Pointwise Bounds for the Solutions of the Smoluchowski Equation with Diffusion

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Abstract

We prove various decay bounds on solutions $(f_n : n > 0)$ of the discrete and continuous Smoluchowski equations with diffusion. More precisely, we establish pointwise upper bounds on $n^\ell f_n$ in terms of a suitable average of the moments of the initial data for every positive ℓ. As a consequence, we can formulate sufficient conditions on the initial data to guarantee the finiteness of $L^p(\mathbb{R}^d \times [0,T])$ norms of the moments $X_a(x,t) := \sum_{m\in\mathbb{N}} m^a f_m(x,t)$, ($\int_0^\infty m^a f_m(x,t)dm$ in the case of continuous Smoluchowski's equation) for every $p \in [1,\infty]$. In previous papers [11] and [5] we proved similar results for all weak solutions to the Smoluchowski's equation provided that the diffusion coefficient $d(n)$ is non-increasing as a function of the mass. In this paper we apply a new method to treat general diffusion coefficients and our bounds are expressed in terms of an auxiliary function $\phi(n)$ that is closely related to the total increase of the diffusion coefficient in the interval $(0,n]$.

1 Introduction

The Smoluchowski equation is a system of partial differential equations that describes the evolving densities of a system of diffusing particles that are prone to coagulate in pairs and fragment into pairs. A family of functions $f_n : \mathbb{R}^d \times [0,\infty) \to [0,\infty)$, $n \in \mathbb{N}$, is a solution of the discrete Smoluchowski equation if it satisfies

$$\frac{\partial}{\partial t} f_n(x,t) = d(n)\Delta f_n(x,t) + Q_n(f)(x,t), \tag{1.1}$$

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with $Q_n = Q_n^C + Q_n^F$, $Q_n^C = Q_n^{C,+} - Q_n^{C,-}$, $Q_n^F = Q_n^{F,+} - Q_n^{F,-}$, where

$$\begin{aligned}
Q_n^{C,+}(f)(x,t) &= \frac{1}{2}\sum_{m=1}^{n-1} \alpha(m, n-m) f_m(x,t) f_{n-m}(x,t), \\
Q_n^{C,-}(f)(x,t) &= \sum_{m=1}^{\infty} \alpha(n,m) f_n(x,t) f_m(x,t), \\
Q_n^{F,+}(f)(x,t) &= \sum_{m=1}^{\infty} \beta(n,m) f_{n+m}(x,t), \\
Q_n^{F,-}(f)(x,t) &= \frac{1}{2}\sum_{m=1}^{n-1} \beta(m, n-m) f_n(x,t).
\end{aligned}$$

Here $\alpha(\cdot,\cdot), \beta(\cdot,\cdot) \geq 0$, $d(\cdot) > 0$, and $d(\cdot)$ is bounded. We will interpret this solution in a weak sense. Namely, we assume that $Q_n^{C,\pm}, Q_n^{F,\pm} \in L^1(\mathbb{R}^d \times [0,T])$ for each $T \in [0,\infty)$ and $n \in \mathbb{N}$, and that

$$f_n(x,t) = S_t^{d(n)} f_n^0(x) + \int_0^t S_{t-s}^{d(n)} Q_n(x,s) ds, \tag{1.2}$$

where $\{f_n^0 : n \in \mathbb{N}\}$ denotes the initial data, S_t^D the semigroup associated with the equation $u_t = D\Delta u$, and where $Q_n(x,s)$ means $Q_n(f)(x,s)$.

In the continuous case, the summations $\sum_{m=1}^{n-1}$, and $\sum_{m=1}^{\infty}$, are replaced with $\int_0^n\, dm$, and $\int_0^\infty\, dm$, respectively. Similarly, a function $f : \mathbb{R}^d \times (0,\infty) \times [0,\infty) \to [0,\infty)$, $f_n(x,t) = f(x,n,t)$, is a solution to the continuous Smoluchowski equation if $f_n, Q_n^{C,\pm}, Q_n^{F,\pm} \in L^1(\mathbb{R}^d \times [0,T])$, for each $T, n \in (0,\infty)$, and (1.2) holds.

The existence of solutions to (1.1) under the assumption

$$\lim_{n\to\infty} \frac{\alpha(n,m)}{n} = 0 \tag{1.3}$$

has been established in Wrzosek [11], [13], Mischler-Rodriguez Ricard [9] and Laurençot–Mischler [8], and in [7] when the equation (1.1) is formulated in a bounded domain. In this case one can prove the existence of a solution by first replacing α and β by a suitable cutoff rates $\alpha^{(N)}$ and $\beta^{(N)}$, and pass to the limit. More precisely, $\alpha^{(N)}$ and $\beta^{(N)}$ are defined by

$$\alpha^{(N)}(n,m) = \begin{cases} \alpha(n,m) & \text{if } n \text{ and } m \leq N, \\ 0 & \text{otherwise,} \end{cases} \tag{1.4}$$

$$\beta^{(N)}(n,m) = \begin{cases} \beta(n,m) & \text{if } n + m \leq N, \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

It is straight forward to prove the existence of a unique solution $f^{(N)}$ associated with $\alpha^{(N)}$ and $\beta^{(N)}$ under a mild assumption on the initial data. We then show that such a sequence $\{f^{(N)}\}_{N\in\mathbb{N}}$ has a convergent subsequence in L^1-sense and that each limit point f is a weak solution to (1.1). We say a solution f to (1.1) is *regular* if f is obtained by the above approximation procedure.

The main goal of this article is to obtain a pointwise and L^p-bounds on weak solutions of the Smoluchowski's equation. Such bounds are obtained for all weak solutions of (1.1) under some regularity assumptions on the initial data and growth conditions on $\alpha(\cdot,\cdot)$ and $d(\cdot)$ provided that there is no fragmentation. In the presence of fragmentation, our bounds are established for regular solutions as described in the previous paragraph.

As a sample of what can be achieved by our approach, we state L^p bounds on solutions when there is no fragmentation and the diffusion coefficient is uniformly positive. Such L^p bounds are achieved in two steps: First we obtain a decay bound on solutions provided that sufficiently large moments of solutions are integrable. We then recall a theorem from [5] and [11] to give sufficient conditions for integrability of any given moment of solutions. Our precise assumptions are described in Hypothesis 1.1.

Hypothesis 1.1

- (i) The function $d(\cdot)$ is uniformly positive and uniformly bounded:

$$0 < \underline{d} = \inf_n d(n) \le \bar{d} = \sup_n d(n) < \infty.$$

- (ii) The total increase variation of $d(\cdot)$ is finite. By this we mean that the total positive variation of $\log d(\cdot)$ is finite. More precisely,

$$\hat{d} := \prod_{n=1}^{\infty} \max\left\{1, \frac{d(n+1)}{d(n)}\right\} < \infty,$$

  in the discrete setting and

$$\hat{d} := \sup_{n_i} \prod_{i=1}^{\infty} \max\left\{1, \frac{d(n_{i+1})}{d(n_i)}\right\} < \infty,$$

  in the continuous setting. Here the supremum is taken over increasing positive sequences $\{n_i\}$.

- (iii) There exists a constant a_0 such that $\alpha(m,n) \le a_0 mn$ for all m and n.

□

We also write

$$X_a = X_a(x,t) = \sum_n n^a f_n(x,t), \quad X_a^0(x) = X_a(x,0),$$

for the moments of f (in the continuous setting, the summation is replaced with an integration). We next define a set D by

$$D := \left\{ (k,\eta) : 2 < k \in \mathbb{N}, \quad \eta > \frac{d+2-2k^{-1}}{2-4k^{-1}} \right\},$$

when the dimension is at least 2, and by $D := \{(4,\eta) : 2\eta > 5\}$, when $d = 1$. We are now ready to state our first result.

Theorem 1.1 *Assume Hypothesis 1.1 and that $\beta \equiv 0$. There exists a constant $C_0 = C_0(\underline{d}, \bar{d}, \hat{d}; k, \eta)$ such that for every $(k,\eta) \in D$ and $p \in [1,\infty]$,*

$$\|f_n(\cdot,t)\|_{L^p} \le \|f_n^0\|_{L^p} + a_0 2^\ell C_0 \; n^{-\ell} \left(\int_0^t \int X_{\ell\eta+1} dx ds \right)^{\eta^{-1}} \left\| X_1^0 * \psi_k \right\|_{L^{p(2-\eta^{-1})}}^{2-\eta^{-1}}, \tag{1.6}$$

where $\psi_k(x) = |x|^{\frac{2}{k}-d}$.

As an example of $(k,\eta) \in D$, we may choose $k = 4$ and $\eta = d+2$. An immediate consequence of Theorem 1.1 is the inequality

$$\|f_n(\cdot,t)\|_{L^p} \le \|f_n^0\|_{L^p} + a_0 2^\ell C_0' \; n^{-\ell}, \tag{1.7}$$

provided that the initial condition satisfies

$$\left\| X_1^0 * \psi_k \right\|_{L^{p(2-\eta^{-1})}} =: A_0 < \infty, \tag{1.8}$$

and that $X_{\ell\eta+1}$ is integrable. We now describe a theorem that gives sufficient conditions for the integrability of moments of f. For this we need another set of condition on α and $d(\cdot)$.
Hypothesis 1.2 The function $d(\cdot)$ is positive and uniformly bounded. Moreover,

$$\lim_{n+m\to\infty} \frac{\alpha(n,m)}{(n+m)(d(n)+d(m))} = 0.$$

□

We also set

$$\tau_0(x) = \begin{cases} |x|^{2-d} & \text{if } d \ge 3, \\ -\frac{1}{2\pi} \log |x| \;\; \mathbb{1}(|x| \le 1) & \text{if } d = 2, \\ \frac{1}{2}(1-|x|) \;\; \mathbb{1}(2|x| \le 1) & \text{if } d = 1. \end{cases}$$

We now recall a result on the L^1 bounds that will be needed in this paper and its proof can be found in [5].

