Kinetic description of scalar conservation laws with Markovian data

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Abstract

We derive a kinetic equation to describe the statistical structure of solutions ρ to scalar conservation laws $\rho_t = H(x,t,\rho)_x$, with certain Markov initial conditions. When the Hamiltonian function is convex and increasing in ρ, we show that the solution $\rho(x,t)$ is a Markov process in x (respectively t) with t (respectively x) fixed. Two classes of Markov conditions are considered in this article. In the first class, the initial data is characterize by a drift b which satisfies a linear PDE, and a jump density f which satisfies a kinetic equation as time varies. In the second class, the initial data is a concatenation of fundamental solutions that are characterized by a parameter y, which is a Markov jump process with a jump density g satisfying a kinetic equation. When H is not increasing in ρ, the restriction of ρ to a line in (x,t) plane is a Markov process of the same type, provided that the slope of the line satisfies an inequality.

1 Introduction

Hamilton–Jacobi equation (HJE) is one of the most popular and studied PDE which enjoys vast applications in numerous areas of science. Originally HJEs were formulated in connection with the completely integrable Hamiltonian ODEs of celestial mechanics. They have also been used to study the evolution of the value functions in control and differential game theory. Several growth models in physics and biology are described by HJEs. In these models, a random interface separates regions associated with different phases and the interface can be locally approximated by the graph of a solution to a HJE. To make up for the lack of exact information or/and the presence of impurity, it is common to assume that the Hamiltonian function which appears in our HJE is random. Naturally we would like to understand how the randomness affects the solutions and how the statistics of solutions are propagated with time.

In dimension one, the differentiated version of a Hamilton–Jacobi equation becomes a scalar conservation law for the inclination of the one-dimensional interface, and may be used to model an one-dimensional fluid. In the context of fluids, we wish to obtain some qualitative information about the structure of shocks and their fluctuations.

The primary purpose of this article is to derive an evolution equation for the statistics of solutions to a HJE in dimension one. We achieve this by utilizing a kinetic description for the shock densities of piecewise smooth solutions.

Given a C^2 Hamiltonian function $H : \mathbb{R} \times [0, \infty) \times \mathbb{R} \to \mathbb{R}$, we consider the HJE

$$u_t = H(x, t, u_x), \qquad t \geq t_0, \tag{1.1}$$

or the corresponding scalar conservation law

$$\rho_t = H(x, t, \rho)_x, \qquad t \geq t_0. \tag{1.2}$$

We assume that the Hamiltonian function $H(x, t, \rho)$ is convex in the *momentum* variable ρ. As our main goal, we show that the statistics of $\rho(x, t)$ admits an exact kinetic description when the initial data $\rho^0(x) = \rho(x, t_0)$ is an inhomogeneous Markov process.

1.1 Main result I

For our first result, we assume that the initial data $\rho^0 = \rho^0(x)$ is a piecewise-deterministic inhomogeneous Markov process (PDMP) Markov process determined by a generator $\mathcal{A}_x^0 = \mathcal{A}_{x,t_0}$ acting on test functions $\psi(\rho)$ according to

$$(\mathcal{A}_x^0 \psi)(\rho) = b^0(x, \rho)\psi'(\rho) + \int_\rho^\infty \big(\psi(\rho_*) - \psi(\rho)\big)\, f^0(x, \rho, \rho_*)\ d\rho_*. \tag{1.3}$$

The random path $\rho^0(x)$ may be constructed by solving (deterministically) the ODE $d\rho^0/dx = b^0(x, \rho^0)$, interrupted by jumps which occur stochastically: the rate density at which ρ^0 makes a jump at x is $f^0(x, \rho^0(x), \rho_*)$. As our main result, we show that the process $x \mapsto \rho(x, t)$ (for fixed $t > t_0$) is again a PDMP, with generator

$$\big(\mathcal{A}_{x,t}\psi\big)(\rho) = b(x, t, \rho)\psi'(\rho) + \int_\rho^\infty \big(\psi(\rho_*) - \psi(\rho)\big)\, f(x, t, \rho, \rho_*)\, d\rho_*. \tag{1.4}$$

Here $b(x, t, \rho)$ and $f(x, t, \rho_-, \rho_+)$ are obtained from their initial ($t = t_0$) conditions

$$b(x, t_0, \rho) = b^0(x, \rho), \qquad f(x, t_0, \rho_-, \rho_+) = f^0(x, \rho_-, \rho_+), \tag{1.5}$$

by solving a semi-linear PDE,

$$b_t + H_x b_\rho - H_\rho b_x = H_{\rho\rho} b^2 + 2H_{\rho x} b + H_{xx}, \tag{1.6}$$

and a kinetic (integro-)PDE

$$f_t - (vf)_x - C(f) = Q(f), \tag{1.7}$$

where

$$v(x,t,\rho_-,\rho_+) := \frac{H(x,t,\rho_-) - H(x,t,\rho_+)}{\rho_- - \rho_+}, \tag{1.8}$$

$Q(f) = Q^+(f) - Q^-(f)$ is a coagulation-like collision operator, and $C(f) = C^+(f) + C^-(f)$ is a linear first order differential operator. More precisely,

(i) Q^+ is a quadratic operator and $Q^+(f)(x,t,\rho_-,\rho_+)$ is defined as

$$\int_{\rho_-}^{\rho_+} \big(v(x,t,\rho_*,\rho_+) - v(x,t,\rho_-,\rho_*)\big) f(x,t,\rho_-,\rho_*) f(x,t,\rho_*,\rho_+)\ d\rho_*. \tag{1.9}$$

(ii) The quadratic operator Q^- is of the form $Q^-(f) = fJf$, for a linear operator J. Given f, the function $(Jf)(x,t,\rho_-,\rho_+)$ is defined as

$$A(vf)(x,t,\rho_+) - A(vf)(x,t,\rho_-) - v(x,t,\rho_-,\rho_+)\big((Af)(x,t,\rho_+) - (Af)(x,t,\rho_-)\big), \tag{1.10}$$

for linear operators A defined by

$$Ah(x,t,\rho_-) = \int_{\rho_-}^{\infty} h(x,t,\rho_-,\rho_+)\ d\rho_+.$$

(iii) Given a C^1 kernel f,

$$\big(C^+ f\big)(x,t,\rho_-,\rho_+) = \left[K(x,t,\rho_+,\rho_-) f(x,t,\rho_-,\rho_+)\right]_{\rho_+}, \tag{1.11}$$

where

$$\begin{aligned} K(x,t,\rho_+,\rho_-) &= b(x,t,\rho_+) v(x,t,\rho_-,\rho_+) - \beta(x,t,\rho_+), \qquad \text{with} \\ \beta(x,t,\rho) &= \big(H_x + bH_\rho\big)(x,t,\rho). \end{aligned}$$

Here and below, by the expression X_a we mean the partial derivative of X with respect to the variable a. For example the right-hand side of (1.11) represents the partial derivative of the expression inside the brackets with respect to ρ_+.

(iv) Given a C^1 kernel f,

$$\begin{aligned}\big(C^-f\big)(x,t,\rho_-,\rho_+) =& b(x,t,\rho_-)(vf)_{\rho_-}(x,t,\rho_-,\rho_+) - \beta(x,t,\rho_-)f_{\rho_-}(x,t,\rho_-,\rho_+)\\ =& b(x,t,\rho_-)\big(v_{\rho_-}f\big)(x,t,\rho_-,\rho_+) + K(x,t,\rho_-,\rho_+)f_{\rho_-}(x,t,\rho_-,\rho_+).\end{aligned} \tag{1.12}$$

Remark 1.1 For a more compact reformulation of our equations (1.6) and (1.7), let us write

$$x_1 = x, \quad x_2 = t, \quad f^1 = f, \quad f^2 = vf^1, \quad b^1 = b, \quad b^2 = \beta. \tag{1.13}$$

Recall $Ag(\rho) = A(g)(\rho) = \int g(\rho,\rho_*)\ d\rho_*$, and define

$$(g\otimes k)(\rho_-,\rho_+) = g(\rho_-,\rho_+)k(\rho_+), \qquad (k\otimes g)(\rho_-,\rho_+) = k(\rho_-)g(\rho_-,\rho_+). \tag{1.14}$$

A more symmetric rewriting of the equations (1.6) and (1.7) read as

$$b^1_{x_2} - b^2_{x_1} = b^1b^2_\rho - b^2b^1_\rho, \qquad f^1_{x_2} - f^2_{x_1} = \mathcal{Q}(f^1,f^2) - \mathcal{Q}(f^2,f^1), \tag{1.15}$$

where

$$\mathcal{Q}(f^j,f^i) = f^j * f^i - A(f^j)\otimes f^i - f^j\otimes A(f^i) + b^j\otimes f^i_{\rho_-} - (f^j\otimes b^i)_{\rho_+}, \tag{1.16}$$

where

$$(f^j * f^i)(\rho_-,\rho_+) = \int f^j(\rho_-,\rho_*)f^i(\rho_*,\rho_+)\ d\rho_*.$$

□

We now formulate our assumptions on the initial drift b^0, the initial jump rate kernel f^0, and the Hamiltonian function $H(x,t,\rho)$.

Hypothesis 1.1(i) The Hamiltonian function $H : \mathbb{R}\times[t_0,T]\times[P_-,P_+] \to \mathbb{R}$ is a C^2 function. Additionally, H is increasing and convex in ρ.

(ii) The PDE (1.6) has a bounded C^1 solution $b \le 0$ for $t \in [t_0,T]$. We set $b^0(x,\rho) := b(x,t_0,\rho)$.

(iii) The PDE (1.7) has a solution $f : \hat\Lambda \to [0,\infty)$, where $\hat\Lambda := \mathbb{R}\times[t_0,T]\times\Lambda(P_-,P_+)$, with

$$\Lambda(P_-,P_+) := \Lambda\cap[-P,P]^2 := \big\{(\rho_-,\rho_+) :\ P_- \le \rho_- \le \rho_+ \le P_+\big\}.$$

We assume that f is C^1 in the interior of $\hat\Lambda$, and that f is continuous in $\hat\Lambda$. Moreover, $f(x,t,\rho_-,\rho_+) > 0$, when $P_- < \rho_- < \rho_+ < P_+$, and $f(x,t,\rho_-,\rho_+) = 0$, whenever ρ_- or $\rho_+ \notin (P_-,P_+)$. To ease our notation, we extend the domain of the definition of f to

$\mathbb{R} \times [t_0, T] \times \mathbb{R}^2$, by setting $f(x, t, \rho_-, \rho_+) = 0$, whenever ρ_- or $\rho_+ \notin \Lambda(P_-, P_+)$. We also write $f^0(x, \rho_-, \rho_+)$ for $f(x, t_0, \rho_-, \rho_+)$.

(iv) We assume that $\rho(x,t)$ is an entropy solution of (1.2), and that its initial condition $\rho^0(x) := \rho(x, t_0)$ is 0 for $x < a_-$, and is a Markov process for $x \geq a_-$ that starts at $\rho^0(a_-) = m_0$. This Markov process has an infinitesimal generator in the form (1.3) for a drift b^0 and a jump rate density f^0. □

Our statistical description consists of a one-dimensional marginal, a drift, and a rate kernel generating the rest of the path. The evolution of the drift and the rate kernel are given by (1.6) and the kinetic equation (1.7). Evolution of the marginal will be described in terms of the solutions to these equations. We continue with some definitions.

Definition 1.1(i) We define the linear operator $\mathcal{A}^i$ by

$$(1.17) \quad \big(\mathcal{A}^i_{x,t}\psi\big)(\rho) = \big(\mathcal{A}^i\psi\big)(\rho) = b^i(x,t,\rho)\psi'(\rho) + \int_\rho^\infty f^i(x,t,\rho,\rho_+)\big(\psi(\rho_+) - \psi(\rho)\big)\, d\rho_+,$$

for $i = 1, 2$. Note that $\mathcal{A}^1 = \mathcal{A}$ of (1.4), and f^i was defined in (1.13). We write $\mathcal{A}^{i*}$ for the adjoint of the operator $\mathcal{A}^i$ which acts on measures. When the measure ν is absolutely continuous with respect to the Lebesgue measure with a C^1 Radon-Nykodym derivative, then $\mathcal{A}^{i*}\nu$ is also absolutely continuous with respect to the Lebesgue measure. The action of the operator $\mathcal{A}^{i*}$ on ν can be described in terms of its action on the corresponding Radon-Nykodym derivative. By a slight abuse of notation, we write $\mathcal{A}^{i*}$ for the corresponding operator that now acts on C^1 functions. More precisely, for a probability density ν, we have

$$\big(\mathcal{A}^{i*}_{x,t}\nu\big)(\rho) = \left[\int_{-\infty}^\rho f^i(x,t,\rho_*,\rho)\ \nu(\rho_*)\ d\rho_*\right] - A(f^i)(x,t,\rho)\nu(\rho) - \big(b^i(x,t,\rho)\nu(\rho)\big)_\rho.$$

(ii) We write $\mathcal{M}$ for the set of measures and $\mathcal{M}_1$ for the set of probability measures. □

Theorem 1.1 *Given a C^1 rate f, and $m_0 \in \mathbb{R}$, assume $\ell : [t_0, \infty) \to \mathcal{M}_1$ satisfies $\ell(t_0, d\rho_0) = \delta_{m_0}(d\rho_0)$, and*

$$(1.18) \qquad \frac{d\ell}{dt} = \mathcal{A}^{2*}_{a_-,t}\ell, \qquad t > t_0.$$

When Hypothesis 1.1 holds, the entropy solution ρ to (1.1) for each fixed $t > t_0$ has $x = a_-$ marginal given by $\ell(t, d\rho_0)$ and for $a_- < x < \infty$ evolves according to a Markov process with the generator $\mathcal{A}^1_{x,t}$. Moreover, the process $t \mapsto \rho(a,t)$ is a Markov process with generator $\mathcal{A}^2_{a,t}$, for every $a \geq a_-$.

Remark 1.2(i) According to Hypothesis 1.1, the function H is increasing. This condition is needed to guarantee that $f^2 \geq 0$, which in turn guarantee that $\mathcal{A}^2$ is a generator of a Markov process. This restriction on H can be relaxed almost completely. The main role of the condition $H_\rho > 0$ is that all shock discontinuities of ρ travel with negative velocities so that they cross any fixed location, say $x = a$ eventually. This allows us to assert that if $\rho(a, t)$ is known, then the law of $\rho(x, t)$ can be determined uniquely for all $x > a$. In general, we may try to determine $\rho(x, t)$ for $x > a(t)$, provided that $\rho(a(t), t)$ is specified. The condition $H_\rho > 0$, allows us to choose $a(t)$ constant. If instead we can find a negative constant c such that $H_\rho > c$, then $\hat{\rho}(x, t) := \rho(x - ct, t)$ satisfies

$$\hat{\rho}_t = \hat{H}(x, t, \hat{\rho})_x,$$

for $\hat{H}(x, t, \rho) = H(x - ct, t, \rho) - c\rho$, which is increasing. Hence, the process $t \mapsto \hat{\rho}(x, t) = \rho(x - ct, t)$ is now Markovian with a generator $\hat{\mathcal{A}}^2$ which is obtained from $\mathcal{A}^2$ by replacing H with $\hat{H}$. Even an upper bound on H' can lead to a result similar to Theorem 1.2. For example if $H_\rho < 0$, then $x \mapsto \rho(x, t)$ is a Markov process but now as we decrease x.

(ii) To guarantee the existence of a solution to (1.6) in an interval $[t_0, T]$, let us assume that $H_{\rho x}$ and H_{xx} are uniformly bounded, and that $H_{xx} \leq 0$ in this interval. Under such assumptions, we claim that if initially at time $t = t_0$ the drift is nonpositive and bounded, then the no blow-up condition of Hypothesis **1.1(ii)** is met because b remains bounded and nonnegative. To see this, assume that the function b solves the equation (1.6), and write $\Theta_s^t(a, m)$ for the flow of the Hamiltonian ODE

$$\dot{x} = -H_\rho(x, t, \rho), \qquad \dot{\rho} = H_x(x, t, \rho). \tag{1.19}$$

In other words $(\rho(t), x(t)) = \Theta_s^t(a, m))$ solves (1.19), subject to the initial conditions $x(s) = a$, and $\rho(s) = m$. To ease the notation, we write $b(x, \rho, t)$ and $H(x, \rho, t)$ for $b(x, t, \rho)$, and $H(x, t, \rho)$ respectively. Evidently, $\hat{b}(x, \rho, t) = b\big(\Theta_s^t(x, \rho), t\big)$ satisfies

$$\hat{b}_t = A\hat{b}^2 + 2B\hat{b} + C, \tag{1.20}$$

where

$$A(x, \rho, t) := H_{\rho\rho}\big(\Theta_s^t(x, \rho), t\big), \quad B(x, \rho, t) := H_{\rho x}\big(\Theta_s^t(x, \rho), t\big), \quad C(x, \rho, t) := H_{xx}\big(\Theta_s^t(x, \rho), t\big).$$

Since the right-hand side of (1.20) is nonpositive when $\hat{b} = 0$, we deduce that $\hat{b}(t) = \hat{b}(x, \rho, t)$ remains nonpositive for $t \in [t_0, T]$, if this is true initially at $t = t_0$. Note that since $\hat{b}_t \geq 2B\hat{b} + C$, with B and C bounded, b is also bounded from below in $[t_0, T]$, if this is so initially.

(iii) The existence of a classical solutions to (1.7) and (1.18) can be found in [KR2] and [OR] when H is independent of (x, t), and b is either constant, or f is independent of (x, t). The same type of arguments can be worked out in our setting.

(iv) As a consequence of Hypothesis 1.1**(iii)**, the density $\rho(x,t) \in [P_-, P_+]$ almost surely. This restriction will be needed in Sections 2 and 3 when we derive a forward equation for the law of $\rho(\cdot,t)$. The boundedness of $\rho(x,t)$ is needed only when we restrict ρ to a bounded set of the form $\Lambda := [a_-, a_+] \times [t_0, T]$ (see Theorem 2.1 below). Note however that for a C^1 drift b, the density is always bounded below in Λ, because the random jump only increases the density. So we only need to require an upper bound on the density, and the requirement $P_- \leq \rho_-$ is redundant. In other words, we can find P_- that depends on b, and a lower bound of ρ^0, such that the condition $P_- \leq \rho_-$ holds in Λ.

In Theorem 1.3 below, we will learn how to relax the boundedness requirement on the density. □

1.2 Main result II

Our Hypothesis 1.1**(ii)** is rather stringent requirement because the right-hand side of the PDE (1.6) is quadratic in b. Our main Theorem 1.1 applies only when no new shock discontinuity is created in the time interval $[t_0, T]$. Indeed a blow-up of the drift occurs exactly when a new jump discontinuity is formed for a local continuous solution that is represented by the ODE $\rho_x = b(x,t,\rho)$. In Remark 1.1**(ii)** we stated conditions that would prevent a blow-up, but these conditions exclude many important stochastic growth models that are governed by HJE associated with random Hamiltonian (see Example 1.1**(ii)** below).

We emphasis that Theorem 1.1 offers a kinetic description for the interaction between the existing shock discontinuities, not those which are created after the initial time. To go beyond what is offered by Theorem 1.1, we need to enlarge the class of Markovian profiles that has been used so far. We offer a way to achieve this by considering profiles that are Markovian concatenations of *fundamental solutions* of (1.2).

Definition 1.2 Given $z = (y,s) \in \mathbb{R}^2$, by a *fundamental solution* $W(\cdot; z) : \mathbb{R} \times (s,\infty) \to \mathbb{R}$ associated with z we mean

$$(1.21) \quad W(x,t;z) = \sup\left\{ \int_s^t L\big(\xi(\theta),\theta,\dot\xi(\theta)\big)\, d\theta :\ \xi \in C^1\big([s,t];\mathbb{R}\big),\ \xi(s) = y,\ \xi(t) = x \right\},$$

where L is the Legendre transform of H in the p-variable:

$$L(x,t,v) = \inf_p \big(p\cdot v + H(x,t,p)\big), \qquad H(x,t,p) = \sup_v \big(L(x,t,v) - p\cdot v\big).$$

We also set $M(x,t;z) = W_x(x,t;z)$ for the x-derivative of W. □

Under our conditions on H, the function W is a Lipschitz function of (x,t) for $t > s$, and $M(x,t) = M(x,t;z)$ is well-defined a.e.. A representation of M is given as follows. For each (x,t), we may find a maximizing piecewise C^1 path $\xi(\theta) = \xi(\theta; x,t;z)$ that is differentiable at time $\theta = t$. The function M is continuous at (x,t) if and only if the maximizing path is unique. When this is the case, we simply have

$$M(x,t) = L_v\big(\xi(t), t, \dot{\xi}(t)\big) = L_v\big(x, t, \dot{\xi}(t)\big). \tag{1.22}$$

In general $M(x,t)$ could be multi-valued; for each maximizing path, the right-hand side of (1.22) offers a possible value for $M(x,t)$.

The Cauchy problem associated with (1.1) has a representation of the form

$$u(x,t) = \sup_y \big(u^0(y) + W(x,t;y,t_0)\big). \tag{1.23}$$

In other words, u given by (1.23), satisfies (1.1) in viscosity sense for $t > t_0$, and $u(x,t_0) = u^0(x)$. The type of stochastic solutions we will be able to describe kinetically would look like

$$u(x,t) = \sup_{i \in I} \big(g_i + W(x,t;z_i)\big),$$

where $\big\{(z_i, g_i) :\ i \in I\big\}$ is a discrete set. Since our Markovian process is $\rho = u_x$, we consider profiles of the form

$$\rho(x,t) = \sum_{i \in I} M(x,t;z_i)\ 1\!\!1\big(x \in [x_i, x_{i+1})\big),$$

for a discrete set $\big\{q_i = (x_i, z_i) :\ i \in I\big\}$. (Note that because of the type of results we have in mind, we switched from (g_i, z_i) to (x_i, z_i).)

We now give the definition of the Markov processes we will work with in this subsection.

Definition 1.3(i) Given s, T, with $s < T$, let $g(x,t;y_-,y_+)$ be a C^1 nonnegative (kernel) function that is defined for $x \in \mathbb{R},\ t \in [s,T],\ y_+ \in (y_-, \infty)$. We also write

$$x_1 = x, \quad x_2 = t, \quad g^1 = g, \quad g^2 = \hat{v} g,$$

where

$$\hat{v}(x,t,y_-,y_+) = \frac{H\big(x,t,M(x,t;y_+,s)\big) - H\big(x,t,M(x,t;y_-,s)\big)}{M(x,t;y_+,s) - M(x,t;y_-,s)}. \tag{1.24}$$

We write $\mathcal{B}^i_{x,t}$ for the operator

$$\mathcal{B}^i_{x_1,x_2} F(y) = \int_y^\infty \big(F(y_*) - F(y)\big)\ g^i(x_1,x_2;y,y_*)\ dy_*.$$

$\mathcal{B}^1$ is the infinitesimal generator of an *inhomogeneous Markov jump* process $\mathbf{y}(x_1)$. When $\hat{v} > 0$, the operator $\mathcal{B}^2$ also generates a Markovian jump process.

(iii) We write $\mathcal{B}^{i*}$ for the adjoint of the operator $\mathcal{B}^i$ which acts on measures. As before, when the measure ν is absolutely continuous with respect to the Lebesgue measure with a C^1 Radon-Nykodym derivative, then $\mathcal{B}^{i*}\nu$ is also absolutely continuous with respect to the Lebesgue measure. By a slight abuse of notation, we write $\mathcal{B}^{i*}$ for the corresponding operator that now acts on C^1 functions. More precisely, for a probability density ν, we have

$$\big(\mathcal{B}_x^{i*}\nu\big)(y) = \left[\int_{-\infty}^{y} g^i(x,t,y_*,y)\nu(y_*)\ dy_*\right] - \hat{A}(g^i)(x,t,y)\nu(y),$$

where

$$\hat{A}(g)(y) = \int_y^\infty g(y,y_*)\ dy_*.$$

(iv) Given $\mathbf{y} : [a_-,a_+] \to \mathbb{R}$, we define

$$\rho(x,t;\mathbf{y},s) := M\big(x,t;\mathbf{y}(x),s\big).$$

□

According to our second main result, if $\rho = u_x$ solves (1.2) with an initial condition which comes from a Markov process associated with a kernel g^0, then at later times $x \mapsto \rho(x,t)$ also comes from a Markov process associated with a kernel g which satisfies a kinetic equation in the form

$$g_t - (\hat{v}g)_x = \hat{Q}(g) = \hat{Q}^+(g) - \hat{Q}^-(g) = \hat{Q}^+(g) - g\hat{J}(g), \tag{1.25}$$

where

$$\hat{Q}^+(g)(y_-,y_+) = \int_{y_-}^{y_+} \big(\hat{v}(y_*,y_+) - \hat{v}(y_-,y_*)\big)g(y_-,y_*)g(y_*,y_+)\ dy_*,$$
$$\hat{J}(g)(y_-,y_+) = \big(\hat{A}(\hat{v}g)(y_+) - \hat{A}(\hat{v}g)(y_-)\big) - \hat{v}(y_-,y_+)\big(\hat{A}(g)(y_+) - \hat{A}(g)(y_-)\big).$$

Here we have not displayed the dependence of our functions on (x,t) for a compact notation.

