}k(\log^+\ell_1)^{-1/2}\tilde\phi(n\ell_0)(r^{2d}\log r+\varepsilon r^3)+c_{12}\frac{k\ell_1}{\ell}r^d\ell_0^{d+1}(np_1\ell_0+\ell_0+p)+\frac{c_{11}r^dk\ell_1}{\ell\log p_1}\ ,$$

for some constant c_{12}. This and (5.12) imply that the expression

$$\int_0^T\iint \Omega_{11}\mathbb{1}(|w|\le\ell_0)dxdwds\ , \tag{5.16}$$

is bounded above by

$$c_{12}k(\log^+\ell_1)^{-1/2}\tilde\phi(n\ell_0)(r^{2d}\log r+\varepsilon r^3)+c_{12}\frac{k\ell_1}{\ell}r^d\ell_0^{d}(np_1\ell_0+\ell_0+p)+\frac{c_{11}r^dk\ell_1}{\ell\log p_1}$$
$$+c_7nkr^d\ell_0^{d+2}p^{-1}\ .$$

We now choose $p = p_1 = \ell_1 = \ell^{1/4}$ and $k = (\log \ell)^{1/4}$ to deduce that the expression (5.16) is bounded above by

$$c_{13}\left(\tilde{\phi}(n\ell_0)(r^{2d}\log r + \varepsilon r^3) + nr^d\ell_0^{d+2}\right)(\log \ell)^{-1/4}\,.$$

This and (5.11) imply that the expression

$$\int_0^T \iint E_N\Omega_1(x, v, w, \mathbf{q}(s))1\!\!1(\Omega_1(x, v, w, \mathbf{q}(s)) \geq \ell)1\!\!1(|w| \leq \ell_0)dxdwds$$

is bounded above by

$$c_{14}\left(\tilde{\phi}(n\ell_0)(r^{2d}\log r + \varepsilon r^3) + nr^d\ell_0^{d+2}\right)(\log \ell)^{-1/4} + c_{14}r^d(\log\log \ell)^{-1}\,,$$

for some constant c_{14}. The term Ω_2 is treated likewise. From this and

$$(\log^+\log^+ X)^{1/2} = \int_e^\infty 1\!\!1(X \geq \ell)\frac{d\ell}{2\ell\log \ell(\log\log \ell)^{1/2}}\,,$$

one can readily deduce that for some constant c_{15},

$$\begin{aligned}E_N\int_0^T \iint \hat{\phi}(D^+(x, v, w, t)) \quad & 1\!\!1(|w| \leq \ell_0)dxdwdt\\ & \leq c_{15}[\tilde{\phi}(n\ell_0)(r^{2d}\log r + \varepsilon r^3) + nr^d\ell_0^{d+2}]\,,\end{aligned}$$

where $\hat{\phi}(z) = z(\log^+\log^+ z)^{1/2}$. This and (5.9) imply

$$\begin{aligned}E_N\int_0^T \iint \hat{\phi}(|D(x, v, w, s)|) \quad & 1\!\!1(|w| \leq \ell_0)dxdwds \\ & \leq c_{16}[\tilde{\phi}(n\ell_0)(r^{2d}\log r + \varepsilon r^3) + nr^d\ell_0^{d+2}]\,,\end{aligned} \tag{5.17}$$

for some constant c_{16}.

Step 7. We now turn to the martingale term. From (5.4) we learn that $E_N\,(M(x, v, w, t) - M(x, v, w, s))^2$ is equal to

$$E_N\int_s^t \sum_{i,j} V^\varepsilon(|x_i - x_j|)B(v_i - v_j, n_{ij})[F(x^\theta, v, w; S^{ij}\mathbf{q}(\theta)) - F(x^\theta, v, w; \mathbf{q}(\theta))]^2 d\theta\,,$$

where $x^\theta := x + \theta w$. This in turn equals to the expected value of

$$\begin{aligned}&\int_s^t \sum_{i,j} V^\varepsilon(|x_i - x_j|)B(v_i - v_j, n_{ij})\left(1 + n^{-1}f(x^\theta, v, w; S^{ij}\mathbf{q})\right)^{-2}\\ &\left(1 + n^{-1}f(x^\theta, v, w; \mathbf{q})\right)^{-2}\\ &\left[\xi\left(\frac{x^\theta - x_j}{\varepsilon}, v - w\right)\zeta\left(\frac{v_i^j - w}{\varepsilon}\right) + \xi\left(\frac{x^\theta - x_i}{\varepsilon}, v - w\right)\zeta\left(\frac{v_j^i - w}{\varepsilon}\right)\right.\\ &\left.-\xi\left(\frac{x^\theta - x_i}{\varepsilon}, v - w\right)\zeta\left(\frac{v_i - w}{\varepsilon}\right) - \xi\left(\frac{x^\theta - x_j}{\varepsilon}, v - w\right)\zeta\left(\frac{v_j - w}{\varepsilon}\right)\right]^2 d\theta.\end{aligned}$$

Here we have simply written $\mathbf{q}$ for $\mathbf{q}(\theta)$. Using $\left(\sum_{r=1}^{4} a_r\right)^2 \le 4\sum_{r=1}^{4} a_r^2$, we bound $E_N\,(M(x,v,w,t)-M(x,v,w,s))^2$ by the sum of four terms Γ_1, Γ_2, Γ_3 and Γ_4. For these terms the square of the expression in the brackets is replaced with

$$4\xi^2\left(\frac{x^\theta - x_i}{\varepsilon}, v-w\right)\zeta^2\left(\frac{v_i^j - w}{\varepsilon}\right),\quad 4\xi^2\left(\frac{x^\theta - x_j}{\varepsilon}, v-w\right)\zeta^2\left(\frac{v_j^i - w}{\varepsilon}\right),$$
$$4\xi^2\left(\frac{x^\theta - x_i}{\varepsilon}, v-w\right)\zeta^2\left(\frac{v_i - w}{\varepsilon}\right),\quad 4\xi^2\left(\frac{x^\theta - x_j}{\varepsilon}, v-w\right)\zeta^2\left(\frac{v_j - w}{\varepsilon}\right),$$

respectively. We start with Γ_3. The term Γ_3 is bounded above by

$$4E_N\int_s^t \sum_{i,j} V^\varepsilon(|x_i - x_j|)B(v_i - v_j, n_{ij})\xi^2\left(\frac{x^\theta - x_i}{\varepsilon}, v-w\right)\zeta^2\left(\frac{v_i - w}{\varepsilon}\right)$$
$$\left(1 + n^{-1}f(x^\theta, v, w; \mathbf{q})\right)^{-2} d\theta$$
$$\le 4c_1 E_N\int_s^t \varepsilon^d \sum_{i,j} 1\!\!1(|x_j - x^\theta| \le c_1 r\varepsilon)(|v_j - w| + c_2\varepsilon)\xi^2\left(\frac{x^\theta - x_i}{\varepsilon}, v-w\right)\zeta^2$$
$$\left(\frac{v_i - w}{\varepsilon}\right)\left(1 + n^{-1}f(x^\theta, v, w; \mathbf{q})\right)^{-2} d\theta\,.$$

(Compare this with (5.8).) From $\xi^2 \le \xi\|\xi\|_{L^\infty}$, $\zeta^2 \le \zeta\|\zeta\|_{L^\infty}$ and $\|\xi\|_{L^\infty} \le 1$ we deduce,

$$\Gamma_3 \le 4c_1\|\zeta\|_{L^\infty}\, nE_N\int_s^t \varepsilon^d \sum_j 1\!\!1(|x_j(\theta) - x^\theta| \le c_1 r\varepsilon)(|v_j(\theta) - w| + c_2\varepsilon)d\theta\,. \tag{5.18}$$

The term Γ_4 is treated likewise.

We now turn to Γ_1. The term Γ_1 is bounded above by

$$4E_N\int_s^t \sum_{i,j} V^\varepsilon(|x_i - x_j|)B(v_i - v_j, n_{ij})\xi^2\left(\frac{x^\theta - x_i}{\varepsilon}, v-w\right)\zeta^2\left(\frac{v_i^j - w}{\varepsilon}\right)$$
$$\left(1 + n^{-1}f(x^\theta, v, w; S^{ij}\mathbf{q})\right)^{-2} d\theta$$
$$= 4\int_s^t \sum_{i,j}\int V^\varepsilon(|x_i - x_j|)B(v_i - v_j, n_{ij})\xi^2\left(\frac{x^\theta - x_i}{\varepsilon}, v-w\right)\zeta^2\left(\frac{v_i^j - w}{\varepsilon}\right)$$
$$\left(1 + n^{-1}f(x^\theta, v, w; S^{ij}\mathbf{q})\right)^{-2} G(\theta, \mathbf{q})d\theta$$
$$= 4\int_s^t \sum_{i,j}\int V^\varepsilon(|x_i - x_j|)B(v_i - v_j, n_{ij})\xi^2\left(\frac{x^\theta - x_i}{\varepsilon}, v-w\right)\zeta^2\left(\frac{v_i - w}{\varepsilon}\right)$$
$$\left(1 + n^{-1}f(x^\theta, v, w; \mathbf{q})\right)^{-2} G(\theta, S^{ij}\mathbf{q})d\theta\,.$$

Fix $k > 1$. As in Step 5 we apply the inequality (5.10) and Lemma 4.7 to deduce that the expression $\iint \Gamma_1 1\!\!1(|w| \leq \ell_0)dxdw$ is bounded above by

$$4k \iint \int_s^t \sum_{i,j} V^\varepsilon(|x_i - x_j|)B(v_i - v_j, n_{ij})\xi^2\left(\frac{x^\theta - x_i}{\varepsilon}, v - w\right)\zeta^2\left(\frac{v_i - w}{\varepsilon}\right)$$
$$\left(1 + n^{-1}f(x, v, w; \mathbf{q})\right)^{-2} 1\!\!1(|w| \leq \ell_0)G(\theta, \mathbf{q})d\theta dxdw + \frac{C_3 r^d \|\zeta\|_{L^\infty}}{\log k - 1}$$
$$\leq 4c_1 nkE_N \iint \int_s^t \varepsilon^d \sum_j 1\!\!1(|x_j - x^\theta| \leq c_1 r\varepsilon)(|v_j - w| + c_2\varepsilon)$$
$$1\!\!1(|w| \leq \ell_0)d\theta dxdw + \frac{C_3 r^d \|\zeta\|_{L^\infty}}{\log k - 1} .$$

The terms Γ_2 is treated likewise. From this, (4.6) and (5.18) we deduce that the expression

$$E_N \iint (M(x, v, w, t) - M(x, v, w, s))^2 1\!\!1(|w| \leq \ell_0)dxdw ,$$

is bounded above by

$$c_{17} nkE_N \iint \int_s^t \varepsilon^d \sum_j 1\!\!1(|x_j(\theta) - x^\theta| \leq c_1 r\varepsilon)(|v_j(\theta) - w| + c_2\varepsilon) \tag{5.19}$$
$$1\!\!1(|w| \leq \ell_0)d\theta dxdw + \frac{c_{17} r^d}{\log k} \leq c_{18} nkr^d \ell_0^{d+1}|t - s| + \frac{c_{18} r^d}{\log k} ,$$

for every $k > 1$. We now choose $k = |t - s|^{-1/2}$ to deduce that for some constant c_{19},

$$E_N \iint (M(x, v, w, t) - M(x, v, w, s))^2 \quad 1\!\!1(|w| \leq \ell_0)dxdw \tag{5.20}$$
$$\leq c_{19} nr^d \ell_0^{d+1} |\log|t - s||^{-1} ,$$

whenever $|t - s| < 1$. If we set $s = 0$ and choose $k = e$ in (5.19) we obtain

$$E_N \iint M(x, v, w, t)^2 1\!\!1(|w| \leq \ell_0)dxdw \leq c_{19}(enr^d \ell_0^{d+1} t + r^d) . \tag{5.21}$$

Step 8. Recall the decomposition (5.5). We fix v and write

$$\begin{aligned}\int g(x, v, w, t)\chi_{\ell_0}(w)dw &= \int g(x - wt, v, w, 0)\chi_{\ell_0}(w)dw \\ &+ \int X(x, v, w, t)\chi_{\ell_0}(w)dw \\ &+ \int Y(x, v, w, t)\chi_{\ell_0}(w)dw \\ &+ \int M(x - wt, v, w, t)\chi_{\ell_0}(w)dw \\ &=: m_1(x, t) + m_2(x, t) + m_3(x, t) + m_4(x, t) ,\end{aligned}$$

where

$$X(x,v,w,t) = \int_0^t A(x - w(t-s), v, w, s)ds,$$
$$Y(x,v,w,t) = \int_0^t D(x - w(t-s), v, w, s)ds\ .$$

Put $g_0(x,v,w) = g(x,v,w,0)$ and $\hat{f}_0^\varepsilon(x,v,w) = \hat{f}^\varepsilon(x,v,w;\mathbf{q}(0))$. By Lemma 5.4,

$$\int_0^T \int_0^T \iint \frac{(m_1(x,t) - m_1(y,s))^2}{|(x,t)-(y,s)|^{d+2}}\, dxdydsdt \le C_6(T)\ell_0^d \|g_0\|^2_{L^2_{\ell_0}} \le c_{20} n \ell_0^d r^d \tag{5.22}$$

because $\int g_0^2 dxdw \le n \int \hat{f}_0^\varepsilon dxdw$. For m_2 we write $m_2 = m_{21} + m_{22}$, where $m_{2i}(x,t) = \int X_i(x,v,w,t)\chi_{\ell_0}(w)dw$ and

$$X_i(x,v,w,t) = \int_0^t A_i(x - w(t-s), v, w, s)ds\ ,$$
$$A_1(x,v,w,t) = A(x,v,w,t)1\!\!1(|A(x,v,w,t)| \ge \ell)\ ,$$
$$A_2(x,v,w,t) = A(x,v,w,t)1\!\!1(|A(x,v,w,t)| < \ell)\ .$$

Using Lemma 5.4 and (5.6), we certainly have

$$E_N \int_0^T \int_0^T \iint \frac{(m_{22}(x,t) - m_{22}(y,s))^2}{|(x,t)-(y,s)|^{d+2}}\, dxdydsdt \tag{5.23}$$
$$\le C_6(T)T\ell_0^d E_N \|A_2\|^2_{L^2_{\ell_0}} \le C_6(T)T\ \ell_0^d \ell E_N \|A\|_{L^1} \le c_{21}\ell_0^d r^d \ell\ .$$

On the other hand,

$$\begin{aligned}
\int_0^T \int |m_{21}(x,t)|dxdt &\le \int_0^T \iint |X_1(x,v,w,t)|1\!\!1(|w| \le \ell_0)dxdwdt \\
&\le T\int_0^T \iint |A(x,v,w,s)|1\!\!1(|A(x,v,w,s)| > \ell) \\
&\quad 1\!\!1(|w| \le \ell_0)dxdwds \\
&\le \frac{T}{\log \ell}\int_0^T \iint \phi(|A(x,v,w,s)|)1\!\!1(|w| \le \ell_0)dxdwds\ .
\end{aligned}$$

This and (5.7) imply

$$E_N \int_0^T \int |m_{21}(x,t)|dxdt \le \frac{c_{22} r^{2d} \log r}{\log \ell}\ . \tag{5.24}$$

One can readily use (5.8), (5.16), and the conservation of the kinetic energy to show

$$\begin{aligned}
&E_N \int_0^T \iint |D(x,v,w,s)|\chi_{\ell_0}(w)dxdwds \\
&\le c_{23} n r^d E_N \int_0^T \int \varepsilon^{2d} \sum_j (|v_j(s)| + \ell_0)\chi_{\ell_0}(w)dwds + c_{23} r^d \\
&\le c_{24} n \ell_0^{d+1} r^d\ .
\end{aligned}$$

We then use this and (5.17) to assert that m_3 can be decomposed as $m_{31} + m_{32}$ with

$$\int_0^T \int_0^T \iint \frac{(m_{32}(x,t) - m_{32}(y,s))^2}{|(x,t)-(y,s)|^{d+2}} \, dxdydsdt \le c_{25} n \ell_0^{2d+1} r^d \ell \,, \tag{5.25}$$

$$E_N \int_0^T \int |m_{31}(x,t)| dxdt \le c_{25} [\tilde{\phi}(n\ell_0)(r^{2d} \log r + \varepsilon r^3) + n r^d \ell_0^{d+1}] (\log\log \ell)^{-\frac{1}{2}} \,,$$

in just the same way we obtained (5.23) and (5.24).

Step 9. We now turn to m_4. Fix $\delta > 0$. We replace M with

$$\hat{M}(x, v, w, t) = \delta^{-1} \int_t^{t+\delta} M(x - wt, v, w, s) ds$$

in the definition of m_4 to yield

$$\hat{m}_4(x,t) = \int \hat{M}(x, v, w, t) \chi_{\ell_0}(w) dw \,.$$

Note that $\hat{M}$ satisfies the equation $\hat{M}_t + w \cdot \hat{M}_x = \tilde{M}$ in the weak sense where,

$$\tilde{M}(x, v, w, t) = \delta^{-1}(M(x - wt, v, w, t + \delta) - M(x - wt, v, w, t)) \,.$$

As a result, we may apply Duhamel's principle to assert,

$$\hat{M}(x, v, w, t) = \delta^{-1} \int_0^\delta M(x - wt, v, w, s) ds + \int_0^t \tilde{M}(x - w(t-s), v, w, s) ds \,.$$

Using this we write

$$\hat{M}(x, v, w, t) = M^0(x - wt, v, w) + M^1(x, v, w, t) \,, \tag{5.26}$$

where $M^0(x, v, w) = \delta^{-1} \int_0^\delta M(x, v, w, s) ds$. We now apply Lemma 5.4 to yield

$$\begin{aligned} \int_0^T \int_0^T \iint & \frac{(\hat{m}_4(x,t) - \hat{m}_4(y,s))^2}{|(x,t)-(y,s)|^{d+2}} \, dxdydtds \\ & \le c_{26} \ell_0^d \left(\|\tilde{M}\|_{L^2_{\ell_0}} \|M^1\|_{L^2_{\ell_0}} + \|M^0\|^2_{L^2_{\ell_0}} \right) \,, \end{aligned} \tag{5.27}$$

By Jensen's inequality,

$$E_N \|M^0\|^2_{L^2_{\ell_0}} \le \delta^{-1} \int_0^\delta \iint E_N M^2(x, v, w, s) \mathbb{1}(|w| \le \ell_0) dxdwds \le c_{26} \ell_0^{d+1} r^d n \,, \tag{5.28}$$

where for the last inequality we have used (5.21). In the same fashion we can show

$$E_N \|\hat{M}\|^2_{L^2_{\ell_0}} \le c_{26} \ell_0^{d+1} r^d n \,.$$

From this, (5.26) and (5.28) we deduce

$$E_N \|M^1\|^2_{L^2_{\ell_0}} \le 2 c_{26} \ell_0^{d+1} r^d n \,. \tag{5.29}$$

By (5.20), $E_N \|\tilde{M}\|^2_{L^2_{\ell_0}}$ is bounded above by

$$\delta^{-2} E_N \int_0^T \iint [M(x,v,w,t+\delta) - M(x,v,w,t)]^2 \ 1\!\!1(|w| \le \ell_0)\, dxdwdt \qquad (5.30)$$
$$\le c_{27} n r^d \ell_0^{d+1} \delta^{-2} |\log \delta|^{-1} \ .$$

From this and (5.27–29) we deduce that there exists a constant c_{28} such that if $\delta \in (0, 1/2)$, then

$$E_N \int_0^T \int_0^T \iint \frac{(\hat{m}_4(x,t) - \hat{m}_4(y,s))^2}{|(x,t)-(y,s)|^{d+2}}\, dxdydtds \qquad (5.31)$$
$$\le c_{28} \ell_0^d \ell_0^{d+1} r^d n \delta^{-1} |\log \delta|^{-1/2} \ .$$

Note that

$$\hat{m}_4(x,t) - m_4(x,t) = \int \left(\delta^{-1} \int_0^\delta N^s(x,v,w,t) ds \right) 1\!\!1(|w| \le \ell_0) dw \ ,$$

where $N^s(x,v,w,t) = M(x-wt, v, w, t+s) - M(x-wt, v, w, t)$. As in (5.30) we may use (5.20) to assert,

$$E_N \|\hat{m}_4 - m_4\|_{L^2_{\ell_0}} \le E_N \delta^{-1} \int_0^\delta \|N^s\|_{L^2_{\ell_0}} ds \le c_{28} \ell_0^{(d+1)/2} r^{d/2} n^{1/2} |\log \delta|^{-1/2} \ . \qquad (5.32)$$

Final Step. From (5.24–25) and (5.32) we learn that if

$$\mathcal{F}_\delta(m) = \int_0^T \int_0^\delta \iint |m(x+h, t+\alpha) - m(x,t)|\ 1\!\!1(|h| \le \delta) \delta^{-d-1}\, dxdhd\alpha dt \ ,$$

then

$$E_N \mathcal{F}_\delta(m^\varepsilon) \le E_N \left(\mathcal{F}_\delta(m_1) + \mathcal{F}_\delta(m_{22}) + \mathcal{F}_\delta(m_{32}) + \mathcal{F}_\delta(\hat{m}_4) \right)$$
$$+ c_{29} \ell_0^d r^{2d} \log r \tilde{\phi}(n\ell_0) (\log\log \ell)^{-1/2} \qquad (5.33)$$
$$+ c_{29} \ell_0^{(d+1)/2} r^{d/2} n^{1/2} |\log \delta|^{-1/2} \ .$$

On the other hand, by Jensen's inequality,

$$\mathcal{F}_\delta(m)^2 \le c_{30} \int_0^T \int_0^\delta \iint |m(x+h,t+\alpha) - m(x,t)|^2\ 1\!\!1(|h| \le \delta) \delta^{-d-1} dxdhd\alpha dt$$
$$\le c_{31} \delta \int_0^T \int_0^\delta \iint \frac{|m(x+h,t+\alpha) - m(x,t)|^2}{|(h,\alpha)|^{d+2}}\ 1\!\!1(|h| \le \delta) dxdhd\alpha dt$$
$$\le c_{31} \delta \int_0^T \int_0^T \iint \frac{|m(x+h,t+\alpha) - m(x,t)|^2}{|(h,\alpha)|^{d+2}}\, dxdhd\alpha dt \ ,$$

whenever $\delta \le T$. As a result, we may apply (5.22–23), (5.25) and (5.31) to assert that the expression

$$E_N \left(\mathcal{F}_\delta(m_1) + \mathcal{F}_\delta(m_{22}) + \mathcal{F}_\delta(m_{32}) + \mathcal{F}_\delta(\hat{m}_4) \right) \ ,$$

is bounded above by a constant multiple of

$$\delta^{1/2}\ell_0^{(d+1)/2}r^{d/2}n^{1/2}\ell^{1/2}+\ell_0^{d+1/2}r^{d/2}n^{1/2}|\log\delta|^{-1/4} .$$

From this and (5.33) we deduce that the expression $E_N\mathcal{F}_\delta(m^\varepsilon)$ is bounded above by a constant multiple of

$$\begin{aligned}&\delta^{1/2}\ell_0^{d+1/2}r^{d/2}n^{1/2}\ell^{1/2}+\ell_0^{d+1/2}r^{d/2}n^{1/2}|\log\delta|^{-1/4}\\&+\ell_0^d r^{2d}\log r\tilde{\phi}(n\ell_0)(\log\log\ell)^{-1/2}+\ell_0^{(d+1)/2}r^{d/2}n^{1/2}|\log\delta|^{-1/2} .\end{aligned}$$

We now choose $\ell=\delta^{-\frac{1}{2}}$ to conclude

$$E_N\mathcal{F}_\delta(m^\varepsilon)\le c_{32}\ell_0^d r^{2d}\log r\tilde{\phi}(n\ell_0)(\log|\log\delta|)^{-\frac{1}{2}} , \tag{5.34}$$

for some constant c_{32}.

Let us write c_{33} for the volume of the unit ball in $\mathbb{R}^d$ and put

$$m^\varepsilon_{\bar\delta}(x,v,t)=\frac{1}{c_{33}}\int_0^{\bar\delta}\int_{|h|\le\bar\delta}m^\varepsilon(x+h,v,t+\alpha)\,\bar\delta^{-d-1}\,dhd\alpha .$$

The bound (5.34) implies

$$E_N\|m^\varepsilon_{\bar\delta}-m^\varepsilon\|_{L^1}\le c_{31}\ell_0^d r^{2d}\log r\tilde{\phi}(n\ell_0)(\log|\log\bar\delta|)^{-\frac{1}{2}} . \tag{5.35}$$

It is not hard to see that the Lipschitz constant of $m^\varepsilon_{\bar\delta}$ in (x,t)-variable is $O(\|m^\varepsilon\|_{L^\infty}\,\bar\delta^{-1})$. Hence,

$$\sup_{|h|,|\alpha|\le\delta}|m^\varepsilon_{\bar\delta}(x+h,v,t+\alpha)-m^\varepsilon_{\bar\delta}(x,v,t)|\le c_{33}n\ell_0^d\bar\delta^{-1}\delta ,$$

for some constant c_{33}. From this and (5.35) we can readily deduce

$$\begin{aligned}E_N\sup_{|h|,|\alpha|\le\delta}&\int|m^\varepsilon(x+h,v,t+\alpha)-m^\varepsilon(x,v,t)|dxdt\\&\le c_{34}n\ell_0^d\bar\delta^{-1}\delta+c_{34}\ell_0^d r^{2d}\log r\tilde{\phi}(n\ell_0)(\log|\log\bar\delta|)^{-\frac{1}{2}} .\end{aligned}$$

We now choose $\bar\delta=\delta^{1/2}$ to complete the proof. □

Proof of Lemma 5.2. First assume $\xi\in\mathcal{C}_r$ and define

$$\hat\rho^\varepsilon_{\ell_0}(x,v,t)=\int\hat f^\varepsilon(x,v,w;\mathbf{q}(t))\chi_{\ell_0}(w)dw .$$

We certainly have

$$\hat f^\varepsilon=g^\varepsilon+\frac{(\hat f^\varepsilon)^2}{\hat f^\varepsilon+n} .$$

Moreover,

$$\begin{aligned}\iint \frac{(\hat f^\varepsilon)^2}{\hat f^\varepsilon+n}\,dxdw &= \iint \frac{(\hat f^\varepsilon)^2}{\hat f^\varepsilon+n}\,\mathbb{1}(\hat f^\varepsilon > n^{\frac12})dxdw + \iint \frac{(\hat f^\varepsilon)^2}{\hat f^\varepsilon+n}\,\mathbb{1}(\hat f^\varepsilon \le n^{\frac12})dxdw\\ &\le \iint \hat f^\varepsilon\ \mathbb{1}(\hat f^\varepsilon > n^{\frac12})dxdw + \iint \frac{n^{\frac12}}{n}\hat f^\varepsilon\,dxdw\\ &\le \frac{2}{\log n}\iint \phi(\hat f^\varepsilon)dxdw + n^{-\frac12}\iint \hat f^\varepsilon dxdw\ ,\end{aligned}$$

where $\phi(f) = f\log^+ f$. Hence we may apply Lemma 5.3 and Lemma 4.4 to assert

$$\begin{aligned}&E_N \sup_{|h|,|\alpha|\le\delta}\int_0^T\int |\hat\rho^\varepsilon_{\ell_0}(x+h,v,t+\alpha)-\hat\rho^\varepsilon_{\ell_0}(x,v,t)|dxdt\\ &\le E_N \sup_{|h|,|\alpha|\le\delta}\int_0^T\int |m^\varepsilon_{n,\ell_0}(x+h,v,t+\alpha)-m^\varepsilon_{n,\ell_0}(x,v,t)|dxdt\\ &+c_1r^{2d}\log r\,(\log n)^{-1}+c_1r^dn^{-\frac12}\\ &\le c_2r^{2d}(\log r)n\ell_0^{d+1}\left(\log^+(n\ell_0)\right)^{1/2}(\log|\log\delta|)^{-1/2}+c_2r^{2d}\log r\,(\log n)^{-1}\ .\end{aligned}$$

We now choose $n=(\log|\log\delta|)^{\frac13}$ to obtain

$$\begin{aligned}&E_N \sup_{|h|,|\alpha|\le\delta}\int_0^T\int |\hat\rho^\varepsilon_{\ell_0}(x+h,v,t+\alpha)-\hat\rho^\varepsilon_{\ell_0}(x,v,t)|dxdt\\ &\qquad\le c_3\ell_0^{d+1}(\log\ell_0)r^{2d}\log r(\log\log|\log\delta|)^{-1}\ .\end{aligned}$$

From this, (4.6), and

$$\int \hat f^\varepsilon(x,v,w;\mathbf{q})\mathbb{1}(|w|\ge\ell_0)dw \le \frac{1}{\ell_0^2}\int \hat f^\varepsilon(x,v,w;\mathbf{q})|w|^2dw\ ,$$

we learn

$$\begin{aligned}&E_N \sup_{|h|,|\alpha|\le\delta}\int_0^T\int |\hat\rho^\varepsilon(x+h,v,t+\alpha)-\hat\rho^\varepsilon(x,v,t)|dxdt\\ &\qquad\le c_4\ell_0^{d+1}(\log\ell_0)r^{2d}\log r(\log\log|\log\delta|)^{-1}+c_4(1+\varepsilon^2r^2)\ell_0^{-2}\ .\end{aligned}$$

By choosing $\ell_0=(\log\log|\log\delta|)^{1/(d+2)}$ we deduce

$$\begin{aligned}&E_N \sup_{|h|,|\alpha|\le\delta}\int_0^T\int |\hat\rho^\varepsilon(x+h,v,t+\alpha)-\hat\rho^\varepsilon(x,v,t)|dxdt\\ &\le c_5r^{2d}\log r(\log\log|\log\delta|)^{-\frac{1}{d+3}}\ ,\end{aligned}$$

whenever $\xi\in\mathcal{C}_r$.

