# Moment Bounds for the Solutions of the Smoluchowski Equation with Coagulation and Fragmentation

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## Abstract

We prove various $L^p(\mathbb{R}^d \times [0,T])$ bounds on moments $X_a(x,t) := \sum_{m \in \mathbb{N}} m^a f_m(x,t)$, (respectively $\int_0^\infty m^a f_m(x,t)dm$,) where $f_m$ is a solution of the discrete (respectively continuous) Smoluchowski coagulation-fragmentation equations with diffusion. In a previous paper [HR1] we proved similar results for all weak solution to the discrete Smoluchowski's equation provided that there is no fragmentation and certain moments are bounded in suitable $L^q$-spaces initially. In this paper we prove the corresponding results in the case of the continuous Smoluchowski's equation. When there is also fragmentation, we need to assume that the solution $f$ is *regular* in the sense that $f$ can be approximated by solutions to Smoluchowski equation for which the coagulation and fragmentation coefficients are 0 when the cluster sizes are large. We also need suitable assumptions on the coagulation rates to avoid gelation. On the fragmentation rate $\beta$, we assume that $\sup_n \sup_{m \le \ell} \beta(m,n)/n < \infty$ for every positive $\ell$, and that there exist constants $a_0 \ge 0$ and $c_0$ such that $\beta(n,m) \le c_0(n+m)^{a_0}$.

# 1   Introduction

The Smoluchowski equation is a coupled system of partial differential equations that describes the evolving densities of a system of diffusing particles that are prone to coagulate

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in pairs and fragment into pairs. A sequence of functions $f_n : \mathbb{R}^d \times [0, \infty) \to [0, \infty)$, $n \in \mathbb{N}$, is a solution of the discrete Smoluchowski equation if it satisfies

$$(1.1) \qquad \frac{\partial}{\partial t} f_n(x, t) = d(n) \Delta f_n(x, t) + Q_n(f)(x, t),$$

with $Q_n = Q_n^C + Q_n^F$, $Q_n^C = Q_n^{C,+} - Q_n^{C,-}$, $Q_n^F = Q_n^{F,+} - Q_n^{F,-}$, where

$$Q_n^{C,+}(f)(x, t) = \sum_{m=1}^{n-1} \alpha(m, n-m) f_m(x, t) f_{n-m}(x, t),$$

$$Q_n^{C,-}(f)(x, t) = 2 \sum_{m=1}^{\infty} \alpha(n, m) f_n(x, t) f_m(x, t),$$

$$Q_n^{F,+}(f)(x, t) = 2 \sum_{m=1}^{\infty} \beta(n, m) f_{n+m}(x, t),$$

and

$$Q_n^{F,-}(f)(x, t) = \sum_{m=1}^{n-1} \beta(m, n-m) f_n(x, t).$$

We will interpret this solution in a weak sense. Namely, we assume that $Q_n^{C,\pm}, Q_n^{F,\pm} \in L^1(\mathbb{R}^d \times (0, T])$ for each $T \in [0, \infty)$ and $n \in \mathbb{N}$, and that

$$(1.2) \qquad f_n(x, t) = S_t^{d(n)} f_n^0(x) + \int_0^t S_{t-s}^{d(n)} Q_n(x, s) ds,$$

where $\{f_n^0 : n \in \mathbb{N}\}$ denotes the initial data, $S_t^D$ the semigroup associated with the equation $u_t = D \Delta u$, and where $Q_n(x, s)$ means $Q_n(f)(x, s)$.

In the continuous case, the summations $\sum_{m=1}^{n-1}$, and $\sum_{m=1}^{\infty}$, are replaced with $\int_0^n dm$, and $\int_0^\infty dm$, respectively. Similarly, a function $f : \mathbb{R}^d \times (0, \infty) \times [0, \infty) \to [0, \infty)$, $f_n(x, t) = f(x, n, t)$, is a solution to the continuous Smoluchowski equation if $f_n, Q_n^{C,\pm}, Q_n^{F,\pm} \in L^1(\mathbb{R}^d \times [0, T])$, for each $T, n \in (0, \infty)$, and (1.2) holds.

In Hammond–Rezakhanlou [HR3], [HR2] and , Yaghouti–Rezakhanlou–Hammond[YRH] the equation (1.1) was derived from a microscopic model of coagulating Brownian particles which of course corresponds to the case $\beta \equiv 0$. In these articles we needed to make suitable assumptions on the microscopic details of the model that led to the property

$$(1.3) \qquad \lim_{n+m \to \infty} \frac{\alpha(n, m)}{(n+m)(d(n) + d(m))} = 0$$

of the macroscopic coagulation rate $\alpha$. This is our principal assumption on $\alpha$ in this article. The existence of solutions to (1.1) under the assumption

$$(1.4) \qquad \lim_{n \to \infty} \frac{\alpha(n, m)}{n} = 0$$

has been established in Wrzosek [W1], [W3], [MR] and Laurençot–Mischler [LM2], and in [LM1] when the equation (1.1) is formulated in a bounded domain. In this case one can prove the existence of a solution by first replacing $\alpha$ and $\beta$ by a suitable cutoff rates $\alpha^{(N)}$ and $\beta^{(N)}$, and pass to the limit. More precisely, $\alpha^{(N)}$ and $\beta^{(N)}$ are defined by

$$(1.5) \qquad \alpha^{(N)}(n, m) = \begin{cases} \alpha(n, m) & \text{if } n \text{ and } m \leq N, \\ 0 & \text{otherwise,} \end{cases}$$

$$(1.6) \qquad \beta^{(N)}(n, m) = \begin{cases} \beta(n, m) & \text{if } n + m \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

We can readily see that for such coagulation and fragmentation rates, there exists a unique solution $f^{(N)}$ for a given initial data. We then show that such a sequence $\{f^{(N)}\}_{N \in \mathbb{N}}$ has a convergent subsequence in $L^1$-sense and that each limit point $f$ is a weak solution to (1.1). We say a solution $f$ to (1.1) is *regular* if $f$ is obtained by the above approximation procedure.

Our results are valid for each dimension $d \geq 1$. Our main results would depend on some regularities in the initial data, and some assumptions on the parameters of the system. We now state the various assumptions that we require. As for the coagulation, fragmentation and diffusion parameters we consider four sets of assumptions.