We are now ready to state our hypotheses and the second main result.

Hypothesis 1.2(i) The Lagrangian function L is C^2 function that is strictly concave in v. Moreover, there are positive constants c_0, c_1 and c_2 such that

$$\begin{aligned}
-c_0 + c_2 v^2 &\le -L(x,t,v) \le c_0 + c_1 v^2,\\
-c_0 + c_2|v| &\le |L_v(x,t,v)| \le c_0 + c_1|v|,\\
|L_{xv}(x,t,v)| &+ |L_{xx}(x,t,v)| \le c_1,\\
|H_x(x,t,v)| &+ |H_{x\rho}(x,t,\rho)| \le c_1.
\end{aligned}$$

(ii) The rate kernel $g(x,t,y_-,y_+)$ is continuous nonnegative solution of (1.25) which is C^1 in (x,t)-variable, and is supported on

$$\big\{(x,t,y_-,y_+):\ x\in\mathbb{R},\ t\in[t_0,T],\ Y_-\le y_-\le y_+\le Y_+\big\},$$

for some constants $Y_\pm$. We write $g^0(x,y_-,y_+)$ for $g(x,t_0,y_-,y_+)$

(iii) $\rho\mapsto H_\rho(a_-,t,\rho)>0$, for every $t\in[t_0,T]$ and $\rho\in[M_-,M_+]$, where

$$M_+=\sup_{t\in[t_0,T]}M(a_-,t;Y_+,s),\qquad M_-=\inf_{t\in[t_0,T]}M(a_-,t;Y_-,s).$$

(iv) Given s and t_0, with $t_0>s$, the initial condition $\rho^0(x)=M(x,t_0;y^0,s)$ for $x<a_-$, and $\rho(x,t_0)=\rho(x,t_0;\mathbf{y}_{t_0},s)$ for $x\ge a_-$, where $\mathbf{y}_{t_0}$ is a Markov process which starts at $\mathbf{y}_{t_0}(a_-)=y^0>a_-$, and has an infinitesimal generator $\mathcal{B}^1_{x,t_0}$, associated with a kernel $g^0(x,y_-,y_+)=g(x,t_0,y_-,y_+)$.

(v) Assume that $\ell:[0,\infty)\to\mathcal{M}_1$ satisfies $\ell(t_0,dy_0)=\delta_{y^0}(dy_0)$, and

$$\frac{d\ell}{dt}=\mathcal{B}^2_{a_-,t}\ell. \tag{1.26}$$

□

Theorem 1.2 *When Hypothesis 1.2 holds, the entropy solution ρ to* (1.2) *for each fixed $t\in[t_0,T]$ has $x=a_-$ marginal given by $M(a_-,t;y_0,s)$, with y_0 distributed according to $\ell(t,dy_0)$ and for $a_-<x$ evolves as $\rho(x,t)=\rho(x,t;\mathbf{y}_t,s)$, with $\mathbf{y}_t$ a Markov process with the generator $\mathcal{B}^1_{x,t}$.*

Remark 1.3(i) Observe that the finiteness of the Lagrangian L implies that Hamilonian function H cannot be monotone ρ. As a consequence, the velocity $\hat v$ can take both negative and positive values, and the process $t\mapsto\rho(x,t)$ many not be a Markov process for every x. However, when $x=a_-$, our Hypothesis 1.2**(iii)** would guarantee that the process $t\mapsto\rho(a_-,t)$ is Markovian. Indeed Hypothesis 1.2**(iii)** is designed to guarantee that no shock discontinuity can cross a_- from left to right. This assumption though can be relaxed for the price of replacing the boundary line segment $\{(a_-,t):\ t\in[t_0,T]\}$ with a suitable line segment which is titled to the right. In other words, part **(i)** of Remark 1.2 is applicable. Moreover, part **(iii)** of Remark 1.2 is also applicable to the kernel g satisfying (1.25).

(ii) As we will see in Proposition 5.2**(iii)** in Section 5, there exist positive constants C_0 and C_1 such that $M(x,t;y,s)\ge -C_1x$ for $x\le -C_0$. Our condition $|L_v|\le c_1(1+|v|)$ in Hypothesis 1.2**(i)**, means

$$|\rho|\le c_1\big(1+|H_\rho(x,t,\rho)|\big).$$

Since $H_\rho(x,t,\rho)$ is an increasing function of ρ, we deduce that $H_\rho(x,t,\rho) \to \pm\infty$ as $\rho \to \pm\infty$. From this we learn that there exists a positive constant C_2 such that $H_\rho(a_-,t,\rho) > 0$ whenever $\rho \geq C_2$. As a consequence, $H_\rho(a_-,t,\rho) > 0$ for $\rho \in [M(a_-,t;y_-,s), M(a_+,t;y_+,s)]$, provided that $a_- \leq -C_2C_1^{-1}$. This means that Hypothesis 1.2**(iii)** is automatically satisfied when $a_- \leq -C_2C_1^{-1}$.

(iii) As a concrete example, when $H(x,t,\rho) = \rho^2/2$, then $L(x,t,v) = -v^2/2$, and $M(x,t;y,s) = (y-x)/(t-s)$. In this case Hypothesis 1.2**(iii)** holds if and only if $a_- < Y_-$. □

Example 1.1 When H does not depend on (x,t), then

$$W(x,t;y,s) = (t-s)L\left(\frac{x-y}{t-s}\right), \qquad M(x,t;y,s) = L'\left(\frac{x-y}{t-s}\right).$$

Remark 1.4 As an example for a stochastic growth model, we may consider $H(x,t,\rho) = H_0(\rho) - V(x,t)$, with H_0 convex, and the potential V given formally as

$$V(x,t) = \sum_{i\in I} \delta_{s_i}(t)1\!\!1(x = a_i), \tag{1.27}$$

where $\omega = \big\{(a_i,s_i) : \ i \in I\big\}$, is a realization of a *Poisson Point Process* in $\mathbb{R}^2$. In practice, we may approximate V by

$$V_\varepsilon(x,t) = \sum_{i\in I} \varepsilon\zeta\left(\frac{t-s_i}{\varepsilon}\right)\eta\left(\frac{x-a_i}{\delta(\varepsilon)}\right),$$

where $\delta(\varepsilon) \to 0$, in small ε-limit, and η and ζ are two smooth functions of compact support such that $\int \zeta(t)\ dt = 1$, and $\eta(x) = 1$ in a neighborhood of the origin. Replacing V with V_ε yields a Hamiltonian function H^ε for which the equation (1.1) is well-defined and its solution u^ε has a limit u as $\varepsilon \to 0$. A variational representation as in (1.21) for u^ε would yield a variational representation for u as well. Indeed the corresponding W still has the form (1.21), where $L(x,t,v) = L_0(v) - V(x,t)$, with L_0 a concave function given by

$$L_0(v) = \inf_p \big(p \cdot v + H_0(p)\big).$$

It is not hard to show that the minimizing path ξ of the variational problem (1.21) is a concatenation of line segments between Poisson points of ω. In other words,

$$W(x,t;y,s) = W(x,t;y,s;\omega) = \sup\left(N(\mathbf{z}) + \sum_{i=0}^{N(\mathbf{z})}(s_{i+1}-s_i)L_0\left(\frac{a_{i+1}-a_i}{s_{i+1}-s_i}\right)\right), \tag{1.28}$$

where the supremum is over sequences $\mathbf{z} = \big((a_0, s_0), (a_1, s_1), \dots, (a_n, s_n), (a_{n+1}, s_{n+1})\big)$, such that $N(\mathbf{z}) = n$, and

$$s_0 < s_1 < \cdots < s_{n+1}, \quad (a_0, s_0) = (y, s), \quad (a_{n+1}, s_{n+1}) = (x, t), \quad (a_1, s_1), \dots, (a_n, s_n) \in \omega.$$

This model was defined and studied in Bakhtin [B] and Bakhtin et al. [BCK] when $H_0(p) = p^2/2$ (which leads to $L_0(v) = -v^2/2$). If $H_0(p) = |p|$, then $L_0(v) = -\infty\, 1\!\!1(|v| > 1)$. In this case,

$$W(x, t; y, s) = W(x, t; y, s; \omega) = \sup N(\mathbf{z}),$$

where the supremum is over sequences $\mathbf{z}$ as in (1.28), with the additional requirement

$$|a_{i+1} - a_i| \le s_{i+1} - s_i.$$

The corresponding $u(x, t)$ is a stochastic growth model that is known as *Polynuclear Growth* (We refer to [PS] for more details).

Our Theorem 1.2 does not directly apply to this model because Hypothesis 1.2**(i)** fails. Also for Hypothesis 1.2**(ii)** to hold, we need to assume that the intensity of ω is 0 outside $[a_-, \infty) \times \mathbb{R}$. Nonetheless, our method of proof can be adopted to treat this model as well. For this model however, it is more natural to consider a concatenation of fundamental solutions $M(x, t; y_i, \theta_i)$, where $\{(y_i, \theta_i) : \ i \in I\}$ is selected randomly. This extension requires developing new techniques and goes beyond the scope of the present article. □

1.3 Unbounded density

In Theorem 1.1 (respectively 1.2), we assumed that the density ρ (respectively y) is bounded. This assumption is technically convenient for the derivation of the forward equation that is carried out in Section 3, and is at the heart of our proofs of Theorems 1.1 and 1.2. Unfortunately it excludes many important models encountered in statistical mechanics, especially when we study stochastic growth models. As an example, if we take the case of the Burgers equation with white noise initial data, the density at later times would be an unbounded Markov jump process (see [Gr], [MS], and [OR]). In this subsection we explain how one can relax this restriction with the aid of an approximation that is related to Doob's h-transform. We carry out this idea in the case of Theorem 1.2 only, though our method of proof is also applicable to the setting of Theorem 1.1.

Imagine that we have a kernel g which satisfies the kinetic equation (1.25), and the arguments $y_\pm$ are not restricted to a bounded interval as in Hypothesis 1.2**(ii)**.

Hypothesis 1.3 We assume that parts **(i)** and **(iii)**-**(iv)** of Hypothesis 1.2 hold, but in part **(ii)**, we allow $Y_+ = \infty$, and assume that $g(x, t, y_-, y_+)$ is a continuous kernel such that the Markov process $\mathbf{y}_{t_0}$ associated with the generator $\mathcal{B}^1_{x,t_0}$ satisfies

$$\limsup_{x\to\infty} x^{-1} y_{t_0}(x) < 1, \tag{1.29}$$

almost surely. ∎

Theorem 1.3 *The conclusion of Theorem 1.2 holds even when g is a kernel which satisfies Hypothesis 1.3.*

Our strategy for proving Theorem 1.3 is by approximating the kernel g with a sequences of kernels g^n for which Theorem 1.2 is applicable. We cannot simply restrict g to a large bounded interval, because the resulting kernel does not satisfy the kinetic equation. However, if $\mathbf{y}$ is a Markov process with the jump kernel density g (associated with the generator $\mathcal{B}^1_{x,t}$ as in the Definition 1.3**(i)**), we may condition $\mathbf{y}$ to remain in a bounded interval. The resulting process is again a Markov process for which the jump kernel $\hat{g}$ is related to g via a Doob's h-transform. In other words, there exists a suitable function $h(x,t,y)$ such that

$$\hat{g}(x,t,y_-,y_+) = \frac{h(x,t,y_+)}{h(x,t,y_-)} g(x,t,y_-,y_+) =: \eta(x,t,y_-,y_+) g(x,t,y_-,y_+). \tag{1.30}$$

Indeed the resulting kernel is again a solution to the kinetic equation as the following result confirms:

Proposition 1.1 *Assume g satisfies* (1.25) *and $h : [a_-,a_+] \times [t_0,T] \times \mathbb{R} \to \mathbb{R}$ is a C^1 function such that*

$$h_x + \mathcal{B}^1_{x,t} h = 0, \qquad h_t + \mathcal{B}^2_{x,t} h = 0. \tag{1.31}$$

Then $\hat{g}$ given by (1.30) *also satisfies* (1.25).

As we will see in Subsection 1.4 below, the two equations that appeared in (1.31) are compatible whenever g satisfies the equation (1.25). This means that one of these equations is redundant:

Proposition 1.2 *Assume that g and h are bounded, and C^1 in (x,t) variable, and that g satisfies* (1.25)*. Also assume that h is uniformly positive, satisfies $h_x + \mathcal{B}^1_{x,t} h = 0$, and*

$$h_t(a_-,t,y) + (\mathcal{B}^2_{a_-,t} h)(a_-,t,y) = 0. \tag{1.32}$$

(In other words, the second equation in (1.31) *holds at $x = a_-$.) Then the second equation in* (1.31) *holds in $[a_-,a_+]$.*

The proof of Proposition 1.2 is similar to the proof of Proposition 4.1 of [OR], and is omitted.

1.4 Heuristics

According to Theorem 1.2, the process $x_i \mapsto \rho(x_1, x_2)$ is a Markov process with the generator

$$\mathcal{A}^i\psi(\rho) = \mathcal{A}^i_{x_1,x_2}\psi(\rho) = b^i\rho_{x_i} + \int_\rho^\infty f^i(x_1, x_2, \rho, \rho_*)\big(\psi(\rho_*) - \psi(\rho)\big)\ d\rho_*.$$

Hence, if $\ell(x_1, x_2, \rho)$ denotes the probability density of $\rho(x_1, x_2)$, then ρ must satisfy the *forward equation*

$$\ell_{x_i} = \mathcal{A}^{i*}\ell = \ell * f^i - A(f^i)\ell - (b^i\ell)_\rho = \ell * f^i - A(\ell \otimes f^i) - (b^i\ell)_\rho, \qquad i = 1, 2,$$

where

$$(\ell * f^i)(\rho) = \int \ell(\rho_*)f^i(\rho_*, \rho)\ d\rho_*.$$

From differentiating both sides we learn

$$\begin{aligned}
\ell_{x_1x_2} =& \mathcal{A}^{1*}(\ell_{x_2}) + \ell * f^1_{x_2} - A(f^1)_{x_2}\ \ell - (b^1_{x_2}\ell)_\rho = \mathcal{A}^{*1}\mathcal{A}^{2*}\ell + \ell * f^1_{x_2} - A(f^1)_{x_2}\ \ell - (b^1_{x_2}\ell)_\rho,\\
\ell_{x_2x_1} =& \mathcal{A}^{2*}(\ell_{x_1}) + \ell * f^2_{x_1} - A(f^2)_{x_1}\ \ell - (b^2_{x_1}\ell)_\rho = \mathcal{A}^{2*}\mathcal{A}^{1*}\ell + \ell * f^2_{x_1} - A(f^2)_{x_1}\ \ell - (b^2_{x_1}\ell)_\rho.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathcal{A}^{i*}\mathcal{A}^{j*}\ell =& \big(\ell * f^j - A(f^j)\ell - (b^j\ell)_\rho\big) * f^i - A(f^i)\left(\ell * f^j - A(f^j)\ell - (b^j\ell)_\rho\right)\\
&- \left[b^i\left(\ell * f^j - A(f^j)\ell - (b^j\ell)_\rho\right)\right]_\rho\\
=& \ell * \left[f^j * f^i - A(f^j) \otimes f^i - f^j \otimes A(f^i) + b^j \otimes f^i_{\rho_-} - (f^j \otimes b^i)_{\rho_+}\right]\\
&+ \ell\left[A(f^i)A(f^j) + A(f^i)b^j_\rho + (A(f^j)b^i)_\rho\right]\\
&+ \ell_\rho\left[A(f^i)b^j + A(f^j)b^i\right] + \left(b^ib^j_\rho\ell\right)_\rho + \left(b^ib^j\ell_\rho\right)_\rho.
\end{aligned}$$

(Here, we have performed an integration by parts to replace $(b^j\ell)_\rho * f^i$ with $\ell * \left(b^j \otimes f^i_{\rho_-}\right)$.) As a result

$$\begin{aligned}
\mathcal{A}^{1*}\mathcal{A}^{2*}\ell - \mathcal{A}^{2*}\mathcal{A}^{1*}\ell =& \ell * \left[\mathcal{Q}(f^2, f^1) - \mathcal{Q}(f^1, f^2)\right] + \left[\left(b^1b^2_\rho - b^2b^1_\rho\right)\ell\right]_\rho\\
&+ \ell\left[A(f^2)_\rho\ b^1 - A(f^1)_\rho\ b^2\right],
\end{aligned}$$

where $\mathcal{Q}(f^j, f^i)$ is given by (1.16). Hence,

$$\begin{aligned}
\ell_{x_1x_2} - \ell_{x_2x_1} =& \ell * [\mathcal{Q}(f^2, f^1) - \mathcal{Q}(f^1, f^2) + f^1_{x_2} - f^2_{x_1}]\\
&- \ell\left[A(f^1)_{x_2} - A(f^2)_{x_1} + A(f^1)_\rho\ b^2 - A(f^2)_\rho\ b^1\right]\\
&+ \left[\left(b^2_{x_1} - b^1_{x_2} + b^1b^2_\rho - b^2b^1_\rho\right)\ell\right]_\rho\ .
\end{aligned} \tag{1.33}$$

It is rather straightforward to show

$$A\left(\mathcal{Q}(f^j,f^i)\right) = -A(f^j)A(f^i) + b^j A(f^i)_\rho.$$

As a consequence,

$$A\left(R(f^1,f^2)\right) = A(f^1)_{x_2} - A(f^2)_{x_1} + A(f^1)_\rho\, b^2 - A(f^2)_\rho\, b^1. \tag{1.34}$$

where

$$R = R(f^1,f^2) = f^1_{x_2} - f^2_{x_1} + \mathcal{Q}(f^2,f^1) - \mathcal{Q}(f^1,f^2).$$

From (1.33) and (1.34) we deduce

$$\ell_{x_1x_2} - \ell_{x_2x_1} = \ell * R - A(R)\ell + (S\ell)_\rho,$$

where

$$S = S(b^1,b^2) = b^2_{x_1} - b^1_{x_2} + b^1 b^2_\rho - b^2 b^1_\rho.$$

Clearly $R = S = 0$ implies the compatibility of the equations $\ell_{x_i} = \mathcal{A}^{i*}\ell$, for $i = 1,2$. Observe that $R = S = 0$ are exactly our equations (1.6) and (1.7).

Various terms in the kinetic equation can be readily explained in terms of the underlying particle system that represents the dynamics of the shock discontinuities of a solution to the PDE (1.1).

(1) According to the generator (1.1), the process $x \mapsto \rho(x,t)$ satisfies the ODE

$$\rho_x(x,t) = b\big(x,t,\rho(x,t)\big), \tag{1.35}$$

in between shock discontinuities. The PDE (1.6), governing the evolution of the velocity b, follows from the consistency of (1.22) with (1.1); differentiating these equations with respect to t and x respectively lead to

$$\rho_{xt} = b_t + b_\rho\ H(x,t,\rho)_x = b_t + b_\rho\big(H_x + H_\rho b\big) = b^1_{x_2} + b^2 b^1_\rho,$$
$$\rho_{tx} = H(x,t,\rho)_{xx} = \big(H_x + H_\rho b\big)_x = H_{xx} + 2H_{x\rho}b + H_{\rho\rho}b^2 + H_\rho b_x + H_\rho b_\rho b = b^2_{x_1} + b^1 b^2_\rho.$$

Matching these two equations yields (1.6). This calculation is simply a repetition of the derivation of the equation $S(b^1,b^2) = 0$.

(2) If a shock discontinuity occurs at a location $x(t)$ with $\rho_\pm(t) = \rho\big(x(t)\pm,t\big)$, then by the classical Rankine-Hugoniot equation

$$\dot{x}(t) = -v\big(x(t),t,\rho_-(t),\rho_+(t)\big), \tag{1.36}$$

where v was defined in (1.8). This equation is responsible for the occurrence of the term $-(vf)_x$ in (1.7).

(3) Since $\rho(x,t)$ solves (1.1) classically away from the jump discontinuities, we have

$$\begin{aligned}\dot\rho_+(t) &= -\rho_x\big(x(t)+,t\big)v\big(x(t),t,\rho_-(t),\rho_+(t)\big)+\rho_t\big(x(t)+,t\big)\\ &= -b\big(x(t),t,\rho_+(t)\big)v\big(x(t),t,\rho_-(t),\rho_+(t)\big)+(H_x+bH_\rho)(x(t),t,\rho_+(t))\\ &= -K\big(x(t),t,\rho_+(t),\rho_-(t)\big).\end{aligned}\tag{1.37}$$

As in **(2)**, this equation is responsible for the occurrence of $-C^+f$ in (1.7) (see (1.11) for the definition of C^+).

(4) A repetition of our calculation in **(3)** yields

$$\dot\rho_-(t) = -K\big(x(t),t,\rho_-(t),\rho_+(t)\big).\tag{1.38}$$

Based on this, we are tempted to guess that C_-f is $\big[K\big(x,t,\rho_-,\rho_+\big)f\big(x,t,\rho_-,\rho_+\big)\big]_{\rho_-}$. This is not what we have in (1.12). The reason behind this has to do with the fact that we regard $\rho(x,t)$ as a Markov process in x as we increase x. As a result, the role of ρ_- and ρ_+ cannot be interchanged. In order to explain the form of C^-f in (1.12), we fix $a\in\mathbb{R}$, and assume that $x(t)$ is the first discontinuity which occurs to the right of a. Now, if we set $\rho_0(t)=\rho(a,t)$, and write

$$\rho(x)=\phi_a^x(m_0;t),$$

for the flow of the ODE (1.35) (in other words $\rho(x)$ solves (1.35) subject to the initial condition $\rho(a)=m_0$), then

$$\rho_-(t)=\phi_a^{x(t)}\big(\rho_0(t);t\big).$$

Since $\rho_0(t)$ satisfies $\dot\rho_0=\beta(a,t,\rho_0)$, its law $\ell(t,\rho_0)$ obeys the equation

$$\ell_t(t,\rho_0)+\big(\beta(a,t,\rho_0)\ell(t,\rho_0)\big)_{\rho_0}=0,$$

away from the shock discontinuity. As it turns out, the function

$$k(x,t,\rho_0,\rho_+):=\ell(t,\rho_0)f\big(x,t,\phi_a^x(\rho_0;t),\rho_+\big),$$

satisfies the identity

$$k_t-(wk)_x-(\beta k)_{\rho_0}=\ell\big(f_t-(vf)_x-C^-f\big),$$

where

$$w(x,t,\rho_0,\rho_+)=v\big(x,t,\phi_a^x(\rho_0;t),\rho_+\big).$$

(5) Observe that if a solution ρ has two jump discontinuities at $x=x(t)$ and $y=y(t)$, with $x<y$, and

$$\rho_-=\rho(x-,t),\qquad \rho_*=\rho(x+,t),\qquad \rho_*'=\rho(y-,t),\qquad \rho_+=\rho(y+,t),$$

then the relative velocity of these two discontinuities is exactly

$$v\big(x,t,\rho_-,\rho_*\big) - v\big(y,t,\rho_*',\rho_+\big).$$

As $y(t)$ catches up with $x(t)$, ρ_*' converges to ρ_* and the relative velocity becomes

$$v\big(x,t,\rho_-,\rho_*\big) - v\big(x,t,\rho_*,\rho_+\big).$$

This explains the form of Q^+ in (1.9). □

1.5 Bibliography and the outline of the paper

Most of the earlier works on stochastic solutions of Hamilton-Jacobi PDEs have been carried out in the Burgers context. For example, Groeneboom [Gr] determined the statistics of solutions to Burgers equation ($H(p) = p^2/2$, $d = 1$) with white noise initial data. Recently Ouaki [O] has extended this result to arbitrary convex Hamiltonian function H. The special cases of $H(p) = \infty 1\!\!1(p \notin [-1,1])$, and $H(p) = p^+$ were already studied in the references Abramson-Evans [AE], Evans-Ouaki [EO], and Pitman-Tang [PW].