Theorem 1.2 *Assume Hypotheses 1.2 and that* $\beta \equiv 0$*. Then for every* $a \geq 2$ *and positive* A *and* T*, there exists a constant* $C_1 = C_1(a, A, T)$ *such that, if* f *is a solution and*

$$\iint X_a^0(X_1^0 * \tau_0)dx \leq A, \quad \operatorname{ess\,sup}(X_a^0 + X_0^0) * \tau_0 \leq A, \quad \int (X_a^0 + X_0^0)dx \leq A, \tag{1.9}$$

then

$$\sup_{t\in[0,T]} \int X_a(x,t)dx \leq C_1.$$

Moreover, when $d \geq 3$*, the constant* C_1 *can be chosen to be independent of* T*.*

Combining Theorems 1.1 and 1.2 yields the following corollary.

Corollary 1.1 *Assume Hypotheses 1.1 and 1.2 and that* $\beta \equiv 0$*. There exists a constant*

$$C_2 = C_2(\underline{d}, \bar{d}, \hat{d}; k, \eta; A_0, A, T),$$

such that if $(k, \eta) \in D$ *and the initial condition* f^0 *satisfies* (1.9) *for* $a = \eta\ell + 1$*and* (1.8)*, then*

$$\|f_n(\cdot, t)\|_{L^p} \leq \|f_n^0\|_{L^p} + a_0 2^\ell C_2 \, n^{-\ell}, \tag{1.10}$$

for every $p \in [1, \infty]$ *and* $t \in [0, T]$*.*

Remark 1.1.

- (i) We note that Hypothesis 1.2 implies Hypothesis 1.1(iii) for sufficiently large a_0.
- (ii) The choice of $(k, \eta) \in D$ only affects the constant C_2 and the nature of bounds we need to assume on the initial data. This is because our assumption (1.8) depends on k via ψ_k and our choice of $a = \eta\ell + 1$ in (1.9) depends on η.
- (iii) In Hammond–Rezakhanlou [6], [7], Rezakhanlou [12] and, Yaghouti–Rezakhanlou–Hammond [16] the equation (1.1) was derived from a microscopic model of coagulating Brownian particles ($\beta \equiv 0$). In these articles the macroscopic coagulation rate α does satisfy Hypothesis 1.2. In the model studied in [12], each particle has a mass m and radius r and the relationship between the mass and radius is given by $r = r(m) = m^\chi$ for a nonnegative parameter χ. In terms of this parameter,

  $$\alpha(m, n) \leq c_0(d(m) + d(n))(r(m) + r(n))^{d-2},$$

  for a suitable positive constant c_0 and whenever $d > 2$. As it was discussed in [6] and [12], the condition $\chi > (d-2)^{-1}$ is equivalent to the "instantaneous" formation of gels and must be excluded for the validity of the macroscopic equation (1.1). In fact the equation (1.1) was derived in [12] under the assumption $\chi < (d-2)^{-1}$, which is exactly what is needed in order to satisfy Hypothesis 1.2.

- (iv) In [5] and [11] we also showed

$$\int_0^T \int \sum_{n,m} (n^{a-1}m + m^{a-1}n)\alpha(n,m)(f_n f_m)(x,t)dxdt < \infty, \tag{1.11}$$

  under the assumptions of Theorem 1.2. As it was shown in [5] and [11], (1.11) implies the conservation of mass when $a = 2$. We also refer to [4] where a different approach is utilized to treat the mass conservation.

- (v) When $\beta \equiv 0$, there is a unique weak solution of (1.1) on the interval $[0, T]$ among those satisfying $X_2 \in L^\infty(\mathbb{R}^d \times [0, T])$. This is a generalization of the uniqueness theorem of Ball and Carr [3], and was shown in [5]. On account of Corollary 1.1, we have $X_2 \in L^\infty$ if the same is true initially and the conditions (1.9) and (1.8) are true for some $\ell > 3$ and some $(k, \eta) \in D$.

$\square$

We now turn to the case of a diffusion coefficient $d(\cdot)$ that is not uniformly positive. When $d(\cdot)$ is not uniformly positive, (1.6) takes a more complicated form that depends on a choice of an auxiliary function ϕ. In the discrete setting we define ϕ by setting $\phi(1) = 1$ and

$$\phi(n) = \prod_{m=1}^{n-1} \min\left\{1, \frac{d(m)}{d(m+1)}\right\}, \tag{1.12}$$

for $n > 1$. The Hypothesis 1.1 in this case is replaced with a set of assumptions that are formulated for a given fixed integer k satisfying $kd > 2$.

Hypothesis 1.3(k)

- (i) The function $d(\cdot)$ is positive and $\bar{d} = \sup_n d(n) < \infty$.
- (ii) There exist constants $a_0, e_0 \geq 0$, such that $\alpha(n,m) \leq a_0(n^{e_0-1} + m^{e_0-1})\gamma_k(n)\gamma_k(m)$, where $\gamma_k(m) = md(m)^{d/2}\phi(m)^{\frac{kd}{2}-1}$.

$\square$

The moments X_a in (1.6) will be replaced with more complicated expressions involving ϕ:

$$\begin{aligned} \hat{X}_a(x,t) &= \hat{X}_{a;k,\phi}(x,t) = \sum_n n^{a-1}\gamma_k(n) f_n(x,t), \quad \hat{X}_a^0(x) = \hat{X}_a(x,0) \\ \bar{X}_a(x,t) &= \bar{X}_{a;k,\phi}(x,t) = \sum_n n^{a-1}d(n)^{-\frac{1}{k}}\gamma_k(n) f_n(x,t). \quad \bar{X}_a^0(x) = \bar{X}_a(x,0), \end{aligned} \tag{1.13}$$

We are now ready to state our generalization of Theorem 1.1 in the discrete setting.

Theorem 1.3 *Assume that $\beta \equiv 0$ and that Hypothesis 1.3(k) holds for some $k \in \mathbb{N}$. Pick η so that $(k, \eta) \in D$. There exists a constant $C_3 = C_3(\bar{d}; k, \eta)$ such that for every $p \in [1, \infty]$,*

$$\begin{aligned} \|f_n(\cdot,t)\|_{L^p} \leq & \|f_n^0\|_{L^p} \\ & + a_0 2^\ell C_3 \; d(n)^{-R(k,\eta)} \; n^{-\ell} \left(\int_0^t \int \hat{X}_{(\ell+e_0-1)\eta+1} dx ds \right)^{\eta^{-1}} \left\| \bar{X}_1^0 * \psi_k \right\|_{L^{p(2-\eta^{-1})}}^{2-\eta^{-1}}, \end{aligned} \tag{1.14}$$

where $\psi_k(x) = |x|^{\frac{2}{k}-d}$ and $R(k, \eta) = 1 - 2k^{-1} - (1 - k^{-1})\eta^{-1}$.

For the analog of Corollary 1.1, we need to make sure that the term $d(n)^{-R(k,\eta)}$ does not annul our polynomial decay term $n^{-\ell}$. A polynomial decay lower bound on the diffusion coefficient would do the job.

Corollary 1.2 *Assume Hypotheses 1.3(k) and 1.2, and that $\beta \equiv 0$. We also assume that there exist constants $r_1 > 0$ and $b_1 \geq 0$ such that*

$$d(n) \geq r_1 n^{-b_1}. \tag{1.15}$$

Then there exists a constant

$$C_4 = C_4(\bar{d}; r_1; k, \eta; A_0, A, T),$$

such that if $(k, \eta) \in D$ and the initial condition f^0 satisfies

$$\left\| \bar{X}_1^0 * \psi_k \right\|_{L^{p(2-\eta^{-1})}} \leq A_0,$$

and (1.9) for

$$a = (\ell + e_0 - 1)\eta + 1,$$

then

$$\|f_n(\cdot,t)\|_{L^p} \leq \|f_n^0\|_{L^p} + a_0 2^\ell C_4 \; n^{-\ell + b_1 R(k,\eta)}, \tag{1.16}$$

for every $p \in [1, \infty]$.

Remark 1.2.

- (i) Observe that ϕ has the following two properties:

  $$\phi(\cdot) \text{ is nonincreasing and } \; d(\cdot)\phi(\cdot) \text{ is nonincreasing.} \tag{1.17}$$

  In fact ϕ is uniquely determined as the largest function satisfying (1.17) and the normalization condition $\phi(1) = 1$.

- (ii) Observe that the total increase variation of $d(\cdot)$ is finite if and only if ϕ is uniformly positive (simply because $\phi(\infty) = \hat{d}^{-1}$). From this we learn that if Parts (i) and (ii) of Hypothesis 1.1 are satisfied then Hypothesis 1.1(iii) is equivalent to Hypothesis 1.3(k)(ii) for the choice of $e_0 = 1$, and that Theorem 1.3 and Corollary 1.2 are equivalent to Theorem 1.1 and Corollary 1.1 respectively. (In Corollary 1.2, we simply choose $b_1 = 0$.)

- (iii) We note that if for positive constants r_1 and r_2, and nonnegative constants b_1 and b_2, we have

$$d(n) \geq r_1 n^{-b_1}, \quad \phi(n) \geq r_2 n^{-b_2}, \tag{1.18}$$

for all n, then Hypothesis 1.3(k)(ii) is satisfied provided that for some constant a_0',

$$\alpha(m,n) \leq a_0' \left(n^{e_0-1} + m^{e_0-1}\right)(nm)^{-b_3}, \tag{1.19}$$

for

$$b_3 = \frac{db_1}{2} + \left(\frac{kd}{2} - 1\right) b_2 - 1.$$

For example, if $\alpha(m,n) \leq a_0 nm$, then (1.19) is true for any $e_0 \geq 2b_3 + 3$.