**Hypothesis 1.1** The function $d(\cdot)$ is positive and uniformly bounded. Moreover,

$$\lim_{n+m \to \infty} \frac{\alpha(n, m)}{(n + m)(d(n) + d(m))} = 0.$$

More precisely, for every $\delta > 0$, there exists $k_0 = k_0(\delta) > 0$ such that if $n + m > k_0$, then

$$\alpha(n, m) \leq \delta(n + m)(d(n) + d(m)).$$

**Hypothesis 1.2** The function $d(n)$ is a non-increasing function of $n$ and that $\alpha(n, m) \leq C_0(n + m)$ for a constant $C_0$. Moreover, there exist positive constants $r_1$ and $r_2$ and nonnegative constants $b_2$ and $b_1$ with $b_2 \leq b_1$, $b_1 > d/2$, such that,

$$r_1 n^{-b_1} \leq d(n) \leq r_2 n^{-b_2},$$

for every $n > 0$. In the case of the continuous Smoluchowski's equation, we also assume that $d(0) = \sup_n d(n) < \infty$.

**Hypothesis 1.3** For every $\ell > 0$, there exists a constant $c_1(\ell)$ such that for every $m$ and $n$ with $m \leq \ell$,
$$\beta(n, m) \leq c_1(\ell)n.$$

**Hypothesis 1.4** There exist constants $a_0 \in [0, \infty)$ and $c_0$ such that

(1.7) $$\beta(n, m) \leq c_0(n + m)^{a_0}.$$

Our notation for the various moments of $f$ in the discrete case will be

(1.8) $$X_a = X_a(x, t) = \sum_n n^a f_n(x, t), \quad \hat{X}_a = \hat{X}_a(x, t) = \sum_n n^a d(n)^{d/2} f_n(x, t).$$

and

$$Y_a(x, t) = \sum_{n,m} nm(n^a + m^a)(d(n) + d(m))f_n(x, t)f_m(x, t),$$

(1.9) $$\hat{Y}_a(x, t) = \sum_{n,m} (n^a m + m^a n)\alpha(n, m)f_n f_m.$$

In the continuous case all the summations are replaced with integration over $n, m \in (0, \infty)$. We also set

(1.10) $$\phi_0(x) = \begin{cases} |x|^{2-d} & \text{if } d \geq 3, \\ -\frac{1}{2\pi} \log |x| \; \mathbb{1}(|x| \leq 1) & \text{if } d = 2, \\ \frac{1}{2}(1 - |x|) \; \mathbb{1}(2|x| \leq 1) & \text{if } d = 1. \end{cases}$$

We now state the three theorems.

**Theorem 1.1** *Assume Hypotheses 1.1 and 1.3. Then for every $a \geq 2$ and positive $A$ and $T$, there exists a constant $C = C(a, A, T)$ such that, if $f$ is a regular solution and*

(1.11) $$\iint X_a(x, 0)X_1(y, 0)\phi_0(x - y)dxdy \leq A,$$

*and*

(1.12) $$\text{ess} \sup_x \int (X_a + X_0)(y, 0)\phi_0(x - y)dy \leq A, \quad \int (X_a + X_0)(x, 0)dx \leq A,$$

*then*

(1.13) $$\int_0^T \int Y_{a-1}dxdt \leq C, \quad \int_0^T \int \hat{Y}_{a-1}dxdt \leq C,$$

*and*

(1.14) $$\sup_{t \in [0,T]} \int X_a(x, t)dx \leq C.$$

*Moreover, when there is no fragmentation, our results are valid for every weak solution.*

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**Theorem 1.2** *Assume Hypotheses 1.2 and 1.4. Given $(b_1, b_2, a_0)$ as in those Hypotheses, there exists a function $\gamma(\cdot; b_1, b_2, a_0) : (1, \infty) \to (0, \infty)$ with $\lim_{a\to\infty} \gamma(a; b_1, b_2, a_0) = \infty$ such that if $f$ is a regular solution to discrete (respectively continuous) Smoluchowski equation and*

(1.15) $\qquad \sum_n n^\gamma \|f_n^0\|_{L^\infty(\mathbb{R}^d)} < \infty, \quad \left( \text{respectively } \int_0^\infty (n+1)^\gamma \|f_n^0\|_{L^\infty(\mathbb{R}^d)} < \infty, \right)$

(1.16) $\qquad X_a \in L^1(\mathbb{R}^d \times [0, T]), \quad \left( \text{respectively } X_a, X_{a_0} \in L^1(\mathbb{R}^d \times [0, T]), \right)$

*then*

(1.17) $\quad \sum_n n^\gamma \|f_n\|_{L^\infty(\mathbb{R}^d \times [0,T])} < \infty, \quad \left( \text{respectively } \int_0^\infty (n+1)^\gamma \|f_n^0\|_{L^\infty(\mathbb{R}^d \times [0,T])} < \infty, \right)$

*where $\gamma = \gamma(a; b_1, b_2, a_0)$. Moreover, when there is no fragmentation, our results are valid for every weak solution.*

**Remark 1.1** In particular, if Hypotheses 1.1–1.4 hold, $X_a(\cdot, 0) \in L^1(\mathbb{R}^d)$, and

$$\sum_n n^a \|f_n^0\|_{L^\infty(\mathbb{R}^d)} < \infty,$$

for every $a \in \mathbb{N}$, then $X_a \in L^\infty(\mathbb{R}^d \times [0, T]) \cap L^1(\mathbb{R}^d \times [0, T])$ for every $a \in \mathbb{N}$ and $T \in (0, \infty)$. We refer to (3.16) for the explicit form of the function $\gamma$. Also note that Theorem 1.1 offers sufficient conditions to ensure $X_a \in L^1(\mathbb{R}^d \times [0, T])$ (so that this theorem has been invoked in Remark 1.1).

We end this section with two results which were carried out in [HR1] in the discrete case. The case of continuous Smoluchowski equation can be treated in a similar way. Note that the uniqueness result is a straightforward generalization of the uniqueness theorem of Ball and Carr [BC].

**Theorem 1.3 (Conservation of Mass)** *Let $f$ be a regular solution of (1.1). Assume that (1.13) holds for $a = 2$. Then $f$ conserves mass on the time interval $[0, T]$. Moreover, when there is no fragmentation, our results are valid for every weak solution.*

**Theorem 1.4 (Uniqueness)** *Suppose that there is no fragmentation and that, for some $c_0 > 0$, we have that $\alpha(n, m) \le c_0 nm$ for each $n, m \in \mathbb{N}$. Then there is a unique weak solution of (1.1) on the interval $[0, T]$ among those satisfying $X_2 \in L^\infty(\mathbb{R}^d \times [0, T])$.*

For some related works, we refer to [A], [AW] (local exitence and uniqueness), [W2] (uniqueness and mass conservation for bounded $\alpha$ and almost constant diffusion coefficient), and [N] (uniqueness and existence for an example with unbounded diffusion coefficient).