Carraro and Duchon [CD1-2] considered *statistical* solutions, which need not coincide with genuine (entropy) solutions, but realized in this context that Lévy process initial data should interact nicely with Burgers equation. Bertoin [Be] showed this intuition was correct on the level of entropy solutions, arguing in a Lagrangian style.

Developing an alternative treatment to that given by Bertoin, which relies less on particulars of Burgers equation and happens to be more Eulerian, was among the goals of Menon and Srinivasan [MS]. Most notably, [MS] formulates an interesting conjecture for the evolution of the infinitesimal generator of the solution $\rho(\cdot,t)$ which is equivalent with our kinetic equation (1.7) when H is independent of (x,t). When the initial data $\rho(x,0)$ is allowed to assume values only in a fixed, finite set of states, the infinitesimal generators of the processes $x \mapsto \rho(x,t)$ and $t \mapsto \rho(x,t)$ can be represented by triangular matrices. The integrability of this matrix evolution has been investigated by Menon [M2] and Li [Li]. For generic matrices—where the genericity assumptions unfortunately exclude the triangular case—this evolution is completely integrable in the Liouville sense. The full treatment of Menon and Srinivasan's conjecture was achieved in papers [KR1] and [KR2] (we also refer to [R] for an overview). The work of [KR1] have been recently extended to higher dimensions in [OR1]. In [OR2], the main result of [KR2] has been used to give a new proof of Groeneboom's results [Gr].

We continue with an outline of the paper:

(i) In Section 2, we show that the evolution of the PDE (1.1) for piecewise smooth solutions is equivalent to a particle system in $\mathbb{R} \times [P_-, P_+]$. We restrict this particle system to a large finite interval $[a_-, a_+]$ and introduce a stochastic boundary condition at a_+. This restriction allows us to reduce Theorem 1.1 to a finite system; the precise statement can be found in Theorem 2.1 of Section 2.

(ii) The strategy of the proof of Theorem 2.1 will be described in Section 3. Our strategy is similar to the one that was utilized in our previous work [KR1-2]: Since we have a candidate for the generator of the process $x \mapsto \rho(x,t)$, we have a candidate measure, say $\mu(\cdot,t)$ for the law of $\rho(\cdot,t)$. We establish Theorem 2.1 by showing that this candidate measure satisfies the *forward equation* associated with Markovian dynamics of the underlying particle system (see the equation (3.2) in Section 3). The particle system has a deterministic evolution inside the interval and a stochastic (Markovian) dynamics at the right end boundary point. The rigorous derivation of the forward equation will be carried out in Section 3.

(iii) Section 4 is devoted to the proof of Theorem 2.1.

(iv) Section 5 is devoted to the proof of Theorem 1.2.

(v) In Section 6, we establish Theorem 1.3 and Proposition 1.1. □

2 Particle System

We assume that the initial condition ρ^0, in the PDE (1.21) is of the following form

- $\rho^0(x) = m_0$ for $x \le x_0 = a_-$.
- There exists a discrete set $I^0 = \{x_i : i \in \mathbb{N}\}$, with $a_- < x_1 < \dots < x_i < \dots$ such that for every $x > 0$ with $x \notin I^0$, we have $\rho^0_x(x) = b^0(x, t_0, \rho^0(x))$.
- If $\rho_i^\pm = \rho^0(x_i \pm)$ denote the right and left values of ρ^0 at x_i, then $\rho_i^- < \rho_i^+$.

Now if ρ is an entropic solution of (1.2) with initial ρ^0, then we may apply the *method of characteristics* to show that for each $t \ge 0$, the function $\rho(\cdot,t)$ has a similar form. To explain this, consider the ODE

$$\frac{d}{dx}\rho(x) = b(x,t,\rho(x)), \tag{2.1}$$

where b is the solution to (1.6), subject to the initial condition $b(x,t_0,\rho) = b^0(x,\rho)$. Recall that we are write $\phi_a^z(m;t)$ for the flow of the ODE (2.1). In other words, if $\rho(x) = \phi_a^x(m;t)$, then (2.1) holds, and $\rho(a) = m$. Then there are pairs $\mathbf{q}(t) = \big((x_i(t), \rho_i(t)) : \ i = 0, 1, \dots\big)$, with

$$a_- = x_0(t) < x_1(t) < \dots < x_i(t) < \dots,$$

such that for $x \ge a_-$, we can write

$$\rho\big(x,t\big) = \sum_{i=0}^{\infty} \phi^x_{x_i(t)}\big(\rho_i(t);t\big)\,\mathbb{1}\big(x_i(t) \le x < x_{i+1}(t)\big). \tag{2.2}$$

Note that $\rho(x_i(t)+,t) = \rho_i(t)$, and the data $\mathbf{q}(t)$ determines $\rho(\cdot,t)$ completely. Because of this, we can fully describe the evolution of $\rho(\cdot,t)$ by describing the evolution of the particle system $\mathbf{q}(t)$. Indeed from the PDE (1.1) and the Rankine-Hugoniot Formula, we have $\dot\rho_0(t) = \beta(a,t,\rho_0(t))$, $\rho_0(t_0) = m_0$, and

$$\dot x_i(t) = -v(x_i(t),t,\hat\rho_{i-1}(t),\rho_i(t)), \qquad \dot\rho_i(t) = -K(x_i(t),t,\rho_i(t),\hat\rho_{i-1}(t)), \tag{2.3}$$

for $i \in \mathbb{N}$, where $\hat\rho_{i-1}(t) = \phi^{x_i(t)}_{x_{i-1}(t)}(\rho_{i-1}(t),t)$ (we refer to Subsection 1.4, especially (1.36) and (1.37) for explanation). Here (2.3) gives a complete description of $\mathbf{q}$ in an inductive fashion; once (x_{i-1},ρ_{i-1}) is determined, then we use (2.3) to write a system of two equations for the pair (x_i,ρ_i). Moreover (2.3) holds so long as x_i's do not collide. When there is a collision between x_i and x_{i+1}, for some $i = 0,1,\dots$, we remove x_{i+1} from the system, replace ρ_i with ρ_{i+1}, and relabel (x_j,ρ_j) as (x_{j-1},ρ_{j-1}) for $j > i+1$. As we will see shortly, the function $\rho(x,t)$, defined by equation (2.2), with $\mathbf{q}(t)$ evolving as above, is the unique entropy solution of (1.2).

According to Theorem 1.1 if $\rho(\cdot,t_0)$ is a PDMP with drift b^0 and jump rate f^0, then $\rho(\cdot,t)$ is also a PDMP with drift $b(x,t,\cdot)$ and $f(x,t,\cdot,\cdot)$. We may translate this into a statement about the law of our particle system $\mathbf{q}(t)$. However, since the dynamics of $\mathbf{q}$ involves infinite number of particles, we may take advantage of the finiteness of propagation speed in (1.2) and reduce Theorem 1.1 to an analogous claim for a finite interval $[a_-,a_+]$.

Since $H_\rho > 0$ by Hypothesis 1.1**(ii)**, all particles travel to left. Because of this, we need to choose appropriate boundary dynamics at the right boundary a_+ only. The involved analysis will all pertain to the following result.

Theorem 2.1 *Assume Hypothesis 1.1. For any fixed $a_+ > a_-$, consider the scalar conservation law (1.2) in $[a_-,a_+] \times [t_0,T)$ with initial condition $\rho(x,t_0) = \rho^0(x)$ (restricted to $[a_-,a_+]$), open boundary at $x = a_-$, and random boundary ζ at $x = a_+$. Suppose the process ζ has initial condition $\zeta(t_0) = \rho^0(a_+)$, and evolves according to the time-dependent rate kernel $f^2(a_+,t,\rho,\rho_+)$ and drift $b^2(a_+,t,\rho)$, independently of ρ^0. Then for all $t > t_0$ the law of $\big(\rho(x,t) : x \in [a_-,a_+]\big)$ is as follows:*

- **(i)** *The $x = a_-$ marginal is $\ell(t,d\rho_0)$, given by $\dot\ell = \mathcal{A}^{2*}_{a_-,t}\ell$.*
- **(ii)** *The rest of the path is a PDMP with generator $\mathcal{A}^1_{x,t}$ (rate kernel $f(x,t,\rho_-,\rho_+)$ and drift $b(x,t,\rho)$).*

To prove our main result Theorem 1.2, we can send $a_+ \to \infty$, applying Theorem 2.1 on each $[a_-,a_+]$, and use bounded speed of propagation. The argument is straightforward and can be found in [KR1].

We prove Theorem 2.1 by showing that the particle system $\mathbf{q}(t)$ restricted to the interval $[a_-,a_+]$ has the correct law predicted by this theorem. We now give a precise description for the evolution of $\mathbf{q}$ restricted to $[a_-,a_+]$. First we make some definitions.

Definition 2.1(i) The configuration space for our particle system $\mathbf{q}$, is the set $\Delta = \cup_{n=0}^{\infty} \bar{\Delta}_n$, where $\bar{\Delta}_n$ is the topological closure of Δ_n, with Δ_n denoting the set

$$\big\{\mathbf{q} = \big((x_i, \rho_i) : i = 0, 1, \dots, n\big) : \ x_0 = a_- < x_1 < \dots < x_n < x_{n+1} = a_+, \quad \rho_0, \dots, \rho_n \in \mathbb{R}\big\}.$$

We write $\mathbf{n}(\mathbf{q})$ for the number of particles i.e., $\mathbf{n}(\mathbf{q}) = n$ means that $\mathbf{q} \in \Delta_n$. What we have in mind is that $\rho_i(t) = \rho(x_i(t)+, t)$ with $x_1, \dots, x_n$ denoting the locations of all shocks in (a_-, a_+).

(ii) Given a realization $\mathbf{q} = \big(x_0, \rho_0, x_1, \rho_1, \dots, x_n, \rho_n\big) \in \bar{\Delta}_n$, we define

$$\rho\big(x, t; \mathbf{q}\big) = R_t(\mathbf{q})(x) = \sum_{i=0}^{n} \phi_{x_{i-1}}^{x_i}\big(\rho_i; t\big) \mathbb{1}\big(x_i \le x < x_{i+1}\big).$$

(iii) The process $\mathbf{q}(t)$ evolves according to the following rules:

(1) So long as x_i remains in (x_{i-1}, x_{i+1}), for some $i \ge 1$, it satisfies $\dot{x}_i = -v(x_i, t, \hat{\rho}_{i-1}, \rho_i)$ with $\hat{\rho}_i(t) = \phi_{x_{i-1}(t)}^{x_i(t)}\big(\rho_{i-1}(t); t\big)$.

(2) We have $\dot{\rho}_0 = \beta(x_0, t, \rho_0)$, and for $i > 0$, we have $\dot{\rho}_i = -K(x_i, t, \rho_i, \hat{\rho}_{i-1})$.

(3) With rate $f^2\big(a_+, t, \hat{\rho}_n, \rho_{n+1}\big)$, the configuration $\mathbf{q}$ gains a new particle (x_{n+1}, ρ_{n+1}), with $x_{n+1} = a_+$. This new configuration is denoted by $\mathbf{q}(\rho_{n+1})$.

(4) When x_1 reaches a_-, we relabel particles (x_i, ρ_i), $i \ge 1$, as (x_{i-1}, ρ_{i-1}).

(5) When $x_{i+1} - x_i$ becomes 0 for some $i \ge 1$, then $\mathbf{q}(t)$ becomes $\mathbf{q}^i(t)$, that is obtained from $\mathbf{q}(t)$ by omitting (ρ_i, x_i) and relabeling particles to the right of the i-th particle. □

As we mentioned before, the function $\rho(x, t; \mathbf{q}(t))$ is indeed an entropic solution of (1.2). We also need a stability inequality for our constructed solutions.

Proposition 2.1 **(i)** *The function $\rho(x, t) = \rho(x, t; \mathbf{q}(t))$, with $\mathbf{q}(t)$ evolving as above, is an entropy solution of $\rho_t = H(x, t, \rho)_x$ in $(a_-, a_+) \times (t_0, T)$.*

(ii) *The process $t \mapsto m(t) := \rho(a_+, t) = \rho(a_+, t; \mathbf{q}(t))$ is a Markov process with generator $\mathcal{A}^2_{a_+,t}$.*

(iii) *Suppose $\rho, \rho' : [a_-, a_+] \times [t_0, T) \to [P_-, P_+]$ are two piecewise C^1 entropy solutions of $\rho_t = H(x, t, \rho)_x$. If $t_0 \le s \le t < T$, then*

$$\begin{aligned} \int_{a_-}^{a_+} |\rho'(x,t) - \rho(x,t)| \, dx \le & e^{C_0(t-s)} \int_{a_-}^{a^+} |\rho'(x,s) - \rho(x,s)| \, dx \\ & + e^{C_0(t-s)} \int_s^t \big|H(a_+, \theta, \rho'(a_+, \theta)) - H(a_+, \theta, \rho(a_+, \theta))\big| \, d\theta, \end{aligned} \tag{2.4}$$

where

$$C_0 = \max_{x\in[a_-,a_+]} \max_{t\in[t_0,T]} \max_{\rho\in[P_-,P_+]} |H_{x\rho}(x,t,\rho)|.$$

Remark 2.1 From Proposition 2.1**(iii)** we learn that two solutions ρ and ρ' equal in $[a_-,a_+]\times[t_0,T]$ if they coincide at $t=0$, and $x=a_+$. This confirms the fact that under the assumption $H_\rho<0$, the boundary a_- is free. In particular, $\rho(x,t;\mathbf{q}(t))$ is the unique solution which satisfies the stochastic boundary condition at $x=a_+$. □

The proof of Proposition 2.1 will be given at the end of this section. We continue with a precise description for the PDMP $\rho(\cdot,t)$ in terms of $\mathbf{q}(t)$ and some preparatory steps toward the proof of Proposition 2.1 and Theorem 2.1.

Definition 2.2(i) To ease the notation, we write $\lambda(x,t,\rho)$ and $A(x,t,\rho)$, for $(Af^1)(x,t,\rho)$ and $(Af^2)(x,t,\rho)$ respectively. Given $\mathbf{q}\in\Delta_n$, we also set

$$\Gamma(x,y,t,\rho) = \int_x^y \lambda(z,t,\phi_x^z(\rho;t))\,dz,$$
$$\Gamma(\mathbf{q},t) = \int_{a_-}^{a_+} \lambda\big(y,t,\rho\big(y,t;\mathbf{q}\big)\big)\,dy = \sum_{i=0}^{n} \Gamma(x_i,x_{i+1},t,\rho_i).$$

(ii) We define a measure $\mu(d\mathbf{q},t)$ on the set Δ that is our candidate for the law of $\mathbf{q}(t)$. The restriction of μ to Δ_n is denoted by $\mu^n(d\mathbf{q},t)$. This measure is explicitly given by

$$\ell(t,d\rho_0)\exp\{-\Gamma(\mathbf{q},t)\}\prod_{i=1}^{n} f\big(x_i,t,\phi_{x_{i-1}}^{x_i}(\rho_{i-1};t),\rho_i\big)\,dx_i d\rho_i,$$

where f solves (1.7) and ℓ solves (1.18). Note that if $\rho(x,t)=R_t(\mathbf{q}(t))(x)$, with R as in Definition **2.1(ii)**, then the process $x\mapsto\rho(x,t), x\geq a_-$ is a Markov process associated with the generator $\mathcal{A}^1_{x,t}$, and an initial law $\ell(t,\cdot)$.

(iii) Let us write $T_x^y g(\rho)=g(\phi_x^y(\rho;t))$ and $(\mathcal{D}_x g)(\rho)=b(x,t,\rho)g'(\rho)$ for its generator (to simplify the notation, we do not display the dependence of T_x^y and $\mathcal{D}_x$ on t). It is straightforward to show

$$T_x^y\circ T_y^z = T_x^z, \qquad \frac{dT_x^y}{dy} = T_x^y\circ\mathcal{D}_y, \qquad \frac{dT_x^y}{dx} = -\mathcal{D}_x\circ T_x^y. \tag{2.5}$$

Indeed

$$T_x^{y+\delta}g = T_x^y\big(g\circ\phi_y^{y+\delta}\big) = T_x^y\big(g+\delta\mathcal{D}_y g+o(\delta)\big) = T_x^y g+\delta\big(T_x^y\circ\mathcal{D}_y\big)g+o(\delta),$$
$$T_{x-\delta}^y g = T_{x-\delta}^x\big(T_x^y g\big) = \big(T_x^y g\big)\circ\phi_{x-\delta}^x = T_x^y g-\delta\big(\mathcal{D}_x\circ T_x^y\big)g+o(\delta).$$

□

In the following Lemma, we derive several identities that we will use for the proof of Proposition 2.1 and Theorem 2.1.

Lemma 2.1 *The following identities are true:*

$$b(x,t,\rho)\Gamma_\rho(x,y,t,\rho) = -\Gamma_x(x,y,t,\rho) + \lambda(x,t,\rho) = -\int_x^y \big[\lambda\big(z,t,\phi_x^z(\rho;t)\big)\big]_x \, dz, \tag{2.6}$$

$$b_t = \beta_x + b\beta_\rho - b_\rho\beta, \tag{2.7}$$

$$\big[\phi_x^y(\rho;t)\big]_t = \beta(y,t,\phi_x^y(\rho;t)) - \beta\big(x,t,\rho\big)\big[\phi_x^y(\rho;t)\big]_\rho, \tag{2.8}$$

$$\lambda_t(x,t,\rho) + \beta(x,t,\rho)\lambda_\rho(x,t,\rho) = b(x,t,\rho)A_\rho(x,t,\rho) + A_x(x,t,\rho), \tag{2.9}$$

$$\Gamma_t(x,y,t,\rho) + \beta(x,t,\rho)\Gamma_\rho(x,y,t,\rho) = A\big(y,t,\phi_x^y(\rho;t)\big) - A(x,t,\rho), \tag{2.10}$$

$$\big[\phi_x^y(\rho;t)\big]_x + b(x,t,\rho)\big[\phi_x^y(\rho;t)\big]_\rho = 0.. \tag{2.11}$$

Proof For the proof of (2.6) use the definition of Γ and (2.5) to assert that the left-hand side of (2.6) equals to

$$\begin{aligned}\int_x^y b(x,t,\rho)\left[\lambda\big(z,t,\phi_x^z(\rho;t)\big)\right]_\rho \, dz &= -\int_x^y \left[\lambda\big(z,t,\phi_x^z(\rho;t)\big)\right]_x dz \\ &= -\left[\int_x^y \lambda\big(z,t,\phi_x^z(\rho;t)\big)\, dz\right]_x + \lambda(x,t,\rho) \\ &= -\Gamma_x(x,y,t,\rho) + \lambda(x,t,\rho).\end{aligned}$$

For (2.7) observe that by (1.6),

$$\begin{aligned}\beta_x + b\beta_\rho - b_\rho\beta &= \big(bH_\rho + H_x\big)_x + b\big(bH_\rho + H_x\big)_\rho - bb_\rho H_\rho - b_\rho H_x \\ &= bH_{\rho x} + b_x H_\rho + H_{xx} + bb_\rho H_\rho + b^2 H_{\rho\rho} + bH_{\rho x} - bb_\rho H_\rho - b_\rho H_x \\ &= 2bH_{\rho x} + b_x H_\rho + H_{xx} + b^2 H_{\rho\rho} - b_\rho H_x = b_t.\end{aligned}$$

We now turn to the proof of (2.8). Set

$$X(x,y,t,\rho) := \big[\phi_x^y(\rho;t)\big]_t - \beta(y,t,\phi_x^y(\rho;t)) + \beta\big(x,t,\rho\big)\big[\phi_x^y(\rho;t)\big]_\rho.$$

We wish to show that $X(\rho,x,y,t) = 0$ for all (x,y,t,ρ). This is trivially true when $x = y$. On the other hand,

$$\begin{aligned}X_y(x,y,t,\rho) =& \big[b(y,t,\phi_x^y(\rho;t))\big]_t - (\beta_y + b\beta_\rho)(y,t,\phi_x^y(\rho;t)) + \beta\big(x,t,\rho\big)\big[b(y,t,\phi_x^y(\rho;t))\big]_\rho \\ =& b_t(y,t,\phi_x^y(\rho;t)) + b_\rho(y,t,\phi_x^y(\rho;t))\big[\phi_x^y(\rho;t)\big]_t - (\beta_y + b\beta_\rho)(y,t,\phi_x^y(\rho;t)) \\ &+ \beta\big(x,t,\rho)b_\rho(y,t,\phi_x^y(\rho;t))\big[\phi_x^y(\rho;t)\big]_\rho \\ =& b_\rho(y,t,\phi_x^y(\rho;t))X(x,y,t,\rho),\end{aligned}$$

where we used (2.7) for the third equality. As a result.

$$X(x,y,t,\rho) = X(x,x,t,\rho)\exp\left[\int_x^y b_\rho\big(z,t,\phi_x^z(\rho;t)\big)\ dz\right] = 0.$$

This completes the proof of (2.8).

For (2.9), we first observe

$$\int_\rho^\infty Q(f)(x,t,\rho,\rho_+)\ d\rho_+ = 0, \tag{2.12}$$

because the left-hand side equals

$$\begin{aligned}
&\int_\rho^\infty \int \mathbb{1}\big(\rho_* \in (\rho,\rho_+)\big)\big(v(x,t,\rho_*,\rho_+) - v(x,t,\rho,\rho_*)\big)f(x,t,\rho,\rho_*)f(x,t,\rho_*,\rho_+)\ d\rho_* d\rho_+ \\
&\quad - \int_\rho^\infty \left[A(x,t,\rho_+) - A(x,t,\rho) - v(x,t,\rho,\rho_+)\big(\lambda(x,t,\rho_+) - \lambda(x,t,\rho)\big)\right] f(x,t,\rho,\rho_+)\ d\rho_+ \\
&= \int_\rho^\infty \big(A(x,t,\rho_*) - v(x,t,\rho,\rho_*)\lambda(x,t,\rho_*)\big)f(x,t,\rho,\rho_*)\ d\rho_* \\
&\quad - \int_\rho^\infty \left[A(x,t,\rho_+) - A(x,t,\rho) - v(x,t,\rho,\rho_+)\big(\lambda(x,t,\rho_+) - \lambda(x,t,\rho)\big)\right] f(x,t,\rho,\rho_+)\ d\rho_+ \\
&= \int_\rho^\infty \big(A(x,t,\rho) - v(x,t,\rho,\rho_+)\lambda(x,t,\rho)\big)f(x,t,\rho,\rho_+)\ d\rho_+ = 0.
\end{aligned}$$

We next integrate both sides of (1.7) with respect to ρ_+ and use (2.12) to assert

$$\begin{aligned}
\lambda_t(x,t,\rho) &= \int_\rho^\infty \left\{(Cf)(x,t,\rho,\rho_+) + \left[(vf)(x,t,\rho,\rho_+)\right]_x\right\}\ d\rho_+ \\
&= \int_\rho^\infty (Cf)(x,t,\rho,\rho_+)\ d\rho_+ + A_x(x,t,\rho).
\end{aligned} \tag{2.13}$$

On the other hand,

$$\begin{aligned}
\int \big(C^- f\big)(x,t,\rho,\rho_+)\ d\rho_+ =& b(x,t,\rho)\int_\rho^\infty (vf)_\rho(x,t,\rho,\rho_+)\ d\rho_+ - \beta(x,t,\rho)\int_\rho^\infty f_\rho(x,t,\rho,\rho_+)\ d\rho_+ \\
=& (bA_\rho - \beta\lambda_\rho)(x,t,\rho) + (bH_\rho - \beta)(x,t,\rho)f(x,t,\rho,\rho), \\
\int \big(C^+ f\big)(x,t,\rho,\rho_+)\ d\rho_+ =& \int_\rho^\infty \left[K(x,t,\rho_+,\rho)f(x,t,\rho,\rho_+)\right]_{\rho_+}\ d\rho_+ \\
=& (\beta - bH_\rho)(x,t,\rho)f(x,t,\rho,\rho),
\end{aligned}$$

because $f(x,t,\rho,\infty)=0$, and

$$\lim_{\rho_+\to\rho} v(x,t,\rho,\rho_+) = H_\rho(x,t,\rho).$$

From this, and (2.13) we deduce (2.9).

