An Invitation to Factorization Algebras

Peter Teichner, Aaron Mazel-Gee
Notes by Qiaochu Yuan

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1 Motivation (Teichner, 1/19)

Let X be a complete Riemannian manifold. For every interval $I \subseteq \mathbb{R}$, let $EL(I)$ be the space of geodesics $I \to X$. (This is a space of critical points of a Lagrangian in a classical field theory; in this case the Lagrangian is the energy functional on paths.) This can be identified with the tangent bundle TX, since a geodesic is determined by a point and a tangent vector, independent of I. In fact if $I \to J$ is an inclusion the pullback map $EL(J) \to EL(I)$ is an isomorphism. Hence the "classical observables" functor

$$I \mapsto \text{Obs}^{cl}(I) = C^\infty(EL(I)) \cong C^\infty(TX) \tag{1}$$

is locally constant. It is an example of a locally constant factorization algebra. It is in particular a cosheaf of rings. Part of the structure it has is that any pair of intervals I_1, I_2 contained in another interval J, there is a map

$$\text{Obs}^{cl}(I_1) \otimes \text{Obs}^{cl}(I_2) \to \text{Obs}^{cl}(J) \tag{2}$$

satisfying the obvious associativiy condition.

What happens when we quantize? Quantum observables $\text{Obs}^q(I)$ will still end up having a factorization algebra structure. This means that we have the above maps when I_1 and I_2 are disjoint. We will also have a more complicated cosheaf condition.

2 The definition (Teichner, 1/21)

Let M be a topological space. The category $\text{Open}(M)$ of open subsets and inclusions has a symmetric multicategory structure where there is a unique multimorphism $U_1, U_2, \ldots U_n \to V$ if the U_i are disjoint and contained in V, and no morphism otherwise. Any symmetric monoidal category also has a symmetric multicategory structure where the multimorphisms $X_1, X_2, \ldots X_n \to Y$ are morphisms $X_1 \otimes X_2 \otimes \cdots \otimes X_n \to Y$. Intuitively, a symmetric multicategory is like a symmetric monoidal category but where the monoidal structure is allowed to be partially defined.

Definition A *prefactorization algebra* on M with values in a symmetric monoidal category $(C, \otimes)$ is a symmetric multifunctor $(\text{Open}(M), \coprod) \to (C, \otimes)$. ($C$ could also be another symmetric multicategory but we will never write down examples like this.)

Explicitly, this means that a prefactorization algebra consists of the following data:

1. For each open set U, an object $F(U) \in C$
2. For each inclusion $U \subseteq V$, a morphism $F(U) \to F(V)$
3. For each disjoint inclusion $U_1 \coprod \cdots \coprod U_n \to V$, a morphism

$$F(U_1) \otimes \cdots \otimes F(U_n) \to F(V) \tag{3}$$

4. A unit map $1_C \to F(\emptyset)$.

This data is subject to various axioms, e.g. a functoriality axiom, an associativity axiom, a symmetry axiom, and a unit axiom. In particular, $F(\emptyset)$ is a commutative algebra object in C (although it might just be the unit 1_C), and everything in sight is a module over it. In examples $(C, \otimes)$ will be vector spaces of some sort, $\otimes$ will be tensor product, and $F(\emptyset)$ will be 1_C, which will be the ground ring, usually $\mathbb{C}$ or $\mathbb{C}[[\hbar]]$. Every $F(U)$ admits a map $F(\emptyset) \to F(U)$.

3 Examples of factorization algebras (Teichner, 2/2)

3.1 Quantum mechanics as factorization algebra on $[0, 1]$

Pick the stratification on $[0, 1]$ given by the endpoints. A constructible (stratified locally constant) factorization algebra associates three different vector spaces V, A, W to open intervals containing 0, neither, and 1 respectively. A is an algebra as usual. V turns out to be a right A-module and W is a left A-module. And V, W both have distinguished points v, w. This is all of the data.

What do we assign to the entire interval? Using the cosheaf condition, we get $V \otimes_A W$.

Now we can relax the constructibility condition and also incorporate a semigroup $h_t \in A$ as before. This will modify the inclusions $F(\emptyset) \to F(U)$ as follows: if U is $[0, s)$ then we assign vh_s, if U is (s, t) then we assign h_{t-s}, and if U is $(s, 1]$ then we assign $h_{1-s}w$. The entire interval gets assigned $vh_1 \otimes w = v \otimes h_1 w \in V \otimes_A W$.

To apply this to quantum mechanics, take V to be a Hilbert space, $A = B(V)$ to be bounded operators on it, $h_t = e^{itH}$ for some self-adjoint H, and $W = \overline{V}$ where the module structure is defined using the adjoint. With the convention that inner products are antilinear in the second variable, we get

$$F([0, 1]) = V \otimes_{B(V)} V^* \cong \mathbb{C} \tag{4}$$

where the isomorphism sends $v \otimes w$ to $\langle v, w \rangle$. This factorization algebra computes "scattering amplitudes"; for example, if after time t we perform an observation a, then after time s we reach the end, then for initial state $v_0 \in V$ and final state $v_1 \in \overline{V}$, the quantity

$$\langle v_0 h_t \mid a \mid h_s v_1 \rangle \in \mathbb{C} \tag{5}$$

is the amplitude that the system is in state v_1 if it started out in state v_0 (where h_t is time evolution), which the factorization algebra knows. Or something like that.

3.2 Universal enveloping algebras

Let $\mathfrak{g}$ be a Lie algebra and M a smooth manifold. First consider

$$F : U \mapsto (\Omega_c^\bullet(U), d_{dR}). \tag{6}$$

This is a covariant functor into chain complexes (because we can extend by zero). It also sends disjoint unions to direct sums, even in the case of infinite disjoint unions. And in fact this is a functor into commutative DGAs.

Now we can apply $(-) \otimes \mathfrak{g}$. This produces DGLAs (differential graded Lie algebras). Finally we can apply the Chevalley-Eilenberg functor $CE_\bullet$, which produces cochain complexes (in fact cocommutative DGCAs) and sends direct sums to tensor products. Altogether we get a functor

$$F : U \mapsto CE_\bullet(\Omega_c^\bullet(U, \mathfrak{g}), d) \in (\mathrm{Ch}, \otimes) \tag{7}$$

In fact $\mathfrak{g}$ could be a DGLA here (in cohomological indexing, so degree 1 like the de Rham differential). The Chevalley-Eilenberg functor takes $\mathfrak{g}$ to the graded vector space $S^\bullet(\mathfrak{g}[1])$ given by the symmetric algebra on $\mathfrak{g}[1]$ (which is $\mathfrak{g}$ shifted to the left), with a differential defined in terms of the Lie bracket and the differential on $\mathfrak{g}$.

This is in fact a locally constant infinitely strong (arbitrary disjoint unions to tensor products) pFA. On $\mathbb{R}$ this means it's some associative algebra, which for $\mathfrak{g}$ an ordinary Lie algebra is the universal enveloping algebra $U(\mathfrak{g})$. But now we can do more interesting things. E.g. on $\mathbb{R}^n$ we get the universal enveloping E_n algebra.

4 The divergence complex and Feynman diagrams (Mazel-Gee and ???, 2/4)

4.1 Motivation

Recall that if M is a d-dimensional compact, orientable, connected manifold, we have the de Rham complex, which ends

$$\dots \Omega^{d-1}(M) \xrightarrow{d_{dR}} \Omega^d(M). \tag{8}$$

Picking an orientation of M, we can integrate against the fundamental class, and this gives an isomorphism

$$H^d(M, \mathbb{R}) \ni [\omega] \mapsto \int_M \omega \in \mathbb{R}. \tag{9}$$

We'd like an analogue of this story if M is infinite-dimensional, but unfortunately there are no top forms in this setting. So we need to do something else. Consider the graded vector space

$$V_\bullet = \Gamma(\wedge^\bullet(TM)) \tag{10}$$

of polyvector fields (sections of exterior powers of the tangent, rather than cotangent, bundle). These are dual to differential forms. If we pick a volume form $\mu \in \Omega^d(M)$, then we can transport the de Rham differential to a differential Δ_μ on polyvector fields, and the homology of this chain complex can be identified with de Rham cohomology. In particular,

$$H^d(M,\mathbb{R}) \cong \mathbb{R} \cong H_0(V_\bullet). \tag{11}$$

Note that although $V_\bullet$ has an algebra structure, Δ_μ is not a differential on it. Measuring its failure to be a differential gives a Poisson bracket.

Physicists would like to calculate integrals, or more precisely quotients of integrals, of the form

$$\langle f \rangle = \frac{\int f e^{-S/\hbar}\mu}{\int e^{-S/\hbar}\mu}. \tag{12}$$

(At least this is a toy model of what we want.) This is the expectation of the observable f, where S is the action. One way to think about this is that it is the quotient of the homology class $[f]$ of f by the homology class $[1]$ of 1 in the divergence complex with differential $\Delta_{e^{-S/\hbar}\mu}$. This suggests some hope for what to do in the infinite-dimensional setting: instead of trying to find a top form, we'll try to find a divergence operator.

Suppose $x_0 \in M$ is the unique minimum of S and that it is a nondegenerate critical point. Then as $\hbar \to 0$ the measure $e^{-S/\hbar}\mu$ is asymptotically supported on an infinitesimal neighborhood of x_0. In local coordinates S takes the form

$$S(x) = S(0) + \frac{1}{2}\sum a_{ij}x_i x_j - b(x) \tag{13}$$

where $b(x) = O(|x|^3)$ and a_{ij} is symmetric and invertible. This lets us write down a formal divergence operator

$$-\hbar\Delta_{e^{-S/\hbar}\mu} = \sum a_{ij}x_i \frac{\partial}{\partial \xi_j} - \sum \frac{\partial b}{\partial x_i}\frac{\partial}{\partial \xi_i}a - \hbar \sum \frac{\partial^2}{\partial x_i \partial \xi_j} \tag{14}$$

acting on formal power series $\mathbb{R}[[x_1, \dots x_d, \xi_1, \dots \xi_d, \hbar]]$, where the ξ_i are degree 1 in the graded commutative sense (locally representing a basis of vector fields).

Example Suppose $d = 1$ and $b = 0$, so we're trying to compute a Gaussian integral, and a_{ij} is just a positive real number a. Formally the divergence complex is $V_0 = \mathbb{R}[[x,\hbar]], V_1 = \mathbb{R}[[x,\hbar]][\xi]$, and the other terms vanish. To understand our divergence operator $Q : V_1 \to V_0$ it suffices to understand what it does to $x^n\xi$ for all n. This gives

$$Q(x^n\xi) = ax^{n+1} - \hbar n x^{n-1} \tag{15}$$

which tells us that in homology we have

$$[x^{n+1}] = \frac{\hbar n}{a}[x^{n-1}]. \tag{16}$$

This gives $[x] = 0$, hence $[x^{2n+1}] = 0$ for all n, and

$$\frac{[x^{2n}]}{[1]} = \left(\frac{\hbar}{a}\right)^n (2n-1)(2n-3)\dots \tag{17}$$

And these are the moments of the Gaussian distribution as desired; that is, we've successfully computed

$$\langle x^{2n} \rangle = \frac{\int_{\mathbb{R}} x^{2n} e^{-\frac{1}{2}\frac{ax^2}{\hbar}}\, dx}{\int_{\mathbb{R}} e^{-\frac{1}{2}\frac{ax^2}{\hbar}}\, dx}. \tag{18}$$

The combinatorial part $(2n-1)(2n-3)\ldots$ is the number of ways to pair up $2n$ elements, and this turns out not to be a coincidence.

Example Now let d be an arbitrary finite dimension, although we will continue to assume that $b = 0$. We'll cleverly compute that

$$Q\left(\sum \xi_k (a^{-1})_{k\alpha} x_\beta\right) = x_\alpha x_\beta - \hbar (a^{-1})_{\beta\alpha} \tag{19}$$

and we'll conclude that

$$[x_\alpha x_\beta] = \hbar (a^{-1})_{\beta\alpha}[1]. \tag{20}$$

More generally, we'll cleverly compute that

$$Q\left(\sum \xi_k (a^{-1})_{k\alpha_1}\right) = x_{\alpha_1} \ldots x_{\alpha_\kappa} - \hbar \sum (a^{-1})_{\alpha_1 \alpha_\lambda} \prod_{others} x_{\alpha_\mu} \tag{21}$$

which gives us that $[x_{\alpha_1} \ldots x_{\alpha_\kappa}]$ is a certain sum over terms corresponding to pairings of the indices. So far the combinatorics aren't so bad, but they get worse when $b \neq 0$.

Example Now let $d = 1$ again, but let's introduce a cubic term $b(x) = \frac{x^3}{3!}$. We get

$$Q(x^n \xi) = ax^{n+1} - \frac{x^{n+2}}{2} - \hbar n x^{n-1} \tag{22}$$

and hence

$$[x^{n+1}] = \frac{1}{2a}[x^{n+2}] + \frac{\hbar n}{a}[x^{n-1}]. \tag{23}$$

Repeatedly applying this recurrence gives something which converges $\hbar$-adically. The result is messy to describe explicitly, which is why we'll start introducing Feynman diagrams.

5 The universal enveloping algebra (Teichner, 2/2, 2/4)

The pFA we constructed out of a Lie algebra last time is not in general a cosheaf, except when $M = \mathbb{R}$. However, it is a homotopy cosheaf. For now let's just observe that in particular all of the functors we wrote down preserve weak equivalences (maps inducing isomorphisms on homology; this makes sense for CDGAs, DGLAs, etc). For the Chevalley-Eilenberg functor the argument involves using a filtration whose associated graded is just the symmetric algebra on the underlying DG vector space, and then we use Kunneth and the fact that over $\mathbb{Q}$ taking coinvariants with respect to the action of a finite group is exact.

We want to show that

$$F(\mathbb{R}) = H^\bullet(CE_\bullet(\Omega_c^\bullet(\mathbb{R}) \otimes \mathfrak{g})) \tag{24}$$

is the universal enveloping algebra $U(\mathfrak{g})$. Let f_0 be a bump function peaked at 0 such that $\int f_0\,dx = 1$. Then $f_0 \mapsto f_0\,dx$ gives a weak equivalence between the complex $\Omega_c^0(\mathbb{R}) \xrightarrow{d} \Omega_c^1(\mathbb{R})$ and the complex $0 \to \mathbb{R}$, which is a DGA with the zero multiplication. This means that

$$F(\mathbb{R}) = H^\bullet(CE_\bullet(\mathfrak{g}[-1]), d_{CE}). \tag{25}$$

This is concentrated in degree 0, and as a vector space it's just $S(\mathfrak{g})$ because d_{CE} vanishes ($\mathfrak{g}$ has no differential, and $\mathfrak{g}[-1]$ has no Lie bracket). In general, on $M = \mathbb{R}^n$ we get $S(\mathfrak{g}[1-n])$ as a vector space (and for $n \geq 2$ we will get it as a commutative algebra equipped with a Poisson bracket of degree $n-1$). This already implies that we can't get a cosheaf when $n \geq 2$ (say on an annulus) because things will show up in the wrong degrees.