On account of Theorem 1.2, we have that $X_2 \in L^\infty$ if the same is true initially and for some (possibly large) $a$, we have that $X_a \in L^1$. On the other hand Theorem 1.1 guarrantees that $X_a \in L^1$ under suitable assumptions on the initial conditions. In summary, there would be a unique solution to (1.1) if there is no fragmentation, $X_2 \in L^\infty$ initially, and certain moments of initial data is bounded with respect to the norms given in Theorem 1.1.

## 2   Proof of Theorem 1.1

We only give a proof of Theorem 1.1 in the case of the discrete Smoluchowski equation because the continuous case can be treated by verbatim arguments.

The case $\beta \equiv 0$ has already been established for all weak solutions in [HR1]. It remains to establish Theorem 1.1 for regular solutions. Since we are considering regular solutions only, it suffices to establish Theorem 1.1 when $\alpha(n, m) = 0$ for $n$ or $m > N$ and $\beta(n, m) = 0$ for $(n, m)$ satisfying $n + m > N$. (Of course our constants will not depend on the cutoff parameter $N$.) Observe that under such assumptions on $\alpha$ and $\beta$, we always have

$$(2.1) \qquad \sum_1^l nQ_n = 0,$$

for $l \geq 2N$, because

$$\sum_{n=1}^l nQ_n = \sum_{n,m}[n\mathbb{1}(n \leq l < n + m) + m\mathbb{1}(m \leq l < n + m)](\beta(n, m)f_{n+m} - \alpha(n, m)f_n f_m).$$

As an immediate consequence of (2.1), we have

$$(2.2) \qquad \int \sum_{n=1}^l nf_n(x, t)dx = \int \sum_{n=1}^l nf_n(x, 0)dx.$$

We first recall a lemma from [HR1].

**Lemma 2.1** *There exist functions $H$ and $K$ such that $H \geq 0$, $K$ is bounded, $K$ is of compact support,*

$$-\Delta H(x) = \delta_0 - K(x),$$

*weakly, and the function $H - \phi_0$ is bounded. (The function $\phi_0$ was defined in (1.10).)*

In fact when $d \geq 3$, we can choose $K \equiv 0$ and $H(x) = c(d)|x|^{2-d}$ with $c(d) = (d-2)^{-1}w_d^{-1}$ where $w_d$ denotees the $(d-1)$-dimensional measure of the unit sphere. We can not afford $K \equiv 0$ in (2.1) when $d = 1$ or $2$, and more work is needed for the construction of $H$. We refer the reader to [HR1] for the proof of Lemma 2.1. As our next preparation for the proof of Theorem 1.1, we state and prove Lemma 2.2 which is the analog of Lemma 3.2 of [HR1].

**Lemma 2.2** *Let $H$ be as in Lemma 2.1. Then there exists a constant $c_1$ such that for every regular solution $f$,*

$$\sup_x \sum_{n=1}^{l} n(f_n^\delta *_x H)(x,t) \leq \sup_x \sum_{n=1}^{l} n(f_n^\delta *_x H)(x,0) + c_1 t \int \sum_{n=1}^{l} n f_n(x,0) dx$$

*for every $\delta > 0$, $l \geq 2N$ and $t > 0$. Here $f^\delta = f *_x \zeta^\delta$ where $\zeta^\delta(z) = \delta^{-d} \zeta\left(\frac{z}{\delta}\right)$ with $\zeta$ a nonnegative smooth function of compact support and total integral $1$. The constant $c_1$ can be chosen to be $0$ when $d > 2$.*

**Proof.** We certainly have

$$\sum_{1}^{l} n(f_n^\delta *_x H)(x,t) = \sum_{1}^{l} n(f_n^\delta *_x H)(x,0) - \int_0^t \sum_{1}^{l} n d(n) f_n^\delta(x,s) ds$$

(2.3)
$$+ \int_0^t \sum_{n=1}^{l} n d(n) (f_n^\delta *_x K)(x,s) ds + \int_0^t \sum_{n=1}^{l} n(Q_{n,\delta} *_x H)(x,s) ds,$$

where $Q_{n,\delta} = Q_n *_x \zeta^\delta$. Since $K$ and $d$ are bounded,

$$\left\| \sum_{n=1}^{l} n d(n)(f_n^\delta *_x K)(\cdot, s) \right\|_{L^\infty(dx)} \leq c_1 \left\| \sum_{n=1}^{l} n f_n^\delta(\cdot, s) \right\|_{L^1(dx)} \leq c_1 \left\| \sum_{n=1}^{l} n f_n(\cdot, s) \right\|_{L^1(dx)} \|\zeta^\delta\|_{L^1(dx)}$$

$$= c_1 \left\| \sum_{n=1}^{l} n f_n(\cdot, 0) \right\|_{L^1(dx)},$$

where we used (2.2) and $\|\zeta^\delta\|_{L^1(dx)} = 1$ for the equality. The Lemma follows from this, (2.3) and (2.1). $\square$

**Proof of Theorem 1.1.** Assume that $l \geq 2N$ and set

$$Z^{l,\delta}(t) = \int \left( \sum_{n=1}^{l} n^a f_n^\delta(x,t) \right) \left( \sum_{n=1}^{l} n f_n^\delta *_x H(x,t) \right) dx$$

$$Z^l(t) = \int \left( \sum_{n=1}^{l} n^a f_n(x,t) \right) \left( \sum_{n=1}^{l} n(f_n *_x H)(x,t) \right) dx.$$

We have that weakly

$$\frac{d}{dt} Z^{l,\delta}(t) = \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 + \Omega_5 + \Omega_6,$$