We now turn to the proof of (2.10). We rewrite (2.10) as

$$\int_x^y \left[\lambda(z,t,\phi_x^z(\rho;t))\right]_t\, dz + \beta(x,t,\rho)\int_x^y \left[\lambda(z,t,\phi_x^z(\rho;t))\right]_\rho\, dz = \int_x^y \left[A\big(z,t,\phi_x^z(\rho;t)\big)\right]_z\, dz.$$

For this, it suffices to check

$$\left[\lambda(z,t,\phi_x^z(\rho;t))\right]_t + \beta(x,t,\rho)\ \left[\lambda(z,t,\phi_x^z(\rho;t))\right]_\rho = \left[A\big(z,t,\phi_x^z(\rho;t)\big)\right]_z, \tag{2.14}$$

for every (x,z,t,ρ). By (2.8), the identity (2.14) is equivalent to

$$\lambda_t(z,t,\phi_x^z(\rho;t)) + \beta(z,t,\phi_x^z(\rho;t))\lambda_\rho(z,t,\phi_x^z(\rho;t)) = \left[A\big(z,t,\phi_x^z(\rho;t)\big)\right]_z.$$

This is an immediate consequence of (2.9) because

$$\left[A\big(z,t,\phi_x^z(\rho;t)\big)\right]_z = b(z,t,\phi_x^z(\rho;t))A_\rho\big(z,t,\phi_x^z(\rho;t)\big) + A_z\big(z,t,\phi_x^z(\rho;t)\big).$$

The proof of (2.10) is complete.

Finally, (2.11) is simply the third equation of (2.5) applied to the function $g(\rho)=\rho$.

□

We are now ready to establish Proposition 2.1.

Proof of Proposition 2.1(i) We first show that ρ solves (1.2) classically away from the shock curves. For this, take a point (x,t) such that $x\in\big(x_i(t),x_{i+1}(t)\big)$, for some nonnegative integer i. Let us write $\hat\phi_x^y(\rho;t)$ and $\tilde\phi_x^y(\rho;t)$ for the partial derivatives $\left[\phi_x^y(\rho;t)\right]_\rho$ and $\left[\phi_x^y(\rho;t)\right]_x$ respectively. From $\rho(x,t)=\phi_{x_i(t)}^x(\rho_i(t);t)$, we learn

$$\begin{aligned}
\rho_t(x,t) =& -v\big(x_i(t),t,\hat\rho_{i-1}(t),\rho_i(t)\big)\tilde\phi_{x_i(t)}^x(\rho_i(t);t)\\
&+\beta(x,t,\rho(x,t)) - \beta\big(x_i(t),t,\rho_i(t)\big)\ \hat\phi_{x_i(t)}^x(\rho_i(t);t)\\
&-K\big(x_i(t),t,\rho_i(t),\hat\rho_{i-1}(t)\big)\hat\phi_{x_i(t)}^x(\rho_i(t);t)\\
=& -v\big(x_i(t),t,\hat\rho_{i-1}(t),\rho_i(t)\big)\tilde\phi_{x_i(t)}^x(\rho_i(t);t) + \beta(x,t,\rho(x,t))\\
&-v\big(x_i(t),t,\hat\rho_{i-1}(t),\rho_i(t)\big)b\big(x_i(t),t,\rho_i(t)\big)\hat\phi_{x_i(t)}^x(\rho_i(t);t)\\
=&\beta(x,t,\rho(x,t)) = H(x,t,\rho(x,t))_x,
\end{aligned} \tag{2.15}$$

as desired. Here we used (2.8) for the first equality, and (2.11) for the third equality. Since the Rankine-Hugoniot Formula is valid at shock curves by our construction, and (1.1) holds classically away from the shock curves, we deduce that ρ is a weak solution of (1.1). On the other hand, since $\rho(x_i(t)-,t) < \rho(x_i(t)+,t)$ by construction, we deduce that ρ is an entropy solution in $(a_-,a_+)\times(t_0,T)$.

(ii) From the way the boundary dynamics is described in **(3)**, the process $m(t)$ depends on the particle system to the left of a_+. Nonetheless we show that if the process $\bar m(t)$ is a Markov process with generator $\mathcal{A}^2_{a_+,t}$, and initial state $m_0=\rho^0(a_+)$, then $m(t)=\bar m(t)$. To verify this, let us construct the process $t\mapsto \bar m(t)$ with the aid of a sequence of independent standard exponential random variables $\big(\tau_i:\ i\in\mathbb{N}\big)$. Let us write $\gamma_s^t(\rho)$ for the flow of the ODE associated with speed $\beta(a_+,t,\rho)$, and define

$$\eta(t,m)=\int_m^\infty f^2(a_+,t,m,\rho_+)\ d\rho_+. \tag{2.16}$$

Now construct a sequence $\mathbf{z}=\big((\sigma_i,m_i):\ i=0,1,\dots\big)$ inductively by the following recipe: $\sigma_0=0$, and given (σ_i,m_i), we set

$$\sigma_{i+1}=\min\left\{s>\sigma_i:\ \int_{\sigma_i}^s \eta\big(\theta,\gamma_{\sigma_i}^\theta(m_i)\big)\ d\theta\ge\tau_{i+1}\right\},\qquad \hat m_i=\gamma_{\sigma_i}^{\sigma_{i+1}}(m_i),$$

and select m_{i+1} randomly according to the probability measure

$$\eta\big(\sigma_{i+1},\hat m_i\big)^{-1}\ f^2\big(a_+,\sigma_{i+1},\hat m_i,m_{i+1}\big)\ dm_{i+1}.$$

Using our sequence $\mathbf{z}$, we construct $\bar m(t)$ by

$$\bar m(t)=\sum_{i=0}^\infty \gamma_{\sigma_i}^t(m_i)1\!\!1\big(t\in[\sigma_i,\sigma_{i+1})\big).$$

By the very construction of the processes $m(t)$ and $\bar m(t)$, the desired equality $m(t)=\bar m(t),\ t\ge t_0$ would follow if we can show that $m(t)=\bar m(t)$ for $t\in(\sigma_{i-1},\sigma_i)$ for every $i\in\mathbb{N}$. This can be checked by induction on i. If there are exactly n particle to the left of a_+, and we already know that $\hat\rho_n(\sigma_i)=\hat m_{i-1}$, then we can guarantee that $\rho_{n+1}(\sigma_i)=m_i$. Moreover, $\hat\rho_{n+1}(t)=\gamma_{\sigma_i}^t(m_i)$ for $t\in(\sigma_i,\sigma_{i+1})$, because the function $\zeta(t)=\phi_{x_n(t)}^{a_+}(\rho_n(t);t)$ satisfies

$$\dot\zeta(t)=\beta(a_+,t,\zeta(t)), \tag{2.17}$$

by (2.15) in the case of $x=a_+$. This completes the proof of part **(ii)**.

(iii) The proof of (2.4) is a standard application of the celebrated Kruzhkov's inequality [K], and we only sketch it. It is not hard to show that the piecewise C^1 entropy solutions ρ and

ρ' can be extended to entropy solutions that are defined on a larger domain $(b_-, b_+) \times [t_0, T)$, with $b_- < a_- < a_+ < b_+$. With a slight abuse of notation, we write ρ and ρ' for these extensions. Given an arbitrary constant c, the following Kruzkov's entropy inequalities hold weakly in $(b_-, b_+) \times [t_0, T)$:

$$|\rho(x,t) - c|_t \le |H(x,t,\rho(x,t)) - H(x,t,c)|_x + \ sgn\big(\rho(x,t) - c\big) H_x(x,t,c),$$
$$|\rho'(x,t) - c|_t \le |H(x,t,\rho'(x,t)) - H(x,t,c)|_x + \ sgn\big(\rho'(x,t) - c\big) H_x(x,t,c).$$

This allows us to use Kruzkov's standard arguments as in [K] to deduce

$$\begin{aligned} |\rho(x,t) - \rho'(x,t)|_t \le & |H(x,t,\rho(x,t)) - H(x,t,\rho'(x,t))|_x \\ & - \ sgn\big(\rho(x,t) - \rho'(x,t)\big)\big(H_x(x,t,\rho(x,t)) - H_x(x,t,\rho'(x,t))\big) \\ \le & |H(x,t,\rho(x,t)) - H(x,t,\rho'(x,t))|_x + C_0|\rho(x,t) - \rho'(x,t)|, \end{aligned}$$

weakly in $(b_-, b_+) \times [t_0, T)$. From this, we can readily deduce

$$\big[e^{-C_0 t}|\rho(x,t) - \rho'(x,t)|\big]_t \le e^{-C_0 t}|H(x,t,\rho(x,t)) - H(x,t,\rho'(x,t))|_x, \tag{2.18}$$

weakly in $(b_-, b_+) \times [0, T)$. We wish to integrate both sides of (2.18) with respect to x from a_- to a_+. To perform such integration rigorously, we take a smooth function γ of compact support with $\int \gamma \ dx = 1$, rescale it as $\gamma_\varepsilon(x) = \varepsilon^{-1}\gamma(x/\varepsilon)$, and choose $\tau_\varepsilon(x)$ so that $\tau_\varepsilon \ge 0$, $\tau'_\varepsilon(x) = \gamma_\varepsilon(x - a_-) - \gamma_\varepsilon(x - a_+)$, and $\tau_\varepsilon(b_-) = 0$. For small ε, the function τ_ε is supported in (b_-, b_+). We can now integrate both sides of (2.18) against τ_ε to deduce that weakly

$$\left[e^{-C_0 t}\int |\rho(x,t) - \rho'(x,t)|\tau_\varepsilon(x) \ dx\right]_t \le -\int e^{-C_0 t}|H(x,t,\rho(x,t)) - H(x,t,\rho'(x,t))| \ \tau'_\varepsilon(x) \ dx.$$

We can now send ε to 0 to arrive at

$$\begin{aligned} \left[e^{-C_0 t}\int_{a_-}^{a_+} |\rho(x,t) - \rho'(x,t)| \ dx\right]_t \le & e^{-C_0 t}|H(a_+,t,\rho(a_+,t)) - H(a_+,t,\rho'(a_+,t))| \\ & - e^{-C_0 t}|H(a_-,t,\rho(a_-,t)) - H(a_-,t,\rho'(a_-,t))|. \end{aligned}$$

Integrating both sides over the time interval $[s,t]$ yields

$$\begin{aligned}
e^{-C_0 t}\int_{a_-}^{a_+} |\rho(x,t)-\rho'(x,t)|\ dx \le & e^{-C_0 s}\int_{a_-}^{a_+} |\rho(x,s)-\rho'(x,s)|\ dx \\
& + \int_s^t e^{-C_0\theta}|H(a_+,\theta,\rho(a_+,\theta)) - H(a_+,\theta,\rho'(a_+,\theta))|\ d\theta \\
& - \int_s^t e^{-C_0\theta}|H(a_-,\theta,\rho(a_-,\theta)) - H(a_-,\theta,\rho'(a_-,\theta))|\ d\theta \\
\le & e^{-C_0 s}\int_{a_-}^{a_+} |\rho(x,s)-\rho'(x,s)|\ dx \\
& + \int_s^t e^{-C_0\theta}|H(x,\theta,\rho(a_+,\theta)) - H(x,\theta,\rho'(a_+,\theta))|\ d\theta.
\end{aligned}$$

This evidently implies (2.4). □

3 Forward Equation

As a preliminary step for establishing Theorem 2.1, we derive a Kolmogorov type forward equation for the measure $\mu(d\mathbf{q},t)$. We first introduce some notation for the particle dynamics.

Definition 3.1(i) For $0 \le s \le t$ and $\mathbf{q} \in \Delta$, we write $\psi_s^t\mathbf{q}$ for the deterministic evolution from time s to t of the configuration $\mathbf{q}$ according to the annihilating particle dynamics of Definition 2.1**(iii)**, *without* random entry dynamics at $x = a_+$.

(ii) Given a configuration $\mathbf{q} = \big((x_0,\rho_0),\dots,(x_n,\rho_n)\big)$ and $\rho_+ \in \mathbb{R}$, write $\epsilon_{\rho_+}\mathbf{q}$ for the configuration $\big((x_0,\rho_0),\dots,(x_n,\rho_n),(a_+,\rho_+)\big)$.

(iii) Write $\Psi_s^t\mathbf{q}$ for the *random* evolution of the configuration according to deterministic particle dynamics interrupted with random entries at $x = a_+$ according to the boundary process as in **(3)** in Definition 2.1**(iii)**, where the latter has been started at time s with value $\phi_{x_n}^{a_+}(\rho_n; s)$. In particular, if the jumps between times s and t occur at times $\tau_1 < \dots < \tau_k$ with values $m_1, \cdots, m_k$, then

$$\Psi_s^t\mathbf{q} = \psi_{\tau_k}^t\epsilon_{m_k}\psi_{\tau_{k-1}}^{\tau_k}\epsilon_{m_{k-1}}\cdots\psi_{\tau_1}^{\tau_2}\epsilon_{m_1}\psi_s^{\tau_1}\mathbf{q}. \tag{3.1}$$

(iv) For $n \ge 1$, and $i \in \{0,\dots,n-1\}$, we write $\partial_i\Delta_n$ for the portion of the boundary Δ_n such that $x_i = x_{i+1}$. Note that $\mathbf{q}(t)$ reaches the boundary set $\partial_0\Delta_n$ at time τ if at this time $x_1(\tau) = a_-$. For time t immediately after τ, the configuration $\mathbf{q}(t)$ belongs to Δ_{n-1} with ρ_0 taking new value. Similarly $\mathbf{q}(t)$ reaches the boundary set $\partial_i\Delta_n$ for some $i > 0$ at time τ if at this time x_{i+1} collides with x_i. For time t immediately after τ, the configuration $\mathbf{q}(t)$ belongs to Δ_{n-1}. We also set

$$\hat\partial\Delta_n = \cup_{i=0}^n \partial_i\Delta_n.$$

(v) We write $\partial_{n+1}\Delta_{n+1}$ for the set of points $\mathbf{q} \in \Delta_{n+1}$ with $x_{n+1} = a_+$. When $\mathbf{q} \in \Delta_n$, and a new particle is created at a_+ at time τ by the stochastic boundary dynamics, the configuration $\mathbf{q}(\tau+)$ is regarded as a boundary point in $\partial_{n+1}\Delta_{n+1}$.

(vi) Given a function $G : \Delta \to \mathbb{R}$, we write G^n for the restriction of the function G to the set Δ_n. Also, given a measure on Δ, we write ν^n for the restriction of a measure ν to Δ_n.

(vii) We write $\mathcal{L} = \mathcal{L}^t$ for the generator of the (inhomogeneoys Markov) process $\mathbf{q}(t)$. This generator can be expressed as $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_b$, where $\mathcal{L}_0$ is the generator of the deterministic part of dynamics, and $\mathcal{L}_b$ represents the Markovian boundary dynamics. The deterministic and stochastic dynamics restricted to Δ_n have generators that are denoted by $\mathcal{L}_{0n}$ and $\mathcal{L}_{bn}$ respectively. While $\mathbf{q}(t)$ remains in Δ_n, its evolution is governed by an ODE of the form

$$\frac{d\mathbf{q}}{dt}(t) = \mathbf{b}\big(\mathbf{q}(t), t\big),$$

with $\mathbf{b} = \mathbf{b}_n : \Delta_n \to \mathbb{R}^{2n+1}$, that can be easily described with the aid of rules **(1)** and **(2)** of Definition 2.1**(iii)**, and (2.3). Given this vector field, the generator $\mathcal{L}_{0n}$ is given by

$$\mathcal{L}_{0n}F = \mathbf{b} \cdot \nabla F,$$

where ∇F is the full gradient of F with respect to variables $\big(\rho_0, x_1, \rho_1, \dots, x_n, \rho_n\big)$. We also write $\mathcal{L}^{t*}_{0n} = \mathcal{L}^*_{0n}$ for the adjoint of $\mathcal{L}_{0n}$ with respect to the Lebesgue measure:

$$\mathcal{L}^*_{0n}\mu = -\nabla \cdot (\mu\mathbf{b}).$$

□

It is not hard to show that $t \mapsto \Psi^t_s\mathbf{q},\ t \geq s$ is indeed a strong Markov process. This is rather straightforward, and we refer to Davis [D] for details.

We establish Theorem 2.1 by verifying the forward equation $\dot{\mu} = \mathcal{L}^*\mu$, or equivalently

$$\dot{\mu}^n = \big(\mathcal{L}^*\mu\big)^n, \tag{3.2}$$

for all $n \geq 0$, where μ was defined in Definition 2.2**(ii)**, and $\mathcal{L}^*$ is the adjoint of the operator $\mathcal{L}$. To explain this, observe that Theorem 2.1 offers a candidate for the law of $\mathbf{q}(t)$, namely the measure $\mu(d\mathbf{q}, t)$. Hence for our Theorem 2.1, it suffices to show

$$\int G\big(\mathbf{q}, t\big)\ \mu\big(d\mathbf{q}, t\big) = \mathbb{E}\int G\big(\Psi^t_0\mathbf{q}, t\big)\ \mu\big(d\mathbf{q}, 0\big), \tag{3.3}$$

for every function G of the form

$$G(\mathbf{q}, t) = \exp\left(i\int_{a_-}^{a_+} \rho(x, t; \mathbf{q})\varphi(x)\ dx\right), \tag{3.4}$$

for some smooth function φ (we refer to the beginning of Section 3 of [KR1] for more details.) Here and below, we write $\mathbb{P}$ and $\mathbb{E}$ for the probability and the expected value for the randomness associated with the boundary dynamics. To ease the notation, we set $\hat{G}(\mathbf{q}, s) = \mathbb{E}\ G(\Psi_s^t \mathbf{q}, t)$. We establish (3.3) by verifying

$$\frac{d}{ds} \int \hat{G}(\mathbf{q}, s)\ \mu(d\mathbf{q}, s) = 0, \tag{3.5}$$

for $t_0 < s < t$. The differentiation of $\mu(d\mathbf{q}, s)$ can be carried out directly and poses no difficulty. As for the contribution of $G(\Psi_s^t \mathbf{q}, t)$ to the s-derivative, we wish to show

$$\int \hat{G}_s(\mathbf{q}, s)\ \mu(d\mathbf{q}, s) = -\int \big(\mathcal{L}^s \hat{G}\big)(\mathbf{q}, s)\ \mu(d\mathbf{q}, s). \tag{3.6}$$

Since the deterministic part of the evolution is discontinuous in time, the justification of (3.6) requires some work. Additionally, to make sense of the right-hand side, we need $\hat{G}$ to be in the domain of the definition of $\mathcal{L}_0^s$. We expect $\hat{G}$ to be weakly differentiable with respect to $\mathbf{q}$. To avoid the differentiability question of $\hat{G}$, we would formally apply an integration by parts to the right-hand side of (3.6), so that the differentiation operator would act on the density of μ, which is differentiable. We also have a boundary contribution that correspond to the collisions between particles. We establish the following variant of the forward equation (3.6).

Theorem 3.1 *We have*

$$\begin{aligned} \lim_{s' \uparrow s} \int_{\Delta_n} \frac{\hat{G}^n(\mathbf{q}, s') - \hat{G}^n(\mathbf{q}, s)}{s - s'}\ \mu^n(\mathbf{q}, s)\ d\mathbf{q} = & \int_{\Delta_n} (\mathcal{L}_b^s \hat{G})^n(\mathbf{q}, s) \mu^n(\mathbf{q}, s)\ d\mathbf{q} \\ & + \int_{\Delta_n} \hat{G}^n(\mathbf{q}, s) \big(\mathcal{L}_{0n}^{s*} \mu^n\big)(\mathbf{q}, s)\ d\mathbf{q} \\ & + \int_{\hat{\partial}\Delta_{n+1}} \hat{G}^{n+1}(\mathbf{q}, s) \mu^{n+1}(\mathbf{q}, s) (\mathbf{b}_{n+1} \cdot \mathbf{N}_{n+1})\ \sigma(d\mathbf{q}), \end{aligned} \tag{3.7}$$

where $\mathbf{N}_{n+1}$ *is the outer unit normal vector of* $\hat{\partial}\Delta_{n+1}$*, and* $\sigma(d\mathbf{q})$ *is the surface measure of* $\partial\Delta_{n+1}$*.*

Note that for the differentiation in (3.7) we will need to compare $\hat{G}(\mathbf{q}, s)$ and $\hat{G}(\mathbf{q}, s')$ for $t_0 < s' < s \le t$. As a warm-up we verify the Lipschitzness of the function $s \mapsto \mathbb{E}\ \hat{G}(\mathbf{q}, s)$.

Lemma 3.1 *Fix* $t > t_0$*. There exists a constant* $C_1 = C_1(\varphi, H, f)$ *such that*

$$\big|\hat{G}(\mathbf{q}, s') - \hat{G}(\mathbf{q}, s)\big| \le C_1 (n+1) |s' - s|, \tag{3.8}$$

for all $\mathbf{q} \in \Delta_n$ *and* $s, s' \in [t_0, t]$*.*

The proof follows from the L^1-stability (2.4) and a coupling argument for the stochastic boundary dynamics. We skip the proof of Lemma 3.1 because it is very similar to the proof of the analogous Lemma 3.1 that appeared in [KR2]. Armed with (3.8), we are now ready for the proof of (3.7).