Now we need to compute the associative algebra structure, and in particular we need to check that the commutator bracket agrees with the Lie bracket on elements of $\mathfrak{g}$. Let $f_t(x) = f_0(x-t)$ be the shift of the bump function from above to t. We'll need a compactly supported function h such that

$$dh = f_1\,dx - f_{-1}\,dx \tag{26}$$

and such that $h = -1$ in a neighborhood of 0. Then we can compute that if $\xi, \eta \in \mathfrak{g}$, then

$$d_{CE}((f_0\,dx \otimes \xi)(h \otimes \eta)) = (f_0\,dx \otimes \xi)(dh \otimes \eta) + (f_0\,dx)h \otimes [\xi, \eta]. \tag{27}$$

Plugging in $dh = f_1\,dx - f_{-1}\,dx$ and using that $hf_0 = -f_0$, we get

$$(f_0\,dx \otimes \xi)(f_1\,dx \otimes \eta) - (f_0\,dx \otimes \xi)(f_{-1}\,dx \otimes \eta) - f_0\,dx[\xi, \eta]. \tag{28}$$

Using the isomorphism

$$\mathfrak{g} \ni \xi \xrightarrow{\alpha} f_0\,dx \otimes \xi \in H^0(S^1(\Omega_c^1(\mathbb{R}) \otimes \mathfrak{g}[1])) \tag{29}$$

this becomes (once we settle left / right conventions)

$$\alpha([\xi, \eta]) = \alpha(\xi)\alpha(\eta) - \alpha(\eta)\alpha(\xi). \tag{30}$$

6 Some rational homotopy theory (Teichner, 2/2, 2/9)

Here everything has the correct grading (usually homological, cohomological for DGCAs).

DGLAs over $\mathbb{Q}$ are important in Quillen's approach to rational homotopy theory.

Definition A *relative category* is a category C equipped with a collection W of morphisms ("weak equivalences"), which stisfy some axioms. Its *homotopy category* $\text{Ho}(C) = C[W^{-1}]$ is the localization of C at W (so we formally invert all weak equivalences).

Morphisms in the homotopy category look like zigzags where some of the legs are weak equivalences. In general it's difficult to tell whether two morphisms are equal. It's also possible that this category can fail to be locally small even if C is.

Quillen considered the localization of the category of simply connected spaces at the *rational equivalences*: those maps $f : X \to Y$ inducing an isomorphism

$$\pi_n(X) \otimes \mathbb{Q} \cong \pi_n(Y) \otimes \mathbb{Q} \tag{31}$$

on rational homotopy. (Equivalently, on rational homology or on rational cohomology.) Remarkably, this homotopy category is equivalent to the homotopy category of connected (concentrated in positive homological degree) DGLAs over $\mathbb{Q}$. Part of the proof involves the Chevalley-Eilenberg functor, which sends a DGLA to a DG cocommutative coalgebra over $\mathbb{Q}$ (which we want to be rational chains on a space), and which is also an equivalence of homotopy categories (with the right connectivity assumptions).

First of all, the ∞-categories of pointed connected spaces and grouplike E_1-spaces (∞-groups) are equivalent: the equivalences are given by taking loop spaces (for best results, maybe take Moore loops or Kan loops) and taking classifying spaces respectively. This continues to be true rationally. Now, if X is a pointed connected rational space we can take rational chains $C_\bullet(X, \mathbb{Q})$. The right version of this is a cocommutative dg coalgebra, and with the right connectivity assumptions this is again an equivalence of ∞-categories. (With finiteness hypotheses we can take rational cochains and, using the right version of this, get a commutative dg algebra, and this is again an equivalence.) Similarly, given a rational grouplike E_1-space G we can take $C_\bullet(X, \mathbb{Q})$, which is a dg Hopf algebra. Its primitive elements form a dg Lie algebra, and with the right connectivity assumptions this is again an equivalence of ∞-categories. These two descriptions are related by the Chevalley-Eilenberg functor.

If X is a pointed connected rational space, then $H_\bullet(\Omega X, \mathbb{Q})$ is a graded Hopf algebra. It is the universal enveloping algebra of its graded Lie algebra of primitive elements, which is $\pi_\bullet(\Omega X, \mathbb{Q})$ equipped with the Samelson bracket. This is the tensor with $\mathbb{Q}$ of the Whitehead bracket, which exists on $\pi_\bullet(\Omega X)$.

Some nice calculations are possible from here.

Example $\pi_\bullet(S^{2n-1}) \otimes \mathbb{Q}$ is $\mathbb{Q}$ when $k = 2n - 1$ and 0 elsewhere, while $\pi_\bullet(S^{2n}) \otimes \mathbb{Q}$ is $\mathbb{Q}$ in degrees $2n$ and $4n - 1$ and 0 elsewhere. This follows from knowing that ΩS^{n+1} is rationally $K(\mathbb{Q}, n)$, whose homology is the free graded algebra on a generator of degree n, and hence whose homotopy is the free graded Lie algebra on a generator of degree n.

We can also do this computation using the rational cohomology, as follows. The rational cohomology is $\mathbb{Q}[x]/x^2$ where $\deg x = n$. We'd like to write down a DGLA $\mathfrak{g}$ such that $CE^\bullet(\mathfrak{g})$ (which is the symmetric algebra on $\mathfrak{g}^*[1]$, with an interesting differential) has this cohomology. This is the same thing as writing down a dg algebra whose underlying graded algebra is free (graded commutative) whose cohomology is the above. When n is odd we can take $\mathbb{Q}[x]$ with trivial differential, and when n is even we can take $\mathbb{Q}[x, y]$ where $\deg y = 2n-1$ and $dy = x^2, dx = 0$. The corresponding Lie algebras are the free graded Lie algebras on generators of degree $n-1$. (Strictly speaking we need some kind of formality result - that various DGAs are equivalent to their cohomology - to complete this argument.)

7 Feynman diagrams and homological integration (Mazel-Gee and ???, 2/11)

Last time we ran into combinatorial difficulties trying to write down homological integrals when the higher-than-quadratic terms in our action functional were nonzero. Today we'll introduce Feynman diagrams as a way to manage this.

Definition A *Feynman diagram* Γ consists of a set E of half-edges and a set V of vertices, together with a fixed-point-free involution $\sigma : E \to E$ on half-edges and a map $\pi : E \to V$ describing which half-edges are connected to which vertices, together with a decomposition of V into a basepoint $\bullet$, some internal vertices (at least trivalent), and some external vertices (univalent). The underlying graph should be connected, and $\pi^{-1}(\bullet)$ is equipped with a total order. The *indegree* is the valence of the basepoint, and the *outdegree* is the number of external vertices. A Feynman diagram is *closed* if it has no external vertices. The *first Betti number* $\beta_1(\Gamma)$ is the number of loops in the underlying graph.

Definition Let $f = \sum f_n x^n \in R[[x]]$ and let Γ be a Feynman diagram. The *evaluation* $\mathrm{ev}_f(\Gamma)$ is

$$\mathrm{ev}_f(\Gamma) = x^{\mathrm{indeg}(\Gamma)} \left(\frac{1}{a}\right)^{\mathrm{intedge}(\Gamma)} f_{\mathrm{indeg}(\Gamma)} \prod_{\mathrm{intvert}} b_{\mathrm{valence}(v)} \tag{32}$$

where $b(x) = \sum b_n \frac{x^n}{n!}$.

Theorem 7.1. *In $H_0(V_\bullet)$ (with respect to the differential Q), we have*

$$\frac{[f]}{[1]} = \sum_{\Gamma \ closed} \frac{ev_f(\Gamma)\hbar^{\beta_1(\Gamma)}}{|Aut(\Gamma)|}. \tag{33}$$

We can prove this by rewriting the recurrence relation we got earlier in terms of evaluations of Feynman diagrams. The general recurrence relation now comes from the computation

$$Q(x^n \xi) = ax^{n-1} - \sum_{m=2}^{\infty} b_m \frac{x^{n+m}}{m!} - n\hbar x^{n-1} \tag{34}$$

Rewriting this in terms of Feynman diagrams gives a way to relate the evaluation of a graph Γ to the evaluation of graphs obtained by adding an internal vertex next to an external vertex or looping down the last external vertex. This gives

$$[x^{n+1}] = \sum_{\Gamma \text{ closed,indeg}=n+1} \mathrm{ev}_{x^{n+1}}(\Gamma)\hbar^{\beta_1(\Gamma)} \frac{\kappa_\Gamma}{\prod_{\text{intvert}}(\mathrm{val}(v)-1)!} \tag{35}$$

where κ_Γ is the number of ways Γ can be constructed by either adding an internal vertex or looping down the last external vertex. Now, consider the set of ways to cyclically order the half-edges around the internal vertices. $\mathrm{Aut}(\Gamma)$ acts freely on this, and the quotient is the set of ways to construct Γ, so it has size κ_Γ. On the other hand, the set of cyclic orders has size the product of factorials above. This gives the desired $\frac{1}{|\mathrm{Aut}(\Gamma)|}$ factor.

8 More rational homotopy (Teichner, 2/11)

Definition A CDGA is *minimal* if it is free as a commutative graded algebra on a graded vector space V (*semifree*) and if its differential, regarded as a map $d : V \to \sum \mathrm{Sym}^k(V)$, has no components landing in degree 0 or 1. A *minimal model* is a minimal CDGA quasi-isomorphic to a given CDGA.

Theorem 8.1. *Any CDGA has a unique (up to isomorphism!) minimal model. If M is the minimal model of $\Omega^\bullet(X, \mathbb{Q})$ (Sullivan forms), then $M = Sym(\pi_\bullet(X, \mathbb{Q})^\bullet)$, and d_2^* is the Whitehead bracket.*

Last time we were really computing minimal models of formal CDGAs.

Definition A CDGA is *formal* if it is quasi-isomorphic to its cohomology. A rational space X is *formal* if $\Omega^\bullet(X, \mathbb{Q})$ is formal.

It follows that the rational homotopy Lie algebra of a formal space can be read off from its cohomology, by writing down a minimal model of it. This explains the computation we were doing for spheres, which are formal because we can write down a quasi-isomorphism using a volume form. $\mathbb{CP}^n$ is also formal because it's a compact Kahler manifold, and this computation is also nice.

The relationship between this Sullivan picture of rational homotopy and the Quillen picture is given by taking the cohomological Chevalley-Eilenberg complex $CE^\bullet(\mathfrak{g})$, which is $\mathrm{Sym}(\mathfrak{g}^*[1])$ with a differential built out of the differential and the Lie bracket on $\mathfrak{g}$. If we imagine modifying the differential to include further terms, this corresponds to giving $\mathfrak{g}$ an L_∞ algebra rather than DGLA structure. This implies that L_∞ structures can be transported along quasi-isomorphisms: in particular, since any dg vector space over $\mathbb{Q}$ is formal, any DGLA can be turned into an L_∞ algebra on its homology. In this language, being a minimal model means being the Chevalley-Eilenberg complex of an L_∞ algebra with trivial differential, and so the above procedure corresponds to taking minimal models. (All of this reflects the fact that the Lie and commutative operads are Koszul dual, and can be used to define E_∞ algebras using certain semifree DGLAs.)

Quillen's actual approach involves a bunch of functors. First we take singular simplicial sets. Next we take the Kan loop group, giving a simplicial group on the nose. Next we take group rings, giving a simplicial Hopf algebra. Next we complete along the augmentation ideal, giving a simplicial complete Hopf algebra. Next we take primitive elements, giving a simplicial Lie algebra. Finally we use the Dold-Kan correspondence, giving a DGLA.

The philosophical lesson is that Chevalley-Eilenberg (co)chains of $\mathfrak{g}$ are like (co)chains on $B\mathfrak{g}$.

9 One-parameter semigroups (Teichner, 2/11)

Last time we constructed a factorization algebra $F_{A,\alpha}$ on $\mathbb{R}$ from a pair of an algebra A and a one-parameter semigroup $\alpha_t : \mathbb{R}_+ \to A$ of elements of A. It turns out that if the α_t are invertible and we restrict our attention to bounded open subsets of $\mathbb{R}$, then $F_{A,\alpha} \cong F_A$. But if we don't, then we need a value α_∞ for the unbounded open subsets, and then if $F_{A,\alpha} \cong F_A$, then α_∞ is invertible, which implies that $\alpha_t = 1$ for all t.

In the first case, the isomorphism φ, on the interval (s,t), sends $a \in A$ to $\alpha_{-s} a \alpha_t$.

10 The main construction (Teichner, 2/16)

Let L be a sheaf of DGLAs on a smooth manifold M. This means that L is first of all (the sheaf of smooth sections of) a levelwise finite-dimensional graded vector bundle on M. Next, there is a morphism $d : L \to L$ of graded vector bundles of degree 1 and a bilinear morphism $[\cdot,\cdot] : L \otimes L \to L$ of degree 0, and these satisfy the DGLA axioms.

Example If $\mathfrak{g}$ is a DGLA, then $L(U) = \Omega^\bullet(U) \otimes \mathfrak{g}$ is a sheaf of DGLAs. This was the example that led to $U(\mathfrak{g})$.

Example On a Riemann surface Σ, we can start with $L(U) = \Omega^{0,\bullet}(U) \otimes \mathfrak{g}$ equipped with the differential $\bar{\partial}$. This admits a central extension given by the 2-cocycle

$$\omega(\alpha,\beta) = \int_U \langle \alpha, \partial\beta \rangle_{\mathfrak{g}} \tag{36}$$

where $\langle \cdot,\cdot \rangle_{\mathfrak{g}}$ is an invariant inner product on $\mathfrak{g}$ (the "level").

Theorem 10.1. *Any sheaf L of DGLAs gives a factorization algebra, the factorization envelope of L, taking values in convenient vector spaces, and given by applying a completed Chevalley-Eilenberg chain functor.*

10.1 Functions on function spaces

Let E be a smooth graded vector bundle over a smooth manifold M. Imagine that the space $E(U)$ of smooth sections of U is the space of fields (or classical solutions) of a classical field theory. Then the space $O(U)$ of classical observables should be the space of functions on $E(U)$, for some value of "functions." Among these the simplest functions are linear functions. This could mean the following:

1. Morphisms $E(U) \to \mathbb{R}$ of convenient vector spaces. These are compactly supported distributions $\overline{E}_c(U)$.
2. Morphisms $E_c(U) \to \mathbb{R}$ of convenient vector spaces, where $E_c(U)$ denotes compactly supported sections. These are distributions $\overline{E}(U)$.
3. Smooth sections of $E^! = E^\vee \otimes \text{Dens}(M)$. These are smeared distributions $E^!(U)$.
4. Compactly supported smooth sections of $E^!$. These are compactly supported smeared distributions $E^!_c(U)$.

Convenient vector spaces are a closed symmetric monoidal complete and cocomplete category of certain topological vector spaces; they contain the spaces $E(U)$ of sections equipped with the Frechet topology (and more generally will include any Frechet space), as well as the spaces $E_c(U)$. All of the above are also convenient vector spaces. Another desirable property of CVSes is that if E_1, E_2 are smooth vector bundles over smooth manifolds M_1, M_2, then the space of (possibly compactly supported) sections of the box product $E_1 \boxtimes E_2$ on $M_1 \times M_2$ is the tensor product of the spaces of (possibly compactly supported) sections of E_1 and E_2.

The point of tensoring with densities above is that there is a dual pairing between E and $E^!$ landing in densities, which we can integrate over M with no extra structure. We'll pick the smallest choice, which is compactly supported smeared distributions.

After linear functions we might like to think about polynomial functions. We'll define homogeneous polynomial functions of degree n to be S_n-coinvariants of compactly supported sections of the box product $\left(E^!\right)^{\boxtimes n}$.

Finally, we'll define functions to be formal functions, so formal power series. Hence

$$O(U) = \prod_n \left(E^!\right)^{\boxtimes n} (U^n)_{S_n} \tag{37}$$

which is a commutative algebra object in CVS. (In fact the product and coproduct agree here.)

10.2 The completed Chevalley-Eilenberg functor

We'll modify the above construction to construct the factorization envelope. This will involve writing down a Chevalley-Eilenberg functor $CE_\bullet(L(U))$ taking values in dg CVSes. In practice we'll have $L(U) = E^!(U)$ for some E, so we'll take

$$CE_\bullet(L(U)) = \prod_n \left(L[1]^{\boxtimes n}(U)\right)_{S_n}. \tag{38}$$

(Assuming products and coproducts agree here, this is the symmetric algebra on $L[1](U)$ in CVSes. But thinking in terms of box powers will make it easier to verify the Weiss condition later.)