- (iii) We note that we can use the bound $\hat{X}_{a;k,\phi} \leq c(k) X_a$ and Theorem 1.2 to bound $\hat{X}_{a;k,\phi}$ in terms of the moments of the initial data. Also note that if we have a decay bound for $d(\cdot)$ and $\phi(\cdot)$ (in case any of these two functions converge to 0), we may use $\hat{X}_{a,k} \leq c'(k) X_{a'}$ instead for a suitably $a' < a$ to use lower moments to bound the right-hand side of (1.14).

□

To extend Theorem 1.3 to the continuous setting, we need a candidate for our auxiliary function ϕ. For our arguments to work, we only need a function ϕ that satisfies the conditions of (1.17). To this end, let us set $\mathcal{A}(d(\cdot))$ to be the set of continuous functions $\phi : (0,\infty) \to (0,\infty)$ such that both $\phi(\cdot)$ and $\phi(\cdot)d(\cdot)$ are non-increasing. Given $\phi \in \mathcal{A}(d(\cdot))$, we define $\hat{X}$ and $\bar{X}$ by replacing n-summations in (1.13) with $\int_0^\infty$-integrations.

Theorem 1.4 *For a given $\phi \in \mathcal{A}(d(\cdot))$, the conclusions of Theorem 1.3 and Corollary 1.2 are true under the assumptions of Theorem 1.3 and Corollary 1.2 respectively.*

We address the question of the existence of ϕ in the next Proposition.

Proposition 1.1

- *(i) Suppose that the function $\log d(\cdot)$ has a finite positive variation in every interval $(0,n]$ with $n > 0$. Then there exists a positive continuous function $\phi \in \mathcal{A}(d(\cdot))$.*

- *(ii) If the function* $\log d(\cdot)$ *has a finite positive variation in* $(0,\infty)$*, then we can find* $\phi \in \mathcal{A}(d(\cdot))$ *that is uniformly positive.*

- *(iii) If the function* $\log d(\cdot)$ *has a finite positive variation in* $(0,\infty)$ *and* $d(\cdot)$ *is uniformly bounded and positive near* 0*, then we can find a bounded* $\phi \in \mathcal{A}(d(\cdot))$*.*

We skip the proof of Proposition 1.1 because Part (i) is exactly Lemma 3.2 of [16] and Parts (ii) and (iii) can be established in a similar way.

We finally turn to the case of nonzero β. As is evident from Corollary 1.2, we can bound L^p norms of moments of solutions in terms of various moments of initial data provided that $\beta \equiv 0$ and we have suitable growth bounds on the diffusion coefficient $d(\cdot)$ and the coagulation rate α. Unfortunately our results in the presence of fragmentation are not as satisfactory and only reduce L^∞ bounds to an appropriate L^r bounds of certain moments of solutions.

We assume

- $\phi \in \mathcal{A}(d(\cdot))$ and ϕ is bounded in the continuous setting.

- ϕ is defined by (1.12) in the discrete setting.

Given such ϕ and a positive integer k, the relevant assumption on the fragmentation rate takes the following form.

Hypothesis 1.4(k) There exist constants $a_1, e_1 \geq 0$, such that

$$\beta(m,n) \leq a_1(m+n)^{e_1-1}\gamma_k(m+n).$$

$\square$

Remark 1.3. What we have in mind is that if for example we are in discrete setting and there exist positive constants r_1, r_2 and a_0, and nonnegative constants b_2 and b_1 such that,

$$r_1 n^{-b_1} \leq d(n), \quad r_2 n^{-b_2} \leq \phi(n), \quad \alpha(m,n) \leq a_0 nm, \quad \beta(m,n) \leq a_1(m+n)^{e_2},$$

for every $m, n > 0$, then we may choose e_0 and e_1 sufficiently large so that Part (ii) of Hypothesis 1.3(k) and Hypothesis 1.4(k) are valid for a given k. $\square$

We set $\chi_k(n) = \gamma_k(n) \max\{d(n)^{-1/k}, n^{e_1+1}\}$ and define

$$\tilde{X}(x,t) = \sum_n \chi_k(n) f_n(x,t). \tag{1.20}$$

Theorem 1.5 *Assume Hypotheses 1.3(k) and 1.4(k). Pick η so that $(k,\eta) \in D$. We also pick $r > kd/2$ and set $b = \frac{r}{r-1}(1 - \frac{2}{kd})$. There exist constants $C_5 = C_5(\bar{d}; k, \eta, \zeta; A)$ and $C_6 = C_6(\bar{d}; k, \eta, \zeta; A)$ such that if*

$$\sup_{s\in[0,t]} \int \left(\hat{X}_{(\ell+e_0-1)\eta+1} + \hat{X}_{(\ell+e_1-1)\eta+1} + \tilde{X}\right)(x,s)\ dx,\ \left\|\bar{X}_1^0 * \psi_k\right\|_{L^\infty} \le A, \tag{1.21}$$

then

$$\|f_n(\cdot,t)\|_{L^\infty} \le \|f_n^0\|_{L^\infty} + (a_0+a_1)2^\ell\ d(n)^{-R'(k,\eta)}\ n^{-\ell}\left(C_5 + C_6 Y^{b(2-\eta)}\right), \tag{1.22}$$

where $\psi_k(x)$ is as in Theorem 1.3, $R'(k,\zeta) = (1-k^{-1})(1-\eta^{-1})$, and

$$Y = \operatorname{esssup}_x \operatorname{esssup}_{s\in[0,t]} \|\tilde{X}(\cdot,t)\|_{L^r(B_1(x))},$$

with $B_1(x)$ denoting the ball of radius 1 and center x.

Theorem 1.6 below would allow us to bound the first expression on the left-hand side of (1.21) in terms of various moments of the initial data. However we do not know how to control Y in terms of initial data. Unfortunately the exponent $b(2-\eta) > 1$ even though $b < 1$. This prevents us to to obtain a bound for Y by multiplying both sides of (1.22) by $\chi_k(n)$ and summing over n.

We now describe a theorem that gives sufficient conditions for the integrability of moments of f. Its proof can be found in [11].

Theorem 1.6 *Assume Hypotheses 1.2 and that for every $\ell > 0$, there exists a constant $c(\ell)$ such that for every m and n with $m \le \ell$,*

$$\beta(n,m) \le c(\ell)n.$$

Then for every $a \ge 2$ and positive A and T, there exists a constant $C_1' = C_1'(a,A,T)$ such that, if f is a regular solution and (1.9) is true, then and

$$\sup_{t\in[0,T]} \int X_a(x,t)dx \le C_1'.$$

In summary, the main results of this article are Theorems 1.1 and 1.3, in which we obtain L^p-bounds for solutions of (1.1) in terms of L^1 norm of various moments of the solution and appropriate bounds of initial data. We then apply Theorems 1.2 to bound L^1 norms of various moments under suitable assumptions of the parameters of our PDE (1.1) and the initial data.

We obtain our L^p-bounds on the solutions from analogous pointwise bounds that are more technical to state and are left for Sections 2 and 4. In fact the main results of this paper are achieved in three steps:

- (i) Pointwise bounds on the total mass $X_1 := \sum_n nf_n$ when $\beta \equiv 0$ and the diffusion coefficient $d(\cdot)$ satisfies Hypothesis 1.1(i)-(ii). If $d(n)$ goes to 0 in large n limit, pointwise bounds are established for a suitable variant of the total mass X_1.

- (ii) We use (i) to find pointwise polynomial decay bounds on f_n as $n \to \infty$.

- (iii) We use (ii) to bound $\|f_n\|_{L^p}$ for $p \in [1, \infty]$.

These three steps are carried out as follows.

- Step (i) was achieved in [5] and [11] provided that the diffusion coefficient is a non-increasing function of the mass size n. One of the main contribution of this work (Theorems 2.1 and 2.2 of Section 2) is a new approach that would allow us to achieve Step (i) for general diffusion coefficients with no monotonicity restriction. However, instead of the full (x,t) pointwise bound on the mass density, we only achieve a pointwise bound on $\int_0^\infty X_1^k(x,t)\, dt$, $k \geq 1$, in terms of the initial data. (As is demonstrated in Remark 2.1 (iii), this bound in turn implies a pointwise bound on $ess \sup_t X_1(x,t)$ if $d(\cdot)$ is also non-increasing.)

- Step (ii) is achieved in Theorems 4.1 and 4.2 of Section 4.

- Step (iii) will be achieved by simply taking the L^p norm of both sides of the pointwise bounds we obtain in Step (ii).

As we mentioned earlier, in the presence of fragmentation, our bounds are valid for regular solutions and not all weak solutions. This restriction however applies to L^1 bounds and our results concerning with Steps (i-iii) are valid for all weak solutions.

For some related works, we refer to Amann [1], Amann and Walker [2] (local existence and uniqueness), Wrzosek [13]-[15] (existence, uniqueness and mass conservation for almost constant diffusion coefficient), Laurençot and Mischler [9]-[10] (existence and regularity of solutions), Mischler and M. Rodriguez Ricard [10] (existence of solutions) and Canizo et al [4] (conservation of mass in bounded domains).

We only present our proofs in the discrete case because the continuous case can be treated by verbatim arguments. The organization of the paper is as follows.

- In Section 2, we state several pointwise bounds on solutions as we described in Step (i) above.

- Section 3 is devoted to the proof of the results of Section 2.

- In Section 4, we state several pointwise bounds on solutions as we discussed in Step (ii) above.

- Section 5 is devoted to the proofs of pointwise bounds we state in Section 4.

- In Section 6 we show how the results of Sections 2 and 4 imply Theorems 1.3 and 1.5. (Recall that by Remark 1.2(ii), Theorem 1.3 implies Theorem 1.1.)