We now would like to relax the restriction on the support of ξ. After a scaling argument we deduce

$$E_N \sup_{|h|,|\alpha|\le\delta} \int_0^T \int |\hat{\rho}^\varepsilon \quad (x+h, v, t+\alpha) - \hat{\rho}^\varepsilon(x, v, t)| dx dt$$
$$\le c_5(\|\xi\|_{L^\infty} + \mathcal{R}_4(\xi)) r^{2d} \log r (\log\log|\log\delta|)^{-\frac{1}{d+3}} , \quad (5.36)$$

whenever $\xi(x, v) = 0$ for $|x| + |v| \ge r$.

We now consider a nonnegative function ξ that satisfies (5.2) only. We write $\hat{\rho}^\varepsilon = \rho_1^\varepsilon + \rho_2^\varepsilon$, where

$$\rho_j^\varepsilon(x, v, t) = \int \hat{f}_j^\varepsilon(x, v, w; \mathbf{q}(t)) dw ,$$
$$\hat{f}_j^\varepsilon(x, v, w; \mathbf{q})) = \sum_i \xi_j \left(\frac{x - x_i}{\varepsilon}, v - w \right) \zeta \left(\frac{v_i - w}{\varepsilon} \right) ,$$

for $j = 1$ and 2, where $\xi_1 + \xi_2 = \xi$ and $\xi_1(x, v) = \xi(x, v)\chi(x/r)\chi(v/r)$, where χ is a smooth function with support inside the ball $\{v : |v| \le 2\}$ and $\chi(v) = 1$ whenever $|v| \le 1$. Since

$$\mathcal{R}_4(\xi_1) \le c_6 r^{b+1} \mathcal{R}_3(\xi) + c_6 \sup_{|x|,|v|\le 2r} \xi(x, v) r^{-1} ,$$

for some constant c_6 and $\mathcal{R}_3(\xi) + \mathcal{R}_0(\xi) \le 1$, we deduce that $\mathcal{R}_4(\xi_1) \le c_7 r^{b+1}$ for some constant c_7. On the other hand, the condition $\xi(x, v) \le (1 + |v|)^{b+1}$ implies that $\|\xi_1\|_{L^\infty} \le c_8 r^{b+1}$ for some constant c_8. We now apply (5.36) to assert

$$E_N \sup_{|h|,|\alpha|\le\delta} \int_0^T \int |\rho_1^\varepsilon(x+h, v, t+\alpha) - \rho_1^\varepsilon(x, v, t)| dx dt$$
$$\le c_9 r^{2d+b+1} \log r (\log\log|\log\delta|)^{-\frac{1}{d+3}} , \quad (5.37)$$

for some constant c_9.

We now turn to ρ_2^ε. First observe that we can write $\xi_2 = \xi_{21} + \xi_{22}$ where $\xi_{21}(x, v) = 0$ if $|v| \le r$ and $\xi_{22}(x, v) = 0$ if $|x| \le r$. With the aid of the decomposition $\xi_2 = \xi_{21} + \xi_{22}$ we write $\rho_2^\varepsilon = \rho_{21}^\varepsilon + \rho_{22}^\varepsilon$. We first treat ρ_{21}^ε. Observe that the condition (5.2) implies that for some function γ with $\int(1 + |x|)\gamma(x) dx \le 1$,

$$|\xi(x, v) - \xi(x, w)| \le \gamma(x)(|v| + |w| + 1)^b |v - w| .$$

This in particular implies

$$\xi(x, v) \le \gamma(x)|v|(|v| + 1)^b + \xi_0(x),$$

where $\xi_0(x) := \xi(x, 0)$. The condition (5.2) implies $\int(1 + |x|)\xi_0 dx \le 1$. As a result,

$$\begin{aligned}
\rho_{21}^{\varepsilon}(x, v, t) &\le c_{10} \int \sum_i \gamma\left(\frac{x - x_i}{\varepsilon}\right) \zeta\left(\frac{v_i - w}{\varepsilon}\right) \left(|v - w|^{b+1} + 1\right) \mathbb{1}(|v - w| \ge r) dw \\
&\quad + c_{10} \int \sum_i \xi_0\left(\frac{x - x_i}{\varepsilon}\right) \zeta\left(\frac{v_i - w}{\varepsilon}\right) \mathbb{1}(|v - w| \ge r) dw \\
&\le \frac{c_{10}}{r^{(1-b)/2}} \int \sum_i \hat{\gamma}\left(\frac{x - x_i}{\varepsilon}\right) \zeta\left(\frac{v_i - w}{\varepsilon}\right) \left(|v - w|^{(b+3)/2} + 1\right) dw \\
&\le \frac{c_{11}}{r^{(1-b)/2}} \int \sum_i \hat{\gamma}\left(\frac{x - x_i}{\varepsilon}\right) \zeta\left(\frac{v_i - w}{\varepsilon}\right) (|v_i|^2 + |v|^{(b+3)/2} + 1) dw \\
&= \frac{c_{11}\varepsilon^d}{r^{(1-b)/2}} \sum_i \hat{\gamma}\left(\frac{x - x_i}{\varepsilon}\right) (|v_i|^2 + |v|^{(b+3)/2} + 1) ,
\end{aligned}$$

where $\hat{\gamma} = \gamma + \xi_0$ and $r \ge 1$. From this and the conservation of energy we deduce

$$\begin{aligned}
E_N \int \rho_{21}^{\varepsilon}(x, v, t) dx &\le E_N c_{12} r^{(b-1)/2} \varepsilon^{2d} \sum_i (|v_i(t)|^2 + |v|^{(b+3)/2} + 1) \\
&= E_N c_{12} r^{(b-1)/2} \varepsilon^{2d} \sum_i (|v_i(0)|^2 + |v|^{(b+3)/2} + 1) \qquad (5.38) \\
&\le c_{13}(1 + |v|^{(b+3)/2}) r^{(b-1)/2} .
\end{aligned}$$

The term ρ_{22}^{ε} is treated likewise;

$$\begin{aligned}
\rho_{22}^{\varepsilon}(x, v, t) &\le c_{14} \int \sum_i \gamma\left(\frac{x - x_i}{\varepsilon}\right) \zeta\left(\frac{v_i - w}{\varepsilon}\right) \left(|v - w|^{b+1} + 1\right) \mathbb{1}(|x - x_i| \ge r\varepsilon) dw \\
&\quad + c_{14} \int \sum_i \xi_0\left(\frac{x - x_i}{\varepsilon}\right) \zeta\left(\frac{v_i - w}{\varepsilon}\right) \mathbb{1}(|x - x_i| \ge r\varepsilon) dw \\
&\le \frac{c_{14}}{r} \int \sum_i \tilde{\gamma}\left(\frac{x - x_i}{\varepsilon}\right) \zeta\left(\frac{v_i - w}{\varepsilon}\right) \left(|v - w|^{b+1} + 1\right) dw \\
&\le \frac{c_{15}}{r} \int \sum_i \tilde{\gamma}\left(\frac{x - x_i}{\varepsilon}\right) \zeta\left(\frac{v_i - w}{\varepsilon}\right) (|v_i|^2 + |v|^{b+1} + 1) dw \\
&= \frac{c_{15}\varepsilon^d}{r} \sum_i \tilde{\gamma}\left(\frac{x - x_i}{\varepsilon}\right) (|v_i|^2 + |v|^{b+1} + 1) ,
\end{aligned}$$

where $\tilde{\gamma}(x) = |x|\gamma(x) + |x|\xi_0(x)$ and $r \ge 1$. As in (5.38) we deduce

$$\begin{aligned}
E_N \int \rho_{22}^{\varepsilon}(x, v, t) dx &\le E_N c_{16} r^{-1} \varepsilon^{2d} \sum_i (|v_i(t)|^2 + |v|^{b+1} + 1) \\
&\le c_{17}(1 + |v|^{b+1}) r^{-1} .
\end{aligned}$$

From this, (5.38) and (5.37) we learn

$$E_N \sup_{|h|,|\alpha|\le\delta} \int_0^T \int |\hat\rho^\varepsilon(x+h,v,t+\alpha)-\hat\rho^\varepsilon(x,v,t)|dxdt$$
$$\le c_{18} r^{2d+b+1}\log r(\log\log|\log\delta|)^{-1/(d+3)} + c_{18}(1+|v|^{(b+3)/2})r^{(b-1)/2}.$$

We now choose $r = (\log\log|\log\delta|)^{\alpha_b}$ to conclude

$$E_N \sup_{|h|,|\alpha|\le\delta} \int_0^T \int |\hat\rho^\varepsilon(x+h,v,t+\alpha)-\hat\rho^\varepsilon(x,v,t)|dxdt$$
$$\le c_{19}(1+|v|^{(b+3)/2})(\log\log|\log\delta|)^{-\alpha_b} .$$

This completes the proof of Lemma 5.2 when $\xi \ge 0$. The proof for general ξ follows from the fact that if (5.2) holds for ξ, then it holds for both the positive and the negative parts of ξ. □

6. Stosszahlensatz for the Loss Term

In this section, we use Theorem 5.1 to establish a variant of Boltzmann's molecular chaos principle for the loss term. Recall the definition of the density $f^{\delta,\varepsilon}$ that was defined by (4.4). Let ζ be a nonnegative continuous function of compact support with $\int \zeta dx = 1$ and define

$$\tilde f^\varepsilon(x,v;\mathbf{q}) := \left(\frac{\varepsilon}{\delta_1(\varepsilon)}\right)^d \left(\frac{\varepsilon}{\delta_2(\varepsilon)}\right)^d \sum_{i=1}^N \zeta\left(\frac{x_i-x}{\delta_1(\varepsilon)}\right)\zeta\left(\frac{v_i-v}{\delta_2(\varepsilon)}\right) ,$$

where $\delta_r(\varepsilon) = \varepsilon\ell_r(\varepsilon)$ for $r = 1, 2$. We assume $\ell_2(\varepsilon) \le \ell_1(\varepsilon) = \ell(\varepsilon)$, where

$$\ell(\varepsilon) := (\log\log\log\log|\log\varepsilon|)^{\frac{1}{2d+1}} . \tag{6.1}$$

Note that we may write

$$\tilde f^\varepsilon(x,v;\mathbf{q}) = \sum_i \tilde\zeta_1^\varepsilon(x_i-x)\tilde\zeta_2^\varepsilon(v_i-v) , \tag{6.2}$$

where

$$\tilde\zeta_r^\varepsilon(z) = \ell_r(\varepsilon)^{-d}\zeta\left(\frac{z}{\varepsilon\ell_r(\varepsilon)}\right) , \tag{6.3}$$

for $r = 1, 2$. Given a smooth function $\alpha : \mathbb{R} \to \mathbb{R}$, we define the renormalized microscopic loss term $Q_-^{\varepsilon,\alpha}$ by

$$Q_-^{\varepsilon,\alpha}(x,v;\mathbf{q}) = \sum_{i,j} V^\varepsilon(|x_i-x_j|)B(v_i-v_j,n_{ij})\,\tilde\zeta_1^\varepsilon(x_i-x)\tilde\zeta_2^\varepsilon(v_i-v)\alpha(\tilde f^\varepsilon(x,v;\mathbf{q})). \tag{6.4}$$

Given a function $g : \mathbb{R}^d \to \mathbb{R}^d$, put

$$Lg(v) = \int\int_{\mathbb{S}} B(v-v_*,n)g(v_*)dndv_* = \int \bar B(v-v_*)g(v_*)dv_*, \tag{6.5}$$

where $\bar B(v) = \int_{\mathbb{S}} B(v,n)dn$. Recall $\alpha_0 = (2d+2)^{-1}(d+3)^{-1}$. Theorem 6.1 is the main result of this section.

Theorem 6.1. *There exists a constant $C_7(T)$ such that if α satisfies*

$$\sup_z (z+1)\alpha(z) \le n \ ,$$

then

$$\begin{aligned} E_N \int_0^T \iint \Big| \tilde{f}^\varepsilon(x,v;\mathbf{q}(s)) \quad & L\tilde{f}^\varepsilon(x,\cdot;\mathbf{q}(s))(v)\,\alpha(\tilde{f}^\varepsilon(x,v;\mathbf{q}(s))) \\ & - Q_-^{\varepsilon,\alpha}(x,v;\mathbf{q}(s))\Big| \mathbb{1}(|v| \le \ell_0)\,dx\,dv\,ds \qquad (6.6) \\ & \le C_7(T) n \ell_0^{d+2} \ell(\varepsilon)^{-1/2}, \end{aligned}$$

provided $\ell_0^{2(d+2)/\alpha_0} \le \log\log|\log\varepsilon|$.

Recall

$$A(x,v) = V(|x|)B\left(v, \frac{x}{|x|}\right) \ ,$$
$$\mathcal{R}(\xi) = \mathcal{R}_0(\xi) + \mathcal{R}_1(\xi) + \mathcal{R}_2(\xi) + \mathcal{R}_3(\xi) \ .$$

Let $\mathcal{B}$ denote the set of functions ξ such that $\mathcal{R}(\xi) < \infty$ for $b = 0$ in (5.1). Recall the space of functions $\mathcal{L}$ and the functional L_k that were defined right before Lemma 4.5. We also define the space $\hat{\mathcal{L}}$ as the space of functions γ for which the following condition holds: There exist two constants $k = k(\gamma)$ and $c = c(\gamma)$ such that for every $\alpha > 0$, we can find a decomposition $\gamma = \gamma_1 + \gamma_2$ with $\|\gamma_1\|_{L^1} \le \alpha$ and $L_k(\gamma_2) \le c$.

Lemma 6.2. *There exist two functions $\hat{A}$ and γ, and a positive constant c_0 such that $\hat{A}$ is of compact support in the x-variable, $\hat{A} \in \mathcal{B}$, $\gamma \in \hat{\mathcal{L}}$, $|\gamma(x)| \le \exp(-c_0|x|)$ for every x with $|x| > 1$, and $A(x,v) = \int \hat{A}(x-y,v)\gamma(y)dy$.*

Proof. The function $\hat{A}$ is simply defined by $\hat{A} = A - \Delta_x A$, where Δ_x denotes the Laplace operator with respect to the x–variable. Recall that by our assumptions on A, the second x–partial derivatives of A are Lipschitz continuous. As a consequence of this we have that $\hat{A} \in \mathcal{B}$. To express A as a convolution involving $\hat{A}$, let us write $\mathcal{F}$ for the Fourier operator in the x–variable. More precisely,

$$\mathcal{F}J(z,v) = \int J(x,v)\exp(2\pi i x\cdot z)dz \ ,$$

where $i = \sqrt{-1}$. Since $\mathcal{F}\hat{A}(z,v) = (1+4\pi^2|z|^2)\mathcal{F}A(z,v) =: \bar{\gamma}(z)\mathcal{F}A(z,v)$, we have $A(x,v) = \int \hat{A}(x-y,v)\gamma(y)dy$ with $\gamma = \mathcal{F}^{-1}\bar{\gamma}$. A straightforward calculation yields

$$\gamma(x) = c_0 \int_0^\infty \exp(-\pi|x|^2/\theta - \theta/(4\pi))\theta^{-d/2}d\theta \ ,$$

for some constant c_0. (See [S], p. 131 for a derivation. It is worth mentioning that $\gamma(x)$ is a constant multiple of $|x|^{-1}e^{-|x|}$ when $d = 3$.) It is not hard to show that $\gamma \in L^1$ and that γ decays exponentially fast as $|x|$ increases. To show $\gamma \in \hat{\mathcal{L}}$, pick a small $\tau > 0$ and define

$$\gamma_2(z) = \begin{cases} \gamma(z) & \text{for } |z| \ge \tau \ , \\ \gamma\left(\frac{\tau z}{|z|}\right) & \text{for } |z| < \tau . \end{cases}$$

Since $\gamma \in L^1$, we have $\lim \|\gamma_1\|_{L^1} = 0$ as $\tau \to 0$. We can readily show that if $k > 0$, then $\sup_\tau L_k(\gamma_2) < \infty$. This completes the proof of the lemma. □

Define

$$K^{\varepsilon}(x,v;\mathbf{q})=\sum_j V^{\varepsilon}(|x-x_j|)B\left(v-v_j,\frac{x-x_j}{|x-x_j|}\right)\ ,$$

$$\hat{K}^{\varepsilon}(x,v;\mathbf{q})=\sum_j \varepsilon^d\hat{A}\left(\frac{x-x_j}{\varepsilon},v-v_j\right)\ ,$$

$$K^{\varepsilon,\delta}(x,v,\mathbf{q})=\int K^{\varepsilon}(x-z,v;\mathbf{q})\zeta^{\delta}(z)\,dz\ ,$$

$$\hat{K}^{\varepsilon,\delta}(x,v,\mathbf{q})=\int \hat{K}^{\varepsilon}(x-z,v;\mathbf{q})\zeta^{\delta}(z)\,dz\ ,$$

where ζ is a smooth nonnegative function of compact support that satisfies $\int\zeta dz=1$, and $\zeta^{\delta}(z)=\delta^{-d}\zeta(z/\delta)$. As a consequence of Theorem 5.1 we have,

Lemma 6.3. *There exists a constant $C_8=C_8(T)$ such that for every $v\in\mathbb{R}^d$ and $\delta>0$,*

$$E_N\int_0^T\int|K^{\varepsilon}(x,v;\mathbf{q}(t))-K^{\varepsilon,\delta}(x,v;\mathbf{q}(t))|dxdt$$
$$\le C_8\left[(1+|v|^2)(\log\log|\log\delta|)^{-\alpha_0}+\varepsilon\right]\ ,$$
$$E_N\int_0^T\int|\hat{K}^{\varepsilon}(x,v;\mathbf{q}(t))-\hat{K}^{\varepsilon,\delta}(x,v;\mathbf{q}(t))|dxdt$$
$$\le C_8\left[(1+|v|^2)(\log\log|\log\delta|)^{-\alpha_0}+\varepsilon\right]\ .$$

Proof of Theorem 6.1. Step 1. To ease the notation, let us write $\alpha(x,v)$ for $\alpha(\tilde{f}^{\varepsilon}(x,v;\mathbf{q}))$, $K(x,v)$ for $K^{\varepsilon}(x,v;\mathbf{q})$ and $K^{\delta}(x,v)$ for $K^{\varepsilon,\delta}(x,v,\mathbf{q})$. We certainly have

$$Q_-^{\varepsilon,\alpha}(x,v;\mathbf{q})=\sum_i\tilde{\zeta}_1^{\varepsilon}(x_i-x)\tilde{\zeta}_2^{\varepsilon}(v_i-v)K(x_i,v_i)\alpha(x,v). \tag{6.7}$$

Our goal is to replace $K(x_i,v_i)$ with $K(x,v)$ in (6.7). For this, we first replace $K(x_i,v_i)$ with $K(x_i,v)$. Since $B(v,n)$ is Lipschitz continuous in v, we have that

$$|K(x_i,v_i)-K(x_i,v)|\le c_0\varepsilon\ell(\varepsilon)\sum_j V^{\varepsilon}(|x_i-x_j|)=:c_0\varepsilon\ell(\varepsilon)g^{\varepsilon}(x_i)\ ,$$

whenever $\tilde{\zeta}_2^{\varepsilon}(v_i-v)\neq 0$. As a result, if we set

$$X(x,v)=\sum_i\tilde{\zeta}_1^{\varepsilon}(x_i-x)\tilde{\zeta}_2^{\varepsilon}(v_i-v)\alpha(x,v)(K(x_i,v_i)-K(x_i,v)),$$

then

$$|X(x,v)|\le c_0\varepsilon\ell(\varepsilon)\sum_i\tilde{\zeta}_1^{\varepsilon}(x_i-x)\tilde{\zeta}_2^{\varepsilon}(v_i-v)\alpha(x,v)g^{\varepsilon}(x_i)$$
$$\le c_1\varepsilon\ell(\varepsilon)\varepsilon^d\sum_{i,j}\tilde{\zeta}_1^{\varepsilon}(x_i-x)\tilde{\zeta}_2^{\varepsilon}(v_i-v)\alpha(x,v)\mathbb{1}(|x_j-x|\le c_1\varepsilon\ell(\varepsilon))$$
$$\le c_1n\varepsilon\ell(\varepsilon)\varepsilon^d\sum_j\mathbb{1}(|x_j-x|\le c_1\varepsilon\ell(\varepsilon))\ ,$$

for some constant c_1. Hence

$$\iint |X(x, v)|\mathbb{1}(|v| \leq \ell_0) dx dv \leq c_2 n \varepsilon \ell(\varepsilon)^{d+1} \ell_0^d. \tag{6.8}$$

Step 2. We would like to show that there exists a function $\psi(\varepsilon)$ with $\lim_{\varepsilon\to 0} \psi(\varepsilon) = 0$ such that

$$E_N \int_0^T \iint |Q_-^{\varepsilon,\alpha}(x, v; \mathbf{q}(t)) - \hat{Q}_-^{\varepsilon,\alpha}(x, v; \mathbf{q}(t))|\mathbb{1}(|v| \leq \ell_0) dx dv dt \leq \psi(\varepsilon), \tag{6.9}$$

where $\hat{Q}_-^{\varepsilon,\alpha}(x, v; \mathbf{q}) = \tilde{f}^\varepsilon(x, v; \mathbf{q})\alpha(x, v)K(x, v)$. To achieve this, let us bound

$$Y(x, v) = \sum_i \tilde{\zeta}_1^\varepsilon(x_i - x)\tilde{\zeta}_2^\varepsilon(v_i - v)\alpha(x, v)(K(x_i, v) - K(x, v)). \tag{6.10}$$

To show that $Y(x, v)$ is small, we write

$$\begin{aligned} K(x_i, v) - K(x, v) = &K^\delta(x_i, v) - K^\delta(x, v) \\ &+K(x_i, v) - K^\delta(x_i, v) \\ &+K^\delta(x, v) - K(x, v). \end{aligned} \tag{6.11}$$

We now replace $K(x_i, v) - K(x, v)$ in the definition of $Y(x, v)$ with each of three differences that appeared on the right-hand side of (6.11). The result will be denoted by $Y_1(x, v)$, $Y_2(x, v)$ and $Y_3(x, v)$ respectively. Evidently,

$$Y(x, v) = Y_1(x, v) + Y_2(x, v) + Y_3(x, v)\ . \tag{6.12}$$

Put $\hat{\zeta} = |\nabla\zeta|$ and $\hat{\zeta}^\delta(z) = \delta^{-d}\hat{\zeta}(z/\delta)$. Evidently,

$$|\zeta^\delta(a) - \zeta^\delta(b)| \leq \delta^{-1}|b - a| \int_0^1 \hat{\zeta}^\delta(a + \theta(b - a)) d\theta\ .$$

From this we learn if $\tilde{\zeta}_1^\varepsilon(x_i - x) \neq 0$ and $\varepsilon\ell(\varepsilon) \leq \delta$, then the expression

$$|K^\delta(x_i, v) - K^\delta(x, v)|\ ,$$

is bounded above by

$$\begin{aligned} &c_3\delta^{-1}\varepsilon\ell(\varepsilon) \int \int_0^1 K(z, v)\hat{\zeta}^\delta(x + \theta(x_i - x) - z) d\theta dz \\ &\leq c_3\|\hat{\zeta}\|_{L^\infty}\, \delta^{-1}\varepsilon\ell(\varepsilon) \int K(z, v)\delta^{-d}\mathbb{1}(|x - z| \leq c_3\delta + c_3\varepsilon\ell(\varepsilon)) dz =: G(x, v) \end{aligned}$$

for some constant c_3. Moreover,

$$\int G(x, v) dx \leq c_4\delta^{-1}\varepsilon\ell(\varepsilon)\varepsilon^{2d} \sum_j |v - v_j| \leq c_4\delta^{-1}\varepsilon\ell(\varepsilon)\varepsilon^{2d} \sum_j (|v| + |v_j|^2 + 1)\ ,$$

for some constant c_4. This and the conservation of the kinetic energy imply that there exists a constant c_5 such that

$$E_N \int_0^T \iint |Y_1(x, v)|\mathbb{1}(|v| \leq \ell_0) dx dv dt \leq c_5\ell_0^{d+1} n\delta^{-1}\varepsilon\ell(\varepsilon)\ , \tag{6.13}$$

whenever $\ell_0 \geq 1$ and $\delta \geq \varepsilon\ell(\varepsilon)$.

To bound Y_3, we write

$$|Y_3(x,v)| \leq \tilde{f}^\varepsilon(x,v;\mathbf{q})\alpha(\tilde{f}^\varepsilon(x,v;\mathbf{q}))|K^\delta(x,v)-K(x,v)| \leq n|K^\delta(x,v)-K(x,v)|.$$

This and Lemma 6.3 imply

$$E_N\int_0^T\iint |Y_3(x,v)|\mathbb{1}(|v|\leq \ell_0)dxdvdt \leq c_6 n\ell_0^{d+2}(\log\log|\log\delta|)^{-\alpha_0}+c_6 n\ell_0^d\varepsilon\ , \tag{6.14}$$

for some constant c_6.