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where,

$$
\begin{aligned}
\Omega_1 &= -\int \left( \sum_{n=1}^{l} n^a f_n^\delta(x,t) \right) \left( \sum_{n=1}^{l} n d(n) f_n^\delta(x,t) \right) dx \\
\Omega_2 &= -\int \left( \sum_{n=1}^{l} n^a d(n) f_n^\delta(x,t) \right) \left( \sum_{n=1}^{l} n f_n^\delta(x,t) \right) dx \\
\Omega_3 &= \int \left( \sum_{n=1}^{l} n^a f_n^\delta(x,t) \right) \left( \sum_{n=1}^{l} n (Q_{n,\delta} *_x H)(x,t) \right) dx \\
\Omega_4 &= \int \left( \sum_{n=1}^{l} n^a Q_{n,\delta}(x,t) \right) \left( \sum_{n=1}^{l} n (f_n^\delta *_x H)(x,t) \right) dx \\
\Omega_5 &= \int \left( \sum_{n=1}^{l} n^a f_n^\delta(x,t) \right) \left( \sum_{n=1}^{l} n d(n) \, (f_n^\delta *_x K)(x,t) \right) dx \\
\Omega_6 &= \int \left( \sum_{n=1}^{l} n^a d(n) f_n^\delta(x,t) \right) \left( \sum_{n=1}^{l} n \, (f_n^\delta *_x K)(x,t) \right) dx,
\end{aligned}
$$

with $Q_{n,\delta} = Q_n *_x \zeta^\delta$. We can readily show

$$
(2.4) \qquad \Omega_1 + \Omega_2 = -\frac{1}{2} \int \sum_{n,m} \mathbb{1}(n, m \leq l) nm(d(n) + d(m))(n^{a-1} + m^{a-1}) f_n^\delta f_m^\delta dx.
$$

By (2.1), we know that $\Omega_3 = 0$. By the boundedness of the functions $K(\cdot)$, $d(\cdot)$, and (2.2) we deduce

$$
(2.5) \qquad |\Omega_5 + \Omega_6| \leq c_1 \int X_a^{l,\delta} dx,
$$

where $X_a^{l,\delta} = X_a^l * \zeta^\delta$ and

$$
X_a^l = \sum_{n=1}^{l} n^a f_n.
$$

It remains to bound $\Omega_4$. Note that

$$\sum_{n=1}^{l} n^a Q_n = \sum_{n,m}[(n+m)^a \mathbb{1}(n+m \le l) - n^a \mathbb{1}(n \le l) - m^a \mathbb{1}(m \le l)]$$

$$(\alpha(n,m)f_n f_m - \beta(n,m)f_{n+m})$$

(2.6)
$$\le \sum_{n,m}[(n+m)^a \mathbb{1}(n+m \le l) - n^a \mathbb{1}(n \le l) - m^a \mathbb{1}(m \le l)]\alpha(n,m)f_n f_m$$

$$\le c_2 \sum_{n,m}(n^{a-1}m + m^{a-1}n)\mathbb{1}(n+m \le l)\alpha(n,m)f_n f_m =: c_2 \hat{Y}_{a-1}^l.$$

Hence, we may apply Lemma 2.2 and (**??**) to deduce

(2.7)
$$\Omega_4 \le c_2 \int \hat{Y}_{a-1}^{l,\delta}\left(\sum_{n=1}^{l} n(f_n^\delta *_x H)(x,t)\right)dx \le c_3 \int \hat{Y}_{a-1}^{l,\delta}dx$$

where $\hat{Y}_{a-1}^{l,\delta} = \hat{Y}_{a-1}^l * \zeta^\delta$. We use Hypothesis 1.1 to assert that for every $\delta_0 > 0$, there exists $k_0 = k_0(\delta_0)$ such that if $k > k_0$, then

$$\hat{Y}_{a-1}^l \le \delta_0 \sum_{n,m} \mathbb{1}(n+m \le l))(n^{a-1}m + m^{a-1}n)(n+m)(d(n)+d(m))f_n f_m$$

(2.8)
$$+ 2k_0(\delta_0)^a \sum_{n,m} \mathbb{1}(n+m \le k_0(\delta_0))\alpha(n,m)f_n f_m.$$

From $(2.4-8)$ we deduce,

$$\frac{d}{dt}Z^{l,\delta}(t) \le -\frac{1}{2}\int \sum_{n,m} \mathbb{1}(n,m \le l)nm(n^{a-1}+m^{a-1})(d(n)+d(m))f_n^\delta f_m^\delta dx$$

(2.9)
$$+ 2c_3\delta_0 \int \sum_{n,m} \mathbb{1}(n,m \le l)nm(n^{a-1}+m^{a-1})(d(n)+d(m))(f_n f_m) * \zeta^\delta dx$$

$$+ 2c_3 k_0(\delta_0)^a \int \sum_{n,m} \mathbb{1}(n+m \le k_0(\delta))\alpha(n,m)((f_n f_m) * \zeta^\delta)dx + c_1 \int X_a^{l,\delta}dx,$$

where we used the inequality

$$(n+m)(n^{a-2}+m^{a-2}) \le 2(n^{a-1}+m^{a-1}),$$

which is valid provided that $a \ge 2$. On the other hand,

(2.10)
$$\frac{d}{dt}\int \sum_{n=1}^{k} n^a f_n(x,t)dx = \int \sum_{n=1}^{k} n^a Q_n(x,t)dx.$$

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Hence, by (2.6),

$$\int \sum_{n=1}^{k} n^a f_n(x,t)dx \leq \int \sum_{n=1}^{k} n^a f_n(x,0)dx + c_4 \int_0^t \int \hat{Y}_{a-1}^k(x,s)dxds$$

$$\leq \int \sum_{n=1}^{k} n^a f_n(x,0)dx + c_4 \int_0^t \int \hat{Y}_{a-1}^k(x,s)dxds.$$

Sending $k \to \infty$ yields

$$\int X_a(x,t)dx \leq \int X_a(x,0)dx + c_4 \int_0^t \int \hat{Y}_{a-1}(x,s)dxds.$$

One more time we use (2.8) to assert

$$(2.11) \qquad \int X_a(x,t)dx \leq c_5 + c_4\delta_0 \int_0^t \int Y_{a-1}dxds$$

$$+ 2c_4 k_0(\delta_0)^a \int_0^t \int \sum_{n,m} \mathbb{1}(n+m \leq k_0(\delta))\alpha(n,m)f_n f_m dxds.$$

In (2.9) we let $\delta \to 0$ and $l \to \infty$, and then apply (2.11) to deduce that the expression $Z^\infty(t) - Z^\infty(0)$ is bounded above by

$$\left(2c_3\delta_0 + 4c_1 c_2 t\delta_0 - \frac{1}{2}\right) \int_0^t \int \sum_{n,m} nm(n^{a-1} + m^{a-1})(d(n) + d(m))f_n f_m dxds$$

$$(2.12) \qquad + 2(c_1 c_2 t + c_3)k_0(\delta_0)^a t \int_0^t \int \sum_{n,m} \mathbb{1}(n+m \leq k_0(\delta))\alpha(n,m)f_n f_m dxds.$$