2 Pointwise Bounds I

In this section we learn how to bound the first moment (or its variant when $d(\cdot)$ does not satisfy Hypothesis 1.1(i,ii)) in terms of the initial data. Ultimately we need an inequality that works in general. However the form of such inequality is rather complicated and for motivational purposes we would rather start from the simplest case, namely when $\beta \equiv 0$ and $d(\cdot)$ satisfies conditions (i) and (ii) of Hypothesis 1.1. Recall $X_1 = \sum_n n f_n$ and set

$$Z_k(x) = \left[\int_0^\infty X_1(x,t)^k dt \right]^{1/k}. \tag{2.1}$$

Theorem 2.1 *Assume $\beta \equiv 0$ and that the diffusion coefficient $d(\cdot)$ satisfies conditions (i) and (ii) of Hypothesis 1.1. Then there exists a constant $C_7 = C_7(\bar{d}, \hat{d}, k)$ such that every weak solution of the discrete Smoluchowski's equation* (1.1) *satisfies*

$$Z_k(x) \le C_7 \int_{\mathbb{R}^d} |z-x|^{\frac{2}{k}-d} \sum_n n f_n(z,0) dz = C_7 \left(\psi_k * X_1\right)(x,0). \tag{2.2}$$

for almost all x.

When $d(\cdot)$ is not uniformly positive, (2.2) takes a more complicated form that depends on our auxiliary function ϕ. For general $d(\cdot)$, we need to replace Z_k in (2.2) with the following variant of the first moment:

$$Z_k^{T,\phi}(x) := \left[\int_0^T \hat{X}_1^k dt \right]^{\frac{1}{k}} = \left[\int_0^T \left(\sum_n \gamma_k(n) f_n(x,t) \right)^k dt \right]^{\frac{1}{k}}, \qquad Z_k^\phi := Z_k^{\infty,\phi}, \tag{2.3}$$

where $\gamma_k(m) = m d(m)^{d/2} \phi(m)^{\frac{kd}{2}-1}$.

We are now ready to state our generalization of Theorem 2.1. Recall the definition of $\bar{X}$, $\hat{X}$ and $\tilde{X}$ that were given in (1.13) and (1.20).

Theorem 2.2 *Assume Hypothesis 1.4(k) for some integer $k > \max\{2/d, d/2\}$. Recall $\psi_k(x) = |x|^{\frac{2}{k}-d}$ and let $\phi \in \mathcal{A}(d(\cdot))$. Then for almost all x,*

$$\begin{aligned} Z_k^{T,\phi}(x)^k \le & c_0(k,d)^k \left(\bar{X}_1 * \psi_k\right)^k (x,0) \\ & + k c_0(k,d)^k c_1(kd) a_1 \int_0^T \left(\tilde{X}_1 * \psi_k\right)^k (x,t) dt. \end{aligned} \tag{2.4}$$

Here c_1 is a suitable constant and $c_0(k,d) = c_0(kd)^{\frac{1}{k}}(k!)^{\frac{1}{k^2}-\frac{d}{2k}}$, where $c_0(kd) = (kd-2)^{-1}\omega_{kd}^{-1}$, with ω_{kd} denoting the surface area of the unit sphere in $\mathbb{R}^{kd}$.

Remark 2.1.

- (i) In the presence of fragmentation, we will not be able to get a pointwise bound on $Z_k^{T,\phi}$ in terms of initial data as in Theorem 2.1 because the right-hand side of (2.4) involves moments of the solution.

- (ii) When $\beta \equiv 0$, or equivalently $a_1 = 0$ in Hypothesis 1.4(k), we may send $T \to \infty$ to obtain

  $$Z_k^{\phi}(x) \le c_0(k,d)\left(\bar{X}_1 * \psi_k\right)(x,0). \tag{2.5}$$

  On the other hand, under the assumptions of Theorem 2.1, we can find a bounded uniformly positive $\phi \in \mathcal{A}(d(\cdot))$ (see Proposition 1.1). If we use such ϕ, then we can readily deduce that Theorem 2.2 implies Theorem 2.1 by sending k to infinity. This is because the constant $c_0(k,d)$ is bounded in k (See (2.7) below), and the expression $d(n)^{-1/k}$ is bounded in (k,n).

- (iii) When $d(\cdot)$ is non-increasing, we may choose $\phi \equiv 1$ in Theorem 2.2. If we also assume that $\beta \equiv 0$ (equivalently $a_1 = 0$), we may send k to infinity on both sides of (2.4) to deduce

  $$\sup_t \sum_n nd(n)^{d/2} f_n(x,t) \le C_1 \int_{\mathbb{R}^d} |z-x|^{-d} \sum_n nd(n)^{d/2} f_n(z,0)dz, \tag{2.6}$$

  for almost all x and a constant C_1. The constant C_1 can be readily calculated with the aid of Stirling's formula:

  $$\log C_1 = \lim_{k\to\infty} k^{-1}\log\left(c_0(kd)(k!)^{\frac{1}{k}-\frac{d}{2}}\right) = \frac{d}{2}\log\frac{d}{2\pi}. \tag{2.7}$$

  In particular, if the right-hand side of (2.6) is in L^p, $p \in [1,\infty]$, then the expression $\sup_t \hat{X}_1 = \sup_t \sum_n nd(n)^{d/2} f_n$ is in L^p.

- (iv) The boundedness of $\hat{X}_1$ was established in [5] and [11] using a different argument. In these references, what we really used was the elementary fact that the operator $D^{d/2}S_t^D$ has a Gaussian kernel that is increasing in the diffusion coefficient D. This means that the operator $d(n)^{d/2}S_t^{d(n)}$ is non-increasing as a function of the mass size n. In the present article we switch to the time averages so that ultimately we are dealing with the operator

  $$d(n)^{d/2-1}\Delta^{-1} = \int_0^\infty d(n)^{d/2}S_t^{d(n)}dt,$$

which is a negative operator when $d \geq 3$. Though for higher $k > 1$ we are really dealing with the operator $\mathbb{A}_k = d(n_1)\Delta_{x_1} + \cdots + d(n_k)\Delta_{x_k}$ and its inverse $\mathbb{A}_k^{-1}$. In this case, we need to introduce the auxiliary function ϕ to preserve certain monotonicity of the operator $\mathbb{A}_k^{-1}$. We refer the reader to the proof of Theorem 2.2 in Section 3 for more details.

- (v) Note that the right-hand sides of (2.2), and (2.4) involve expressions of the form $h * \psi_k$ with h a suitable moment of the initial data. It is straight forward to bound $\|h * \psi_k\|_{L^p}$ for every $p \in [1, \infty]$ in terms of h. Let us write $\|h\|_{r_1,r_2}$ for $\|h\|_{L^{r_1}} + \|h\|_{L^{r_2}}$. Now using Young's inequality and the fact that $\psi_k \mathbb{1}(|x| \leq 1) \in L^{r_1}, \psi_k \mathbb{1}(|x| \geq 1) \in L^{r_2}$ whenever $r_2^{-1} < 1 - 2(kd)^{-1} < r_1^{-1}$, we learn that if
$$1 + p^{-1} = r_1^{-1} + s_1^{-1} = r_2^{-1} + s_2^{-1},$$
then
$$\|h * \psi_k\|_{L^p} \leq c(p, k, r_1, r_2)\|h\|_{s_1,s_2},$$
with $c(p, k, r_1, r_2)$ a finite constant. Hence, if (p, k, s_1, s_2) satisfies $s_1^{-1} < p^{-1}+2(dk)^{-1} < s_2^{-1}$, then
$$\|h * \psi_k\|_{L^p} \leq c'(p, k, s_1, s_2)\|h\|_{r_1,r_2}.$$

- (vi) A microscopic analog of (2.4) was established in [16] in order to control the correlation among particles in the particle system studied in this reference.

□

As we mentioned in Remark 2.1 (ii) Theorem 2.2 implies Theorem 2.1. Theorem 2.2 will be established in the next section.

3 Proof of Theorem 2.2

Step1. To ease the notation, let us write $\mathbf{x}, \mathbf{n}$ for $(x_1, \dots, x_k)$ and $(n_1, \dots, n_k)$ respectively, and define $\Lambda^{\mathbf{n}}K(\mathbf{x})$ to be
$$c_0(kd) \int \left(\frac{|x_1 - z_1|^2}{d(n_1)} + \cdots + \frac{|x_k - z_k|^2}{d(n_k)} \right)^{1-\frac{kd}{2}} K(z_1, \dots, z_k) \prod_{r=1}^{k} d(n_r)^{-d/2} \, dz_r.$$
The operator $\Lambda^{\mathbf{n}}$ is defined so that $\Delta_{\mathbf{n}}\Lambda^{\mathbf{n}}K = -K$, for
$$\Delta_{\mathbf{n}} = d(n_1)\Delta_{x_1} + \cdots + d(n_k)\Delta_{x_k}.$$

Pick a positive integer ℓ and a bounded non-negative smooth function $K : (\mathbb{R}^d)^k \to \mathbb{R}$, and set

$$G(t) = \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \sum_{\mathbf{n}} \Lambda^{\mathbf{n}} K\big(\mathbf{x}\big) \prod_{r=1}^{k} \gamma_k^{\ell}\big(n_r\big) f_{n_r}(x_r, t) dx_r.$$

where $\gamma_r^\ell(n) = \gamma_r(n)\mathbb{1}(n \le \ell)$. We wish to calculate dG/dt. For this we use $\Delta_{\mathbf{n}}\Lambda^{\mathbf{n}}K = -K$ and the identity

$$\sum_n a_n Q_n = \sum_{n,m} \big(\alpha(m,n) f_n f_m - \beta(m,n) f_{m+n}\big)\big(a_{m+n} - a_m - a_m\big).$$