Proof of Theorem 3.1 *(Step 1)* Let $t_0 < s' < s \le t$. We first show that we can separate the deterministic and stochastic portions of the dynamics over the time interval $[s', s]$, when $s - s'$ is small. Write $\tau = \tau(\mathbf{q}, s')$ for the first time a jump occurs at $x = a_+$ after the time s', and let E denote the event that $\tau \in (s', s)$. We also write

$$\hat{\rho}_n = R_s(\mathbf{q})(a_+) = \phi_{x_n}^{a_+}(\rho_n; s), \qquad \hat{\rho}'_n = R_{s'}(\mathbf{q})(a_+) = \phi_{x_n}^{a_+}(\rho_n; s').$$

By the Lipschitz regularity of b, we can show that $\hat{\rho}_n - \hat{\rho}'_n = O(s - s')$ (see also (3.14) below). Recall that γ denotes the flow associated with the ODE (2.17), and η was defined in (2.16). Observe that by the Lipschitz regularity of η (which is the consequence of the Lipschitz regularity of v and f),

$$\mathbb{P}(E) = \int_{s'}^{s} \eta\big(\theta, \gamma_{s'}^{\theta}(\theta, \hat{\rho}'_n)\big)\, d\theta + O((s - s')^2) = (s - s')\eta\big(s, \hat{\rho}_n\big) + O((s - s')^2), \tag{3.9}$$

with both errors bounded uniformly over $\mathbf{q}$. We claim that there exists a constant c_1 so that for $\mathbf{q} \in \Delta_n$,

$$\begin{aligned} \hat{G}(\mathbf{q}, s') =&(s - s') \int_{\hat{\rho}_n}^{\infty} \Big(\mathbb{E}\left[\hat{G}\big(\epsilon_{\rho_+}\psi_{s'}^{\tau}\mathbf{q}, s\big) \;\big|\; E\right] - \hat{G}\big(\psi_{s'}^{s}\mathbf{q}, s\big)\Big) f^2(a_+, s, \hat{\rho}_n, \rho_+)\, d\rho_+ \\ &+ \hat{G}\big(\psi_{s'}^{s}\mathbf{q}, s\big) + (s - s')^2 R(\mathbf{q}, s', s), \end{aligned} \tag{3.10}$$

with $\big|R(\mathbf{q}, s', s)\big| \le c_1(n + 1)$. To prove (3.10), first observe that by the Markov property of the random flow Ψ,

$$\hat{G}(\mathbf{q}, s') = \mathbb{E}\; G\big(\Psi_s^t \Psi_{s'}^s \mathbf{q}, t\big) = \mathbb{E}\; \hat{G}\big(\Psi_{s'}^s \mathbf{q}, s\big).$$

On E^c (the complement of E), we see only the deterministic flow ψ over the time interval (s', s):

$$\begin{aligned} \mathbb{E}\; \hat{G}\big(\Psi_{s'}^s \mathbf{q}, s\big)1\!\!1_{E^c} &= \hat{G}\big(\psi_{s'}^s \mathbf{q}, s\big)\; \mathbb{P}\big(E^c\big) = \hat{G}\big(\psi_{s'}^s \mathbf{q}, s\big) - \hat{G}\big(\psi_{s'}^s \mathbf{q}, s\big)\; \mathbb{P}(E) \\ &= \hat{G}\big(\psi_{s'}^s \mathbf{q}, s\big) - (s - s')\hat{G}\big(\psi_{s'}^s \mathbf{q}, s\big)\eta\big(s, \hat{\rho}_n\big) + O((s - s')^2) \\ &= \hat{G}\big(\psi_{s'}^s \mathbf{q}, s\big) - (s - s')\hat{G}\big(\psi_{s'}^s \mathbf{q}, s\big) \int_{\rho_+}^{\infty} f^2(a_+, s, \hat{\rho}_n, \rho_+)\, d\rho_+ + O((s - s')^2), \end{aligned} \tag{3.11}$$

where (3.9) is used for the third equality. Moreover, using the strong Markov property for the random boundary at the stopping time τ,

$$\mathbb{E}\; \hat{G}\big(\Psi_{s'}^s \mathbf{q}, s\big)1\!\!1_E = \mathbb{E}\; \hat{G}\big(\Psi_{\tau}^s \epsilon_{\rho_+}\psi_{s'}^{\tau}\mathbf{q}, s\big)1\!\!1_E = \mathbb{E}\; \hat{G}\big(\epsilon_{\rho_+}\psi_{s'}^{\tau}\mathbf{q}, \tau\big)1\!\!1_E. \tag{3.12}$$

By (3.8),

$$\left|\mathbb{E}\ \hat G\big(\epsilon_{\rho_+}\psi^\tau_{s'}\mathbf{q},\tau\big)\mathbb{1}_E - \mathbb{E}\ \hat G\big(\epsilon_{\rho_+}\psi^\tau_{s'}\mathbf{q},s\big)\mathbb{1}_E\right| \le C_1(n+1)(s-s')\mathbb{P}(E). \tag{3.13}$$

Next we modify the distribution from which ρ_+ is selected; at present, ρ_+ is selected according to a random measure with density

$$\hat f^2\big(a_+,\tau,\tilde\rho_n,\rho_+\big) := \eta\big(\tau,\tilde\rho_n\big)^{-1} f^2\big(a_+,\tau,\tilde\rho_n,\rho_+\big),$$

where $\tilde\rho_n := \gamma^\tau_{s'}(\hat\rho_n')$. From $H \in C^2$, and the Lipshitzness of $b^i(x,s,\rho)$ for $i=1,2$, it is not hard to show that there exists a constant c_2 such that

$$\big|\hat\rho_n' - \hat\rho_n\big| \le c_2|s'-s|, \qquad \big|\hat\rho_n' - \tilde\rho_n\big| \le c_2|s'-s|. \tag{3.14}$$

Let us write $\hat\rho_+$ for an independent random variable distributed as $\hat f(a_+,s,\hat\rho_n,\rho_+)\ d\rho_+$. Observe

$$\eta(\theta,m) = \int_m^\infty f^2(a_+,\theta,m,\rho_+)\ d\rho_+ \ge \left[\min_{\theta'\in[t_0,T]} H_\rho(a_+,\theta',P_-)\right]\int_m^\infty f(a_+,\theta,m,\rho_+)\ d\rho_+.$$

From this, (3.14), and the Lipschitzness of $f^2 = vf$ we can readily show

$$\big|\hat f^2(a_+,s,\hat\rho_n,\rho_+) - \hat f^2\big(a_+,\tau,\tilde\rho_n,\rho_+\big)\big| \le c_3|s'-s|, \tag{3.15}$$

for a constant c_3. We then use (3.14) and (3.15) to assert that there exists a constant c_4 such that the expression

$$\left|\mathbb{E}\ \left[\hat G\big(\epsilon_{\rho_+}\psi^\tau_{s'}\mathbf{q},s\big) - \hat G\big(\epsilon_{\hat\rho_+}\psi^\tau_{s'}\mathbf{q},s\big)\right]\mathbb{1}_E\right|,$$

is bounded above by

$$\begin{aligned}&\left|\mathbb{E}\ \mathbb{1}_E\int_{\tilde\rho_n\vee\hat\rho_n}^\infty \hat G\big(\epsilon_{\rho_+}\psi^\tau_{s'}\mathbf{q},s\big)\big(\hat f^2\big(a_+,\tau,\tilde\rho_n,\rho_+\big) - \hat f^2(a_+,s,\hat\rho_n,\rho_+)\big)\ d\rho_+\right| \\ &\quad+\left|\mathbb{E}\ \mathbb{1}_E\ \mathbb{1}\big(\hat\rho_n \le \tilde\rho_n\big)\int_{\hat\rho_n}^{\tilde\rho_n} \hat G\big(\epsilon_{\rho_+}\psi^\tau_{s'}\mathbf{q},s\big)\ \hat f^2\big(a_+,s,\hat\rho_n,\rho_+\big)\ d\rho_+\right| \\ &\quad+\left|\mathbb{E}\ \mathbb{1}_E\ \mathbb{1}\big(\hat\rho_n > \tilde\rho_n\big)\int_{\tilde\rho_n}^{\hat\rho_n} \hat G\big(\epsilon_{\rho_+}\psi^\tau_{s'}\mathbf{q},s\big)\ \hat f^2\big(a_+,\tau,\tilde\rho_n,\rho_+\big)\ d\rho_+\right| \le c_4(s'-s)\mathbb{P}(E).\end{aligned}$$

From this, (3.12), (3.13), and (3.9) we learn

$$\begin{aligned}\mathbb{E}\ \hat G\big(\Psi^s_{s'}\mathbf{q},s\big)\mathbb{1}_E =&\mathbb{E}\left[\hat G\big(\epsilon_{\hat\rho_+}\psi^\tau_{s'}\mathbf{q},s\big)\ \big|\ E\right]\mathbb{P}(E) + (s-s')^2R_1 \\ =&\mathbb{E}\left[\int_{\hat\rho_n}^\infty \hat G\big(\epsilon_{\rho_+}\psi^\tau_{s'}\mathbf{q},s\big)\ f^2(a_+,s,\hat\rho_n,\rho_+)\ d\rho_+\ \big|\ E\right]\eta(s,\hat\rho_n)^{-1}\ \mathbb{P}(E) + (s-s')^2R_1 \\ =&(s-s')\int_{\hat\rho_n}^\infty \mathbb{E}\left[\hat G\big(\epsilon_{\rho_+}\psi^\tau_{s'}\mathbf{q},s\big)\ f^2(a_+,s,\hat\rho_n,\rho_+)\ \big|\ E\right]\ d\rho_+ + (s-s')^2R_2.\end{aligned}$$

where R_1 and R_2 are bounded by a constant multiple of $n+1$. This and (3.11) complete the proof of (3.10).

(Step 2) We wish to establish (3.7) with the aid of (3.10). Observe that $\mu(d\mathbf{q}, s)$ is the law of a Markov process with a bounded jump rates. For such a Markov process, we can readily show that if $\mathbf{n}(\mathbf{q})$ denotes the number of jumps/particles of $\mathbf{q}$ in the interval $[a_-, a_+]$, then

$$\sup_{s\in[t_0,T]} \int \mathbf{n}(\mathbf{q})^k \ \mu(d\mathbf{q}, s) < \infty, \tag{3.16}$$

for every $k \in \mathbb{N}$. Indeed if we choose δ_0 so that $\lambda(x, t_0, \rho) \geq \delta_0$ for all $(x, \rho) \in [a_-, a_+] \times [P_-, P_+]$, then there exists a Poisson random variable N_{δ_0} of intensity $\delta_0^{-1}(a_+ - a_-)$ such that $\mathbf{n}(\mathbf{q}) \leq N_{\delta_0}$ almost surely. From (3.16), and (3.10), we can write

$$(s-s')^{-1}\big(\hat{G}(\mathbf{q}, s') - \hat{G}(\mathbf{q}, s)\big) = \sum_{r=1}^{5} \Omega_r(s', s),$$

where

$$\begin{aligned}
\Omega_1(s', s) =& \int_{\hat{\rho}_n}^{\infty} \Big(\mathbb{E}\left[\hat{G}\big(\epsilon_{\rho_+}\psi_{s'}^{\tau}\mathbf{q}, s\big) \mid E\right] - \hat{G}\big(\epsilon_{\rho_+}\mathbf{q}, s\big)\Big) f^2(a_+, s, \hat{\rho}_n, \rho_+) \ d\rho_+ \\
=& \int_{\hat{\rho}_{\mathbf{n}(\mathbf{q})}}^{\infty} \mathbb{E}\left[\hat{G}\big(\varepsilon_{\rho_+}\psi_{s'}^{\tau}\mathbf{q}, s\big) - \hat{G}\big(\varepsilon_{\rho_+}\mathbf{q}, s\big) \middle| E\right] f^2(a_+, s, \hat{\rho}_{\mathbf{n}(\mathbf{q})}, \rho_+) \ d\rho_+, \\
\Omega_2(s', s) =& \int_{\hat{\rho}_n}^{\infty} \Big(\hat{G}\big(\epsilon_{\rho_+}\mathbf{q}, s\big) - \hat{G}(\mathbf{q}, s) \ \Big) f^2(a_+, s, \hat{\rho}_n, \rho_+) \ d\rho_+ = (\mathcal{L}_{bn}^{s}\hat{G})(\mathbf{q}, s), \\
\Omega_3(s', s) =& \int_{\hat{\rho}_n}^{\infty} \Big(\hat{G}(\mathbf{q}, s) - \hat{G}\big(\psi_{s'}^{s}\mathbf{q}, s\big)\Big) f^2(a_+, s, \hat{\rho}_n, \rho_+) \ d\rho_+ = \Big(\hat{G}(\mathbf{q}, s) - \hat{G}\big(\psi_{s'}^{s}\mathbf{q}, s\big)\Big) \eta(s, \hat{\rho}_n), \\
\Omega_4(s', s) =& (s-s')^{-1} \Big(\hat{G}\big(\psi_{s'}^{s}\mathbf{q}, s\big) - \hat{G}(\mathbf{q}, s)\Big),
\end{aligned}$$

and $|\Omega_5(s', s)| = (s - s')|R(\mathbf{q}, s', s)| \leq c_1(n + 1)(s - s')$. (For the second equality, we used the fact that the event E depends only on the stochastic boundary that is independent from the law of ρ_+.) By (3.16),

$$\int \big|\Omega_5(s', s)\big| \ \ \mu(d\mathbf{q}, s) = O(s - s').$$

As a result, (3.7) would follow if we can show

$$\lim_{s'\uparrow s} \int \Omega_1(s', s) \ \mu(d\mathbf{q}, s) = 0, \tag{3.17}$$

$$\lim_{s'\uparrow s} \int \Omega_3(s', s) \ \mu(d\mathbf{q}, s) = 0. \tag{3.18}$$

and that the limit

$$\lim_{s'\uparrow s} \int \Omega_4(s',s)\ \mu(d\mathbf{q},s), \tag{3.19}$$

equals to the sum of the last two terms on the right-hand side of (3.7). The proof of this will be carried out in the last step. A slight modification of this proof can be carried out to establish (3.18).

(Step 3) We turn out attention to (3.17). Recall that $\mu(d\mathbf{q},s)$ is the law of a Markov process $(\rho(x,s):\ x\in[a_-,a_+])$ with generator $\mathcal{A}^1_{x,s}$. For our proof we will need a lower bound on the density f. Since $f(x,s,\rho_-,\rho_+)>0$ only when (ρ_-,ρ_+) is in the interior of $\Lambda(P_-,P_+)$, we wish to estimate the probability of the set $B(\delta,s)$ consisting of those $\mathbf{q}$ such that for some $x\in[a_-,a_+]$, we have either $\rho(x,s)=R_s(\mathbf{q})(x)\in[P_+-\delta,P_+]$, or $\rho(x+,s)-\rho(x-,s)\in(0,\delta)$. If we write $\mathbb{P}^m_s$ for the law of our Markov process $\rho(x,s)$ associated with the generator $\mathcal{A}^1_{x,s}$, and the initial condition $\rho(a_-,s)=m$, and if $m<P_+-\delta$, then it is not hard to show

$$\mathbb{P}^m_s\big(B(\delta,s)\big)=\mathbb{E}^m_s\int_{a_-}^{a_+}\left[\int_{\rho(x,s)}^{\rho(x,s)+\delta}+\int_{\rho(x,s)\vee(P_+-\delta)}^{P_+}\right]f(x,s,\rho(x,s),\rho_+)\ d\rho_+\ dx\le c_5\delta.$$

From this, we learn that (3.17) would follow if we can show

$$\lim_{s'\uparrow s}\int \Omega_1(s',s)\ \hat\mu(d\mathbf{q},s)=0, \tag{3.20}$$

where

$$\hat\mu(d\mathbf{q},s)=1\!\!1\big(\mathbf{q}\notin B(\delta,s)\big)\ \mu(d\mathbf{q},s).$$

(Step 4) To verify (3.20), write $\sigma(\mathbf{q},s')$ for the first time $\sigma>s'$ at which $\psi^\sigma_{s'}\mathbf{q}$ experiences a collision between particles of $\mathbf{q}$. We claim

$$\int 1\!\!1\big(\sigma(\mathbf{q},s')\le s\big)\ \mu(d\mathbf{q},s)\le c_5(s-s')\int \mathbf{n}(\mathbf{q})\ \mu(d\mathbf{q},s)\le c_6(s-s'), \tag{3.21}$$

for constants c_6 and c_5. This is an immediate consequence of (3.16) and the following fact: If $\mathbf{q}=(x_0,\rho_0,x_1,\rho_1,\dots,x_n,\rho_n)$, and $\sigma(\mathbf{q},s')\le s$, then for some i, we have $|x_i-x_{i+1}|\le 2c_6|s-s'|$, where c_6 is an upper bound on the speed of particles. Because of (3.21), the claim (3.20) is equivalent to

$$\lim_{s'\uparrow s}\left|\sum_{n=0}^{\infty}X_n(s')\right|=0, \tag{3.22}$$

where $X_n(s')$ is the expression

$$\int\int_{\hat\rho_n}^{\infty} \mathbb{E}\ \left[\hat G\big(\varepsilon_{\rho_+}\psi_{s'}^{\tau}\mathbf{q}, s\big) - \hat G\big(\varepsilon_{\rho_+}\mathbf{q}, s\big)\big|\ E\right] 1\!\!1\big(\sigma(\mathbf{q}, s') > s\big)\ f(a_+, s, \hat\rho_n, \rho_+)\hat\mu^n(\mathbf{q}, s)\ d\rho_+ d\mathbf{q}.$$

On account of (3.9), the claim (3.22) would follow if we can show

$$\lim_{s'\uparrow s}(s-s')^{-1}\left|\sum_{n=0}^{\infty} Y_n(s')\right| = 0, \tag{3.23}$$

where $Y_n(s') = Y_n^+(s') - Y_n^-(s')$, with

$$Y_n^+(s') = \int\int_{\hat\rho_n}^{\infty} \mathbb{E}\ \hat G\big(\varepsilon_{\rho_+}\psi_{s'}^{\tau}\mathbf{q}, s\big) 1\!\!1\big(\sigma(\mathbf{q}, s') > s > \tau(\mathbf{q}, s')\big)\ f^2(a_+, s, \hat\rho_n, \rho_+)\hat\mu^n(\mathbf{q}, s)\ d\rho_+ d\mathbf{q},$$

$$Y_n^-(s') = \int\int_{\hat\rho_n}^{\infty} \mathbb{E}\ \hat G\big(\varepsilon_{\rho_+}\mathbf{q}, s\big) 1\!\!1\big(\sigma(\mathbf{q}, s') > s > \tau(\mathbf{q}, s')\big)\ f^2(a_+, s, \hat\rho_n, \rho_+)\hat\mu^n(\mathbf{q}, s)\ d\rho_+ d\mathbf{q}.$$

(Step 5) The expected value in the definition of $Y_n^{\pm}$ is for the random variable $\tau = \tau(\mathbf{q}, s')$. As was explained in the proof of Proposition 2.1**(ii)**, the variable τ can be expressed in terms of $\hat\rho_n$ and a standard exponential random variable. More precisely,

$$\tau = \tau(\mathbf{q}, s') = \ell(r, \hat\rho_n, s'),$$

with $r > 0$ a random variable with distribution $e^{-r}\ dr$, and $\ell(r, \hat\rho_n, s')$ denoting the inverse of the map

$$\tau \mapsto r = \int_{s'}^{\tau} \eta\big(\theta, \gamma_{s'}^{\theta}(\theta, \hat\rho_n)\big)\ d\theta, \qquad \tau \in (s', \infty).$$

As a result, we may replace the expected values in (3.23) with an integration with respect to $e^{-r}\ dr$. On the other hand,

$$1\!\!1(r > 0)\ e^{-r}\ dr = 1\!\!1(\tau > s')\ e^{-r}\eta\big(\tau, \gamma_{s'}^{\tau}(\hat\rho_n)\big)\ d\tau = 1\!\!1(\tau > s')\ \big(\eta\big(s', \hat\rho_n\big) + O(\tau - s')\big)\ d\tau,$$

by the Lipschitz regularity of η. Because of this, (3.23) would follow if we can show

$$\lim_{s'\uparrow s}(s-s')^{-1}\left|\sum_{n=0}^{\infty} Z_n(s')\right| = 0, \tag{3.24}$$

where $Z_n(s') = Z_n^+(s') - Z_n^-(s')$, with

$$Z_n^+(s') = \int\int_{\hat\rho_n}^{\infty}\int_{s'}^{s} \hat G\big(\varepsilon_{\rho_+}\psi_{s'}^{\tau}\mathbf{q}, s\big) 1\!\!1\big(\sigma(\mathbf{q}, s') > s\big)\eta\big(\hat\rho_n, s'\big) f^2(a_+, s, \hat\rho_n, \rho_+)\hat\mu^n(\mathbf{q}, s)\ d\tau d\rho_+ d\mathbf{q},$$

$$Z_n^-(s') = \int\int_{\hat\rho_n}^{\infty}\int_{s'}^{s} \hat G\big(\varepsilon_{\rho_+}\mathbf{q}, s\big) 1\!\!1\big(\sigma(\mathbf{q}, s') > s\big)\eta\big(\hat\rho_n, s'\big) f^2(a_+, s, \hat\rho_n, \rho_+)\hat\mu^n(\mathbf{q}, s)\ d\tau d\rho_+ d\mathbf{q}.$$

To prove (3.24), we carry out the $d\mathbf{q}$ integration first. Fix $\tau > 0$ and ρ_+, and make a change of variables $\mathbf{q}' = \psi^{\tau}_{s'}\mathbf{q}$ for $d\mathbf{q}$ integration in $Z^+_n(s')$. For this, we wish to replace $\hat{\mu}^n(\mathbf{q}, s)$, with $\hat{\mu}^n\big(\psi^{\tau}_{s'}\mathbf{q}, s\big)$. Observe that by Hypothesis 1.1**(iii)**, then kernel $f(x, s, \rho_-, \rho_+) > 0$ in the interior of $\Lambda(P_-, P_+)$. As a result, we can find $\delta_0 > 0$ such that

$$\mathbf{q} = (x_0, \rho_0, \dots, x_n, \rho_n) \notin B(\delta, s) \quad \Longrightarrow \quad f(x_i, \hat{\rho}_i, \rho_{i+1}) \geq \delta_1. \tag{3.25}$$

Since $\hat{\mu}^n$ is supported on the complement of the event $B(\delta, s)$, we use

$$\big[\log \hat{\mu}^n\big(\psi^{\tau}_{s'}\mathbf{q}, s\big)\big]_\tau = \mathbf{b}\big(\psi^{\tau}_{s'}\mathbf{q}, \tau\big) \cdot \nabla \left[\log \hat{\mu}^n\big(\psi^{\tau}_{s'}\mathbf{q}, s\big)\right],$$

our assumption $f \in C^1$, and (3.25) to assert

$$\big[\log \hat{\mu}^n\big(\psi^{\tau}_{s'}\mathbf{q}, s\big)\big]_\tau = O(n),$$

which in turn implies

$$\hat{\mu}^n\big(\psi^{\tau}_{s'}\mathbf{q}, s\big) = \mu^n(\mathbf{q}, s)\big(1 + (s' - s)O(n)\big). \tag{3.26}$$

Since the map $\mathbf{q} \mapsto \psi^{\tau}_{s'}\mathbf{q}$ is the flow of the ODE associated with vector field $\mathbf{b}$, its Jacobian has the expansion

$$1 + (\tau - s')div(\mathbf{b}) + \mathbf{n}(\mathbf{q})\ o(\tau - s').$$

Since $div(\mathbf{b}) = O(\mathbf{n}(\mathbf{q}))$, a change of variable $\mathbf{q}' = \psi^{\tau}_{s'}\mathbf{q}$ causes a Jacobian factor of the form

$$1 + \mathbf{n}(\mathbf{q})O(\tau - s') = 1 + \mathbf{n}(\mathbf{q})O(s - s').$$

From this, (3.26), and (3.14) we learn

$$\eta\big(\hat{\rho}_n, s'\big)f^2(a_+, s, \hat{\rho}_n, \rho_+)\mu^n(\mathbf{q}, s)\ d\mathbf{q} = \eta\big(\hat{\rho}'_n, s'\big)f^2(a_+, s, \hat{\rho}'_n, \rho_+)\mu^n(\mathbf{q}', s)\big(1 + nO(s - s')\big)\ d\mathbf{q}'.$$

From all this we deduce that $Z^+_n(s') = \hat{Z}^+_n(s') + R_n$, where $\hat{Z}^+_n(s')$ is given by

$$\int\int\int_{\hat{\rho}_n}^{\infty}\int_{s'}^{s} \hat{G}\big(\varepsilon_{\rho_+}\mathbf{q}', s\big)1\!\!1\big(\sigma(\psi^{s'}_{\tau}\mathbf{q}', s') > s\big)\eta\big(\hat{\rho}'_n, s'\big)f^2(a_+, s, \hat{\rho}_n, \rho'_+)\hat{\mu}^n(\mathbf{q}', s)\ d\tau d\rho_+ d\mathbf{q}'.$$

and the R_n is an error term that satisfies

$$\sum_{n=0}^{\infty}|R_n| \leq c_2(s - s')^2\int \mathbf{n}(\mathbf{q})^2\ \mu(d\mathbf{q}, s) = c_3(s - s')^2.$$

By $\psi^{s'}_{\tau}$ we mean the inverse of $\psi^{\tau}_{s'}$. After renaming $\mathbf{q}'$ as $\mathbf{q}$ and comparing $\hat{Z}^+_n(s')$ with $Z^-_n(s')$, we learn that $\hat{Z}^+_n(s') - Z^-_n(s')$ equals to

$$\int\int\int_{\hat{\rho}_n}^{\infty}\int_{s'}^{s} \hat{G}\big(\varepsilon_{\rho_+}\mathbf{q}, s\big)\chi(\mathbf{q}; s', \tau, s)\eta\big(\hat{\rho}_n, s'\big)f^2(a_+, s, \hat{\rho}_n, \rho_+)\mu^n(\mathbf{q}, s)\ d\tau d\rho_+ d\mathbf{q},$$

where $\chi(\mathbf{q}; s', \tau, s) = \big|1\!\!1\big(\sigma(\psi_\tau^{s'}\mathbf{q}, s') > s\big) - 1\!\!1\big(\sigma(\mathbf{q}, s') > s\big)\big|$. After replacing $\hat{G}$ with an upper bound, and carrying out the $d\rho_+$ integration, we obtain

$$\sum_{n=0}^{\infty} \big|\hat{Z}_n^+(s') - Z_n^-(s')\big| \le c_4 \int_{s'}^{s} \int \chi(\mathbf{q}; s', \tau, s)\ \mu(d\mathbf{q}, s) d\tau.$$

Finally, since $\chi(\mathbf{q}; s, s) = 0$, we can readily show

$$\lim_{s'\uparrow s} (s - s')^{-1} \sum_{n=0}^{\infty} \big(\hat{Z}^+(s') - Z^-(s')\big) = 0,$$

completing the proof of (3.24), that in turn completes the proof of (3.17).