Let us simply write $k_0$ for $k_0(\delta_0)$. Note that from

$$\frac{d}{dt} \int \sum_{n=1}^{k_0} f_n dx = \int [\mathbb{1}(n+m \leq k_0) - \mathbb{1}(n \leq k_0) - \mathbb{1}(m \leq k_0)](\alpha(n,m)f_n f_m - \beta(n,m)f_{n+m})dx,$$

we learn that the expression

$$\int_0^t \int \sum_{n,m \leq k_0} \alpha(n,m)f_n f_m dxds$$

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is bounded above by

$$\int \sum_{n=1}^{k_0} f_n(x,0)dx - \int_0^t \int \sum_{n,m} [\mathbb{1}(n+m \le k_0) - \mathbb{1}(n \le k_0) - \mathbb{1}(m \le k_0)]\beta(n,m)f_{n+m}dxds$$

$$\le \int \sum_{n=1}^{k_0} f_n(x,0)dx + \int_0^t \int 2\sum_{n,m} \mathbb{1}(n \text{ or } m \le k_0)\beta(n,m)f_{n+m}dx$$

$$\le c_5 + 4\int_0^t \int \sum_{r=1}^{\infty} \left[ \sum_{n+m=r} \mathbb{1}(n \le k_0)\beta(n,m) \right] f_r dxds$$

$$\le c_5 + 4k_0 c_1(k_0) \int_0^t \int \sum_{r=1}^{\infty} r f_r dxds,$$

where for the last inequality we have used Hypothesis 1.3. From this and (2.2) we deduce,

(2.13) $$\int_0^t \int \sum_{n,m \le k_0} \alpha(n,m)f_n f_m dxds \le c_4 + c_6(k_0)t.$$

From this and (2.12) we learn that the expression $Z^\infty(t) - Z^\infty(0)$ is bounded above by

$$\left(2c_3\delta_0 + 4c_1 c_2 t\delta_0 - \frac{1}{2}\right) \int_0^t \int \sum_{n,m} nm(n^{a-1} + m^{a-1})(d(n) + d(m))f_n f_m dxds$$

$$+ c_7(k_0(\delta_0))(1 + t^2),$$

for a constant $c_7(\cdot)$. We now choose $\delta_0 = \delta_0(t)$ so that $\frac{1}{2} > (2c_3 + 4c_1 c_2 t)\delta_0$. With this choice the bounds in (1.13) follow with the help of (1.11). To prove (1.14), we simply use (2.11), (2.13) and (1.13). □

## 3  $L^\infty$ bounds

In this section we establish Theorem 1.2. To ease the notation, we do not display the dependence on the $x$-variable. For example we simply write $f_n(t)$ for $f_n(\cdot, t)$ and $Q_n(t)$ for

$Q_n(\cdot, t)$. We first claim

$$\sum_1^l nd(n)^{d/2} f_n(t) \le d(1)^{d/2} S_t^{d(1)} \left( \sum_1^l n f_n^0 \right)$$

(3.1)
$$+ d(l)^{d/2} \int_0^t S_{t-s}^{d(l)} \left( \sum_1^l n Q_n^C(s) \right) ds$$

$$+ \int_0^t \sum_1^l nd(n)^{d/2} \left( S_{t-s}^{d(n)} Q_n^{+,F}(s) \right) ds.$$

The proof of (3.1) is identical to the proof of (4.2) of [HR1] and we do not repeat it here. However, the proof given in [HR1] uses induction and is not applicable in the continuous case. Here is the corresponding statement and its proof in the continuous case.

**Lemma 3.1** *Suppose that $f$ is a solution to the continuous Smoluchowski equation. Then for every $l \ge 0$, we have*

$$\int_0^l (n+1)d(n)^{d/2} f_n(t) dn \le d(0)^{d/2} S_t^{d(0)} \left( \int_0^l (n+1) f_n^0 dn \right)$$

(3.2)
$$+ d(l)^{d/2} \int_0^t S_{t-s}^{d(l)} \left( \int_0^l (n+1) Q_n^C(s) dn \right) ds$$

$$+ \int_0^t \left[ \int_0^l (n+1)d(n)^{d/2} \left( S_{t-s}^{d(n)} Q_n^{+,F}(s) \right) dn \right] ds,$$

*almost everywhere, where $d(0) = \sup_n d(n)$.*

**Proof.** First we rewrite (3.2) as

(3.3)
$$A(l) \le B(l) + C(l, l) + D(l),$$

where

$$A(l) = \int_0^l (n+1)d(n)^{d/2} f_n(t) dn,$$

$$B(l) = d(0)^{d/2} S_t^{d(0)} \left( \int_0^l (n+1) f_n^0 dn \right),$$

$$C(k, l) = d(k)^{d/2} \int_0^t S_{t-s}^{d(k)} \left( \int_0^l (n+1) Q_n^C(s) dn \right) ds,$$

$$D(l) = \int_0^t \left[ \int_0^l (n+1)d(n)^{d/2} S_{t-s}^{d(n)} Q_n^{+,F}(s) dn \right] ds.$$

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Note that both sides of (3.2) are functions of $(x, t)$ and (3.2) is equivalent to saying

(3.4) $$A(l, J) \leq B(l, J) + C(l, l, J) + D(l, J),$$

for every smooth function $J \geq 0$ of compact support, where $A(l, J), B(l, J), C(k, l, J)$, and $D(l, J)$, are obtained by multiplying $A(l), B(l), C(k, l)$ and $D(l)$, by $J$ and integrating with respect to $dx\, dt$. Set

$$\eta(k, l) = B(l, J) + C(k, l, J) + D(l, J) - A(l, J), \quad \gamma(l) = \eta(l, l).$$

The inequality (3.4) means that $\gamma(l) \geq 0$ and it suffices to show that $\gamma$ is non-decreasing because $\gamma(0) = 0$. This is equivalent to saying that the weak derivative of $\gamma$ is non-negative which would follow if we can show that $\eta$ is non-decreasing in $k$-variable and that $\frac{\partial \eta}{\partial l}(l, l) \geq 0$. The former is an immediate consequence of