As a result, if $(f_n : n \in \mathbb{N})$ is a solution to (1.1), then we have

$$\frac{dG}{dt}(t) = -\Omega_1(t) + \sum_{j=1}^{k} \Omega_{2,j}^{C}(t) - \sum_{j=1}^{k} \Omega_{2,j}^{F}(t), \tag{3.1}$$

where

$$\begin{aligned}
\Omega_1(t) &= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} K\big(\mathbf{x}\big) \sum_{\mathbf{n}} \prod_{r=1}^{k} \gamma_k^\ell\big(n_r\big) f_{n_r}(x_r, t) dx_r, \\
\Omega_{2,j}^{C}(t) &= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \sum_{\mathbf{n}_j, n_j', n_j''} \Gamma^C(\mathbf{x}, \mathbf{n}_j, n_j', n_j'')\alpha(n_j', n_j'') dx_j \prod_{r \ne j} \gamma_k^\ell\big(n_r\big) f_{n_r}(x_r, t) dx_r, \\
\Omega_{2,j}^{F}(t) &= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \sum_{\mathbf{n}_j, n_j', n_j''} \Gamma^F(\mathbf{x}, \mathbf{n}_j, n_j', n_j'')\beta(n_j', n_j'') dx_j \prod_{r \ne j} \gamma_k^\ell\big(n_r\big) f_{n_r}(x_r, t) dx_r,
\end{aligned}$$

where

$$\begin{aligned}
\Gamma^C(\mathbf{x}, \mathbf{n}_j, n_j', n_j'') &= \Gamma(\mathbf{x}, \mathbf{n}_j, n_j', n_j'') f_{n_j'}(x_j, t) f_{n_j''}(x_j, t), \\
\Gamma^F(\mathbf{x}, \mathbf{n}_j, n_j', n_j'') &= \Gamma(\mathbf{x}, \mathbf{n}_j, n_j', n_j'') f_{n_j'+n_j''}(x_j, t).
\end{aligned}$$

with

$$\Gamma(\mathbf{x}, \mathbf{n}_j, n_j', n_j'') = \Lambda^{\mathbf{n}^j(n_j'+n_j'')} K\big(\mathbf{x}\big)\gamma_k^\ell(n_j' + n_j'') - \Lambda^{\mathbf{n}^j(n_j')} K\big(\mathbf{x}\big)\gamma_k^\ell(n_j') - \Lambda^{\mathbf{n}^j(n_j'')} K\big(\mathbf{x}\big)\gamma_k^\ell(n_j''),$$

By $\mathbf{n}^j(m)$ we mean that the j-th component n_j of $\mathbf{n}$ is replaced with m and by $\mathbf{n}_j$ we mean that the j-th component n_j of $\mathbf{n}$ is dropped. We now claim that $\Omega_{2,j}^C \le 0$. This would follow provided that we can show $\Gamma(\mathbf{x}, \mathbf{n}_j, n_j', n_j'') \le 0$. Since $n_j' + n_j'' \le \ell$ implies that $n_j' \le \ell$ and $n_j'' \le \ell$, and since $K \ge 0$, it suffices to show

$$\begin{aligned}
(m+n)\phi(m+n)^{\frac{kd}{2}-1} \left(\frac{A}{d(m+n)} + B\right)^{1-\frac{kd}{2}} \le & m\phi(m)^{\frac{kd}{2}-1} \left(\frac{A}{d(m)} + B\right)^{1-\frac{kd}{2}} \\
& + n\phi(n)^{\frac{kd}{2}-1} \left(\frac{A}{d(n)} + B\right)^{1-\frac{kd}{2}},
\end{aligned}$$

for every pair of positive numbers A and B. This would be the case if we can show

$$\phi(m+n)^{\frac{kd}{2}-1}\left(\frac{A}{d(m+n)}+B\right)^{1-\frac{kd}{2}} \le \phi(m)^{\frac{kd}{2}-1}\left(\frac{A}{d(m)}+B\right)^{1-\frac{kd}{2}},$$

for every pairs of positive numbers (A,B) and (m,n). For this, it suffices to show that for any pair of positive numbers A' and B,

$$\phi(m+n)^{\frac{kd}{2}-1}\left(A'\frac{d(m)}{d(m+n)}+B\right)^{1-\frac{kd}{2}} \le \phi(m)^{\frac{kd}{2}-1}(A'+B)^{1-\frac{kd}{2}},$$

or equivalently,

$$\phi(m+n)(A'+B) \le \phi(m)\left(A'\frac{d(m)}{d(m+n)}+B\right). \tag{3.2}$$

We are done because the assertion (3.2) for fixed m,n and all positive A' and B is equivalent to the inequalities

$$\phi(m)d(m) \ge \phi\big(m+n\big)d\big(m+n\big),$$

and

$$\phi(m) \ge \phi(m+n),$$

both being satisfied, and these are true for all choices of m and n because $\phi \in \mathcal{A}(d(\cdot))$ (see (1.17)). From $\Omega^C_{2,j} \le 0$ and (3.1) we deduce

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} \sum_{\mathbf{n}} K(\mathbf{x}) \prod_{r=1}^k \gamma_k^\ell\big(n_r\big) f_{n_r}(x_r,t)dx_r\ dt \\
&\quad \le \int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} \sum_{\mathbf{n}} \Lambda^{\mathbf{n}} K(\mathbf{x}) \prod_{r=1}^k \gamma_k^\ell\big(n_r\big) f_{n_r}(x_r,0)dx_r \\
&\qquad + 2\int_0^T \int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} \sum_j \sum_{\mathbf{n}_j,n'_j,n''_j} f_{n'_j+n''_j}(x_j,t)\Lambda^{\mathbf{n}^j(n'_j)} K\big(\mathbf{x}\big)\gamma_k^\ell(n'_j)\beta(n'_j,n''_j)dx_j \\
&\qquad\qquad \cdot \prod_{r\neq j} \gamma_k^\ell\big(n_r\big) f_{n_r}(x_r,t)dx_r\ dt,
\end{aligned} \tag{3.3}$$

because

$$-\Gamma(\mathbf{x},\mathbf{n}_j,n'_j,n''_j) \le \Lambda^{\mathbf{n}^j(n'_j)} K\big(\mathbf{x}\big)\gamma_k^\ell(n'_j) + \Lambda^{\mathbf{n}^j(n''_j)} K\big(\mathbf{x}\big)\gamma_k^\ell(n''_j).$$

Step 2. By choosing K to approximate the measure $\delta_x(dx_1)\delta_x(dx_2)\dots\delta_x(dx_k)$ in (2.3) and sending $\ell \to \infty$, we obtain that for almost all x, the expression

$$\int_0^T \left(\sum_n \gamma_k(n) f_n(x,t)\right)^k dt,$$

is bounded above by

$$\begin{aligned}
&\sum_{\mathbf{n}} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \int \lambda^{\mathbf{n}}(x_1 - x, \ldots, x_k - x) \prod_{r=1}^k \gamma_k\big(n_r\big) f_{n_r}(x_r, 0) dx_r \\
&\quad + 2\int_0^T \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \sum_j \sum_{\mathbf{n}_j, n'_j, n''_j} f_{n'_j + n''_j}(x_j, t) \lambda^{\mathbf{n}^j(n'_j)}(x_1 - x, \ldots, x_k - x)\gamma_k(n'_j)\beta(n'_j, n''_j) dx_j \\
&\qquad \cdot \prod_{r \neq j} \gamma_k\big(n_r\big) f_{n_r}(x_r, t) dx_r \; dt,
\end{aligned}$$

where

$$\lambda^{\mathbf{n}}(z_1, \ldots, z_k) = c_0(kd) \left(\prod_{r=1}^k d(n_r)^{-\frac{d}{2}}\right) \left(\frac{|z_1|^2}{d(n_1)} + \cdots + \frac{|z_k|^2}{d(n_k)}\right)^{1-\frac{kd}{2}}$$

From the elementary inequality $k! a_1^2 \ldots a_k^2 \le (a_1^2 + \cdots + a_k^2)^k$, we deduce

$$\lambda^{\mathbf{n}}(z_1, \ldots, z_k) \le c_0(kd)(k!)^{\frac{1}{k} - \frac{d}{2}} \prod_{r=1}^k |z_r|^{\frac{2}{k} - d} d(n_r)^{-\frac{1}{k}}.$$

This implies that $Z_k^{T,\phi}(x)^k$ is bounded above by

$$\begin{aligned}
&c_0(k,d)^k \left[\int_{\mathbb{R}^d} |z - x|^{\frac{2}{k} - d} \sum_n n d(n)^{\frac{d}{2} - \frac{1}{k}} \phi(n)^{\frac{kd}{2} - 1} f_n(z, 0) dz\right]^k \\
&\quad + 2k c_0(k,d)^k \int_0^T \left[\int_{\mathbb{R}^d} |z - x|^{\frac{2}{k} - d} \sum_n n d(n)^{\frac{d}{2} - \frac{1}{k}} \phi(n)^{\frac{kd}{2} - 1} f_n(z, t) dz\right]^{k-1} X(x,t) dt, \\
&= \big(c_0(k,d) \left(\bar{X}_1 * \psi_k\right)(x, 0)\big)^k + 2k c_0(k,d)^k \int_0^T \big(\left(\bar{X}_1 * \psi_k\right)(x, t)\big)^{k-1} X(x,t) dt, \qquad (3.4)
\end{aligned}$$

where

$$X(x,t) = \int_{\mathbb{R}^d} |z - x|^{\frac{2}{k} - d} \sum_n \left(\sum_{m=1}^{n-1} m d(m)^{\frac{d}{2} - \frac{1}{k}} \phi(m)^{\frac{kd}{2} - 1} \beta(m, n - m)\right) f_n(z, t) dz$$