Step 3. We now concentrate on Y_2. By Lemma 6.2, $A(x,v)=\int \hat{A}(x-y,v)\gamma(y)dy$ for a function $\hat{A}\in\mathcal{B}$ and $\gamma\in\hat{\mathcal{L}}$. As a result,

$$K(x,v)=\int \hat{K}(x-\varepsilon y,v)\gamma(y)dy=\int \hat{K}(x-y,v)\gamma^\varepsilon(y)dy\ , \tag{6.15}$$

where $\gamma^\varepsilon(y)=\varepsilon^{-d}\gamma(y/\varepsilon)$. We certainly have

$$K^\delta(x,v)=\int \hat{K}^\delta(x-y,v)\gamma^\varepsilon(y)dy\ ,$$

for $\hat{K}^\delta=\hat{K}^{\varepsilon,\delta}$. Write

$$\begin{aligned} K-K^\delta=(\hat{K}-\hat{K}^\delta)*_x\gamma^\varepsilon &= (\hat{K}-\hat{K}^\delta)\mathbb{1}(\hat{K}-\hat{K}^\delta\leq \ell)*_x\gamma^\varepsilon \\ &\quad +(\hat{K}-\hat{K}^\delta)\mathbb{1}(\hat{K}-\hat{K}^\delta> \ell)*_x\gamma^\varepsilon\ , \end{aligned} \tag{6.16}$$

where $*_x$ denotes the convolution in the x-variable. Replace $K-K^\delta$ in the definition of Y_2 with the two terms which appeared on the right-hand side of (6.16). The result will be denoted by Y_{21} and Y_{22}. As a result

$$Y_2(x,v)=Y_{21}(x,v)+Y_{22}(x,v), \tag{6.17}$$

where,

$$Y_{21}(x,v)=\sum_i \tilde{\zeta}_1^\varepsilon(x_i-x)\tilde{\zeta}_2^\varepsilon(v_i-v)\alpha(x,v)(H^\delta *_x\gamma^\varepsilon)(x_i,v)\ , \tag{6.18}$$

where $H^\delta=(\hat{K}-\hat{K}^\delta)\mathbb{1}(\hat{K}-\hat{K}^\delta\leq\ell)$. Note that we may write $\gamma=\gamma_{1,\tau}+\gamma_{2,\tau}$, where $\sup_\tau L_{k_0}(\gamma_{2,\tau})<\infty$ for each $k_0>0$, and $\lim\|\gamma_{1,\tau}\|_{L^1}=0$ as $\tau\to 0$. To ease the notation, we simply write γ_r for $\gamma_{r,\tau}$. Set $\gamma_r^\varepsilon(x)=\varepsilon^{-d}\gamma_r(x/\varepsilon)$ for $r=1,2$. We replace γ^ε in (6.18) with γ_r^ε for $r=1$ and 2 and denote the result by Y_{211} and Y_{212} respectively. Evidently,

$$Y_{21}(x,v)=Y_{211}(x,v)+Y_{212}(x,v). \tag{6.19}$$

We certainly have

$$\lim_{\tau\to 0}\|\gamma_2^\varepsilon\|_{L^1}=\lim_{\tau\to 0}\|\gamma_2\|_{L^1}=0\ . \tag{6.20}$$

From this we learn

$$\lim_{\tau\to 0}\|H^\delta *_x\gamma_2^\varepsilon\|_{L^\infty}\leq \limsup_{\tau\to 0}\|H^\delta\|_{L^\infty}\|\gamma_2^\varepsilon\|_{L^1}\leq \ell\limsup_{\tau\to 0}\|\gamma_2^\varepsilon\|_{L^1}=0\ .$$

This implies

$$\lim_{\tau\to 0}\sup_{x,v}|Y_{212}(x,v)| \le n \lim_{\tau\to 0}\|H^\delta *_x \gamma_2^\varepsilon\|_{L^\infty} = 0. \tag{6.21}$$

On the other hand,

$$\begin{aligned}|Y_{211}(x,v)| &\le n\sum_i \tilde\zeta_1^\varepsilon(x_i-x)\tilde\zeta_2^\varepsilon(v_i-v)|H^\delta *_x \gamma_1^\varepsilon|(x_i,v)\\ &\le n\sum_i \tilde\zeta_1^\varepsilon(x_i-x)\tilde\zeta_2^\varepsilon(v_i-v)(|H^\delta| *_x \gamma_1^\varepsilon)(x_i,v)\ .\end{aligned}$$

As a result, the expression $\iint |Y_{211}(x,v)|\mathbb{1}(|v|\le \ell_0)dxdv$ is bounded above by

$$n\int \varepsilon^d \sum_i \tilde\zeta_2^\varepsilon(v_i-v)(|H^\delta| *_x \gamma_1^\varepsilon)(x_i,v)\mathbb{1}(|v|\le\ell_0)dv = n\varepsilon^{2d}\sum_i G^{\varepsilon,\delta}(x_i,v_i)\ , \tag{6.22}$$

where

$$\begin{aligned}G^{\varepsilon,\delta}(x,w) &= \iint \varepsilon^{-d}|H^\delta(x-y,v)|\mathbb{1}(|v|\le\ell_0)\gamma_1^\varepsilon(y)\zeta\left(\frac{w-v}{\varepsilon\ell_2(\varepsilon)}\right)\ell_2(\varepsilon)^{-d}dydv\\ &= \iint \varepsilon^{-d}|H^\delta(x-y,w-v)|\mathbb{1}(|v-w|\le\ell_0)\gamma_1^\varepsilon(y)\zeta\left(\frac{v}{\varepsilon\ell_2(\varepsilon)}\right)\ell_2(\varepsilon)^{-d}dydv\\ &= \varepsilon^{-2d}\iint \rho^\delta(x-y,w-v)\tilde\eta_\varepsilon\left(\frac{y}{\varepsilon},\frac{v}{\varepsilon}\right)dydv\ ,\end{aligned}$$

where

$$\begin{aligned}\rho^\delta(x,v) &= |H^\delta(x,v)|\mathbb{1}(|v|\le\ell_0)\ ,\\ \tilde\eta_\varepsilon(x,v) &= \gamma_1(x)\ell(\varepsilon)^{-d}\zeta\left(\tfrac{v}{\ell(\varepsilon)}\right)\ .\end{aligned}$$

We are now in a position to apply Lemma 4.6. Recall $\sup_\tau \|L_{k_0}(\gamma_2)\|_{L^1} < \infty$. From this, it is not hard to deduce

$$\sup_{\tau,\varepsilon}\|L_{k_0}(\tilde\eta_\varepsilon)\|_{L^1} < \infty\ . \tag{6.23}$$

Observe that $\|\rho^\delta\|_{L^\infty}\le\ell$. From (6.22-23) and Lemma 4.6 we deduce that the expression

$$E_N\int_0^T\iint |Y_{211}(x,v)|\chi_{\ell_0}(v)dxdv\ ,$$

is bounded above by

$$\begin{aligned}&c_7 nE_N\int_0^T \|\rho^\delta\|_{L^\infty}h(\|\rho^\delta\|_{L^1})(1+N^{-1}\Phi(\mathbf{q}(t)))dt\\ &\le c_7 n\ell\left(E_N\int_0^T h^2\left(\|\rho^\delta\|_{L^1}\right)dt\right)^{1/2}\left(E_N\int_0^T\left(1+N^{-1}\Phi(\mathbf{q}(t))\right)^2dt\right)^{1/2}\\ &\le c_8 n\ell\, h\left(\int_0^T E_N(\|\rho^\delta\|_{L^1})dt\right)\ ,\end{aligned} \tag{6.24}$$

where for the last inequality we have used Proposition 4.3(i), Jensen's inequality and the concavity of the function h^2. Also, we may apply Lemma 6.3 to assert

$$\begin{aligned} E_N \int_0^T \|\rho^\delta\|_{L^1}\,dt &\le \int_0^T \|(\hat K - \hat K^\delta)\chi_{\ell_0}\|_{L^1}\,dt \\ &\le c_9(\log\log|\log\delta|)^{-\alpha_0}\int (1+|v|^2)\chi_{\ell_0}(v)dv + c_9\varepsilon\ell_0^d \\ &\le c_{10}\ell_0^{d+2}(\log\log|\log\delta|)^{-\alpha_0} + c_9\varepsilon\ell_0^d\ , \end{aligned}$$

where $\chi_{\ell_0}(v) = \mathbb{1}(|v| \le \ell_0)$. This and (6.24) yield

$$E_N \iint |Y_{211}(x,v)|\chi_{\ell_0}(v)dxdv \le c_{11}n\ell\left[(\log\log\log|\log\delta|)^{-1} + |\log\varepsilon|^{-1}\right] \tag{6.25}$$

for some constant c_{11} that is independent of τ and so long as $\ell_0^{2(d+2)/\alpha_0} \le \log\log|\log\delta|$ and $\ell_0^d \le \varepsilon^{-1/2}$. (Here we are using the fact that if $\ell_0^{2(d+2)/\alpha_0} \le \log\log|\log\delta|$ and $\ell_0^d \le \varepsilon^{-1/2}$, then

$$\ell_0^{d+2}(\log\log|\log\delta|)^{-\alpha_0} \le (\log\log|\log\delta|)^{-\alpha_0/2}\ ,\ \text{and } \varepsilon\ell_0^d \le \varepsilon^{1/2}\ .)$$

Using (6.20), (6.21) and the fact that the constant c_{11} in (6.25) is independent of τ, we deduce,

$$E_N \iint |Y_{21}(x,v)|\chi_{\ell_0}(v)dxdv \le c_{11}n\ell\left[(\log\log\log|\log\delta|)^{-1} + |\log\varepsilon|^{-1}\right]\ , \tag{6.26}$$

so long as $\ell_0^{2(d+2)/\alpha_0} \le \log\log|\log\delta|$, $\ell_0^d \le \varepsilon^{-1/2}$.

Step 4. We now turn to Y_{22}. Observe that if $\tilde\zeta_1^\varepsilon(x_i - x) \ne 0$, then $|x_i - x| \le c_{12}\varepsilon\ell(\varepsilon)$ for some constant c_{12}. Also, since $\hat A(x,v)$ is of compact support in the x-variable, we have that the expression

$$|\hat K(x_i - \varepsilon y, v) - \hat K^\delta(x_i - \varepsilon y, v)| = \left|\int (\hat K(x_i - \varepsilon y, v) - \hat K(x_i - \varepsilon y - z, v))\zeta^\delta(z)dz\right|\ ,$$

is bounded above by

$$\begin{aligned} &\int \varepsilon^d \sum_j \left|\hat A\left(\frac{x_i - x_j}{\varepsilon} - y, v - v_j\right) - \hat A\left(\frac{x_i - x_j - z}{\varepsilon} - y, v - v_j\right)\right|\zeta^\delta(z)dz \\ &\quad\le c_{13}\int \varepsilon^d \sum_j \mathbb{1}(|x_i - x_j - z| \text{ or } |x_i - x_j| \le c_{14}\varepsilon|y| + c_{14}\varepsilon)|v - v_j|\zeta^\delta(z)dz \\ &\quad\le c_{13}\int \varepsilon^d \sum_j \mathbb{1}(|x - x_j - z| \text{ or } |x - x_j| \le c_{15}\varepsilon|y| + c_{15}\varepsilon\ell(\varepsilon))|v - v_j|\zeta^\delta(z)dz\ , \end{aligned}$$

whenever $\tilde{\zeta}_1^\varepsilon(x_i - x) \neq 0$. In particular, if $|y| \leq \ell(\varepsilon)$, then

$$|\hat{K}(x_i - \varepsilon y, v) - \hat{K}^\delta(x_i - \varepsilon y, v)| \leq c_{16}\ell(\varepsilon)^d \int p^\varepsilon(x, z, v)\zeta^\delta(z)dz, \qquad (6.27)$$

where $p^\varepsilon(x, z, v) = p^\varepsilon(x, v) + p^\varepsilon(x - z, v)$ and

$$p^\varepsilon(x, v) = \varepsilon^d \ell(\varepsilon)^{-d} \sum_j \mathbb{1}(|x - x_j| \leq 2c_{15}\varepsilon\ell(\varepsilon))|v - v_j| .$$

Because of this, we decompose $\gamma = \hat{\gamma}_1 + \hat{\gamma}_2$ with $\hat{\gamma}_1(z) = \gamma(z)\mathbb{1}(|z| \leq \ell(\varepsilon))$. Set

$$R_r = (\hat{K} - \hat{K}^\delta)\mathbb{1}(\hat{K} - \hat{K}^\delta \geq \ell) *_x \hat{\gamma}_r^\varepsilon,$$
$$Y_{22r}(x, v) = \sum_i \tilde{\zeta}_1^\varepsilon(x_i - x)\tilde{\zeta}_2^\varepsilon(v_i - v)R_r(x_i, v)\alpha(x, v) ,$$

where $\hat{\gamma}^\varepsilon(z) = \varepsilon^{-d}\hat{\gamma}(z/\varepsilon)$ for $r = 1$ and 2. We certainly have

$$Y_{22}(x, v) = Y_{221}(x, v) + Y_{222}(x, v) . \qquad (6.28)$$

Recall that there exists a constant c_{17} such that the function γ satisfies $|\gamma(z)| \leq c_{17}e^{-c_{17}|z|}$ for $|z| > 1$. Set $\tilde{\gamma}(z) = c_{17}e^{-c_{17}|z|/2}$, $\tilde{\gamma}^\varepsilon(x) = \varepsilon^{-d}\tilde{\gamma}^\varepsilon(x/\varepsilon)$. We have

$$|R_2| \leq e^{-c_{17}\ell(\varepsilon)/2}\left(|\hat{K}| + |\hat{K}^\delta|\right) *_x \tilde{\gamma}^\varepsilon .$$

As in the derivation of (6.27), we can easily show that if $|x_i - x| \leq c_{13}\varepsilon\ell(\varepsilon)$, then

$$|\hat{K}(x_i - y, v)| \leq q^\varepsilon(x - y, v) , \quad |\hat{K}^\delta(x_i - y, v)| \leq (q^\varepsilon *_x \zeta^\delta)(x - y, v) ,$$

where

$$q^\varepsilon(x, v) = c_{18}(1 + |v|)\varepsilon^d \sum_j \mathbb{1}(|x - x_j| \leq c_{18}\varepsilon\ell(\varepsilon))(|v_j| + 1) ,$$

for a constant c_{18}. As a result,

$$|Y_{222}(x, v)|\chi_{\ell_0}(v) \leq c_{19}n\ell_0\chi_{\ell_0}(v)e^{-c_{17}\ell(\varepsilon)/2}(q^\varepsilon *_x \tilde{\gamma}^\varepsilon + q^\varepsilon *_x \tilde{\gamma}^\varepsilon *_x \zeta^\delta)(x, v) ,$$

for a constant c_{19}. From this we deduce

$$E_N \int_0^T \iint |Y_{222}(x, v)|\chi_{\ell_0}(v)dxdvdt \leq c_{20}n\ell_0^{d+1}e^{-c_{17}\ell(\varepsilon)/2}\ell(\varepsilon)^d , \qquad (6.29)$$

for some constant c_{20}.

We now turn to Y_{221}. So far ℓ has been an arbitrary positive number. We now assume that $\hat{\ell}(\varepsilon) := \ell\ell(\varepsilon)^{-d}/c_{16} > 1$. The inequality (6.27) and Jensen's inequality imply that the expression $|Y_{221}(x, v)|$ is bounded above by

$$
\begin{aligned}
c_{16} & \|\hat{\gamma}_1\|_{L^1}\, \ell(\varepsilon)^d \sum_i \tilde{\zeta}_1^\varepsilon(x_i - x)\tilde{\zeta}_2^\varepsilon(v_i - v)\alpha(x, v) \int p^\varepsilon(x, z, v)\zeta^\delta(z)dz \\
& \cdot \mathbb{1}\left(c_{16}\ell(\varepsilon)^d \int p^\varepsilon(x, z, v)\zeta^\delta(z)dz \geq \ell\right) \\
& \leq c_{16}\|\hat{\gamma}_1\|_{L^1}\, n\ell(\varepsilon)^d \left(\int p^\varepsilon(x, z, v)\zeta^\delta(z)dz\right) \\
& \quad \cdot \mathbb{1}\left(c_{16}\ell(\varepsilon)^d \int p^\varepsilon(x, z, v)\zeta^\delta(z)dz \geq \ell\right) \\
& \leq c_{16}\|\hat{\gamma}_1\|_{L^1}\, n\ell(\varepsilon)^d (\log \hat{\ell}(\varepsilon))^{-1/2}\tilde{\phi}\left(\int p^\varepsilon(x, z, v)\zeta^\delta(z)dz\right) \\
& \leq c_{16}\|\hat{\gamma}_1\|_{L^1}\, n\ell(\varepsilon)^d (\log \hat{\ell}(\varepsilon))^{-1/2} \int \tilde{\phi}(p^\varepsilon(x, z, v))\zeta^\delta(z)dz,
\end{aligned}
$$

where $\tilde{\phi}(z) = z(\log^+ z)^{1/2}$. As a result,

$$
E_N \int_0^T \iint |Y_{221}(x, v)|\mathbb{1}(|v| \leq \ell_0)dxdvdt \leq c_{21}n\ell(\varepsilon)^d(\log \hat{\ell}(\varepsilon))^{-1/2}\,. \quad (6.30)
$$

Here we are using (4.8), Jensen's inequality, and the fact that the density p can be expressed as an average of f^ε–like densities. More precisely, the function $p^\varepsilon(x, v)$ is bounded above by a constant multiple of

$$
\int \left(\varepsilon^d \sum_i \mathbb{1}(|a - x_i| \leq \varepsilon)|v_i - v|\right)\beta(x - a)da\,,
$$

where $\beta(a) = (\varepsilon\ell(\varepsilon))^{-d}\mathbb{1}(|a| \leq 2c_{15}\varepsilon\ell(\varepsilon))$.

Step 5. From (6.8), (6.12–14), (6.17), (6.26) and (6.28–30) we deduce,

$$
\begin{aligned}
E_N & \int_0^T \iint \left|Q_-^{\varepsilon,\alpha}(x, v; \mathbf{q}(t)) - \hat{Q}_-^{\varepsilon,\alpha}(x, v; \mathbf{q}(t))\right|\mathbb{1}(|v| \leq \ell_0)dxdvdt \\
& \leq c_{22}n\big[\varepsilon\ell(\varepsilon)^{d+1}\ell_0^d + \delta^{-1}\varepsilon\ell(\varepsilon)\ell_0^{d+1} + \ell_0^{d+2}(\log\log|\log\delta|)^{-\alpha_0} + \varepsilon\ell_0^d \\
& + \ell(\log\log\log|\log\delta|)^{-1} + \ell|\log\varepsilon|^{-1} + \ell_0^{d+1}e^{-c_{17}\ell(\varepsilon)/2}\ell(\varepsilon)^d + \ell(\varepsilon)^d(\log\hat{\ell}(\varepsilon))^{-1/2}],
\end{aligned}
$$

so long as $\ell_0^{2(d+2)/\alpha_0} \leq \log\log|\log\delta|$ and $\ell_0^d \leq \varepsilon^{-1/2}$. We now choose $\ell = (\log\log\log|\log\varepsilon|)^{1/2}$ and $\delta = \varepsilon\ell(\varepsilon)^2$ to derive (6.9). More precisely,

$$
\begin{aligned}
E_N \int_0^T \iint & \left|Q_-^{\varepsilon,\alpha}(x, v; \mathbf{q}(t)) - \hat{Q}_-^{\varepsilon,\alpha}(x, v; \mathbf{q}(t)\right|\mathbb{1}(|v| \leq \ell_0)dxdvdt \quad (6.31) \\
& \leq c_{23}n\ell_0^{d+2}(\log\log\log\log|\log\varepsilon|)^{-1/(4d+2)}\,,
\end{aligned}
$$

provided $\ell_0^{2(d+2)/\alpha_0} \leq \log\log|\log\varepsilon|$.

To complete the proof, we need to study

$$
\begin{aligned}
&|\hat{Q}_-^{\varepsilon,\alpha}(x,v;\mathbf{q}) - \tilde{f}^\varepsilon(x,v;\mathbf{q})\alpha(x,v)L\tilde{f}^\varepsilon(x,\cdot;\mathbf{q})(v)| \\
&\le n|K^\varepsilon(x,v;\mathbf{q}) - L\tilde{f}^\varepsilon(x,\cdot;\mathbf{q})(v)|. \qquad (6.32)
\end{aligned}
$$

As a consequence of Lemma 6.3,

$$
\begin{aligned}
E_N\int_0^T \iint |K^\varepsilon(x,v;\mathbf{q}(t)) - K^{\varepsilon,\delta}(x,v;\mathbf{q}(t))|\mathbb{1}(|v|\le \ell_0)dxdvdt \qquad (6.33)\\
\le c_{24}\ell_0^{d+2}\left[(\log\log|\log\delta|)^{-\alpha_0}+\varepsilon\right].
\end{aligned}
$$

Because of this, we may compare Lf^ε with $K^{\varepsilon,\delta}$. Indeed

$$
\begin{aligned}
K^{\varepsilon,\delta}(x,v) &= \int \varepsilon^d \sum V\left(\frac{|x-x_j-z|}{\varepsilon}\right) B\left(v-v_j,\frac{x-x_j-z}{|x-x_j-z|}\right)\zeta^\delta(z)dz \\
&= \int \varepsilon^d \sum_j V\left(\frac{|y|}{\varepsilon}\right) B\left(v-v_j,\frac{y}{|y|}\right)\zeta^\delta(x-y-x_j)dy \\
&= \int_0^\infty \int_{\mathbb{S}} \varepsilon^{2d}\sum_j V(\rho)B(v-v_j,n)\zeta^\delta(x-\varepsilon\rho n-x_j)\rho^{d-1}dnd\rho \\
&= \int_0^\infty \int_{\mathbb{S}} \varepsilon^{2d}\sum_j V(\rho)\rho^{d-1}B(v-v_j,n)(\zeta^\delta(x-\varepsilon\rho n-x_j) \\
&\quad -\zeta^\delta(x-x_j))dnd\rho \\
&\quad +\int_0^\infty \int_{\mathbb{S}} \varepsilon^{2d}\sum_j V(\rho)\rho^{d-1}B(v-v_j,n)\zeta^\delta(x-x_j)dnd\rho \\
&=: \Omega_1(x,v)+\Omega_2(x,v).
\end{aligned}
$$

Evidently

$$
\Omega_2(x,v) = \varepsilon^{2d}\sum_j \bar{B}(v-v_j)\zeta^\delta(x-x_j). \qquad (6.34)
$$

On the other hand, if $\delta_r = \varepsilon\ell_r(\varepsilon)$ for $r=1,2$ and $\delta=\delta_1$, then

$$
\begin{aligned}
L\tilde{f}^\varepsilon(x,\cdot;\mathbf{q})(v) &= \int \bar{B}(v-v_*)\sum_j \tilde{\zeta}_1^\varepsilon(x_j-x)\tilde{\zeta}_2^\varepsilon(v_j-v_*)dv_* \\
&= \varepsilon^{2d}\sum_j \zeta^{\delta_1}(x-x_j)\int \bar{B}(v-v_*)\zeta^{\delta_2}(v_j-v_*)dv_* \\
&= \varepsilon^{2d}\sum_j \zeta^{\delta_1}(x-x_j)\int (\bar{B}(v-v_*)-\bar{B}(v-v_j))\zeta^{\delta_2}(v_j-v_*)dv_* \\
&\quad +\Omega_2(x,v) \qquad (6.35)\\
&=: \Omega_3(x,v)+\Omega_2(x,v).
\end{aligned}
$$

By the Lipschitzness of $\bar{B}$,

$$|\Omega_3(x,v)| \le c_{25}\varepsilon^{2d}\sum_j \zeta^{\delta_1}(x-x_j)\int |v_*-v_j|\zeta^{\delta_2}(v_j-v_*)dv_*$$
$$\le c_{26}\delta_2\varepsilon^{2d}\sum_j \zeta^{\delta_1}(x-x_j),$$

for some constants c_{25} and c_{26}. This and (6.35) yield,

$$\iint |\Omega_2(x,v)-L\tilde{f}^{\varepsilon}(x,\cdot;\mathbf{q})(v)|1\!\!1(|v|\le \ell_0)dxdv \le c_{27}\delta_2\ell_0^d. \tag{6.36}$$

Moreover,

$$\Omega_1(x,v) = -\int_0^{\varepsilon}\int_0^{\infty}\int_{\mathbb{S}} \varepsilon^{2d}\sum_j V(\rho)\rho^{d-1}B(v-v_j,n)\rho$$
$$\nabla\zeta^{\delta_1}(x-\theta\rho n-x_j)\cdot n\, dnd\rho d\theta.$$

If $V(\rho)=0$ for $\rho > c_{28}$, then

$$|\Omega_1(x,v)| \le c_{28}\int_0^{\varepsilon}\int_0^{\infty}\int_{\mathbb{S}} \varepsilon^{2d}\sum_j V(\rho)\rho^{d-1}B(v-v_j,n)$$
$$|\nabla\zeta^{\delta_1}(x-\theta\rho n-x_j)|dnd\rho d\theta,$$
$$\int |\Omega_1(x,v)|dx \le c_{28}\varepsilon\delta_1^{-1}\|\nabla\zeta\|_{L^1}\,\varepsilon^{2d}\sum_j \bar{B}(v-v_j)$$
$$\le c_{29}\varepsilon\delta_1^{-1}\varepsilon^{2d}\sum_j |v-v_j|.$$

Hence, we can use the conservation of the kinetic energy to assert

$$E_N\iint |\Omega_1(x,v)|1\!\!1(|v|\le \ell_0)dxdv \le c_{30}\varepsilon\delta_1^{-1}\ell_0^{d+1}. \tag{6.37}$$

From $K^{\varepsilon,\delta}=\Omega_1+\Omega_2$, (6.36) and (6.37) we learn

$$E_N\int_0^T\iint |K^{\varepsilon,\delta}(x,v)-L\tilde{f}^{\varepsilon}(x,\cdot;\mathbf{q})(v)|1\!\!1(|v|\le \ell_0)dxdv$$
$$\le c_{30}\ell_0^{d+1}\ell_1(\varepsilon)^{-1}+c_{27}\ell_0^d\varepsilon\ell_2(\varepsilon)\ .$$

This, (6.32) and (6.33) imply

$$\int_0^T\iint E_N|\hat{Q}^-_{\varepsilon,\alpha}(x,v;\mathbf{q}(t)) - \tilde{f}^{\varepsilon}(x,v;\mathbf{q}(t))(v)\alpha(x,v)Lf^{\varepsilon}(x,\cdot;\mathbf{q}(t))(x)|$$
$$1\!\!1(|v|\le \ell_0)dxdvdt \le c_{31}n\ell_0^{d+2}\ell_1(\varepsilon)^{-1}.$$

This and (6.31) complete the proof. □

We end this section with two consequences of Theorem 6.1 that will be used in Sect. 9. For our first corollary, we obtain a bound on the renormalized loss term.