10.3 Convenient vector spaces

Let V be a complete locally convex Hausdorff topological vector space. This is the same as saying the topology is defined by a set of seminorms which are all zero only for the zero vector, and V is complete with respect to these. For example, if K is a compact smooth manifold, $C^\infty(K)$ has a (Frechet) topology determined by the Sobolev seminorms

$$\|f\|_k = \sum_{i=0}^{k} \max_{x \in K} \|f^{(i)}(x)\|^2. \tag{39}$$

If K is a noncompact smooth manifold, we can exhaust it with compact subspaces, and take the Sobolev seminorms on these.

There are many topologies we might place on the algebraic tensor product of LCTVSes, and completing them gives many symmetric monoidal structures. Unfortunately, none of them are closed.

Definition A subset $B \subseteq V$ of a LCTVS is *bounded* if for all open $0 \in U \subseteq V$, there is $\lambda > 0$ such that $B \subseteq \lambda U$. A linear map is *bounded* if it sends bounded sets to bounded sets. (In particular, continuous linear maps are bounded.)

Bounded sets can be axiomatized into structures called bornologies, and a bornological vector space is a vector space with this structure. Morphisms are bounded linear maps.

11 States and vacua (???, 2/18)

The free scalar field theory on a Riemannian manifold M with mass m has fields

$$E(U) = \left[C^\infty(U) \xrightarrow{\Delta + m^2} C^\infty(U)[-1] \right]. \tag{40}$$

(This is the derived space of solutions to the differential equation $(\Delta + m^2)f = 0$. When $m = 0$ these are just harmonic functions.) This is a chain complex of convenient vector spaces, so we can take its CVS dual

$$E^\vee(U) = \left[D_c(U)[1] \xrightarrow{\Delta + m^2} D_c(U) \right] \tag{41}$$

where D_c stands for compactly supported distributions. We can also consider smeared distributions

$$E_c^!(U) = \left[C_c^\infty(U)[1] \xrightarrow{\Delta + m^2} C_c^\infty(U) \right] \tag{42}$$

which is chain homotopy equivalent (e.g. by the Atiyan-Bott lemma) but nicer to work with because e.g. we can multiply smeared distributions. Now we'll define classical observables to be the symmetric algebra on this:

$$\mathrm{Obs}^{cl}(U) = \mathrm{Sym}(E_c^!(U)) \tag{43}$$
$$= \cdots \to \Lambda^2 C_c^\infty(U)[2] \otimes \mathrm{Sym}(C_c^\infty(U)) \to C_c^\infty(U)[1] \otimes Sym(C_c^\infty(U)) \to \mathrm{Sym}(C_c^\infty(U)) \tag{44}$$

This should look like an infinite-dimensional version of the divergence complex. It is a PFA in a straightforward way.

The divergence complex with differential $\hbar\Delta_{e^{-S/\hbar}\mu}$, where S is a quadratic form on $\mathbb{R}^n$, can be defined as the Chevalley-Eilenberg algebra of a graded Lie algebra built as a Heisenberg algebra using a central extension determined by S. We can use this idea to define quantum observables.

Definition Let F be a PFA on $\mathbb{R}^n$. A *translation equivariant* structure on F is the choice, for every $x \in \mathbb{R}^n$, of an isomorphism α_x between F and F translated by x such that $\alpha_{x+y} = \alpha_x \alpha_y$.

$U \mapsto C^\infty(U)$ is translation equivariant in an obvious way, and hence so is anything built functorially out of it, including the free scalar field theory above.

Definition Let F be a PFA on M. A *state* on M is a smooth linear function $\langle \cdot \rangle : H^\bullet(F(\mathbb{R}^n)) \to R[[\hbar]]$ (where $\mathbb{R}[[\hbar]]$ is concentrated in degree zero, so this factors through H^0) such that $\langle 1 \rangle = 1$ (where $1 \in H^0(F(\mathbb{R}^n))$ is the element corresponding to the inclusion of $F(\emptyset)$). If $M = \mathbb{R}^n$ and F is a translation equivariant PFA, then a state is *translation invariant* if it commutes with the action of $\mathbb{R}^n$ (including the infinitesimal action). A state is *vacuum* if it is translation invariant and, for any two observables $O_1, O_2 \in F(M)$, we have

$$\langle O_1 \tau_x O_1 \langle \to \langle O_1 \rangle \langle O_2 \rangle \tag{45}$$

as $x \to \infty$, where τ_x denotes translation by x.

For example, consider the state in the free scalar field theory on $\mathbb{R}^n$ which assigns to an element of $\mathrm{Obs}^c(\mathbb{R}^n)[[\hbar]]$ its degree 0 component in $\mathbb{R}[[\hbar]]$. When $m > 0$ this is a vacuum. When $m = 0$ this is not a vacuum for $n \leq 2$, but is a vacuum for $n \geq 3$.

12 More about convenient vector spaces , and classical field theories (Teichner, 2/18)

Definition Let V be a LCTVS. A map $\rho : \mathbb{R} \to V$ is *differentiable* if the derivative

$$\dot{\rho}(t) = \lim_{\epsilon \to 0} \frac{\rho(t+\epsilon) - \rho(t)}{\epsilon} \in V \tag{46}$$

always exists. ρ is *smooth* if all derivatives of all orders exist.

Theorem 12.1. *A linear map $F : V \to W$ between LCTVSes is smooth (in the sense that it takes smooth paths to smooth paths) iff F is bounded in the sense that it takes bounded sets to bounded sets.*

Consider the category of LCTVSes with bounded maps. (There's another definition in the literature involving developing the notion of bornologies.) There's a forgetful functor from the category of LCTVSes with continuous maps, and it has a right adjoint $R : BVS \to LCTVS$ which equips an LCTVS with the finest topology with the same bounded sets. It embeds the category of LCTVSes with bounded maps as a reflective subcategory of the category of LCTVSes with continuous maps.

Definition An LCTVS is *bornological* if it is in the essential image of this right adjoint (equivalently, lies in the above reflective subcategory).

Theorem 12.2. *Frechet spaces are bornological.*

The convenient vector spaces are a full subcategory of LCTVSes and bounded linear maps, although we'll think of them as smooth maps. This inclusion will have a left adjoint which performs a certain completion.

Definition An LCTVS V is *C^∞-complete* if every smooth path $\mathbb{R} \to V$ has an antiderivative.

Definition Let M be a smooth manifold. A map $f : M \to V$ to an LCTVS V is *smooth* if for every smooth map $\rho : \mathbb{R} \to M$, the composition $f \circ \rho : R \to V$ is smooth.

Definition The category of *convenient vector spaces* is the category of C^∞-complete LCTVSes and smooth (equivalently, bounded) linear maps.

Theorem 12.3. *CVS is complete, cocomplete, and closed symmetric monoidal with tensor product the C^∞-completion of the algebraic tensor product. If $\pi_i : E_i \to B_i$ are smooth vector bundles over smooth manifolds, then this tensor product satisfies $E_1(M_1) \otimes E_2(M_2 0 \cong (E_1 \boxtimes E_2)(M_1 \times M_2)$.*

12.1 Free classical field theories

Definition A *free classical field theory* consists of the following data:

1. A smooth manifold M ("spacetime"),
2. A graded vector bundle $E \to M$ (so sections $E(U)$ are "fields"),
3. A differential $d : E \to E$ of grading 1 which is a differential operator,
4. A quadratic map $S(U) : E(U) \to \mathbb{C}$ (the "action functional") which has the form

$$S(U)(\varphi) = \int_U (d\varphi, \varphi) = \langle d\varphi, \varphi \rangle \qquad (47)$$

   for some nondegenerate pairing $(\cdot, \cdot) : E[1] \otimes E \to \mathrm{Dens}(M)$. (Here $E[1]$ means $E[1]^i = E^{i+1}$.)

These should satisfy the following properties:

1. (E, d) is an elliptic complex.
2. d is self-adjoint with respect to the pairing $(\cdot, \cdot)$.
3. The pairing $(\cdot, \cdot)$ is symplectic of degree -1 (antisymmetric, keeping Koszul sign rule in mind).

Example (The *free boson*) Let M be an oriented Riemannian manifold and let

$$E(U) = \left[C^\infty(U) \xrightarrow{\Delta+m^2} C^\infty(U)[-1]\right] \tag{48}$$

(so the second term is concentrated in degree 1). This is the derived space of solutions to the differential equation $(\Delta + m^2)f = 0$. The shriek dual is

$$E_c^!(U) = \left[C_c^\infty(U)[1] \xrightarrow{\Delta+m^2} C_c^\infty(U)\right]. \tag{49}$$

Taking formal functions on fields gives

$$\mathrm{Obs}^{cl}(U) = \mathrm{Sym}(E_c^!(U), d^!). \tag{50}$$

The orientation trivializes the density bundle, so now we can define the pairing $(\cdot, \cdot)$ as the adjoint to the obvious identification between $E[1]$ and $E^!$. This induces a Poisson algebra structure on $\mathrm{Obs}^{cl}(U)$. It is determined by what it does to linear obserables, as follows: if $\beta_1, \beta_2 \in E_c^!(U)$, then

$$\{\beta_1, \beta_2\} = \int_U \alpha^{-1}(\beta_1)\beta_2 \in \mathbb{C} = \mathrm{Sym}^0. \tag{51}$$

To write down quantum observables we'll use α to write down a Heisenberg-type (graded) Lie algebra as a central extension

$$0 \to \mathbb{C}[-1]\hbar \to L_c(U) \to E_c(U) \to 0. \tag{52}$$

This is a DGLA in CVS, and so we can apply the Chevalley-Eilenberg functor to it, giving

$$\mathrm{Obs}^q(U) = (CE_\bullet(L_c(U)), d_{CE}). \tag{53}$$

The Chevalley-Eilenberg differential is a deformation of the differential on the classical observables. Both classical and quantum observables are factorization algebras.

13 Homotopy factorization algebras (Mazel-Gee, 2/23)

13.1 Towards homotopy (co)sheaves

In many situations it's natural to work not with (co)sheaves of sets but with (co)sheaves of more complicated objects such as groupoids. This modifies the sheaf condition. For example, the assignment $U \mapsto \text{Bun}_G(U)$ can be interpreted as assigning to an open set U the set of isomorphism classes of G-bundles. This is not a sheaf of sets. However, if it is interpreted as assigning to an open set U the groupoid of G-bundles, then it becomes a sheaf of groupoids, or a stack. This requires a cocycle condition on triple intersections.

More precisely, if U_i is an open cover of U, then descent data for a G-bundle on U is

1. An object in $\prod_i \text{Bun}_G(U_i)$,
2. A morphism in $\prod_{ij} \text{Bun}_G(U_{ij})$ (where $U_{ij} = U_i \cap U_j$), and
3. A triangle of morphisms in $\prod_{ijk} \text{Bun}_G(U_{ijk})$ witnessing compatibilities.

This is a certain homotopy limit which generalizes the equalizer usually used to state the sheaf condition. The claim that Bun_G is a stack is precisely the claim that the groupoid of descent data for the cover U_i of U is naturally equivalent to $\text{Bun}_G(U)$.

Here is a definition of homotopy (co)limits that is good enough for our purposes. Let (C, W) be a category with weak equivalences W. For any diagram category J, we get an induced (pointwise) notion of weak equivalences on the diagram category $[J, C]$. There is a constant diagram functor $C \to [J, C]$ giving a constant diagram functor $\text{Ho}(C) \to \text{Ho}([J, C])$.

Definition A *homotopy colimit* resp. homotopy limit is a left resp. right adjoint to $\text{Ho}(C) \to \text{Ho}([J, C])$.

Homotopy limits and colimits in this sense actually correspond to homotopy coherent cones and cocones respectively. In practice we will attempt to compute homotopy limits and colimits by resolving diagrams appropriately.

Recall that the simplex category Δ can be described as the category of nonempty finite total orders and order-preserving maps. The object $n \in \Delta$ is the total order with $n+1$ elements. We can present Δ using special morphisms between these objects called coface d^i and codegeneracy s^i maps (which either skip elements or repeat elements), which satisfy some straightforward relations. A simplicial object in a category C is a functor $\Delta^{op} \to C$, and a cosimplicial object is a functor $\Delta \to C$. So simplicial objects are sequences of objects related by face d_i and degeneracy s_i maps.

Let $X_\bullet$ be a simplicial topological space (so a functor $\Delta^{op} \to \text{Top}$). Its *geometric realization* is

$$|X| = \text{coeq}\left(\coprod_{j \to i} X_i \times \Delta^j \rightrightarrows \coprod_n X_n \times \Delta^n \right) \tag{54}$$

where Δ^i here denotes the standard i-simplex, or more abstractly the standard cosimplicial object $\Delta^\bullet : \Delta \to \text{Top}$. (Abstractly it can be thought of as the functor tensor product of $X_\bullet$ and $\Delta^\bullet$ over Δ.) In good cases, this models the homotopy colimit of X over Δ^{op}. (We can think of $\Delta^\bullet$ as a "projective resolution" of the constant cosimplicial object, so we are computing a "derived tensor product" using a resolution.)

13.2 The case of factorization algebras

Let F be a PFA with values in dg-CVS on X, thought of as a lax symmetric monoidal functor on the completion $\text{Open}(X)^{\amalg}$ of $\text{Open}(X)$ with respect to disjoint unions. What is the correct homotopy version of the Weiss cosheaf condition? If $U_i, i \in I$ covers U, we get a Cech simplicial object $C(U)$, the *Cech nerve*, with

$$C(U)_n = \coprod_{(i_0,\dots i_n)\in I^{n+1}} U_{i_0,\dots i_n} \tag{55}$$

with various face and degeneracy maps.

Definition The (Cech) codescent object $C(U, F)$ is the homotopy colimit of the composite $\Delta^{op} \xrightarrow{C(-)} \text{Open}(X)^{\amalg} \xrightarrow{F} \text{Ch}$. F is a *homotopy cosheaf* with respect to this cover if the natural map $C(U, F) \to F(U)$ is a weak equivalence.

14 Wick's lemma (Yuan, 2/25)

15 Homotopy factorization algebras II (Mazel-Gee, 2/25)

When we considered descent for a (co)sheaf valued in groupoids, we needed to go up to triple intersections. This is because groupoids form a 2-category. When we move to chain complexes, they form an ∞-category, so we need to go to all finite intersections. This is what the Cech codescent object from earlier accomplishes.

We can relate the simplicial stuff we were doing earlier to chain complexes as follows. The Dold-Kan correspondence asserts that the category $sA = [\Delta^{op}, A]$ of simplicial objects in an abelian category A is equivalent to the category $\text{Ch}_{\geq 0}(A)$ of connective chain complexes (or coconnective cochain complexes) in A. Actually we care about a more homotopical statement than this. If $X_\bullet$ is a simplicial object, the corresponding complex is the objects X_n with differentials the alternating sum $\sum(-1)^i d_i$ of the face maps. This functor sends taking homotopy "groups" (objects of A in general) to taking homology and sends weak equivalences of simplicial objects (suitably defined) to quasi-isomorphisms, and induces an equivalence on homotopy categories (and can even be upgraded to a Quillen equivalence between model categories). This gives rise to the intuition that simplicial objects can be used as "nonabelian resolutions" in categories that aren't abelian.