(Final Step) It remains to find the limit in (3.19). The proof we present is very general, and works whenever $\hat{G}^n$ is continuous, μ^n is C^1, the vector field $\mathbf{b} = \mathbf{b}_n$ is C^1, and the boundary of Δ_n is piecewise C^1. Fix s and for $s' < s$ we write

$$\Delta_n(s', s) = \big\{\mathbf{q} \in \Delta_n : \ \psi_{s'}^{s}\mathbf{q} \in \Delta_n\big\}, \qquad \hat{\Delta}_n(s', s) = \psi_{s'}^{s}\big(\Delta_n(s', s)\big).$$

We make a change of variables to write

$$\begin{aligned} \int_{\Delta_n} \hat{G}^n\big(\psi_{s'}^{s}\mathbf{q}, s\big)\mu^n(\mathbf{q}, s)\ d\mathbf{q} = &\int_{\Delta_n\setminus\Delta_n(s',s)} \hat{G}^n\big(\psi_{s'}^{s}\mathbf{q}, s\big)\mu^n(\mathbf{q}, s)\ d\mathbf{q} \\ &+ \int_{\hat{\Delta}_n(s',s)} \hat{G}^n\big(\mathbf{q}, s\big)\mu^n(\psi_s^{s'}\mathbf{q}, s)\ \det\big(D\psi_s^{s'}\big)(\mathbf{q})\ d\mathbf{q}, \end{aligned} \tag{3.27}$$

where $\psi_s^{s'}$ denotes the inverse of the function $\psi_{s'}^{s}$. For $s - s'$ small, the volume $|\Delta_n \setminus \Delta_n(s', s)|$ is of order $O(n(s - s'))$. From this, and the continuity of $\hat{G}$ we learn

$$\begin{aligned} \int_{\Delta_n\setminus\Delta_n(s',s)} \hat{G}^n\big(\psi_{s'}^{s}\mathbf{q}, s\big)\mu^n(\mathbf{q}, s)\ d\mathbf{q} = &\int_{\Delta_n\setminus\Delta_n(s',s)} \big(\hat{G}^n\mu^n\big)\big(\mathbf{q}, s\big)\ d\mathbf{q} + o(n(s - s')) \\ =&(s - s')\int_{\partial\Delta_n} \big(\hat{G}^n\mu^n\big)\big(\mathbf{q}', s\big)\big(\mathbf{N}_n(\mathbf{q}')\cdot\mathbf{b}_n(\mathbf{q}', s)\big)\ \sigma(d\mathbf{q}') \\ &+ o(n(s - s')). \end{aligned} \tag{3.28}$$

Here we have used the fact that we may parametrize the set $\Delta_n \setminus \Delta_n(s', s)$ by the map

$$\zeta : \partial\Delta_n \times [s', s] \to \Delta_n \setminus \Delta_n(s', s), \qquad \zeta(\mathbf{q}', \theta) = \psi_s^{\theta}\mathbf{q}',$$

with $1\!\!1\big(\mathbf{q} \in \Delta_n \setminus \Delta_n(s', s)\big)\ d\mathbf{q}$ equals to

$$1\!\!1\big((\mathbf{q}', \theta) \in \partial\Delta_n \times [s', s]\big)\ \big(1 + (s - s')\big(\mathbf{N}_n(\mathbf{q}')\cdot\mathbf{b}_n(\mathbf{q}', s)\big) + o(n(s - s'))\big)\ \sigma(d\mathbf{q}')d\theta.$$

The map ζ is one-to-one if $\partial\Delta_n$ is C^1, and $s-s'$ is sufficiently small. This is no longer the case when C^1 is only piecewise C^1. Though the set of $\mathbf{q}$ for which $\zeta^{-1}(\mathbf{q})$ is multivalued, is of volume $O(n(s-s')^2)$ (this is the set of $\mathbf{q}$ such that for some i, we have $x_{i+1}-x_i, x_i-x_{i-1} = O(s-s')$).

As for the second term on the right-hand side of (3.27), we use

$$\begin{aligned}\mu^n(\psi_s^{s'}\mathbf{q},s) =&\mu^n(\mathbf{q},s)+(s'-s)\big(\mathbf{b}_n\cdot\nabla\mu^n\big)(\mathbf{q},s)+o(n(s-s'))\\ \det\big(D\psi_s^{s'}\big)(\mathbf{q}) =&\mu^n(\mathbf{q},s)+(s'-s)\big(\mu^n div\ \mathbf{b}_n\big)(\mathbf{q},s)+o(n(s-s')),\end{aligned}$$

to assert

$$\begin{aligned}\mu^n(\psi_s^{s'}\mathbf{q},s)\ \det\big(D\psi_s^{s'}\big)(\mathbf{q}) =&\mu^n(\mathbf{q},s)+(s'-s)\big(\mu^n div\ \mathbf{b}_n\big)(\mathbf{q},s)\\ &+(s'-s)\big(\mathbf{b}_n\cdot\nabla\mu^n\big)(\mathbf{q},s)+o(n(s-s'))\\ =&\mu^n(\mathbf{q},s)+(s-s')\big(\mathcal{L}_{0n}^{s*}\mu^n\big)(\mathbf{q},s)+o(n(s-s')).\end{aligned}$$

From this and (3.28) we deduce that the second term on the right-hand side of (3.27) equals to

$$\begin{aligned}&\int_{\hat\Delta_n(s')}\hat G^n\big(\mathbf{q},s\big)\hat G^n\big(\mathbf{q},s\big)\big[\mu^n(\mathbf{q},s)+(s-s')\big(\mathcal{L}_{0n}^{s*}\mu^n\big)(\mathbf{q},s)\big]\ d\mathbf{q}+o(n(s-s'))\\ &\quad=\int_{\Delta_n}\big[\mu^n(\mathbf{q},s)+(s-s')\big(\mathcal{L}_{0n}^{s*}\mu^n\big)(\mathbf{q},s)\big]\ d\mathbf{q}+o(n(s-s')).\end{aligned}$$

This, (3.27), and (3.28) complete the proof. □

4 Proof of Theorem 2.1

The proof of Theorem 2.1 is carried out in five steps.

(Step 1) As we explained in Section 3, we only need to prove (3.5). For this, it suffices to show

(4.1) $$\lim_{s'\uparrow s}(s-s')^{-1}\big(\mathbb{G}_n(s)-\mathbb{G}_n(s')\big)=0,$$

where $\mathbb{G}_n(s)=\int\hat G^n(\mathbf{q},s)\ \mu^n(d\mathbf{q},s)$. Evidently

$$(s-s')^{-1}\big(\mathbb{G}_n(s)-\mathbb{G}_n(s')\big)=\Omega_1(s')+\Omega_2(s')-\Omega_3(s'),$$

where

$$\Omega_1(s') = (s-s')^{-1} \int \big(\hat{G}^n(\mathbf{q}, s) - \hat{G}^n(\mathbf{q}, s')\big)\mu^n(d\mathbf{q}, s)$$
$$\Omega_2(s') = (s-s')^{-1} \int \hat{G}^n(\mathbf{q}, s) \ \big(\mu^n(d\mathbf{q}, s) - \mu^n(d\mathbf{q}, s')\big)$$
$$\Omega_3(s') = (s-s')^{-1} \int \big(\hat{G}^n(\mathbf{q}, s) - \hat{G}^n(\mathbf{q}, s')\big) \ \big(\mu^n(d\mathbf{q}, s) - \mu^n(d\mathbf{q}, s')\big).$$

We claim

$$\lim_{s'\uparrow s} \big|\Omega_3(s')\big| = 0. \tag{4.2}$$

By Lemma 3.1,

$$\limsup_{s\to 0} (s-s')^{-1}\big|\Omega_3(s')\big| \le C_0(n+1) \limsup_{s\to 0} \int \big|\big(\mu^n(d\mathbf{q}, s) - \mu^n(d\mathbf{q}, s')\big)\big| \,. \tag{4.3}$$

As we will see in Step 2 below, $\mu^n_s = X^n\mu^n$ for a term X^n that is explicit. From this and (4.3), it is not hard to deduce (4.2). From (4.2), and Theorem 3.1 we deduce that (4.1) would follow if we can show

$$\begin{aligned} \int_{\Delta_n} \hat{G}^n(\mathbf{q}, s) \ \mu^n_s(d\mathbf{q}, s) = &\int_{\Delta_n} (\mathcal{L}^s_b\hat{G})^n(\mathbf{q}, s)\mu^n(\mathbf{q}, s) \ d\mathbf{q} + \int_{\Delta_n} \hat{G}^n(\mathbf{q}, s)\big(\mathcal{L}^{s*}_{0n}\mu^n\big)(\mathbf{q}, s) \ d\mathbf{q} \\ &+ \int_{\hat{\partial}\Delta_{n+1}} \hat{G}^{n+1}(\mathbf{q}, s)\mu^{n+1}(\mathbf{q}, s)(\mathbf{b}_{n+1}\cdot\mathbf{N}_{n+1}) \ \sigma(d\mathbf{q}). \end{aligned} \tag{4.4}$$

(Step 2) To simplify our presentation, we assume that ℓ has a density with respect to the Lebesgue measure. With a slight abuse of notation, we write $\ell(\rho, s)$ for this density: $\ell(d\rho, s) = \ell(\rho, s) \ d\rho$. To verify (4.4), we start with finding a tractable expression for the left-hand side. We claim

$$\mu^n_s = X^n\mu^n = \left(X_1 + X_2 + \sum_{i=3}^{11} X^n_i\right)\mu^n, \tag{4.5}$$

where

$$X_1 = \frac{(\ell * f^2)(x_0, s, \rho_0)}{\ell(s, \rho_0)}, \qquad X_2 = -\frac{\big(\beta(x_0, s, \rho_0)\ell(s, \rho_0)\big)_{\rho_0}}{\ell(s, \rho_0)},$$

$$X_3^n = \sum_{i=1}^{n} \frac{Q^+(f)(x_i, s, \hat\rho_{i-1}, \rho_i)}{f(x_i, s, \hat\rho_{i-1}, \rho_i)}, \qquad X_4^n = \sum_{i=1}^{n} \frac{f_x^2\big(x_i, t, \hat\rho_{i-1}, \rho_i\big)}{f\big(x_i, t, \hat\rho_{i-1}, \rho_i\big)}$$

$$X_5^n = \beta(x_0, t, \rho_0)\Gamma_\rho(x_0, x_1, s, \rho_0) - A(a_+, s, \hat\rho_n)$$

$$X_6^n = \sum_{i=1}^{n} \beta(x_i, t, \rho_i)\Gamma_\rho(x_i, x_{i+1}, s, \rho_i)$$

$$X_7^n = \sum_{i=1}^{n} v(x_i, s, \hat\rho_{i-1}, \rho_i)\big(\lambda(x_i, s, \rho_i) - \lambda(x_i, s, \hat\rho_{i-1})\big)$$

$$X_8^n = \sum_{i=1}^{n} \frac{\big[K(x_i, s, \rho_i, \hat\rho_{i-1}) f(x_i, s, \hat\rho_{i-1}, \rho_i)\big]_{\rho_i}}{f\big(x_i, s, \hat\rho_{i-1}, \rho_i\big)}$$

$$X_9^n = \sum_{i=1}^{n} b(x_i, s, \hat\rho_{i-1}) v_{\rho_-}(x_i, s, \hat\rho_{i-1}, \rho_i)$$

$$X_{10}^n = \sum_{i=1}^{n} v(x_i, s, \hat\rho_{i-1}, \rho_i) b(x_i, s, \hat\rho_{i-1}) \frac{f_{\rho_-}(x_i, s, \hat\rho_{i-1}, \rho_i)}{f(x_i, s, \hat\rho_{i-1}, \rho_i)}$$

$$X_{11}^n = -\sum_{i=1}^{n} \beta(x_{i-1}, s, \rho_{i-1}) \frac{\big[f(x_i, s, \hat\rho_{i-1}, \rho_i)\big]_{\rho_{i-1}}}{f(x_i, s, \hat\rho_{i-1}, \rho_i)},$$

where $\hat\rho_{i-1} = \phi_{x_{i-1}}^{x_i}\big(\rho_{i-1}; s\big)$. To verify (4.5), observe that by direct differentiation (see Definition 2.2**(ii)** for the definition of μ^n)

$$X^n = -\Gamma_s(\mathbf{q}, s) + \frac{\ell_s(s, \rho_0)}{\ell(s, \rho_0)} + \sum_{i=1}^{n} \frac{\big[f(x_i, s, \hat\rho_{i-1}, \rho_i)\big]_s}{f(x_i, s, \hat\rho_{i-1}, \rho_i)}. \tag{4.6}$$

By (2.10), (1.18), and (2.8) (in this order)

$$\begin{aligned}
-\Gamma_s(\mathbf{q},s) &= -\sum_{i=0}^{n}\big\{\big(A(x_{i+1},s,\hat\rho_i)-A(x_i,s,\rho_i)\big)-\beta(x_i,t,\rho_i)\Gamma_\rho(x_i,x_{i+1},s,\rho_i)\big\}\\
&= -\sum_{i=0}^{n}\big(A(x_{i+1},s,\hat\rho_i)-A(x_i,s,\rho_i)\big)+\beta(x_0,t,\rho_0)\Gamma_\rho(x_0,x_1,s,\rho_0)+X_6^n\\
&= \sum_{i=1}^{n}\big(A(x_i,s,\rho_i)-A(x_i,s,\hat\rho_{i-1})\big)+A(x_0,s,\rho_0)-A(x_{n+1},s,\hat\rho_n)\\
&\quad +\beta(x_0,t,\rho_0)\Gamma_\rho(x_0,x_1,s,\rho_0)+X_6^n,\\
\frac{\ell_s(s,\rho_0)}{\ell(s,\rho_0)} &= X_1+X_2-A(x_0,s,\rho_0),\\
\frac{\big[f(x_i,s,\hat\rho_{i-1},\rho_i)\big]_s}{f(x_i,s,\hat\rho_{i-1},\rho_i)} &= \frac{f_s(x_i,s,\hat\rho_{i-1},\rho_i)}{f(x_i,s,\hat\rho_{i-1},\rho_i)}+U^n(i),
\end{aligned}$$

where

$$\begin{aligned}
U^n(i) &= \Big[\beta(x_i,s,\hat\rho_{i-1})-\beta(x_{i-1},s,\rho_{i-1})\big[\phi_{x_{i-1}}^{x_i}(\rho_{i-1};s)\big]_{\rho_{i-1}}\Big]\frac{f_{\rho_-}(x_i,s,\hat\rho_{i-1},\rho_i)}{f(x_i,s,\hat\rho_{i-1},\rho_i)}\\
&=\beta(x_i,s,\hat\rho_{i-1})\frac{f_{\rho_-}(x_i,s,\hat\rho_{i-1},\rho_i)}{f(x_i,s,\hat\rho_{i-1},\rho_i)}-\beta(x_{i-1},s,\rho_{i-1})\frac{\big[f(x_i,s,\hat\rho_{i-1},\rho_i)\big]_{\rho_{i-1}}}{f(x_i,s,\hat\rho_{i-1},\rho_i)}.
\end{aligned}$$

From this, (4.6), and the kinetic equation we deduce

$$X^n = X_1+X_2+X_3^n+X_4^n+X_5^n+X_6^n+X_7^n+U^n+W^n,$$

where $U^n=\sum_{i=1}^n U^n(i)$, and

$$W^n=\sum_{i=1}^{n}\frac{(Cf)(x_i,s,\hat\rho_{i-1},\rho_i)}{f(x_i,s,\hat\rho_{i-1},\rho_i)}=X_8^n+X_9^n+\sum_{i=1}^{n}K(x_i,s,\hat\rho_{i-1},\rho_i)\frac{f_{\rho_-}(x_i,s,\hat\rho_{i-1},\rho_i)}{f(x_i,s,\hat\rho_{i-1},\rho_i)}.$$

We are done because $U^n+W^n=X_8^n+X_9^n+X_{10}^n+X_{11}^n$.

(Step 3) We now turn our attention to the right-hand side of (4.4). We certainly have

$$\int(\mathcal{L}_b^s\hat G)^n(\mathbf{q},s)\mu^n(\mathbf{q},s)\ d\mathbf{q}=Y_{b,+}^n-Y_{b,-}^n, \tag{4.7}$$

where

$$\begin{aligned}
Y_{b,+}^n &= \int\int_{\hat\rho_n}^{\infty} f^2(a_+,s,\hat\rho_n,\rho_+)\hat G^{n+1}\big(\varepsilon_{\rho_+}\mathbf{q},s\big)\mu^n(\mathbf{q},s)\ d\rho_+d\mathbf{q},\\
Y_{b,-}^n &= \int A(a_+,s,\hat\rho_n)\hat G^n(\mathbf{q},s)\mu^n(\mathbf{q},s)\ d\mathbf{q}.
\end{aligned}$$

As for the second term on the right-hand side of (4.4), we write $\mathcal{L}_{0n}^*\mu^n = Z^n\mu^n$, with

$$Z^n = \sum_{j=1}^{3} Z_{1j} + \sum_{i=2}^{3}\sum_{j=1}^{3} Z_{ij}^n, \tag{4.8}$$

where

$$Z_{11} = \beta(x_0, s, \rho_0)\Gamma_\rho(x_0, x_1, s, \rho_0)$$

$$Z_{12} = -\beta(x_0, s, \rho_0)\frac{\big[f\big(x_1, s, \hat\rho_0, \rho_1\big)\big]_{\rho_0}}{f\big(x_1, s, \hat\rho_0, \rho_1\big)}, \qquad Z_{13} = -\frac{\big(\beta(x_0, s, \rho_0)\ell(s, \rho_0)\big)_{\rho_0}}{\ell(s, \rho_0)},$$

$$Z_{21}^n = -\sum_{i=1}^{n} K(x_i, s, \rho_i, \hat\rho_{i-1})\Gamma_\rho(x_i, x_{i+1}, s, \rho_i),$$

$$Z_{22}^n = \sum_{i=1}^{n-1} \frac{\big[K(x_i, s, \rho_i, \hat\rho_{i-1})f(x_i, s, \hat\rho_{i-1}, \rho_i)f(x_{i+1}, s, \hat\rho_i, \rho_{i+1})\big]_{\rho_i}}{f(x_i, s, \hat\rho_{i-1}, \rho_i\big)f(x_{i+1}, s, \hat\rho_i, \rho_{i+1})}$$

$$Z_{23}^n = \frac{\big[K(x_n, s, \rho_n, \hat\rho_{n-1})f(x_n, s, \hat\rho_{n-1}, \rho_n)\big]_{\rho_n}}{f(x_n, s, \hat\rho_{n-1}, \rho_n)},$$

$$Z_{31}^n = \sum_{i=1}^{n-1} \frac{\big[f^2\big(x_i, s, \hat\rho_{i-1}, \rho_i\big)f\big(x_{i+1}, s, \hat\rho_i, \rho_{i+1}\big)\big]_{x_i}}{f\big(x_i, s, \hat\rho_{i-1}, \rho_i\big)f\big(x_{i+1}, s, \hat\rho_i, \rho_{i+1}\big)}, \qquad Z_{32}^n = \frac{\big[f^2\big(x_n, s, \hat\rho_{n-1}, \rho_n\big)\big]_{x_n}}{f\big(x_n, s, \hat\rho_{n-1}, \rho_n\big)},$$

$$Z_{33}^n = -\sum_{i=1}^{n} v\big(x_i, s, \hat\rho_{i-1}, \rho_i\big)\big[\Gamma(x_{i-1}, x_i, s, \rho_{i-1}) + \Gamma(x_i, x_{i+1}, s, \rho_i)\big]_{x_i}.$$

Recall that $\mathcal{L}_{0n}^*$ is the adjoint of $\mathcal{L}_{0n}$, and is obtained by an integration by parts. More specifically,

- The sum $Z_{11}+Z_{12}+Z_{13}$ comes from an integration by parts with respect to the variable ρ_0, and the i-terms in Z_{21}^n, Z_{22}^n come from an integration by parts with respect to the variable ρ_i for $i \in \{1, \dots, n-1\}$, and Z_{23}^n comes from an integration by parts with respect to the variable ρ_n. The dynamics of ρ_i as in rule **(2)** of Definition 2.1**(iii)** is responsible for these contributions.

- The i-th terms in Z_{31}^n, Z_{32}^n and Z_{33}^n come from an integration by parts with respect to the variable x_i. The dynamics of x_i as in rule **(1)** of Definition 2.1**(iii)** is responsible for this contribution.

(Step 4) We next focus on the third term on the right-hand side of (4.4). This term can be expressed as

$$Y_0^n = \sum_{i=0}^{n} Y_{0i}^n + \hat Y_0^n, \tag{4.9}$$

where Y_{0i}^n is the boundary contribution coming from the condition $x_i = x_{i+1}$, and $\hat{Y}_0^n$ is the boundary contribution coming from the condition $x_{n+1} = x_{n+2} = a_+$. For $i = 0, \dots, n$

$$Y_{0i}^n = \int_{\Delta_n} \hat{G}^n(\mathbf{q}, s) W_i(\mathbf{q}, s) \mu^n(\mathbf{q}, s) \, d\mathbf{q}, \tag{4.10}$$

where

$$W_0 = \frac{\int f^2(x_0, s, \rho_*, \rho_0) \; \ell(s, d\rho_*)}{\ell(s, \rho_0)}, \qquad W_i = \frac{Q^+(f)\big(x_i, s, \hat{\rho}_{i-1}, \rho_i\big)}{f\big(x_i, s, \hat{\rho}_{i-1}, \rho_i\big)},$$

for $i = 1, \dots, n$. Here,

- The term W_0 comes from the boundary term $x_1 = x_0 = a_-$ in the integration by parts with respect to the variable x_1. This boundary condition represents the event that x_1 has reached x_0 after which ρ_0 becomes ρ_1, and (x_i, ρ_i) is relabeled as (x_{i-1}, ρ_{i-1}) for $i \geq 2$.
- The term W_i comes from the boundary term $x_i = x_{i+1}$. The relative distance $x_{i+1} - x_i$ travels with speed
$$-\big[v(x_{i+1}, s, \hat{\rho}_i, \rho_{i+1}) - v(x_i, s, \hat{\rho}_{i-1}, \rho_i)\big],$$
As x_{i+1} catches up with x_i, the particle x_i disappears and its density $\rho_i = \hat{\rho}_i$ is renamed ρ_*, and is integrated out. (The resulting integral is $Q^+(f)\big(x_i, s, \hat{\rho}_{i-1}, \rho_i\big)$.) We then relabel (x_j, ρ_j), $j > i$, as (x_{j-1}, ρ_{j-1}).

As for $\hat{Y}_0^n$, we simply have

$$\hat{Y}_0^n = -Y_{b,+}^n, \tag{4.11}$$

where $Y_{b,+}^n$ was defined in (4.7).