It remains to bound X. From $\phi \in \mathcal{A}(d(\cdot))$ and Hypothesis 1.4(k) we deduce

$$\sum_{m=1}^{n-1} md(m)^{\frac{2}{d}-\frac{1}{k}}\phi(m)^{\frac{kd}{2}-1}\beta(m,n-m) \le c_1 a_1 n^{e_1+1}\gamma_k(n),$$

where $c_1 = \max\left\{(\sup_n d(n)+1)^{2/d}, \sup_n \phi(n)^{(kd)/2-1}\right\}$. Hence

$$X \le c_1 a_1 \left(\hat{X}_{e_1+2} * \psi_k\right).$$

This and (3.4) imply (2.4) because $\hat{X}_1, \bar{X}_1 \le \tilde{X}$ by the definition of $\tilde{X}$. □

4 Pointwise Bounds II

This section is devoted to Step (ii) of our strategy. More precisely we state pointwise bounds on the solution as a preparation for the proof of our L^p bounds. Since we already know how to bound Z_k or Z_k^ϕ by Theorems 2.1 and 2.2, we find a pointwise bound on a solution f_n in terms of L^1 bounds of moments (X_a or its variant $\hat{X}_a$ of (1.13)) and a convolution of Z_k with a suitable kernel. Let us define a set D' that is related to the set D of Theorems 1.1 and 1.3, but happen to be larger: When dimension $d \ge 2$, we set

$$D' := \left\{(k,\eta) : 2 < k, \quad \eta > 1 + (k-2)^{-1}\right\},$$

and when $d = 1$, $D' = D_1' \cup D_2'$, where

$$\begin{aligned} D_1' &:= \left\{(k,\eta) : 2 < k \le 4, \quad \eta > 1 + (k-2)^{-1}\right\}, \\ D_2' &:= \left\{(k,\eta) : 4 < k, \quad 2 + 6(k-4)^{-1} > \eta > 1 + (k-2)^{-1}\right\}. \end{aligned}$$

For simplicity, we first discuss the case $\beta \equiv 0$. Recall $R(k,\eta)$ that was defined in Theorem 1.3. Also recall $\bar{d} = \sup_n d(n)$.

Theorem 4.1 *Assume that $\beta \equiv 0$ and that for some constant a_0, we have $\alpha(n,m) \le a_0 nm$. Pick any pair $(k,\eta) \in D'$. Then there exists a constant $C_8 = C_8(k,\eta)$ such that every weak solution f of (1.1) satisfies*

$$\begin{aligned} (4.1) \qquad f_n(x,t) \le{}& \left(S_t^{d(n)} f^0\right)(x) \\ &+ C_8 a_0 2^\ell n^{-\ell} d(n)^{-R(k,\eta)} \left[\int_0^t \int X_{\ell\eta+1}\, dy\, ds\right]^{\eta^{-1}} \left(Z_k^{1+\frac{\eta}{\eta-1}} * \xi_{k,\eta}^{t\bar{d}}(x)\right)^{1-\eta^{-1}}, \end{aligned}$$

for almost all x and every t. Here $\ell > 0$ and

$$\xi^a_{k,\eta}(y) = |y|^{\theta(k,\eta)} \exp\left(\frac{-|y|^2}{8a(1-\eta^{-1})}\right), \tag{4.2}$$

with

$$\theta(k,\eta) = \frac{2R(k,\eta) - d}{1-\eta^{-1}}.$$

Remark 4.1. We note that $\xi^a_{k,\eta} \in L^1$ if and only if $\theta(k,\eta) + d > 0$. This is equivalent to assuming

$$\frac{d+2-2k^{-1}}{2-4k^{-1}} < \eta.$$

This is exactly the condition we used for the definition of D right before the statement of Theorem 1.1. □

In fact Theorem 4.1 can be easily used to establish Theorem 1.1. To treat Theorems 1.3 and 1.5, we need to formulate a pointwise bound involving modified moments $\hat{X}_a$ and Z^ϕ_k. Recall $R(k,\eta)$ and $R'(k,\eta)$ of Theorem 1.5 and to ease the notation, we write $\hat{Z}_k$ for $Z^{t,\phi}_k$.

Theorem 4.2 *Assume Hypotheses 1.3(k) and 1.4(k). Let (k,η,ζ) be any triple such that both $(k,\eta),(k,\zeta) \in D'$. Then there exists a constant $C_9 = C_9(k,\eta,\zeta)$ such that every weak solution f of (1.1) satisfies*

$$\begin{aligned} f_n(x,t) &\le \left(S^{d(n)}_t f^0\right)(x) \\ &+ C_9 a_0 2^\ell n^{-\ell} d(n)^{-R(k,\eta)} \left[\int_0^t \int \hat{X}_{(\ell+e_0-1)\eta+1} dy ds\right]^{\eta^{-1}} \left(\hat{Z}_k^{1+\frac{\eta}{\eta-1}} * \xi^{t\bar{d}}_{k,\eta}(x)\right)^{1-\eta^{-1}} \\ &+ C_9 a_1 2^\ell n^{-\ell} d(n)^{-R'(k,\zeta)} \left[\int_0^t \int \hat{X}_{(\ell+e_1-1)\zeta+1} dy ds\right]^{\zeta^{-1}} \left(\hat{Z}_k * \hat{\xi}^{t\bar{d}}_{k,\zeta}(x)\right)^{1-\zeta^{-1}}, \end{aligned} \tag{4.3}$$

for every $t, \ell > 0$ and almost all x. Here $\xi^a_{k,\eta}$ is as in Theorem 4.1, and

$$\hat{\xi}^a_{k,\zeta}(y) = |y|^{\theta'(k,\zeta)} \exp\left(\frac{-|y|^2}{8a(1-\zeta^{-1})}\right),$$

with

$$\theta'(k,\zeta) = 2\left(1-k^{-1}\right) - d\left(1-\zeta^{-1}\right)^{-1}.$$

5 Proofs of Theorems 4.1 and 4.2

Proof of Theorem 4.1. We certainly have

$$f_n(x,t) = (S_t^{d(n)} f_n^0)(x) + (p_{d(n)} * Q_n^C)(x,t) \le (S_t^{d(n)} f_n^0)(x) + (p_{d(n)} * Q_n^{C,+})(x,t), \tag{5.1}$$

where the convolution is in (x,t) variable and

$$p_D(x,t) = (4\pi t D)^{-d/2} \exp\left(-\frac{|x|^2}{4tD}\right) \mathbb{1}(t>0).$$

On the other hand, since $\alpha(n_1,n_2) \le a_0 n_1 n_2$,

$$\begin{aligned} Q_n^{C,+} &= \sum_{n_1+n_2=n} \alpha(n_1,n_2) f_{n_1} f_{n_2} \\ &\le \sum_{n_1,n_2} \mathbb{1}\left(n_1 \ge \frac{n}{2} \text{ or } n_2 \ge \frac{n}{2}\right) \alpha(n_1,n_2) f_{n_1} f_{n_2} \\ &\le 2a_0 X_1(1) X_1\left(\frac{n}{2}\right) \le 2a_0 2^\ell n^{-\ell} X_1(1) X_{\ell+1}\left(\frac{n}{2}\right) \le 2a_0 2^\ell n^{-\ell} X_1 X_{\ell+1}. \end{aligned} \tag{5.2}$$

where $X_a(r) = \sum_{n\ge r} n^a f_n$ (recall that we simply write X_a for $X_a(1)$.) To bound $X_1 X_{\ell+1}$, observe that for $\eta > 1$,

$$X_{\ell+1} = X_1 \left(\sum_n n^\ell \frac{n f_n}{X_1}\right)^{\eta\eta^{-1}} \le X_1 \left(\sum_n n^{\ell\eta} \frac{n f_n}{X_1}\right)^{\eta^{-1}} = X_{\ell\eta+1}^{\eta^{-1}} X_1^{1-\eta^{-1}}. \tag{5.3}$$

Furthermore, the expression

$$\left(p_{d(n)} * (X_1 X_{\ell+1})\right)(x,t), \tag{5.4}$$

is equal to

$$\begin{aligned} &\int\int_0^t p_{d(n)}(x-y,t-s)\,(X_1 X_{\ell+1})(y,s)\;dsdy \\ &\quad \le \int\int_0^t p_{d(n)}(x-y,t-s)\left(X_{\ell\eta+1}^{\eta^{-1}} X_1^{2-\eta^{-1}}\right)(y,s)\;dsdy \\ &\quad \le \int \left[\int_0^t (4\pi(t-s)d(n))^{-dr_1/2} \exp\left(-\frac{r_1|x-y|^2}{4(t-s)d(n)}\right)\;ds\right]^{\frac{1}{r_1}} \left[\int_0^t X_{\ell\eta+1}(y,s)ds\right]^{\eta^{-1}} \\ &\qquad\qquad \cdot \left[\int_0^t X_1(y,s)^{(2-\eta^{-1})r_2} ds\right]^{\frac{1}{r_2}}\;dy. \end{aligned}$$