We'll be looking at the case where $A = \text{Ch}(CVS)$ is chain complexes of convenient vector spaces, which itself already has an internal notion of weak equivalence. This complicates the story. So sA, simplicial objects in A, get related to chain complexes in $\mathrm{Ch}(CVS)$, so certain double complexes (here we mean the vertical and horizontal complexes commute). At this point we'll switch to cochain complexes. A weak equivalence in $s\,\mathrm{Ch}(CVS)$ is a levelwise weak equivalence. There should be a hocolim functor $s\,\mathrm{Ch}(CVS) \to \mathrm{Ch}(CVS)$ sending weak equivalences to weak equivalences. Explicitly, thinking of objects in $s\,\mathrm{Ch}(CVS)$ as double complexes $C^{\bullet,\bullet}$ of CVSes, hocolim can be computed as

$$\mathrm{Tot}(C)^n = \bigoplus_{i+j=n} C^{i,j} \tag{56}$$

with differential $D : \mathrm{Tot}^n \to \mathrm{Tot}^{n+1}$ given by $d_{vert} + (-1)^j d_{hor}$. This is isomorphic to tensoring over Δ with simplicial chains on the standard cosimplicial set in simplicial sets, which is the Yoneda embedding of Δ into simplicial sets. (Simplicial chains means we take a simplicial set, apply the free abelian group functor levelwise to get a simplicial abelian group, and then apply Dold-Kan to get a chain complex.)

Now a homotopy cosheaf in chain complexes is a functor $F : \mathrm{Open}(X) \to \mathrm{Ch}$ such that for any open cover U_i of an open U, $F(U)$ is naturally quasi-isomorphic to the totalization of the simplicial object obtained by applying F to the Cech simplicial object of the cover.

To show that various PFAs are homotopy factorization algebras the key input will be the following.

Theorem 15.1. *Let V be a smooth vector bundle on M (identified with its presheaf of smooth sections), and let V_c be its presheaf of compactly supported sections.*

1. *V is a sheaf in CVS.*
2. *V is a homotopy sheaf in* $\mathrm{Ch}(CVS)$*.*
3. *V_c is a cosheaf in CVS.*
4. *V_c is a homotopy cosheaf in* $\mathrm{Ch}(CVS)$*.*

Proof. Let $U_i \to U$ be an open cover, and let ρ_i be a smooth partition of unity subordinate to the cover. To show that V_c is a cosheaf we need to show that

$$\bigoplus_{i,j} V_c(U_{ij}) \rightrightarrows \bigoplus_k V_c(U_k) \xrightarrow{e} V_c(U) \tag{57}$$

is a coequalizer diagram. The second map has a section ρ taking $v \in V_c(U)$ to $\sum \rho_i v \in \bigoplus_k V_c(U_k)$, so $e \circ \rho = \mathrm{id}_{V_c(U)}$. Using this section we can write down a map from $V_c(U)$ to the coequalizer, and to show that this map is an isomorphism it suffices to show that every element in the image of $\rho \circ e$ is in the image of the difference of the coequalizer maps d_0, d_1.

Suppose $v \in \bigoplus_i V_c(U_i)$ witih components v_i. We'll check the above for the special case where $v_i = 0$ for $i \neq k$, where k is fixed but arbitrary. Define

$$\widetilde{v} = \rho \circ e(v), \widetilde{v}_i = \rho_i v_k \tag{58}$$

and

$$w_{ij} = \begin{cases} \rho_i v_k & \text{if } j = k \\ 0 & \text{otherwise} \end{cases} \in \bigoplus_{ij} V_c(U_{ij}). \tag{59}$$

Then we can check that $d_0(w) = v$ and $d_1(w) = \widetilde{v}$, which gives the desired result.

This gives that V_c is a cosheaf. To check that it's a homotopy cosheaf, we can consider the canonical map from the Cech codescent object $C(U, V_c)$ to $V_c(U)$ (concentrated in degree 0), and to prove that it's a quasi-isomorphism it suffices to prove that its mapping cone is acyclic. This is

$$\cdots \to C(U, V_c)^1 \to C(U, V_c)^0 \to V_c(U) \tag{60}$$

and we'll prove that it's acyclic by writing down a contracting homotopy, namely a sequence of maps $K^n : C^{n-1} \to C^n$ such that $Kd + dK = \text{id}$. This is the same as the data of a homotopy between the identity endomorphism and the zero endomorphism. And we can write these down using partitions of unity, in a way that generalizes the above argument. (This is a version of the standard argument showing that partitions of unity imply that higher sheaf cohomology vanishes.) □

Corollary 15.2. *(de Rham theorem) The de Rham cohomology $H^\bullet_{dR}(M)$ of a smooth manifold is the Cech cohomology of the constant sheaf with value $\mathbb{R}$ on M.*

Proof. Pick a finite cover U_i of M. This gives a double complex given by applying Dold-Kan to differential forms on the Cech nerve. After taking the mapping cone, the fact that taking smooth sections of vector bundles (such as the exterior powers of the cotangent bundles) is a homotopy sheaf implies that the rows of the augmented double complex are exact. A little homological algebra shows that $\Omega^\bullet(M)$ is quasi-isomorphic to the total complex of this double complex. Now, if U_i is a good cover (so all finite intersections are empty or contractible), then the Poincare lemma implies that the cohomology of the columns are concentrated in degree 0, where they recover locally constant functions. So augmented by the Cech complex for $\mathbb{R}$, we get that Cech cohomology is also quasi-isomorphic to the total complex of this double complex. □

Dually, compactly supported de Rham cohomology gives homology with a degree shift; this is a version of Poincare duality.

16 The Chevalley-Eilenberg and divergence complexes (Teichner, 3/1)

Any local DGLA gives a (homotopy) factorization algebra. Recall that a local DGLA L consists of the data of a smooth graded vector bundle together with a differential differential of degree 1 and a bidifferential Lie bracket in the graded sense. The space of compactly supported sections of such a thing is a DGLA in CVS, and the homotopy factorization algebra takes the form

$$U \mapsto CE_\bullet(L_c(U)). \tag{61}$$

For example, we could take $L = \Omega^\bullet \otimes \mathfrak{g}$ where $\mathfrak{g}$ is an ordinary Lie algebra. When $M = \mathbb{R}$ this recovers $U(\mathfrak{g})$. When M is a complex manifold, we can also take a certain central extension of $\Omega^{0,\bullet} \otimes \mathfrak{g}$ (where $\Omega^{0,\bullet}$ is the Dolbeault complex); this recovers generalizations of Kac-Moody algebras. Finally, starting from a field free theory we can take a certain central extension coming from the symplectic form. When $M = \mathbb{R}$ this recovers the Weyl algebra.

In the free field theory we have both classical and quantum observables, one of which is a Sym and one of which is a CE. As graded vector spaces, we have $\text{Obs}^{cl} \cong \text{Obs}^q \otimes \mathbb{C}[[\hbar]]$, but the Chevalley-Eilenberg differential is different. In particular, it is not a derivation. However, it reduces to the Sym differential $\bmod \hbar$, and it is a BV differential.

What does this have to do with the divergence complex? Suppose Φ is an n-dimensional manifold (later it will be the infinite-dimensional space of fields) with a volume form $\omega \in \Omega^n(\Phi)$. Then we can write down a complex, the divergence complex $PV^\bullet(\Phi)$ by dualizing the de Rham complex using integration against ω. This takes the form

$$\cdots \to \Lambda^2(\text{Vect})(\Phi) \xrightarrow{\text{Div}_\omega} \text{Vect}(\Phi) \xrightarrow{\text{Div}_\omega} C^\infty(\Phi) \tag{62}$$

where Div_ω is the divergence operator. $\Lambda^n(\text{Vect})(\Phi)$ is regarded as sitting in degree $-n$. (We need some restrictions here; for example, Φ should be compact, or we should consider compactly supported sections. We'll consider polynomial sections, which will make sense because for us Φ will be a vector space.) There's an integration map

$$C^\infty(\Phi) \ni f \mapsto \int f\omega \in \mathbb{R} \tag{63}$$

continuing this complex which computes expectation values, and it also computes the map to H^0.

Example Suppose $\Phi = V$ is a finite-dimensional vector space. (So we are considering fields on a point.) Let ω_0 be Lebesgue measure, normalized so that if $\omega = e^{S/\hbar}\omega_0$, then $\int_V \omega = 1$, where S is a quadratic form on V for simplicity. Then, taking polynomial functions and vector fields everywhere,

$$PV^\bullet(V) = \Lambda^\bullet(V) \otimes \text{Sym}^\bullet(V^*). \tag{64}$$

We can compute that

$$\text{Div}_\omega = (-) \vee \hbar^{-1}\, dS + \text{Div}_{\omega_0}. \tag{65}$$

So we can consider the family of complexes with differential $\hbar \,\text{Div}_\omega = (-) \vee dS + \hbar\, \text{Div}_{\omega_0}$, depending on a parameter $\hbar$. When $\hbar \neq 0$ this is quasi-isomorphic to the same complex with differential Div_ω. When $\hbar = 0$ the differential becomes $(-) \vee dS$. The resulting complex describes the derived critical locus of S. In particular, H^0 is functions on the critical locus. It deserves to be called classical observables, and correspondingly the divergence complex deserves to be called quantum observables.

In the infinite-dimensional case it's unclear what Div_{ω_0} ought to mean becaue there is no Lebesgue measure. However, it turns out we can still make sense of Div_ω.

Now think of S as a symmetric bilinear form $S = q : V \otimes V \to \mathbb{C}$. Take E to be the complex

$$E = V \xrightarrow{S} V^*[-1] \tag{66}$$

with differential determined by S. This gives us an isomorphism $E[1] \cong E^!$, which allows us to write down a symplectic form on E of degree -1 and a corresponding central extension $\widehat{E}$ (a Heisenberg Lie algebra). In this setting, the Chevalley-Eilenberg complex of $\widehat{E}$ turns out to be the divergence complex of S (over $\mathbb{C}[\hbar]$, with differential $\hbar \,\mathrm{Div}_\omega$). This is one way to motivate the definition of quantum observables for a free field theory.

17 Non-free field theories (Teichner, 3/2)

Kevin's 3 books are about perturbative QFT. This involves fixing a classical solution and perturbing around it. Mathematically we do formal geometry in a formal neighborhood of the classical solution. The physicists know how to do this already but Kevin is describing it in mathematical language. Perturbatively we just get sections of some vector bundle, with the zero section corresponding to our original classical solution, and eventually we start from a classical solution φ_0 and quantize it to get a factorization algebra $FA(\varphi_0)$. To do things non-perturbatively we should try to do something like turning this into a sheaf of factorization algebras over the space of classical solutions and taking its global sections. This involves thinking of classical solutions as a derived stack and doing everything in the setting of derived geometry.

A classical field theory in the perturbative setting is a commutative FA (valued in commutative algebras), and we want to deform it to an interesting FA (valued in chain complexes). Recall that for a free field theory the action functional is quadratic and gives rise to a (-1)-shifted symplectic form, which we turned into a Heisenberg algebra and took Chevalley-Eilenberg chains on. In a non-free theory the action functional has an interaction term

$$S(e) = \langle Q(e), e \rangle + I(e) \tag{67}$$

which might e.g. be cubic or higher. So what do we do?

We can express classical observables as a divergence complex. This involves writing down the divergence of $\omega = e^{S/\hbar}\omega_0$ as (in coordinates)

$$\mathrm{Div}_\omega \left(\sum f_i \frac{\partial}{\partial x_i} \right) = -\frac{1}{\hbar} \sum f_i \frac{\partial S}{\partial x_i} + \sum \frac{\partial f}{\partial x_i} \tag{68}$$

and we can try to write down an infinite-dimensional version of this. On $\text{Sym}^n(E_c^1(U)) \otimes E_C(U)[1]$ and with a quadratic action functional we can take

$$e_1 \dots e_n \otimes e \mapsto -\frac{1}{\hbar} e_1 \dots e_n Q(e) + \sum_{i=1}^{n} e_1 \dots \hat{e}_i \dots e_n \int_U e(x) e_i(x) \in \text{Sym}(E_c^!(U)) \tag{69}$$

because of compact support. But this doesn't generalize to interaction terms.

In finite dimensions the $\hbar \to 0$ limit of the divergence complex was functions on the derived critical locus, where the differential is given by wedging with dS. This has a Poisson structure $\{\cdot, \cdot\}$ given by the Schouten-Nijenhuis bracket (extending the Lie bracket of vector fields and the Lie derivative of functions), which has degree 1 (where vector fields live in degree -1), which lets us rewrite the differential as $\{S, -\}$. IIn the non-free case we also need to assume the classical master equation $\{S, S\} = 0$.

Now we'd like to deform this complex. This is done in book 3 using effective quantizations from book 1. These may not exist (due to gauge symmetry) or be unique. For each length cutoff $\ell > 0$ there is a propagator (using a heat kernel for Q), and we can require that these are related by effective interactions I_ℓ, a renormalization group equation relating length cutoffs, a quantum master equation, and a strong locality condition as $\ell \to 0$.

18 Geometric factorization algebras (Teichner, 3/8)

Now we'd like to talk about how a geometric factorization algebra (defined on manifolds with geometric structure) gives rise to a twisted geometric field theory (in the Atiyah-Segal sense of a functor on bordisms with geometric structure).

By a geometric structure on manifolds we mean e.g. a smooth structure, an orientation, a spin structure, a Riemannian metric, a conformal metric, a complex structure (in odd dimensions), etc. Whatever geometric structures are, they need to glue along open subsets. If $\mathcal{G}$ is a geometric structure, then $\mathcal{G}\,\text{Man}_d$ is the symmetric monoidal (under disjoint union) category whose objects are $\mathcal{G}$-manifolds (without boundary) of dimension d and morphisms are $\mathcal{G}$-embeddings, and we can define $\mathcal{G}$-factorization algebras as symmetric monoidal functors out of this category satisfying a suitable cosheaf condition.

Similarly, $\mathcal{G}\,\text{Bord}_d$ is the symmetric monoidal (under disjoint union) category whose objects are $\mathcal{G}$-manifolds of dimension $d-1$ and whose morphisms are (isomorphism classes of) $\mathcal{G}$-manifolds of dimension d with boundary, where composition is given by gluing (although we need to say something about how this interacts with geometric structures; we asked for gluing along open subsets but now want gluing along boundaries). A $\mathcal{G}$-field theory is a symmetric monoidal functor out of this category. The motivating example is when $d = 2$ and $\mathcal{G}$ is a conformal structure, in which case we get Segal's definition of conformal field theory. Segal worked out in a nontrivial way how to glue cobordisms in this case.

Most examples of geometric field theories have anomalies in the sense that they do not really assign vector spaces but only assign vector spaces up to scale (in the sense that the corresponding linear maps are only well-defined up to scale). We can change the target to account for this but then we forget things about the anomaly that we might want to remember. In order to prevent this we'll upgrade our categories to bicategories.

18.1 Bicategories

Definition A bicategory consists of the following data:

1. A collection of objects
2. For each pair X, Y of objects, a category $\mathrm{Hom}(X, Y)$ (with objects 1-morphisms and morphisms 2-morphisms; composition here is called vertical composition)
3. For each triple X, Y, Z of objects, a functor $\circ_{X,Y,Z} : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ (horizontal composition)
4. For each triple $f : X \to Y, g : Y \to Z, h : Z \to W$ of composable 1-morphisms, an associator 2-morphism $\alpha : (f \circ g) \circ h \cong f \circ (g \circ h)$

together with some units satisfying some unit and coherence axioms.

What we really need is a good notion of symmetric monoidal bicategory.
Below let V be a closed monoidal category.

Example There is a nice bicategory whose objects are algebras, morphisms are bimodules, 2-morphisms are bimodule homomorphisms, and composition is tensor product of bimodules. Equivalence in this bicategory is Morita equivalence. We can do this internal to any V.

Example Cat is a bicategory whose objects are categories, morphisms are functors, 2-morphisms are natural transformations, and composition is composition of functors. Here horizontal composition is strict: that is, we have an equality $(f \circ g) \circ h = f \circ (g \circ h)$, so we can take the associators to be identities.

Example Let V be a monoidal category. V Cat is a bicategory whose objects are V-enriched categories (V-categories), morphisms V-bimodules, and 2-morphisms are morphisms of V-bimodules. In the special case of V-categories with one object we get algebras and bimodules as above. We'll be interested in the case that V is vector spaces or dg vector spaces.