(3.5) $$\int_0^l (n + 1) Q_n^C dn \leq 0,$$

the assumption that $d(\cdot)$ is non-increasing, and the elementary fact that the function $a \mapsto a^{d/2} S_t^a g$ is decreasing for every non-positive function $g$. For the latter observe that $\frac{\partial \eta}{\partial l}(l, l)$ equals

$$d(0)^{d/2}(l+1) \iint \left( S_t^{d(0)} f_l^0 \right) J dx dt + d(l)^{d/2}(l+1) \iint \int_0^t \left( S_{t-s}^{d(l)} Q_l^C(s) \right) ds\, J\, dx dt$$

$$+ d(l)^{d/2}(l+1) \iint \int_0^t \left( S_{t-s}^{d(l)} Q_l^{+,F}(s) \right) ds\, J\, dx dt - d(l)^{d/2}(l+1) \iint f_l J\, dx dt$$

$$\geq d(l)^{d/2}(l+1) \iint S_t^{d(l)} f_l^0\ J dx dt + d(l)^{d/2}(l+1) \iint \int_0^t S_{t-s}^{d(l)} Q_l^C(s) ds\ J\ dx dt$$

$$+ d(l)^{d/2}(l+1) \iint \int_0^t \left[ S_{t-s}^{d(l)} Q_l^{+,F}(s) \right] ds\ J\ dx dt - d(l)^{d/2}(l+1) \iint f_l J\ dx dt$$

$$\geq 0,$$

where for the first inequality we used the monotonicity of $d(\cdot)$ and the fact that the function $a \mapsto a^{d/2} S_t^a$ is increasing, and for the second inequality we used (1.2) and $Q^{+,F} \geq Q^F$. This completes the proof. $\square$

**Remark 3.1**. When there is no fragmentation, the bounds (3.1) and (3.2), and the inequality (3.5) imply

$$\text{esssup}_x \left( \sum_{n=1}^\infty n d(n)^{d/2} f_n(x, t) \right) \leq d(1)^{d/2} \text{esssup}_x \left( \sum_{n=1}^\infty n f_n(x, 0) \right),$$

$$\text{esssup}_x \left( \int_0^\infty (n + 1) d(n)^{d/2} f_n(x, t) dn \right) \leq d(0)^{d/2} \text{esssup}_x \left( \int_0^\infty (n + 1) f_n(x, 0) dn \right),$$

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for the solutions to the Smoluchowski equation in the discrete and continuous case respectively. This yields a $L^\infty$ bound at time $t$ provided that the associated bound is assumed initially. This and the $L^1$ bound of Section 2 can be used to establish Theorem 1.2 when there is no fragmentation. This was carried out in Section 4 of [HR1] in the discrete case. A similar proof is applicable in the continuous case. $\qquad\square$

As our next step towards the proof of Theorem 1.2, we use Lemmas 3.1 and 3.2 to deduce a $L^\infty$ bound on certain moments of solutions. When there is fragmentation, our $L^\infty$ bounds are expressed in terms of suitable $L^r$-bounds. We first state our result in the discrete case.

**Lemma 3.2** *For every $r > 1 + \frac{d}{2}$ and $b \geq 0$, there exists a finite constant $C_0(r, T)$ such that for every $t \leq T$,*

$$
\mathrm{esssup}_x \left( \sum_{n=1}^{\infty} n d(n)^{d/2} f_n(x,t) \right) \leq d(1)^{d/2} \, \mathrm{esssup}_x \left( \sum_{n=1}^{\infty} n f_n(x,0) \right)
$$

(3.6)
$$
+ C_0(r,T) \left( \sum_{n=1}^{\infty} n^{1-b} d(n)^{d - b_0 d/2} \right)
$$
$$
\cdot \left( \int_0^T \int \left( \sum_n n^{b+a_0} f_n(x,s) \right)^r dx ds \right)^{1/r},
$$

*where $b_0 = r/(r-1)$.*

**Proof.** We note that the middle term on the right-hand side of (3.1) is non-positive because $\sum_1^l n Q_n^C \leq 0$. As for the fragmentation term, observe that $\int_0^t S_{t-s}^D h_s ds = p_D * h$ where the convolution is in both $x$ and $t$ variables and

(3.7)
$$
p_D(x,t) = \begin{cases} (4\pi D t)^{-d/2} \exp\left( \frac{-|x|^2}{4Dt} \right) & \text{if } t > 0 \\ 0 & \text{if } t < 0. \end{cases}
$$

Note that

$$
\int_0^T \int p_D(x,s)^{b_0} dx ds = (4\pi)^{\frac{d}{2}(1-b_0)} b_0^{-\frac{d}{2}} \int_0^T (Ds)^{\frac{d}{2}(1-b_0)} ds
$$
$$
= c_1(T, b_0) D^{\frac{d}{2}(1-b_0)}
$$

is finite if and only if $b_0 < \frac{2}{d} + 1$. Hence if $\frac{1}{b_0} + \frac{1}{r} = 1$ with $b_0 < \frac{2}{d} + 1$ and $r > 1 + \frac{d}{2}$, then

$$
\left\| \int_0^t S_{t-s}^{d(n)} Q_n^{+,F}(s) ds \right\|_{L^\infty} \leq \|p_{d(n)}\|_{L^{b_0}} \|Q_n^{+,F}\|_{L^r}
$$

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where all the $L^p$-spaces are defined over the space $\mathbb{R}^d \times [0, T]$. As a result, $L^\infty(\mathbb{R}^d \times [0, T])-$ norm of the third term on the right-hand side of (3.1) is bounded above by

$$c_1 \sum_1^l nd(n)^{d/2}d(n)^{\frac{d}{2}(1-b_0)}\|Q_n^{+,F}\|_{L^r} \le c_2 \sum_1^l nd(n)^{d-b_0d/2}\left\|\sum_{m=1}^l (n+m)^{a_0}f_{n+m}\right\|_{L^r}$$

$$\le c_2 \sum_1^l n^{1-b}d(n)^{d-b_0d/2}\left\|\sum_{m=1}^l (n+m)^{a_0+b}f_{n+m}\right\|_{L^r}$$

$$\le c_2 \left(\sum_1^l n^{1-b}d(n)^{d-b_0d/2}\right)\|X_{a_0+b}\|_{L^r}.$$

From this and (3.1) we deduce (3.6) because $\|S_t^D h\|_{L^\infty(\mathbb{R}^d)} \le \|h\|_{L^\infty(\mathbb{R}^d)}$. $\qquad\square$