(Step 5) Recall that we wish to establish (4.4). The identities (4.5), and (4.7)-(4.11) allow us to rewrite (4.4) as

$$X_1 + X_2 + \sum_{i=3}^{11} X_i^n = \sum_{j=1}^{3} Z_{1j} + \sum_{i=2}^{3} \sum_{j=1}^{3} Z_{ij}^n - A(a_+, s, \hat{\rho}_n) + W_0 + W^n,$$

where $W^n = \sum_{i=1}^n W_i^n$. For this we only need to verify

$$X_4^n + \sum_{i=6}^{11} X_i^n = Z_{12} + \sum_{i=2}^{3} \sum_{j=1}^{3} Z_{ij}^n, \tag{4.12}$$

because

$$X_1 = W_0, \qquad X_2 = Z_{13}, \qquad X_3^n = W^n, \qquad X_5^n = Z_{11} - A(a_+, s, \hat{\rho}_n).$$

We use the definition of K to write $Z_{21}^n = Z_{211}^n + Z_{212}^n$, where $Z_{212}^n = X_6^n$, and

$$Z_{211}^n = -\sum_{i=1}^{n} b(x_i, s, \rho_i) v(x_i, s, \hat{\rho}_{i-1}, \rho_i)\big)\Gamma_\rho(x_i, x_{i+1}, s, \rho_i).$$

Hence (4.12) is equivalent to

$$X_4^n + \sum_{i=7}^{11} X_i^n = Z_{12} + Z_{211}^n + Z_{22}^n + Z_{23}^n + \sum_{j=1}^{3} Z_{3j}^n. \tag{4.13}$$

Also observe that the expression

$$\big[\Gamma(x_{i-1}, x_i, s, \rho_{i-1}) + \Gamma(x_i, x_{i+1}, s, \rho_i)\big]_{x_i},$$

equals

$$\begin{aligned}\bigg[\int_{x_{i-1}}^{x_i} &\lambda\big(z, s, \phi^z_{x_{i-1}}(\rho_{i-1}; s)\big)\ dz + \int_{x_i}^{x_{i+1}} \lambda\big(z, s, \phi^z_{x_i}(\rho_i; s)\big)\ dz\bigg]_{x_i} \\ &= \lambda(x_i, s, \hat{\rho}_{i-1}) - \lambda(x_i, s, \rho_i) + \int_{x_i}^{x_{i+1}} \big[\lambda\big(z, s, \phi^z_{x_i}(\rho_i; s)\big)\big]_{x_i}\ dz\end{aligned}$$

From this and (2.6) we learn

$$Z_{211}^n + Z_{33}^n = -\sum_{i=1}^{n} v\big(x_i, s, \hat{\rho}_{i-1}, \rho_i\big)\big(\lambda(x_i, s, \hat{\rho}_{i-1}) - \lambda(x_i, s, \rho_i)\big) = X_7^n.$$

This reduces (4.13) to

$$X_4^n + \sum_{i=8}^{11} X_i^n = Z_{12} + Z_{22}^n + Z_{23}^n + Z_{31}^n + Z_{32}^n. \tag{4.14}$$

Observe that $Z_{22}^n + Z_{23}^n = \widehat{Z}_{22}^n + \widehat{Z}_{23}^n$, and $Z_{31}^n + Z_{32}^n = Z_{311}^n + Z_{312}^n + Z_{313}^n$, where

$$\widehat{Z}_{22}^n = \sum_{i=1}^{n} \frac{\big[K(x_i, s, \rho_i, \hat\rho_{i-1}) f(x_i, s, \hat\rho_{i-1}, \rho_i)\big]_{\rho_i}}{f(x_i, s, \hat\rho_{i-1}, \rho_i)},$$

$$\widehat{Z}_{23}^n = \sum_{i=1}^{n-1} K(x_i, s, \rho_i, \hat\rho_{i-1}) \frac{\big[f(x_{i+1}, s, \hat\rho_i, \rho_{i+1})\big]_{\rho_i}}{f(x_{i+1}, s, \hat\rho_i, \rho_{i+1})},$$

$$Z_{311}^n = \sum_{i=1}^{n} \frac{f_x^2\big(x_i, s, \hat\rho_{i-1}, \rho_i\big)}{f\big(x_i, s, \hat\rho_{i-1}, \rho_i\big)},$$

$$Z_{312}^n = \sum_{i=1}^{n} b\big(x_i, s, \hat\rho_{i-1}\big) \frac{f_{\rho_-}^2\big(x_i, s, \hat\rho_{i-1}, \rho_i\big)}{f\big(x_i, s, \hat\rho_{i-1}, \rho_i\big)},$$

$$Z_{313}^n = -\sum_{i=1}^{n-1} v\big(x_i, s, \hat\rho_{i-1}, \rho_i\big) b\big(x_i, s, \rho_i\big) \frac{\big[f\big(x_{i+1}, s, \hat\rho_i, \rho_{i+1}\big)\big]_{\rho_i}}{f\big(x_{i+1}, s, \hat\rho_i, \rho_{i+1}\big)},$$

where we used (2.5) for the last equation. Observe that by the definition of K,

$$\hat Z_{23}^n + Z_{313}^n = -\sum_{i=1}^{n-1} \beta(x_i, s, \rho_i) \frac{\big[f(x_{i+1}, s, \hat\rho_i, \rho_{i+1})\big]_{\rho_i}}{f(x_{i+1}, s, \hat\rho_i, \rho_{i+1})}.$$

The equation (4.14) follows because

$$X_8^n = \hat Z_{22}^n, \qquad X_9^n + X_{10}^n = Z_{312}^n, \qquad X_4^n = Z_{311}^n, \qquad X_{11}^n = Z_{12} + \hat Z_{23}^n + Z_{313}^n.$$

□

5 Proof of Theorem 1.2

According to Theorem 1.2 if $\rho(\cdot, t_0) = \rho(\cdot, t_0; s, \mathbf{y}^0)$, with $\mathbf{y}^0$ a Markov jump process with jump rate $g^0(x, y_-, y_+)$, then for $t > t_0$, we can express $\rho(\cdot, t) = \rho(\cdot, t; s, \mathbf{y}_t)$, where $\mathbf{y}_t$ is also a jump process with a jump rate $g(x, t; y_-, y_+)$, where g is a solution to the kinetic equation (1.25). There is a one-to-one correspondence between the realization

$$\mathbf{y}(x) = \sum_{i=0}^{\infty} y_i \mathbb{1}\big(x \in [x_i, x_{i+1})\big),$$

and the particle configuration

$$\mathbf{q} = \big((x_0, y_0), (x_1, y_1), \dots\big),$$

with

$$x_0 = a_- < x_1 < \cdots < x_n < \dots, \qquad y_0 < y_1 < \cdots < y_n < \dots.$$

We may translate this into a statement about the law of our particle system $\mathbf{q}(t)$. As before, it suffices to establish a variant of Theorem 1.2 for a finite interval $[a_-, a_+]$. The condition $H_\rho(a_-, t, \rho) > 0$ means that particles can cross a_- only from left to right. Because of this, we can treat a_- as a free boundary point. As it turns out, the point a_+ will be free boundary point (no particle can cross a_+ from right to left) if a_+ is sufficiently large. Indeed as we will see in Proposition 5.2**(iii)**, there are positive constants C_0 and C_1 such that $M(x, t; y, s) \le -C_1 x$ for $x \ge C_0$. On the other hand, by Hypothesis 1.2**(i)**, we know

$$|\rho| \le c_1\big(1 + |H_\rho(x, t, \rho)|\big),$$

which in turn implies that $H_\rho(x, t, \rho) \to -\infty$ as $\rho \to -\infty$. As a result, there exists a positive constant C_2 such that $H_\rho(x, t, \rho) \le 0$, whenever $\rho \le -C_2$. From this we deduce that $\hat{v}(a_+, t, y_-, y_+) > 0$ if $a_+ \ge \max\{C_2 C_1^{-1}, C_0\} =: C_3$. From all this we learn that Theorem 1.2 would follow if we can establish the following result.

Theorem 5.1 *Assume Hypothesis 1.2. For any fixed a_+ such that $a_+ > \max\{a_-, C_3\}$, consider the scalar conservation law* (1.2) *in $[a_-, a_+] \times [t_0, T)$ with initial condition $\rho(x, t_0) = M(x, t_0; \mathbf{y}^0(x), s)$ (restricted to $[a_-, a_+]$), open boundary at $x = a_\pm$. Then for all $t > t_0$, we have $\rho(x, t) = M(x, t; \mathbf{y}_t(x), s)$, where the law of $\big(\mathbf{y}_t(x) :\ x \in [a_-, a_+]\big)$ is as follows:*

- **(i)** *The $x = a_-$ marginal is $\ell(t, dy_0)$, given by $\dot{\ell} = \mathcal{B}^{2*}_{a_-,t}\ell$.*
- **(ii)** *The rest of the path is a PDMP with generator $\mathcal{B}^1_{x,t}$.*

To prove our main result Theorem 1.2, we send $a_+ \to \infty$.

We continue with a preparatory definition.

Definition 5.1(i) The configuration space for our particle system $\mathbf{q}$, is the set $\Delta = \cup_{n=0}^{\infty} \bar{\Delta}_n$, where $\bar{\Delta}_n$ is the topological closure of Δ_n, with Δ_n denoting the set

$$\big\{\mathbf{q} = \big((x_i, y_i) : i = 0, 1, \dots, n\big) :\ x_0 = a_- < x_1 < \cdots < x_n < x_{n+1} = a_+, \quad y_0 < \cdots < y_n\big\}.$$

We write $\mathbf{n}(\mathbf{q})$ for the number of particles i.e., $\mathbf{n}(\mathbf{q}) = n$ means that $\mathbf{q} \in \Delta_n$.

(ii) Given a realization $\mathbf{q} = \big(x_0, y_0, x_1, y_1, \dots, x_n, y_n\big) \in \bar{\Delta}_n$, we define

$$\rho\big(x, t; \mathbf{q}\big) = R_t(\mathbf{q})(x) = \sum_{i=0}^{n} M(x, t; y_i, s) 1\!\!1\big(x_i \le x < x_{i+1}\big).$$

(iii) The process $\mathbf{q}(t)$ evolves according to the following rules:

(1) So long as x_i remains in (x_{i-1}, x_{i+1}), it satisfies $\dot{x}_i = -\hat{v}(x_i, t, y_{i-1}, y_i)$.

(2) When x_1 reaches a_-, we relabel particles (x_i, y_i), $i \geq 1$, as (x_{i-1}, y_{i-1}).

(3) When x_n reaches a_+, a particle is lost and $\mathbf{q}$ enters Δ_{n-1}.

(4) When $x_{i+1} = x_i$, then $\mathbf{q}(t)$ becomes $\mathbf{q}^i(t)$, that is obtained from $\mathbf{q}(t)$ by omitting (x_i, y_i) and relabeling particles to the right of the i-th particle.

□

Some care is needed for the rule **(1)** because $\hat{v}$ given by (1.24) is not a continuous function of x. Recall that $x_i(t)$ represents the location of a shock discontinuity that separates two fundamental solutions. However, the fundamental solution $\big(M(x,t;y_i,s):\ x \in (x_{i-1}(t), x_i(t)\big)$ may also include some shock discontinuities. When $x_i(t)$ catches up with a shock discontinuity of $M(\cdot,t;y_i,s)$, or $M(\cdot,t;y_{i+1},s)$, $\dot{x}_i(\cdot)$ fails to exists. Nonetheless off of such a discrete set of moments, the ODE of **(1)** is well-defined, and this is good enough to determine the evolution of x_i fully.

We write $\widehat{\mathcal{L}} = \widehat{\mathcal{L}}^t$ for the generator of the (inhomogeneous Markov) process $\mathbf{q}(t)$. This generator can be expressed as $\widehat{\mathcal{L}} = \widehat{\mathcal{L}}_0 + \widehat{\mathcal{L}}_b$, where $\widehat{\mathcal{L}}_0$ is the generator of the deterministic part of dynamics, and $\widehat{\mathcal{L}}_b$ represents the Markovian boundary dynamics. The deterministic and stochastic dynamics restricted to Δ_n have generators that are denoted by $\widehat{\mathcal{L}}_{0n}$ and $\widehat{\mathcal{L}}_{bn}$ respectively. While $\mathbf{q}(t)$ remains in Δ_n, its evolution is governed by an ODE of the form

$$\frac{d\mathbf{q}}{dt}(t) = \hat{\mathbf{b}}\big(\mathbf{q}(t), t\big),$$

where $\hat{\mathbf{b}}$ can be easily described with the aid of rule **(1)** of Definition 5.1**(iii)**. We establish Theorem 5.1 by verifying the forward equation

$$\dot{\mu}^n = \big(\widehat{\mathcal{L}}^* \mu\big)^n, \tag{5.1}$$

for all $n \geq 0$, where $\widehat{\mathcal{L}}^*$ is the adjoint of the operator $\widehat{\mathcal{L}}$. We follow our strategy as in Section 3 and use a test function $G(\mathbf{q}, t)$ with is the analog of what we had in (3.4). Again, our Theorem 5.1 would follow if we can show the analog of (3.5). We follow our notation as in (3.2), and the analog of Theorem 3.1 is also valid when $\mathcal{L}$ is replaced with $\widehat{\mathcal{L}}$.

The following variant of Proposition 2.1 ensures that our particle system produces the unique entropy solution of (1.1) in the interval $[a_-, a_+]$.

Proposition 5.1 *The function $\rho(x,t) = \rho(x,t;\mathbf{q}(t))$, with $\mathbf{q}(t)$ evolving as in Definition 5.1***(iii)***, is the unique entropy solution of $\rho_t = H(x,t,\rho)_x$ in $(a_-, a_+) \times (0, \infty)$.*

Proof As in Section 2, we can readily check that $\rho(x,t;\mathbf{q}(t))$ is a weak solution of (1.2) because the Rankin-Hugoniot condition is satisfied. To satisfy the entropy condition, we need to make sure that $\rho(x-,t;\mathbf{q}(t)) < \rho(x+,t;\mathbf{q}(t))$ at each discontinuity point. This is an immediate consequence of the monotonicity of the fundamental solution that is stated in Proposition 5.2**(ii)** below. The uniqueness of the entropy solution follows from the fact that the end points $a_\pm$ are both free. To see this, assume that ρ and ρ' are two solutions that are both concatenations of fundamental solutions. We use Kruzkov's inequality [K] (as in the proof of Proposition 2.1**(iii)**) to assert that weakly,

$$\begin{aligned}|\rho(x,t)-\rho'(x,t)|_t \leq &\big(Q(x,t,\rho(x,t),\rho'(x,t))\big)_x\\ &- \ sgn\big(\rho(x,t)-\rho'(x,t)\big)\big(H_x(x,t,\rho(x,t))-H_x(x,t,\rho'(x,t))\big)\\ \leq&\big(Q(x,t,\rho(x,t),\rho'(x,t))\big)_x + c_1|\rho(x,t)-\rho'(x,t)|,\end{aligned}$$

where $Q(x,t,\rho,\rho') = sgn(\rho-\rho')\big(H(x,t,\rho)-H(x,t,\rho')\big)$. Here we have used Hypothesis 1.2**(i)** for the second inequality. As a consequence

$$\big[e^{-c_1t}|\rho(x,t)-\rho'(x,t)|\big]_t \leq e^{-c_1t}\big(Q(x,t,\rho(x,t),\rho'(x,t))\big)_x.$$

As in the proof of Proposition 2.1**(iii)**, we can integrate over $[a_-,a_+]$ to assert

$$\begin{aligned}\left[e^{-c_1t}\int_{a_-}^{a_+}|\rho(x,t)-\rho'(x,t)|\ dx\right]_t \leq& e^{-c_1t}Q(a_+,t,\rho(a_+,t),\rho'(a_+,t))\\ &- e^{-c_1t}Q(a_-,t,\rho(a_-,t),\rho'(a_-,t)).\end{aligned}$$

We claim that our free boundary conditions at $a_\pm$ imply that the right-hand side is nonpositive. Indeed, $\mp H_\rho(a_\pm,t,M(a_\pm,t;y_-,y_+))\geq 0$ implies

$$Q(a_\pm,t,\rho(a_\pm,t),\rho'(a_\pm,t)) = \mp\big|H(a_\pm,t,\rho(a_\pm,t))-H(a_\pm,t,\rho'(a_\pm,t))\big|.$$

This allows us to assert

$$\left[e^{-c_1t}\int_{a_-}^{a_+}|\rho(x,t)-\rho'(x,t)|\ dx\right]_t \leq 0.$$

As an immediate consequence we learn that if $\rho(x,t_0)=\rho'(x,t_0)$ for all $x\in[a_-,a_+]$, then $\rho(x,t)=\rho'(x,t)$ for all $(x,t)\in[a_-,a_+]\times[t_0,T]$. □

Proposition 5.2 **(i)** *If $x_1 < x_2$, and $\xi(\theta;x_i,t;y,s)$ is a maximizing path in* (1.21) *for $x = x_i$, then $\xi(\theta;x_1,t;y,s) < \xi(\theta;x_2,t;y,s)$ for $\theta\in(s,t]$.*

(ii) *The fundamental solution $M(x,t;y,s)$ is increasing in y.*

(iii) *Given $s < T$, and $\delta \in (0,1)$, there exist positive constants $C_0 = C_0(s,\delta,T)$, and $C_1 = C_1(s,\delta,T)$ such that if $|x| \geq C_0$, and $|y| \leq (1-\delta)|x|$, then $M(x,t;y,s)$ and $-x$ have the same sign, and*

$$C_1|x| \leq |M(x,t;y,s)|. \tag{5.2}$$

Proof(i) It is well known that under Hypothesis 2.1**(i)** the following statements are true (see for example [Go]):

(1) In (1.21) we may take the supremum over those $\xi : [t_0, T]$ such that $\xi(s) = y,\ \xi(t) = x$, and ξ is weakly differentiable with $\dot{\xi} \in L^2\big([s,t];\mathbb{R}\big)$.

(2) If the supremum in attained at ξ, then necessarily $\xi \in C^2$.

(3) The maximizing path ξ satisfies the Euler-Lagrange equation

$$\big(L_v(\xi(\theta),\theta,\dot{\xi}(\theta))\big)_\theta = L_x(\xi(\theta),\theta,\dot{\xi}(\theta)). \tag{5.3}$$

Equivalently, if $p(\theta) = L_v(\xi(\theta),\theta,\dot{\xi}(\theta))$, then the pair (ξ,p) satisfies the Hamiltonian ODE

$$\dot{\xi}(\theta) = -H_\rho(\xi(\theta),\theta,p(\theta)), \quad \dot{p}(\theta) = H_x(\xi(\theta),\theta,p(\theta)). \tag{5.4}$$

We now take $x_1 < x_2$, and write $\xi^1, \xi^2 : [s,t] \to \mathbb{R}$ for the maximizing paths in (1.21) for $x = x_1$ and $x = x_2$ respectively. We wish to show that $\xi^1(\theta) \neq \xi^2(\theta)$ for every $\theta \in (s,t)$. We argue by contradiction. Suppose to the contrary $\xi^1(\theta_0) = \xi^2(\theta_0)$ for some $\theta_0 \in (s,t)$. We define

$$\eta^2(\theta) = \begin{cases} \xi^1(\theta), & \theta \in [s,\theta_0], \\ \xi^2(\theta), & \theta \in [\theta_0,t], \end{cases}, \qquad \eta^1(\theta) = \begin{cases} \xi^2(\theta), & \theta \in [s,\theta_0], \\ \xi^1(\theta), & \theta \in [\theta_0,t]. \end{cases} \tag{5.5}$$

Since ξ^i maximizes the action, and η^i is weakly differentiable with square integrable derivative for $i = 1,2$, we learn

$$\begin{aligned} \int_s^t L\big(\eta^1(\theta),\theta,\dot{\eta}^1(\theta)\big)\,d\theta &\leq \int_s^t L\big(\xi^1(\theta),\theta,\dot{\xi}^1(\theta)\big)\,d\theta, \\ \int_s^t L\big(\eta^2(\theta),\theta,\dot{\eta}^2(\theta)\big)\,d\theta &\leq \int_s^t L\big(\xi^2(\theta),\theta,\dot{\xi}^2(\theta)\big)\,d\theta. \end{aligned} \tag{5.6}$$

Expressing the integrals on the left in terms of ξ^i would lead to

$$\int_s^{\theta_0} L\big(\xi^1(\theta),\theta,\dot{\xi}^1(\theta)\big)\,d\theta = \int_s^{\theta_0} L\big(\xi^2(\theta),\theta,\dot{\xi}^2(\theta)\big)\,d\theta.$$

This in turn implies that we have equality in (5.6). As a result, η^i is also a maximizing path with $\eta^i(t) = x_i$. Hence by (2) above, η^i must be C^1. This means that we must have $\dot\xi^1(\theta_0) = \dot\xi^2(\theta_0)$. Since we also have $\xi^1(\theta_0) = \xi^2(\theta_0)$, we may use the uniqueness of the solutions to Euler-Lagrange equation (5.3), to deduce that $\xi^1 = \xi^2$ on $[s,t]$. This contradicts $x_1 = \xi^1(t) < x_2 = \xi^2(t)$. As a result, ξ^1 and ξ^2 cannot intersect in $(s,t]$. Since $x_1 < x_2$, we must have $\xi^1(\theta) < \xi^2(\theta)$ for $\theta > s$.

(ii) We take $y_1 < y_2$, and write $\xi^1, \xi^2 : [s,t] \to \mathbb{R}$ for the maximizing paths in (1.21) for $z = (y_1, s)$ and $z = (y_2, s)$ respectively. We wish to show that $\xi^1(\theta) \neq \xi^2(\theta)$ for every $\theta \in (s,t)$. We again argue by contradiction. Suppose to the contrary $\xi^1(\theta_0) = \xi^2(\theta_0)$ for some $\theta_0 \in (s,t)$. We define η^1 and η^2 as in (5.5). Again, since ξ^1 (respectively ξ^2) is a maximizer in (1.21), and that η^2 (respectively η^1) is weakly differentiable with square integrable derivative, we learn

$$\begin{aligned}
&\int_s^t L\big(\eta^2(\theta),\theta,\dot\eta^2(\theta)\big)\, d\theta \le \int_s^t L\big(\xi^1(\theta),\theta,\dot\xi^1(\theta)\big)\, d\theta,\\
&\int_s^t L\big(\eta^1(\theta),\theta,\dot\eta^1(\theta)\big)\, d\theta \le \int_s^t L\big(\xi^2(\theta),\theta,\dot\xi^2(\theta)\big)\, d\theta.
\end{aligned} \tag{5.7}$$

Expressing the integrals on the left in terms of ξ^i would lead to

$$\int_{\theta_0}^t L\big(\xi^1(\theta),\theta,\dot\xi^1(\theta)\big)\, d\theta = \int_{\theta_0}^t L\big(\xi^2(\theta),\theta,\dot\xi^2(\theta)\big)\, d\theta.$$

This in turn implies that we have equality in (5.7), and that η^1 (respectively η^2) is also a maximizing path with $\eta^1(s) = y_2$ (respectively $\eta^2(s) = y_1$). Hence by (2) above, η^i must be C^1 for $i = 1,2$. This means that we must have $\dot\xi^1(\theta_0) = \dot\xi^2(\theta)$. By the uniqueness of the corresponding Euler-Lagrange equation, we must have $\xi^1 = \xi^2$ on $[s,t]$. This contradicts $y_1 = \xi^1(s) < y_2 = \xi^2(s)$. As a result, ξ^1 and ξ^2 cannot intersect in $(s,t]$. Since $y_1 < y_2$, we must have $\xi^1(\theta) < \xi^2(\theta)$ for $\theta \le t$. This in particular implies that $\dot\xi^1(t) \ge \dot\xi^2(t)$. Moreover $\dot\xi^1(t) = \dot\xi^2(t)$ would imply $\dot\xi^1 = \dot\xi^2$ by the uniqueness of the corresponding (5.3). As a result, we must have $\dot\xi^1(t) > \dot\xi^2(t)$. This, (1.22), and the strict concavity of L in v imply the desired inequality $M(x,t;y_1,s) < M(x,t;y_2,s)$.

(iii) Recall that the pair (ξ, p) satisfies (5.4), and the boundary conditions

$$\xi(s) = y, \qquad \xi(t) = x, \qquad p(t) = M(x,t;y,s). \tag{5.8}$$

From (5.3) and Hypothesis 1.2**(i)**, we learn

$$|\dot p(\theta)| = \left|\big(L_v(\xi(\theta),\theta,\dot\xi(\theta))\big)_\theta\right| = \left|L_x(\xi(\theta),\theta,\dot\xi(\theta))\right| \le c_1,$$

which in turn implies

$$|p(\theta) - p(t)| \le c_1(t-s), \tag{5.9}$$

for $\theta \in [s,t]$. This, and (5.4) imply

$$|\dot{\xi}(\theta)| = \big|H_\rho(\xi(\theta),\theta,p(\theta))\big| \le c_0c_2^{-1} + c_2^{-1}|p(\theta)| \le c(1+|p(t)|),$$

for a constant $c = c(s,T)$. Here we used

$$-c_0 + c_2\big|H_\rho(x,\theta,\rho)\big| \le |\rho| \le c_0 + c_1\big|H_\rho(x,\theta,\rho)\big|, \tag{5.10}$$

which follows from Hypothesis 1.2**(i)**. As a result,

$$|x-y| = |\xi(t)-\xi(s)| \le c'(1+|p(t)|),$$

for a positive constant $c' = c'(s,T)$. On the other hand, if $|y| \le (1-\delta)|x|$, then we deduce

$$|x| \le c'\delta^{-1}(1+|p(t)|). \tag{5.11}$$

We next claim that there exists a constant C_0 such that

$$|x| \ge C_0 \quad\Longrightarrow\quad xp(t) < 0. \tag{5.12}$$

To see this, observe that by (5.10) and the monotonicity of $\rho \mapsto H_\rho(x,\theta,\rho)$, we can find a constant c'' such that

$$|\rho| \ge c'' \quad\Longrightarrow\quad H_\rho(x,\theta,\rho)\rho \ge 0, \tag{5.13}$$

for all (x,θ). Let us assume that $x \ge C_0$, for a positive constant C_0 (to be determined later). Suppose contrary to (5.12), we have $p(t) \ge 0$. From (5.11) we deduce

$$p(t) \ge (c')^{-1}\delta x - 1.$$

This and (5.9) imply

$$p(\theta) \ge (c')^{-1}\delta x - 1 - c_1(t-s) \ge (c')^{-1}\delta C_0 - 1 - c_1(T-s), \tag{5.14}$$

for all $\theta \in (s,t)$. Choose C_0 large enough so that the right-hand side of (5.14) is at least c''. This would guarantee

$$H_\rho(\xi(\theta),\theta,p(\theta)) \ge 0, \qquad \text{for } \ \theta \in [s,t],$$

by (5.13). From this and (5.4) we deduce that $\dot{\xi}(\theta) \le 0$ for $\theta \in [s,t]$. As a result, $x - y = \xi(t) - \xi(s) \le 0$. But this is impossible if $y \le (1-\delta)x$. Hence the condition $x \ge C_0$ implies that $p(t) > 0$. In the same fashion, we can show that the condition $x \le -C_0$ implies that $p(t) < 0$. This completes the proof of (5.12). From this, (5.8), and (5.11), we can readily deduce (5.2). □

We next give a recipe for the law of the process $\mathbf{y}_t$.