Definition Let A, B be V-categories. An (A, B)-bimodule is a collection of objects $M(a, b) \in V$ (where $a \in A, b \in B$) and a collection of action maps

$$A(a', a) \otimes M(a, b) \otimes M(b, b') \to M(a', b') \tag{70}$$

satisfying the obvious compatibility conditions. Equivalently, it's a V-functor $A \otimes B^{op} \to V$ (or maybe the other way around).

Bimodules have a composition given by tensor product of bimodules, which is computed using a certain coend. If V is an ordinary category (e.g. vector spaces), the tensor product $M \otimes_B N$ of an (A, B)-bimodule M and a (B, C)-bimodule N is the coequalizer of the two morphisms

$$\coprod_{b,b' \in B} M(a,b) \otimes B(b,b') \otimes N(b',c) \rightrightarrows \coprod_{b} M(a,b) \otimes N(b,c) \tag{71}$$

given by the left and right action of B respectively. If V is a higher category (e.g. dg vector spaces) we need to take the geometric realization of a simplicial object (a homotopy coend, computing a derived tensor product).

19 More about geometric factorization algebras (Teichner, 3/10

19.1 The topological and locally constant case

Consider locally constant geometric factorization algebras on smooth framed manifolds. By the cosheaf condition this is the same as a locally constant factorization algebra on $\mathbb{R}^d$, and then by a theorem (of Lurie?) these are the same as E_d-algebras (internal to some V, say symmetric monoidal). On the other hand, by the cobordism hypothesis (Lurie) fully dualizable objects in symmetric monoidal (∞, d)-categories give rise to fully extended TFTs on smooth framed bordisms. There is a symmetric monoidal (∞, d)-category whose objects are E_d-algebras, morphisms are bimodules, etc., and every object is fully dualizable, so defines a fully extended TFT. These can be computed using factorization homology.

When $d = 1$ this construction takes as input a locally constant factorization algebra on $\mathbb{R}$ (so an algebra A) and returns as output a TFT which takes value A, or equivalently $\text{Mod}(A)$, on a point. This is a categorical shift from what Atiyah-Segal had in mind, where we should assign a vector space to a point. We can get back down a categorical level using the idea of twists. The trivial twist T_0 assigns the underlying ring k (or equivalently $\text{Mod}(k)$; e.g. Vect if k is a field) to a point, and a twisted theory in the sense of a natural transformation $T_0 \to T_0$ assigns to a point a (k, k)-bimodule over k, or equivalently a k-module, and assigns linear maps to 1-morphisms.

For an example of a nontrivial twist, consider what a 2d CFT assigns to conformal tori. It should assign to every conformal torus a number, and so should define a function on the moduli space of conformal tori, or in other words a modular function. But in fact what one actually finds in examples is modular forms, not modular functions. This is because of a nontrivial twist. Here a twist T assigns to each conformal torus a vector space, and this assignment organizes itself into a vector bundle on the moduli space. Then T-twisted theories assign to each conformal torus an element of the corresponding vector space, and this assignment organizes itself into a holomorphic section of the vector bundle. These give (weak) modular forms when we take powers of the determinant line bundle. T assigns to a circle a modular tensor category, e.g. the category of positive energy representations of a loop group LG at a particular level k, which is the twist relevant to the Wess-Zumino-Witten model.

To get a twisted field theory, rather than just a twist, from an E_d-algebra we can think about it as a left module over itself. This gives a T-twisted field theory where T assigns A to a point.

19.2 In general

Let V be a closed monoidal category, such as Vect or dgVect. Today the convention is that if A, B are V-categories, then an (A, B)-bimodule is a V-functor $A \otimes B^{op} \to V$, where V is regarded as a V-category via its closed structure.

If $\mathcal{G}$ is a geometric structure, the bicategory $\mathcal{G}\,\mathrm{Bord}_d$ has objects which consist of pairs of a closed $(d-1)$-manifold Y_c together with a $\mathcal{G}$-structure on $Y_c \times (0, \varepsilon)$ for some $\varepsilon > 0$ (the "collar"). Eventually we hope to really take germs of such manifolds (so take $\varepsilon \to 0$). If Y_0, Y_1 are two objects, a morphism between them is a compact d-manifold Σ with boundary $Y_0 \coprod Y_1$ and a $\mathcal{G}$-structure on the interior of Σ (the "core") together with isomorphisms of the restriction of this $\mathcal{G}$-structure to small neighborhoods of Y_0, Y_1 with the existing $\mathcal{G}$-structures on Y_0, Y_1. The 2-morphisms between these are $\mathcal{G}$-isomorphisms of bordisms. Incoming and outgoing objects are not treated symmetrically: the collars are always pointing the same way. This is to allow for compositions. (Also, really we want things to vary smoothly in some sense as we vary $\mathcal{G}$-structures. One way to do this is to work fibered over smooth manifolds. The corresponding factorization algebras are called smooth.)

20 Geometric factorization algebras and field theories (Teichner, 3/15)

Below V is closed monoidal and cocomplete.

Suppose $F : \mathcal{G}\,\mathrm{Man} \to V$ is a geometric factorization algebra. We're going to use this to construct a twisted geometric field theory. The objects of our $\mathcal{G}$-bordism category are thickened compact $d-1$-manifolds $Y = Y_c \times (0, 1)$ equipped with $\mathcal{G}$-structure. The object we'll assign to these is

$$A_Y(t_1, t_0) = \begin{cases} F(Y_c \times (t_1, t_0) & \text{if } t_0 \geq t_1 \\ 0 & \text{if } t_0 < t_1 \end{cases}. \tag{72}$$

which is a V-algebra.

New conventions for the bordism category: for the bordisms, all of $Y_c \times (0, 1)$ needs to appear.

Now if Σ is a bordism between Y_1 and Y_0 and $t_0 \geq t_1$, let $\Sigma(t_1, t_0)$ denote the portion of Σ between $Y_1 \times \{t_1\}$ and $Y_0 \times \{t_0\}$. Then we assign

$$M_\Sigma(t_1,t_0) = \begin{cases} F(\Sigma(t_1,t_0)) & \text{if } t_0 \geq t_1 \\ 0 & \text{if } t_0 < t_1 \end{cases}. \tag{73}$$

which is an (A_{Y_1}, A_{Y_0})-bimodule using the factorization structure.

The main lemma is that composition of bimodules works out: we have an equivalence or weak equivalence

$$M_{\Sigma_{21}} \otimes_{A_{Y_1}} \otimes M_{\Sigma_{10}} \sim M_{\Sigma_{20}}. \tag{74}$$

of (A_{Y_2}, A_{Y_0})-bimodules. This requires using the sheaf condition.

The problem with this is that we really only want germs of geometries of Y_c near $Y_c \times \{0\}$. We can do this by inverting certain morphisms. Or we can do the following.

Definition An A_Y-module is *germlike* if the natural map

$$M/A_{Y_\varepsilon} \otimes_{A_{Y_\varepsilon}} A_Y \sim M \tag{75}$$

is an equivalence for all $\varepsilon \in (0,1)$.

Lemma 20.1. *There is an adjunction*

$$res : \mathrm{Mod}^{gl}_{A_Y} \leftrightarrow Mod^{gl}_{A_{Y_\varepsilon}} : (-) \otimes A. \tag{76}$$

Definition The *twist* T is

$$T(Y_i) = \mathrm{Mod}^{gl}_{A_{Y_i}} \tag{77}$$

$$T(\Sigma) : \mathrm{Mod}^{gl}_{A_{Y_1}} \ni M \mapsto M \otimes_{A_{Y_1}} M_\Sigma \in \mathrm{Mod}^{gl}_{A_{Y_0}}. \tag{78}$$

This is a contravariant functor to V-categories but this turns out not to matter.

Next we'll start defining the T-twisted theory E.

Definition $M_Y \in \mathrm{Mod}^{gl}_{A_Y}$ is the A_Y-module whose objects are the points in $[0,1)$ and where A_Y acts via the factorization structure on $Y \times [0,1)$ the right. (Checking that this is germlike involves the sheaf condition again, with the Weiss cover $Y_t = Y \setminus (Y_c \times \{t\})$.)

Definition $E(Y) : T_0(Y) = V \to T_1(Y) = \mathrm{Mod}^{gl}_{A_Y}$ is the functor

$$X \mapsto X \otimes M_Y. \tag{79}$$

21 Overview of factorization homology (Mazel-Gee, 3/17)

Recall that a homology theory for spaces can be described as a functor

$$F : \mathrm{Space}^{fin} \to \mathrm{Ch} \tag{80}$$

from finite CW complexes to chain complexes which is additive in the sense that it sends disjoint unions to direct sums and which satisfies excision in the sense that a (homotopy) pushout square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \coprod_X Z \end{array} \tag{81}$$

gets sent to a (homotopy) pushout of chain complexes.

We can replace Space^{fin} with smooth or topological manifolds and all maps, and we get the same theories in that they all factor through spaces. Let $H(\mathrm{Space}^{fin}, \mathrm{Ch})$ denote the ∞-category of homology theories.

Theorem 21.1. *(Eilenberg-Steenrod) The functor*

$$H(\mathrm{Space}^{fin}, \mathrm{Ch}) \ni F \mapsto F(\bullet) \in \mathrm{Ch} \tag{82}$$

is an equivalence (of ∞-categories). The inverse is

$$\mathrm{Ch} \ni V \mapsto C_\bullet(-, V) \cong C_\bullet(-, \mathbb{Z}) \otimes V \in H(\mathrm{Space}^{fin}, \mathrm{Ch}). \tag{83}$$

To get a more refined theory we'll instead look at Man_n, the category of manifolds of a fixed dimension n and embeddings (rather than all maps). This has a symmetric monoidal structure given by disjoint union, which is no longer the coproduct. We can now define a homology theory for manifolds valued in an arbitrary symmetric monoidal ∞-category $(C, \otimes)$ to be a functor

$$F : \mathrm{Man}_n \to C \tag{84}$$

which is symmetric monoidal (sending disjoint unions to tensor products) and satisfies excision in the following sense: if

$$M = M_0 \coprod_{N \times \mathbb{R}} M_1 \tag{85}$$

then

$$F(M_0) \otimes_{F(N \times \mathbb{R})} F(M_1) \cong F(M) \tag{86}$$

where $F(N \times \mathbb{R})$ has an algebra structure coming from the $\mathbb{R}$ factor, and the tensor product above is (derived) tensor product of modules over this algebra.

The point appeared in Eilenberg-Steenrod because spaces are built up from points (by homotopy colimits). Similarly, manifolds are built up from $\mathbb{R}^n$. Here the analogue of Eilenberg-Steenrod is the following. We'll state it for manifolds equipped with the following more general kind of tangential data. Any topological n-manifold has a tangent microbundle, classified by a map $M \to B\,\mathrm{Top}(n)$, where $\mathrm{Top}(n)$ is the group of homeomorphisms $\mathbb{R}^n \to \mathbb{R}^n$. If $B \to B\,\mathrm{Top}(n)$ is any map, then a B-framed manifold is a (homotopy) lift of the classifying map to B. For example, we can take B to be a point, in which case we get topologically framed manifolds. And if $n \neq 4$ and $B = BO(n)$, then we get smooth manifolds.

Let Man_n^B denote the ∞-category of B-framed n-manifolds and embeddings and let $H(\mathrm{Man}_n^B, C)$ denote the ∞-category of homology theories in the above sense for B-framed manifolds valued in a symmetric monoidal ∞-category C.

Theorem 21.2. *(Ayala-Francis) The functor evaluating a homology theory $\mathrm{Man}_n^B \to C$ on disjoint unions of copies of B-framed $\mathbb{R}^n$ induces an equivalence of ∞-categories*

$$H(\mathrm{Man}_n^B, C) \cong \mathrm{Alg}_{\mathrm{Disk}_n^B}(C) \tag{87}$$

where $\mathrm{Alg}_{\mathrm{Disk}_n^B}(C)$ (B-framed n-disk algebras) denotes functors from disjoint unions of copies of B-framed $\mathbb{R}^n$ to C satisfying the above axioms.

For example, for $B = \bullet$, we get E_n-algebras. For $B = B\,\mathrm{Top}(n)$ we get n-disk algebras, which are E_n-algebras with homotopy fixed point data for the action of $\mathrm{Top}(n)$.

Definition Let A be a B-framed n-disk algebra. Then it determines a homology theory for B-framed n-manifolds M. Evaluating this homology theory on M gives the factorization homology $\int_M A$ of M with coefficients in A.

It's also possible to give a more explicit description as a homotopy colimit.

21.1 Relation to QFT

Suppose we have a notion of fields F given by a functor from the opposite of the category of Riemannian n-manifolds to vector spaces of some sort (e.g. convenient vector spaces). In addition, for each manifold M we're given an action functional $S : F(M) \to \mathbb{R}$. We'd like to take solutions to the Euler-Lagrange equations, which means taking the intersection of dS with the zero section in $T^*F(M)$. Sometimes, we can quantize by replacing this intersection with a derived intersection, then taking quantum observables Obs^q to be functions on this.

In good cases, Obs^q will satisfy the Weiss cosheaf condition. In no interesting cases does Obs^q factor through Man_n. But if both of these conditions hold, then restricting Obs^q to disjoint unions of $\mathbb{R}^n$s, we get an E_n-algebra A such that

$$\mathrm{Obs}^q(M) = \int_M A \tag{88}$$

is factorization homology.

21.2 Nonabelian Poincaré duality

Suppose M is a closed oriented n-manifold and A is an abelian group. Most simply, Poincaré duality asserts that the orientation gives an isomorphism

$$H_k(M, A) \cong H^{n-k}(M, A). \tag{89}$$

for all k. We'd like to relate this to factorization homology in some way.

First, the LHS is $\pi_k C_\bullet(M, A)$. We can think about $C_\bullet(M, A)$ as a space (via Dold-Kan) which can be modeled as the configuration space of points on M labeled by A. Meanwhile, $H^{n-k}(M, A)$ is $\pi_k[M, B^n A]$ where $[-, -]$ denotes mapping spaces and $B^n A = K(A, n)$ is the Eilenberg-MacLane space classifying n^{th} cohomology with coefficients in A.

Poincaré duality asserts that these spaces have the same homotopy groups. It's reasonable to expect that there's in fact a natural map between them which is a weak equivalence. So we want to relate A-labeled configurations in M with maps $M \to B^n A$. In fact this configuration space is $\int_M A$, and (at least if M is framed) we can restate Poincaré duality as the weak equivalence

$$\int_M A \cong [M, B^n A]. \tag{90}$$

More generally, if X is an n-connected pointed space, then $\Omega^n X$ is an E_n-algebra in spaces, so we can consider factorization homology with coefficients in it.

Theorem 21.3. *(Nonabelian Poincaré duality) (Salvatore, Segal, Lurie) We have a weak equivalence*

$$\int_M \Omega^n X \cong [M, X]. \tag{91}$$

This will in fact follow from the classification theorem for homology theories: we just need to check that $[M, X]$ comes from a homology theory on manifolds, which will in general be given by compactly supported maps, which has value $\Omega^n X$ on $\mathbb{R}^n$.

21.3 Poincaré-Koszul duality

Nonabelian Poincaré duality concerns factorization homology with values in spaces. What can we say in general?

Take $n = 1$. Suppose A is an E_1 algebra. Its factorization algebra over a circle

$$\int_{S^1} A = HH(A) \tag{92}$$

is the Hochschild homology of A. We might guess that Poincaré duality in this setting will say something to the effect that the dual of $HH(A)$ is $HH(B)$ for some other B. This is usually false. However, working in chain complexes (so A is a dg algebra), if A is sufficiently connected, then

$$HH(A)^\vee \cong HH(DA) \tag{93}$$

where DA is the Koszul dual of A, which is another E_1 algebra.

