275 Quantum Field Theory

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1 Introduction

Quantum field theory models certain physical processes. Whether this is mathematics or physics is unclear. Analogously, classical mechanics is physics, but the study of ODEs is mathematics. Classical mechanics is in some sense a special case of the theory of ODEs but it is more than this. Quantum mechanics is physics, but the study of the Schrodinger equation and C*-algebras, etc. is mathematics. Taking the classical limit turns quantum mechanics into classical mechanics. Mathematically, the Schrodinger equation has a short-wave limit which gives Hamiltonian dynamics. Classical electrodynamics may be described as follows: there is a vector potential A, which is a connection on a line bundle over spacetime. This potential satisfies a PDE (Maxwell's equations), possibly with boundary conditions. General relativity may be described as follows: there is a Lorentzian metric g on a 4-dimensional manifold, and it satisfies a nonlinear PDE (Einstein's equations), possibly with boundary conditions.

Classical and quantum mechanics of N particles is well-understood. In the limit as $N \to \infty$, on the classical side we obtain effective field theory, which unfortunately is not covariant under suitable diffeomorphisms. On the quantum side we obtain quantum field theory. Unlike other theories, it does not have a rigorous mathematical foundation (say using PDEs). There are various competing proposals, none of which are entirely satisfactory. Physicists use a high-energy approximation, which gives a formal power series whose terms involve Feynman diagrams. This power series has zero radius of convergence, and physicists generally only take the first few terms. The mathematical meaning of this is unclear, and making sense of it mathematically is extremely difficult.

The standard model is a quantum field theory. It admits a gauge symmetry, which is in some vague sense an infinite-dimensional Lie group symmetry.

Whatever quantum field theory is, there are certain things which should be models of it. Integrable systems (the classical ones and their quantum counterparts) should be examples. Conformal field theories (which admit enough symmetries to completely describe the theory) should also be examples. These are closely related to the representation theory of affine Lie algebras $\hat{\mathfrak{g}}$. Topological field theories (in which the Hamiltonian is zero and spacetime only evolves topologically) is physically trivial but mathematically interesting.

Constructive field theory is also mathematically interesting. Quantum field theory often proceeds via the Feynman path integral, which is not mathematically rigorous, and constructive field theorists attempt to rigorize this. The Weiner integral in probability is one approach to doing this. But this field is now more or less dead; the remaining open problems are highly technical. Constructive field theorists generally focused on $\mathbb{R}^n$ but it would be interesting to work with more general manifolds (this is open).

A brief outline of this course:

1. Classical field theory (done functorially). We will work with a category of cobordisms whose morphisms are spacetimes with spacelike boundary. The functor will assign fields and classical action data, and this data produces solutions to the Euler-Lagrange equations. We will also attach boundary conditions. This is the Lagrangian formulation; we will also look at the Hamiltonian formulation.

2. Gauge symmetries in classical field theory.

3. Quantization. We will first quantize formally using Feynman diagrams. We will then see what we really want.

4. Renormalization.

2 Classical field theories

QFT may be regarded as a functor from a suitable category of cobordisms to a suitable category of vector spaces. Exactly which cobordisms and vector spaces are suitable is subtle. Classical field theory may be reformulated in this way.

We first consider categories of smooth n-dimensional cobordisms. The objects are $n-1$-dimensional compact oriented smooth manifolds (without boundary), possibly with extra structure such as a fiber bundle; these describe space. It is helpful to think of such manifolds N in terms of their n-dimensional collars $N \times (-\epsilon, \epsilon)$. The morphisms $N_1 \to N_2$ are (diffeomorphism classes of?) n-dimensional compact oriented smooth manifolds M, possibly with extra structure, whose boundary is $\overline{N_1} \sqcup N_2$ with induced orientation (where $\overline{N_1}$ denotes the opposite orientation). If M is equipped with a fiber bundle, its restriction to the boundary determines the corresponding fiber bundles over N_1, N_2. Composition is described by gluing along a common boundary.

We also consider Riemannian cobordisms, obtained by attaching Riemannian metrics to morphisms. The corresponding metrics on objects, thought of in terms of their collars, is flat in the collar direction. Similarly we consider Lorentzian cobordisms. The objects are the same as in the Riemannian case but the morphisms have Lorentzian metrics such that the boundaries are spacelike; this is in accordance with general relativity.

In cases of interest, we attach spaces of fields (sections of fiber bundles) which are infinite-dimensional and it is unclear how to handle these. One way is to approximate by finite-dimensional things, which we can do by considering combinatorial cobordisms, obtained by restricting our attention to finite oriented CW-complexes. We may attach a discrete analogue of a Riemannian metric by assigning a number (roughly its volume) to each cell. This is related to the finite element method in numerical analysis and to lattice methods in physics. There are some beautiful theorems here: for example, combinatorial Laplacians converge in a suitable sense to smooth Laplacians. However, we will work primarily with the smooth category.

A field theory contains three basic pieces of data: the structure of spacetime, the space of fields, and the action functional.

Example Scalar fields. Here spacetime is smooth, oriented, compact, and Riemannian. Fields are smooth maps $M \to \mathbb{R}$ (sections of the trivial line bundle). (More general fiber bundles give nonlinear σ-models.) The action functional is

$$S[\varphi] = \frac{1}{2}\int_M \left(\langle d\varphi, d\varphi\rangle + V(\varphi)\right)\, d\mathrm{Vol} \tag{1}$$

where φ is a function, $d\varphi$ the associated 1-form, $\langle -, - \rangle$ the metric, dVol indicates integration with respect to the volume form, and V a real-valued function of a real variable, usually a polynomial. We may also think of the first term as the integral of $d\varphi \wedge *d\varphi$ where $*$ is Hodge star. The special case

$$V(\varphi) = -\frac{m^2}{2}\varphi^2 \tag{2}$$

is the free or Gaussian theory.

Given a functional we want to study its critical points. We do this by taking, at least in some formal sense, directional derivative $D_\epsilon S[\varphi]$ of S in the direction $\epsilon \in T_\varphi F_M$ (F_M the space of fields). This is

$$\frac{1}{2}\int_M \left(\langle d\epsilon, d\varphi\rangle + \langle d\varphi, d\epsilon\rangle + V'(\varphi)\epsilon\right) \mathrm{dVol}. \tag{3}$$

The first term can be computed using integration by parts (Stokes' theorem), giving

$$\int_M d\epsilon \wedge *d\varphi = \int_{\partial M} \epsilon(\iota^*(*d\varphi)) - \int_M \epsilon(d * d\varphi) \tag{4}$$

where $\iota^* : \Omega^\bullet(M) \to \Omega^\bullet(\partial M)$ is pullback along the inclusion $\partial M \to M$. But $d * d\varphi = (d^* d\varphi)\, d\mathrm{Vol} = \Delta\varphi\, d\mathrm{Vol}$, hence

$$D_\epsilon S = \int_M \epsilon\left(-\Delta\varphi + V'(\varphi)\right)\, d\mathrm{Vol} + \int_{\partial M} \epsilon\iota^*(*d\varphi). \tag{5}$$

We want this to be zero for every ϵ. The critical points are then the set of solutions to the Euler-Lagrange equation

$$EL_M = \{\varphi : -\Delta\varphi + V'(\varphi) = 0\} \subset F_M. \tag{6}$$

This space is generally infinite-dimensional and we can cut it down to a finite-dimensional space using Dirichlet boundary conditions, where we fix $\varphi|_{\partial M} = \eta$ on the boundary. When V is a constant, these are just harmonic functions on M with Dirichlet boundary conditions.

Example Pure (Euclidean) Yang-Mills. Spacetime is smooth, compact, oriented and Riemannian (really it should be Lorentzian), and it is also equipped with a principal G-bundle E_M, where G is a compact simple Lie group. The space of fields F_M is the space of connections on E_M. Assume further that $E_M = M \times G$ is trivial and that G is a matrix group, so it is equipped with a Killing form $\mathrm{tr}(ab)$. Then connections on E_M may be identified with the space of $\mathfrak{g}$-valued 1-forms $\Omega^1(M, \mathfrak{g})$ (we need to choose a connection, e.g. the trivial connection, to make this identification). The action functional is

$$S[A] = \frac{1}{2}\int_M \mathrm{tr}(F(A) \wedge *F(A))\, dx \tag{7}$$

where $F(A) = dA + \frac{1}{2}A \wedge A$ is the curvature of the connection A. In local coordinates this is

$$F(A) = \sum_{i,j}(\partial_i A_j - \partial_j A_i + [A_i, A_j])dx^i \wedge dx^j \tag{8}$$

where $A_i \in \mathfrak{g}$.

3 Yang-Mills

What are the critical points of the action functional of Yang-Mills? We compute:

$$D_\epsilon S[A] = \int_M \mathrm{tr}(D_\epsilon F(A) \wedge *F(A)) \tag{9}$$

where $\epsilon \in T_A(F_M)$. We have $D_\epsilon(F(A)) = d\epsilon + [A \wedge \epsilon] = d_A(\epsilon)$, where d_A is the covariant derivative with respect to A. By Stokes' theorem,

$$D_\epsilon S[A] = \pm \int_M \mathrm{tr}(\epsilon \wedge d_A * F(A)) + \int_{\partial M} \mathrm{tr}(\epsilon \wedge *F(A)). \tag{10}$$

Setting the bulk term to zero gives

$$EL_M = \{A : d_A * F(A) = 0\} \tag{11}$$

(the Yang-Mills equation), and setting the boundary term to zero gives

$$\int_{\partial M} \mathrm{tr}(\epsilon \wedge *F(A)) = 0 \tag{12}$$

(where ϵ is the pullback of ϵ to the boundary). This means that if we impose a boundary condition of the form $A \subset \Lambda \subset F_{\partial M}$ then $\epsilon \in T_A(\Lambda) \subset T_A(F_{\partial M})$.

Yang-Mills has an important gauge symmetry. The *gauge group* is the automorphism group of the bundle E_M. Since $E_M = M \times G$ is trivial, this gauge group may be identified with the group G_M of smooth maps $M \to G$. This is an infinite-dimensional Lie group, and its action induces an action of an infinite-dimensional Lie algebra $\mathfrak{g}_M$ whose elements are smooth maps $M \to \mathfrak{g}$. This action sends a connection A thought of as an element of $\Omega^1(M, \mathfrak{g})$ to

$$D_\lambda(A) = [A, \lambda] + d\lambda = d_A\lambda \tag{13}$$

where d_A is the covariant derivative. (This is not quite the usual action on $\Omega^1(M, \mathfrak{g})$ because our identification relies on a choice of connection.) Letting $\epsilon = d_A\lambda$, we compute

$$\partial_\epsilon S[A] = \int_M \mathrm{tr}(d_A\epsilon \wedge *F(A)) \tag{14}$$

where $d_A\epsilon = d_A^2\lambda = [F(A), \lambda]$. So we conclude that

$$\partial_\epsilon S[A] = \int_M \mathrm{tr}([F(A), \lambda] \wedge *F(A)) = 0 \tag{15}$$

by the cyclicity of the trace. Hence the Yang-Mills functional has a gauge symmetry.

We impose Dirichlet boundary conditions as follows. Fix $a \in F_{\partial M}$ (a connection on $\partial M \times G$) and impose the conditions $i^*(A) = a$ (pullback) and $i^*(\epsilon) = 0$. Recall that in the case of scalar fields for vanishing potential, imposing Dirichlet boundary conditions gives a unique solution. Can we expect an analogous result here?

With Dirichlet boundary conditions there is an action of a certain gauge group. There is a map $\tilde{\pi} : G_M \to G_{\partial M}$ whose kernel $\tilde{G_M}$ consists of gauge transformations which fix the boundary. This group acts on the space of solutions to the Euler-Lagrange equations with Dirichlet boundary conditions. We might hope that the space of orbits of $\tilde{G_M}$ on the solutions is finite-dimensional. This is open.

Let M be a smooth 4-dimensional manifold and A a connection. Its curvature $F(A)$ is a 2-form, and the Hodge star $*F(A)$ is an $n-2$-form, which is also a 2-form; this is specific to the 4-dimensional case. This allows us to talk about self-dual and anti-self-dual connections satisfying $F(A) = \pm * F(A)$; the corresponding solutions of the Yang-Mills equation are called instantons. Duality gives $d_A * F(A) = \pm d_A F(A) = 0$ by the Bianchi identity, so it implies the Yang-Mills equation.

Exercise 3.1. *Instantons are local minima of the Yang-Mills functional.*

This is important for quantizing Yang-Mills.

There is a moduli space of instantons given by the quotient of the self-dual and anti-self-dual connections by gauge equivalence, and this moduli space is finite-dimensional. When $M = \mathbb{R}^4$ a complete description of this moduli space is known.

Answer to the exercise: consider the inner product

$$0 \leq \int_M \langle F - *F, F - *F \rangle \, dx = 2S[A] - 2 \int_M \mathrm{tr}(F \wedge F). \tag{16}$$

The last term is, up to some scalar, the second Chern class $c_2(E)$; in particular it is independent of A. Hence the second Chern class gives a lower bound on the Yang-Mills action. On the other hand, equality is achieved precisely when $F = *F$, hence precisely when F is self-dual.

4 Geometry review

Let M be a smooth manifold and $d : \Omega^i(M) \to \Omega^{i+1}(M)$ be the exterior derivative. If M is Riemannian, there is an inner product on forms given by

$$(\omega, \omega') = \int_M \langle \omega, \omega' \rangle_x \, dx = \int_M \omega \wedge *\omega' \tag{17}$$

where dx is the Riemannian volume form and $*$ is the Hodge star $* : \Omega^i \to \Omega^{n-i}$. The square of the Hodge star acting on Ω^k is not the identity: instead it is multiplication by $(-1)^{k(n-k)}$. The exterior derivative then admits a formal adjoint

$$d^* = \pm * d* : \Omega^i \to \Omega^{i-1}. \tag{18}$$

This allows us to define the Hodge Laplacian $\Delta = dd^* + d^*d$. (Project idea!) If M is compact without boundary, then $\Omega(M)$ canonically decomposes into a direct sum

$$\Omega(M) = \Omega_{\text{ex}}(M) \oplus H(M) \oplus \Omega_{\text{coex}}(M) \tag{19}$$

of the exact forms (those of the form $d\omega$), the harmonic forms (those satisfying $d\omega = d^*\omega = 0$, and the coexact forms (those of the form $d^*\omega$). The middle term is canonically isomorphic to de Rham cohomology.

If M has a boundary, the corresponding decomposition is more interesting. (Project idea!)

Similarly, if A is a connection and d_A the corresponding covariant derivative, we can write down a formal adjoint

$$d_A^* = \pm * d_A * . \tag{20}$$

5 Classical mechanics

Newtonian mechanics is described by the second-order differential equation $m\frac{\partial^2 x}{\partial t^2} = F(x)$ where x is a parameterized path and F is the force (we can allow everything to be vector-valued). We will concentrate on the case that $F(x) = -\nabla V(x)$ for some scalar potential V. Newtonian mechanics admits a Lagrangian formulation using the action functional

$$S[\gamma] = \int_{t_1}^{t_2} \left(\frac{m}{2} \left| \frac{\partial x}{\partial t} \right|^2 - V(x(t)) \right) dt. \tag{21}$$

Computing the variation and finding the Euler-Lagrange equations here reproduce Newtonian mechanics, although we need to impose a boundary condition $x(t_1) = q_1, x(t_2) = q_2$.