Corollary 6.4. *There exists a constant* $\hat{C}_7(T)$ *such that*

$$E_N \int_0^T \iint Q_-^{\varepsilon,\alpha}(x, v; \mathbf{q}(s))1\!\!1(|v| \leq \ell_0)dxdvdt \leq \hat{C}_7(T)n\ell_0^{d+2} . \tag{6.38}$$

Proof. Observe that Theorem 6.1 allows us to replace $Q^-_{\varepsilon,\alpha}$ with $\tilde{f}^\varepsilon L\tilde{f}^\varepsilon\alpha(\tilde{f}^\varepsilon)$. Since $\tilde{f}^\varepsilon\alpha(\tilde{f}^\varepsilon) \leq n$ and $\int L\tilde{f}^\varepsilon(x, v, t)dx \leq c_0(1 + |v|)$, we conclude (6.38). □

A review of the proof of Theorem 6.1 reveals that there is a slight room for improvement in the bound (6.38). Indeed, our arguments involved momentum-type bounds whereas the conservation of the kinetic energy implies a bound like (4.6). To take advantage of this, we may replace $B(v, n)$ with $\hat{B}(v, n) := B(v, n)J(v)$ in the proof of Theorem 6.1, where J is a nonnegative smooth function such that $\limsup J(v)|v|^{-b} < \infty$ as $|v| \to \infty$ for a constant $b < 1$. Using Theorem 5.1, (4.8) and Lemma 4.5 for $a = b+1$, one can readily check that Theorem 6.1 is still valid for $\hat{B}$. As a corollary to this we have:

Corollary 6.5. *There exists a constant* $\hat{C}_7(T, b)$ *such that for* $b \in [0, 1)$*,*

$$E_N \int_0^T \iint Q_\varepsilon^{\varepsilon,\alpha,b}(x, v; \mathbf{q}(s))1\!\!1(|v| \leq \ell_0)dxdvdt \leq \hat{C}_7(T, b)n\ell_0^{d+2} ,$$

where

$$Q_-^{\varepsilon,\alpha,b}(x, v; \mathbf{q}) = \sum_{i,j} V^\varepsilon(|x_i - x_j|)B(v_i - v_j, n_{ij})|v_i - v_j|^b\, \tilde{\zeta}_1^\varepsilon(x_i - x)\tilde{\zeta}_2^\varepsilon(v_i - v)$$
$$\alpha(\tilde{f}^\varepsilon(x, v; \mathbf{q})) .$$

7. Stosszahlensatz for the Gain Term, Part I

In this section, we establish some type of Stosszahlensatz for the gain term. Our formulation however differs from what we had in Sect. 6. Instead of an inequality analogous to inequality (6.6), we prove two alternative inequalities for the gain term. These inequalities are the content of Theorem 7.1 of this section and Theorem 8.1 of the next section. Theorem 7.1 will be used in Sect. 9 when we show that the macroscopic densities are supersolutions. Theorem 8.1 will be used in Sect. 10 to show that the macroscopic densities are subsolutions.

To prepare for the statement of the main result of this section, let us start with some definitions. Assume that ζ is a nonnegative smooth function of compact support that satisfies $\int \zeta dz = 1$. Using this ζ, define $\tilde{\zeta}_1^\varepsilon$ and $\tilde{\zeta}_2^\varepsilon$ as in Sect. 6. Recall the function $\tilde{f}^\varepsilon(x, v; \mathbf{q}(s)) = \tilde{f}^\varepsilon(x, v, s)$ that was given right before (6.1). Define

$$\begin{aligned} Q_+^\varepsilon(x, v; \mathbf{q}) &= \sum_{i,j} V^\varepsilon(|x_i - x_j|)B(v_i - v_j, n_{ij})\tilde{\zeta}_1^\varepsilon(x_i - x)\tilde{\zeta}_2^\varepsilon(v_i^j - v) ,\\ \bar{u}^\varepsilon(x; \mathbf{q}) &= \sum_j V^\varepsilon(|x - x_j|)(|v_j|^{3/2} + 1) ,\\ \hat{u}^\varepsilon(x; \mathbf{q}) &= \varepsilon^d \sum_j \tilde{\zeta}_1^\varepsilon(x - x_j)(|v_j|^{3/2} + 1) . \end{aligned} \tag{7.1}$$

In Sect. 9, we need to study

$$Q^{\varepsilon}_{+,n}(x;\mathbf{q};J) =: \int Q^{\varepsilon}_{+}(x,v;\mathbf{q})(1+n^{-1}\tilde{f}^{\varepsilon}(x,v;\mathbf{q}))^{-2}J(v)dv,$$

where J is a continuous function of compact support. Define

$$\begin{aligned} Q_+(g)(v) &= \int\int_{\mathbb{S}} B(v-v_*,n)g(v')g(v'_*)dndv_*\ , \\ Q_+(g;J) &= \int Q_+(g)(v)J(v)dv\ , \\ \Gamma(g) &= \int g(v)(1+|v|)dv\ . \end{aligned}$$

Also define

$$Q_+(g;\mathbf{q};J;\alpha_1,\alpha_2) := Q_+(g;J)(1+\alpha_1\bar{u}^{\varepsilon}(x;\mathbf{q}))^{-1}(1+\alpha_2\hat{u}^{\varepsilon}(x;\mathbf{q}))^{-2}\ . \quad (7.2)$$

We are now ready to state the main result of this section, Theorem 7.1. In this section, we reduce the proof of Theorem 7.1 to Theorem 8.1 of Sect. 8. For both Theorems 7.1 and 8.1, we need to assume that the size of the support of ζ is sufficiently large. This assumption is not used in the part of the proof of Theorem 7.1 that is presented in this section, and is needed only for the proof of Theorem 8.1.

Theorem 7.1. *There exists a constant $C_8(T,J)$ such that for every $\ell \geq 1$ and every nonnegative continuous function J of compact support,*

$$\begin{aligned} E_N\int_0^T\int \Big[Q^{\varepsilon}_{+,n}(x;\mathbf{q}(s);J) - Q_+(\tilde{f}^{\varepsilon}(x,\cdot,\mathbf{q}(s));\mathbf{q};J;\ell^{-1},\ell^{-1})\Big]^- dxds \\ \leq C_8(T,J)(\ell(\log\log n)^{-1} + n^{-1/2}\ell_2(\varepsilon)^{-2d} + \ell\ell(\varepsilon)^{-1/4})\ . \end{aligned}$$

Proof. Step 1. Define

$$\begin{aligned} Q^{\varepsilon}_{+,n,\ell}(x;\mathbf{q};J) = \int \sum_{i,j} V^{\varepsilon}(|x_i-x_j|)B(v_i-v_j,n_{ij})\tilde{\zeta}^{\varepsilon}_1(x_i-x)\tilde{\zeta}^{\varepsilon}_2(v^j_i-v) \\ (1+n^{-1}\tilde{f}^{\varepsilon}(x,v;\mathbf{q}))^{-2}(1+\ell^{-1}\bar{u}^{\varepsilon}(x_i;\mathbf{q}))^{-1}J(v)dv\ . \end{aligned}$$

We certainly have

$$Q^{\varepsilon}_{+,n}(x;\mathbf{q};J) \geq Q^{\varepsilon}_{+,n,\ell}(x;\mathbf{q};J). \quad (7.3)$$

Also define $\hat{Q}^{\varepsilon}_{+,\ell}(x;\mathbf{q};J)$ to be,

$$\begin{aligned} \int\sum_{i,j} V^{\varepsilon}(|x_i-x_j|)B(v_i-v_j,n_{ij})\tilde{\zeta}^{\varepsilon}_1(x_i-x)\tilde{\zeta}^{\varepsilon}_2(v^j_i-v) \\ (1+\ell^{-1}\bar{u}^{\varepsilon}(x_i;\mathbf{q}))^{-1}J(v)dv\ . \end{aligned}$$

We would like to show that $Q^{\varepsilon}_{+,n,\ell} - \hat{Q}^{\varepsilon}_{+,\ell}$ is small whenever n is large. To show this, we first observe that if $\tilde{\zeta}^{\varepsilon}_1(x_i-x)\tilde{\zeta}^{\varepsilon}_2(v^j_i-v) \neq 0$ then

$$|x_i-x| \leq c_0\varepsilon\ell_1(\varepsilon),\ |v^j_i-v| \leq c_0\varepsilon\ell_2(\varepsilon) \quad (7.4)$$

for some positive constant c_0. Take a nonnegative smooth function β of compact support with $\beta(z) = 1$ for $|z| \le 2c_0$ and define

$$g^\varepsilon(x, v; \mathbf{q}) = c_1 \ell_1(\varepsilon)^{-d} \ell_2(\varepsilon)^{-d} \sum_k \beta\left(\frac{x_k - x}{\varepsilon \ell_1(\varepsilon)}\right) \beta\left(\frac{v_k - v}{\varepsilon \ell_2(\varepsilon)}\right) ,$$

for $c_1 = \|\zeta\|^2_{L^\infty}$. We then have that if (7.4) occurs, then

$$\tilde{f}^\varepsilon(x, v; \mathbf{q}) \le g^\varepsilon(x_i, v_i^j; \mathbf{q}). \tag{7.5}$$

Using this we deduce

$$1 - (1 + n^{-1} \tilde{f}^\varepsilon(x, v; \mathbf{q}))^{-2} \le 1 - (1 + n^{-1} g^\varepsilon(x_i, v_i^j; \mathbf{q}))^{-2}$$

whenever $\tilde{\zeta}_1^\varepsilon(x_i - x)\tilde{\zeta}_2^\varepsilon(v_i^j - v) \neq 0$. As a result,

$$\begin{aligned} |Q^\varepsilon_{+,n,\ell}(x) - \hat{Q}^\varepsilon_{+,\ell}(x)| \le & \int \sum_{i,j} V^\varepsilon(|x_i - x_j|) B(v_i - v_j, n_{ij}) \tilde{\zeta}_1^\varepsilon(x_i - x) \tilde{\zeta}_2^\varepsilon(v_i^j - v) \\ & \left[1 - (1 + n^{-1} g^\varepsilon(x_i, v_i^j; \mathbf{q}))^{-2}\right] (1 + \ell^{-1} \bar{u}^\varepsilon(x_i; \mathbf{q}))^{-1} J(v) dv \\ = & \ \varepsilon^d \sum_{i,j} V^\varepsilon(|x_i - x_j|) B(v_i - v_j, n_{ij}) \tilde{\zeta}_1^\varepsilon(x_i - x) J^\varepsilon(v_i^j) \\ & \left[1 - (1 + n^{-1} g^\varepsilon(x_i, v_i^j; \mathbf{q}))^{-2}\right] (1 + \ell^{-1} \bar{u}^\varepsilon(x_i; \mathbf{q}))^{-1}, \end{aligned}$$

where

$$J^\varepsilon(v) = \varepsilon^{-d} (J *_v \tilde{\zeta}_2^\varepsilon)(v) = \varepsilon^{-d} \ell_2(\varepsilon)^{-d} \int J(v - w) \zeta\left(\frac{w}{\varepsilon \ell_2(\varepsilon)}\right) dw .$$

This and the elementary inequality

$$1 - (1 + n^{-1} g^\varepsilon(x, v; \mathbf{q}))^{-2} \le \frac{2n^{-1} g^\varepsilon(x, v; \mathbf{q})}{1 + n^{-1} g^\varepsilon(x, v; \mathbf{q})} =: g_n^\varepsilon(x, v)$$

imply that the expression

$$\int |Q^\varepsilon_{+,n,\ell}(x; \mathbf{q}; J) - \hat{Q}^\varepsilon_{+,\ell}(x; \mathbf{q}; J)| dx \tag{7.6}$$

is bounded above by

$$\begin{aligned} & c_1 \varepsilon^{2d} \sum_{i,j} V^\varepsilon(|x_i - x_j|) B(v_i - v_j, n_{ij}) g_n^\varepsilon(x_i, v_i^j; \mathbf{q}) (1 + \ell^{-1} \bar{u}^\varepsilon(x_i; \mathbf{q}))^{-1} \\ & \mathbb{1}(|v_i^j| \le \ell_0) , \end{aligned} \tag{7.7}$$

where $c_1 = \|J\|_{L^\infty}$ and ℓ_0 is chosen so that $J^\varepsilon(w) = 0$ for any w with $|w| > \ell_0$.

Step 2. Put $c_2 = 4\|V\|_{L^\infty}$. Using (7.7), we can certainly assert that the expression (7.6) is bounded above by $\Omega_1(\mathbf{q}) + \Omega_2(\mathbf{q})$, where

$$\begin{aligned}\Omega_1(\mathbf{q}) &= c_1\varepsilon^{2d}\sum_{i,j} V^\varepsilon(|x_i - x_j|)B(v_i - v_j, n_{ij})g_n^\varepsilon(x_i, v_i^j; \mathbf{q})(1 + \ell^{-1}\bar{u}^\varepsilon(x_i; \mathbf{q}))^{-1}\\ &\qquad 1\!\!1(|v_i^j| \le \ell_0)1\!\!1\left(c_2\varepsilon^d\ell^{-1}|v_i - v_j|^{3/2} > 1\right),\\ \Omega_2(\mathbf{q}) &= c_1\varepsilon^{2d}\sum_{i,j} V^\varepsilon(|x_i - x_j|)B(v_i - v_j, n_{ij})g_n^\varepsilon(x_i, v_i^j; \mathbf{q})(1 + \ell^{-1}\bar{u}^\varepsilon(x_i; \mathbf{q}))^{-1}\\ &\qquad 1\!\!1(|v_i^j| \le \ell_0)1\!\!1\left(c_2\varepsilon^d\ell^{-1}|v_i - v_j|^{3/2} \le 1\right).\end{aligned}$$

Using the assumption $B(v_i - v_j, n_{ij}) \le c_3|v_i - v_j|$, the bound $g_n^\varepsilon \le 2$, and the elementary inequalities

$$1\!\!1\left(c_2\varepsilon^d\ell^{-1}|v_i - v_j|^{3/2} > 1\right) \le c_2^{1/3}\varepsilon^{d/3}\ell^{-1/3}|v_i - v_j|^{1/2}, \quad |v_i - v_j|^{3/2} \le 2|v_i|^{3/2} + 2|v_j|^{3/2},$$

we deduce that the term $\Omega_1(\mathbf{q})$ is bounded above by

$$\begin{aligned}&4c_1c_2^{1/3}c_3\varepsilon^{2d}\ell^{-1/3}\varepsilon^{d/3}\sum_{i,j} V^\varepsilon(|x_i - x_j|)(|v_i|^{3/2} + |v_j|^{3/2})(1 + \ell^{-1}\bar{u}^\varepsilon(x_i; \mathbf{q}))^{-1}\\ &\le 4c_1c_2^{1/3}c_3\varepsilon^{2d}\ell^{-1/3}\varepsilon^{d/3}\sum_{i,j} V^\varepsilon(|x_i - x_j|)|v_i|^{3/2}(1 + \ell^{-1}\bar{u}^\varepsilon(x_i; \mathbf{q}))^{-1}\\ &\quad + 4c_1c_2^{1/3}c_3\varepsilon^{2d}\ell^{-1/3}\varepsilon^{d/3}\sum_{i,j} V^\varepsilon(|x_i - x_j|)|v_j|^{3/2}(1 + \ell^{-1}\bar{u}^\varepsilon(x_i; \mathbf{q}))^{-1}\\ &\le 4c_1c_2^{1/3}c_3\varepsilon^{2d}\ell^{-1}\varepsilon^{d/3}\sum_{i} |v_i|^{3/2}\bar{u}^\varepsilon(x_i, \mathbf{q})(1 + \ell^{-1}\bar{u}^\varepsilon(x_i; \mathbf{q}))^{-1}\\ &\quad + 4c_1c_2^{1/3}c_3\varepsilon^{2d}\ell^{-1}\varepsilon^{d/3}\sum_{j} \bar{u}^\varepsilon(x_i, \mathbf{q})(1 + \ell^{-1}\bar{u}^\varepsilon(x_i; \mathbf{q}))^{-1}\\ &\le 8c_1c_2^{1/3}c_3\varepsilon^{2d}\varepsilon^{d/3}\sum_{i}(|v_i|^{3/2} + 1).\end{aligned} \tag{7.8}$$

From this and the conservation of the kinetic energy we deduce that for some constant c_4,

$$\sup_s E_N\Omega_1(\mathbf{q}(s)) \le c_4\varepsilon^{d/3}. \tag{7.9}$$

We now turn to the second term. We have,

$$\begin{aligned}E_N\Omega_2(\mathbf{q}(s)) = &\int c_1\varepsilon^{2d}\sum_{i,j} V^\varepsilon(|x_i-x_j|)B(v_i-v_j,n_{ij})g_n^\varepsilon(x_i,v_i^j;\mathbf{q})\\ &(1+\ell^{-1}\bar{u}^\varepsilon(x_i;\mathbf{q}))^{-1}\\ &1\!\!1(|v_i^j|\le \ell_0)1\!\!1\left(c_2\varepsilon^d\ell^{-1}|v_i-v_j|^{3/2}\le 1\right)G(s,\mathbf{q})\nu_\beta(d\mathbf{q})\\ = &\int c_1\varepsilon^{2d}\sum_{i,j} V^\varepsilon(|x_i-x_j|)B(v_i-v_j,n_{ij})g_n^\varepsilon(x_i,v_i;S^{ij}\mathbf{q})1\!\!1(|v_i|\le\ell_0)\\ &(1+\ell^{-1}\bar{u}^\varepsilon(x_i;S^{ij}\mathbf{q}))^{-1}1\!\!1\left(c_2\varepsilon^d\ell^{-1}|v_i-v_j|^{3/2}\le 1\right)G(s,S^{ij}\mathbf{q})\nu_\beta(d\mathbf{q}).\end{aligned}$$

One can easily verify that for some constant c_5,

$$\begin{aligned}\bar{u}^\varepsilon(x_i;S^{ij}\mathbf{q}) &\ge \frac{1}{2}\bar{u}^\varepsilon(x_i;\mathbf{q}) - 2\|V\|_{L^\infty}\varepsilon^d|v_i-v_j|^{3/2},\\ g_n^\varepsilon(x_i,v_i;S^{ij}\mathbf{q}) &\le g_n^\varepsilon(x_i,v_i;\mathbf{q}) + c_5n^{-1}\ell_1(\varepsilon)^{-d}\ell_2(\varepsilon)^{-d}\ ,\end{aligned}\tag{7.10}$$

where for the first inequality we have used the elementary inequality $|v_i^j|^{3/2}\ge \frac{1}{2}|v_i|^{3/2} - |v_i-v_j|$. The first inequality in (7.10) implies that if $c_2\varepsilon^d\ell^{-1}|v_i-v_j|^{3/2}\le 1$, then

$$1+\ell^{-1}\bar{u}^\varepsilon(x_i;S^{ij}\mathbf{q}) \ge \frac{1}{2}+\frac{1}{2}\ell^{-1}\bar{u}^\varepsilon(x_i;\mathbf{q}).$$

From this and (7.10) we deduce

$$\begin{aligned}E_N\Omega_2(\mathbf{q}(s)) \le c_6 &\int \varepsilon^{2d}\sum_{i,j} V^\varepsilon(|x_i-x_j|)B(v_i-v_j,n_{ij})g_n^\varepsilon(x_i,v_i;\mathbf{q})\\ &1\!\!1(|v_i|\le\ell_0)\left(1+\ell^{-1}\bar{u}^\varepsilon(x_i;\mathbf{q})\right)^{-1}G(s,S^{ij}\mathbf{q})\nu_\beta(d\mathbf{q})\\ &+c_6n^{-1}\ell_2(\varepsilon)^{-2d}\int\varepsilon^{2d}\sum_{i,j}V^\varepsilon(|x_i-x_j|)B(v_i-v_j,n_{ij})\\ &1\!\!1(|v_i|\le\ell_0)\left(1+\ell^{-1}\bar{u}^\varepsilon(x_i;\mathbf{q})\right)^{-1}G(s,S^{ij}\mathbf{q})\nu_\beta(d\mathbf{q})\\ =:\ &\Omega_{21}(s)+\Omega_{22}(s)\ .\end{aligned}$$

Fix $k\ge 2$. We now apply (5.10) to deduce that the term $\Omega_{21}(s)$ is bounded above by

$$\begin{aligned}&k\int c_6\varepsilon^{2d}\sum_{i,j}V^\varepsilon(|x_i-x_j|)B(v_i-v_j,n_{ij})g_n^\varepsilon(x_i,v_i;\mathbf{q})\\ &1\!\!1(|v_i|\le\ell_0)\left(1+\ell^{-1}\bar{u}^\varepsilon(x_i;\mathbf{q})\right)^{-1}G(s,\mathbf{q})\nu_\beta(d\mathbf{q})\\ &+\frac{2c_6}{\log k-1}\int\varepsilon^{2d}\sum_{i,j}V^\varepsilon(|x_i-x_j|)B(v_i-v_j,n_{ij})\psi\left(\frac{G(s,S^{ij}\mathbf{q})}{G(s,\mathbf{q})}\right)\nu_\beta(d\mathbf{q})\\ &=:\Omega_{211}(s)+\Omega_{212}(s)\ ,\end{aligned}\tag{7.11}$$

because $g_n^\varepsilon \le 2$. We use Lemma 4.7 to claim

$$\int_0^T \Omega_{212}(s)ds \le \frac{c_7}{\log k} \ , \tag{7.12}$$

for some constant c_7. On the other hand, the inequality

$$\varepsilon^d \sum_j V^\varepsilon(|x_i - x_j|)|v_i - v_j| \le (|v_i| + 1) \sum_j V^\varepsilon(|x_i - x_j|)(1 + |v_j|) \ , \tag{7.13}$$

implies,

$$\int_0^T \Omega_{211}(s)ds \le c_8 \ell k \int_0^T \int \varepsilon^{2d} \sum_i g_n^\varepsilon(x_i, v_i; \mathbf{q}) G(s, \mathbf{q}) \nu_\beta(d\mathbf{q}) ds \ ,$$

for a constant c_8. This and (7.12) imply that for every $k \ge 2$,

$$\int_0^T \Omega_{21}(s)ds \le c_8 \ell k \int_0^T \int \varepsilon^{2d} \sum_i g_n^\varepsilon(x_i, v_i; \mathbf{q}) G(s, \mathbf{q}) \nu_\beta(d\mathbf{q}) ds + \frac{c_7}{\log k} \ . \tag{7.14}$$

Repeating (7.11–14) for the term Ω_{22} leads to

$$\int_0^T \Omega_{22}(s)ds \le c_9 n^{-1} \ell_2(\varepsilon)^{-2d} k_1 + \frac{c_7}{\log k_1} \ ,$$

for some constant c_9 and every $k_1 \ge 2$. By choosing $k_1 = \sqrt{n}$ we deduce

$$\int_0^T \Omega_{22}(s)ds \le c_9 n^{-1/2} \ell_2(\varepsilon)^{-2d} + \frac{2c_7}{\log n} \ . \tag{7.15}$$

Step 3. We certainly have

$$g_n^\varepsilon = \frac{2g^\varepsilon}{n + g^\varepsilon} \le \frac{2r}{n + r} + 2\mathbb{1}(g^\varepsilon > r) \ ,$$

for every positive r. This implies

$$\int_0^T \Omega_{211}(s)ds \le \Omega_{2111} + \Omega_{2112} \ , \tag{7.16}$$

where

$$\Omega_{2111} = 2c_8 \ell k \frac{r}{n + r} \varepsilon^{2d} N = 2c_8 Z \ell k \frac{r}{n + r} =: c_9 \ell k \frac{r}{n + r} \ ,$$
$$\Omega_{2112} = 2c_8 \ell k \int_0^T \varepsilon^{2d} \sum_i \mathbb{1}\left((x_i, v_i) \in A_r^\varepsilon(\mathbf{q})\right) G(s, \mathbf{q}) \nu_\beta(d\mathbf{q}) ds,$$

where

$$A_r^\varepsilon(\mathbf{q}) = \{(x, v) : g^\varepsilon(x, v; \mathbf{q}) > r\} \ .$$

We certainly have that for some constants c_{10} and c_{11},

$$g^\varepsilon(x,v;\mathbf{q}) \le c_{10}\ell_1(\varepsilon)^{-d}\ell_2(\varepsilon)^{-d}\sum_k \mathbb{1}(|x_k-x|\le c_{11}\varepsilon\ell_1(\varepsilon),\,|v_k-v|\le c_{11}\varepsilon\ell_2(\varepsilon))\ .$$

Also, if

$$\begin{aligned}\hat g^\varepsilon(x,v;\mathbf{q}) &:= c_{10}\ell_1(\varepsilon)^{-d}\ell_2(\varepsilon)^{-d}\sum_k \mathbb{1}(|x_k-x|\le 2c_{11}\varepsilon\ell_1(\varepsilon),\,|v_k-v|\\ &\qquad \le 2c_{11}\varepsilon\ell_2(\varepsilon))\ ,\\ \hat A_r^\varepsilon(\mathbf{q}) &:= \{(x,v):\hat g^\varepsilon(x,v;\mathbf{q})>r\}\ ,\end{aligned}$$

then we can find a positive constant c_{12} such that

$$A_r^\varepsilon(\mathbf{q})+c_{12}\varepsilon[-1,1]^{2d}\subseteq \hat A_r^\varepsilon(\mathbf{q})\ .$$

We can now apply Proposition 4.3(iii) to deduce

$$\Omega_{2112}\le c_{13}\ell k E_N\int_0^T h(|\hat A_r^\varepsilon(\mathbf{q}(s))|)(1+N^{-1}\Phi^\varepsilon(\mathbf{q}(s)))ds \tag{7.17}$$

for some constant c_{13}. By Chebeyshev's inequality

$$|\hat A_r^\varepsilon(\mathbf{q})|\le \frac{1}{r}\iint \hat g^\varepsilon(x,v;\mathbf{q})dxdv=\frac{c_{14}}{r}\ ,$$

for some constant c_{14}. From this and (7.17) we learn

$$\Omega_{2112}\le c_{13}\ell k E_N\int_0^T h\left(\frac{c_{14}}{r}\right)(1+N^{-1}\Phi^\varepsilon(\mathbf{q}(s)))ds\ .$$

This and Proposition 4.3(i) imply

$$\Omega_{2112}\le c_{15}\ell k T\ h\left(\frac{c_{14}}{r}\right)\ .$$

From this and (7.16) we deduce

$$E_N\int_0^T \Omega_{211}(s)ds\le c_9\ell k\frac{r}{n+r}+c_{15}\ell k\ h\left(\frac{c_{14}}{r}\right).$$

By choosing $r=\sqrt{n}$ we deduce

$$E_N\int_0^T \Omega_{211}(s)ds\le c_{16}\ell k(\log n)^{-1}.$$

This and (7.12) (or (7.14)) imply

$$E_N\int_0^T \Omega_{21}(s)ds\le c_{16}\ell k(\log n)^{-1}+c_7(\log k)^{-1}\ .$$

By choosing $k=(\log n)^{\frac12}$ we learn

$$E_N\int_0^T \Omega_{21}(s)ds\le c_{17}\ell(\log\log n)^{-1}.$$

We now use this and (7.15) to obtain

$$E_N \int_0^T \Omega_2(s)ds \le c_{17}\ell(\log\log n)^{-1} + c_9 n^{-1/2}\ell_2(\varepsilon)^{-2d} + \frac{2c_7}{\log n} .$$

From this and (7.9) we conclude

$$\begin{aligned} &E_N \int_0^T \int |Q^\varepsilon_{+,n,\ell}(x;\mathbf{q};J) - \hat{Q}^\varepsilon_{+,\ell}(x;\mathbf{q}(s),J)|dxds \\ &\le c_{18}\ell(\log\log n)^{-1} + c_{18}n^{-1/2}\ell_2(\varepsilon)^{-2d} + c_{18}\varepsilon^{d/3} . \end{aligned} \tag{7.18}$$

Step 4. In view of (7.18) and (7.3) we would like to study $\hat{Q}^\varepsilon_{+,\ell}$. Evidently

$$\begin{aligned} \hat{Q}^\varepsilon_{+,\ell}(x;\mathbf{q};J) &= \varepsilon^d \sum_{i,j} V^\varepsilon(|x_i - x_j|)B(v_i - v_j, n_{ij}) \\ &\quad \times \tilde{\zeta}^\varepsilon_1(x_i - x)(1 + \ell^{-1}\bar{u}^\varepsilon(x_i;\mathbf{q}))^{-1} J^\varepsilon(v_i^j) \\ &= \varepsilon^d \sum_i \tilde{\zeta}^\varepsilon_1(x_i - x)K_J(x_i, v_i)(1 + \ell^{-1}\bar{u}^\varepsilon(x_i;\mathbf{q}))^{-1} , \end{aligned}$$

where

$$J^\varepsilon = J * \hat{\zeta}^\varepsilon , \quad \hat{\zeta}^\varepsilon(v) = \varepsilon^{-d}\ell_2(\varepsilon)^{-d}\zeta\left(\frac{v}{\varepsilon\ell_2(\varepsilon)}\right) ,$$

and $K_J(x, v)$ is equal to

$$\sum_j V^\varepsilon(|x - x_j|)B\left(v - v_j, \frac{x - x_j}{|x - x_j|}\right) J^\varepsilon\left(v - (v - v_j)\cdot\frac{x - x_j}{|x - x_j|}\frac{x - x_j}{|x - x_j|}\right) . \tag{7.19}$$

Let us define

$$\begin{aligned} Q^\varepsilon_+(x;\mathbf{q};J;\alpha_1,\alpha_2) := \varepsilon^d \sum_i \tilde{\zeta}^\varepsilon_1(x_i - x)K_J(x_i, v_i)(1 + \alpha_1\bar{u}^\varepsilon(x_i;\mathbf{q}))^{-1} \\ \cdot(1 + \alpha_2\hat{u}^\varepsilon(x;\mathbf{q}))^{-2} . \end{aligned} \tag{7.20}$$

We certainly have

$$\hat{Q}^\varepsilon_{+,\ell}(x;\mathbf{q};J) \ge Q^\varepsilon_+(x;\mathbf{q};J;\ell^{-1},\ell^{-1}) . \tag{7.21}$$

On the other hand, it follows from Theorem 8.1 of the next section that for a constant c_{19},

$$\begin{aligned} &E_N \int_0^T \int \left| Q^\varepsilon_+(x;\mathbf{q}(s);J;\ell^{-1},\ell^{-1}) - Q_+(\tilde{f}^\varepsilon(x,\cdot;\mathbf{q}(s));\mathbf{q};J;\ell^{-1},\ell^{-1})\right| dxds \\ &\le c_{19}\ell\ell(\varepsilon)^{-1/4} . \end{aligned}$$

This, (7.3), (7.18) and (7.21) complete the proof of the theorem. □

8. Stosszahlensatz for the Gain Term, Part II

In the previous section, we reduced the proof of Theorem 7.1 to a claim that is the main goal of this section, namely Theorem 8.1. This theorem will also be used in Sect. 10 to show that the macroscopic densities are subsolutions.

