Definable Cohomology

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In this note we review the construction of definable Galois cohomology groups due to Anand Pillay, with some slight modifications and relaxations of the various hypotheses.

Insert background definitions

The main result is the following

Theorem 0.1. There is a natural isomorphism between $P_{def}(G)$ and $H^1_{def}(\mathcal{G}, G)$.

Proof. We exhibit an isomorphism between the two. It suffices to give an isomorphism

$$\mu : P_{def}(G) \to H^1_{def}(\mathcal{G}, G).$$

Let X be an A-definable PHS. Pick $x_0 \in X$. Then for all $\sigma \in \mathcal{G}$ there is a unique $g_\sigma \in G$ such that

$$x_0 \cdot g_\sigma = \sigma(x_0)$$

The map $\sigma \mapsto g_\sigma$ is a definable cocycle f_{x_0}, and if $x_1 = x_0 \cdot g_{01}$ is another point, then the cocycle f_{x_1} associated to x_1 satisfies

$$f_{x_1} = g_{01} \cdot f_{x_0} \cdot \sigma(g_{01}) \sim f_{x_0}$$

so that in terms of cohomology classes,

$$[f_{x_0}] = [f_{x_1}] \in H^1_{def}(\mathcal{G}, G).$$

We now exhibit the map μ^{-1}. Given a definable cocycle f we wish to construct a definable PHS X_f. Let f be such a cocycle, and suppose that f is represented by the definable function θ. That is, there is a finite tuple $\exists \overline{a}_f \in M$ such that

$$f(\sigma) = \theta(\overline{a}_f, \sigma(\overline{a}_f))$$

for all $\sigma \in \mathcal{G}$. Consider the type-definable set

$$Z = \mathrm{tp}(\overline{a}_f/A) \times G$$

and define an equivalence relation R on Z by setting

$$(b, g)\, R\, (c, h) \iff \theta(\overline{a}_f, b) \cdot g = \theta(\overline{a}_f, c) \cdot h.$$

R is a type-definable equivalence relation **JUSTIFY**. Defining an action of G on Z/R by setting

$$(b, h) \cdot g = (b, h \cdot g)$$

gives the structure of an A-definable action of G on Z/R making it have the structure of an abstract principal homogeneous space. But then Z/R is the image of a definable map on G and therefore is representable by a definable set. □