**Remark 3.2.** By a verbatim argument, we can show

$$\operatorname{esssup}_x \left(\int_0^\infty (n+1)d(n)^{d/2}f_n(x,t)dn\right) \le d(0)^{d/2}\operatorname{esssup}_x \left(\int_0^\infty (n+1)f_n(x,0)dn\right)$$

$$+ C_0(r,T)\left(\int_0^\infty (n+1)^{1-b}d(n)^{d-b_0d/2}\right)$$

$$\cdot \left(\int_0^T \int \left(\int_0^\infty (n+1)^{b+a_0}f_n(x,s)dn\right)^r dxds\right)^{1/r},$$

in the case of the continuous Smoluchowski equation. $\qquad\square$

***Proof of Theorem 1.2.*** We first present the proof in the discrete case. To ease the notation, we simply write $L^p$ for $L^p(\mathbb{R}^d \times [0, T])$. Set

$$\hat{X}_\theta = \sum_n n^\theta d(n)^{d/2}f_n.$$

Given any real number $l > 1$, we use Jensen's inequality to assert,

$$\hat{X}_{\theta+1} = \sum_n n^\theta \frac{nd(n)^{d/2}f_n}{\hat{X}_1}\hat{X}_1,$$

$$\hat{X}_{\theta+1}^l \le \hat{X}_1^{l-1}\sum_n n^{\theta l+1}d(n)^{d/2}f_n.$$

From this and Lemma 3.2 we deduce that for every $r > 1 + \frac{d}{2}$,

$$\|\hat{X}_{\theta+1}\|_{L^l} \le \|\hat{X}_1\|_{L^\infty}^{1-\frac{1}{l}}\|\hat{X}_{\theta l+1}\|_{L^1}^{\frac{1}{l}}$$

$$\le \left\{c_1 + C_0(r,T)^{1-\frac{1}{l}}\left(\sum_n n^{1-b}d(n)^{d-b_0d/2}\right)^{1-\frac{1}{l}}\|X_{b+a_0}\|_{L^r}^{1-\frac{1}{l}}\right\}\|\hat{X}_{\theta l+1}\|_{L^1}^{\frac{1}{l}},$$

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with $c_1$ a constant depending on the initial densities and $b_0 = r/(r-1)$. Choose $r = l$ and choose $b$ so that

$$\sum_n n^{1-b} d(n)^{d-b_0 d/2} = \sum_n n^{1-b} d(n)^{d(l-2)/(2l-2)} < \infty.$$

We are now assuming that $l > 1 + \frac{d}{2}$ and we choose any $\theta$ such that

$$n^{\theta+1} d(n)^{d/2} \geq n^{b+a_0}$$

so that

$$X_{b+a_0} \leq \hat{X}_{\theta+1}.$$

For such a pair of $\theta$ and $b$ we obtain

(3.8)
$$\|\hat{X}_{\theta+1}\|_{L^l} \leq c_2 + c_2 \|\hat{X}_{\theta l+1}\|_{L^1},$$

for a suitable constant $c_2$. In summary, for sufficiently large $\theta$, we always have (3.8) provided that $l > 1 + d/2$. Hence $L^l$-norm of $\hat{X}_{\theta+1}$ is finite if $L^1$-norm of sufficiently large moment of $f$ is finite. Recall that we are assuming Hypothesis 1.2. To have (3.8), we may choose $\theta$ and $b$ any pair of numbers satisfying

$$\theta \geq b + a_0 + \frac{d}{2}b_1 - 1, \quad b > 2 - b_2 d \frac{l-2}{2l-2},$$

provided that $l \geq 2$ and $l > 1 + d/2$. Hence, given such a number $l$, we may assume that $\theta$ satisfies

$$\theta > 1 + a_0 + \frac{d}{2}\left(b_1 - b_2 \frac{l-2}{l-1}\right),$$

in order to have (3.8). From this we deduce

(3.9)
$$X_a \in L^1 \Rightarrow X_B \in L^{\frac{2a+b_2 d-2}{2B+b_1 d-2}}$$

provided that

$$B > 2 - b_2 d \frac{l-2}{2l-2} + a_0 = 2 + a_0 - \frac{b_2 d}{2} + \frac{b_2 d}{2l-2}, \quad l > \max\left\{1 + \frac{d}{2}, 2\right\},$$

with

$$l = \frac{2a + b_2 d - 2}{2B + b_1 d - 2}.$$

We note that the second condition implies the first one if we assume

$$B \geq B_0 := 2 + a_0 - b_2(1 - d/2)^-.$$

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Also note that by our Hypothesis 1.2, we have that $b_1 d - 2 > 0$.

Let us write $X_a(N) = \sum_{n \geq N} n^a f_n$. We certainly have

$$Q_n^{C,+} = \sum_{n_1+n_2=n} \alpha(n_1, n_2) f_{n_1} f_{n_2}$$

$$\leq \sum_{n_1,n_2} \mathbb{1}\left(n_1 \geq \frac{n}{2} \text{ or } n_2 \geq \frac{n}{2}\right) \alpha(n_1, n_2) f_{n_1} f_{n_2}$$

$$\leq c_1 \left[X_1(1)X_0\left(\frac{n}{2}\right) + X_0(1)X_1\left(\frac{n}{2}\right)\right]$$

because $\alpha(n_1, n_2) \leq c_0(n_1 + n_2)$ by assumption. Moreover

$$Q_n^{F,+} = 2\sum_{m=1}^{\infty} \beta(n, m) f_{n+m}$$

$$\leq c_2 \sum_{m=1}^{\infty} (n+m)^{a_0} f_{n+m} \leq c_2 X_{a_0}(n).$$

Hence, for any $K \geq 0$,

$$Q_n^+ \leq c_3 n^{-K}[X_1 X_K + X_{K+1} X_0 + X_{a_0+K}],$$
$$\|Q_n^+\|_{L^p} \leq c_3 n^{-K}[\|X_1\|_{L^{l_1}}\|X_K\|_{L^{l_2}} + \|X_{K+1}\|_{L^{l_3}}\|X_0\|_{L^{l_4}} + \|X_{a_0+K}\|_{L^p}]$$

provided that $\frac{1}{p} = \frac{1}{l_1} + \frac{1}{l_2} = \frac{1}{l_3} + \frac{1}{l_4}$. To bound the right-hand side, we would like to apply (3.9) with $B = 0, 1, K, K+1$ and $K + a_0$. Note that $B$ can not be smaller than $B_0$. So, we need to assume that $K \geq B_0^+$. However since in general we may not have $0 \geq B_0$, we need to choose $B = B_0^+$ and $(B_0 - 1)^+ + 1$ in (3.9) in place of $B = 0$ and $B = 1$ respectively. Hence, if $X_a \in L^1$, then by (3.9),

$$X_0 \in L^{\frac{2a+b_2d-2}{2B_0^+ +b_1d-2}}, \quad X_1 \in L^{\frac{2a+b_2d-2}{2(B_0-1)^+ +b_1d}}, \quad X_K \in L^{\frac{2a+b_2d-2}{2K+b_1d-2}},$$
$$X_{K+1} \in L^{\frac{2a+b_2d-2}{2K+b_1d}}, \quad X_{a_0+K} \in L^{\frac{2a+b_2d-2}{2a_0+2K+b_1d-2}},$$

provided that $a$ and $K$ satisfy

(3.10) $$\frac{2a + b_2d - 2}{2K + b_1d}, \quad \frac{2a + b_2d - 2}{2a_0 + 2K + b_1d - 2} > \max(2, 1 + d/2), \quad K \geq B_0^+.$$