Recall the functions $\tilde{\zeta}_1^\varepsilon$, $\tilde{\zeta}_2^\varepsilon$, $\bar{u}^\varepsilon$, $\hat{u}^\varepsilon$, and the density $\tilde{f}^\varepsilon$ of Sect. 6 and 7. Also recall K_J that was given by (7.19) and

$$Q_+^\varepsilon(x; \mathbf{q}; J; \alpha_1, \alpha_2) = \varepsilon^d \sum_i \tilde{\zeta}_1^\varepsilon(x_i - x) K_J(x_i, v_i)(1 + \alpha_1 \bar{u}^\varepsilon(x_i; \mathbf{q}))^{-1}(1 + \alpha_2 \hat{u}^\varepsilon(x; \mathbf{q}))^{-2} ,$$

$$Q_+(g; \mathbf{q}; J; \alpha_1, \alpha_2) = Q_+(g; J)(1 + \alpha_1 \bar{u}^\varepsilon(x; \mathbf{q}))^{-1}(1 + \alpha_2 \hat{u}^\varepsilon(x; \mathbf{q}))^{-2} .$$

Theorem 8.1. *There exists a constant $C_9 = C_9(T, J)$ such that for every continuous function J of compact support,*

$$E_N \int_0^T \int \left| Q_+^\varepsilon(x; \mathbf{q}(s); J; \alpha_1, \alpha_2) - Q_+(\tilde{f}^\varepsilon(x, \cdot, \mathbf{q}(s)); \mathbf{q}; J; \alpha_1, \alpha_2) \right| dx ds$$
$$\le C_9(1 + \alpha_1 \alpha_2^{-1})(1 + \alpha_2^{-1})\ell(\varepsilon)^{-1/4} .$$

To prepare for the proof of Theorem 8.1, we state two lemmas that are the analogs of Lemmas 6.2 and 6.3. Define,

$$A(v, \bar{x}, \bar{v}) = V(|\bar{x}|) B\left(\bar{v}, \frac{\bar{x}}{|\bar{x}|}\right) J\left(v - \bar{v} \cdot \frac{\bar{x}}{|\bar{x}|} \frac{\bar{x}}{|\bar{x}|}\right) .$$

Evidently

$$\begin{aligned} K_J(x, v) &= \sum_j \int A\left(v - w, \frac{x - x_j}{\varepsilon}, v - v_j\right) \hat{\zeta}^\varepsilon(w) dw \\ &=: \varepsilon^d \sum_j A^\varepsilon\left(v, \frac{x - x_j}{\varepsilon}, v - v_j\right) , \end{aligned}$$

where $\hat{\zeta}^\varepsilon(v) = \varepsilon^{-d} \ell_2(\varepsilon)^{-d} \zeta\left(\frac{v}{\varepsilon \ell_2(\varepsilon)}\right)$ and $A^\varepsilon = A *_v \hat{\zeta}^\varepsilon$.

Lemma 8.2. *There exist three functions $\hat{A} = \hat{A}(v, \bar{x}, \bar{v})$, $\eta = \eta(x)$ and $\gamma = \gamma(x)$, and two constants c and R such that $\hat{A}(v, \bar{x}, \bar{v}) = \eta(\bar{x}) = 0$ if $|\bar{x}| > R$, $\sup_v \mathcal{R}(\hat{A}(v, \cdot, \cdot)) < \infty$,*

$$|\hat{A}(v, \bar{x}, \bar{v}) - \hat{A}(w, \bar{x}, \bar{v})||v - w|^{-1} + |\hat{A}(v, \bar{x}, \bar{v}) - \hat{A}(v, \bar{x}, \bar{w})||\bar{v} - \bar{w}|^{-1} \le \eta(x) ,$$

$\gamma \in \hat{\mathcal{L}}$, $|\gamma(x)| \le \exp(-c|x|)$ for x with $|x| > 1$, $\eta(x) \le c$ for all x, and

$$A(v, \bar{x}, \bar{v}) = \int \hat{A}(v, \bar{x} - y, \bar{v}) \gamma(y) dy .$$

The proof Lemma 8.2 is very similar to the proof of Lemma 6.2 and is omitted.

Define

$$K_J^\varepsilon(v,\bar{x},\bar{v};\mathbf{q}) = \sum_j \varepsilon^d A\left(v, \frac{\bar{x}-x_j}{\varepsilon}, \bar{v}-v_j\right) ,$$

$$\hat{K}_J^\varepsilon(v,\bar{x},\bar{v};\mathbf{q}) = \sum_j \varepsilon^d \hat{A}\left(v, \frac{\bar{x}-x_j}{\varepsilon}, \bar{v}-v_j\right) ,$$

$$K_J^{\varepsilon,\delta}(v,\bar{x},\bar{v};\mathbf{q}) = \int K_J^\varepsilon(v,\bar{x}-z,\bar{v};\mathbf{q})\zeta^\delta(z)\,dz ,$$

$$\hat{K}_J^{\varepsilon,\delta}(v,\bar{x},\bar{v};\mathbf{q}) = \int \hat{K}_J^\varepsilon(v,\bar{x}-z,\bar{v};\mathbf{q})\zeta^\delta(z)dz .$$

As a consequence of Theorem 5.1 we have,

Lemma 8.3. *There exists a constant $C_{10} = C_{10}(T, J)$ such that the expressions*

$$E_N \int_0^T \int |K_J^\varepsilon(v,\bar{x},\bar{v};\mathbf{q}(s)) - K_J^{\varepsilon,\delta}(v,\bar{x},\bar{v},s;\mathbf{q}(s))|d\bar{x}ds ,$$

$$E_N \int_0^T \int |\hat{K}_J^\varepsilon(v,\bar{x},\bar{v};\mathbf{q}(s)) - \hat{K}_J^{\varepsilon,\delta}(v,\bar{x},\bar{v},s;\mathbf{q}(s))|d\bar{x}ds ,$$

are bounded above by

$$C_{10}(1+|\bar{v}|^2)(\log\log|\log\delta|)^{-\alpha_0} + C_{10}\varepsilon ,$$

for every $v, \bar{v} \in \mathbb{R}^d$ and $\delta > 0$.

We are now ready to give a proof for Theorem 8.1. The proof of this theorem is similar to the proof of Theorem 6.1. Because of this, some of the steps are only sketched.

Proof of Theorem 8.1. Step 1. To ease the notation, let us write $U(x) = U(x, \mathbf{q})$ for $(1+\alpha_2\hat{u}^\varepsilon(x;\mathbf{q}))^{-1}$ and $S(x) = S(x,\mathbf{q})$ for $(1+\alpha_1\bar{u}^\varepsilon(x;\mathbf{q}))^{-1}$. Using these abbreviations we have

$$Q_+^\varepsilon(x;\mathbf{q};J;\alpha_1,\alpha_2) = \varepsilon^d \sum_i \tilde{\zeta}_1^\varepsilon(x_i - x)K_J(x_i,v_i)S(x_i)U^2(x) . \tag{8.1}$$

We first would like to replace $K_J(x_i, v_i)$ with $K_J(x, v_i)$ in (8.1). Recall that $*_x$ denotes the convolution in the x-variable. Define $K_J^\delta = K_J *_x \zeta^\delta$, where $\zeta^\delta(z) = \delta^{-d}\zeta(z/\delta)$. Let us write

$$Y(x) = Y(x,\mathbf{q}) := \varepsilon^d \sum_i \tilde{\zeta}_1^\varepsilon(x_i - x)(K_J(x_i,v_i) - K_J^\delta(x_i,v_i))S(x_i)U^2(x) . \tag{8.2}$$

We write $A = \hat{A} *_{\bar{x}} \gamma$ for functions $\hat{A}(v,\cdot,\cdot) \in \mathcal{B}$ and $\gamma \in \hat{\mathcal{L}}$ that satisfy the assumptions of Lemma 8.2. As a result, $A^\varepsilon = \hat{A}^\varepsilon *_{\bar{x}} \gamma$, where

$$\hat{A}^\varepsilon(v,\bar{x},\bar{v}) = \int \hat{A}(v-w,\bar{x},\bar{v})\hat{\zeta}^\varepsilon(w)dw .$$

We certainly have $K_J = \hat{K}_J *_x \gamma^\varepsilon$, where

$$\hat{K}_J(x, v) = \varepsilon^d \sum_j \hat{A}^\varepsilon \left(v, \frac{x - x_j}{\varepsilon}, v - v_j \right) , \quad \gamma^\varepsilon(y) = \varepsilon^{-d} \gamma \left(\frac{y}{\varepsilon} \right) . \tag{8.3}$$

Also, $K_J^\delta = \hat{K}_J^\delta *_x \gamma^\varepsilon$, where $\hat{K}_J^\delta = \hat{K}_J *_x \zeta^\delta$. Write

$$K_J - K_J^\delta = (\hat{K}_J - \hat{K}_J^\delta) *_x \gamma^\varepsilon = T_1 *_x \gamma^\varepsilon + T_2 *_x \gamma^\varepsilon , \tag{8.4}$$

where $T_1 = \min(\hat{K}_J - \hat{K}_J^\delta, \ell)$. Replace $K_J - K_J^\delta$ in the definition of Y with the two terms which appeared on the right-hand side of (8.4). The result will be denoted by $Y_1 = Y_1(x, \mathbf{q})$ and $Y_2 = Y_2(x, \mathbf{q})$. As a result

$$Y(x) = Y_1(x) + Y_2(x) , \tag{8.5}$$

where

$$Y_1(x) = \varepsilon^d \sum_i \tilde{\zeta}_1^\varepsilon(x_i - x)(T_1 * \gamma^\varepsilon)(x_i, v_i) S(x_i) U^2(x). \tag{8.6}$$

Step 2. By Lemma 8.2, we may write $\gamma = \gamma_{1,\tau} + \gamma_{2,\tau}$, where $\sup_\tau L_{k_0}(\gamma_{2,\tau}) < \infty$ for some $k_0 > 0$, and $\lim \|\gamma_{1,\tau}\|_{L^1} = 0$ as $\tau \to 0$. To ease the notation, we simply write γ_r for $\gamma_{r,\tau}$. Set $\gamma_r^\varepsilon(x) = \varepsilon^{-d}\gamma_r(x/\varepsilon)$ for $r = 1, 2$. We replace γ^ε in (8.6) with γ_r^ε for $r = 1$ and 2 and denote the result by Y_{11} and Y_{12} respectively. Evidently

$$Y_1(x) = Y_{11}(x) + Y_{12}(x) . \tag{8.7}$$

We certainly have

$$\lim_{\tau \to 0} \|\gamma_2^\varepsilon\|_{L^1} = \lim_{\tau \to 0} \|\gamma_2\|_{L^1} = 0 .$$

From this we learn

$$\lim_{\tau \to 0} \|T_1 *_x \gamma_2^\varepsilon\|_{L^\infty} \le \limsup_{\tau \to 0} \|T_1\|_{L^\infty} \|\gamma_2^\varepsilon\|_{L^1} \le \ell \limsup_{\tau \to 0} \|\gamma_2^\varepsilon\|_{L^1} = 0 .$$

This implies

$$\lim_{\tau \to 0} \int |Y_{12}(x)| dx \le \ell \lim_{\tau \to 0} \|T_1 *_x \gamma_2^\varepsilon\|_{L^\infty} = 0 . \tag{8.8}$$

On the other hand, if we write G^ε for $|T_1| *_x \gamma_1^\varepsilon$, then

$$|Y_{11}(x)| \le \varepsilon^d \sum_i \tilde{\zeta}_1^\varepsilon(x_i - x) G^\varepsilon(x_i, v_i) U(x) . \tag{8.9}$$

Take a nonincreasing smooth function $\beta : [0, \infty) \to [0, 1]$ such that $\beta(a) = 1$ for $a \in [0, 1]$ and $\beta(a) = 0$ for $a \in [2, \infty)$. Fix a positive k and put $\beta_k(a) = \beta(a/k)$. Define

$$Y_{111}(x) = \varepsilon^d \sum_i \tilde{\zeta}_1^\varepsilon(x_i - x) G^\varepsilon(x_i, v_i) \beta_k(|v_i|) U(x) ,$$

$$Y_{112}(x) = \varepsilon^d \sum_i \tilde{\zeta}_1^\varepsilon(x_i - x) G^\varepsilon(x_i, v_i)(1 - \beta_k(|v_i|)) U(x) .$$

Evidently,

$$|Y_{11}| \le Y_{111} + Y_{112} \,. \tag{8.10}$$

Moreover,

$$\begin{aligned} Y_{112}(x) &\le \ell \varepsilon^d \sum_i \tilde{\zeta}_1^\varepsilon(x_i - x) 1\!\!1(|v_i| \ge k) U(x) \\ &\le \ell k^{-3/2} \varepsilon^d \sum_i \tilde{\zeta}_1^\varepsilon(x_i - x) |v_i|^{3/2} U(x) \le \ell k^{-3/2} \alpha_2^{-1} \,. \end{aligned} \tag{8.11}$$

As in the proof of Theorem 6.1, we would like to apply Lemma 4.6 to bound Y_{111}. For this, we need to have a convolution in both x and v variables. We already have a convolution in the x-variable. To produce a v-convolution, we first estimate the Lipschitz constant of T_1 in the v–variable. For this, we only need to bound the Lipschitz constant of the function $\hat{K}_J$ in the v-variable. For this, we apply Lemma 8.2 to obtain

$$|\hat{K}_J(x, v) - \hat{K}_J(x, w)| \le c_0 |v - w| \varepsilon^d \sum_j 1\!\!1(|x - x_j| \le c_0 \varepsilon) \,, \tag{8.12}$$

for some constant c_0. Let us write $G_k^\varepsilon(x, v)$ for $G^\varepsilon(x, v)\beta_k(v)$. Using (8.12) one can readily obtain,

$$|G_k^\varepsilon(x, v) - G_k^\varepsilon(x, w)| \le c_1 |v - w| \varepsilon^d \sum_j 1\!\!1(|x - x_j| \le c_0 \varepsilon) + c_1 k^{-1} G^\varepsilon(x, w) |v - w| \,,$$

for some constant c_1. From this, it is not hard to deduce that the expression

$$|\varepsilon^d \sum_i \tilde{\zeta}_1^\varepsilon(x_i - x)(G_k^\varepsilon(x_i, v_i) - G_k^\varepsilon(x_i, v_i + z)) U(x)| \,, \tag{8.13}$$

is bounded above by

$$\begin{aligned} & c_1 |z| \varepsilon^{2d} \sum_{i,j} \tilde{\zeta}_1^\varepsilon(x_i - x) 1\!\!1(|x_i - x_j| \le c_0 \varepsilon) U(x) + c_1 |z| \ell k^{-1} \sum_i \tilde{\zeta}_1^\varepsilon(x_i - x) U(x) \\ &\le c_1 |z| \varepsilon^{2d} \sum_{i,j} \tilde{\zeta}_1^\varepsilon(x_i - x) 1\!\!1(|x - x_j| \le c_2 \varepsilon \ell(\varepsilon)) U(x) + c_1 \ell k^{-1} \alpha_2^{-1} |z| \\ &\le c_1 \varepsilon^d |z| \alpha_2^{-1} \sum_j 1\!\!1(|x - x_j| \le c_2 \varepsilon \ell(\varepsilon)) + c_1 \ell k^{-1} \alpha_2^{-1} |z| \,. \end{aligned}$$

Because of this, we may define $\hat{G}_k^\varepsilon = G_k^\varepsilon *_v \zeta^\varepsilon$ and assert

$$\begin{aligned} &\int \Big| \varepsilon^d \sum_i \tilde{\zeta}_1^\varepsilon(x_i - x) G_k^\varepsilon(x_i, v_i) U(x) - \varepsilon^d \sum_i \tilde{\zeta}_1^\varepsilon(x_i - x) \hat{G}_k^\varepsilon(x_i, v_i) U(x) \Big| dx \\ &\quad \le c_3 \alpha_2^{-1} \varepsilon \ell(\varepsilon)^d + c_3 \ell k^{-1} \alpha_2^{-1} \varepsilon \,, \end{aligned} \tag{8.14}$$

for a constant c_3. Since $\hat{G}_k^\varepsilon = (|T_1|\beta_k) * \bar{\eta}^\varepsilon$ for $\bar{\eta}^\varepsilon(x, v) = \varepsilon^{-2d} \bar{\eta}(x/\varepsilon, v/\varepsilon)$, $\bar{\eta}(x, v) = \gamma_1(x)\zeta(v)$, we can now apply Lemma 4.6 to deduce that the expression

$$E_N \int_0^T \int |Y_{111}(x)| dx ds \,, \tag{8.15}$$

is bounded above by,

$$c_4E_N\int_0^T \|L_{k_0}(\bar\eta)\|_{L^1}\ \|T_1\|_{L^\infty}\, h(\|T_1\beta_k\|_{L^1})(1+N^{-1}\Phi(\mathbf{q}(s)))ds \\ +c_3\alpha_2^{-1}\varepsilon\ell(\varepsilon)^d + c_3k\ell\alpha_2^{-1}\varepsilon\ . \tag{8.16}$$

On the other hand, we may apply Lemma 8.3 and the definition T_1 to assert,

$$\begin{aligned}&\|L_{k_0}(\bar\eta)\|_{L^1} \le c_5(k_0)\ , \quad \|T_1\|_{L^\infty}\le \ell\ ,\\ &E_N\int_0^T \|T_1\beta_k\|_{L^1}ds \le E_N\int_0^T \|(\hat K_J-\hat K_J^\delta)\beta_k\|_{L^1}ds\\ &\quad\le c_6(\log\log|\log\delta|)^{-\alpha_0}\int(1+|v|^2)\beta_k(v)dv + c_6k\varepsilon\\ &\quad\le c_7k^{d+2}(\log\log|\log\delta|)^{-\alpha_0}+c_6k\varepsilon\ ,\end{aligned}$$

for constants $c_5(k_0)$, c_6 and c_7 that are independent of τ and ε. We now use this and (8.16), and repeat (6.24) to deduce that the expression (8.15) is bounded above by

$$c_8\ell(\log\log\log|\log\delta|)^{-1}+c_8|\log(k\varepsilon)|^{-1}+c_3\alpha_2^{-1}\varepsilon\ell(\varepsilon)^d+c_3\ell k^{-1}\alpha_2^{-1}\varepsilon\ , \tag{8.17}$$

for a constant c_8 that is independent of τ. Choose $k=(\log\log|\log\delta|)^{1/2}$ and assume that $k\varepsilon\le\varepsilon^{1/2}$. Using (8.8), (8.10–11), (8.17) and the fact that the constants c_3 and c_5 in (8.16) are independent of τ, we deduce that the expression $E_N\int_0^T\int|Y_1(x)|dxds$ is bounded above by

$$c_8\ell(\log\log\log|\log\delta|)^{-1}+2c_8|\log\varepsilon|^{-1}+c_3\alpha_2^{-1}\varepsilon\ell(\varepsilon)^d \\ +c_3(\log\log|\log\delta|)^{-1/2}\ell\alpha_2^{-1}\varepsilon+c_3\ell(\log\log|\log\delta|)^{-3/4}\alpha_2^{-1}\ . \tag{8.18}$$

Step 3. We now turn to Y_2. In this step, we mostly follow Step 4 of the proof of Theorem 6.1. Observe that if $\tilde\zeta_1^\varepsilon(x_i-x)\ne 0$, then $|x_i-x|\le c_9\varepsilon\ell(\varepsilon)$ for some constant c_9. Also, since $\hat A(v,\bar x,\bar v)$ is of compact support in the $\bar x$-variable, we can repeat the proof of (6.27) to assert that whenever $\tilde\zeta_1^\varepsilon(x_i-x)\ne 0$ and $|y|\le\ell(\varepsilon)$,

$$|\hat K_J(x_i-\varepsilon y,v)-\hat K_J^\delta(x_i-\varepsilon y,v)|\le c_{10}\ell(\varepsilon)^d(|v|+1)\int p^\varepsilon(x,z)\zeta^\delta(z)dz\ , \tag{8.19}$$

for a constant c_{10}, where $p^\varepsilon(x,z)=p^\varepsilon(x)+p^\varepsilon(x-z)$ and

$$p^\varepsilon(x)=c_{11}\varepsilon^d\ell(\varepsilon)^{-d}\sum_j 1\!\!1(|x_j-x|\le c_{11}\varepsilon\ell(\varepsilon))(|v_j|+1)\ ,$$

for a constant c_{11}. Because of this, we decompose $\gamma=\hat\gamma_1+\hat\gamma_2$ with $\hat\gamma_1(z)=\gamma(z)1\!\!1(|z|\le\ell(\varepsilon))$. Set $R_r=T_2*_x\hat\gamma_r^\varepsilon$, for $\hat\gamma^\varepsilon(z)=\varepsilon^{-d}\hat\gamma_r(z/\varepsilon)$, fix a positive k, and define

$$\begin{aligned}Y_{21}(x,\mathbf{q})&=\varepsilon^d\sum_i\tilde\zeta_1^\varepsilon(x_i-x)1\!\!1(|v_i|\le k)R_1(x_i,v_i)S(x_i)U^2(x)\ ,\\ Y_{22}(x,\mathbf{q})&=\varepsilon^d\sum_i\tilde\zeta_1^\varepsilon(x_i-x)1\!\!1(|v_i|> k)R_1(x_i,v_i)S(x_i)U^2(x)\ ,\\ Y_{23}(x,\mathbf{q})&=\varepsilon^d\sum_i\tilde\zeta_1^\varepsilon(x_i-x)R_2(x_i,v_i)S(x_i)U^2(x)\ .\end{aligned}$$

We certainly have

$$Y_2(x,\mathbf{q}) = Y_{21}(x,\mathbf{q}) + Y_{22}(x,\mathbf{q}) + Y_{23}(x,\mathbf{q})\ .$$

Recall that the function γ satisfies $|\gamma(z)| \le c_{12}e^{-c_{12}|z|}$ for $|z| > 1$ and a positive constant c_{12}. We can show

$$E_N \int_0^T \int |Y_{23}(x,\mathbf{q}(s))|dxds \le c_{13}\alpha_2^{-1}e^{-c_{12}\ell(\varepsilon)/2}\ell(\varepsilon)^d\ , \tag{8.20}$$

in just the same way we showed (6.29).

We now assume $k+1 = \ell^{1/2}$. As in the proof of Theorem 6.1 we assume that $\hat{\ell}(\varepsilon) := \ell^{1/2}(\ell(\varepsilon))^{-d} > 1$. The inequality (8.19) implies that the expression $|Y_{21}(x,\mathbf{q})|$ is bounded above by

$$\begin{aligned}
&c_{14}\int \ell(\varepsilon)^d\varepsilon^d\sum_i \tilde{\zeta}_1^\varepsilon(x_i - x)(|v_i|+1)U(x)p^\varepsilon(x,z)\zeta^\delta(z)dz \\
&\quad\times \mathbb{1}\left((1+k)\ell(\varepsilon)^d\int p^\varepsilon(x,z)\zeta^\delta(z)dz \ge \ell\right) \\
&\quad\le c_{14}\alpha_2^{-1}\ell(\varepsilon)^d\left(\int p^\varepsilon(x,z)\zeta^\delta(z)dz\right)\mathbb{1}\left(\ell(\varepsilon)^d\int p^\varepsilon(x,z)\zeta^\delta(z)dz \ge \ell^{1/2}\right) \\
&\quad\le c_{14}\alpha_2^{-1}\ell(\varepsilon)^d(\log\hat{\ell}(\varepsilon))^{-1/2}\tilde{\phi}\left(\int p^\varepsilon(x,z)\zeta^\delta(z)dz\right) \\
&\quad\le c_{14}\alpha_2^{-1}\ell(\varepsilon)^d(\log\hat{\ell}(\varepsilon))^{-1/2}\int \tilde{\phi}(p^\varepsilon(x,z))\zeta^\delta(z)dz\ ,
\end{aligned}$$

where $\tilde{\phi}(z) = z(\log^+ z)^{1/2}$. As a result, we may apply (4.8) to deduce

$$E_N \int |Y_{21}(x,\mathbf{q}(s))|dx \le c_{15}\ell(\varepsilon)^d(\log\hat{\ell}(\varepsilon))^{-1/2}\alpha_2^{-1}\ . \tag{8.21}$$

(Compare this with (6.30).) Similarly, we use (8.19) to assert that the expression $|Y_{22}(x,\mathbf{q})|$ is bounded above by

$$\begin{aligned}
&c_{16}\ell(\varepsilon)^d\varepsilon^d\sum_i \tilde{\zeta}_1^\varepsilon(x_i - x)(|v_i|+1)\mathbb{1}(|v_i|>k)U(x)\int p^\varepsilon(x,z)\zeta^\delta(z)dz \\
&\quad\le c_{16}k^{-1/2}\ell(\varepsilon)^d\varepsilon^d\sum_i \tilde{\zeta}_1^\varepsilon(x_i - x)(|v_i|+1)^{3/2}U(x)\int p^\varepsilon(x,z)\zeta^\delta(z)dz \\
&\quad\le c_{17}\ell(\varepsilon)^dk^{-1/2}\alpha_2^{-1}\int p^\varepsilon(x,z)\zeta^\delta(z)dz\ ,
\end{aligned}$$

from constants c_{16} and c_{17}. Recall $k+1 = \ell^{1/2}$. As a result,

$$E_N \int_0^T \int |Y_{22}(x,\mathbf{q}(s))|dxds \le c_{18}\ell(\varepsilon)^d\ell^{-1/4}\alpha_2^{-1}\ , \tag{8.22}$$

for a constant c_{18}.