Definition 5.2(i) We set

$$\Gamma(a,b,t,\rho)=\int_a^b \hat{A}(g)(z,t,y)\ dz,\qquad \Gamma(\mathbf{q},t)=\sum_{i=0}^{n}\Gamma(x_i,x_{i+1},t,y_i).$$

(ii) We define a measure $\mu(d\mathbf{q},t)$ on the set Δ that is our candidate for the law of $\mathbf{q}(t)$. The restriction of μ to Δ_n is given by

$$\mu^n(d\mathbf{q},t):=\ell(t,dy_0)\exp\left\{-\Gamma(\mathbf{q},t)\right\}\prod_{i=1}^{n}\ g\big(x_i,t,y_{i-1},y_i\big)\ dx_idy_i,$$

where g solves (1.25) and ℓ solves (1.26). To simplify our presentation, we assume that $\ell(t,dy_0)=\ell(t,y_0)\ dy_0$ is absolutely continuous with respect to the Lebesgue measure. Such an assumption would allow us to express $\mu^n(d\mathbf{q},t):=\mu^n(\mathbf{q},t)\ d\mathbf{q}$, where

$$d\mathbf{q}=dy_0\ \prod_{i=1}^{n}dx_idy_i,\qquad \mu^n(\mathbf{q},t)=\ell(t,y_0)\exp\left\{-\Gamma(\mathbf{q},t)\right\}\prod_{i=1}^{n}\ g\big(x_i,t,y_{i-1},y_i\big).$$

□

Proposition 5.3 *Let g be a solution of* (1.25)*. Then* $\hat{A}(g^1)_t=\hat{A}(g^2)_x$.

Proof From integrating both sides of (1.25) with respect y_+ we learn

(5.15) $$\hat{A}(g^1)_t-\hat{A}(g^2)_x=\hat{A}\big(\hat{Q}^+(g)\big)-\hat{A}\big(g\hat{J}(g)\big).$$

On the other hand,

$$\begin{aligned}\hat{A}\big(\hat{Q}^+(g)\big)(y_-)&=\int g^1(y_-,y_*)\hat{A}(g^2)(y_*)\ dy_*-\int g^2(y_-,y_*)\hat{A}(g^1)(y_*)\ dy_*,\\ \hat{A}\big(g\hat{J}(g)\big)(y_-)&=\int g^1(y_-,y_*)\hat{A}(g^2)(y_*)\ dy_*-\hat{A}(g^2)(y_-)\hat{A}(g^1)(y_-)\\ &\quad-\int g^2(y_-,y_*)\hat{A}(g^1)(y_*)\ dy_*+\hat{A}(g^1)(y_-)\hat{A}(g^2)(y_-).\end{aligned}$$

This implies that the right-hand side of (5.15) is 0. □

We are now ready to present the proof of Theorem 5.1, which is similar to the proof of Theorem 2.1.

Proof of Theorem 5.1 We wish to establish the analog of (4.1) in our setting. Theorem 3.1, and a repetition of the first step of the proof of Theorem 2.1 allow us to reduce the proof of Theorem 5.1 to the verification of an analog of (4.4), namely

$$\begin{aligned}\int_{\Delta_n} \hat{G}^n(\mathbf{q},s)\ \mu_s^n(d\mathbf{q},s) = &\int_{\Delta_n} \hat{G}^n(\mathbf{q},s)\big(\widehat{\mathcal{L}}_{0n}^{s*}\mu^n\big)(\mathbf{q},s)\ d\mathbf{q} \\ &+ \int_{\partial\Delta_{n+1}} \hat{G}^{n+1}(\mathbf{q},s)\mu^{n+1}(\mathbf{q},s)(\mathbf{b}_{n+1}\cdot\mathbf{N}_{n+1})\ \sigma(d\mathbf{q}),\end{aligned} \tag{5.16}$$

with $\hat{G}$ as in (3.5). For a more tractable expression for the left hand side of (5.16), we write

$$\mu_s^n = X^n\mu^n, \tag{5.17}$$

where

$$X^n = -\Gamma_s(\mathbf{q},s) + \frac{\ell_s(s,y_0)}{\ell(s,y_0)} + \sum_{i=1}^{n} \frac{g_s(x_i,s,y_{i-1},y_i)}{g(x_i,s,y_{i-1},y_i)}. \tag{5.18}$$

On the other hand, by Proposition 5.3,

$$\begin{aligned}\Gamma_s(\mathbf{q},s) =& \sum_{i=0}^{n}\int_{x_i}^{x_{i+1}} \hat{A}(g^1)_s(z,s,y_i)\ dz = \sum_{i=0}^{n}\int_{x_i}^{x_{i+1}} \hat{A}(g^2)_z(z,s,y_i)\ dz \\ =& \sum_{i=0}^{n}\big(\hat{A}(g^2)(x_{i+1},s,y_i) - \hat{A}(g^2)(x_i,s,y_i)\big) \\ =& \hat{A}(g^2)(x_{n+1},s,y_n) - \hat{A}(g^2)(x_0,s,y_0) - \sum_{i=1}^{n}\big(\hat{A}(g^2)(x_i,s,y_i) - \hat{A}(g^2)(x_i,s,y_{i-1})\big).\end{aligned}$$

From this, (5.18), (1.26), and (1.25) we deduce

$$\begin{aligned}X^n =& -\hat{A}(g^2)(a_+,s,y_n) + \frac{(\ell * g^2)(a_-,s,y_0)}{\ell(s,y_0)} + \sum_{i=1}^{n}\frac{\hat{Q}^+(g)(x_i,s,y_{i-1},y_i)}{g(x_i,s,y_{i-1},y_i)} \\ &+ \sum_{i=1}^{n}\hat{v}(x_i,s,y_{i-1},y_i)\big(\hat{A}(g)(x_i,s,y_i) - \hat{A}(g)(x_i,s,y_{i-1})\big).\end{aligned} \tag{5.19}$$

We can rewrite the right-hand side of (5.16) as

$$\int_{\Delta_n} \hat{G}^n(\mathbf{q},s)Y^n(\mathbf{q})\mu^n(\mathbf{q},s)\ d\mathbf{q},$$

where $Y^n = Y_1^n + Y_2^n$, with Y_1^n and Y_2^n corresponding to two terms on the right-hand side of (5.16). Indeed, an integration by parts yields

$$
\begin{aligned}
Y_n^1(\mathbf{q}) &= -\sum_{i=1}^{n} \hat{v}_i(x_i, s, y_{i-1}, y_i)\Gamma(s, \mathbf{q})_{x_i} \\
&= \sum_{i=1}^{n} \hat{v}_i(x_i, s, y_{i-1}, y_i)\big(\hat{A}(g)(x_i, s, y_i) - \hat{A}(g)(x_i, s, y_{i-1})\big).
\end{aligned} \tag{5.20}
$$

As for Y_2^n, we write $Y_2^n = Y_{2-}^n + Y_{2*}^n + Y_{2+}^n$, where the terms Y_{2-}^n, Y_{2*}^n, and Y_{2+}^n correspond to the boundary contributions associated with the conditions $x_1 = a_-$, $x_i = x_{i+1}$, with $i \in \{1, \dots, n\}$, and $x_{n+1} = a_+$, respectively. More precisely,

$$
\begin{aligned}
Y_{2-}^n(\mathbf{q}) &= \frac{(\ell * g^2)(a_-, s, y_0)}{\ell(s, y_0)}, \qquad Y_{2+}^n(\mathbf{q}) = -\hat{A}(g^2)(a_+, s, y_n), \\
Y_{2*}^n(\mathbf{q}) &= \sum_{i=1}^{n} \frac{\hat{Q}^+(g)(x_i, s, y_{i-1}, y_i)}{g(x_i, s, y_{i-1}, y_i)}.
\end{aligned}
$$

This, (5.17), (5.19), and (5.20) complete the proof of (5.16). □

6 Proofs of Proposition 1.1 and Theorem 1.3

Proof of Proposition 1.1 Let us write

$$
\mathcal{K}(g) = \nabla \cdot (\nu g) - \hat{Q}^+(g) + \hat{Q}^-(g).
$$

where $\nu = (-\hat{v}, 1)$, and $\nabla = (\partial_x, \partial_t)$. To ease the notation, we do not display the dependence of g, $\hat{g}, \eta$, and h on (x, t). We certainly have

$$
\begin{aligned}
&\frac{\hat{Q}^+(\hat{g})}{\hat{g}} - \frac{\hat{Q}^+(g)}{g} = 0, \\
&\left(\frac{\nabla \cdot (\nu \hat{g})}{\hat{g}} - \frac{\nabla \cdot (\nu g)}{g}\right)(y_-, y_+) = \left(\nu \cdot \frac{\nabla \eta}{\eta}\right)(y_-, y_+) = \nu \cdot \frac{\nabla h}{h}(y_+) - \nu \cdot \frac{\nabla h}{h}(y_-), \\
&\frac{\hat{Q}^-(g)}{g}(y_-, y_+) = \hat{A}(\hat{v}g)(y_+) - \hat{A}(\hat{v}g)(y_-) - \hat{v}(y_-, y_+)\left(\hat{A}(g)(y_+) - \hat{A}(g)(y_-)\right) \\
&\frac{\hat{Q}^-(\hat{g})}{\hat{g}}(y_-, y_+) = \frac{\hat{A}(\hat{v}g \otimes h)}{h}(y_+) - \frac{\hat{A}(\hat{v}g \otimes h)}{h}(y_-) \\
&\qquad - \hat{v}(y_-, y_+)\left(\frac{\hat{A}(g \otimes h)}{h}(y_+) - \frac{\hat{A}(g \otimes h)}{h}(y_-)\right).
\end{aligned}
$$

Hence

$$\begin{aligned}\left(\frac{\mathcal{K}(\hat g)}{\hat g}-\frac{\mathcal{K}(g)}{g}\right)(y_-,y_+) =&\nu(y_-,y_+)\cdot\left(\frac{\nabla h}{h}(y_+)-\frac{\nabla h}{h}(y_-)\right)+\left(\frac{\hat Q^-(\hat g)}{\hat g}-\frac{\hat Q^-(g)}{g}\right)(y_-,y_+)\\ =&\frac{h_t+\mathcal{B}^2h}{h}(y_+)-\frac{h_t+\mathcal{B}^2h}{h}(y_-)\\ &-\hat v(y_-,y_+)\left[\frac{h_x+\mathcal{B}^1h}{h}(y_+)-\frac{h_x+\mathcal{B}^1h}{h}(y_-)\right].\end{aligned}$$

The right-hand side is 0, when h satisfies (1.31). This completes the proof because g (respectively $\hat g$) solves (1.25) if and only if $\mathcal{K}(g)=0$ (respectively $\mathcal{K}(\hat g)=0$). □

The proof of Theorem 1.3, uses Doob's h-transform that we now recall.

Proposition 6.1 *Let $\mathbb{P}$ be the law of Markov jump process $\big(\mathbf{y}(x):\ x\in[a_-,a_+]\big)$, with the jump kernel density $g(x,y_-,y_+)$, and the generator $\mathcal{L}_x$. Assume that g is C^1 in x. Let U be an interval, and let $\widehat{\mathbb{P}}$ denote the law of $\mathbb{P}$, conditioned on the event $\mathbf{y}(x)\in U$ for all $x\in[a_-,a_+]$. Then $\widehat{\mathbb{P}}$ is the law of a Markov jump process with a jump kernel density $\hat g$, given by*

$$\hat g(x,y_-,y_+)=\frac{h(x,y_+)}{h(x,y_-)}\,g(x,y_-,y_+),\qquad x\in[a_-,a_+),\quad y_\pm\in U, \tag{6.1}$$

where

$$h(x,y)=\mathbb{P}\big(\mathbf{y}(a)\in U\quad for\ \ a\in[x,a_+]\mid \mathbf{y}(x)=y\big). \tag{6.2}$$

Moreover, h is C^1 in x, and satisfies

$$h_x+\mathcal{L}_xh=0. \tag{6.3}$$

Proof *(Step 1)* We can write

$$h(x,y)=\sum_{n=0}^{\infty}h_n(x,y), \tag{6.4}$$

where

$$h_n(x,y)=\int_{X_n(x,y)}\mu^n(\mathbf{q},x,y)\ d\mathbf{q}_n \tag{6.5}$$

where for $n \geq 1$,

$$\mathbf{q}_n = (x_1, y_1, \ldots, x_n, y_n), \qquad d\mathbf{q}_n = \prod_{i=1}^{n} dx_i dy_i,$$

$$\mu^n(\mathbf{q}_n, x, y) = \exp\{-\Gamma(\mathbf{q}_n, x, y)\} \prod_{i=1}^{n} g\big(x_i, y_{i-1}, y_i\big), \quad \text{with} \quad y_0 = y,$$

$$\Gamma(\mathbf{q}_n, x, y) = \sum_{i=0}^{n} \Gamma(x_i, x_{i+1}, y_i), \qquad \text{with} \qquad x_0 = x,\ y_0 = y,$$

$$\Gamma(a, b, y) = \int_a^b (\hat{A}g)(z, y)\ dz,$$

and the set $X_n(x, y)$ consists of $\mathbf{q}$, satisfying

$$x < x_1 < \cdots < x_n < a_+, \qquad y_1, \ldots, y_n \in U.$$

When $n = 0$, we simply have $h_0(x, y) = \exp\big\{-\Gamma(x, a_+, y)\big\}$. It is straightforward to verify continuous differentiability of h, and deduce (6.3) from (6.4) and (6.5).

(Step 2) The law $\hat{P}$ is simply given by

$$\hat{P} = \sum_{n=1}^{\infty} \hat{\mu}^n,$$

where $\hat{\mu}_n(d\mathbf{q}_n) = \hat{\mu}_n(\mathbf{q}_n)\ d\mathbf{q}_n$, with

$$\hat{\mu}_n(\mathbf{q}_n) = h(a_-, y)^{-1}\ \mu_n(\mathbf{q}_n)\ \mathbb{1}\big(y_1, \ldots, y_n \in U\big). \tag{6.6}$$

We wish to show that $\hat{P}$ is the law of a jump process associated with the jump density $\hat{g}$. To achieve this, we rewrite $\hat{\mu}_n$ using the fact that h satisfies (6.3). Indeed, (6.3) implies

$$e^{-\int_a^b \hat{A}(g)(z, y_-)dz}\ \frac{h(b, y_+)}{h(a, y_-)} = e^{-\int_a^b \hat{A}(\hat{g})(z, y_-)dz}\ \frac{h(b, y_+)}{h(b, y_-)}. \tag{6.7}$$

This is equivalent to asserting

$$\begin{aligned} \frac{h(b, y_-)}{h(a, y_-)} &= \exp\left(-\int_a^b \big(\hat{A}(\hat{g}) - \hat{A}(g)\big)(z, y_-)\ dz\right) \\ &= \exp\left(-\int_a^b \left(\frac{\hat{A}(g \otimes h) - \hat{A}(g)h}{h}\right)(z, y_-)\ dz\right) \\ &= \exp\left(\int_a^b \frac{-(\mathcal{L}_z h)(z, y_-)}{h(z, y_-)}\ dz\right) = \exp\left(\int_a^b \frac{h_z(z, y_-)}{h(z, y_-)}\ dz\right) \\ &= \exp\left(\int_a^b (\log h)_z(z, y_-)\ dz\right), \end{aligned} \tag{6.8}$$

which is evidently true. We set, $x_0 = a_-$, $y_0 = y$ as before. Observe that $\hat{\mu}_n(\mathbf{q}_n)$ of (6.6) can be written as

$$\frac{1}{h(a_+, y_n)} e^{-\int_{x_n}^{a_+} \hat{A}(g)(z,y_n)dz} \frac{h(a_+, y_n)}{h(x_n, y_n)} \prod_{i=1}^{n} e^{-\int_{x_{i-1}}^{x_i} \hat{A}(g)(z,y_{i-1})dz} \frac{h(x_i, y_i)}{h(x_{i-1}, y_{i-1})} g(x_i, y_{i-1}, y_i)$$
$$= \frac{1}{h(a_+, y_n)} e^{-\int_{x_n}^{a_+} \hat{A}(\hat{g})(z,y_n)dz} \prod_{i=1}^{n} e^{-\int_{x_{i-1}}^{x_i} \hat{A}(\hat{g})(z,y_{i-1})dz} \frac{h(x_i, y_i)}{h(x_i, y_{i-1})} g(x_i, y_{i-1}, y_i)$$
$$= e^{-\int_{x_n}^{a_+} \hat{A}(\hat{g})(z,y_n)dz} \prod_{i=1}^{n} e^{-\int_{x_{i-1}}^{x_i} \hat{A}(\hat{g})(z,y_{i-1})dz} \hat{g}(x_i, y_{i-1}, y_i),$$

where we used (6.8) and (6.7) for the first equality, and for the last equality we used the definition of $\hat{g}$, and $h(a_+, y_n) = 1$, which follows from the definition of h. The right-hand side is the law of a Markov jump process associated with the kernel density $\hat{g}$, as desired. □

Proof of Theorem 1.3 *(Step 1)* Recall that $\rho(x,t)$ is the solution of (1.2) with the initial condition $\rho(x,t_0) = \rho(x,t_0;\mathbf{y}_{t_0}, s)$, where $\mathbf{y}_{t_0}$ is a jump process associated with the kernel $g(x,t_0,y_-,y_+)$. We wish to show that $\rho(x,t) = \rho(x,t;\mathbf{y}_t,s)$ for $(x,t) \in [a_-,\infty) \times [t_0,T]$. It suffices to verify this for $(x,t) \in [a_-,a_+] \times [t_0,T]$, where a_+ is any large number in (a_-,∞).

Pick any $\delta \in (0,1)$. From Proposition 5.2**(iii)**, and (5.13), we learn that there exist constants $C_0 = C_0(\delta)$, $C_1 = C_1(\delta)$, and C_2 such that

(6.9) $$|y_+| \le (1-\delta)a_+, \quad a_+ \ge C_0 \implies M(a_+, t_0; y_+, s) < -C_1 a_+,$$

(6.10) $$(x,\theta) \in \mathbb{R} \times [s,T], \quad |\rho| \ge C_2 \implies \rho H_\rho(x,t,\rho) > 0,$$

for every $t \in [t_0,T]$. Note that (6.9) and Proposition 5.2**(ii)** imply that if $a_+ \ge C_0$, then

$$y_- < y_+, \ |y_+| < (1-\delta)a_+, \implies M(a_+,t_0;y_-,s) < M(a_+,t_0;y_+,s) < -C_1 a_+.$$

From this, and (6.2) we can readily deduce

(6.11) $$Y_- \le y_- < y_+ \le (1-\delta)a_+, \quad a_+ \ge C_3 \implies \hat{v}(x,t,y_-,y_+) > 0,$$

for every $t \in [t_0,T]$, where
$$C_3 = \max\left\{C_0, C_1^{-1}C_2, |Y_-|\right\}.$$
We pick any $a_+ \ge \max\{a_-, C_3\}$.

(Step 2) We write $\mathbb{W}$ for the law of the Markov process $(\mathbf{w}(t) : \ t \in [t_0,T])$, associated with the generator $\mathcal{B}^2_{a_-,t}$, such that $\mathbf{w}(t_0) = y^0$. We also define a family of probability measures $\big(\mathbb{P}_t : t \in [t_0,T]\big)$ with the following recipe: For each t, $\mathbb{P}_t$ is the law of the Markov process

$\mathbf{y}_t : [a_-, a_+] \to [Y_-, \infty)$, associated with the generator $\mathcal{B}^1_{x,t}$, satisfying the initial condition $\mathbf{y}_t(a_-) = \mathbf{w}(t)$. We define θ_δ to be the smallest $t > t_0$ such that $\mathbf{w}(t) \notin U(\delta)$, where

$$U(\delta) := [Y_-, (1-\delta)a_+) =: [Y_-, Y^\delta).$$

We also define $\tau_\delta(t)$ to be the smallest $x > a_-$ such that $\mathbf{y}_t(x) \notin U(\delta)$. We set

$$\mathbb{W}^\delta(A) = \mathbb{W}(A \mid \theta_\delta > T), \qquad \mathbb{P}^\delta_t(A) = \mathbb{P}_t(A \mid \tau_\delta(t) > a_+).$$

We write $\mathbf{w}^\delta(t),\ t \in [t_0, T]$ for the jump process that is distributed according to $\mathbb{W}^\delta$. For each $t \in [t_0, T]$, we write $\mathbf{y}^\delta_t$ for the jump process that is distributed according to $\mathbb{P}^\delta_t$. By Proposition 6.1, the process $t \to \mathbf{w}^\delta(t)$, and the processes $x \mapsto \mathbf{y}^\delta_t(x),\ t \in [t_0, T]$ are again Markov jump processes. We set

$$\begin{aligned} h^\delta(x, t, y) &= \mathbb{P}^\delta_t\big(\tau_\delta(t) > a_+ \mid \mathbf{y}_t(x) = y\big), \\ \ell^\delta(t, y) &= \mathbb{W}\big(\theta_\delta > T \mid \mathbf{w}(t) = y\big). \end{aligned}$$

By (6.3), we have the following equations for h^δ and ℓ^δ:

$$h^\delta_x(x, t, y) + \big(\mathcal{B}^1_{x,t} h^\delta\big)(x, t, y) = 0, \qquad x \in (a_-, a_+),\ y \in U(\delta),\ t \in [t_0, T], \tag{6.12}$$

$$\ell^\delta_t(t, y) + \big(\mathcal{B}^2_{a_-,t} \ell^\delta\big)(s, y) = 0, \qquad t \in (t_0, T),\ y \in U(\delta). \tag{6.13}$$

Since, $h^\delta(a_-, t, y) = \ell^\delta(t, y)$, the equations (6.12) and (6.13) allow us to apply Proposition 1.2 to assert that h^δ satisfies

$$h^\delta_x(x, t, y) + \big(\mathcal{B}^1_{x,t} h^\delta\big)(x, t, y) = 0, \qquad h^\delta_t(x, t, y) + \big(\mathcal{B}^2_{x,t} h^\delta\big)(x, t, y) = 0. \tag{6.14}$$

(Step 3) Since the jump process $\mathbf{y}^\delta_{t_0}$ takes value in $[Y_-, (1-\delta)a_+)$, our Theorem 1.2 or Theorem 5.1 is applicable. More precisely, if the initial data of (1.2) is given by $\rho(x, t_0; \mathbf{y}^\delta_{t_0}, s)$, for some $s < t_0$, and with $y^\delta_{t_0}$ a Markov process distributed according to $\mathbb{P}^{a_+,\delta}_{t_0}$, then the solution $\rho(x, t)$ at a later time $t > t_0$ is given by $\mathbb{P}^\delta_t$. As a consequence, Theorem 1.3 is true when the initial process satisfies $\mathbf{y}_{t_0}(a_+) \in U(\delta)$. This condition is true with probability density $h^\delta(a_-, t_0, y^0)$. The condition (1.29) would implies that $h^\delta(a_-, t_0, y^0) \to 1$, and $\mathbf{y}^\delta_t \to \mathbf{y}_t$ in small δ limit, when restricted to the interval $[a_-, a_+]$. This completes the proof. □

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