We can generalize this to function on the tangent bundle $T(N)$ of a smooth manifold N. When N is Riemannian the analogous choice is to take

$$\mathcal{L}(v, x) = \frac{m}{2} |v|^2 - V(x) \tag{22}$$

for some scalar potential V. The corresponding action functional is

$$S[\gamma] = \int_{t_1}^{t_2} \mathcal{L}(\frac{\partial x}{\partial t}, x) \, dt \tag{23}$$

and computing the variation gives

$$\begin{aligned} \delta S[\gamma] &= \int_{t_1}^{t_2} \left(\partial \dot{x}^i \frac{\partial \mathcal{L}}{\partial v^i}(\dot{x}, x) + \partial x^i \frac{\partial \mathcal{L}}{\partial x^i}(\dot{x}, x) \right) dt && (24) \\ &= \partial x^i(t) \frac{\partial \mathcal{L}}{\partial v^i}(\dot{x}, x)|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(-\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial v^i}(\dot{x}, x) + \frac{\partial \mathcal{L}}{\partial x^i} \right) \delta x^i \, dt && (25) \end{aligned}$$

giving the corresponding Euler-Lagrange equation

$$\sum_j \ddot{x}^j \frac{\partial^2 \mathcal{L}}{\partial v^i \partial v^j}(\dot{x}, x) + \dot{x}^j \frac{\partial^2 \mathcal{L}}{\partial v^i \partial x^j}(\dot{x}, x) - \frac{\partial \mathcal{L}}{\partial x^i} = 0. \tag{26}$$

This equation is nice if the matrix of second partial derivatives of $\mathcal{L}$ in the v_i has nonzero determinant, and nicer if the matrix is in fact positive-definite. The best scenario is if it is independent of $\dot{x}$, which occurs when the Lagrangian is quadratic in the tangent directions. If N is Riemannian and the Lagrangian is chosen as above then in fact $g_{ij} = \frac{\partial^2 \mathcal{L}}{\partial v_i \partial v_j}$.

When the matrix is degenerate there is a different story. (Project idea!) In the most degenerate case all of the second partial derivatives are zero and the Lagrangian has the form

$$\mathcal{L}(v,x) = \sum_i v^i \alpha_i(x) - V(x). \tag{27}$$

The Euler-Lagrange equation then takes the form

$$\omega_{ij}(x)\dot{x}^j = \frac{\partial V}{\partial x^i} \tag{28}$$

where $\omega_{ij} = -\partial_j \alpha_i + \partial_i \alpha_j$. In order for solutions to exist locally ω must give a nondegenerate 2-form on N, giving N the structure of a symplectic manifold, and the Euler-Lagrange equations then have solutions given by Hamiltonian vector fields with respect to the Hamiltonian $-V$.

6 Symplectic manifolds

The Euler-Lagrange equations are second-order, but we can replace them with a first-order theory. An analogous strategy will be useful in field theory.

Definition A *symplectic manifold* (M,ω) is a smooth manifold M together with a closed nondegenerate 2-form ω. In particular, if $\dim M = 2n$, then $\Lambda^n(\omega)$ does not vanish anywhere.

$\Lambda^n(\omega)$ in fact defines a volume form and an orientation. We can think of ω as a section of $\Lambda^2(T^*(M))$, but we can also think of it as an isomorphism $T(M) \to T^*(M)$. The inverse of this isomorphism ω^{-1} is an isomorphism $T^*(M) \to T(M)$, which we can also think of as a section of $\Lambda^2(T(M))$.

Definition The *Hamiltonian vector field* X_H generated by H is given by $\omega^{-1}(dH)$.

X_H defines flow lines which are the trajectories of the corresponding dynamical system.

The symplectic form ω induces an algebraic structure on the smooth functions $C^\infty(M)$. Given two functions f, g, their *Poisson bracket* is

$$\{f,g\} = \omega^{-1}(df \wedge dg). \tag{29}$$

Exercise 6.1. *Show that $\{f,g\}$ is a Lie bracket if and only if ω is closed.*

In deformation quantization we want to deform the multiplication on $C^\infty(M)$ in such a way that the first-order deformation (more precisely its commutator) is given by the Poisson bracket.

If $f \in C^\infty(M)$, let $f_t(x) = f(x(t))$ where $x(t)$ is a flow line such that $x(0) = x$. Then

$$\frac{df_t}{dt}(x) = \{H, f_t\}(x). \tag{30}$$

So the Hamiltonian flow induces a one-parameter family of automorphisms of $C^\infty(M)$ (or at least $C^\infty(U)$ for suitable neighborhoods U) described by linear differential equations.

We return to the case of a degenerate Lagrangian. Let N be a smooth oriented manifold and α a 1-form on N such that $\omega = d\alpha$ is nondegenerate (recall that we needed this assumption to get local solutions to the Euler-Lagrange equations). Then (N, ω) is symplectic. The action functional is

$$S[\gamma] = \int_{t_1}^{t_2} \alpha_i(x(t))\dot{x}^i(t)\, dt - \int_{t_1}^{t_2} V(x(t))\, dt \tag{31}$$

where V is some potential. The variation is

$$\delta S[\gamma] = \int_{t_1}^{t_2} \sum_i \left(\sum_j \omega_{ij}(x(t))\dot{x}^j(t) - \frac{\partial V}{\partial x^i}(x(t)) \right) \delta x^i(t)\, dt + \sum_i \alpha_i(x(t))\partial x^i(t)|_{t_1}^{t_2} \tag{32}$$

by integration by parts. The space of solutions to the Euler-Lagrange equations is

$$EL = \{\gamma : \dot{x}^j = \omega^{jk}\partial_k V\} \tag{33}$$

or equivalently $\dot{x}(t) = \omega^{-1}(dV)(x(t)) = X_{dV}(x(t))$. This is precisely the space of flow lines of the Hamiltonian vector field generated by V on (N, ω).

It remains to determine boundary conditions. Consider $\tilde{\alpha} = (\alpha, -\alpha)$ on $N \times N$. Write

$$\tilde{\alpha}(x_1, x_2) = \alpha(x_2) - \alpha(x_1) \in T^*_{x_1}(N) \oplus T^*_{x_2}(N) \cong T^*_{(x_1,x_2)}(N \times N). \tag{34}$$

We want to set the variation of $S[\gamma]$ on the boundary to $(\tilde{\alpha}, \delta\tilde{x})$ where $\delta\tilde{x} = (\delta x(t_2), \delta x(t_1)) \in T^*_{(x_1,x_2)}(N \times N)$. If $B \subset N \times N$ is some submanifold, then we want $\iota^*(\tilde{\alpha}) = 0$ where $\iota : B \to N \times N$ is the inclusion. Since we want the freedom to set as many boundary conditions as we want, we want a maximal B with this property.

Definition Let (M, ω) be a symplectic manifold. $S \subset M$ is an *isotropic* submanifold if $T^\perp(S) \supseteq T(S)$, where $T^\perp_x(S) = \{\xi \in T_x(M) : \omega(\xi, \eta) = 0 \forall \eta \in T_x(S)\}$. Equivalently, $\omega_x(\xi, \eta) = 0$ for all $\xi, \eta \in T_x(S)$ and for all x. S is *coisotropic* if $T^\perp(S) \subseteq T(S)$. Finally, S is *Lagrangian* if it is both isotropic and coisotropic, so $T^\perp(S) = T(S)$.

We have $\dim S \leq \frac{\dim M}{2}$ if S is isotropic. S is Lagrangian if and only if S is isotropic of dimension $\frac{\dim M}{2}$ (the maximal possible such dimension).

Example In Darboux coordinates (p_i, q^i) we can write $\omega = \sum_i dp_i \wedge q^i$. (The weak Darboux theorem asserts that such coordinates always exist.) Let S be given locally by the p_i. Then $T(S) = \text{span}(\partial_{p_i})$ and $T^\perp(S) = \{\partial_{p_i}\} = T(S)$, so S is Lagrangian. If S is given locally by $\{p_1, ...p_k, q^n\}$ where $k < n$, then $T(S) \subseteq T^\perp(S)$ and S is isotropic. If $k = n - 1$, then S is also Lagrangian.

The B that we want to impose boundary conditions on are certain Lagrangian submanifolds of $N \times N$ with $\omega = (-d\alpha, d\alpha)$; more precisely we want exact Lagrangian submanifolds (submanifolds such that the pullback of α is zero).

Question from the audience: why did we choose a Lagrangian which is first-order in tangent vectors when the natural physical Lagrangian is second-order?

Answer: we are mathematicians, and as mathematicians we can fantasize.

Question: do physicists need to use such a Lagrangian?

Answer: physicists need to know about the Hamilton-Jacobi action in the context of quantization. It also appears in probability in the context of large deviations. Suppose that we flip a coin 1000 times and let M_N denote the mean number of coins after N trials. The probability that this is $x < \frac{1}{2}$ is approximately $\exp(-NS(x))$ where $S(x) = -x \ln x - (1 - x) \ln(1 - x)$ is the entropy function. This is the analogue of the Hamilton-Jacobi action.

Question: do physicists need the Hamiltonian framework vs. the Lagrangian framework?

Answer: Lagrangians are important for understanding quantization, but Hamiltonians are important for understanding integrability.

Back to boundary conditions. Fix an exact Lagrangian submanifold B. Then we want to consider solutions of the Euler-Lagrange equations $x(t)$ such that $(x(t_2), x(t_1)) \in B$.

Example Newtonian mechanics on a Riemannian manifold $\mathcal{N}$. Take $\mathcal{L}(\xi, x) = \frac{1}{2}(\xi, \xi)_x - V(x) \in C^\infty(T(\mathcal{N}))$ (second-order Lagrangian); then the Euler-Lagrange equations give Newtonian mechanics. We want to reformulate this with a first-order Lagrangian on $T^*(\mathcal{N})$.

To do this we need to move from a vector space to its dual. So let W be a vector space and W^* its dual. Let $f \in C^\infty(W)$ be a smooth function. Its Legendre transform $f^* \in C^\infty(W^*)$ is

$$f^*(p) = \max_{v \in W}(p(v) - f(v)) \tag{35}$$

if this maximum exists. A sufficient condition for the existence of the Legendre transform is that f is convex; when f is smooth, this is equivalent to the matrix of second partial derivatives being positive-definite. In this case, $f^*(p) = p(v^*) - f(v^*)$ where v^* is the unique extremizing point of $p(v) - f(v)$, hence $p_i = \frac{\partial f}{\partial v_i}(v^*)$. We also obtain a diffeomorphism $W^* \ni p \mapsto v^* \in W$. For example, if $f(v) = \frac{1}{2}\langle v, v\rangle$, then $f^*(p) = \frac{1}{2}\langle p, p\rangle$ (induced inner product on the dual).

We return to $\mathcal{N}$. The Hamiltonian $H \in C^\infty(T^*(\mathcal{N}))$ is the Legendre transform of $\mathcal{L}(\xi, x)$. Explicitly, if $q \in \mathcal{N}, p \in T^*_q(\mathcal{N})$, then

$$H(p, q) = \frac{1}{2}\langle p, p\rangle_q + V(q). \tag{36}$$

This allows us to recast Newtonian mechanics in terms of symplectic mechanics on $T^*(\mathcal{N})$; in particular, the latter has a symplectic structure. In local coordinates, if q^i are local coordinates on $\mathcal{N}$ and p_i the dual coordinates on $T^*_q(\mathcal{N})$, then the symplectic form is given by

$$\omega = \sum_{i=1}^{\dim \mathcal{N}} dp_i \wedge dq^i = d\alpha \tag{37}$$

where $\alpha = \sum_i p_i dq^i$ is the Poincaré 1-form.

Theorem 6.2. *The Legendre transformation $T(\mathcal{N}) \to T^*(\mathcal{N})$ is a diffeomorphism sending flow lines $(\dot{x}(t), x(t))$ (solutions of the Euler-Lagrange equations) to Hamiltonian flow lines with respect to the Hamiltonian above.*

Exercise 6.3. *Give a coordinate-free definition of* ω.

Why should we care about arbitrary Riemannian manifolds rather than just $\mathbb{R}^3$? The answer is configuration spaces. For example, the configuration space of N particles is $\mathbb{R}^{3N}$, and the Lagrangian is

$$\mathcal{L}(\xi, q) = \sum_{i=1}^{N} \frac{m}{2} \langle \xi_i, \xi_i \rangle_{\mathbb{R}^3} - \sum_{i<j} V(q_i - q_j) - \sum_i U_i(q_i) \tag{38}$$

where the V describe interactions between the particles and the U describe forces on the particles. If the particles are in addition constrained in some way, then the configuration space is a Riemannian submanifold of $\mathbb{R}^{3N}$. (Project idea: discuss the Lagrangians and Hamiltonians of constrained systems.)

7 Classical mechanics as a field theory

We can think of classical mechanics as a 1-dimensional field theory where spacetime $[t_1, t_2]$ is a flat 1-dimensional Riemannian manifold. The target space N is a smooth manifold with a 1-form $\alpha \in \Omega^1(N)$ such that $\omega = d\alpha$ is a symplectic form on N. The space of fields is the space of smooth maps $[t_1, t_2] \to N$, and the action functional is

$$S_{[t_1,t_2]}(\gamma) = \int_\gamma \alpha - \int_{t_1}^{t_2} V(x(t))\, dt \tag{39}$$

where V is some potential. The set of solutions to the Euler-Lagrange equations is

$$EL = \{x(t) : \dot{x}(t) = \omega^{-1}(dV(x(t)))\} \tag{40}$$

which is precisely the space of flow lines of the associated Hamiltonian vector field.

Boundary conditions are described by exact Lagrangian submanifolds of $N \times N$ ($\tilde{\alpha} = (-\alpha, \alpha)$ restricts to 0). A special case includes those submanifolds of the form $B_1 \times B_2$ where B_i is an exact Lagrangian submanifold of N. These are related to gluing of time intervals: suppose we consider the motion of the system from t_1 to t_2 and then from t_2 to t_3. The corresponding action functionals are related by

$$S_{[t_1,t_2]}(\gamma) + S_{[t_2,t_3]}(\gamma) = S_{[t_1,t_3]}(\gamma). \tag{41}$$

Suppose we furthermore want to glue solutions to the Euler-Lagrange equations. We can do this if we write down three exact Lagrangian submanifolds B_i and require that $x(t_i) \in B_i$. However, there is a problem at t_2: there is no reason to expect a gluing of two solutions to be smooth. We can avoid this problem by considering the evolution of the system from t_1 to t_3 and hope that at t_2 it passes through B_2. In general to do this we need to vary B_2, so we need families of Lagrangian submanifolds. This is provided by a Lagrangian fiber bundle

$$\begin{tikzcd} N \arrow[d] & \text{Lagrangian fibers} \arrow[l] \\ X \end{tikzcd} . \tag{42}$$

By varying points in the base X we thereby obtain families of Lagrangian submanifolds.

Example Dirichlet boundary conditions, taking $N = T^*(\mathcal{N})$, $X = \mathcal{N}$, and $N \to X$ the bundle projection.

The goal of the above discussion was to motivate the following definition.

Definition *Local variational boundary conditions* on N are given by a Lagrangian fiber bundle $N \to X$. The fiber over $x \in X$ will be denoted B_x.

Note that there is a natural projection $\pi : C^\infty([t_1, t_2] \to N) \to N \times N$ sending a path γ to its endpoints $(x(t_1), x(t_2))$. The image of the solutions to the Euler-Lagrange equations under this projection form a subset $\pi(EL) \subset N \times N$.

Theorem 7.1. *Under mild assumptions, $\pi(EL)$ is a Lagrangian submanifold of $N \times N$.*

(Project idea: spell out these assumptions.)

$\pi(EL)$ and the boundary conditions $B \subset N \times N$ are now both Lagrangian submanifolds of $N \times N$. Generically, the intersection of two Lagrangian submanifolds (each of which has dimension half the dimension of the whole space) has dimension 0, so we obtain a discrete set of solutions to the Euler-Lagrange equations with boundary conditions B. The assumption that B is Lagrangian is crucial: if B is isotropic of smaller dimension, generically there are no solutions. (We want B to be exact isotropic so that the boundary terms in the variation vanish.)