Step 5. From (8.5), (8.18), and (8.20-22) we learn that the expression

$$E_N \int_0^T \int |Y(x, \mathbf{q}(s))| dx ds \ ,$$

is bounded above by

$$\begin{aligned}&c_8\ell(\log\log\log|\log\delta|)^{-1} + 2c_8|\log\varepsilon|^{-1/2} + c_3\alpha_2^{-1}\varepsilon\ell(\varepsilon)^d\\ &\quad + c_3(\log\log|\log\delta|)^{1/2}\ell\alpha_2^{-1}\varepsilon + c_3\ell(\log\log|\log\delta|)^{-3/4}\alpha_2^{-1}\\ &\quad + c_{13}\alpha_2^{-1}e^{-c_{12}\ell(\varepsilon)/2}\ell(\varepsilon)^d + c_{15}\ell(\varepsilon)^d(\log\hat{\ell}(\varepsilon))^{-1/2}\alpha_2^{-1} + c_{18}\ell(\varepsilon)^d\ell^{-1/4}\alpha_2^{-1} \ .\end{aligned}$$

Choose $\ell = (\log\log\log|\log\varepsilon|)^{1/2}$ and $\delta = \varepsilon\ell(\varepsilon)^2$ to deduce

$$E_N \int_0^T \int |Y(x, \mathbf{q}(s))| dx ds \le c_{19}(1 + \alpha_2^{-1})(\log\log\log\log|\log\varepsilon|)^{-\frac{1}{4d+2}} \ . \quad (8.23)$$

We now bound

$$\int \varepsilon^d \sum_i \tilde{\zeta}_1^\varepsilon(x_i - x)|K_J^\delta(x_i, v_i) - K_J^\delta(x, v_i)|S(x_i)U^2(x)dx \ . \quad (8.24)$$

This can be treated in just the same way we established (6.13). Indeed whenever $\tilde{\zeta}_1^\varepsilon(x_i - x) \neq 0$, the expression

$$|K_J^\delta(x_i, v_i) - K_J^\delta(x, v_i)| \ ,$$

is bounded above by

$$\begin{aligned}&c_{20}\delta^{-1}\varepsilon\ell(\varepsilon)\int K_J(z, v_i)\mathbb{1}(|x - z| \le c_{20}\delta + c_{20}\varepsilon\ell(\varepsilon))dz\\ &\quad \le c_{20}\delta^{-1}\varepsilon\ell(\varepsilon)\int K_J(z, v_i)\mathbb{1}(|x - z| \le 2c_{20}\delta)dz,\end{aligned}$$

for some constant c_{20}. Moreover, from $|v_i - v_j| \le (|v_i| + 1)(|v_j| + 1)$ we learn,

$$K_J(z, v_i) \le c_{21}(|v_i| + 1)\sum_j V^\varepsilon(|z - x_j|)(|v_j| + 1) =: c_{21}(|v_i| + 1)u^\varepsilon(z) \ .$$

Hence the term $|K_J^\delta(x_i, v_i) - K_J^\delta(x, v_i)|$ is bounded above by

$$c_{22}\delta^{-1}\varepsilon\ell(\varepsilon)(|v_i| + 1)\delta^{-1}\varepsilon\ell(\varepsilon)\int u^\varepsilon(z)\mathbb{1}(|x - z| \le 2c_{20}\delta)dz =: \delta^{-1}\varepsilon\ell(\varepsilon)(|v_i| + 1)G^\varepsilon(x).$$

As a result, the expression (8.24) is bounded above by

$$\begin{aligned}&c_{22}\delta^{-1}\varepsilon\ell(\varepsilon)E_N\int \varepsilon^d \sum_i \tilde{\zeta}_1^\varepsilon(x_i - x)(|v_i| + 1)G^\varepsilon(x)S(x_i)U^2(x)dx\\ &\quad \le c_{22}\delta^{-1}\varepsilon\ell(\varepsilon)\alpha_2^{-1}E_N\int G^\varepsilon(x)dx\\ &\quad \le c_{23}\delta^{-1}\varepsilon\ell(\varepsilon)\varepsilon^{2d}E_N\varepsilon^{2d}\sum_j(|v_j| + 1) \le c_{24}\delta^{-1}\varepsilon\ell(\varepsilon) \ ,\end{aligned}$$

where for the last inequality, we have used the conservation of the kinetic energy. From this and (8.23) we learn

$$E_N \int_0^T \int \Big| \sum_i \tilde{\zeta}_1^\varepsilon(x_i(s)-x)(K_J(x_i(s),v_i(s)) - K_J^\delta(x,v_i(s)))S(x_i(s))U^2(x)\Big| dxds$$
$$\leq c_{25}(1+\alpha_2^{-1})(\log\log\log\log|\log\varepsilon|)^{-\frac{1}{4d+2}}, \tag{8.25}$$

for $\delta = \varepsilon\ell(\varepsilon)^2$ and a constant c_{25}. However, this δ is not what we need. We would rather have $\delta_1 = \delta_1(\varepsilon) = \varepsilon\ell(\varepsilon)$ in place of δ. The reason we were forced to choose such a δ was because when we replaced x_i with x in K_J^δ, we had an error of order $O(\varepsilon\ell(\varepsilon)\delta^{-1})$. Otherwise a choice of $\delta = \delta_1$, would have led to the same estimate (8.23). Based on this observation, we can repeat the proof of (8.23) to assert that the expression

$$E_N \int_0^T \int \Big| \sum_i \tilde{\zeta}_1^\varepsilon(x_i(s)-x)(K_J^{\delta_1}(x,v_i(s)) - K_J^\delta(x,v_i(s)))S(x_i(s))U^2(x)\Big| dxds\ ,$$

is bounded above by

$$c_{26}(1+\alpha_2^{-1})(\log\log\log\log|\log\varepsilon|)^{-\frac{1}{4d+2}}\ . \tag{8.26}$$

(In showing this, some of the steps of the proof of (8.23) can be skipped.) We can certainly write

$$\varepsilon^d \sum_i \tilde{\zeta}_1^\varepsilon(x_i-x)K_J^{\delta_1}(x,v_i)S(x_i)U^2(x)$$
$$= \int \sum_i \tilde{\zeta}_1^\varepsilon(x_i-x)\tilde{\zeta}_2^\varepsilon(v_i-v)K_J^{\delta_1}(x,v_i)S(x_i)U^2(x)dv\ .$$

Moreover, using a bound similar to (8.12), it is not hard to show that the expression

$$\iint \varepsilon^d \sum_i \tilde{\zeta}_1^\varepsilon(x_i-x)\tilde{\zeta}_2^\varepsilon(v_i-v)|K_J^{\delta_1}(x,v_i) - K_J^{\delta_1}(x,v)|S(x_i)U^2(x)dxdv\ ,$$

is bounded above by

$$c_{27}\iint \sum_i \tilde{\zeta}_1^\varepsilon(x_i-x)\tilde{\zeta}_2^\varepsilon(v_i-v)U(x)\int \sum_j V^\varepsilon(|x_i-x_j+z|)|v-v_i|\zeta^{\delta_1}(z)dzdxdv$$
$$\leq c_{28}\varepsilon\ell(\varepsilon)\iint \varepsilon^d \sum_i \tilde{\zeta}_1^\varepsilon(x_i-x)\tilde{\zeta}_2^\varepsilon(v_i-v)U(x)\sum_j \int \mathbb{1}(|x_j-x+z|$$
$$\leq c_{28}\varepsilon\ell(\varepsilon))\zeta^{\delta_1}(z)dzdx$$
$$\leq c_{28}\iint \alpha_2^{-1}\ell(\varepsilon)\sum_j \mathbb{1}(|x_j-x+z| \leq c_{28}\varepsilon\ell(\varepsilon))\zeta^{\delta_1}(z)dzdx$$
$$\leq c_{29}\alpha_2^{-1}\varepsilon\ell(\varepsilon)^{d+1}\ .$$

From this and (8.25–26) we learn that there exists a constant c_{30} such that

$$E_N \int_0^T \iint \left| Q_+^\varepsilon(x;\mathbf{q}(s);J;\alpha_1,\alpha_2) - \hat{Q}_+^\varepsilon(x;\mathbf{q}(s);J;\alpha_1,\alpha_2)\right| dxds$$
$$\leq c_{30}(1+\alpha_2^{-1})(\log\log\log\log|\log\varepsilon|)^{-\frac{1}{4d+2}}, \tag{8.27}$$

where

$$\hat{Q}_+^\varepsilon(x;\mathbf{q};J;\alpha_1,\alpha_2) = \int \sum_i \tilde{\zeta}_1^\varepsilon(x_i - x)\tilde{\zeta}_2^\varepsilon(v_i - v)K_J^{\delta_1}(x,v)S(x_i)U^2(x)dv\ ,$$

and $\delta_1 = \varepsilon\ell(\varepsilon)$.

Step 6. Next we would like to replace $S(x_i)$ with $S^{\delta_1}(x)$, where $S^{\delta_1} = (1+\alpha_1\tilde{u}^{\delta_1})^{-1}$ with $\tilde{u}^{\delta_1} = \bar{u}^\varepsilon * \zeta^{\delta_1}$ and $\delta_1 = \varepsilon\ell(\varepsilon)$. Define

$$\begin{aligned}\bar{Q}_+^\varepsilon(x;\mathbf{q};J;\alpha_1,\alpha_2) &= \int \sum_i \tilde{\zeta}_1^\varepsilon(x_i - x)\tilde{\zeta}_2^\varepsilon(v_i - v)K_J^{\delta_1}(x,v)S^{\delta_1}(x)U^2(x)dv \\ &= \int \tilde{f}^\varepsilon(x,v;\mathbf{q})K_J^{\delta_1}(x,v)S^{\delta_1}(x)U^2(x)dv\ .\end{aligned}$$

We would like to show that for some constant c_{31},

$$E_N \int_0^T \iint \left| \bar{Q}_+^\varepsilon(x;\mathbf{q}(s);J;\alpha_1,\alpha_2) - \hat{Q}_+^\varepsilon(x;\mathbf{q}(s);J;\alpha_1,\alpha_2)\right| dxds$$
$$\leq c_{31}[\alpha_2^{-1} + \alpha_1\alpha_2^{-1}(1+\alpha_2^{-1})]\ell(\varepsilon)^{-1/4}\ . \tag{8.28}$$

The proof of (8.28) can be carried out in the same way we showed (8.27). Indeed, we first restrict v to a bounded set. This is done by defining

$$\hat{Q}_{+,k_0}^\varepsilon(x;\mathbf{q};J;\alpha_1,\alpha_2) = \int \sum_i \tilde{\zeta}_1^\varepsilon(x_i - x)\tilde{\zeta}_2^\varepsilon(v_i - v)K_J^{\delta_1}(x,v)\mathbb{1}(|v|\leq k_0)S(x_i)U^2(x)dv\ ,$$
$$\bar{Q}_{+,k_0}^\varepsilon(x;\mathbf{q};J;\alpha_1,\alpha_2) = \int \tilde{f}^\varepsilon(x,v;\mathbf{q})K_J^{\delta_1}(x,v)\mathbb{1}(|v|\leq k_0)S^{\delta_1}(x)U^2(x)dv\ ,$$

for a large positive k_0. It is not hard to see that the term $K_J^{\delta_1}(x,v)U(x)$ is bounded by a constant multiple of $\alpha_2^{-1}(|v|+1)$ provided that the support of ζ is sufficiently large. As a result,

$$\begin{aligned}&E_N \int_0^T \iint \left| \hat{Q}_+^\varepsilon(x;\mathbf{q}(s);J;\alpha_1,\alpha_2) - \hat{Q}_{+,k_0}^\varepsilon(x;\mathbf{q}(s);J;\alpha_1,\alpha_2)\right| dxds \\ &\leq c_{32}\alpha_2^{-1}E_N \int_0^T \iint \sum_i \tilde{\zeta}_1^\varepsilon(x_i - x)\tilde{\zeta}_2^\varepsilon(v_i - v)(|v|+1)\mathbb{1}(|v|\leq k_0)dxdvds \\ &\leq c_{33}\alpha_2^{-1}k_0^{-1}E_N \int_0^T \varepsilon^{2d}\sum_i(1+|v_i|^2)ds \\ &= c_{34}\alpha_2^{-1}k_0^{-1}\ ,\end{aligned}$$

where for the last equality we have used the conservation of the kinetic energy. This means that if we replace $\hat{Q}_+^\varepsilon$ with $\hat{Q}_{+,k_0}^\varepsilon$ in (8.28), we cause an error that is bounded above by $c_{34}\alpha_2^{-1}k_0^{-1}$. In the same fashion we can argue that if we replace $\bar{Q}_+^\varepsilon$ with $\bar{Q}_{+,k_0}^\varepsilon$

in (8.28), we cause an error that is bounded above by $c_{34}\alpha_2^{-1}k_0^{-1}$. After this, we first replace $\bar{u}^\varepsilon$ with $\tilde{u}^\delta$ for $\delta = \varepsilon\ell(\varepsilon)^2$. Note

$$S(x_i) - S^\delta(x_i) = \alpha_1(\tilde{u}^\delta(x_i) - \bar{u}^\varepsilon(x_i))(1 + \alpha_1\tilde{u}^\delta(x_i))^{-1}(1 + \alpha_1\bar{u}^\varepsilon(x_i))^{-1} .$$

Now $\tilde{u}^\delta(x_i) - \bar{u}^\varepsilon(x_i)$ plays the role of $K_J^\delta - K_J$ in the proof of (8.25). To follow the proof of (8.25) line by line, observe that the term $(1 + \alpha_1\tilde{u}^\delta)^{-1}(1 + \alpha_1\bar{u}^\varepsilon)^{-1}$ is bounded by 1 and that the term $K_J^{\delta_1}(x, v)U(x)$ is bounded by a constant multiple of $\alpha_2^{-1}k_0$ provided that the support of ζ is sufficiently large. Hence we spare one U to control $K_J^{\delta_1}$ and use the other U to repeat the proof of (8.25). We then repeat the proof of (8.26) to replace S^δ with S^{δ_1}. Finally we choose $k_0 = \ell(\varepsilon)^{1/4}$ to complete the proof of (8.28).

Final Step. To complete the proof of the theorem, we first observe that $Q_+(g; J) = \int gL(g, J)dv$, where

$$L(g, J)(v) = \int\int_{\mathbb{S}} B(v - v_*, n)g(v_*)J(v - (v - v_*)n \cdot n)dndv_* .$$

As in the final step of the proof of Theorem 6.1, we have

$$|K_J^\delta(x, v) - L(\tilde{f}^\varepsilon(x, .; \mathbf{q}), J)(v)| \le (\varepsilon\delta_1^{-1} + \delta_2)X(x, v) , \tag{8.29}$$

where

$$X(x, v) = \varepsilon^{2d}\sum_j \gamma^{\delta_1}(x - x_j)(|v - v_j| + 1) ,$$

where $\gamma^{\delta_1}(x) = \delta_1^{-d}\gamma(x/\delta_1)$ for a suitable function γ of compact support. It is not hard to see that XU is bounded above by a constant multiple of $|v| + 1$ if the support of ζ is sufficiently large. This and (8.27–29) complete the proof of the theorem. □

9. Supersolutions

In this section we establish one half of Theorem 2.1 by showing that any limit point of $\mathcal{P}_N$ is concentrated on the space of supersolutions of the Boltzmann equation (1.1).

An integrable function f is called a *supersolution* of (1.1) with initial data f^0 if for every $t \in [0, T]$,

$$f(x, v, t) \ge f^0(x - vt, v) + \int_0^t Q(f, f)(x - v(t - s), v, s)ds ,$$

for almost all (x, v).

It is not hard to show that f is a supersolution if and only if

$$\begin{aligned} f(x + vt, v, t) \ge\ & f(x, v, 0)\exp\left(-\int_0^t Lf(x + v\theta, v, \theta)d\theta\right) \\ &+ \int_0^t Q_+(f(x, \cdot, s))(v)\exp\left(-\int_s^t Lf(x + v\theta, v, \theta)d\theta\right)ds, \end{aligned} \tag{9.1}$$

for almost all (x, v). (See for example [DLi1], p. 345.)

Recall that

$$\tilde{f}^{\varepsilon}(x,v;\mathbf{q}) = \varepsilon^{2d}\sum_i \zeta^{\delta_1(\varepsilon)}(x_i - x)\zeta^{\delta_2(\varepsilon)}(v_i - v)\ ,$$

where $\zeta^{\delta}(z) = \delta^{-d}\zeta(z/\delta)$, $\delta_r(\varepsilon) = \varepsilon\ell_r(\varepsilon)$ for $r = 1, 2$, $\ell_2(\varepsilon) \le \ell_1(\varepsilon) = \ell(\varepsilon)$ and $\ell(\varepsilon)$ is as in (6.1). We now assume

$$\lim_{\varepsilon\to 0}\ell_2(\varepsilon)\ell(\varepsilon)^{-1} = 0\ , \quad \lim_{\varepsilon\to 0}\ell_2(\varepsilon) = \infty\ .$$

The transformation

$$\mathbf{q}(\cdot) \mapsto \tilde{f}^{\varepsilon}(x,v,t)dxdvdt = \tilde{f}^{\varepsilon}(x,v;\mathbf{q}(t))dxdvdt =: \pi(dx,dv,dt)\ ,$$

assigns a measure to each realization of $\mathbf{q}(\cdot)$. We regard this measure as a member of

$$\hat{\mathcal{M}} := \{\pi : \pi(\mathbb{T}^d \times \mathbb{R}^d \times [0,T]) = ZT\}\ . \tag{9.2}$$

The transformation $\mathbf{q}(\cdot) \mapsto \pi$ induces a probability measure $\mathcal{Q}_N$ on $\hat{\mathcal{M}}$. The main result of this section is Theorem 9.1.

Theorem 9.1. *The sequence $\{\mathcal{Q}_N\}$ is tight and if $\mathcal{Q}$ is a limit point, then $\mathcal{Q}$ is concentrated on the space of measures π such that $\pi(dx,dv,dt) = f(x,v,t)dxdvdt$ for a nonnegative integrable function f such that $\frac{Q^{\pm}(f,f)}{1+f} \in L^1(\mathbb{T}^d \times \mathbb{R}^d \times [0,T])$,*

$$\sup_{t\in[0,T]}\iint f(1+|x|^2+|v|^2+\log^+ f)dxdv < \infty, \tag{9.3}$$

and f is a supersolution of (1.1) with initial data f^0.

Proof. Step 1. As in the proof of Theorem 5.1, let us write $F(x,v;\mathbf{q}) = F_n(x,v;\mathbf{q}) = \frac{n\tilde{f}^{\varepsilon}(x,v;\mathbf{q})}{n+\tilde{f}^{\varepsilon}(x,v;\mathbf{q})} = \beta_n(\tilde{f}^{\varepsilon}(x,v;\mathbf{q}))$, where $\beta_n(r) = \frac{nr}{n+r}$. Recall that the process

$$M(x,v,t) = F(x+vt,v;\mathbf{q}(t)) - F(x,v;\mathbf{q}(0)) - \int_0^t \left(\frac{\partial}{\partial s} + \mathcal{A}\right)F(x+vs,v;\mathbf{q}(s))ds$$

is a martingale and that its quadratic variation is given by

$$\begin{aligned} E_N M(x,v,t)^2 = E_N\Gamma(t) &:= E_N\int_0^t (\mathcal{A}F^2 - 2F\mathcal{A}F)(x+vs,v;\mathbf{q}(s))ds \\ &= E_N\int_0^t (\mathcal{A}_c F^2 - 2F\mathcal{A}_c F)(x+vs,v;\mathbf{q}(s))ds. \end{aligned} \tag{9.4}$$

As a result, we may write

$$\begin{aligned} F(x,v;\mathbf{q}(t)) = F(x-vt,v;\mathbf{q}(0)) &+ \int_0^t A(x-v(t-s),v,s)ds \\ &+ \int_0^t D(x-v(t-s),v,s) + M(x-vt,v,t), \end{aligned} \tag{9.5}$$

where $A = \left(\frac{\partial}{\partial x} \cdot v + \mathcal{A}_0\right) F$ and $D(x, v, t) = \mathcal{A}_c F(x, v; \mathbf{q}(t))$. A straightforward calculation yields

$$
\begin{aligned}
&(\mathcal{A}_c F^2 - 2F\mathcal{A}_c F)(x, v) \\
&\quad = \frac{1}{2} \sum_{i,j} V^\varepsilon(|x_i - x_j|) B(v_i - v_j, n_{ij}) (\tilde{f}^\varepsilon(x, v; S^{ij}\mathbf{q}) - \tilde{f}(x, v; \mathbf{q}))^2 \\
&\qquad \times \left(1 + n^{-1} \tilde{f}^\varepsilon(x, v; \mathbf{q})\right)^{-2} \left(1 + n^{-1} \tilde{f}^\varepsilon(x, v; S^{ij}\mathbf{q})\right)^{-2}.
\end{aligned}
$$

Evidently $(\tilde{f}^\varepsilon(x, v; S^{ij}\mathbf{q}) - \tilde{f}(x, v; \mathbf{q}))^2$ is equal to

$$
\begin{aligned}
&\ell_1(\varepsilon)^{-2d} \ell_2(\varepsilon)^{-2d} \left[\zeta\left(\frac{x_i - x}{\delta_1(\varepsilon)}\right) \zeta\left(\frac{v_i^j - v}{\delta_2(\varepsilon)}\right) + \zeta\left(\frac{x_j - x}{\delta_1(\varepsilon)}\right) \zeta\left(\frac{v_j^i - v}{\delta_2(\varepsilon)}\right) \right. \\
&\quad \left. - \zeta\left(\frac{x_i - x}{\delta_1(\varepsilon)}\right) \zeta\left(\frac{v_i - v}{\delta_2(\varepsilon)}\right) - \zeta\left(\frac{x_j - x}{\delta_1(\varepsilon)}\right) \zeta\left(\frac{v_j - v}{\delta_2(\varepsilon)}\right) \right]^2 .
\end{aligned}
$$

Define

$$
m_{ij}^1 = 8\zeta^2\left(\frac{x_i - x}{\delta_1(\varepsilon)}\right) \zeta^2\left(\frac{v_i^j - v}{\delta_2(\varepsilon)}\right), \quad m_{ij}^2 = 8\zeta^2\left(\frac{x_i - x}{\delta_1(\varepsilon)}\right) \zeta^2\left(\frac{v_i - v}{\delta_2(\varepsilon)}\right) .
$$

Using this we can write,

$$
E_N M^2(x, v, t) \le E_N \int_0^t M_1(x + vs, v, \mathbf{q}(s)) ds + E_N \int_0^t M_2(x + vs, v, \mathbf{q}(s)) ds , \tag{9.6}
$$

where

$$
\begin{aligned}
M_r(x, v, \mathbf{q}) = \ell_1(\varepsilon)^{-2d} \ell_2(\varepsilon)^{-2d} &\sum_{i,j} V^\varepsilon(|x_i - x_j|) B(v_i - v_j, n_{ij}) m_{ij}^r(x, v; \mathbf{q}) \\
&\times \left(1 + n^{-1} \tilde{f}^\varepsilon(x, v; \mathbf{q})\right)^{-2} \left(1 + n^{-1} \tilde{f}^\varepsilon(x, v; S^{ij}\mathbf{q})\right)^{-2}
\end{aligned}
$$

for $r = 1, 2$. For some constants c_0, c_1 and c_2 we have that the term $M_2(x, v, \mathbf{q})$ is bounded above by

$$
\begin{aligned}
&c_0\ell_1(\varepsilon)^{-2d}\ell_2(\varepsilon)^{-2d}\sum_{i,j}V^\varepsilon(|x_i-x_j|)|v_i-v_j| \\
&\quad\times\zeta^2\left(\frac{x_i-x}{\delta_1(\varepsilon)}\right)\zeta^2\left(\frac{v_i-v}{\delta_2(\varepsilon)}\right)\left(1+n^{-1}\tilde f^\varepsilon(x,v;\mathbf{q})\right)^{-1} \\
&\le c_1\ell_1(\varepsilon)^{-2d}\ell_2(\varepsilon)^{-2d}\varepsilon^d\sum_{i,j}\mathbb{1}(|x_j-x|\le c_1\delta_1(\varepsilon))(|v_j-v|+c_1\delta_2(\varepsilon)) \\
&\quad\times\zeta^2\left(\frac{x_i-x}{\delta_1(\varepsilon)}\right)\zeta^2\left(\frac{v_i-v}{\delta_2(\varepsilon)}\right)\left(1+n^{-1}\tilde f^\varepsilon(x,v;\mathbf{q})\right)^{-1} \\
&\le c_2\ell_1(\varepsilon)^{-2d}\ell_2(\varepsilon)^{-2d}\varepsilon^d\sum_{i,j}\mathbb{1}(|x_j-x|\le c_1\delta_1(\varepsilon))(|v_j-v|+c_1\delta_2(\varepsilon)) \\
&\quad\times\zeta\left(\frac{x_i-x}{\delta_1(\varepsilon)}\right)\zeta\left(\frac{v_i-v}{\delta_2(\varepsilon)}\right)\left(1+n^{-1}\tilde f^\varepsilon(x,v;\mathbf{q})\right)^{-1} \\
&\le c_2\ell_1(\varepsilon)^{-d}\ell_2(\varepsilon)^{-d}n\varepsilon^d\sum_{j}\mathbb{1}(|x_j-x|\le c_1\delta_1(\varepsilon))(|v_j-v|+c_1\delta_2(\varepsilon))\ .
\end{aligned}
$$

From this and the conservation of the kinetic energy we learn

$$
\begin{aligned}
&E_N\int_0^t\iint M_2(x+vs,v,\mathbf{q}(s))\mathbb{1}(|v|\le\ell_0)dxdvds \\
&\quad\le c_3n\ell_2(\varepsilon)^{-d}\varepsilon^{2d}\ell_0^dE_N\int_0^t\sum_j(|v_j|^2+\ell_0)\mathbb{1}(|v|\le\ell_0)ds\le c_4n\ell_0^{d+1}\ell_2(\varepsilon)^{-d},
\end{aligned}
\tag{9.7}
$$

for some constants c_3 and c_4. On the other hand the term $E_NM_1(x,v,\mathbf{q}(s))$ is bounded above by

$$
\begin{aligned}
&8\ell_1(\varepsilon)^{-2d}\ell_2(\varepsilon)^{-2d}\int\sum_{i,j}V^\varepsilon(|x_i-x_j|)B(v_i-v_j,n_{ij}) \\
&\quad\times\zeta^2\left(\frac{x_i-x}{\delta_1(\varepsilon)}\right)\zeta^2\left(\frac{v_i^j-v}{\delta_2(\varepsilon)}\right)\left(1+n^{-1}\tilde f^\varepsilon(x,v;S^{ij}\mathbf{q})\right)^{-1}G(s,\mathbf{q})\nu_\beta(d\mathbf{q}) \\
&=8\ell_1(\varepsilon)^{-2d}\ell_2(\varepsilon)^{-2d}\int\sum_{i,j}V^\varepsilon(|x_i-x_j|)B(v_i-v_j,n_{ij}) \\
&\quad\times\zeta^2\left(\frac{x_i-x}{\delta_1(\varepsilon)}\right)\zeta^2\left(\frac{v_i-v}{\delta_2(\varepsilon)}\right)\left(1+n^{-1}\tilde f^\varepsilon(x,v;\mathbf{q})\right)^{-1}G(s,S^{ij}\mathbf{q})\nu_\beta(d\mathbf{q})\ .
\end{aligned}
$$

As in Step 8 of the proof of Lemma 5.3, we can use (5.10), (9.7) and Lemma 4.7 to show that for every $k>1$,

$$
\begin{aligned}
&E_N\int_0^t\iint M_1(x+vs,v;\mathbf{q}(s))\mathbb{1}(|v|\le\ell_0)dxdvds \\
&\quad\le c_5kn\ell_0^{d+1}\ell_2(\varepsilon)^{-d}+c_5\ell_1(\varepsilon)^{-d}\ell_2(\varepsilon)^{-d}(\log k)^{-1}.
\end{aligned}
$$