More generally, suppose A is an E_n algebra in chain complexes. Then it has an E_n-Koszul dual D^nA, which is another E_n algebra. There is a Poincaré duality map

$$\int_M D^nA \to \left(\int_M A\right)^\vee \tag{94}$$

but it is generally not an equivalence unless A is very connected.

To fix this, we can define a notion of E_n formal moduli problems given by functors of some sort from artinian E_n-algebras to spaces. For example, if A is an augmented E_n-algebra (so equipped with a map $A \to 1$, 1 the unit) then it defines a moduli problem MC_A (Maurer-Cartan) describing the formal neighborhood of the point $\operatorname{Spec} 1 \to \operatorname{Spec} A$. The key fact is that global sections of MC_A are given by the Koszul dual D^nA. However, in general MC_A is not affine: that is, it is not $\operatorname{Spec} D^nA$.

We can define factorization homology with coefficients in MC_A, which turns out to be the correct replacement for $\int_M D^nA$. More generally, if X is a formal moduli problem, then

$$\int_M X = \Gamma\left(X, \int_M \mathcal{O}_X\right). \tag{95}$$

That is, X has some structure sheaf of E_n algebras, and we take the factorization homology of this and then global sections. (Physically these correspond to observables in a non-perturbative QFT.)

Theorem 21.4. *(Poincaré-Koszul duality) (Ayala, Francis) The duality map* $\int_M D^nA \to \left(\int_M A\right)^\vee$ *factors through an equivalence*

$$\int_M MC_A \cong \left(\int_M A\right)^\vee. \tag{96}$$

This is supposed to be an avatar of S-duality in QFT, which exchanges perturbative and non-perturbative phenomena.

22 ∞-categories (Mazel-Gee, 3/29)

22.1 Simplicial sets

Recall that the simplex category Δ is the category of finite nonempty total orders, with morphisms order-preserving maps. We'll write $[n]$ to denote the ordered set $\{0 < 1 < \cdots < n\}$. The category $\text{sSet} = [\Delta^{op}, \text{Set}]$ of simplicial sets behaves like a category of spaces. Most basically, there are the representable functors $\Delta^n = \operatorname{Hom}(-, [n])$ which we can think of as n-simplices. When $n = 0$, the object $[0]$ is the terminal object, so Δ^0 is the terminal functor.

If X is a simplicial set, we'll write $X_n = X([n])$, and think of it as the set of "n-simplices in X." So Δ^0 has a single (usually "degenerate") n-simplex for each n. We should think of it as a point.

Δ^1 is supposed to look like an interval. Explicitly, $(\Delta^1)_0 = \text{Hom}([0],[1])$ has 2 elements, corresponding to the endpoints. $(\Delta^1)_1 = \text{Hom}([1],[1])$ has 3 elements, corresponding to the endpoints and the interval between them (two of these 1-simplices are "degenerate"). $(\Delta^1)_2 = \text{Hom}([2],[1])$ has 4 elements, corresponding to the endpoints (which are completely degenerate 2-simplices) and two other partially degenerate 2-simplices. This pattern continues, with various levels of degeneracy.

One reason to keep track of degeneracies is that it makes products nicer. Simplicial sets have a pointwise product (which is the categorical product), meaning that $(X \times Y)_n = X_n \times Y_n$, and this turns out to model the product of spaces. In particular, looking at $\Delta^1 \times \Delta^1$, whose 2-simplices are pairs of 2-simplices in Δ^1 and Δ^1, we're supposed to get something that looks like a square. This is because some of the degenerate 2-simplices in Δ^1 become nondegenerate in $\Delta^1 \times \Delta^1$ once we take products.

Recall that there's a functor called geometric realization

$$\text{sSet} \ni X \mapsto |X| \in \text{Top} \tag{97}$$

from simplicial sets to topological spaces given by

$$|X| = \text{coeq}\left(\coprod_{j \to i} X_i \times \Delta^j \rightrightarrows \coprod_n X_n \times \Delta^n \right) \tag{98}$$

where the two maps are given by either applying face and degeneracy maps to $X_\bullet$ or to $\Delta^\bullet$, where now Δ^n denotes the topological n-simplex in Top; these organize into a functor $\Delta \to \text{Top}$ (a cosimplicial object). Geometric realization can also be described as the left adjoint of the singular simplicial set functor $\text{Top} \to \text{sSet}$, which explicitly is the restricted Yoneda embedding

$$\text{Sing} : \text{Top} \ni X \mapsto ([n] \mapsto \text{Hom}(\Delta^n, X) \in \text{sSet} \tag{99}$$

where again Δ^n denotes the topological n-simplex.

Theorem 22.1. *(Quillen) Define a (Kan-Quillen) weak equivalence of simplicial sets $f : X \to Y$ to be a map whose geometric realization $|f| : |X| \to |Y|$ is a weak equivalence. Then the adjunction between geometric realization and Sing induces an equivalence on homotopy categories (inverting weak equivalences).*

Not all simplicial sets behave like spaces. The ones coming from singular simplicial sets are special in the following way.

The i^{th} *horn* Λ^n_i is the subobject of Δ^n given by "removing the i^{th} top face": there are $n+1$ of them, and they all include into Δ^n.

Definition A simplicial set $X_\bullet$ is a *Kan complex* if every map $\Lambda^n_i \to X$ extends to a map $\Delta^n \to X$.

In particular, the singular simplicial set always takes values in Kan complexes. (With suitable model structures, every topological space is fibrant, and the Kan complexes are the fibrant simplicial sets.)

The horns Λ_i^n themselves are examples of simplicial sets which are not Kan complexes: the identity map does not admit an extension, meaning that Λ_i^n is not a retract of Δ^n. However, $|\Lambda_i^n|$ is a retract of $|\Delta^n|$ in Top, which is why singular simplicial sets are Kan complexes.

Theorem 22.2. *Let X be a simplicial set and Y be a Kan complex. Then the natural map from $Hom_{\text{sSet}}(X,Y)$ to $Hom_{\text{Ho(sSet)}}(X,Y)$ is surjective.*

This is not true in general. For example, if $X = Y = \Delta^1/\partial\Delta^1$, then $|X| = |Y| = S^1$, so homotopy classes of maps between them is $\mathbb{Z}$, but there are only two endomorphisms of $\Delta^1/\partial\Delta^1$. The problem is that we cannot compose loops, which is the sort of thing that being a Kan complex lets us do (up to homotopy); the extensions (fillers of horns) describe compositions up to homotopy.

22.2 Nerves

In addition to describing spaces, simplicial sets can also be used to describe categories. There is a cosimplicial object $\Delta \to \text{Cat}$ given by taking each n-simplex to the corresponding poset, regarded as a category. This gives us a restricted Yoneda embedding called the nerve

$$N : \text{Cat} \ni C \mapsto ([n] \mapsto \text{Hom}([n], C)) \in \text{sSet}\,. \tag{100}$$

Explicitly, $\text{Hom}([n], C)$ is n-tuples of composable morphisms in C. It has a left adjoint $\tau_1 : \text{sSet} \to \text{Cat}$.

In Kan complexes, it doesn't matter what direction edges are going. But it certainly matters in nerves.

Theorem 22.3. *A simplicial set X is isomorphic to the nerve of a category iff for all* inner *horns Δ_i^n (meaning that $0 < i < n$), maps $\Delta_i^n \to X$ have unique extensions to Δ^n (fill uniquely).*

These unique extensions describe compositions.

22.3 The homotopy hypothesis

Taking the fundamental groupoid $\Pi_{\leq 1}$ of a topological space gives an equivalence between groupoids and homotopy 1-types (spaces with no homotopy above π_1 in any of their connected components). Grothendieck wanted to generalize this to homotopy n-types (spaces with no homotopy above π_n in any of their connected components). He suggested that homotopy n-types should be modeled by certain n-categories called n-groupoids. There should be a fundamental n-groupoid functor $\Pi_{\leq n}$ from spaces to n-groupoids implementing this equivalence. This functor should have a left adjoint in a homotopical sense.

It's tempting to define n-categories inductively as categories enriched over $n-1$-categories. This gives strict n-categories. But strict n-groupoids don't model all homotopy n-types starting when $n = 3$.

To model all spaces, we need to talk about ∞-categories and ∞-groupoids. But how should we describe these explicitly? The idea is to describe corresponding simplicial sets.

Definition (Joyal, Boardmann-Vogt) A *quasicategory* or *weak Kan complex* is a simplicial set in which every inner horn has a filling (not necessarily unique).

Restricting to inner horns corresponds to allowing directed compositions (so categories, with morphisms that aren't necessarily invertible), while not asking for uniqueness corresponds to allowing Kan complexes (compositions up to homotopy). These are supposed to model $(\infty, 1)$-categories, which are categories enriched (in a suitable homotopical sense) over ∞-groupoids (spaces up to weak equivalence).

Two other models for $(\infty, 1)$-categories are categories enriched over Top and over sSet. There's a functor from the former to the latter given by taking singular simplicial sets, and a functor from sSet-enriched categories to sSet called the homotopy coherent nerve. This is again a restricted Yoneda embedding along a cosimplicial object

$$\Delta \to \text{Cat}_{\text{sSet}} \tag{101}$$

23 More about ∞-categories (Mazel-Gee, 3/31)

Let C be a quasicategory. We can think of C_0 as the objects and C_1 as the 1-morphisms. Given two objects $x, y \in C_0$, we can define the hom space $\text{Hom}_C(x, y)$ as the pullback of the diagram (in simplicial sets)

$$\begin{array}{ccc} & & [\Delta^1, C] \\ & & \downarrow \\ \{(x, y)\} & \longrightarrow & [\partial\Delta^1, C] \end{array} \tag{102}$$

(where $[-, -]$ is the internal hom in simplicial sets) which turns out to be a Kan complex, hence we can think of it as an ∞-groupoid.

Now we can start doing category theory.

Definition An object $c \in C_0$ is an *initial object* if $\text{Hom}_C(c, d)$ is weakly equivalent to Δ^0 for all $d \in C_0$.

Definition If J is a category, a *diagram* in C of shape J is a map $N(J) \to C$ of simplicial sets. A *colimit* of a diagram $F : N(J) \to C$ is an initial object of the quasicategory of extensions of F to the right cone $N(J) * \Delta^0$ (where $*$ denotes the join).

Definition Given a quasicategory C and a subcategory W, the *quasicategorical localization* $C[[W^{-1}]]$ is a pushout, in $N^{hc}(\mathrm{QCat})$ (the homotopy coherent nerve of quasicategories, regarded as enriched over Kan complexes), of the diagram

$$\begin{array}{ccc} W & \longrightarrow & C \\ \downarrow & & \\ W^{gpd} & & \end{array} \tag{103}$$

where W^{gpd} is the Kan complex completion of W (inverting all morphisms homotopy coherently).

Definition An ∞-*category* is an object in the quasicategory Cat_∞ which is the localization of $N^{hc}(\mathrm{QCat})$ (as a quasicategory, in the above sense) at maps which are fully faithful and essentially surjective (meaning surjective on isomorphism classes and weak equivalences on hom spaces).

23.1 Manifolds

Definition Man_n is the ∞-category underlying the topologically enriched category whose objects are n-manifolds admitting finite good covers (a finite cover by open sets all of whose intersections are contractible or empty) and whose morphisms are embeddings (with the compact-open topology).

Man_n is symmetric monoidal under disjoint union $\sqcup$, which is not the coproduct $\coprod$ (which doesn't exist). Write Euc_n for the full subcategory of Man_n on $\mathbb{R}^n$ (so $\mathbb{R}^n$ and embeddings).

Theorem 23.1. *(Kister-Mazur)* Euc_n *is equivalent (as an* ∞*-category) to the* ∞*-groupoid* $B\,\mathrm{Top}(n)$*, where* $\mathrm{Top}(n)$ *is the topological group of homeomorphisms* $\mathbb{R}^n \to \mathbb{R}^n$*.*

Definition The *tangent microbundle* functor is the composite

$$\tau : \mathrm{Man}_n \xrightarrow{Yo} Fun(\mathrm{Euc}_n^{op}, \mathrm{Space}) \cong \mathrm{Fun}(B\,\mathrm{Top}(n), \mathrm{Space}) \cong \mathrm{Space}_{/B\,\mathrm{Top}(n)} . \tag{104}$$

Here the first map is the restricted Yoneda embedding, where we send a manifold M to the functor $\mathbb{R}^n \mapsto \mathrm{Hom}(\mathbb{R}^n, M)$. The last equivalence is given by the Grothendieck construction, which is an equivalence between functors $X \to \mathrm{Space}$ and spaces over X, for X a space (given by taking the colimit over the functor, regarded as a diagram). We end up with a map from the fundamental ∞-groupoid $\Pi_{\leq\infty}(M)$ to $B\,\mathrm{Top}(n)$, which is the classifying map of the tangent microbundle.

Definition If B is a space equipped with a map to $B\,\mathrm{Top}(n)$, the ∞-category Man_n^B of *B-framed n-manifolds* is the pullback

$$\mathrm{Man}_n^B \times_{\mathrm{Space}_{/B\,\mathrm{Top}(n)}} \mathrm{Space}_{/B} . \tag{105}$$

Its objects are n-manifolds M together with lifts of the tangent microbundle $\Pi_{\leq\infty}(M) \to B\,\mathrm{Top}(n)$ to B (up to homotopy).

The action of $O(n)$ on $\mathbb{R}^n$ gives a map $BO(n) \to B\,\mathrm{Top}(n)$, and typical examples of B are given by maps into $BO(n)$, e.g. $BSO(n), B\,\mathrm{Spin}(n)$, which correspond to orientations and spin structures respectively.

Theorem 23.2. *(Kirby-Siebemann) $BO(n)$-framings give smooth structures when $n \neq 4$.*

In fact, the ∞-category Man_n^{sm} of smooth n-manifolds is equivalent to the ∞-category of $BO(n)$-framings when $n \neq 4$.

If M, N are B-framed n-manifolds, then the space of morphisms between them (B-framed embeddings) fits into a (homotopy) pullback square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Man}_n^B}(M,N) & \longrightarrow & \mathrm{Hom}_{\mathrm{Space}_{/B}}(M,N) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{Man}_n}(M,N) & \longrightarrow & \mathrm{Hom}_{\mathrm{Space}_{/B\,\mathrm{Top}(n)}}(M,N) \end{array} \tag{106}$$

so can be understood in terms of homotopy-theoretic data and ordinary embeddings.

Write Disk_n^B for the full subcategory of Man_n^B on disjoint unions of B-framed $\mathbb{R}^n$s, and $\left(\mathrm{Disk}_n^B\right)_{/M}$ for the ∞-category of B-framed $\mathbb{R}^n$s together with B-framed maps into M.

Definition Let C be a symmetric monoidal ∞-category. A Disk_n^B-*algebra* in C is a symmetric monoidal functor $\mathrm{Disk}_n^B \to C$.

For example, when $n = 1$ and $B = BO(1)$, a Disk_1^B-algebra is an E_1 algebra (associative algebra, in a homotopy coherent sense) in C.

Definition Let A be a Disk_n^B-algebra in C and let $M \in \mathrm{Man}_n^B$. The *factorization homology* $\int_M A$ of M with coefficients in A is the colimit of the diagram

$$\left(\mathrm{Disk}_n^B\right)_{/M} \to \mathrm{Disk}_n^B \xrightarrow{A} C \tag{107}$$

in C.

Loosely speaking, this looks like a configuration space of points (really, disks) in M labeled by A, which are "multiplied" using the structure maps of A when they collide.