Question from the audience: why do we want both the non-boundary and boundary terms to vanish separately when we talk about Euler-Lagrange equations?

Answer: if M is any manifold with boundary, the inclusion $\partial M \to M$ induces a pullback $\pi : \Omega^\bullet(M) \to \Omega^\bullet(\partial M)$. This induces a short exact sequence

$$0 \to \ker(\pi) \to \Omega^\bullet(M) \to \Omega^\bullet(\partial M) \to 0. \tag{43}$$

If we choose a splitting of this sequence, we can describe variation of forms on M in terms of bulk variation and boundary variation, and it is natural to consider these separately.

Example Let $N = T^*(\mathcal{N})$ and consider Newtonian mechanics on $\mathcal{N}$. Let α be the canonical 1-form, which can be defined as follows. There is a bundle projection $\pi : T^*(\mathcal{N}) \to \mathcal{N}$. Differentiating induces a map $T\pi : T(T^*(\mathcal{N})) \to T(\mathcal{N})$. Then we define

$$\alpha_{(p,q)}(\xi) = p(d\pi(\xi)). \tag{44}$$

The boundary conditions we want to consider are given by the Lagrangian fibration $T^*(\mathcal{N}) \to \mathcal{N}$; these give the Lagrangian submanifolds $B_{q_1} \times B_{q_2} = T^*_{q_1}(\mathcal{N}) \times T^*_{q_2}(\mathcal{N})$. The action functional here is the *Hamilton-Jacobi action*

$$S_{[t_1,t_2]}(\gamma) = \int_\gamma \alpha - \int_{t_1}^{t_2} H(p(t), q(t))\, dt \tag{45}$$

where $H(p,q) = \frac{1}{2}\langle p, p\rangle + V(q)$ is the corresponding Hamiltonian.

This idea of passing from a Lagrangian to a Hamiltonian formulation will be useful later on.

8 First-order scalar field theory

Scalar field theory is a second-order theory, but based on what we did above we will now convert it to a first-order theory. Here spacetime is a smooth oriented compact Riemannian manifold M with $\dim M = n$. In the first-order theory, the fields are now

$$F_M = \Omega^0(M) \oplus \Omega^{n-1}(M). \tag{46}$$

We write a scalar field as φ and an element of $\Omega^{n-1}(M)$ as $p(x)$. The action functional is

$$S_M(p,\varphi) = \int_M p \wedge d\varphi - \frac{1}{2}\int_M p \wedge *p - \int_M V(\varphi)\, dx. \tag{47}$$

The first term is topological and analogous to $\int_\gamma \alpha$ in the Hamilton-Jacobi action. The second two terms together use the metric and are analogous to the integral of the Hamiltonian in the Hamilton-Jacobi action.

Question from the audience: how did physicists discover this formalism? What is its physical meaning?

Answer: physicists don't know this formalism. They know it in the very special case $M = [t_1, t_2] \times N$ (a cylinder) where N is usually $\mathbb{R}^3$ where it is due to Fock. Graeme Segal uses a similar formalism. Scalar field theories in some form are due to Heisenberg and were used by Yukawa to describe mesons.

The variation is

$$\int_M \delta p \wedge (d\varphi - *p) - (-1)^{n-1}\int_M dp \wedge \delta\varphi + (-1)^{n-1}\int_{\partial M} p\,\delta\varphi - \int_M V'(\varphi)\,\delta\varphi\, dx. \tag{48}$$

The Euler-Lagrange equations are therefore

$$d\varphi - *p = 0, (-1)^{n-1} dp + V'(\varphi)\, dx = 0. \tag{49}$$

The first equation gives $p = (-1)^{n-1} * d\varphi$, and substituting this into the second equation gives

$$d * d\varphi + V'(\varphi)\, dx = 0. \tag{50}$$

Taking the Hodge star a final time,

$$\Delta\varphi + V'(\varphi) = 0. \tag{51}$$

(Perhaps $\Delta\varphi - V'(\varphi) = 0$ instead?) To recover the original second-order Lagrangian we can substitute $p = (-1)^{n-1} * d\varphi$ to obtain

$$S_M((-1)^{n-1} * d\varphi, \varphi) = \int_M (-1)^{n-1} * d\varphi \wedge d\varphi - \frac{1}{2}\int *d\varphi \wedge *^2 d\varphi - \int_M V(\varphi)\, dx \tag{52}$$

$$= \frac{1}{2}\int_M d\varphi \wedge *d\varphi - \int_M V(\varphi)\, dx \tag{53}$$

$$= \int_M \left(\frac{1}{2}(d\varphi, d\varphi) - V(\varphi)\right) dx. \tag{54}$$

Boundary conditions are as follows. Pulling back our fields gives us an element of $\Omega^0(\partial M) \oplus \Omega^{n-1}(\partial M)$. The boundary term in the variation is

$$\delta_{\text{bound}} S_M = (-1)^{n-1} \int_{\partial M} p\, \delta\varphi = \langle \alpha_{\partial M}, (\delta p, d\varphi)\rangle \tag{55}$$

where

$$\alpha_{\partial M} = \int_{\partial M} p\, D\varphi \in \Omega^1(F_{\partial M}) \tag{56}$$

where D is the de Rham differential on $\Omega^\bullet(F_{\partial M})$, defined roughly as follows. (Project idea! Make this rigorous.) For simplicity we will only discuss $\Omega^0_{\partial M}$. We have

$$\Omega^i(\Omega^0_{\partial M}) = \left\{ \int_{\partial M^i} f(x_1, ...x_i) D\varphi(x_1) \wedge ... \wedge D\varphi(x_i) \right\} \tag{57}$$

where $\varphi \in \Omega^0(\partial M)$, $f(x_1, ...x_i)$ is a top-degree form on ∂M^i, and $D\varphi(x_1) \wedge ... \wedge D\varphi(x_i)$ is a 0-form on ∂M^i. Roughly speaking the elements $D\varphi(x_1) \wedge ... \wedge D\varphi(x_i)$ are a basis of $\Lambda^k(T^*_\varphi(\Omega^0(\partial M)))$.

Question: can we be more explicit about what α does?

Answer: if $\xi \in T_\varphi(F_{\partial M}) \cong F_{\partial M}$ is a tangent vector, then we can describe the evaluation, which is

$$\iota_\xi \alpha_{\partial M} = \int_{\partial M} p\xi. \tag{58}$$

We can think of the space $F_{\partial M}$ of boundary fields as $T^*(\Omega^0(\partial M))$ (thinking of $\Omega^{n-1}(\partial M)$ as the cotangent space) in the following manner: if $\delta\varphi \in T_\varphi(\Omega^0(\partial M)) \cong \Omega^0(\partial M)$ is a tangent vector and $A \in \Omega^{n-1}(\partial M)$ is a cotangent vector, then we can pair them via

$$A(\delta\varphi) = \int_{\partial M} A \wedge \delta\varphi. \tag{59}$$

So $F_{\partial M}$ is an infinite-dimensional symplectic manifold modulo some analytic details with symplectic form $\omega_{\partial M} = D\alpha_{\partial M}$. We can impose Dirichlet boundary conditions using the natural Lagrangian fibration $F_{\partial M} \cong T^*(\Omega^0(\partial M)) \to \Omega^0(\partial M)$. We will fix the value $\varphi|_{\partial M}$ of φ on the boundary to lie in the fibers of the above fibration and impose no conditions on $p|_{\partial M}$.

As before, we have a canonical map $\pi : F_M \to F_{\partial M}$ given by restriction, and we have a subspace $EL_M \subset F_M$ of solutions to the Euler-Lagrange equations.

Exercise 8.1. *Suppose there is a unique solution to $\Delta\varphi - V'(\varphi) = 0$ with given boundary conditions $\varphi|_{\partial M} = \eta$. Then $\pi(EL_M)$ is a Lagrangian submanifold of $F_{\partial M}$.*

(Hint: it suffices to prove that the image is the graph of a map $\Omega^0(\partial M) \to F_{\partial M}$.)

In infinite dimensions we cannot define Lagrangian submanifolds as isotropic of maximal dimension; we instead need to define Lagrangian submanifolds as isotropic and coisotropic. Now boundary conditions (another Lagrangian submanifold) generically gives a discrete set of solutions as before.

In Yang-Mills we will no longer get a discrete set of solutions due to gauge symmetry, so we will instead consider gauge classes of fields. Our goal is to construct a quantum field theory out of a given classical field theory, and we will do this as follows: we will always have a space of fields and a space of boundary fields $F_M \xrightarrow{\Pi} F_{\partial M}$ which is exact symplectic. We will fix boundary conditions given by a Lagrangian fibration on $F_{\partial M}$. We will quantize using ideas from geometric quantization and path integral quantization: we want to assign to the boundary some vector space $H(\partial M)$ and, in this vector space, some vector $Z_M \in H(\partial M)$. This assignment should satisfy certain natural gluing axioms.

$H(\partial M)$ will be something like the space of functionals on the base $B_{\partial M}$ of the Lagrangian fibration on $F_{\partial M}$. The vector Z_M will be something like the Feynman integral

$$Z_M(b) = \int_{f \in F_M, \Pi(f) \in \pi^{-1}(b)} e^{\frac{i}{\hbar} S_M(f)} Df \tag{60}$$

where Df is some fantasy measure. Applying geometric quantization ideas we will replace some functions with sections of line bundles.

In the future we will see what goes wrong when we try to quantize field theories in general. In a very special case we can write down a quantum field theory in 1 dimension without any functional integrals or Feynman diagrams, and the reason we can do this is because the correseponding classical field theory is an infinite-dimensional integrable system. (Project idea!)

Question: why do we want to ensure that the set of solutions is discrete?

Answer: ideally we would want to set boundary conditions so that the set of solutions is unique (in engineering, etc.). Sometimes this is not possible, and setting boundary conditions so that the set of solutions is discrete is the next best thing.

9 First-order Yang-Mills

Spacetime is again a smooth compact oriented Riemannian manifold M. If G is a compact simple simply-connected Lie group with Lie algebra $\mathfrak{g}$, fix an embedding of G into $\text{Aut}(V)$, hence an embedding of $\mathfrak{g}$ into $\text{End}(V)$, and take fields to be

$$F_M = \Omega^1(M, \mathfrak{g}) \oplus \Omega^{n-2}(M, \mathfrak{g}) \tag{61}$$

where we think of $\Omega^1(M, \mathfrak{g})$ as the space of connections on a trivial G-bundle over M. (If we use a nontrivial G-bundle over M then the first term should be replaced with the corresponding space of connections.) We denote an element of F_M by an ordered pair (A, B). The action functional is

$$S_M(A, B) = \int_M \text{tr}(B \wedge F(A)) - \frac{1}{2} \int_M \text{tr}(B \wedge *B) \tag{62}$$

where $F(A)$ is the curvature of A as a connection. The first Euler-Lagrange equation is $F(A) = *B$ or equivalently $B = *F(A)$. The second comes from the following computation of the variation with respect to A: it is

$$\pm \int_M \text{tr}(d_A B \wedge \delta A) + \int_{\partial M} \text{tr}(B \wedge \delta A) \tag{63}$$

so we have $d_A B = 0$. The space of boundary fields is $F_{\partial M} = \Omega^1(\partial M, \mathfrak{g}) \oplus \Omega^{n-2}(\partial M, \mathfrak{g})$. On this space we have a 1-form $\alpha_{\partial M}$ which requires some complicated analysis to make sense of rigorously. It may be written $\int_{\partial M} \text{tr}(B \wedge DA)$ and it contracts with vector fields on $F_{\partial M}$ (which can be identified with $F_{\partial M}$) as follows:

$$\iota_\xi \alpha_{\partial M} = \mathrm{tr} \int_{\partial M} B \wedge \xi_1 \tag{64}$$

where $\xi = (\xi_1, \xi_2)$. We get a corresponding symplectic form $\omega_{\partial M} = D\alpha_{\partial M}$ on $F_{\partial M}$. We also have the projection map $\pi : F_M \to F_{\partial M}$ and the subspace $L_M = \pi(EL_M)$ of images of solutions to the Euler-Lagrange equations. Letting $\partial M_\epsilon = [0, \epsilon) \times \partial M$, we have that L_M is naturally a subspace of the *Cauchy subspace*

$$C_{\partial M} = \pi(EL_{\partial M_\epsilon}) \tag{65}$$

of solutions to the Euler-Lagrange equations on a small neighborhood of the boundary. The claim is that L_M is Lagrangian in $C_{\partial M}$ and $F_{\partial M}$ and also that $C_{\partial M}$ is coisotropic in $F_{\partial M}$, which is symplectic. $C_{\partial M}$ also makes sense in scalar field theory, where explicitly it consists of pairs $(p, \varphi) \in \Omega^{n-2}(\partial M) \oplus \Omega^0(\partial M)$ where p is the pullback of some p_0 defined in a neighborhood of the boundary and φ is the pullback of some φ_0 defined in a neighborhood of the boundary, and moreover $p_0 = *d\varphi_0$ and $\Delta\varphi_0 - V'(\varphi_0) = 0$.

Exercise 9.1. *Sort out the relationship between these spaces.*

Yang-Mills has the following gauge symmetry. Thinking of $A \in \Omega^1(M, \mathfrak{g})$ as a connection on the trivial G-bundle over M, this bundle $G \times M$ has automorphism group the gauge group $G_M = M \Rightarrow G$ (the space of smooth maps from M to G). For a nontrivial bundle this is only true locally. An element of the gauge group acts on connections by

$$g : A \mapsto A^g = gAg^{-1} + g^{-1}dg, B \mapsto B^g = gBg^{-1}. \tag{66}$$

The Yang-Mills functional is invariant under this symmetry:

$$S_M(A^g, B^g) = \int_M \mathrm{tr}(B^g \wedge F(A^g)) - \frac{1}{2} \int_M \mathrm{tr}(B^g \wedge *B^g) \tag{67}$$

where $F(A^g) = gF(A)g^{-1}$, so every term transforms under conjugation and due to the traces we conclude that the above is equal to $S_M(A, B)$. The action of G_M induces an infinitesimal action of $\mathfrak{g}$-valued vector fields on M, denoted by $\mathfrak{g}_M$, as follows: $\lambda \in \mathfrak{g}_M$ acts by

$$\delta_\lambda A = [\lambda, A] + d\lambda = d_A\lambda, \delta_\lambda B = [\lambda, B] \tag{68}$$

where the commutator is the pointwise commutator.

There is a projection map of gauge groups $\tilde{\pi} : G_M \to G_{\partial M}$ which is a group homomorphism. This map is surjective, so we obtain an exact sequence

$$0 \to \mathrm{Ker}(\tilde{\pi}) \to G_M \to G_{\partial M} \to 0 \tag{69}$$

where $\mathrm{Ker}(\tilde{\pi})$ is the group of gauge transformations fixing the boundary. The claim is that $G_{\partial M}$ preserves the symplectic form $\omega_{\partial M}$, which may be thought of as $\int_{\partial M} \mathrm{tr}(DA \wedge DB)$.

Clarification: when $\mathfrak{g} = \mathbb{R}$, Yang-Mills becomes electromagnetism (in Riemannian signature). The Euler-Lagrange equations become Maxwell's equations.

Returning to $C_{\partial M}$, for the scalar field theory we have $C_{\partial M} = F_{\partial M}$. For Yang-Mills we instead have the following.

Exercise 9.2. $C_{\partial M} = \{(a, b) : d_a b = 0\}$

In addition we have the following.

Exercise 9.3. *For each $(a, b) \in C_{\partial M}$ there exists a unique gauge class (with respect to $\tilde{G}_{\partial M_\epsilon}$) of solutions in $EL_{\partial M_\epsilon}$ with Cauchy data (a, b) (that is, which restricts to (a, b) on $\{0\} \times \partial M$).*

We also have the following.