Here (r_1, r_2) is any pair satisfying $r_1, r_2 > 1$ and $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{\eta} = 1$. Choose r_2 so that $(2 - \eta^{-1})r_2 = k$. As a result

$$r_1^{-1} = 1 - 2k^{-1} - \left(1 - k^{-1}\right)\eta^{-1} =: R(k, \eta).$$

Such r_1 exists because the right-hand side is positive by our assumptions on k and η. Observe,

$$\begin{aligned}
\int_0^t (4\pi\theta d(n))^{-dr_1/2} \exp\left(-\frac{r_1 a^2}{4\theta d(n)}\right) d\theta &= d(n)^{-1}\left(r_1 a^2\right)^{1-\frac{dr_1}{2}} \int_0^{\frac{td(n)}{r_1 a^2}} (4\pi\theta)^{-dr_1/2} e^{-\frac{1}{4\theta}}\, d\theta \\
&\le d(n)^{-1}\left(r_1 a^2\right)^{1-\frac{dr_1}{2}} \int_0^{\frac{t\bar{d}}{r_1 a^2}} (4\pi\theta)^{-dr_1/2} e^{-\frac{1}{4\theta}}\, d\theta \\
&=: d(n)^{-1}\left(r_1 a^2\right)^{1-\frac{dr_1}{2}} \phi_{r_1}\left(\frac{r_1 a^2}{t\bar{d}}\right). \qquad (5.5)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\phi_{r_1}(z) = \int_0^{z^{-1}} (4\pi\theta)^{-dr_1/2} e^{-\frac{1}{4\theta}}\, d\theta &\le e^{-\frac{z}{8}} \int_0^{z^{-1}} (4\pi\theta)^{-dr_1/2} e^{-\frac{1}{8\theta}}\, d\theta \\
\le e^{-\frac{z}{8}} \int_0^{\infty} (4\pi\theta)^{-dr_1/2} e^{-\frac{1}{8\theta}}\, d\theta &\le c_1(r_1) e^{-\frac{z}{8}},
\end{aligned}$$

for a constant $c_1(r_1) < \infty$. For the finiteness of $c_1(r_1)$, we are using the fact that $dr_1/2 > 1$, which is obvious when $d \ge 2$, and is a consequence of our assumption on (k, η) when $d = 1$. (The set D' is defined so that the condition $dr_1 > 1$ holds in all dimensions.) Hence, we may use (5.5) and Hölder Inequality to assert that the expression (5.4) is bounded above by

$$\begin{aligned}
&c_2(r_1) d(n)^{-r_1^{-1}} \int Z_k(y)^{2-\eta^{-1}} \left[\int_0^t X_{\ell\eta+1}(y, s)\, ds\right]^{\eta^{-1}} |x - y|^{\frac{2}{r_1} - d} \exp\left(-\frac{|x-y|^2}{8t\bar{d}}\right) dy \\
&\le c_2(r_1) d(n)^{-r_1^{-1}} \left[\int\int_0^t X_{\ell\eta+1}(y, s)\, ds dy\right]^{\eta^{-1}} \\
&\quad \cdot \left[\int Z_k(y)^{\frac{2-\eta^{-1}}{1-\eta^{-1}}} |x - y|^{\frac{2r_1^{-1} - d}{1-\eta^{-1}}} \exp\left(-\frac{|x-y|^2}{8t\bar{d}(1-\eta^{-1})}\right) dy\right]^{1-\eta^{-1}},
\end{aligned}$$

This, (5.1) and (5.2) imply (4.1). □

Proof of Theorem 4.2. The proof is very similar to the proof of Theorem 4.1 and we only explain the differences. We certainly have

$$f_n(x, t) = (S_t^{d(n)} f_n^0)(x) + (p_{d(n)} * Q_n)(x, t) \le (S_t^{d(n)} f_n^0)(x) + (p_{d(n)} * Q_n^+)(x, t), \qquad (5.6)$$

with $Q_n^+ = Q_n^{C,+} + Q_n^{F,+}$. We first bound $Q_n^{C,+}$; by Hypothesis 1.2,

$$
\begin{aligned}
Q_n^{C,+} &= \sum_{n_1+n_2=n} \alpha(n_1,n_2) f_{n_1} f_{n_2} \le \sum_{n_1,n_2} \mathbb{1}\left(n_1 \ge \frac{n}{2} \text{ or } n_2 \ge \frac{n}{2}\right) \alpha(n_1,n_2) f_{n_1} f_{n_2} \\
&\le 4a_0 \sum_{n_1,n_2} \mathbb{1}\left(n_1 \ge \frac{n}{2}\right) n_1^{e_0-1} \gamma_k(n_1)\gamma_k(n_2) f_{n_1} f_{n_2} \qquad (5.7)\\
&\le 4a_0 \hat{X}_{e_0}\left(\frac{n}{2}\right)\hat{X}_1(1) \le 4a_0 2^\ell n^{-\ell} \hat{X}_{e_0+\ell}\left(\frac{n}{2}\right)\hat{X}_1(1) \\
&\le 4a_0 2^\ell n^{-\ell} \hat{X}_{e_0+\ell} \hat{X}_1.
\end{aligned}
$$

where $\hat{X}_a(r) = \sum_{n \ge r} n^{a-1}\gamma_k(n) f_n$ and by definition, $\hat{X}_a(1) = \hat{X}_a$. As for $Q_n^{F,+}$, we use Hypothesis 1.4(k) to assert

$$
\begin{aligned}
Q_n^{F,+} &= \sum_{m=1}^{\infty} \beta(n,m) f_{n+m} \le a_1 \sum_{m=1}^{\infty} (n+m)^{e_1-1}\gamma_k(n+m) f_{n+m} \\
&= a_1 \hat{X}_{e_1}(n+1) \le a_1 n^{-\ell}\ \hat{X}_{e_1+\ell}.
\end{aligned}
$$

From this, (5.6) and (5.7) we deduce

$$
\begin{aligned}
f_n(x,t) \le & (S_t^{d(n)} f_n^0)(x) + a_1 n^{-\ell}\left(p_{d(n)} * \hat{X}_{e_1+\ell}\right)(x,t) \\
& + 4a_0 2^\ell n^{-\ell}\left(p_{d(n)} * \left(\hat{X}_1 \hat{X}_{e_0+\ell}\right)\right)(x,t), \qquad (5.8)
\end{aligned}
$$

We wish to show that the three terms appearing on the right-hand side of (5.8) are bounded above by the three terms appearing on the right-hand side of (4.3). To bound $\hat{X}$, observe that as in (5.3)

$$\hat{X}_{r+1} \le \hat{X}_{r\eta+1}^{\eta^{-1}}\ \hat{X}_1^{1-\eta^{-1}},$$

for $\eta > 1$. Hence

$$\hat{X}_1 \hat{X}_{e_0+\ell} \le \hat{X}_{(\ell+e_0-1)\eta+1}^{\eta^{-1}} \hat{X}_1^{2-\eta^{-1}}, \quad \hat{X}_{e_1+\ell} \le \hat{X}_{(\ell+e_1-1)\zeta+1}^{\zeta^{-1}} \hat{X}_1^{1-\zeta^{-1}}. \qquad (5.9)$$

We then repeat the proof of Theorem 4.1 after (5.3) and use (5.9) to assert that the expression

$$4a_0 2^\ell n^{-\ell}\left(p_{d(n)} * \left(\hat{X}_1 \hat{X}_{e_0+\ell}\right)\right)(x,t),$$

is bounded above by the third term on the right-hand side of (4.3). Similarly, we can show that the expression

$$a_1 n^{-\ell}\left(p_{d(n)} * \hat{X}_{e_1+\ell}\right)(x,t),$$

is bounded above by

$$
\begin{aligned}
&a_1 n^{-\ell} c_2(r_1) d(n)^{-r_1^{-1}} \int \hat{Z}_k(y)^{1-\zeta^{-1}} \left[\int_0^t \hat{X}_{\ell\zeta+1}(y,s)\, ds\right]^{\zeta^{-1}} |x-y|^{\frac{2}{r_1}-d} \exp\left(-\frac{|x-y|^2}{8t\bar{d}}\right) dy \\
&\quad \le a_1 n^{-\ell} c_2(r_1) d(n)^{-r_1^{-1}} \left[\int \int_0^t \hat{X}_{\ell\zeta+1}(y,s)\, dsdy\right]^{\zeta^{-1}} \\
&\qquad \cdot \left[\int \hat{Z}_k(y) |x-y|^{\frac{2r_1^{-1}-d}{1-\zeta^{-1}}} \exp\left(-\frac{|x-y|^2}{8t\bar{d}(1-\eta^{-1})}\right) dy\right]^{1-\zeta^{-1}},
\end{aligned}
$$

where this time r_1 is given by $r_1^{-1} = (1-k^{-1})(1-\zeta^{-1}) = R'(k,\zeta)$. This expression is exactly the second term on the right-hand side of (4.3), and this complete the proof. □

6 Proofs of Theorems 1.3 and 1.5

Proof of Theorem 1.3. This is a straight forward consequence of Theorem 4.2. To simplify the notation, write $\ell' = (\ell + e_0 - 1)\eta + 1$ and

$$
A_h = \int_0^T \int \hat{X}_{h+1} dyds. \tag{6.1}
$$

Observe that if $\beta \equiv 0$, then $a_1 = 0$, and if we take the L^p norm of both sides (4.3), we obtain

$$
\|f_n(\cdot,t)\|_{L^p} \le \|f_n^0\|_{L^p} + C_9 a_0 2^\ell n^{-\ell} d(n)^{-R(k,\eta)} A_{\ell'}^{\eta^{-1}} \left\| \hat{Z}_k^{1+\frac{\eta}{\eta-1}} * \xi_{k,\eta}^{t\bar{d}}(x) \right\|_{L^{p(1-\eta^{-1})}}^{1-\eta^{-1}} \tag{6.2}
$$

for $p \in [1,\infty]$. Here we write L^r for $L^r(\mathbb{R}^d)$. Since $(k,\eta) \in D$, we have that $\xi_{k,\eta}^{t\bar{d}} \in L^1$ by Remark 4.1. From (6.2) and Young's inequality,

$$
\begin{aligned}
\|f_n(\cdot,t)\|_{L^p} &\le \|f_n^0\|_{L^p} + C_9' a_0 2^\ell n^{-\ell} d(n)^{-R(k,\eta)} A_{\ell'}^{\eta^{-1}} \left\| \hat{Z}_k^{1+\frac{\eta}{\eta-1}} \right\|_{L^{p(1-\eta^{-1})}}^{1-\eta^{-1}} \\
&= \|f_n^0\|_{L^p} + C_9' a_0 2^\ell n^{-\ell} d(n)^{-R(k,\eta)} A_{\ell'}^{\eta^{-1}} \left\| \hat{Z}_k \right\|_{L^{p(2-\eta^{-1})}}^{2-\eta^{-1}},
\end{aligned} \tag{6.3}
$$

where C_9' is a constant that depends on (k,η) and $\bar{d}$ only. Now (1.14) is an immediate consequence of Theorem 2.2 with $a_1 = 0$, and (6.3). □

The proof of Theorem 1.5 is more involved because when a_1 is not zero, the right-hand side of (2.4) now depends on the solution $(f(\cdot,t); t \in [0,T])$. Fortunately the second term on the right-hand side is a convolution and as a preparation, let us learn how to take advantage of this.