24 More about factorization homology (Mazel-Gee, 4/5)

Previously we defined factorization homology for B-framed manifolds with coefficients in a Disk_n^B algebra. For example, a $\mathrm{Disk}_1^{\mathrm{pt}}$ algebra is an associative algebra, and a $\mathrm{Disk}_n^{\mathrm{pt}}$ algebra is an E_n algebra. We'll call a $\mathrm{Disk}_n^{BSO(n)}$ algebra a ribbon E_n algebra. When $n = 1$ this reproduces a version of *-algebras. (The orientation-reversing map from $\mathbb{R}^1$ to $\mathbb{R}^1$ gives an isomorphism from the algebra to its opposite.)

For example, take C = Space (equipped with product). Examples of E_n algebras are n-fold based loop spaces $\Omega^n X$ of pointed spaces X. As a Disk_n^{pt} algebra, the corresponding functor assigns to a disjoint union of framed disks M the space $\text{Maps}_c(M, X)$ of compactly supported maps $M \to X$, where compactly supported means that they take value the basepoint outside of a compact subspace. In particular, $\text{Maps}_c(\mathbb{R}^n, X) \cong \Omega^n X$. Compactly supported maps are covariantly functorial (in M) with respect to embeddings: we take the extension by zero, where if $M_1 \to M_2$ is an embedding, then everything in M_2 not in M_1 gets sent to the basepoint.

In fact we can replace M with an arbitrary framed manifold, and this is the factorization homology $\int_M \Omega^n X$ ("nonabelian Poincaré duality").

For spaces we can think about the colimit defining factorization homology by first talking about its points. A point in the colimit describing $\int_M A$, where A is an E_n algebra in spaces, is a list of k disks in M which are labeled by elements of A. These are glued together by structure maps where these disks collide and the E_n algebra structure of A is used to relabel the resulting disks. It turns out ("nonabelian Poincaré duality") that

$$\int_M \Omega^n X \cong \text{Maps}_c(M, X). \tag{108}$$

We'll prove this by checking that both the LHS and the RHS satisfy some properties characterizing factorization homology.

As a variation, instead of manifolds we might talk about manifolds with boundary and disks with boundary ($\mathbb{R}^n$ or half spaces). When $n = 1$, and with framings, a $\text{Disk}_1^{\partial,or}$-algebra (oriented / framed 1-disks with boundary) is a tuple consisting of an algebra A, a right module M, and a left module N. We'd like to compute the factorization homology of this thing on a closed interval $[-1, 1]$ (not a disk with boundary).

Definition A functor $F : I \to J$ is *final* if for all diagrams $G : J \to D$ admitting a colimit, the composite $I \xrightarrow{F} J \xrightarrow{G} D$ admits a colimit, and the natural map between them is an isomorphism.

For example, the inclusion of a terminal (final) object is final. A complete characterization (Quillen's Theorem A) of final functors is known. There is a functor from Δ^{op} to $\left(\text{Disk}_1^{\partial,or}\right)_{/[-1,1]}$ (oriented disks with boundary in $[-1, 1]$) sending $[n]$ to $n + 2$ intervals in $[-1, 1]$ including the endpoints, and this functor is final. We can use this to compute the factorization homology of the tuple (A, M, N) above, and what we get is the geometric realization of the two-sided bar construction describing the (derived) tensor product $M \otimes_A N$. Symbolically,

$$\int_{[-1,1]} (M \curvearrowleft A \curvearrowright N) = |\text{Bar}(M, A, N)| = M \otimes_A N. \tag{109}$$

25 Towards excision (Mazel-Gee, 4/7)

Definition Let $M \in \mathrm{Man}_n^B$. A *collar gluing* of M is a continuous map $f : M \to [-1, 1]$ such that the pullback to $(-1, 1)$ is a fiber bundle of manifolds.

Write

$$M_0 = f^{-1}(0), M_+ = f^{-1}([-1, 1)), M_- = f^{-1}((-1, 1]). \tag{110}$$

The B-framing on M induces B-framings on $M_0 \times \mathbb{R}, M_+, M_-$. We have

$$M \cong M_- \coprod_{M_0 \times (-1,1)} M_+ \tag{111}$$

as B-framed manifolds. f^{-1} organizes into a functor

$$f^{-1} : \left(\mathrm{Disk}_1^{\partial,or}\right)_{/[-1,1]} \to \left(\mathrm{Man}_n^B\right)_{/M}. \tag{112}$$

If $F : \mathrm{Man}_n^B \to C$ is a symmetric monoidal functor, it gives us a functor

$$\left(\mathrm{Man}_n^B\right)_{/M} \to C_{/F(M)}. \tag{113}$$

f^{-1} uniquely extends to a symmetric monoidal functor $\mathrm{Disk}_1^{\partial,or} \to \mathrm{Man}_n^B$, and the composite with F is a $\mathrm{Disk}_1^{\partial,or}$-algebra in C, so a triple (M, A, N) of an algebra, a left module, and a right module over it. This means we can contemplate its factorization homology over $[-1, 1]$, which is

$$\int_{[-1,1]} f_*(F) = F(M_-) \otimes_{F(M_0 \times \mathbb{R})} F(M_+). \tag{114}$$

The reason this deserves to be called a tensor product is that $M_0 \times \mathbb{R}$ is an algebra in Man_n^B, and $F(M_-)$ and $F(M_+)$ are left and right modules over it respectively. This comes equipped with a natural map to $F(M)$.

Definition A symmetric monoidal functor $F : \mathrm{Man}_n^B \to C$ is a *homology theory* if it satisfies *tensor excision*: for all collar gluings on all $M \in \mathrm{Man}_n^B$, the natural map

$$F(M_-) \otimes_{F(M_0 \times \mathbb{R})} F(M_+) \to F(M) \tag{115}$$

is an equivalence.

Theorem 25.1. *(Ayala-Francis) If C is* tensor presentable*, the pullback from* $\mathrm{Fun}^{\otimes}(\mathrm{Man}_n^B, C)$ *to* $\mathrm{Fun}^{\otimes}(\mathrm{Disk}_n^B, C)$ *has a left adjoint called factorization homology*

$$\int_{(=)} (-) : \mathrm{Fun}^{\otimes}(\mathrm{Disk}_n^B, C) \to \mathrm{Fun}^{\otimes}(\mathrm{Man}_n^B, C) \tag{116}$$

which induces an equivalence from Disk_n^B*-algebras in C to homology theories.*

This is a version of the Eilenberg-Steenrod classification of homology theories for spaces.

Presentable means a category is generated under colimits by a small set of compact objects (whatever that means). Tensor presentable means presentable, and $\otimes$ distributes over colimits in both variables. (Given presentability, this is equivalent to admitting an internal hom.)

Proof. Step 0: using tensor presentability, we get that the left adjoint exists and takes values in symmetric monoidal functors.

Step 1: we want to show that factorization homology is a homology theory. Let A be a Disk_n^B-algebra, $M \in \mathrm{Man}_n^B$, and $f : M \to [-1,1]$ a collar gluing. The idea is that factorization homology $\int_M A$ can be computed by taking the pushforward of A to a point, and we can factor this map through f. This gives an equivalence

$$\int_{M_{-1}} A \otimes_{\int_{M_0\times\mathbb{R}} A} \int_{M_1} A \cong \int_{[-1,1]} f_* A \cong \int_A M. \tag{117}$$

which is the desired excision property.

Step 2: the inclusion $i : \mathrm{Disk}_n^B \to \mathrm{Man}_n^B$ is fully faithful, hence so is the left adjoint to pullback (left Kan extension i_L). It remains to show that if F is a homology theory, it's computed by the factorization homology of F restricted to Disk_n^B. □

26 Excision (Mazel-Gee, 4/12)

26.1 Strictness

There is a strict version of this whole story where we take a 1-categorical version of Man_n (so homs are sets of embeddings), and Disk_n, and with B-framings, etc. It's a difficult statement that the map from the 1-categorical version of $\left(\mathrm{Disk}_n^B\right)_{/M}$ to $\left(\mathrm{Disk}_n^B\right)_{/M}$ is an ∞-categorical localization (at every map that becomes an equivalence). This has the effect of identifying Disk_n^B-algebras with strict (1-categorical) Disk_n^B-algebras which are locally constant in a suitable sense. Localizations are also final, so we can compute factorization homology as a colimit over the 1-categorical version of $\left(\mathrm{Disk}_n^B\right)_{/M}$ (which is a poset) for locally constant strict Disk_n^B-algebras.

26.2 Pushforwards

Suppose M is a B-framed m-manifold, N is a B-framed n-manifold with boundary, and $f : M \to N$ is continuous. Taking preimages gives a functor

$$f^{-1} : \left(\mathrm{Disk}_n^{\partial,B}\right)_{/N} \to \left(\mathrm{Man}_m^B\right)_{/M} \xrightarrow{\text{loc}} \left(\mathrm{Man}_m^B\right)_{/M} \tag{118}$$

where the first two things are strict and the third one is ∞-categorical. If f restricts to a bundle of manifolds on the interior and the boundary of N, then f^{-1} takes isotopies to equivalences. This strict construction allows us to take the pushforward of a factorization algebra on M to a factorization algebra on N, which we used above, in such a way that taking factorization homology is given by pushforward to a point, and pushforward is functorial (this is hard).

This allows us to justify pushing forward to $[-1, 1]$ to show that factorization homology satisfies excision.

26.3 Excision

What's left in the proof of the equivalence between Disk_n^B-algebras and homology theories is to show that if $F : \text{Man}_n^B \to C$ is a homology theory, then the natural map

$$\int_M F(\mathbb{R}^n) \to F(M) \tag{119}$$

is an equivalence. As functors in M, this is an equivalence on $M = \mathbb{R}^n$. Both sides satisfy excision, so by induction, it's an equivalence on handles $S^i \times \mathbb{R}^{n-i}$. If M admits a handle decomposition, then by excision again and induction on handles, it's an equivalence on M. Every manifold admits a handle decomposition except when $n = 4$. If M is a connected 4-manifold, then $M \setminus \{m\}$ admits a handle decomposition, and we can write

$$M \cong (M \setminus \{m\}) \coprod_{S^3 \times \mathbb{R}} \mathbb{R}^4. \tag{120}$$

This gives the result by excision again.

27 Applications of excision (Mazel-Gee, 4/14)

27.1 Commutative coefficients

How do we write down Disk_n^B-algebras, anyway?

Let $\text{Com} = \text{Fin Set}$ be the symmetric monoidal category of finite sets under disjoint union. If C is another symmetric monoidal category (or ∞-category, if we set things up properly), then the category of symmetric monoidal functors $\text{Fun}^{\otimes}(\text{Com}, C)$ is equivalent to the category of commutative algebras in C. Com can be thought of as the terminal operad, so any other operad admits a map to it. Disk_n^B, in particular, can be thought of as an operad, and the corresponding map

$$\text{Disk}_n^B \to \text{Com} \tag{121}$$

is given by taking π_0. This gives a map

$$\text{Alg}_{\text{Com}}(C) \to \text{Alg}_{\text{Disk}_n^B}(C) \tag{122}$$

from commutative algebras to Disk_n^B-algebras. Hence any commutative algebra gives a Disk_n^B-algebra and can be taken as coefficients for factorization homology.

Theorem 27.1. *Let $M \in \mathrm{Man}_n^B$ be a B-framed n-manifold and let A be a commutative algebra in C. If C admits finite colimits, then the factorization homology $\int_M A$ is given by the tensoring / copowering $\Pi_{\leq\infty}(M) \otimes A$ (computed in commutative algebras, then forgotten down to C).*

In particular, $\int_M A$ only sees the homotopy type of M, and is insensitive to any further information like its framing.

For an object c in an ∞-category and a space X, the tensoring / copowering $X \otimes c$, if it exists, is the space fitting into an adjunction

$$\mathrm{Hom}(X \otimes c, d) \cong \mathrm{Hom}(X, \mathrm{Hom}(c, d)) \tag{123}$$

(recall that $\mathrm{Hom}(-,-)$ takes values in spaces). Equivalently, it is the colimit of the constant diagram with shape X and constant value c.

Example When X is discrete, this is the coproduct of X copies of c.

Example When $X = BG$ and c is a chain complex, this is the group homology of G with trivial coefficients in c.

Example When $M = S^1$ and A is a commutative algebra (e.g. a DGCA, so C is DG vector spaces), $\int_{S^1} A \cong S^1 \otimes A$ is the Hochschild homology of A. If we present S^1 as a suitable pushout, we get that Hochschild homology is similarly a pushout

$$S^1 \otimes A \cong A \otimes_{A \otimes A} A \tag{124}$$

which recovers the more familiar description of Hochschild homology.

One way to think about tensoring / copowering is that Space is the free cocomplete ∞-category on a point. This means that the ∞-category of cocontinuous functors Space $\to C$ from Space into a cocomplete ∞-category C is equivalent to C, with the equivalence given by evaluating the functor on a point. Given $c \in C$, the corresponding cocontinuous functor is $X \mapsto X \otimes c$, which is uniquely determined by the condition that it takes value c on a point and is cocontinuous in X. (This is an ∞ version of a corresponding story for Set: it is the free cocomplete category on a point.)

Now let's prove the theorem.

Proof. It suffices to check that $\Pi_{\leq\infty}(M) \otimes A$ is symmetric monoidal and satisfies excision. Symmetric monoidality is just the observation that disjoint union of manifolds gets sent to coproducts of spaces, which gets sent to coproducts of commutative algebras (by compatibility with colimits); but this is just the tensor product in C. (We really need commutativity here.) Excision follows from compatibility with pushouts (collar gluings give homotopy pushouts because the relevant maps are (Hurewicz) cofibrations) together with the observation that pushouts of commutative algebras are also computed by (balanced) tensor products. □

Example Suppose the symmetric monoidal structure on C is the coproduct. Then every object is canonically a commutative algebra, where the multiplication is the fold / codiagonal map $c \coprod c \to c$. Hence $\int_M c \cong M \otimes c$ for any object $c \in C$. In particular, C could be chain complexes under direct sum; this reproduces ordinary homology.

Example There is an adjunction between commutative algebras in C and C where the right adjoint is the forgetful functor; we'll call the left adjoint Sym. It takes coproducts to tensor products, and we can use this to show that

$$\int_M \mathrm{Sym}(c) \cong \mathrm{Sym}\left(M \otimes c\right). \tag{125}$$

27.2 (Topological) Hochschild homology

Definition Let A be an algebra / associative algebra / A_∞ algebra / E_1 algebra / Disk_1^{or}-algebra. Its *(topological) Hochschild homology* is

$$HH(A) = \int_{S^1} A. \tag{126}$$

We can relate this to usual Hochschlid homology if A is an algebra over a field by writing down a final functor from Δ^{op} to $(\mathrm{Disk}_1^{or})_{/S^1}$, using this to compute the colimit defining $\int_{S^1} A$ (in chain complexes), and then taking normalized chains to get the usual Hochschild complex.

If A is a commutative algebra corresponding to an affine scheme $X = \mathrm{Spec}\, A$, we can think of $\mathrm{Spec} \int_{S^1} A$ as a (homotopy) limit in (derived) affine schemes (the cotensor / power $[S^1, \mathrm{Spec}\, A]$), or equivalently in all (derived) schemes, which computes the (derived) free loop space

$$[S^1, X] \cong LX \cong \mathrm{Spec}\, HH(A). \tag{127}$$

Explicitly, the limit is given by the pullback of the diagonal $X \xrightarrow{\Delta} X \times X$ along itself, so computes the (derived) self-intersection of the diagonal.

28 Poincaré-Koszul duality (Mazel-Gee, 4/19)

Let X be a grouplike E_n-algebra in spaces and let M be a framed n-manifold.