Exercise 9.4. *$C_{\partial M} \subset F_{\partial M}$ is a coisotropic subspace of the symplectic space $F_{\partial M}$.*

We will show this for $\mathfrak{g} = \mathbb{R}$. In this case $C_{\partial M}^{\perp}$ consists of all $(\alpha, \beta) \in \Omega^1(\partial M) \oplus \Omega^{n-2}(\partial M)$ such that

$$\int_{\partial M} \alpha \wedge \beta + \int_{\partial M} \beta \wedge \alpha = 0 \tag{70}$$

for all $(a, b) \in C_{\partial M} \subset \Omega^1(\partial M) \oplus \Omega^{n-2}(\partial M)$. Requiring this condition for all b gives that $\beta = 0$ and requiring this condition for all a gives that α is exact, so we have $C_{\partial M}^{\perp} = \Omega^1_{\mathrm{ex}}(\partial M) \subset C_{\partial M}$ as desired.

10 Hamiltonian reduction

First, a digression. If (M, ω) is a symplectic manifold, then recall that on $C^\infty(M)$ we can define a Poisson bracket $\{f, g\} = \omega^{-1}(df \wedge dg)$. In local coordinates, if $\omega = \sum_{i,j} \omega_{ij}(x) dx^i \wedge dx^j$, then

$$\omega^{-1} = \sum_{i,j} (\omega^{-1}(x))^{ij} \partial_i \wedge \partial_j. \tag{71}$$

The Poisson bracket is a Lie bracket. More generally, a pair (M, p) where $p \in \Gamma(\Lambda^2(T(M)))$ (the *Poisson bivector*) is a *Poisson manifold* if $\{f, g\} = p(df \wedge dg)$ is a Lie bracket. Every symplectic manifold is a Poisson manifold, but p is allowed to be degenerate. In other words, p defines a map $T^*(M) \to T(M)$ whose image is a proper subbundle in general. We get a distribution $\mathrm{im}(p_x) \subset T_x(M)$.

In general, a ring with a Poisson bracket can be thought of as an infinitesimal version of a noncommutative algebra. Thinking of only the smooth functions $C^\infty(M)$ themselves we can really only talk about points. Because we also have the Poisson bracket, we have in addition $\mathrm{im}(p)$.

Definition The leaves of $\mathrm{im}(p)$ are the *symplectic leaves* of (M, p).

Symplectic leaves should be thought of as a geometric analogue of irreducible representations. This is related to geometric quantization and coadjoint orbits.

We now return to symplectic manifolds. Let (M, ω) be a symplectic manifold admitting an action of a Lie group G which fixes ω. Equivalently, if $\xi \in \mathfrak{g}$, we have $L_v \omega = 0$ (where L_v is the Lie derivative with respect to the image v of ξ in $\Gamma(T(M))$). In the rare situation that the quotient M/G exists as a manifold, it is a Poisson manifold; we have $C^\infty(M/G) \cong C^\infty(M)^G$ and the Poisson bracket on $C^\infty(M)$ is invariant under G, hence descends to the Poisson subalgebra $C^\infty(M)^G$.

Exercise 10.1. *Show that M/G is not symplectic.*

Suppose in addition that the action of G is *Hamiltonian*. This means that for each $\xi \in \mathfrak{g}$, the corresponding vector field v_ξ, there exists a function $H_\xi \in C^\infty(M)$ such that $L_{v_\xi} f = \{H_\xi, f\}$; moreover, the map $\xi \mapsto H_\xi$ is a linear map $\mathfrak{g} \to C^\infty(M)$. For convenience we will also assume that this action is *strongly Hamiltonian*, so the map $\xi \mapsto H_\xi$ is a Lie algebra homomorphism. (Without this assumption, we can only conclude that $\{H_{\xi_1}, H_{\xi_2}\} - H_{[\xi_1, \xi_2]}$ lies in the center of $C^\infty(M)$, which consists of constants, so we have to deal with a central extension.) Dualizing, we obtain the *moment map*

$$\mu : M \ni m \mapsto (\xi \mapsto H_\xi(m)) \in \mathfrak{g}^*. \tag{72}$$

Theorem 10.2. *If $\mathcal{O} \subset \mathfrak{g}^*$ is a* coadjoint orbit *(an orbit under the dual of the adjoint action of G on $\mathfrak{g}$), then*

1. *$\mu^{-1}(\mathcal{O})$ is a coisotropic submanifold of M, and*
2. *$\mu^{-1}(\mathcal{O})/G \subset M/G$ is a symplectic leaf.*

This is *Hamiltonian reduction* (relative to $\mathcal{O}$).

Exercise 10.3. *Prove this theorem.*

(Project idea! Go through the details of Hamiltonian and symplectic reduction. Look at various examples. Examine what happens when the quotient M/G is a manifold or is not a manifold.)

Let $C \subset M$ be a coisotropic submanifold. Consider the space $C^\infty(M)_C$ of smooth functions which vanish on C. Hamiltonian vector fields generated by such functions form a distribution, the *characteristic distribution* on C. The characteristic distribution can alternately be defined as the kernel of $\omega : T(M) \to T^*(M)$ when restricted to C.

Definition The leaves of the characteristic distribution on C form the *symplectic reduction* $\underline{C}$ of C, which has a natural symplectic structure.

Exercise 10.4. *When the action of G is Hamiltonian, $\underline{\mu^{-1}(\mathcal{O})} \cong \mu^{-1}(\mathcal{O})/G$.*

Example The coadjoint action of G on $\mathfrak{g}^*$ is almost Hamiltonian, except that $\mathfrak{g}^*$ is only a Poisson manifold and not a symplectic manifold. To get a Hamiltonian action, consider the action of G on $T^*(G)$ which extends left multiplication of G on G.

Exercise 10.5. *Check that the action of G on $T^*(G)$ is Hamiltonian with respect to the natural symplectic structure.*

Exercise 10.6. *Prove that $T^*(G)/G \cong \mathfrak{g}^*$.*

Exercise 10.7. *Prove that if $\{e_i\}$ is a basis of $\mathfrak{g}$ with structure constants $[e_i, e_j] = c_{ij}^k e_k$, then thinking of e_i as coordinates on $\mathfrak{g}^*$, defining $\{e_i, e_j\} = c_{ij}^k e_k$ yields a Poisson bracket on $\mathfrak{g}^*$ (Kirillov-Kostant-Lie).*

Exercise 10.8. *If $f, g \in C^\infty(\mathfrak{g}^*)$, then the Poisson bracket can be defined as*

$$\{f, g\}(x) = ([df(x), dg(x)])\,(x) \tag{73}$$

where we identify $df(x) \in T_x^*(\mathfrak{g}^*) \cong \mathfrak{g}$.

Exercise 10.9. *Check that the coadjoint orbits in $\mathfrak{g}^*$ are symplectic leaves.*

In representation theory, this leads to the orbit method and geometric quantization.

11 Yang-Mills and reduction

Recall that we have the chain of subspaces $L_M \subset C_{\partial M} \subset F_M$ where L_M is Lagrangian, $C_{\partial M}$ is coisotropic, and F_M is symplectic.

Theorem 11.1. *The action of $G_{\partial M}$ on $F_{\partial M}$ is Hamiltonian.*

Proof. (Outline) Let f be a function on $F_{\partial M}$ and let $\lambda \in \mathfrak{g}_{\partial M}$. Let $\delta_\lambda f$ denote the corresponding Lie derivative. Then

$$\delta_\lambda f((A, B)) = \int_{\partial M} \mathrm{tr}\left(\frac{\delta f}{\delta A} \wedge (d_A\lambda = \delta_\lambda A) + \frac{\delta f}{\delta B} \wedge [\lambda, B]\right) \tag{74}$$

and we want to show that this is equal to the Poisson bracket $\{H_\lambda, f\}$ where

$$H_\lambda = \int_{\partial M} \mathrm{tr}(\lambda d_A B). \tag{75}$$

The Poisson bracket on functions on $F_{\partial M}$ is given by

$$\{f, g\} = \int_{\partial M} \mathrm{tr}\left(\frac{\delta f}{\delta A} \wedge \frac{\delta g}{\delta B} - \frac{\delta g}{\delta A} \wedge \frac{\delta f}{\delta B}\right). \tag{76}$$

We compute that

$$\frac{\delta H_\lambda}{\delta A} = \frac{\delta}{\delta A}\left(\int_{\partial M} \mathrm{tr}(\lambda\, dB + \lambda[A \wedge b])\right) = [\lambda, B] \tag{77}$$

and we compute, using integration by parts, that

$$\frac{\delta H_\lambda}{\delta B} = d_A B = dB + [A \wedge B] \tag{78}$$

where dB is the covariant derivative with respect to the trivial connection. (Here A and B were denoted by a and b earlier.) Combining our calculations we get the result. $\square$

Question from the audience: what does $[A \wedge B]$ mean?

Answer: let $\omega, \omega' \in \Omega^\bullet(M, \mathfrak{g})$. Writing $\omega = \sum_I \omega_I dx^I$ and $\omega' = \sum_I \omega'_I dx^I$, we have

$$[\omega \wedge \omega'] = \sum_{I,J} [\omega_I, \omega_J] dx^I \wedge dx^J \tag{79}$$

where I, J are mult-indices. This is a Lie superbracket.

The Hamiltonian action above induces a moment map $\mu : F_{\partial M} \to \mathfrak{g}^*_{\partial M}$, and the symplectic reduction $\mu^{-1}(0)/G_{\partial M}$, if it exists in a reasonable sense, is a symplectic manifold.

If $S_M(A, B)$ is the action functional, then the de Rham differential DS_M can be written as the sum of a bulk term and a boundary term $\int_{\partial M} \mathrm{tr}(B \wedge DA)$ defining the 1-form $\alpha_{\partial M}$. The bulk term vanishes by definition for solutions to the Euler-Lagrange equations, so we have

$$DS_M|_{EL_M} = \pi^*(\alpha_{\partial M}|_{L_M}) \tag{80}$$

where $\pi : F_M \to F_{\partial M}$ is the usual projection and $L_M = \pi(EL_M)$. This is analogous to the following situation in classical mechanics. Let N be a configuration space (such as $\mathbb{R}^n$) and $T^*(N)$ be the corresponding phase space. Let γ be a parameterized path in $T^*(N)$ such that, writing $\gamma(t) = (p(t), q(t))$ (where p is momenta and q is position), we have $q(t_i) = q_i$ for two fixed points q_1, q_2. If γ_{cl} is a solution to the Euler-Lagrange equations, then

$$dS^{\gamma_{\mathrm{cl}}}_{t_1,t_2}(q_1, q_2) = \pi^*(p_1\, dq_1 - p_2\, dq_2) \tag{81}$$

where $p_1 = p(t_1), p_2 = p(t_2)$ are determined by t_1, t_2, q_1, q_2. This is the Hamilton-Jacobi function.

We now replace the chain of inclusions $L_M \subset C_{\partial M} \subset F_{\partial M}$ with the chain of inclusions of gauge equivalence classes

$$L_M/G_{\partial M} \subset C_{\partial M}/G_{\partial M} \subset F_{\partial M}/G_{\partial M}. \tag{82}$$

The rightmost space is a Poisson manifold since the action of $G_{\partial M}$ is Hamiltonian. The middle space is the Hamiltonian reduction of $C_{\partial M}$ and is a symplectic leaf in the rightmost space. (We are ignoring the other symplectic leaves for now. We may have to deal with them if part of the action is not gauge invariant.) The leftmost space is still Lagrangian by the following.

Exercise 11.2. *Let $L \subset C \subset M$ be a chain of inclusion of a Lagrangian submanifold into a coisotropic submanifold into a symplectic manifold. Passing to reductions, $\underline{L} \subset \underline{C}$ is still Lagrangian.*

We have $L_M/G_{\partial M} = \pi(EL_M/G_M)$, and since S_M descends to a function on F_M/G_M by gauge invariance, we have as before

$$DS_M|_{\underline{EL}_M} = \pi^*(\alpha_{\partial M}|_{\underline{L}_M}). \tag{83}$$

So we get a setup very similar to our previous setups by passing to gauge equivalence classes. The price we pay for doing this is that the corresponding quotient spaces are not very nice. We want to avoid dealing with these quotients as much as possible.

Question from the audience: do these quotient spaces have geometric significance analogous to the corresponding spaces in Chern-Simons?

Answer: probably $L_M/G_{\partial M}$ has no geometric significance. It is more likely that $C_{\partial M}/G_{\partial M}$ has some geometric significance.

One way to avoid dealing with quotients is the following. We can try to find sections of the quotient map $F_M \to F_M/G_M$; in other words, we can try to find subspaces of F_M which intersect each orbit at one point. These usually do not exist globally, but for semiclassical quantization it suffices to find local sections since the asymptotics of the corresponding integrals (which don't exist) only depend on small neighborhoods of critical points.

In the special case of electromagnetism ($G = \mathrm{U}(1), \mathfrak{g} = \mathbb{R}$), the space of fields is $F_M = \Omega^1(M) \oplus \Omega^{n-2}(M)$ and similarly for the boundary. If M has no boundary, the gauge group $G_M = \Omega^0(M)$ acts on fields as follows: $A \mapsto A + d\alpha, B \mapsto B$. We can construct a global section of the corresponding quotient using Hodge decomposition: we know that

$$\Omega^\bullet(M) \cong \Omega^\bullet_{\text{exact}} \oplus H^\bullet \oplus \Omega^\bullet_{\text{coexact}} \tag{84}$$

where the middle term consists of harmonic forms. In particular,

$$\Omega^1(M) = d\Omega^0 \oplus H^1 \oplus d^*\Omega^2 \tag{85}$$

where the last two terms gives a global section. In physics, choosing a global section is called gauge fixing, and this particular choice of gauge is called the Laurenz gauge, where $d^*A = 0$ (or $\mathrm{div}(A) = 0$).

12 Chern-Simons

Spacetime is a smooth, compact, oriented 3-manifold M. Fields F_M are connections on the trivial G-bundle $E_M = M \times G$ with G a compact, simple, simply connected Lie group, which we will identify with 1-forms $\Omega^1(M, \mathfrak{g})$. The action functional is

$$S(A) = \int_M \mathrm{tr}\left(\frac{1}{2}A \wedge dA + \frac{1}{3}A \wedge A \wedge A\right) \tag{86}$$

where A is a connection. Where the Yang-Mills functional was quadratic in the derivative of the connection, Chern-Simons is linear, and this is important. The variation is

$$S_M(A) = \int_M \mathrm{tr}(F(A) \wedge \delta A) + \int_{\partial M} \mathrm{tr}(A \wedge \delta A) \tag{87}$$

so the space of solutions EL_M to the Euler-Lagrange equations is the space of flat connections (those satisfying $F(A) = 0$). The boundary term here comes from a 1-form

$$\alpha_{\partial M} = \int_{\partial M} \mathrm{tr}(A \wedge DA) \tag{88}$$

on the space $\Omega^1(\partial M, \mathfrak{g})$ of fields $F_{\partial M}$ on the boundary.