From this we learn

(3.11) $$X_a \in L^1 \Rightarrow \|Q_n^+\|_{L^p} \leq c n^{-K}$$

provided that $a$ and $K$ satisfy (3.10) and $p$ is chosen to be

$$
(3.12) \qquad p = \frac{2a + b_2 d - 2}{2K + b_1 d - 2 + \max(2B_0^+ + b_1 d, 2a_0)}.
$$

We now follow [HR1]. We certainly have

$$
f_n(x, t) = (S_t^{d(n)} f_n^0)(x) + (p_{d(n)} * Q_n)(x, t) \leq (S_t^{d(n)} f_n^0)(x) + (p_{d(n)} * Q_n^+)(x, t).
$$

So,

$$
(3.13) \qquad \|f_n\|_{L^\infty} \leq \|(S_t^{d(n)} f_n^0)(x)\|_{L^\infty} + \|p_{d(n)}\|_{L^r} \|Q_n^+\|_{L^p}
$$

provided that $\frac{1}{r} + \frac{1}{p} = 1$. Since by (3.7) we have $p_D \in L^r$ with $r < \frac{2}{d} + 1$, it suffices to have

$$
\frac{1}{1 + 2/d} + \frac{1}{p} < 1,
$$

or equivalently $p > 1 + d/2$. With $p$ as in (3.12), it suffices to have

$$
\frac{2a + b_2 d - 2}{2K + b_1 d - 2 + \max(2B_0^+ + b_1 d, 2a_0)} > \max(2, 1 + d/2).
$$

More precisely,

$$
(3.14) \qquad K < K_0(a; b_1, b_2, a_0) := \frac{a + b_2 d/2 - 1}{\max(2, 1 + d/2)} - \frac{b_1 d}{2} + 1 + \max\left(B_0^+ + \frac{b_1 d}{2}, a_0\right).
$$

Note that this condition implies the first condition in (3.10). In summary, we need $K$ to satisfy (3.14) and $K \geq B_0^+$, and $r$ to satisfy $\frac{d}{2}(1 - r) > -1$. (We remark that by our assumptions $a > 1$ and $b_1 d > 2$, so (3.15) is compatible with the requirement $K \geq B_0^+$, because $K_0 > B_0^+$.) Hence, if $X_a \in L^1$, $K \geq B_0^+$ and $K$ satisfies (3.14), then

$$
\|f_n\|_{L^\infty} \leq A_n + c_3 \, n^{-K} d(n)^{d(1-r)/2} \leq A_n + c_3 \, n^{-K} d(n)^{-1},
$$

where

$$
A_n = \|(S_t^{d(n)}) f_n^0(x)\|_{L^\infty} \leq \|f_n^0\|_{L^\infty(\mathbb{R}^d)}.
$$

As a result, we would have that $\sum_n n^e \|f_n\|_{L^\infty} < \infty$ if $X_a \in L^1$,

$$
(3.15) \qquad \sum_n n^{e-K} d(n)^{-1} < \infty,
$$

and,

$$
\sum_n n^e \|f_n^0\|_{L^\infty(\mathbb{R}^d)} < \infty.
$$

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For (3.15) it suffices to have

$$e - K + b_1 < -1.$$

Such a number $K$ can be found if we have

(3.16) $$e < \gamma(a; b_1, b_2, a_0) := K_0(a; b_1, b_2, a_0) - b_1 + 1$$

In summary, if

$$X_a \in L^1, \quad \sum_n n^e \|f_n^0\|_{L^\infty(\mathbb{R}^d)} < \infty, \quad e < \gamma(a; b_1, b_2, a_0),$$

then

$$\sum_n n^e \|f_n\|_{L^\infty(\mathbb{R}^d \times [0,T])} < \infty$$

where

$$\gamma(a; b_1, b_2, a_0) = \frac{2a + b_2 d - 2}{\max(4, d+2)} - b_1 \left(1 + \frac{d}{2}\right) + \max\left(B_0^+ + \frac{b_1 d}{2}, a_0\right).$$

We now turn to the proof in the the continuous case. The proof in this case is almost identical to the proof in the discrete case. The only step which requires an explanation is the fact that in the continuous case we can not assert that $X_a \le X_b$ whenever $a \le b$, because now all integrations are over $n \in (0, \infty)$. However, we can repeat all the steps if we replace $n$ with $n + 1$. More precisely we switch from $X_a$ and $\hat{X}_a$ in the proof to $X_a' = \int_0^\infty (n+1)^a f_n dn$, and $\hat{X}_a' = \int_0^\infty (n+1)^a d(n)^{d/2} f_n dn$. For this, we need to make sure that $X_a, X_{a_0} \in L^1$ imply that $X_a' \in L^1$. This would follow if we can show

(3.17) $$\sup_t \int_0^1 \int f_n(x, t) \, dx \, dn < \infty.$$

This can be readily establish because the expression

$$\int_0^1 f_n(x, 0) \, dn \, dx - \int_0^1 f_n(x, t) \, dn \, dx,$$

equals

$$\int_0^t \int \iint [\mathbb{1}(n + m \le 1) - \mathbb{1}(n \le 1) - \mathbb{1}(m \le 1)](\alpha(m, n) f_n f_m - \beta(m, n) f_{n+m}) \, dm \, dn \, dx \, ds$$

$$\le 2 \int_0^t \int \iint \mathbb{1}(n \text{ or } m \le 1) \beta(m, n) f_{n+m} \, dm \, dn \, dx \, ds \le 4 c_0 X_{a_0},$$

where we have used Hypothesis 1.3 for the second inequality. This completes the proof. $\square$

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