Lemma 6.1 *Under the assumptions of Theorem 2.2, we can find a constant $C_{10} = C_{10}(k,r)$ such that for every $r > kd/2$,*

$$
\begin{aligned}
(6.4) \qquad \hat{Z}_k(x) \leq & c_0(k,d)(\bar{X}_1^0 * \psi_k)(x) \\
& + C_{10} \sup_{t\in[0,T]} \left[\|\tilde{X}(\cdot,t)\|_{L^1(\mathbb{R}^d)}^{1-b} \|\tilde{X}(\cdot,t)\|_{L^r(B_1(x))}^{b} + \|\tilde{X}(\cdot,t)\|_{L^1(\mathbb{R}^d)} \right],
\end{aligned}
$$

where $b = \frac{r}{r-1}(1-\frac{2}{kd}) \in (1-\frac{2}{kd}, 1)$ and $B_1(x)$ denotes the ball of radius 1 and center x.

Proof. Pick $\delta \in (0,1]$ and write $\psi_k = \hat{\psi}_k + \bar{\psi}_k$, where $\hat{\psi}_k(x) = \psi_k(x)1\!\!1(|x| \leq \delta)$. We certainly have $\bar{\psi}_k \leq \delta^{\frac{2}{k}-d}$. From this we learn

$$
(6.5) \qquad (\tilde{X}(\cdot,t) * \bar{\psi}_k)(x) \leq \delta^{\frac{2}{k}-d} \|\tilde{X}(\cdot,t)\|_{L^1(\mathbb{R}^d)}.
$$

On the other hand, if r' satisfies $r'^{-1} + r^{-1} = 1$, with $r > kd/2$, then $(2k^{-1} - d) + dr'^{-1} > 0$, and

$$
\|\hat{\psi}_k\|_{L^{r'}(B_\delta(0))} = \left(r'\left(2k^{-1}-d\right)+d\right)^{-1} \delta^{(2k^{-1}-d)+dr'^{-1}} = \left(r'\left(2k^{-1}-d\right)+d\right)^{-1} \delta^{2k^{-1}-dr^{-1}}.
$$

From this and Hölder Inequality we deduce

$$
(\tilde{X}(\cdot,t) * \hat{\psi}_k)(x) \leq \left(r'\left(2k^{-1}-d\right)+d\right)^{-1} \delta^{2k^{-1}-dr^{-1}} \|\tilde{X}(\cdot,t)\|_{L^r(B_\delta(x))}.
$$

From this and (6.5) we learn

$$
\begin{aligned}
(6.6) \qquad (\tilde{X} * \psi_k)(x) \leq & \delta^{\frac{2}{k}-d} \|\tilde{X}(\cdot,t)\|_{L^1(\mathbb{R}^d)} \\
& + \left(r'\left(2k^{-1}-d\right)+d\right)^{-1} \delta^{2k^{-1}-dr^{-1}} \|\tilde{X}(\cdot,t)\|_{L^r(B_1(x))}.
\end{aligned}
$$

To ease the notation, write

$$
\theta_1 = d - 2k^{-1}, \quad \theta_2 = (2k^{-1}-d) + dr'^{-1}, \quad A = \left(r'\left(2k^{-1}-d\right)+d\right)^{-1}.
$$

We now optimize the bound (6.6) over $\delta \in (0,1]$. Either

$$
(6.7) \qquad A\theta_2 \|\tilde{X}(\cdot,t)\|_{L^r(B_1(x))} \leq \theta_1 \|\tilde{X}(\cdot,t)\|_{L^1(\mathbb{R}^d)},
$$

is true or the opposite inequality. If the latter case occurs, then the inequality (6.6) is optimized for some $\delta \in (0,1)$ and we deduce that for some constant $c = c(\theta_1, \theta_2, A)$,

$$
(6.8) \qquad (\tilde{X} * \psi_k)(x) \leq c \left\|\tilde{X}(\cdot,t)\right\|_{L^1(\mathbb{R}^d)}^{\frac{\theta_2}{\theta_1+\theta_2}} \left\|\tilde{X}(\cdot,t)\right\|_{L^r(B_1(x))}^{\frac{\theta_1}{\theta_1+\theta_2}}.
$$

If (6.7) occurs instead, then the inequality (6.6) is optimized for $\delta = 1$ and we deduce that for some constant $c' = c'(\theta_1, \theta_2, A)$,

$$(\tilde{X} * \psi_k)(x) \le c' \left\| \tilde{X}(\cdot, t) \right\|_{L^1(\mathbb{R}^d)} . \tag{6.9}$$

We now use (6.8) and (6.9) to bound $\tilde{X} * \psi_k$ in (2.4) and conclude (6.4). □

Proof of Theorem 1.5. Recall that by Remark 1.2, $\theta(k, \zeta) + d > 0$ because $(k, \zeta) \in D$. On the other hand, we can readily check that $\theta'(k, \zeta) > \theta(k, \zeta)$, which in turn implies that $\theta'(k, \zeta) + d > 0$ and $\xi^{T\bar{d}}_{k,\eta}, \hat{\xi}^{T\bar{d}}_{k,\zeta} \in L^1$. Also, $\xi^{t\bar{d}}_{k,\eta} \le \xi^{T\bar{d}}_{k,\eta}$ and $\hat{\xi}^{t\bar{d}}_{k,\zeta} \le \hat{\xi}^{T\bar{d}}_{k,\zeta}$ for $t \in [0, T]$. As a result, for some constant $c_1 = c_1(k, \eta, \zeta, T)$,

$$\begin{aligned} &\left(\hat{Z}_k^{1+\frac{\eta}{\eta-1}} * \xi^{t\bar{d}}_{k,\eta}\right)^{1-\eta^{-1}} (x) \le c_1 \left\| \hat{Z}_k^{1+\frac{\eta}{\eta-1}} \right\|_{L^\infty}^{1-\eta^{-1}} = c_1 \left\| \hat{Z}_k \right\|_{L^\infty}^{2-\eta^{-1}} , \\ &\left(\hat{Z}_k * \hat{\xi}^{t\bar{d}}_{k,\zeta}\right)^{1-\zeta^{-1}} (x) \le c_1 \left\| \hat{Z}_k \right\|_{L^\infty}^{1-\zeta^{-1}} , \end{aligned} \tag{6.10}$$

for every $t \in [0, T]$ and every x . Recall that by (6.1), we write A_h for the L^1-norm of $\hat{X}_h$. To ease the notion, we set $\ell' = (\ell + e_0 - 1)\eta + 1$, and $\ell'' = (\ell + e_1 - 1)\zeta + 1$. From Theorem 4.2, Lemma 6.1 and (6.10) we learn

$$\|f_n(\cdot, t)\|_{L^\infty} \le \|f^0\|_{L^\infty} + c_2(\Omega_1 + \Omega_2 + \Omega_3 + \Omega_4), \tag{6.11}$$

for a constant $c_2 = c_2(k, \eta, \zeta, T)$ and every $t \in [0, T]$, where

$$\begin{aligned} \Omega_1 =& a_1 2^\ell n^{-\ell} d(n)^{-R'(k,\zeta)} A_{\ell''}^{\zeta^{-1}} \sup_{t\in[0,T]} \left\| \tilde{X}(\cdot, t) \right\|_{L^1(\mathbb{R}^d)}^{(1-b)(1-\zeta^{-1})} Y^{b(1-\zeta^{-1})}, \\ \Omega_2 =& a_0 2^\ell n^{-\ell} d(n)^{-R(k,\eta)} A_{\ell'}^{\eta^{-1}} \sup_{t\in[0,T]} \left\| \tilde{X}(\cdot, t) \right\|_{L^1(\mathbb{R}^d)}^{(1-b)(2-\eta^{-1})} Y^{b(2-\eta^{-1})}, \\ \Omega_3 =& 2^\ell n^{-\ell} \left[a_0 d(n)^{-R(k,\eta)} A_{\ell'}^{\eta^{-1}} \left\| \bar{X}_1^0 * \psi_k \right\|_{L^\infty}^{2-\eta^{-1}} + a_1 d(n)^{-R'(k,\zeta)} A_{\ell''}^{\zeta^{-1}} \left\| \bar{X}_1^0 * \psi_k \right\|_{L^\infty}^{1-\zeta^{-1}} \right], \\ \Omega_4 =& 2^\ell n^{-\ell} \left[a_0 d(n)^{-R(k,\eta)} A_{\ell'}^{\eta^{-1}} \sup_t \left\| \tilde{X}(\cdot, t) \right\|_{L^1(\mathbb{R}^d)}^{2-\eta^{-1}} + a_1 d(n)^{-R'(k,\zeta)} A_{\ell''}^{\zeta^{-1}} \sup_t \left\| \tilde{X}(\cdot, t) \right\|_{L^1(\mathbb{R}^d)}^{1-\zeta^{-1}} \right]. \end{aligned}$$

We finally choose $\zeta = \eta$ and observe that $R'(k, \eta) > R(k, \eta)$. This allows us to deduce (1.22) from (6.11).

□

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