The gauge group G_M of smooth maps $M \to G$ acts as follows. If M is closed,

$$S_M(A^g) = S_M(A) + \text{const} \int_M \mathrm{tr}(dgg^{-1} \wedge dgg^{-1} \wedge dgg^{-1}). \tag{89}$$

Exercise 12.1. *Compute the constant.*

Strictly speaking, the Chern-Simons functional is therefore not invariant under gauge transformations. However, the form we are integrating is an integral form (with the appropriate constant), so it is invariant up to an integer, and

$$e^{2\pi i k S_M(A)} \tag{90}$$

will be genuinely gauge invariant if $k \in \mathbb{Z}$. In any case, getting integers is a global phenomenon, so S_M is invariant under the connected component of the identity of G_M, hence under the action of the Lie algebra $\mathfrak{g}_M$. We say that S_M is *infinitesimally invariant.*

If M has boundary, we can use our computation of the variation to compute the action of an infinitesimal transformation by setting $\delta A = d_A\lambda$ where $\lambda \in \mathfrak{g}_M$. This gives

$$\delta_\lambda S_M(A) = \int_M \operatorname{tr}(F(A) \wedge d_A\lambda) = \int_{\partial M} \operatorname{tr}(A \wedge d_A\lambda). \tag{91}$$

By Stokes' theorem, we can write this as

$$\int_M \operatorname{tr}(d_A F(A) \wedge \lambda) + \int_{\partial M} \operatorname{tr}(F(A)\lambda). \tag{92}$$

The bulk term is 0 by infinitesimal invariance. The boundary term is 0 if the curvature of A vanishes at the boundary, but not otherwise. We will return to this. For now, it is enough to note that the action functional is gauge invariant when restricted to EL_M (flat connections).

The 1-form $\alpha_{\partial M} \in \Omega^1(F_{\partial M})$ induces a symplectic form

$$\omega_{\partial M} = D\alpha_{\partial M} = \int_{\partial M} \operatorname{tr}(DA \wedge DA) \tag{93}$$

on $F_{\partial M}$.

Exercise 12.2. *The action of $\mathfrak{g}_{\partial M}$ on $F_{\partial M}$ is Hamiltonian with respect to this symplectic form. The Hamiltonian is*

$$H_\lambda(A) = \int_{\partial M} tr(F(A)\lambda). \tag{94}$$

This induces a moment map $\mu : F_{\partial M} \to \mathfrak{g}^*_{\partial M}$ given by $\mu(A)(\lambda) = H_\lambda(A)$.

As for Yang-Mills, let $C_{\partial M}$ be the space of boundary values on a small neighborhood of the boundary. This can be identified with the space of flat G-connections on

∂M and with $\mu^{-1}(0)$. Hence $C_{\partial M}$ is a coisotropic submanifold of $F_{\partial M}$. We have a chain of inclusions

$$L_M = \pi(EL_M) \subset C_{\partial M} \subset F_{\partial M} \tag{95}$$

where L_M is the space of flat connections on ∂M which extend to flat connections on M.

Exercise 12.3. *L_M is Lagrangian.*

This is not obvious. It is obvious that L_M is isotropic but not obvious that it is coisotropic. (Project idea!) Hint: consider the abelian case first.

From the above we conclude that

$$DS_M|_{EL_M} = \pi^*(\alpha_{\partial M}|_{L_M}) \tag{96}$$

which is the Hamilton-Jacobi property.

We now want to pass to gauge classes

$$L_M/G_{\partial M} \subset C_{\partial M}/G_{\partial M} \subset F_{\partial M}/G_{\partial M} \tag{97}$$

where the middle term is the symplectic / Hamiltonian reduction $\mu^{-1}(0)/G_{\partial M} \cong \underline{C}_{\partial M}$, which is symplectic. The left term is Lagrangian, and the right term is Poisson. (These quotients are highly singular in general, and these statements should be interpreted in a neighborhood of a smooth point.)

The middle term $C_{\partial M}/G_{\partial M}$ is the moduli space $\mathcal{M}_{\partial M}^G$ of flat G-connections on ∂M. This is very similar to EL_M/G_M, which is the moduli space $\mathcal{M}_M^G$ of flat G-connections on M. Unlike in Yang-Mills, these spaces are finite-dimensional.

Exercise 12.4. *There is a natural isomorphism*

$$\mathcal{M}_M^G \cong \mathit{Hom}(\pi_1(M), G)/G \tag{98}$$

where G acts on $\mathit{Hom}(\pi_1(M), G)$ by conjugation. The map from the LHS to the RHS is given by taking holonomy and the map from the RHS to the LHS is given by taking an associated bundle.

There is a natural projection $\pi : \mathcal{M}_M^G \to \mathcal{M}_{\partial M}^G$ with image $\underline{L}_M$. The codomain is still symplectic, but before reduction it was exact and after reduction the 1-form

$\alpha_{\partial M}$ becomes a connection and hence $\mathcal{M}^G_{\partial M}$ is not exact. However, $S_M|_{EL_M}$ is gauge invariant, so we still have a reduced Hamilton-Jacobi identity

$$D\tilde{S}_M = \pi^*(\underline{\alpha_{\partial M}|_{L_M}}) \tag{99}$$

and this gives the *reduced Hamiltonian structure* of Chern-Simons.

13 BF-theory

In th first-order formulation of Yang-Mills, one of the terms depended on the metric; it is analogous to the term $\frac{p^2}{2m}$ in a classical Hamiltonian. By sending $m \to \infty$, we remove the dependence on the metric and get only a topological action. This is how we get BF-theory.

Spacetime M is smooth, oriented, and compact, and is equipped with a trivial G-bundle where G is connected, simple or abelian, simply connected, and compact. (Project idea! There is a theory for nontrivial bundles, but it is more involved.) Fields are

$$F_M = \Omega^1(M, \mathfrak{g}) \oplus \Omega^{n-2}(M, \mathfrak{g}) \tag{100}$$

where $\Omega^1(M, \mathfrak{g})$ describes connections on the trivial G-bundle. As usual, the gauge group G_M is the space of smooth maps $M \to G$, and it acts on $A \in \Omega^1(M, \mathfrak{g})$ by $A \mapsto g^{-1}Ag + g^{-1}dg$ and on $B \in \Omega^{n-2}(M, \mathfrak{g})$ by $B \mapsto g^{-1}Bg$.

This is all as in Yang-Mills, except without the assumption that M is Riemannian. The action functional is

$$S_M(A, B) = \int_M \operatorname{tr}(B \wedge F(A)) \tag{101}$$

which is the topological term of Yang-Mills. This is clearly gauge-invariant, but we can also send $A \mapsto A, B \mapsto B + d\beta$ where $\beta \in \Omega^{n-3}(M, \mathfrak{g})$. This gives

$$S_M(A, B + d\beta) = S_M(A, B) + \int_M \operatorname{tr}(d_A\beta \wedge F(A)) \tag{102}$$

where the second term is

$$\int_M \operatorname{tr}(\beta \wedge d_A F(A)) + \int_{\partial M} \operatorname{tr}(\beta \wedge F(A)). \tag{103}$$

The bulk term vanishes and the only additional contribution is a boundary term.

The solutions to the Euler-Lagrange equations are

$$EL_M = \{(A, B) : F(A) = 0, d_A B = 0\}. \tag{104}$$

(In Yang-Mills we instead had $F(A) = *B$, so roughly speaking in BF-theory we are sending $* \to 0$.) The variation of S_M is

$$\delta S_M = \text{bulk} + \int_{\partial M} \operatorname{tr}(B \wedge \delta A) \tag{105}$$

which gives a 1-form

$$\alpha_{\partial M} = \int_{\partial M} \operatorname{tr}(B \wedge DA) \tag{106}$$

on boundary fields and a corresponding exact symplectic form

$$\omega_{\partial M} = D\alpha_{\partial M} = \int_{\partial M} \operatorname{tr}(DB \wedge DA). \tag{107}$$

Boundary fields are given by $F_{\partial M} = \Omega^1(\partial M, \mathfrak{g}) \oplus \Omega^{n-2}(\partial M, \mathfrak{g})$. Our additional symmetry $B \mapsto B + d\beta$ gives us a larger gauge group

$$G_M^{\text{tot}} = G_M \times \Omega_M^{n-3} \tag{108}$$

and a corresponding boundary gauge group

$$G_{\partial M}^{\text{tot}} = G_{\partial M} \times \Omega_{\partial M}^{n-3}. \tag{109}$$

Theorem 13.1. *The action of $G_{\partial M}^{tot}$ is Hamiltonian.*

Proof. (Sketch) If $\alpha \in \Omega^0(\partial M, \mathfrak{g})$ (an element of the Lie algebra) and $\beta \in \Omega^{n-3}(\partial M, \mathfrak{g}$, then we can take

$$H_\alpha(A, B) = \int_{\partial M} \operatorname{tr}(B \wedge d_A \alpha) \tag{110}$$

$$H_\beta(A, B) = \int_{\partial M} \operatorname{tr}(A \wedge d_A \beta). \tag{111}$$

□

This defines a moment map $\mu : F_{\partial M} \to \Omega^0(\partial M, \mathfrak{g}) \oplus \Omega^{n-3}(\partial M, \mathfrak{g})$. As before, set $C_{\partial M} = \mu^{-1}(0)$ and $L_M = \pi(EL_M) \subset F_{\partial M}$, so that we have a chain of inclusions

$$L_M \subset C_{\partial M} \subset F_{\partial M} \tag{112}$$

as in Yang-Mills. Also as in Yang-Mills, we have the Hamilton-Jacobi relation

$$DS_M|_{EL_M} = \pi^*(\alpha_{\partial M}|_{L_M}). \tag{113}$$

This is the unreduced theory. After reduction, we get as in Yang-Mills an inclusion

$$\underline{L}_M \subset \underline{C}_{\partial M} \tag{114}$$

where $\underline{L}_M$ is Lagrangian and $\underline{C}_{\partial M}$ is symplectic. (There is a subtlety here. The action functional is not gauge invariant, so when passing to reductions the action functional is not a function but a section of a line bundle.)

Question from the audience: why is it important to study BF-theory?

Answer: consider the abelian case $\mathfrak{g} = \mathbb{R}$. Then

$$S_M(A, B) = \int_M B \wedge dA. \tag{115}$$

To quantize this theory very naively, we would like to compute an integral like

$$\int e^{\frac{i}{\hbar} S_M(A,B)} DA\, DB. \tag{116}$$

This doesn't make sense as written, but quantizing in a more reasonable way gives us the theory of Reidemeister torsion. On the classical level, gauge fixing leads to the theory of Hodge decomposition. The nonabelian case is a generalization of this.

So this is mathematically important. Physically it is probably not important.

We now perturb BF-theory as follows. When $n = 4$, we can modify the action functional to

$$S_M(A, B) = \int_M \mathrm{tr}(B \wedge F(A)) + \frac{1}{2} \int_M \mathrm{tr}(B \wedge B). \tag{117}$$

This theory in some sense describes Chern classes. The Euler-Lagrange equations are

$$EL_M = \{(A,B) : B = F(A), d_A B = 0\}. \tag{118}$$

By the Bianchi identity we always have $d_A F(A) = 0$, so A is arbitrary. (Project idea! Report more on this.)

We can also perturb BF-theory as follows. When $n = 3$, we can modify the action functional to

$$S_M(A,B) = \frac{1}{2}\int_M \operatorname{tr}(B \wedge F(A)) \pm \frac{1}{3}\int_M \operatorname{tr}(\Lambda^3(B)). \tag{119}$$

Equivalently, $\int_M \operatorname{tr}(B \wedge F(A)) = \int_M \operatorname{tr}(B \wedge (dA + \frac{1}{2}[A \wedge A]))$.

Write $A_1 = A + B, A_2 = A - B$. Then we can write the action functional as $CS_M(A_1) + CS_{\bar{M}}(A_2)$ when the sign is chosen to be positive, and we can write it as $\operatorname{Im}(CS_M(A + iB))$ when the sign is chosen to be negative, where CS refers to the Chern-Simons functional.

Next week we will start talking about quantization via the path integral and Feynman diagrams. (Project idea! Someone should talk about Wiener integrals.)

14 Quantum mechanics

Recall that in Hamiltonian mechanics we use a symplectic manifold (M, ω) together with a Hamiltonian $H \in C^\infty(M)$ (where we think of $C^\infty(M)$ as the observables). The Hamiltonian induces a vector field $v_H = \omega^{-1}(dH)$ which is time evolution. As an action on smooth functions, it can be written

$$\frac{df_t}{dt} = \{H, f_t\} \tag{120}$$

where $\{-,-\}$ is the Poisson bracket.

Thinking of Hamiltonian mechanics as a 1-dimensional field theory, let $[t_1, t_2]$ be a time interval and let $\varphi_{[t_1,t_2]} : M \to M$ be the symplectomorphism given by $\varphi_{[t_1,t_2]}(x_1) = x_2$ where $\gamma(t_1) = x_1, \gamma(t_2) = x_2$ and γ is a flow line of v_H. Any symplectomorphism such as $\varphi_{[t_1,t_2]}$ induces a Lagrangian submanifold of $M \times M$, which in this case is

$$L_{[t_1,t_2]} = \{(x_1, x_2) : \gamma(t_1) = x_1, \gamma(t_2) = x_2\} \subset M \times M \tag{121}$$

where γ is a flow line of v_H.

A (probabilistic) state may be defined as a probability distribution on M, which may be identified with its expectation $f \mapsto S(f)$. More precisely, this is a positive linear functional $C^\infty(M) \to \mathbb{R}$ with value 1 at 1. Simple examples include the distribution supported at x_0 given by $S_{x_0}(f) = f(x_0)$ and the Boltzmann distribution

$$S(f) = \int_M f(x) e^{-\frac{E(x)}{T}}\, d\mu(x) \tag{122}$$

where $\mu(x)$ is a measure such that $\int e^{-\frac{E(x)}{T}}\, d\mu = 1$, $E(x)$ can be interpreted as energy, and T can be interpreted as temperature.

$C^\infty(M)$ is the space of functions on a symplectic manifold, so it is not only a commutative algebra but a Poisson algebra with Poisson bracket induced by the symplectic form. Moreover, the only functions in the Poisson center are scalars.

In quantum mechanics, we replace $C^\infty(M)$ by a noncommutative algebra. More precisely, we would like to use a *-algebra A, which is a $\mathbb{C}$-algebra equipped with a $\mathbb{C}$-antilinear involution $a \mapsto a^*$ which reverses the order of multiplication. This is our algebra of observables. Within A there is a real subspace $A_{\mathbb{R}}$ of self-adjoint elements (which represent observable quantities), which are those fixed by the involution. The corresponding classical theory should be thought of as $A = C^\infty(M) \otimes \mathbb{C}$ with pointwise complex conjugation as the involution.

In quantization, we want to choose an algebra A which is in some sense a deformation of $C^\infty(M) \otimes \mathbb{C}$ in a direction determined by the Poisson bracket. We will return to this later.

Time evolution is determined by a Hamiltonian $H \in A_{\mathbb{R}}$, which defines a one-parameter group of automorphisms of A as follows:

$$a \mapsto \exp\left(\frac{iHt}{\hbar}\right) a \exp\left(-\frac{iHt}{\hbar}\right). \tag{123}$$

Alternately,

$$\frac{df_t}{dt} = [H, f_t] \tag{124}$$

where $[-,-]$ is a suitable multiple of the commutator.

We may choose to set $\hbar = 1$ by reparameterizing time, but eventually we will want to take the classical limit $\hbar \to 0$ to relate classical and quantum mechanics.

Symmetries of a quantum system are described by a group G acting by automorphisms of A (as a *-algebra) which fixes H.

Example Finite-dimensional quantum mechanics is the case where $A = \text{End}(\mathbb{C}^n)$ with conjugate transpose as the *-operation (thinking of $\mathbb{C}^n$ as equipped with the usual inner product). The self-adjoint elements here are the self-adjoint matrices.

In the particular case that $\mathbb{C}^n$ is the tensor product of N copies of $\mathbb{C}^2$ (so $n = 2^N$), an example of a Hamiltonian is to take $H = \sum_{i=1}^N s_i s_{i+1}$ (the Heisenberg spin system) where

$$s_i = 1 \otimes ... \otimes s \otimes ... \otimes 1 \tag{125}$$

and $s = (s^1, s^2, s^3)$ are the three Pauli spin matrices. SU(2) acts diagonally on this tensor product and this Hamiltonian is SU(2)-invariant. This is a nice example because it is in some sense an integrable system.