Theorem 28.1. *(Nonabelian Poincaré duality) There is a natural equivalence*

$$\int_M X \cong \mathrm{Maps}^c(M, B^n X) \tag{128}$$

where $B^n X$ is the n-fold delooping and Maps^c *is compactly supported maps (maps which are the basepoint of $B^n X$ outside of a compact subspace).*

For example, when $X = A$ is an abelian group we have $B^n A = K(A, n)$, so taking π_0 gives an isomorphism

$$H_k(M, A) \cong H_c^{n-k}(M, A) \tag{129}$$

between homology and compactly supported cohomology with coefficients in A; this is ordinary Poincaré duality (for framed manifolds). Moreover, the LHS describes a computation of homology / cohomology given by configurations of disks in M labeled by elements of A. This is related to the Dold-Thom theorem (in the case $A = \mathbb{Z}$) and some classical material around scanning maps.

Poincaré duality is strange because the LHS is a covariant functor of the manifold M but the RHS is contravariant. We can write down an ∞-category of manifolds which describes both of these kinds of functoriality.

28.1 Zero-pointed manifolds

Definition The ∞-category of *zero-pointed n-manifolds* ZMan_n is the underlying ∞-category of the Top-enriched category whose objects are locally compact Hausdorff based spaces M_* such that $M = M_* \setminus \{*\}$ is a topological n-manifold, and whose spaces of morphisms are based continuous maps $f : M_* \to N_*$ such that the restriction of f to $f^{-1}(N)$ is an embedding of topological n-manifolds.

Zero-pointed manifolds are symmetric monoidal via wedge sum and have a zero object, namely the point. A large class of examples is given by quotienting a manifold M with boundary by its boundary ∂M (so identifying all of the boundary to a point).

An important class of morphisms in ZMan_n is given by Pontryagin-Thom collapse maps. If $\mathbb{R}^n \to M$ is an embedding, we get an induced collapse map

$$M_+ \to (\mathbb{R}^n)^+ \tag{130}$$

which collapses the complement of the embedding to the basepoint of $(\mathbb{R}^n)^+ \cong S^n$.

Zero-pointed manifolds have a contravariant "negation" involution given by

$$(-)^\neg : M_* \mapsto (M_*)^+ \setminus \{*\} \tag{131}$$

where $(-)^+$ denotes one-point compactification. On maps, it performs a version of Pontryagin-Thom collapse. There are two natural functors $\text{Man}_n \to \text{ZMan}_n$ given by either adding a disjoint basepoint $M \mapsto M_+$ or taking the one-point compactification M^+, and negation defines a contravariant equivalence between the full subcategories given by the images of these constructions:

$$(-)\neg : (\text{Man}_{n,+})^{op} \cong \text{Man}_n^+ . \tag{132}$$

For example, $\neg$ intertwines the inclusion $\mathbb{R}^n \sqcup \mathbb{R}^n \to \mathbb{R}^n$ of two disks (after applying $(-)_+$) into a bigger disk and the based map $S^n \to S^n \vee S^n$ corepresenting the addition on π_n.

In $\mathrm{Man}_{n,+}$ and Man_n^+ there are subcategories given by the images of $\mathrm{Disk}_n \in \mathrm{Man}_n$ which are sent to each other under negation. Explicitly, $\mathrm{Disk}_{n,+}$ is given by disjoint unions of disks with a disjoint basepoint, and Disk_n^+ is given by wedges of spheres. $\mathrm{Disk}_{n,+}$ algebras (symmetric monoidal functors out) end up being augmented E_n algebras: there is an extra map collapsing everything to the basepoint, giving a corresponding map from an E_n algebra to the unit of C, regarded as a constant E_n algebra.

An augmented E_n coalgebra in C is similarly a symmetric monoidal functor from $(\mathrm{Disk}_{n,+})^{op}$ to C, which by negation is the same as a Disk_n^+ algebra. Explicitly, when $n = 1$, we get that Disk_1^+ consists of wedges of circles, and various maps between wedges of circles give the augmentation and the comultiplication.

28.2 Factorization cohomology and homology

Definition *Factorization homology* $\int_{M_*} A$ of an augmented E_n algebra is, when it exists, the left adjoint of the forgetful functor

$$\mathrm{Fun}^{\otimes}(\mathrm{ZMan}_n, C) \to \mathrm{Fun}^{\otimes}(\mathrm{Disk}_{n,+}, C) \tag{133}$$

and similarly, *factorization cohomology* $\int^{M_*^{\neg}} A$ of an augmented E_n coalgebra is the left adjoint of the forgetful functor

$$\mathrm{Fun}^{\otimes}(\mathrm{ZMan}_n, C) \to \mathrm{Fun}^{\otimes}(\mathrm{Disk}_n^+, C). \tag{134}$$

Example Every space X has a canonical augmented E_n coalgebra structure where the augmentation is the terminal map $X \to \mathrm{pt}$ and the comultiplication is the diagonal $\Delta : X \to X \times X$. Factorization cohomology (with the $\neg$ above removed) takes the form

$$\int^{M_+} X \cong \mathrm{Maps}_*(M_*, X). \tag{135}$$

The two adjunctions above, when they both exist, give a composite adjunction between augmented E_n algebras and augmented E_n coalgebras. This is the Koszul duality adjunction. If A is an augmented E_n algebra then on objects this adjunction gives the Disk_n^+-algebra

$$(\sqcup_k \mathbb{R}^n)^+ \mapsto \int_{(\sqcup_k \mathbb{R}^n)^+} A. \tag{136}$$

We'll call this the n-fold bar construction $B^n A$. This reproduces the usual n-fold bar construction for spaces. For example, when $n = 1$, we get

$$A \mapsto 1 \otimes_A 1 \cong \int_{\mathbb{R}^+} A \tag{137}$$

where 1 is the unit. The augmented coalgebra structure on this comes from the augmented coalgebra structure on $\mathbb{R}^+$ in ZMan_1. We might want to iterate this construction, which corresponds to doing the same thing for augmented E_n algebras over

$$(\mathbb{R}^n)^+ \cong \left(\mathbb{R}^+\right)^{\vee n} \tag{138}$$

where $(-)^{\vee n}$ denotes iterated smash product.

Dually, the n-fold cobar construction sends an augmented E_n coalgebra D to

$$\Omega^n : D \mapsto \int^{\left(\mathbb{R}^n_+\right)^\neg} D. \tag{139}$$

This adjunction gives rise to a unit map for any augmented E_n algebra A, the *Poincaré duality map*

$$\int_{(-)} A \to \int^{(-)^\neg} B^n A. \tag{140}$$

We'd like to know when this is an equivalence. In spaces, this unit map is $X \mapsto \Omega^n B^n X$ (given by evaluation on $\mathbb{R}^n_+$), which is an equivalence iff X is grouplike.

Theorem 28.2. *If C is either an ∞-topos or a stable ∞-category and $\otimes$ is the categorical product, then the Poincaré duality map is an equivalence iff the unit map $A \to \Omega^n B^n A$ is an equivalence.*

It turns out that factorization cohomology of $B^n A$ is always the analytification, in the sense of Goodwillie calculus, of factorization homology of A. So Poincaré duality fails in this setting iff factorization homology of A fails to be analytic. Later we'll see how to fix this failure of Poincaré duality by replacing $B^n A$ with a Maurer-Cartan functor, which is a non-affine object.

29 Fixing Poincaré-Koszul duality (Mazel-Gee, 4/21)

How do we fix Poincaré-Koszul duality in general?

29.1 Formal geometry

First we need to say something about formal completions. If A is a commutative k-algebra (k a field), the corresponding affine scheme $\operatorname{Spec} A$ has functor of points

$$\mathrm{CAlg}(k) \ni R \mapsto \mathrm{Hom}_k(A, R). \tag{141}$$

If I is an ideal of A, the quotient map $A \to A/I$ induces a map $\operatorname{Spec} A/I \to \operatorname{Spec} A$ of affine schemes; the correspnoding map on functors of points is the natural map $\mathrm{Hom}_k(A/I, R) \to \mathrm{Hom}_k(A, R)$ given by pulling back along the quotient map. For example, if $A = k[x, y]$, then $\operatorname{Spec} A$ is the affine plane: its functor of points sends R to the set R^2, and the ideal $I = (y - x^2)$ is the parabola inside the affine plane: its functor of points, as a subfunctor of the affine plane, sends R to the set of points $(r, r^2), r \in R$.

Definition The *formal neighborhood* of Spec A/I in Spec A is the I-adic completion

$$A_I^\vee \cong \lim_n A/I^n \tag{142}$$

(regarded as an object in the category of commutative pro-k-algebras, or possibly its opposite).

The idea is that this describes geometry infinitesimally close to Spec A/I inside Spec A.

Definition A commutative k-algebra A is *Artin* if it is finite-dimensional and local, with a unique maximal ideal m such that $R/m \cong k$.

A typical example is $A = k[x]/x^2$. This algebra has the property that a map $R \to A$, or equivalently a map Spec $A \to$ Spec R, is the same thing as a pair consisting of a k-point of Spec A and a Zariski tangent vector to it.

What kind of object is the formal neighborhood, exactly? One way to say it is that it has a functor of points defined, not only arbitrary k-algebras, but only on Artin k-algebras.

Definition The *formal spectrum* of $A_I^\vee$ is the functor of points

$$\operatorname{Spf}(A_I^\vee) : \operatorname{Artin}(k) \ni R \mapsto \operatorname{colim}_n \operatorname{Hom}(A/I^n, R) \in \operatorname{Set}\,. \tag{143}$$

29.2 Derived deformation theory

If k has characteristic 0, we'll take CAlg(k) in the derived sense to mean the ∞-category given by localizing CDGAs over k at quasi-isomorphisms. In general we want to consider E_∞ algebras in Mod(k), which in the derived sense means the ∞-category persented by Ch(k).

Definition An object $A \in \operatorname{CAlg}(k)$ is *Artin* if it is connective (meaning $\pi_{<0}(A) = 0$), perfect as an object of Mod(k) (equivalently, because k is a field, $\pi_\bullet(A)$ is finite-dimensional), and $\pi_0(A)$ has a unique maximal ideal m such that $\pi_0(A)/m \cong k$.

Loosely speaking, Spec of an Artin algebra is a thickened version of Spec k.

Definition A *moduli problem* is a functor

$$X : \operatorname{CAlg}(k) \to \operatorname{Space} \tag{144}$$

Definition Given a moduli problem X and a point $x \in X(k)$, the *formal completion* of X at x is the functor

$$X_x^\vee : \operatorname{Artin}(k) \ni R \mapsto \lim \left(\operatorname{pt} \xrightarrow{x} X(k) \leftarrow X(R)\right) \in \operatorname{Space}\,. \tag{145}$$

We can axiomatize the sort of object we get this way.

Definition A *formal moduli problem* is a functor $Y : \mathrm{Artin}(k) \to \mathrm{Space}_*$ such that

$$Y(k) \cong \mathrm{pt} \tag{146}$$

and Y takes fiber products to fiber products.

We'll call this Moduli(k).

A formal moduli problem has a notion of tangent complex in Mod(k) which, in the classical case, produces the dual of the cotangent complex. In the smooth case, the cotangent complex is concentrated in degree 0 and gives Kahler differentials; in general, it's a "derived functor" of this.

Theorem 29.1. *(Lurie) If k has characteristic 0, the tangent complex functor gives rise to an equivalence*

$$\mathrm{Moduli}(k) \ni Y \mapsto \Sigma^{-1} T_Y \in \mathrm{Lie}(k) \tag{147}$$

,

where Lie(k) *denotes the ∞-category of L_∞ algebras over k, presented by DGLAs over k. (The Lie bracket is, loosely speaking, the bracket of vector fields.) The inverse sends a Lie algebra $\mathfrak{g} \in \mathrm{Lie}(k)$ (so we really mean a DGLA or L_∞ algebra) to a formal moduli problem, the* Maurer-Cartan *functor*

$$MC(\mathfrak{g}) : \mathrm{Artin}(k) \ni R \mapsto \{x \in m_R \otimes \mathfrak{g} : dx = [x, x]\} \in \mathrm{Space}\,. \tag{148}$$

29.3 E_n formal geometry

Now we can repeat all of the above definitions but with E_∞ algebras replaced with E_n algebras, giving us E_n formal moduli problems Moduli_n defined in terms of functors out of Artinian E_n algebras Artin_n, as well as formal Specs of augmented E_n algebras. The forgetful functor from E_∞ algebras to E_n algebras induces a map $\mathrm{Moduli}_n \to \mathrm{Moduli}$, but E_n formal moduli problems have more structure since they can be tested on a wider class of algebras.

Theorem 29.2. *For k any field, there is an equivalence*

$$\mathrm{Moduli}_n(k) \ni Y \mapsto \Phi_n(Y) \in \mathrm{Alg}_{E_n}^{aug}(k) \tag{149}$$

with inverse given by a Maurer-Cartan functor $MC(-)$.

Define the Koszul duality functor

$$D^n : \left(\mathrm{Alg}_{E_n}^{aug}\right)^{op} \xrightarrow{B^n} \left(\mathrm{CoAlg}_{E_n}^{aug}\right)^{op} \xrightarrow{(-)^\vee} \mathrm{Alg}_{E_n}^{aug}\,. \tag{150}$$

Then the Maurer-Cartan functor is

$$MC(A) : \mathrm{Artin}(k) \ni R \mapsto \mathrm{Hom}_{\mathrm{Alg}^{aug}} (k, A \otimes D^n R) . \tag{151}$$

In general this is a non-affine object with affinization $D^n A$. Φ_n can be interpreted as the sheafification of Koszul duality. On Artin algebras, it is (some version, up to linear duality, of) Koszul duality D^n. If $R = \lim R_\alpha$ is a pro-Artin algebra, then $\mathrm{Spf}\, R$ makes sense as a formal moduli problem, and

$$\Phi_n(\mathrm{Spf}\, R) \cong \mathrm{colim}_\alpha \Phi_n(\mathrm{Spec}\, R_\alpha). \tag{152}$$

Generally, Φ_n is determined by this condition using descent along smooth hypercovers.

There is a shifted tangent complex functor $X \mapsto \Sigma^{n-1} T_X$ from formal moduli problems to k-modules, and the corresponding Koszul dual functor sends A to the maximal ideal m_A.

30 Poincaré-Koszul duality via formal moduli problems (Mazel-Gee, 4/28)

The Maurer-Cartan functor $MC(A)$ has an affinization, (the formal Spec of) $O(MC(A))$, which we claimed above is (the formal Spec of) the Koszul dual $D^n A$. Let's see this. By definition, if Z is an augmented E_n algebra, then the space of maps from Z to $O(MC(A))$ is

$$\lim_{R \in \mathrm{Artin}_n, f : \mathrm{Hom}(D^n R, A)} \mathrm{Hom}_{\mathrm{Alg}_n^{aug}}(Z, R) \tag{153}$$

We'd like to set $R = D^n A$, but unfortunately this need not be Artin. But we can take a cofinal / initial part of this diagram approximating $D^n A$.

Now we want a notion of factorization homology (on zero-pointed manifolds) with coefficients in an object $X \in \mathrm{Moduli}_n$. This is

$$\int_{M_*} X = \lim_{R \in \mathrm{Artin}_n, f : \mathrm{Spf}\, R \to X} \int_{M_*} R. \tag{154}$$

On an affine thing, this produces usual factorization homology. There's an affinization map

$$\int_{M_*} MC_A \to \int_{M_*} \mathrm{Spf}\, D^n A \tag{155}$$

which is in general not an equivalence. MC_A usually fails to be affine for stacky reasons.

Theorem 30.1. *(Poincaré-Koszul duality) For any $A \in \mathrm{Alg}_{E_n}^{aug}$ and any $M_* \in \mathrm{ZMan}_n$, we have an equivalence*

$$\left(\int_{M_*} A \right)^\vee \cong \int_{M_*^\neg} MC_A. \tag{156}$$

Proof. (Sketch) We can resolve A by finitely presented and $(-n)$-coconnective algebras. If F is such an algebra, MC_F turns out to be affine, and we have Poincaré-Koszul duality between F and D^nF. Koszul duality swaps these finitely presented algebras with Artin algebras. This gives the result. □