In just the same way we derived (9.7) and (5.19). By choosing $k=e$ we learn

$$
E_N\int_0^t\iint M_1(x+vs,v;\mathbf{q}(s))\mathbb{1}(|v|\le\ell_0)dxdvds\le c_6n\ell_0^{d+1}\ell_2(\varepsilon)^{-d}\ ,
$$

for some constant c_6. This and (9.6–7) imply

$$E_N M(x,v,t)^2 \le c_7 n \ell_0^{d+1} \ell_2(\varepsilon)^{-d} , \tag{9.8}$$

for some constant c_7. Now we use Doob's inequality to deduce

$$E_N \iint \sup_{0\le t\le T} M(x,v,t)^2 1\!\!1(|v|\le \ell_0)\,dx\,dv \le 4c_7 \ell_0^{d+1} \ell_2(\varepsilon)^{-d} . \tag{9.9}$$

Step 2. We certainly have $D(x,v,s) = D^+(x,v,s) - D^-(x,v,s)$ where $D^\pm(x,v,s) = D^\pm(x,v;\mathbf{q}(s))$ and

$$\begin{aligned} D^+(x,v,\mathbf{q}) = \varepsilon^{2d} \sum_{i,j} & V^\varepsilon(|x_i - x_j|) B(v_i - v_j, n_{ij}) \zeta^{\delta_1(\varepsilon)}(x_i - x) \zeta^{\delta_2(\varepsilon)}(v_i^j - v) \\ & \times \left(1 + n^{-1} \tilde f^\varepsilon(x,v;\mathbf{q})\right)^{-1} \left(1 + n^{-1}\tilde f^\varepsilon(x,v;S^{ij}\mathbf{q})\right)^{-1}, \\ D^-(x,v,\mathbf{q}) = \varepsilon^{2d} \sum_{i,j} & V^\varepsilon(|x_i - x_j|) B(v_i - v_j, n_{ij}) \zeta^{\delta_1(\varepsilon)}(x_i - x) \zeta^{\delta_2(\varepsilon)}(v_i - v) \\ & \times \left(1 + n^{-1} \tilde f^\varepsilon(x,v;\mathbf{q})\right)^{-1} \left(1 + n^{-1}\tilde f^\varepsilon(x,v;S^{ij}\mathbf{q})\right)^{-1}. \end{aligned}$$

It is not hard to see that there exists a constant c_8 such that,

$$|\tilde f^\varepsilon(x,v;\mathbf{q}) - \tilde f^\varepsilon(x,v;S^{ij}\mathbf{q})| \le c_8 \ell_1(\varepsilon)^{-d} \ell_2(\varepsilon)^{-d} . \tag{9.10}$$

On account of this, let us define

$$\hat D^\pm(x,v;\mathbf{q}) = Q^\varepsilon_\pm(x,v;\mathbf{q}) \left(1 + n^{-1}\tilde f^\varepsilon(x,v;\mathbf{q})\right)^{-2}, \tag{9.11}$$

where Q^ε_+ was defined by (7.1) and

$$Q^\varepsilon_-(x,v;\mathbf{q}) = \sum_{i,j} V^\varepsilon(|x_i - x_j|) B(v_i - v_j, n_{ij})\, \tilde\zeta_1^\varepsilon(x_i - x) \tilde\zeta_2^\varepsilon(v_i - v).$$

From (9.10) and $\ell_1(\varepsilon) \ge \ell_2(\varepsilon)$ we deduce

$$|D^\pm(x,v;\mathbf{q}) - \hat D^\pm(x,v;\mathbf{q})| \le c_8 \ell_2(\varepsilon)^{-2d} Q^\varepsilon_\pm(x,v;\mathbf{q}) \left(1 + n^{-1}\tilde f^\varepsilon(x,v;\mathbf{q})\right)^{-1} . \tag{9.12}$$

We now claim

$$\begin{aligned} E_N \int_0^T \iint |D^\pm(x,v;\mathbf{q}(s)) - \hat D^\pm(x,v;\mathbf{q}(s))| 1\!\!1(|v|\le \ell_0)\,dx\,dv\,ds \\ \le c_9 n \ell_0^{d+2} \ell_2(\varepsilon)^{-2d} . \end{aligned} \tag{9.13}$$

for a constant c_9. For this, it suffices to show that there exists a constant c_{10} such that

$$\begin{aligned} E_N \int_0^T \iint Q^\varepsilon_\pm(x,v;\mathbf{q}(s)) \left(1 + n^{-1}\tilde f^\varepsilon(x,v;\mathbf{q}(s))\right)^{-1} 1\!\!1(|v|\le \ell_0)\,dx\,dv\,ds \\ \le c_{10} \ell_0^{d+2} n . \end{aligned} \tag{9.14}$$

In the case of Q_-^ε, the bound (9.14) is a consequence of Corollary 6.4. We delay the proof of the inequality (9.14) in the case Q_+^ε because we need something stronger, namely a uniform integrability of the renormalized loss and gain terms. More precisely, we prove that there exists a constant c_{11} such that

$$E_N \int_0^T \iint X_\pm(x, v; \mathbf{q}(s)) \mathbb{1}(X_\pm(x, v; \mathbf{q}(s)) \geq \ell) dx dv ds \\ \leq c_{11} n \left[\ell_0^{d+2} (\log \ell)^{1/4} \ell_2(\varepsilon)^{-1/2} + (\log \log \ell)^{-1} \right] \tag{9.15}$$

for $\ell > e$ and small ε, where $X_\pm$ is a short-hand for $Q_\pm^\varepsilon \left(1 + n^{-1} \tilde{f}^\varepsilon\right)^{-1}$.

We establish (9.15) with the aid of Theorem 6.1 and (4.8). To this end let us write Y_- for $\tilde{f}^\varepsilon \left(1 + n^{-1} \tilde{f}^\varepsilon\right)^{-1} L \tilde{f}^\varepsilon$. Fix $k > 1$. We certainly have

$$\begin{aligned} X_- \mathbb{1}(X_- \geq \ell) &\leq Y_- \mathbb{1}(X_- \geq \ell) + |X_- - Y_-| \\ &\leq Y_- \mathbb{1}(X_- - Y_- \geq \ell/2) + Y_- \mathbb{1}(Y_- \geq \ell/2) + |X_- - Y_-| \\ &\leq k \mathbb{1}(X_- - Y_- \geq \ell/2) + \frac{1}{(\log k)^{1/2}} \tilde{\phi}(Y_-) \\ &\quad + \frac{1}{(\log(\ell/2))^{1/2}} \tilde{\phi}(Y_-) + |X_- - Y_-| \\ &\leq \left(\frac{2k}{\ell} + 1\right) |X_- - Y_-| + [(\log k)^{-1/2} + (\log(\ell/2))^{-1/2}] \tilde{\phi}(Y_-) \ , \end{aligned}$$

where $\tilde{\phi}(z) = z(\log^+ z)^{1/2}$. From this, the inequality $Y_- \leq nL\tilde{f}^\varepsilon$, (4.8) and Theorem 6.1 we deduce,

$$E_N \int_0^T \iint X_-(x, v; \mathbf{q}(s)) \mathbb{1}(X_-(x, v; \mathbf{q}(s)) \geq \ell) \mathbb{1}(|v| \leq \ell_0) \\ \leq c_{12} \left(\frac{2k}{\ell} + 1\right) n \ell_0^{d+2} \ell(\varepsilon)^{-1/2} + c_{12} \tilde{\phi}(\ell_0) \left[(\log k)^{-1/2} + (\log \ell)^{-1/2} \right] .$$

Choosing $k = \ell$ yields

$$E_N \int_0^T \iint X_-(x, v; \mathbf{q}(s)) \mathbb{1}(X_-(x, v; \mathbf{q}(s)) \geq \ell) \mathbb{1}(|v| \leq \ell_0) dx dv ds \\ \leq 3 c_{12} n \ell_0^{d+2} \ell(\varepsilon)^{-1/2} + 2 c_{12} \tilde{\phi}(\ell_0) (\log \ell)^{-1/2} .$$

This implies (9.15) in the case of X_-.

We can use (5.10) and Lemma 4.7 to establish a similar bound for X_+. First observe that (9.10) implies

$$\left(1 + n^{-1} \tilde{f}^\varepsilon(x, v; S^{ij} \mathbf{q})\right)^{-1} \leq c_{13} \left(1 + n^{-1} \tilde{f}^\varepsilon(x, v; \mathbf{q})\right)^{-1} , \tag{9.16}$$

for small ε. From (5.10) we deduce that we can find a constant c_{14} such that for every $k > 1$,

$$\begin{aligned}
&\int X_+(x,v;\mathbf{q})1\!\!1(X_+(x,v;\mathbf{q}) \geq \ell)G(s,\mathbf{q})\nu_\beta(d\mathbf{q}) \\
&= \int \varepsilon^{2d} \sum_{i,j} V^\varepsilon(|x_i - x_j|)B(v_i - v_j, n_{ij})\zeta^{\delta_1(\varepsilon)}(x_i - x)\zeta^{\delta_2(\varepsilon)}(v_i - v) \\
&\quad \times \left(1 + n^{-1}\tilde{f}^\varepsilon(x,v;S^{ij}\mathbf{q})\right)^{-1} 1\!\!1(X_+(x,v;S^{ij}\mathbf{q}) \geq \ell)G(s,S^{ij}\mathbf{q})\nu_\beta(d\mathbf{q}) \\
&\leq k \int \varepsilon^{2d} \sum_{i,j} V^\varepsilon(|x_i - x_j|)B(v_i - v_j, n_{ij})\zeta^{\delta_1(\varepsilon)}(x_i - x)\zeta^{\delta_2(\varepsilon)}(v_i - v) \\
&\quad \times \left(1 + n^{-1}\tilde{f}^\varepsilon(x,v;S^{ij}\mathbf{q})\right)^{-1} 1\!\!1(X_+(x,v;S^{ij}\mathbf{q}) \\
&\geq \ell)G(s,\mathbf{q})\nu_\beta(d\mathbf{q}) + (\log k)^{-1}\Omega(x,v,s)\ ,
\end{aligned}$$

where the function $\Omega(x, v, s)$ satisfies

$$\int_0^T \iint \Omega(x,v,s)dxdvds \leq c_{14}\ ,$$

by Lemma 4.7. In the case of $\ell = e$, we simply use (9.16) to deduce

$$\begin{aligned}
&E_N \int_0^T \iint X_+(x,v;\mathbf{q}(s))1\!\!1(X_+(x,v;\mathbf{q}(s)) \geq e\ ,\ |v| \leq \ell_0)dxdvds \\
&\leq c_{15}kE_N \int_0^T \iint X_-(x,v;\mathbf{q}(s))1\!\!1(|v| \leq \ell_0)dxdvds + c_{14}(\log k)^{-1}\ . \quad (9.17)
\end{aligned}$$

We now choose $k = e$ in (9.17) to deduce (9.14) in the case of Q_+^ε from (9.14) in the case of Q_-^ε.

Going back to (9.15), we apply (9.16) to assert that the expression

$$\int_0^T \iiint X_+(x,v;\mathbf{q})1\!\!1(X_+(x,v;\mathbf{q}) \geq \ell)1\!\!1(|v| \leq \ell_0)G(s,\mathbf{q})\nu_\beta(d\mathbf{q})dxdvds\ , \quad (9.18)$$

is bounded above by

$$c_{16}\int_0^T \iint \Omega_1 1\!\!1(|v| \leq \ell_0)dxdvds + c_{16}\int_0^T \iint \Omega_2 1\!\!1(|v| \leq \ell_0)dxdvdt + c_{14}(\log k)^{-1}\ ,$$

for some constant c_{16}, where

$$\begin{aligned}
\Omega_1 = k &\int \varepsilon^{2d} \sum_{i,j} V^\varepsilon(|x_i - x_j|)B(v_i - v_j, n_{ij})1\!\!1(|v_i - v_j| \leq p)\zeta^{\delta_1(\varepsilon)}(x_i - x)\zeta^{\delta_2(\varepsilon)}(v_i - v) \\
&\times \left(1 + n^{-1}\tilde{f}^\varepsilon(x,v;\mathbf{q})\right)^{-1} 1\!\!1(X_+(x,v;S^{ij}\mathbf{q}) \geq \ell)G(s,\mathbf{q})\nu_\beta(d\mathbf{q})\ ,
\end{aligned}$$

and Ω_2 is obtained from Ω_1 by replacing $1\!\!1(|v_i - v_j| \leq p)$ with $1\!\!1(|v_i - v_j| > p)$. Here p is a fixed positive number that will be chosen later. We now use Chebyshev's inequality to assert that the term

$$\int_0^T \iint \Omega_2 1\!\!1(|v| \leq \ell_0) dx dv ds \ ,$$

is bounded above by

$$kp^{-1/2} \int_0^T \iiint \int \varepsilon^{2d} \sum_{i,j} V^\varepsilon(|x_i - x_j|) B(v_i - v_j, n_{ij}) |v_i - v_j|^{1/2} \zeta^{\delta_1(\varepsilon)}(x_i - x) \zeta^{\delta_2(\varepsilon)}(v_i - v)$$
$$\times \left(1 + n^{-1} \tilde{f}^\varepsilon(x, v; \mathbf{q})\right)^{-1} 1\!\!1(X_+(x, v; S^{ij}\mathbf{q}) \geq \ell) G(s, \mathbf{q}) \nu_\beta(d\mathbf{q}) dx dv ds \ .$$

This and Corollary 6.5 imply

$$\int_0^T \iint \Omega_2 1\!\!1(|v| \leq \ell_0) dx dv dt \leq c_{17} k p^{-1/2} n \ell_0^{d+2} \tag{9.19}$$

for a constant c_{15}.

We now turn to Ω_1. Observe that by (9.16), $X^+ \leq \hat{X}^+$, where,

$$\hat{X}_+(x, v; \mathbf{q}) \leq \sqrt{c_{13}}\ Q_+^\varepsilon(x, v; \mathbf{q}) \left(1 + n^{-1} \tilde{f}^\varepsilon(x, v; \mathbf{q})\right)^{-1/2} \left(1 + n^{-1} \tilde{f}^\varepsilon(x, v; S^{ij}\mathbf{q})\right)^{-1/2} .$$

Evidently we can find a constant c_{18} such that if

$$V^\varepsilon(|x_i - x_j|) \zeta^{\delta_1(\varepsilon)}(x_i - x) \zeta^{\delta_2(\varepsilon)}(v_i - v) 1\!\!1(|v_i - v_j| \leq p) \neq 0,$$

then $|x_i - x|, |x_j - x| \leq c_{18}\delta_1(\varepsilon)$, $|v_i - v| \leq c_{18}\delta_2(\varepsilon)$ and $|v_i - v_j| \leq p$. From this we learn that the expression

$$|\hat{X}_+(x, v; S^{ij}\mathbf{q}) - \hat{X}_+(x, v; \mathbf{q})| \ ,$$

is bounded above by

$$c_{19}\varepsilon^d \ell_1(\varepsilon)^{-d} \ell_2(\varepsilon)^{-d} \sum_k 1\!\!1(|x_k - x| \leq c_{19}\delta_1(\varepsilon))(|v_k - v| + p)) =: \ell_2(\varepsilon)^{-d} R(x, v; \mathbf{q}) \ ,$$

for a constant c_{17}. The proof of this is very similar to what was presented in the beginning of Step 6 of the proof of Lemma 5.3 and is omitted. Hence, for every $p_1 > 1$, the expression

$$\int_0^T \iint \Omega_1 1\!\!1(|v| \leq \ell_0)\ dx dv ds \ ,$$

is bounded above by

$$k \int_0^T \iiint \int X_-(x, v; \mathbf{q}) 1\!\!1(X_+(x, v; \mathbf{q}) + \ell_2(\varepsilon)^{-d} R(x, v; \mathbf{q})$$
$$\geq \ell\ ,\ |v| \leq \ell_0) G(s, \mathbf{q}) \nu_\beta(d\mathbf{q}) dx dv ds$$

$$
\begin{aligned}
&\le kp_1 E_N \int_0^T \iint \mathbb{1}(X_+(x,v;\mathbf{q}(s)) + \ell_2(\varepsilon)^{-d} R(x,v;\mathbf{q}(s)) \ge \ell)dxdvds \\
&\quad + kE_N \int_0^T \iint \mathbb{1}(X_-(x,v;\mathbf{q}(s)) \ge p_1) X_-(x,v;\mathbf{q}(s))\mathbb{1}(|v| \le \ell_0)dxdvds \\
&\le \frac{kp_1}{\ell} E_N \int_0^T \iint \Big(X_+(x,v;\mathbf{q}(s)) + \ell_2(\varepsilon)^{-d} R(x,v;\mathbf{q}(s))\Big) \mathbb{1}(|v| \le \ell_0)dxdvds \\
&\quad + c_{20} k \left(n\ell_0^{d+2}\ell(\varepsilon)^{-1/2} + \tilde{\phi}(\ell_0)(\log p_1)^{-1/2}\right) \\
&\le c_{21}\frac{kp_1}{\ell}(\ell_0^{d+2} n + p\ell_0^{d+1}\ell_2(\varepsilon)^{-d}) + c_{20}k\left(n\ell_0^{d+2}\ell(\varepsilon)^{-1/2} + \tilde{\phi}(\ell_0)(\log p_1)^{-1/2}\right),
\end{aligned}
$$

where for the second inequality we used Chebyshev's inequality and (9.15) and for the third inequality we used (9.14). From this and (9.19) we learn that the expression (9.18) is bounded above by

$$
\begin{aligned}
&c_{22}\frac{kp_1}{\ell}(\ell_0^{d+2} n + p\ell_0^{d+1}\ell_2(\varepsilon)^{-d}) + c_{22}k\left(n\ell_0^{d+2}\ell(\varepsilon)^{-1/2} + \tilde{\phi}(\ell_0)(\log p_1)^{-1/2}\right) \\
&\quad + c_{22}kp^{-1/2} n\ell_0^{d+2} + c_{14}(\log k)^{-1} \ .
\end{aligned}
$$

We choose $p = \log \ell$, $p_1 = \ell^{1/2}$ and $k = (\log p_1)^{1/4}$ to deduce (9.14) in the case of X_+.

We now discuss a consequence of (9.14) that is easier to use. Define

$$
\bar{\ell}(\varepsilon) = \exp(\ell(\varepsilon)^{8/5}) \ .
$$

Note that if $\ell \le \bar{\ell}(\varepsilon)$, then

$$
(\log \ell)^{1/4}\ell_2(\varepsilon)^{-2} + (\log\log \ell)^{-1} \le 2(\log\log \ell)^{-1} \ ,
$$

for sufficiently small ε. From this, (9.14) and the identity

$$
\left(\log^+ \log^+ \min(X, \bar{\ell}(\varepsilon))\right)^{1/2} = \int_e^{\bar{\ell}(\varepsilon)} \mathbb{1}(X \ge \ell)\frac{d\ell}{2\ell \log \ell (\log\log \ell)^{1/2}} \ ,
$$

one can readily deduce that for some constant c_{20},

$$
\begin{aligned}
&E_N \int_0^T \iint \hat{\phi}\left(\min\left(|\hat{D}^{\pm}(x,v,t)|, \bar{\ell}(\varepsilon)\right)\right) \mathbb{1}(|v| \le \ell_0)dxdvdt \\
&\le c_{23} n\ell_0^{d+2} \ , \\
&E_N \int_0^T \iint |\hat{D}^{\pm}(x,v,t)|\mathbb{1}(|\hat{D}^{\pm}(x,v,t)| \ge \bar{\ell}(\varepsilon))\mathbb{1}(|v| \le \ell_0)dxdvdt \\
&\le c_{23} n\ell_0^{d+2}(\log\log \bar{\ell}(\varepsilon))^{-1} \ ,
\end{aligned} \tag{9.20}
$$

where $\hat{\phi}(z) = z(\log^+ \log^+ z)^{1/2}$.

Step 3. Consider the process

$$
F(x+vt, v; \mathbf{q}(t)) \exp\left(\int_0^t L\tilde{f}^{\varepsilon}(x+v\theta, \cdot; \mathbf{q}(\theta))(v)d\theta\right) \ .
$$

This is a product of a semimartingale and a monotone process. More precisely, fix (x, v) and consider the process $X(t) = F(x + vt, v; \mathbf{q}(t))$. We have $dX = (A + D)dt + dM$, where A, D and M are as in Step 1. To ease the notation, let us simply write $L\tilde{f}^{\varepsilon}(x + v\theta, v, \theta)$ for $L\tilde{f}^{\varepsilon}(x + v\theta, \cdot, \mathbf{q}(\theta))(v)$ and $Y(t) = \int_0^t W(\theta)d\theta$ for the increasing process $\int_0^t L\tilde{f}^{\varepsilon}(x + v\theta, v, \theta)d\theta$. By a standard stochastic calculation,

$$d(Xe^Y) = (A + D + XW)e^Y dt + e^Y dM \ .$$

As a result, the function $F(x + vt, v; \mathbf{q}(t))$ equals to

$$\begin{aligned} &F(x, v; \mathbf{q}(0)) \exp\left(-\int_0^t L\tilde{f}^{\varepsilon}(x + v\theta, v, \theta)d\theta\right) + \hat{M}(x, v, t) \\ &+ \int_0^t \left[\left(\frac{\partial}{\partial s} + \mathcal{A}\right) F(x + vs, v; \mathbf{q}(s)) + F(x + vs, v; \mathbf{q}(s))L\tilde{f}^{\varepsilon}(x + vs, v; \mathbf{q}(s))\right] \\ &\quad \cdot \exp\left(-\int_s^t L\tilde{f}^{\varepsilon}(x + v\theta, v, \theta)d\theta\right) ds, \end{aligned}$$

where $\hat{M}(x, v, \cdot)$ is a martingale with quadratic variation

$$E_N \hat{M}(x, v, t)^2 = E_N \int_0^t \exp\left(-2\int_s^t L\tilde{f}^{\varepsilon}(x + v\theta, v, \theta)d\theta\right) \Gamma(ds) \ , \tag{9.21}$$

where the function Γ was defined in (9.4). From this we learn

$$\begin{aligned} \beta_n(\tilde{f}^{\varepsilon}(x + vt, v, t)) = \beta_n(\tilde{f}^{\varepsilon}(x, v, 0)) &\exp\left(-\int_0^t L\tilde{f}^{\varepsilon}(x + v\theta, v, \theta)d\theta\right) \\ &+ \int_0^t A(x + vs, v, s) \exp\left(-\int_s^t L\tilde{f}^{\varepsilon}(x + v\theta, v, \theta)d\theta\right) ds \\ &+ \int_0^t D^+(x + vs, v, s) \exp\left(-\int_s^t L\tilde{f}^{\varepsilon}(x + v\theta, v, \theta)d\theta\right) ds \\ &+ R_1(x, v, t) + \hat{M}(x, v, t) \ , \end{aligned} \tag{9.22}$$

where

$$\begin{aligned} R_1(x, v, t) = \int_0^t &\Big(F(x + vs, v; \mathbf{q}(s))L\tilde{f}^{\varepsilon}(x + vs, v, s) - D^-(x + vs, v, s)\Big) \\ &\cdot \exp\left(-\int_s^t L\tilde{f}^{\varepsilon}(x + v\theta, v, \theta)d\theta\right) ds \ . \end{aligned}$$

From (9.21) we learn that $E_N \hat{M}(x, v, t)^2 \leq E_N \Gamma(t)$. This, (9.8) and Doob's inequality imply

$$E_N \sup_{t \in [0,T]} \hat{M}(x, v, t)^2 \leq 4E_N \Gamma(T) \leq 4c_7 n \ell_0^{d+1} \ell_2(\varepsilon)^{-d} \ . \tag{9.23}$$

Also, observe $FL\tilde{f}^{\varepsilon} \geq \tilde{f}^{\varepsilon} L\tilde{f}^{\varepsilon}(1+n^{-1}\tilde{f}^{\varepsilon})^{-2}$. We now use this, (9.13) and Theorem 6.1 to assert

$$E_N \sup_{t \in [0,T]} \iint [R_1(x, v, t)]^- \mathbb{1}(|v| \leq \ell_0) dx dv \leq c_{24} n \ell_0^{d+2} \ell_2(\varepsilon)^{-1/2} \ , \tag{9.24}$$

for a constant c_{24}. Here and below, we write a^- for $\max(-a, 0)$. On the other hand, we have

$$A(x,v,t) = \beta'(\tilde{f}^\varepsilon(x,v,t))\ell_1(\varepsilon)^{-d}\ell_2(\varepsilon)^{-d}\sum_i \frac{v_i - v}{\delta_1(\varepsilon)}\cdot\nabla\zeta\left(\frac{x_i - x}{\delta_1(\varepsilon)}\right)\zeta\left(\frac{v_i - v}{\delta_2(\varepsilon)}\right) ,$$

which implies

$$|A(x,v,t)| \le c_{25}\ell_1(\varepsilon)^{-d}\ell_2(\varepsilon)^{-d}\sum_i |\nabla\zeta|\left(\frac{x_i - x}{\delta_1(\varepsilon)}\right)\zeta\left(\frac{v_i - v}{\delta_2(\varepsilon)}\right)\frac{\delta_2(\varepsilon)}{\delta_1(\varepsilon)} .$$

Hence,

$$\iint |A(x,v,t)|dxdv \le c_{26}\frac{\ell_2(\varepsilon)}{\ell_1(\varepsilon)} . \tag{9.25}$$

Fix $k > 0$ and put $\tilde{f}^\varepsilon_k = \min(\tilde{f}^\varepsilon, k)$. We now would like to replace $D^+(x,v,t)$ with

$$Q_+(\tilde{f}^\varepsilon_k(x,\cdot,t))(v)(1+\ell^{-1}\hat{u}^\varepsilon(x,t))^{-2}(1+\ell^{-1}\bar{u}^\varepsilon(x,t))^{-2} ,$$

where $\hat{u}^\varepsilon(x,t) = \int(1+|v|^{3/2})\tilde{f}^\varepsilon(x,v,t)dv$ and $\bar{u}^\varepsilon(x,t)$ are as in (7.1), and ℓ is a fixed positive number that will be sent to infinity in the end. Recall that by (9.12), the replacement of D^+ with $\hat{D}^+$ causes a small error. In view of (9.20), let us define $Z^\varepsilon(x,v,t)$ to be

$$\min\left(\hat{D}^+(x,v,t), \bar{\ell}(\varepsilon)\right) - Q_+(\tilde{f}^\varepsilon_k(x,\cdot,t))(v)(1+\ell^{-1}\hat{u}^\varepsilon(x,t))^{-2}(1+\ell^{-1}\bar{u}^\varepsilon(x,t))^{-2} .$$

From (9.12), (9.20) and (9.23–25) we deduce

$$\begin{aligned}
\tilde{f}^\varepsilon(x+vt,v,t) &\ge \beta_n(\tilde{f}^\varepsilon(x+vt,v,t)) \\
&= \beta_n(\tilde{f}^\varepsilon(x,v,0))\exp\left(-\int_0^t L\tilde{f}^\varepsilon(x+v\theta,v,\theta)d\theta\right) \\
&\quad + \int_0^t Q_+(\tilde{f}^\varepsilon_k(x,\cdot,s))(v)(1+\ell^{-1}\hat{u}^\varepsilon(x,t))^{-2}(1+\ell^{-1}\bar{u}^\varepsilon(x,t))^{-2} \\
&\quad \times \exp\left(-\int_s^t L\tilde{f}^\varepsilon(x+v\theta,v,\theta)d\theta\right)ds \\
&\quad + \int_0^t Z^\varepsilon(x,v,s)\exp\left(-\int_s^t L\tilde{f}^\varepsilon(x+v\theta,v,\theta)d\theta\right)ds \\
&\quad + R_2(x,v,t) ,
\end{aligned}$$

with R_2 satisfying

$$\begin{aligned}
&E_N\int_0^T\iint [R_2(x,v,s)]^- \mathbb{1}(|v| \le \ell_0)dxdvds \\
&\quad \le c_{26}n\ell_0^{d+2}\left[\ell_2(\varepsilon)^{-1/2} + (\log\log\bar{\ell}(e))^{-1}\right] .
\end{aligned} \tag{9.26}$$