States are positive *-linear functionals $\lambda : A \to \mathbb{C}$ (where positive means that $\lambda(a^*a) \geq 0$). In the example $A = \text{End}(\mathbb{C}^n)$, every state can be written in the form

$$\lambda_\rho(a) = \text{tr}(\rho a) \tag{126}$$

where ρ is a positive self-adjoint matrix with trace 1, the density matrix of the state. The pure states occur when ρ is a rank-1 idempotent; if $v \in \mathbb{C}^n$ is a unit vector, then the corresponding pure state ρ_v is projection onto v, and

$$\lambda_{\rho_v}(a) = \langle v, av \rangle. \tag{127}$$

Writing $\mathbb{C}^n = V$ and $\text{End}(V) \cong V \otimes V^*$, the corresponding state is $v \otimes v^*$ if v is a unit vector.

More generally, if e_i are the (unit) eigenvectors of ρ with eigenvalues ρ_i, then we may write

$$\lambda_\rho(a) = \sum \rho_i \langle e_i, ae_i \rangle. \tag{128}$$

So we may think of a general state as a mixture of pure states.

Note that if we replace v with $e^{i\alpha} v$, then the corresponding state does not change. So pure states should be identified with the space of lines in $\mathbb{C}^n$ rather than vectors in Hilbert space.

Next time we will discuss quantization of classical mechanics, for which we take A to be complex-valued differential operators on a smooth manifold N. This space acts on $L^2(N)$ (once we have discussed what this means; there is a nice way to do this using half-forms), which induces a *-operation on A.

Example Let $H = L^2(\mathbb{R}^n)$ with inner product

$$\langle x, y \rangle = \int_{\mathbb{R}^n} \bar{x}(q) y(q) \, dq \tag{129}$$

and let $A = \mathrm{End}(H)$. (If we wanted to be careful we would restrict to bounded operators, but many operators of interest are not bounded.) The *-operation is the adjoint (which is subtle for bounded operators).

States in this case are in bijection with trace-class operators on H which are positive self-adjoint and have trace 1; if ρ is such a state, we take $\lambda_\rho(a) = \mathrm{tr}(\rho a)$ (when the trace exists).

15 Deformation quantization

Quantum mechanics can be thought of as a deformation of classical mechanics, although in some sense it should be the other way around: classical mechanics is a limiting case of quantum mechanics, which is the more fundamental theory, as $\hbar \to 0$. The latter operation is functorial and the former is not.

To make sense of deformation, suppose A_0 is a commutative algebra over $\mathbb{C}$ and $A_\hbar$ is a family of deformations of A_0 depending on a parameter $\hbar$ in some sense. Suppose that we have fixed linear isomorphisms $A_\hbar \cong A_0$. Then the multiplications on $A_\hbar$ are linear maps $m_\hbar : A_\hbar \otimes A_\hbar \to A_\hbar$, and composing with the isomorphism $\varphi_\hbar : A_\hbar \to A_0$ gives linear maps $A_0 \otimes A_0 \to A_0$ of the form

$$a \star_\hbar b = \varphi_\hbar(\varphi_h(a)^{-1} \varphi_h(b)^{-1}). \tag{130}$$

Assume that this family of maps can be written in the form

$$a \star_\hbar b = ab + \hbar m^{(1)}(a, b) + \hbar^2 m^{(2)}(a, b) + O(\hbar^3) \tag{131}$$

in some suitable sense. Requiring that this product is associative places constraints on the maps $m^{(i)}$. In particular,

Exercise 15.1. $\{a,b\} = m^{(1)}(a,b) - m^{(1)}(b,a)$ *is a Poisson bracket on* A_0.

In other words, any family of deformations of a commutative algebra A_0 induces a Poisson bracket on A_0. Doing this analytically is difficult, but working formally over $\mathbb{C}[[\hbar]]$ simplifies the problem.

Conversely, given a Poisson algebra A_0, deformation quantization of A_0 means finding all families $A_\hbar$ as above whose corresponding Poisson brackets are the Poisson bracket on A_0. This problem was solved for symplectic manifolds by Lecomte and Fedosov and for Poisson manifolds by Kontsevich: there is a unique family of operators $h^{(n)}$ such that

$$a \star_\hbar b = ab + \frac{\hbar}{2}\{a,b\} + \sum_{n=2}^{\infty} \hbar^n m^{(n)}(a,b) \tag{132}$$

up to equivalence under a family of automorphisms of the form

$$a \mapsto a + \sum_{n=1}^{\infty} \hbar^n \varphi_n(a). \tag{133}$$

Remarkably, this problem is related to a certain 2-dimensional Poisson σ-model.

Example Let $A_0 = \mathbb{C}[p_i, q^i]$ with Poisson bracket $\{p_i, q^j = \delta_i^j$. Define

$$A_\hbar = \mathbb{C}\langle p_i, q^i\rangle/(p_i q^i - q^i p_i = \delta_i^j \hbar). \tag{134}$$

A special case of the Poincaré-Birkhoff-Witt theorem (straightforward to prove here) shows that $A_\hbar$ has a basis given by products of the form $p_{i_1}^{m_1} p_{i_2}^{m_2} ... (q^{j_1})^{n_1} (q^{j_2})^{n_2} ...$ where the i and j indices are in increasing order. Since A_0 also has such a basis, we have a family of deformations of A_0.

For quantum mechanics we also need *-operations. These should satisfy

$$(a \star_\hbar b)^{*_\hbar} = b^{*_\hbar} \star_\hbar a^{*_\hbar} \tag{135}$$

and be $\mathbb{C}$-antilinear. (As $\hbar \to 0$ these should give a real structure on A_0.)

We can also talk about deformations with torsion. In this case we do not require that $A_\hbar$ is isomorphic to A_0 (some Poisson algebra) as a vector space; instead, we only require this to hold asymptotically as $\hbar \to 0$. More precisely, we want maps $\varphi_\hbar : A_\hbar \to A$ and $\psi_\hbar : A \to A_\hbar$ such that

$$\lim_{\hbar\to 0} \varphi_\hbar(\psi_\hbar(a)) = a \tag{136}$$

$$\lim_{\hbar\to 0} \varphi_\hbar(\psi_\hbar(a)\psi_\hbar(b)) = ab \tag{137}$$

$$\lim_{\hbar\to 0} \frac{1}{\hbar}\varphi_\hbar\left([\psi_\hbar(a), \psi_\hbar(b)]\right) = \{a, b\}. \tag{138}$$

Examples occur in geometric quantization of compact symplectic manifolds; here $\hbar = \frac{1}{2}, \frac{1}{3}, ...$ and the $A_\hbar$ are matrix algebras.

Example Let $A = C^\infty_{\text{pol}}(T^*(\mathbb{R}^n))$ be the space of smooth functions on the cotangent bundle of $\mathbb{R}^n$ which are polynomial in the cotangent variables p_i. A family of quantizations is given by the subalgebra of (complex) differential operators generated by $-i\hbar\frac{\partial}{\partial q^i}, f \in C^\infty(\mathbb{R}^n)$. The *-operation is induced by formal adjoint, which is

$$\left(-i\hbar\frac{\partial}{\partial q^i}\right)^* = -i\hbar\frac{\partial}{\partial q^i}, f^*(q) = \overline{f(q)}. \tag{139}$$

Classically, we used the Hamiltonian

$$H = \frac{1}{2m}\langle p, p\rangle + V(q) \tag{140}$$

to describe Newtonian mechanics, whereas in quantum mechanics we use the quantum Hamiltonian

$$H = -\frac{\hbar^2}{2m}\Delta + V(q) \tag{141}$$

and we can relate them by the classical limit $\hbar \to 0$. To see this, we first work with the Heisenberg picture of time evolution, where observables evolve by

$$a(t) = e^{\frac{iHt}{\hbar}} a e^{-\frac{iHt}{\hbar}} \tag{142}$$

where $a(t)$ is a family of operators acting on $L^2(\mathbb{R}^n)$. We can also switch to the Schrödinger picture, where states in L^2 evolve by

$$\psi(t) = e^{-\frac{iHt}{\hbar}}\psi. \tag{143}$$

As an operator on L^2, it can be written as an integral kernel

$$(e^{-\frac{iHt}{\hbar}}\psi)(x) = \int_{\mathbb{R}^n} U_t(x,x')\psi(x')\,dx' \tag{144}$$

where $U_t(x,x')$ is some distribution. This kernel should satisfy

$$i\hbar\frac{\partial U_t(x,y)}{\partial t} = H_x U_t(x,y) \tag{145}$$

where H_x is the Hamiltonian acting on the x variable, and

$$\lim_{t\to 0^+} U_t(x,y) = \delta(x-y). \tag{146}$$

Consider the special case that $V=0$. Then the Schrödinger equation is

$$i\hbar\frac{\partial\psi_t(x)}{\partial t} = -\frac{\hbar^2\Delta}{2m}\psi_t(x). \tag{147}$$

Taking Fourier transforms, we have

$$\psi(x) = \int_{\mathbb{R}^n} e^{i\frac{xp}{\hbar}}\hat{\psi}(p)\,dp \tag{148}$$

where the Fourier transform satisfies

$$\hat{\psi}(p) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i\frac{xp}{\hbar}}\psi(x)\,dx. \tag{149}$$

The Fourier-transformed Schrödinger equation is

$$i\hbar\frac{\partial\hat{\psi}_t(p)}{\partial t} = \frac{p^2}{2m}\hat{\psi}_t(p) \tag{150}$$

which gives

$$\hat{\psi}_t(p) = e^{\frac{ip^2}{2m\hbar}t}\hat{\psi}(p). \tag{151}$$

Taking inverse Fourier transforms gives

$$\psi_t(x) = \int_{\mathbb{R}^n} e^{i\frac{xp}{\hbar}-i\frac{p^2t}{2m\hbar}}\hat{\psi}(p)\,dp = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} e^{i\frac{(x-y)p}{\hbar}-i\frac{p^2t}{2m\hbar}}\,dp\right)\psi(y)\,dy. \tag{152}$$

The expression being exponentiated can be written

$$\frac{i}{\hbar}\left((x-y)p - \frac{p^2 t}{2m}\right) = \frac{i}{\hbar}\left(-\frac{\left(p - \frac{(x-y)m}{t}\right)^2 t}{2m} - \frac{m(x-y)^2}{2t}\right). \tag{153}$$

Write $p' = p - \frac{(x-y)m}{t}$. Then we conclude that

$$U_t(x,y) = \frac{e^{-i\frac{m(x-y)^2}{2\hbar t}}}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i\frac{p'^2 t}{2m\hbar}}\, dp'. \tag{154}$$

We now need to compute a Gaussian integral. These will become very important. A change of variables gives

$$\int_{\mathbb{R}^n} e^{-i\left(\frac{t}{m\hbar}\right)\frac{p'^2}{2}}\, dp' = \left(\frac{m\hbar}{t}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\frac{p'^2}{2}}\, dp \tag{155}$$

where

$$\int_{\mathbb{R}^n} e^{-i\frac{p'^2}{2}}\, dp = e^{-i\frac{\pi}{4}n}(2\pi)^{\frac{n}{2}}. \tag{156}$$

This gives

$$U_t(x,y) = \frac{1}{(2\pi i)^{\frac{n}{2}}}\left(\frac{m\hbar}{t}\right)^{\frac{n}{2}} e^{-i\frac{m}{2\hbar t}(x-y)^2}. \tag{157}$$

Exercise 15.2. *As $t \to 0^+$, we have $U_t(x-y) \to \delta(x-y)$ as distributions in the weak sense.*

What can we learn from this formula? Let's return to classical mechanics. In the absence of a potential, the trajectory $\gamma_c(\tau)$ of a classical particle satisfies $m\ddot{\gamma}_c = 0$, hence

$$\gamma_c(\tau) = y(t-\tau) + \tau x = y + \frac{(x-y)\tau}{t} \tag{158}$$

where $\gamma_c(0) = y, \gamma_c(t) = x$. The corresponding momentum is $m\dot{\gamma}_c = \frac{m(x-y)}{t}$. The Hamilton-Jacobi action is therefore

$$S(\gamma_c) = \frac{m}{2}\int_0^t \dot{\gamma}_c^2(\tau)\, d\tau = \frac{m(x-y)^2}{2t}. \tag{159}$$

We also have

$$\frac{\partial^2 S(\gamma_c)}{\partial x \partial y} = -\frac{m}{t}. \tag{160}$$

This lets us rewrite $U_t(x,y)$ as

$$U_t(x,y) = \frac{\hbar^{\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}} \left(\left| \frac{\partial^2 S_t(\gamma_c)}{\partial x \partial y} \right| \right)^{\frac{n}{2}} e^{-\frac{i}{\hbar} S_t(\gamma_c)}. \tag{161}$$

In other words, the propagator can be rewritten in terms of classical data.

We return more generally to the Schrödinger equation

$$i\hbar \frac{\partial \psi_t(x)}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi_t(x) + V(x) \psi_t(x) \tag{162}$$

with initial value a *semiclassical state*

$$\psi_0(x) = e^{i \frac{f(x)}{\hbar}} \varphi(x). \tag{163}$$

Analytically we would like to describe the behavior of the solution as $\hbar \to 0$. Algebraically we are happy to describe formally a power series expansion. We use the ansatz

$$\psi_t = e^{\frac{iS}{\hbar}} \left(\psi^{(0)} + \hbar \psi^{(1)} + ... \right). \tag{164}$$

Up to zeroth order, the Schrödinger equation then becomes

$$\frac{\partial S}{\partial t} = \frac{1}{2m} \left(\frac{\partial S}{\partial x} \right)^2 + V(x). \tag{165}$$

Later we will see how we can formally write $U_t(x,y)$ as a path integral

$$U_t(x,y) = \int_{\gamma(0)=y, \gamma(t)=x} e^{-\frac{i}{\hbar} S(\gamma)} \, D\gamma. \tag{166}$$

This path integral does not really exist, but we can write down an asymptotic expansion of it which agrees with the asymptotic expansion coming from the Schrödinger equation.

16 Semiclassics

Let $\psi(q) = e^{\frac{if(q)}{\hbar}}\varphi(q)$ be a semiclassical state. We want to compute the expectation of observables like $p = -i\hbar\frac{\partial}{\partial q}, q = q$ with respect to such a state. We compute that

$$\langle \psi, p\psi\rangle = \int_{\mathbb{R}} \overline{\varphi(q)} \left(\frac{\partial f}{\partial q} - i\hbar\frac{\partial}{\partial q}\right)\varphi(q) = \int_{\mathbb{R}} |\varphi(q)|^2 \frac{\partial f}{\partial q}\, dq + O(\hbar). \tag{167}$$

More generally, if $P(p,q)$ is a polynomial,

$$\langle \psi, P(p,q)\psi\rangle = \int_{\mathbb{R}} |\varphi(q)|^2 P\left(\frac{\partial f}{\partial q}, q\right)\, dq + O(\hbar). \tag{168}$$

In other words, as $\hbar \to 0$ the semiclassical state $\psi(q)$ tends to the probabilistic measure on $\mathbb{R}^2 = (p,q)$ supported on the Lagrangian subspace $L = \{p = \frac{\partial f}{\partial q}\}$ with density $|\varphi(q)|^2\, dq$.

Exercise 16.1. *Construct a semiclassical mixed state converging to a classical state described by a density $\rho(p,q)\, dp\, dq$.*

Exercise 16.2. *Describe the time evolution of this state and of expected values as $|t| \to \infty$.*

We return again to the Schrödinger equation

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial q^2} + V(q)\psi(q) \tag{169}$$

and look for solutions of the form

$$\psi = e^{i\frac{S}{\hbar}}A \tag{170}$$

where ψ is a formal power series in $\hbar$. It need not converge; we only require that it is an asymptotic expansion in the sense that $|A - (A^0 + ... + \hbar^n A^n)| < C\hbar^{n+1}$ as $\hbar \to 0$. Substitution gives

$$-\partial_t S + i\hbar\frac{\partial_t A}{A} = \frac{1}{2m}\left(\partial_q S\right)^2 + V(q) - i\frac{\hbar}{2m}\partial_q^2 S - i\frac{\hbar}{m}\partial_q S\partial_q A - \frac{\hbar^2}{2m}\frac{\partial_q^2 A}{A}. \tag{171}$$

Up to zeroth order this gives

$$-\partial_t S = \frac{(\partial_q S)^2}{2m} + V(q). \tag{172}$$

This implies that A satisfies the eikonal equation

$$\partial_t A + \frac{1}{2m}\partial_q^s S A + \frac{1}{m}\partial_q S \partial_q A = i\hbar \partial_q^2 A. \tag{173}$$

For A^0, the zeroth-order term in A, this gives the transport equation

$$\left(\partial_t + \frac{1}{m}\partial_q S \partial_q + \frac{1}{2m}\partial_q S\right) A^0 = 0. \tag{174}$$

More generally, letting D denote the above differential operator, we have

$$DA^{n+1} = i\partial_q^2 A^n. \tag{175}$$

There is a more explicit description of A^n in terms of Feynman diagrams.

Next time we will prove the following.

Theorem 16.3. *As $\hbar \to 0$, we have in 1 dimension the asymptotic expansion*

$$U_t(q, q') = \sqrt{2\pi\hbar}\sqrt{\left|\frac{\partial^2 S_t(q, q')}{\partial q \partial q'}\right|} e^{\frac{i}{\hbar}S_t(q,q') + i\frac{\pi}{4}t} \left(1 + O(\hbar)\right). \tag{176}$$

For now we have the following lemma. Recall that in 1 dimension Newtonian mechanics is described by the action

$$S(\gamma) = \int_0^t \left(\frac{m}{2}\dot{\gamma}^2(\tau) - V(\gamma(\tau))\right) d\tau. \tag{177}$$

We would like to approximate this integral by a sum

$$S(\varphi) = \sum_{i=0}^{N} \frac{1}{2} a_i (\varphi_i - \varphi_{i+1})^2 - \sum_{i=1}^{N} \left(V(\varphi_i)\Delta t_i\right) \tag{178}$$

where we have boundary values $\varphi_0 = b_0, \varphi_{N+1} = b_1$. Roughly speaking we have $a_i = \frac{m}{\Delta t_i^2}$ and $\varphi_i = \gamma(t_i)$.

Assme that for each (b_0, b_1) there is a unique extremum of $S(\varphi)$. This holds for exampe if V is convex, since then S is convex. Then there exists a unique solution to $\frac{\partial S}{\partial \varphi_i} = 0$, which is

$$(a_{i+1}+a_i)\varphi_i - a_i\varphi_{i-1} - a_{i+1}\varphi_{i+1} - V'(\varphi_i)\Delta t_i = 0. \tag{179}$$

Call this unique solution φ^c and let

$$\Delta_{ij} = \frac{\partial^2 S}{\partial\varphi_i\partial\varphi_j}(\varphi^c) \tag{180}$$

which is a certain tridiagonal matrix.

Lemma 16.4.

$$\frac{\partial^2 S(\varphi^c)}{\partial b_0 \partial b_1} = (-1)^N \frac{\prod_{i=0}^N a_i}{\det(\Delta)}. \tag{181}$$

Proof. We compute that

$$\frac{\partial S}{\partial b_0} = \sum_{i=1}^N \frac{\partial \varphi_i}{\partial b_0}\frac{\partial S}{\partial \varphi_i} + a_0 b_0 - a_0\varphi_1 \tag{182}$$

and differentiating the above with respect to φ_1, then specializing to φ^c, gives

$$\sum_{i=1}^N \frac{\partial^2\varphi_i}{\partial b_0\partial b_1}\frac{\partial S}{\partial\varphi_i} + \sum_{i=1}^N \frac{\partial\varphi_i}{\partial b_0}\Delta_{ij}\frac{\partial\varphi_j}{\partial b_1} - a_0\frac{\partial\varphi_1}{\partial b_1} - a_N\frac{\partial\varphi_N}{\partial b_0} \tag{183}$$

where the first term is 0 at φ^c by assumption. We further compute that

$$\frac{\partial}{\partial b_0}\left(\frac{\partial S}{\partial\varphi_i}\right) = \sum_{j=1}^N \Delta_{ij}\frac{\partial\varphi_j}{\partial b_0} - a_0\delta_{i,1} = 0 \tag{184}$$

and

$$\frac{\partial}{\partial b_1}\left(\frac{\partial S}{\partial\varphi_i}\right) = \sum_{j=1}^N \Delta_{ij}\frac{\partial\varphi_j}{\partial b_1} - a_N\delta_{i,N} = 0 \tag{185}$$

hence that

$$\frac{\partial\varphi_i}{\partial b_0} = a_0\left(\Delta^{-1}\right)_{i,1} \tag{186}$$

and

$$\frac{\partial\varphi_i}{\partial b_1} = a_N\left(\Delta^{-1}\right)_{i,N}. \tag{187}$$

We also have

$$\frac{\partial S}{\partial b_0 \partial b_1} = -a_N \frac{\partial \varphi_N}{\partial b_0} = -a_0 a_N \left(\Delta^{-1}\right)_{1,N} \tag{188}$$

where $(\Delta^{-1})_{1,N} = (-1)^N \frac{a_1 \ldots a_{N-1}}{\det(\Delta)}$, and the conclusion follows. $\square$

This lemma allows us to conclude in the limit that we have

$$\left| \frac{\partial^2 S_t^{\gamma^c}(q, q')}{\partial q \partial q'} \right| = \frac{1}{\left| \det'_\zeta(-\frac{d^2}{d\tau^2} + V''(\gamma^c(\tau))) \right|} \tag{189}$$

where $\det'_\zeta$ is the zeta-regularized determinant. There is a more analytic proof using properties of the zeta-regularized determinant which can be found in Takhtajan.

We can now do the following to the path integral. Writing a path γ as $\gamma^c + \alpha$ where γ^c is the classical path, we have

$$\int e^{\frac{i}{\hbar} S(\gamma)} \, D\gamma = \int \exp\left(\frac{i}{\hbar} \left(S(\gamma^c) + \int_0^\tau \left(\frac{m}{2} \dot{\alpha}(\tau)^2 - V(\gamma^c(\tau)) \alpha^2(\tau) \right) \right) \right) D\alpha \tag{190}$$

$$= e^{\frac{i}{\hbar} S(\gamma^c)} \int e^{\frac{i}{2}(\alpha, B\alpha)} \, D\alpha = \frac{C}{\sqrt{|\det(B)|}} \tag{191}$$

$$\tag{192}$$

where $B = -\frac{d^2}{d\tau^2} + V''(\gamma^c(\tau))$.

17 More semiclassics (Theo)

With the Hamiltonian $H = -\frac{\hbar^2}{2m}\Delta + V(q)$, we wanted to study the time evolution operator $U(t) = e^{\frac{i}{\hbar}Ht}$. This has an integral kernel

$$U(t, q_0, q_1) = \langle q_1 | U(t) | q_0 \rangle \tag{193}$$

where q_0 is the Dirac delta supported at q_0 which should satisfy the following properties:

1. as $t \to 0$ we should have $U(t, q_0, q_1) \to \delta(q_1 - q_0)$ (in an appropriate sense),

2. $-i\hbar \partial_t U = H_{q_1} U$ (Schrödinger equation),

3. $U(t_1+t_2, q_0, q_2) = \int U(t_1, q_0, q_1) U(t_2, q_1, q_2)\, dq_1$ (composition law).

What are the $\hbar \to 0$ asymptotics of $U(t, q_0, q_1)$? This is quite subtle. However, if they exist, we can say something about what they are. Namely, we guess

$$U(t, q_0, q_1) = e^{\frac{i}{\hbar} S(t, q_0, q_1)} \sqrt{\left| \det_{i,j} \frac{\partial^2 S}{\partial q_0^i \partial q_1^j} \right|} (2\pi\hbar)^{\frac{n}{2}} e^{i \frac{\pi}{4} \text{something}} (1 + O(\hbar)) \tag{194}$$

where $S(t, q_0, q_1)$ is the action of the classical path (solution to the Euler-Lagrange equation) connecting q_0 to q_1 with duration t (we should actually make a smooth choice of such a path).

Question from the audience: what is the distinction between semiclassical and classical?

Answer: semiclassically we keep asymptotic information in $\hbar$.

Our guess satisfies Schrödinger's equation to zeroth order. To satisfy Schrödinger's equation to higher order we continue plugging in terms to see what differential equations the higher-order corrections satisfy. This is the WKB approximation.

But only knowing that the higher-order corrections are given as solutions to various PDEs is somewhat indirect. We would prefer to have an integral expression for these higher-order corrections instead. To do this, we will use the composition law. Divide the interval $[0, t]$ into subintervals of step size $\frac{t}{N}$. Then by iterating the composition law, we can write

$$U(t, q_0, q_1) = \int_{\gamma : \{0, \frac{t}{N}, \dots \frac{(N-1)t}{N}\} \to \mathbb{R}^2} \prod_{\tau \in \{0, \frac{t}{N}, \dots \frac{(N-1)t}{N}\}} U\left(\frac{t}{N}, \gamma(\tau), \gamma\left(\tau + \frac{t}{N}\right)\right) d\gamma \tag{195}$$

where $d\gamma = \prod d\gamma(\tau)$. If the potential $V(q)$ satisfies some nice conditions, then we should have in some sense

$$U\left(\frac{t}{N}, q_0, q_1\right) = \delta(q_1 - q_0) + O\left(\frac{1}{N}\right). \tag{196}$$

This is an acceptable estimate if q_0 and q_1 are far apart, but if not, we want a sharper estimate (in some sense)

$$U\left(\frac{t}{N}, q, q+\frac{t}{N}v\right) = \exp\left(\frac{i}{\hbar}\frac{t}{N}L(v,q) + O\left(\frac{1}{N^2}\right)\right) \tag{197}$$

where $L(v,q)$ is the classical Lagrangian. By multiplying all of the exponentials, in the limit as $N \to \infty$ we should have, in some sense,

$$U(t, q_0, q_1) = \int_{\gamma:[0,t]\to\mathbb{R}^2, \gamma(0)=q_0, \gamma(t)=q_1} \exp\left(\frac{i}{\hbar}\int_0^t L(\dot{\gamma}(\tau), \gamma(\tau))\, d\tau\right) d\gamma \tag{198}$$

where the measure $d\gamma$ does not exist. There are some ways of computing this integral by passing to imaginary time, which gives a well-defined measure and integral (the Wiener measure and integral). If the corresponding integral is analytic in t, we can analytically continue back to real time, which requires some hypotheses on the potential.

Rather than making sense of this integral, we will pretend that this integral exists. As $\hbar \to 0$, it becomes a rapidly oscillating integral, and asymptotic behavior of such things are known in finite dimensions. Consider an integral of the form

$$\int_{\mathbb{R}^N} \exp\left(\frac{i}{\hbar}A(x)\right) dx \tag{199}$$

where A is smooth. This integral is not absolutely convergent, but it might be conditionally convergent. We will make it converge conditionally using smooth compactly supported bump functions; that is, we will study integrals of the form

$$\int_{\mathbb{R}^N} f(x) \exp\left(\frac{i}{\hbar}A(x)\right) dx \tag{200}$$

where f is a smooth bump function. We will take the limit as the support of f increases.

The idea of the asymptotic analysis as $\hbar \to 0$ is that most of the rapid oscillation cancels out.

Theorem 17.1. *(Fundamental theorem of oscillating integrals) If f is a smooth compactly supported function such that $\{dA = 0\} \cap supp(f) = \emptyset$, then the above integral is $O(\hbar^\infty)$ as $\hbar \to 0$.*

(See, for example, Evans and Zworski.)

Here $O(\hbar^\infty) = \bigcap_N O(\hbar^N)$. In other words, the asymptotic contribution to the integral comes from arbitrarily small neighborhoods of the critical locus $\{dA = 0\}$ of A. If the critical locus is compact, then increasing the support of bump functions will give a well-defined conditionally convergent integral.

We now make the strong assumption that the critical locus is finite and that the critical points are nondegenerate (the Hessian determinant is nonzero). We might as well use a bump function f supported in a neighborhood of one critical point; we can then sum the contributions from all critical points in general.

Assume without loss of generality that $A(x)$ has a nondegenerate critical point at 0 and that $f = 1$ in a neighborhood of the origin. Then we take a Taylor series expansion

$$A(x) = A^{(0)} + A^{(2)}\frac{x^2}{2} + ... \tag{201}$$

where $A^{(n)}$ is a symmetric $(n, 0)$ tensor and x is a $(0, 1)$ tensor so the powers above should be interpreted as tensor powers. Thus we can write the desired integral as

$$\exp\left(\frac{i}{\hbar}A^{(0)}\right)\int \exp\left(\frac{i}{\hbar}A^{(2)}\frac{x^2}{2} + \sum_{n\geq 3} A^{(n)}\frac{x^n}{n!}\right) dx. \tag{202}$$

We don't want to divide by $\hbar$ since we want to take it to zero, so substituting $\frac{x}{\sqrt{\hbar}}$ for x we get

$$\exp\left(\frac{i}{\hbar}A^{(0)}\right)\sqrt{\hbar}^N \int \exp\left(iA^{(2)}\frac{x^2}{2} + i\sum_{n\geq 3} \hbar^{\frac{n}{2}-1}A^{(n)}\frac{x^n}{n!}\right) dx. \tag{203}$$

We can now write this as

$$\exp\left(\frac{i}{\hbar}A^{(0)}\right)\sqrt{\hbar}^N \int \exp\left(iA^{(2)}\frac{x^2}{2}\right)\sum_{\ell\geq 0}\frac{1}{\ell!}\left(i\sum_{k\geq 3} \hbar^{\frac{k}{2}-1}A^{(k)}\frac{x^k}{k!}\right) \tag{204}$$

to reduce the problem to integration of a polynomial against a Gaussian.

As a tensor we should think of $A^{(k)}$ as a box, or perhaps a single vertex, with k inputs and x as a box with one output. That is, we should associate to a vertex with k inputs the term $i\hbar^{\frac{k}{2}-1}A^{(k)}x^k$, and we should think of the sum

$$\sum i\hbar^{\frac{k}{2}-1}A^{(k)}\frac{x^k}{k!} \tag{205}$$

as the integral of the function above over the groupoid G of vertices of valence at least 3, where we sum over the groupoid by dividing by the size of the automorphism group. Since we actually want to exponentiate this, we can do this by integrating over the exponential $\exp(G)$ of the groupoid, which is the disjoint union of the symmetric powers $\frac{G^\ell}{\ell!}$ obtained by taking products and dividing by the action of S_ℓ.

An integral over $\exp(G)$ is a sum over unordered collections of vertices of valence at least 3, weighted by automorphisms, and the function we want to integrate over it assigns to a k-valent vertex the function $i\hbar^{\frac{k}{2}-1}A^{(k)}x^k$ and is multiplicative under disjoint union. We want to integrate all of these functions against a Gaussian.

So we should figure out how to integrate $(m,0)$-tensors against a Gaussian. Given such a tensor B, consider the function of two $(0,1)$-tensors x,y given by $B(yx^{n-1})$.