Final Step. Define $\bar{f}^\varepsilon(x, v; \mathbf{q}) = \sum V^\varepsilon(|x_i - x|)V^\varepsilon(|v_i - v|)$. The transformation

$$\begin{aligned}\mathbf{q}(\cdot) &\mapsto (\tilde{f}^\varepsilon(x, v, t)dxdvdt, \bar{f}^\varepsilon(x, v, t)dxdvdt) \\ &= (\tilde{f}^\varepsilon(x, v; \mathbf{q}(t))dxdvdt, \bar{f}^\varepsilon(x, v; \mathbf{q}(t))dxdvdt \\ &=: (\pi(dx, dv, dt), \pi'(dx, dv, dt)) ,\end{aligned}$$

assigns a measure to each realization of $\mathbf{q}(\cdot)$. We regard this measure as a member of $\hat{\mathcal{M}}^2$. The transformation $\mathbf{q}(\cdot) \mapsto (\pi, \pi')$ induces a probability measure $\hat{\mathcal{Q}}_N$ on $\hat{\mathcal{M}}^2$. Let us define $\mathcal{S}(m_1, m_2, m_3)$ to be the set of nonnegative measurable functions (f, f'), such that $f, f' : \mathbb{T}^d \times \mathbb{R}^d \times [0, T] \to [0, \infty)$,

$$\begin{aligned}\sup_{0\le t\le T} \iint \phi(f(x, v, t) + f'(x, v, t))dxdv &\le m_1 , \\ \sup_{0\le t\le T} \iint (f(x, v, t) + f'(x, v, t))dxdv &\le m_1 , \\ \sup_{0\le t\le T} \int \bar{\phi}(u(x, t) + u'(x, t))dx &\le m_1 ,\end{aligned}$$

for $\bar{\phi}(z) = z(\log^+ z)^{1/4}$,

$$u(x, t) = \int (|v|^{3/2} + 1)f(x, v, t)dv , \quad u'(x, t) = \int (|v|^{3/2} + 1)f'(x, v, t)dv ,$$

and that we can find a pair of functions g and r such that

$$\begin{aligned}&f(x, v, t) = f^0(x - vt, v) + \int_0^t g(x - v(t - s), v, s)ds + r(x, v, t) , \\ &\int_0^T \iint \hat{\phi}(|g(x, v, t)|)\mathbb{1}(|v| \le \ell_0)dxdvdt \le m_2\ell_0^{d+2} , \\ &\int_0^T \iint |r(x, v, t)|\mathbb{1}(|v| \le \ell_0)dxdvdt \le m_3^{-1}\ell_0^{d+2} .\end{aligned}$$

As in the proof of Lemma 5.2,

$$\begin{aligned}\iint |\beta_n(f) - f|dxdv &= \iint \frac{f^2}{f + n}dxdv \\ &\le \frac{2}{\log n} \iint \phi(f)dxdv + n^{-\frac{1}{2}} \iint fdxdv \\ &\le \frac{2}{\log n} \iint (\phi(f) + f)\, dxdv . \end{aligned} \tag{9.27}$$

From this, Lemma 4.4, (9.5), (9.9), (9.12), (9.20), (9.24) and Chebyshev's inequality we deduce

$$\begin{aligned}&\hat{\mathcal{Q}}_N(\mathcal{S}(m_1, m_2, m_3)^c) \le c_{27}\left(m_1^{-1} + nm_2^{-1}\right) \\ &\quad +c_{27}\left(m_3^{-1} - 4m_1/\log n\right)^{-1}\left(n(\log\log \bar{\ell}(\varepsilon))^{-1} + \ell_2(\varepsilon)\ell(\varepsilon)^{-1}\right) ,\end{aligned}$$

where A^c denotes the complement of a set A. We choose $n = m_2^{1/2}$ to obtain

$$\hat{\mathcal{Q}}_N(\mathcal{S}(m_1, m_2, m_3)^c) \le c_{27}(m_1^{-1} + m_2^{-1/2})$$
$$+c_{27}\left(m_3^{-1} - 8m_1/\log m_2\right)^{-1}\left(m_2^{1/2}(\log\log\bar{\ell}(\varepsilon))^{-1} + \ell_2(\varepsilon)\ell(\varepsilon)^{-1}\right) .$$

We now choose $m_2 = \exp m_1^2$ and $m_3 = m_1/9$ to yield

$$\hat{\mathcal{Q}}_N(\mathcal{S}(m_1, m_2, m_3)^c) \le c_{28}m_1^{-1} + c_{28}\left((m_2\log\log\bar{\ell}(\varepsilon))^{-1} + \ell_2(\varepsilon)(\ell(\varepsilon))^{-1}\right) .$$

From this we learn that there exists an integer $N(m_1)$ such that

$$\limsup_{m_1\to\infty}\ \sup_{N\ge N(m_1)} \hat{\mathcal{Q}}_N(\mathcal{S}(m_1, \exp m_1^2, m_1/9)^c) = 0 . \tag{9.28}$$

We now consider the space $\mathcal{E}$ consisting of measurable functions (f, f', Z) such that $f, f', Z : \mathbb{T}^d \times \mathbb{R}^d \times [0, T] \to [0, \infty)$ and

$$\int_0^T \iint (f + f' + |Z|)\, dxdvdt < \infty .$$

The transformation

$$\mathbf{q} \mapsto \left(\tilde{f}^\varepsilon(x, v, t), \bar{f}^\varepsilon(x, v, t), Z^\varepsilon(x, v, t)\right) ,$$

defines an augmented probability measure $\tilde{\mathcal{Q}}_N^n$ on the space $\mathcal{E}$. Let us define $\tilde{\mathcal{S}}(m_1, m_2)$ to be the set of (f, f', Z) such that $(f, f') \in \mathcal{S}(m_1, \exp m_1^2, m_1/9)$ and

$$\int_0^T \iint \hat{\phi}\,(|Z(x, v, t)|)\, 1\!\!1(|v| \le \ell_0) dxdvdt \le m_2\ell_0^{d+2} .$$

From (9.28) and (9.20) we learn

$$\limsup_{m_1\to\infty}\ \sup_{N\ge N(m_1)} \tilde{\mathcal{Q}}_N^n(\tilde{\mathcal{S}}(m_1, m_1)^c) = 0 . \tag{9.29}$$

Pick a nonnegative continuous function J of compact support and define

$$\mathcal{F}_J^n(f, f', Z) = \int_0^T \iint \left[\mathcal{G}^n(f, f', Z)(x, v, t)\right]^- J(x, v, t)dxdvdt ,$$

where $\mathcal{G}^n(f, f', Z)(x, v, t)$ is defined to be

$$f(x + vt, v, t) - \beta_n(f(x, v, 0))\exp\left(-\int_0^t Lf(x + v\theta, v, \theta)d\theta\right)$$
$$-\int_0^t Q_+(f_k)(v)\left(1 + \ell^{-1}u(x, s)\right)^{-2}\left(1 + \ell^{-1}u'(x, s)\right)^{-2}$$
$$\times\ \exp\left(-\int_s^t Lf(x + v\theta, v, \theta)d\theta\right) ds$$
$$-\int_0^t Z(x, v, s)\exp\left(-\int_s^t Lf(x + v\theta, v, \theta)d\theta\right) ds ,$$

where $f_k = \min(f, k)$ and,

$$u(x,t) = \int (|v|^{3/2} + 1) f(x,v,t)dv \ , \quad u'(x,t) = \int (|v|^{3/2} + 1) f'(x,v,t)dv \ .$$

Then we may apply (9.26) to assert

$$\lim_{N\to\infty} \int \mathcal{F}_J^n(f, f', Z)\tilde{\mathcal{Q}}_N^n(df, df', dZ) = 0 \ . \tag{9.30}$$

We would like to study the integrand as a functional of (f, f', Z) when $(f, f', Z) \in \tilde{\mathcal{S}}(m_1, m_1)$. In fact $\mathcal{F}_J^n$ restricted to $\tilde{\mathcal{S}}(m_1, m_1)$ is a continuous functional with respect to the weak topology. This follows from DiPerna–Lions' work [DLi1] because if (f_m, f'_m, Z_m) is a sequence in $\tilde{\mathcal{S}}(m_1, m_1)$ such that $(f_m, f'_m, Z_m) \to (f, f', Z)$ weakly in L^1, then $\int_s^t Lf_m(x + v\theta, v, \theta)d\theta$ converges to $\int_s^t Lf(x + v\theta, v, \theta)d\theta$ strongly in L^1 and

$$Q_+ \left(\min(f_m(x,\cdot,s), k)\right)(v) \left(1 + \ell^{-1}u_m(x,s)\right)^{-2} \left(1 + \ell^{-1}u'_m(x,s)\right)^{-2} \ ,$$

converges weakly in L^1–sense to

$$Q_+ \left(\min(f(x,\cdot,s), k)\right)(v) \left(1 + \ell^{-1}u(x,s)\right)^{-2} \left(1 + \ell^{-1}u'(x,s)\right)^{-2} \ .$$

(See for example Lemma 5.3.11 of [CIP].) Choose a sequence $\{N_r\}$ such that $\tilde{\mathcal{Q}}_{N_r}^n$ is convergent as $r \to \infty$ for every n. As a result, if $\tilde{\mathcal{Q}}^n$ is the limit of $\tilde{\mathcal{Q}}_{N_r}^n$, then we apply (9.30) to deduce that the measure $\tilde{\mathcal{Q}}^n$ is concentrated on the space of functions (f, f', Z) for which $\mathcal{F}_J^n(f, f', Z) = 0$. On the other hand, we can now use Theorem 7.1 and (9.20) to assert that if $J(\cdot)$ is a nonnegative continuous function of compact support, then $\int Z(x,v,s)J(v)dv \geq 0$ almost surely with respect to $\tilde{\mathcal{Q}}^n$. We then send $k \to \infty$, and $\ell \to \infty$ in this order and use the montone convergence theorem to deduce that if $(f, f', Z) \in \tilde{\mathcal{S}}(m_1, m_1)$, then f satisfies

$$\begin{aligned} f(x + vt, v, t) \geq \beta_n(f(x,v,0)) \exp\left(-\int_0^t Lf(x + v\theta, v, \theta)d\theta\right) \\ + \int_0^t Q_+(f_k(x,\cdot,s))(v) \exp\left(-\int_s^t Lf(x + v\theta, v, \theta)d\theta\right) ds \ , \end{aligned} \tag{9.31}$$

with probability one with respect to the measure $\tilde{\mathcal{Q}}^n$. We send $m_1 \to \infty$ and use (9.29) to deduce that the measure $\mathcal{Q}^n$ is concentrated on the space of functions (f, f', Z) for which (9.31) holds. The statement (9.31) does not involve (f', Z) and the f–marginal of $\tilde{\mathcal{Q}}^n$, say $\mathcal{Q}$, is independent of n and is a limit point of $\mathcal{Q}_N$. As a result, (9.31) is valid with probability one with respect to any limit point $\mathcal{Q}$ of the sequence $\{\mathcal{Q}_N\}$. We finally send $n \to \infty$ to conclude that the measure $\mathcal{Q}$ is concentrated on the space of supersolutions. □

10. Subsolutions

In this section we establish the other half of Theorem 2.1, namely any limit point of the sequence $\{\mathcal{P}_N\}$ is concentrated on the space of supersolutions of the Boltzmann equation (1.1). As in Sect. 9, it is more convenient to work with the sequence $\{\mathcal{Q}_N\}$. Let us start with a definition for subsolutions.

An integrable function f is called a *subsolution* of (1.1) with initial data f^0, if for every $t \in [0, T]$,

$$f(x, v, t) \leq f^0(x - vt, v) + \int_0^t Q(f, f)(x - v(t - s), v, s)ds \ ,$$

for almost all (x, v).

It is not hard to show that f is a subsolution if and only if

$$\begin{aligned} f(x + vt, v, t) \leq f(x, v, 0) \exp\left(-\int_0^t Lf(x + v\theta, v, \theta)d\theta\right) \\ + \int_0^t Q_+(f(x, \cdot, s))(v) \exp\left(-\int_s^t Lf(x + v\theta, v, \theta)d\theta\right) ds \ , \end{aligned} \tag{10.1}$$

for almost all (x, v). (See for example [DLi1], p. 350.)

The main result of this section is Theorem 10.1.

Theorem 10.1. *If $\mathcal{Q}$ is a limit point of the sequence $\{\mathcal{Q}_N\}$, then $\mathcal{Q}$ is concentrated on the space of measures $\pi(dx, dv, dt) = f(x, v, t)dxdvdt$ with f a nonnegative subsolution of (1.1) with initial data f^0.*

Proof. Let us simply write $Q^\varepsilon_\pm(x, v, s)$ for $Q^\varepsilon_\pm(x, v; \mathbf{q}(s))$ and $\tilde{f}^\varepsilon(x, v, s)$ for $\tilde{f}^\varepsilon(x, v; \mathbf{q}(s))$. As in the proof of Theorem 9.1, we apply (9.12), Theorem 6.1, (9.22–23) and (9.25) to assert

$$\begin{aligned} \beta_n(\tilde{f}^\varepsilon(x + vt, v, t)) = \beta_n(\tilde{f}^\varepsilon(x, v, 0)) \exp\left(-\int_0^t L\tilde{f}^\varepsilon(x + v\theta, v, \theta)d\theta\right) \\ + \int_0^t Q^\varepsilon_+(x + vs, v, s)\left(1 + n^{-1}\tilde{f}^\varepsilon(x, v, s)\right)^{-2} \\ \cdot \exp\left(-\int_s^t L\tilde{f}^\varepsilon(x + v\theta, v, \theta)d\theta\right) ds \\ + n^{-1}\int_0^t \beta_n(\tilde{f}^\varepsilon(x + vs, v, s))\frac{L\tilde{f}^\varepsilon(x + vs, v, s)}{1 + n^{-1}\tilde{f}^\varepsilon(x + vs, v, s)} \\ \cdot \exp\left(-\int_s^t L\tilde{f}^\varepsilon(x + v\theta, v, \theta)\right) ds \\ + R^\varepsilon(x, v, t), \end{aligned}$$

where R^ε satisfies

$$E_N \int_0^T \iint |R^\varepsilon(x, v, s)| 1\!\!1(|v| \leq \ell_0)dxdvds \leq c_0 n \ell_0^{d+2} \ell(\varepsilon)^{-1/2} + c_0 \ell_2(\varepsilon)\ell(\varepsilon)^{-1} \ . \tag{10.2}$$

We now consider the transformation $\mathbf{q}(\cdot) \mapsto (\tilde{f}^\varepsilon, Z_1^\varepsilon, Z_2^\varepsilon)$ for

$$Z_1^\varepsilon = \min\left\{ Q_+^\varepsilon \left(1 + n^{-1}\tilde{f}^\varepsilon\right)^{-2}, \bar{\ell}(\varepsilon)\right\} ,$$
$$Z_2^\varepsilon = n^{-1}\beta_n(\tilde{f}^\varepsilon)\left(1 + n^{-1}\tilde{f}^\varepsilon\right)^{-1} ,$$

and denote the distribution of this transformation by $\tilde{\mathcal{Q}}_N^n$. If

$$\mathcal{F}_J^n(f, Z_1, Z_2) = \int_0^T \iint \left|\mathcal{G}^n(f, Z_1, Z_2)\right| J \, dxdvdt ,$$

for

$$\begin{aligned}\mathcal{G}^n(f, Z_1, Z_2) &= \beta_n(f(x + vt, v, t)) - \beta_n(f(x, v, 0)) \exp\left(-\int_0^t Lf(x + v\theta, v, \theta)d\theta\right)\\ &+ \int_0^t Z_1(x + vs, v, s) \exp\left(-\int_s^t Lf(x + v\theta, v, \theta)d\theta\right) ds\\ &+ \int_0^t Z_2(x+vs, v, s)Lf(x+vs, v, s) \exp\left(-\int_s^t Lf(x + v\theta, v, \theta)\right) ds,\end{aligned}$$

then we use (10.2) and (9.20) to assert that for every continuous function J of compact support,

$$\limsup_{N\to\infty} \int \mathcal{F}_J^n(f, Z_1, Z_2)\tilde{\mathcal{Q}}_N^n(df, dZ_1, dZ_2) = 0 . \tag{10.3}$$

Let us define $\hat{\mathcal{S}}(m_1)$ to be the set of functions (f, Z_1, Z_2) such that $Z_2 \in [0, 1]$ and $(f, Z_1) \in \tilde{\mathcal{S}}(m_1, m_1)$ with $\tilde{\mathcal{S}}$ as in (9.28). Evidently (9.28) implies

$$\limsup_{m_1\to\infty} \sup_{N\geq N(m_1)} \tilde{\mathcal{Q}}_N^n(\hat{\mathcal{S}}(m_1)^c) = 0 . \tag{10.4}$$

Note that $\mathcal{F}_J^n$, restricted to the set $\hat{\mathcal{S}}(m_1)$, is a continuous functional with respect to the weak L^1–convergence. This is because $Z_2 \in [0, 1]$ and that by the velocity averaging lemma, if f_m is a sequence of functions such that $f_m \to f$ weakly, then $Lf_m \to Lf$ strongly in L^1–sense. Given a subsequence of $\{N\}$, we can find a subsequence of it, say $\{N_r\}$, such that the sequence $\{\tilde{\mathcal{Q}}_{N_r}^n\}$ converges for every n as $N_r \to \infty$. As in the proof of Theorem 9.1, we can use the continuity of $\mathcal{F}_J^n$, (10.4) and (10.3) to deduce that if $\tilde{\mathcal{Q}}^n$ is the limit of the sequence $\{\tilde{\mathcal{Q}}_{N_r}^n\}$ as $r \to \infty$, then $\tilde{\mathcal{Q}}^n$ is concentrated on the set of (f, Z_1, Z_2) such that $\mathcal{F}_J^n(f, Z_1, Z_2) = 0$. To complete the proof, we need to identify Z_1 and Z_2. First we can claim that for any continuous function J of compact support,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \int \left\{\int_0^T \iint Z_2 \; J \; dxdvdt\right\} \tilde{\mathcal{Q}}_N^n(df, dZ_1, dZ_2) = 0 . \tag{10.5}$$

To see this, observe that for every positive k, $Z_2^\varepsilon \le Z_{21}^{\varepsilon,k} + Z_{21}^{\varepsilon,k}$, where

$$Z_{21}^{\varepsilon,k} = Z_2^\varepsilon 1\!\!1(\tilde{f}^\varepsilon \le k) \ , \quad Z_{22}^{\varepsilon,k} = 1\!\!1(\tilde{f}^\varepsilon \ge k) \ .$$

From this we can readily deduce (10.5) because

$$\lim_{n\to\infty} \sup_N E_N \int_0^T \iint Z_{21}^{\varepsilon,k} J \ dxdvdt = 0 \ ,$$
$$\lim_{k\to\infty} \sup_N E_N \int_0^T \iint 1\!\!1(\tilde{f}^\varepsilon \ge k) J \ dxdvdt = 0 \ .$$

From (10.5) we deduce that $Z_2 = 0$ almost surely with respect to $\tilde{\mathcal{Q}}$, where $\tilde{\mathcal{Q}}$ is any limit point of $\tilde{\mathcal{Q}}^n$.

On the other hand, for every nonnegative continuous function J of compact support, we may apply Theorem 8.1 with $\alpha_1 = 0$ and $\alpha_2 = 1$ to assert that for every n,

$$\lim_{N\to\infty} \int \left\{ \int_0^T \iint \int (Z_1 - Q_+(f))(1+u)^{-2} J dxdvdt \right\}^+ \tilde{\mathcal{Q}}_N^n(df, dZ_1, dZ_2) = 0 \ ,$$

where $u = u(x,t) = \int (1+|v|^{3/2}) f(x,v,t) dv$. Again the expression inside the curly brackets is a continuous functional of (f, Z_1, Z_2) if we restrict it to the set $\hat{\mathcal{S}}(m_1)$. From this and (10.4) we can readily deduce that $\tilde{\mathcal{Q}}^n$ is concentrated on the set of triplets (f, Z_1, Z_2) such that

$$Z_1 \le Q^+(f).$$

This, (10.5), (9.27) and Lemma 4.4 imply the f–marginal of $\tilde{\mathcal{Q}}$ is concentrated on the space of subsolutions. □

11. Entropy Production Bound Revisited

In this section we establish a variant of (2.6). The method of the proof is similar to [DLi2]. Define $\beta(a,b) = (a-b)\log\frac{a}{b}$ for $a, b > 0$. We also put $\beta(a,b) = +\infty$ whenever a or $b \le 0$.

Theorem 11.1. *Let $\mathcal{Q}$ be a limit point of the sequence $\{\mathcal{Q}_N\}$. Then*

$$\int_0^\infty \int \int \int_{\mathbb{S}} \beta(F(x,v,v_*,t), F(x,v',v'_*,t)) B(v-v_*,n) dndvdv_*dxdt < \infty, \tag{11.1}$$

where

$$F(x,v,v_*,t) = \int f(x,v,t) f(x,v_*,t) \mathcal{Q}(df). \tag{11.2}$$

Proof. The proof is similar to what has been presented in previous sections and we only sketch it. To ease the notation, we simply write $\{\mathcal{Q}_N\}$ for a convergent subsequence of

$\{\mathcal{Q}_N\}$. Let $\psi(z) = z \log z - z + 1$ for $z > 0$ and $\psi(z) = +\infty$ for $z \leq 0$. Recall the function G of (4.1). Since ψ is convex,

$$\psi\left(\frac{G(t, S^{ij}\mathbf{q})}{G(t, \mathbf{q})}\right) \geq \psi(a) + \psi'(a)\left(\frac{G(t, S^{ij}\mathbf{q})}{G(t, \mathbf{q})} - a\right),$$

for every positive a. As a result,

$$\psi\left(\frac{G(t, S^{ij}\mathbf{q})}{G(t, \mathbf{q})}\right) G(t, \mathbf{q}) \geq \hat{\psi}(a)G(t, \mathbf{q}) + \psi'(a)G(t, S^{ij}\mathbf{q}), \tag{11.3}$$

where $\hat{\psi}(a) = \psi(a) - a\psi'(a) = 1 - a$. By Lemma 4.7, there exists a constant c_0 such that for every N and $T > 0$,

$$\int_0^T \int \varepsilon^{2d} \sum_{i,j} V^\varepsilon(|x_i - x_j|)B(v_i - v_j, n_{ij})\psi\left(\frac{G(t, S^{ij}\mathbf{q})}{G(t, \mathbf{q})}\right) G(t, \mathbf{q})\nu_\beta(d\mathbf{q})dt \leq c_0. \tag{11.4}$$

We would like to derive (11.1) from (11.4). For this we would rather have a linear expression in G for the integrand of (11.4). Let us take a smooth bounded nonnegative function $a = a(t, x, v, v_*, n)$ and use (11.4) and (11.3) to assert

$$\begin{aligned}\Omega := \int_0^T \int \varepsilon^{2d} \sum_{i,j} &V^\varepsilon(|x_i - x_j|)B(v_i - v_j, n_{ij})\zeta^{\delta_1(\varepsilon)}(x_i - x)(1 + \alpha\hat{u}^\varepsilon(x; \mathbf{q}))^{-2} \\ &\times \left[\hat{\psi}(a(x, v_i, v_j, n_{ij})) + \psi'(a(x, v_i^j, v_j^i, n_{ij}))\right] G(t, \mathbf{q})\nu_\beta(d\mathbf{q})dxdt \leq c,\end{aligned}$$

where $\delta_1(\varepsilon)$ is as in Sect. 9, the function $\hat{u}^\varepsilon$ is as in Theorem 8.1, and $\alpha > 0$ is a fixed constant that will be sent to 0 in the end. Using the proof of Theorem 8.1, it is not hard to establish

$$\begin{aligned}\lim_{N\to\infty} \Omega &= \lim_{N\to\infty} \int \int_0^T \iint \int \int_{\mathbb{S}} B(v - v_*, n)f(x, v, t)f(x, v_*, t)\,(1 + \alpha u(x, t))^{-2} \\ &\quad \cdot \left[\hat{\psi}(a(x, v, v_*, n)) + \psi'(a(x, v', v'_*, n))\right] dndxdvdv_*dt\ \mathcal{Q}_N(df) \\ &=: \lim_{N\to\infty} \int X_\alpha(f)\mathcal{Q}_N(df),\end{aligned} \tag{11.5}$$

where $u(x, t) = \int f(x, w, t)(|w|^{3/2} + 1)dw$. Using the proof of Theorem 9.1 we can readily deduce

$$\lim_{N\to\infty} \Omega = \int X_\alpha(f)\mathcal{Q}(df). \tag{11.6}$$

From (11.4–6) we learn

$$\int \left[\int_0^T \iint \int \int_{\mathbb{S}} B(ff_*\hat{\psi}(a) + f'f'_*\psi'(a))(1 + \alpha u)^{-2}dndxdvdv_*dt\right] \mathcal{Q}(df) \leq c_0.$$

So far we have assumed that a is smooth, bounded and nonnegative. The smoothness condition can be relaxed by approximating a measurable function a by smooth functions and applying the dominated convergence theorem. From this we deduce

$$\int_0^T \iint \int \int_{\mathbb{S}} B(v - v_*, n)[F_\alpha(x, v, v_*, t)\hat{\psi}(a(x, v, v_*, n, t)) \\ +F_\alpha(x, v', v'_*, t)\psi'(a(x, v, v_*, n, t))]dndxdvdv_*dt \le c_0 \tag{11.7}$$

for every bounded uniformly measurable function a, where

$$F_\alpha(x, v, v_*, t) = \int f(x, v, t)f(x, v_*, t)(1 + \alpha u(x, t))^{-2}\mathcal{Q}(df) \ .$$

Ideally we would like to choose $a(x, v, v_*, n, t)$ to be $F_\alpha(x, v', v'_*, t)/F_\alpha(x, v, v_*, t)$. Since a is supposed to be bounded, we first put a to be $\min(\hat{F}_\alpha(x, v', v'_*, t)/(\hat{F}_\alpha(x, v, v_*, t), \ell)$. Using such a choice for a in (11.7) we obtain

$$\int_0^T \iint \int \int_{\mathbb{S}} B(v - v_*, n)\psi_\ell(F_\alpha(x, v', v'_*, t)/F_\alpha(x, v, v_*, t))F_\alpha(x, v, v_*, t)dndxdvdv_*dt \le c_0 \ ,$$

where $\psi_\ell(z) = \psi(z)1\!\!1(z \le \ell)$. We now send $\ell \to \infty, \alpha \to 0$ and use Fatou's lemma to deduce

$$\int_0^T \iint \int \int_{\mathbb{S}} B(v - v_*, n)\psi(F(x, v', v'_*, t)/F(x, v, v_*, t))F(x, v, v_*, t)dndxdvdv_*dt \le c_0 \ .$$

From this we can readily deduce

$$\int_0^T \iint \int \int_{\mathbb{S}} B(v - v_*, n)\psi(F(x, v, v_*, t)/F(x, v', v'_*, t))F(x, v', v'_*, t)dndxdvdv_*dt \le c_0 \ .$$

This completes the proof of (11.1) because $\beta(a, b) = \psi(a/b)b + \psi(b/a)a$. □

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