Exercise 17.2. *(Wick, Isserlis) The Gaussian integral of B is the Gaussian integral of*

$$\sum_{i,j}\frac{\partial}{\partial x^j}\frac{\partial}{\partial y^i}B(yx^{n-1})\left((A^{(2)})^{-1}\right)^{i,j}. \tag{206}$$

Alternatively, the Gaussian integral may be computed by choosing ways to contract $(A^{(2)})^{-1}$ with B. In total, we find that $\langle B\rangle = 0$ if m is odd, and if m is even it is a sum over ways to contract B with copies of $(A^{(2)})^{-1}$ (with suitable weights).

This gives our original integral as

$$\exp\left(\frac{i}{\hbar}A^{(0)}\right)\sqrt{\hbar}^N\int\exp\left(iA^{(2)}\frac{x^2}{2}\right)\,dx\int_{G'}F \tag{207}$$

where G' is the groupoid of all closed diagrams with valences at least 3. The Feynman rules assign $i\hbar^{\frac{k}{2}-1}A^{(k)}$ to a vertex with valence k, $i(A^{(2)})^{-1}$ to a vertex with valence 2, disjoint union to tensor product, and contraction of graphs to contraction of tensors. We also have to divide by automorphism groups, which gives

$$\int_{G'}F = \sum_{\text{diagrams }\Gamma}\frac{F(\Gamma)}{|\mathrm{Aut}(\Gamma)|}. \tag{208}$$

Exercise 17.3. *The power of $\hbar$ which appears in such a term is $\hbar^{\chi(\Gamma)}$ where $\chi(\Gamma)$ is the Euler characteristic.*

It remains to compute one Gaussian integral. This integral does not converge absolutely, and we will regularize it by pushing the contour $\mathbb{R}^N$ into $\mathbb{C}^N$ so that $\text{Re}(iA^{(2)})$ is negative definite. This gives

$$\int \exp\left(iA^{(2)}\frac{x^2}{2}\right)\,dx = \sqrt{2\pi}^N \frac{1}{\sqrt{|\det A^{(2)}|}} e^{i\frac{\pi}{4}N}(-i)^{\eta(A^{(2)})} \tag{209}$$

where $\eta(A^{(2)})$ (the *Morse index*) is the maximal dimension of any subspace on which $A^{(2)}$ is negative definite. (Everything above was an asymptotic expansion, so there should be $O(\hbar^\infty)$s in everything.)

Recall that our original goal was to compute a path integral

$$\int_{\gamma:[0,t]\to\mathbb{R}^n,\gamma(0)=q_0,\gamma(t)=q_1} \exp\left(\frac{i}{\hbar}S(\gamma)\right)\,d\gamma. \tag{210}$$

The WKB expansion turned out to be

$$\exp\left(\frac{i}{\hbar}S(\gamma_{\text{cl}})\right)\sqrt{2\pi i\hbar}^{-n}\sqrt{\left|\det\frac{\partial^2 S}{\partial q_0 \partial q_1}\right|}(-i)^{\text{something}}\,(1+O(\hbar)) \tag{211}$$

which is quite close to the above. It suggests that we should think of the space of paths as having dimension $-n$. Comparing the determinants that appear, the analogue of $A^{(2)}$ is the differential equation defining Jacobi fields along a solution to the Euler-Lagrange equations. The exponent of $-i$ should be a Morse index, and the remaining terms should have something to do with Feynman diagrams, where $(A^{(2)})^{-1}$ is the Green's function for Jacobi vector fields.

18 Even more semiclassics

To summarize, the asymptotic expansion of an oscillatory integral in finite dimensions is

$$\int_{\mathbb{R}^N} e^{i\frac{A(x)}{\hbar}} f(x)\,dx \cong \sum_{x_0\in C_A} \frac{e^{i\frac{A(x_0)}{\hbar}}(2\pi\hbar)^{\frac{N}{2}}}{\sqrt{|\det(A^{(2)}(x_0))|}} e^{i\frac{\pi}{4}\text{sgn}(A^{(2)}(x_0))}\left(1+\sum_\Gamma \frac{(i\hbar)^{-\chi(\Gamma)+1}F(\Gamma)}{|\text{Aut}(\Gamma)|}\right) \tag{212}$$

where

1. A is smooth with finitely many critical points C_A, all of which are nondegenerate,

2. $A^{(2)}(x_0)$ is the second term in the Taylor series expansion of A about a critical point x_0,

3. $f(x)$ is analytic, rapidly decaying, and equal to 1 in a neighborhood of the critical points of A,

4. sgn denotes the signature,

5. Γ is a Feynman diagram.

A Feynman diagram is a graph with two kinds of vertices: a filled vertex has valence at least 3 and a blank vertex (there is exactly one such vertex) has any valence. The contribution coming from a given Feynman diagram is

$$F(\Gamma) = \sum_i \prod_{\text{edges}} (A^{(2)})^{i_e j_e} \prod_{\text{filled vertices } v} \frac{\partial^n A(x_0)}{\partial x^{\{i_v\}}} \frac{\partial^k f(x_0)}{\partial x^i}. \tag{213}$$

Abstractly one should think of F as a functor from graphs to vector spaces. In a more pedestrian manner, F is a sum over *states* of weights, where a state is an assignment of indices to both ends of each edge, the weight of a state is a product over weights assigned to vertices and edges, and

1. The weight of a filled vertex with adjacent indices $i_1, ...i_n$ is $\partial_{i_1}...\partial_{i_n} A(x_0)$,

2. The weight of an edge with adjacent indices i, j is $(A^{(2)})^{ij}$,

3. The weight of a blank vertex with adjacent indices $i_1, ...i_n$ is $\partial_{i_1}...\partial_{i_n} f(x_0)$.

The terms of order $\hbar$ come from Feynman diagrams of Euler characteristic 0. There are four such diagrams with automorphism groups of size $2^3, 3!2, 2^3, 2$. Trying to sum the contributions from Feynman diagrams of higher order gives a more accurate asymptotic expansion but in a smaller interval; in practice, the first-order or zeroth-order terms suffice.

19 More about Feynman diagrams

Now suppose that A and f above also depend on a time parameter t. Write

$$Z(t) = \int_{V_1} e^{\frac{i}{\hbar}A(x_1,t)} f(x_1,t)\,dx_1 \approx_{\hbar\to 0} \mathrm{Feynman}_1(t) \tag{214}$$

where $\mathrm{Feynman}_1(t)$ resembles the above expansion but with an explicit time dependence. Consider an integral of the form

$$\int_W \left(\int_{V_1} e^{\frac{i}{\hbar}A_1(x_1,t)} f_1(x_1,t)\,dx_1\right)\left(\int_{V_2} e^{\frac{i}{\hbar}A_2(x_2,t)} f_2(x_2,t)\,dx_2\right) g(t) e^{\frac{i}{\hbar}A_0(t)}\,dt \tag{215}$$

which by Fubini's theorem is

$$\int_{V_1\oplus V_2\oplus W} e^{i\frac{A(x)}{\hbar}} f(x)\,dx \tag{216}$$

where

1. $x = (x_1, x_2, t)$,
2. $dx = dx_1\,dx_2\,dt$,
3. $f(x) = f_1(x_1,t) f_2(x_2,t) g(t)$,
4. $A(x) = A_1(x_1,t) + A_2(x_2,t) + A_0(t)$.

On the other hand, we can apply the semiclassical expansion $\hbar \to 0$ to both the first integrand above and the second integrand above. The first integrand is a certain sum over three types of Feynman diagrams while the second integrand is another sum over Feynman diagrams. This gives a somewhat complicated identity involving Feynman diagrams which looks schematically like

$$\mathrm{Feyn}_1 \star \mathrm{Feyn}_2 \star \mathrm{Feyn}_0 = \mathrm{Feyn}_{\mathrm{tot}} \tag{217}$$

where $\star$ is defined in terms of the semiclassical expansion.

A remark about critical points. We assume that for fixed t all of the above functionals have isolated critical points. Let $(x_1(t_0), x_2(t_0), t_0)$ be a critical point of A. Then we can find functions $x_1(t), x_2(t)$ in a neighborhood of t of t_0 extending

$x_1(t_0), x_2(t_0)$ such that $x_1(t)$ is a critical point of $A_1(-,t)$ and $x_2(t)$ is a critical point of $A_2(-,t)$. Moreover, t_0 is a critical point of $A(x_1(t), x_2(t), t)$. This is how we can compare the above semiclassical expansions.

Exercise 19.1. *Show that the determinants appearing in the denominators of the semiclassical expansions above match. More precisely,*

$$\det(\partial_{x_1}^2 A_1(x_1(t),t)) \det(\partial_{x_2}^2 A_2(x_2(t),t)) \det\left(\frac{\partial^2}{\partial t^2}\left(A_1(x_1(t),t) + A_2(x_2(t),t) + A_0(t)\right)\right) \tag{218}$$

is equal to

$$\det \begin{bmatrix} \partial_{x_1}^2 A_1(x_1,t) & 0 & \partial_{x_1}\partial_t A_1(x_1,t) \\ 0 & \partial_{x_2}^2 A_2(x_2,t) & \partial_{x_2}\partial_t A_2(x,t) \\ \partial_{x_1}\partial_t A_1(x_1,t) & \partial_{x_2}\partial_t A_2(x_2,t) & \partial_t^2(A_1(x_1,t) + A_2(x_2,t) + A_0(t)) \end{bmatrix}. \tag{219}$$

Integration led to an interesting fact about Feynman diagrams. Now we will differentiate. With

$$Z(t) = \int_V e^{\frac{i}{\hbar}A(x,t)} f(x,t)\, dx \tag{220}$$

we have

$$\frac{\partial Z}{\partial t} = \int_V e^{\frac{i}{\hbar}A(x,t)} \left(\frac{i}{\hbar}\partial_t A(x,t) f + \partial_t f\right) dx. \tag{221}$$

Writing the term in parentheses as $\tilde{f}$ we get an identity describing derivatives of a sum over Feynman diagrams as a different sum over Feynman diagrams; schematically,

$$\frac{\partial}{\partial t}\mathrm{Feyn}(f) = \mathrm{Feyn}(\tilde{f}). \tag{222}$$

20 Quantum mechanics via Feynman diagrams

Recall that heuristically we saw that the propagator could be written as a Feynman integral

$$U_t(q,q') = \int_{\gamma:q'\to q} e^{\frac{i}{\hbar}S[\gamma]}\,D\gamma. \tag{223}$$

Unfortunately, we don't know how to define this integral, so we don't know how to write down an asymptotic expansion of it. However, assuming formally that it resembles the asymptotic expansion of the finite-dimensional integrals above, one term should be a (zeta-regularized) determinant of a matrix of second partial derivatives. To understand what this is we should compute the second variation

$$\begin{aligned}
\delta^2 S[\gamma] &= \delta^2\left(\int_0^t \left(\frac{m}{2}\dot\gamma(\tau) - V(\gamma(\tau))\right)d\tau\right) && (224)\\
&= \int_0^t \left(m\delta\dot\gamma\delta\dot\gamma - \frac{\partial^2 V}{\partial q^2}(\gamma(\tau))\delta\gamma(\tau)^2\right)d\tau && (225)\\
&= \int_0^t \left(-m\frac{d^2}{d\tau^2} - \frac{\partial^2 V}{\partial q^2}(\gamma(\tau))\right)\delta\gamma(\tau)\delta\gamma(\tau)\,d\tau && (226)
\end{aligned}$$

hence we should consider as a substitute for the matrix of second partial derivatives the operator

$$S_t^{(2)} = -m\frac{d^2}{d\tau^2} - \frac{\partial^2 V}{\partial q^2}(\gamma(\tau)). \tag{227}$$

We also need to compute the signature of $S^{(2)}$. This is the number of positive eigenvalues minus the number of negative eigenvalues, or the number of eigenvalues minus twice the number of negative eigenvalues, and the latter is finite so we will use it instead.

Finally we need to decide on Feynman rules. Writing the action as

$$S[\gamma_c+\delta\gamma] = S[\gamma_c] + \frac{1}{2}(S^{(2)},\delta\gamma^2) + \sum_{n\ge 3}\int_0^t -\frac{\partial^n V}{\partial q^n}(\gamma_c(\tau))(\delta\gamma(\tau))^2\,d\tau \tag{228}$$

we will think of the first two terms as a Gaussian integral and then compute the contribution of the remaining terms using Wick's theorem.

With boundary conditions $\delta\gamma(0)=0$ we will assume that there is a Green's function $G=(S^{(2)})^{-1}$. This is a function of two variables τ,σ such that $S_\tau^{(2)}(G) =$

$\delta(\tau - \sigma)$. Our Feynman rules assign to an edge with labels τ_1, τ_2 the value $G(\tau_1, \tau_2)$ and assigns to a vertex with labels $\tau_1, ...\tau_n$ the value

$$\delta(\tau_1 = \tau_2 = ... = \tau_n = \tau) \frac{\partial^n V(\gamma_c(\tau))}{\partial q^n(\tau)}. \tag{229}$$

We therefore assign to a Feynman diagram a product of distributions, which turns out to be a distribution and can therefore be integrated, and this is the corresponding value.

This gives, formally,

$$U_t(q, q') = \frac{C e^{i \frac{\pi}{2} \operatorname{neg}(S_t^{(2)})}}{\sqrt{\det'_\zeta(S_t^{(2)}(\gamma_c))}} e^{\frac{i}{\hbar} S_t[\gamma_c]} (1 + \text{Feynman diagrams}) \tag{230}$$

where γ_c is the classical path (which we will assume is unique).

A warning about working in coordinates. If $f : M \to \mathbb{R}$ is a smooth function, then in local coordinates we have a Taylor series expansion

$$f(x) \approx f(0) + \frac{\partial f}{\partial x^i}(0) x^i + \frac{1}{2} \frac{\partial^2 f}{\partial x^i \partial x^j} x^i x^j + ... \tag{231}$$

and the first term transforms like a tensor (a section of the cotangent bundle) but the second term does not. However, if x is a critical point, then the second partial derivatives exist as a tensor. Thus the contractions above in our Feynman rules are not guaranteed to behave nicely under change of coordinates. Morally speaking this is because we need to insert some infinite-dimensional Jacobian determinant when we change coordinates.

21 The framework on quantum field theory

Now that we have discussed classical field theory and quantum mechanics, we will discuss quantum field theory. This is closely related to statistical mechanics through statistical field theory.

Consider a spacetime category whose objects are $(n-1)$-dimensional spaces (manifolds) and whose morphisms are n-dimensional spacetimes (cobordisms of manifolds between them). We require that everything is oriented. An n-dimensional quantum field theory assigns

1. Vector spaces $H(N)$ (spaces of boundary states) to objects (spaces) and
2. Vectors $Z_M \in H(\partial M)$ (partition functions) to morphisms (spacetimes).

These assignments should satisfy various axioms (where $\bar{N}$ is the manifold N with the opposite orientation):

1. Locality of the space of states: $H(\emptyset) = \mathbb{C}$, $H(N_1 \sqcup N_2) = H(N_1) \otimes H(N_2)$.
2. For each N there is a nondegenerate pairing $\langle \cdot, \cdot \rangle_N : H(N) \otimes H(\bar{N}) \to \mathbb{C}$ such that $\langle \cdot, \cdot \rangle_{N_1 \sqcup N_2} = \langle \cdot, \cdot \rangle_{N_1} \otimes \langle \cdot, \cdot \rangle_{N_2}$.
3. The orientation-reversing map $\sigma : N \to \bar{N}$ induces a $\mathbb{C}$-antilinear isomorphism $\hat{\sigma}_N : H(N) \to H(\bar{N})$ such that $\hat{\sigma}_{N_1 \sqcup N_2} = \hat{\sigma}_{N_1} \otimes \hat{\sigma}_{N_2}$ and $\hat{\sigma}_{\bar{N}} \hat{\sigma}_N = \mathrm{id}_N$.
4. Any orientation-preserving isomorphism $f : N_1 \to N_2$ lifts to an isomorphism $T_f : H(N_1) \to H(N_2)$ such that $T_{f \sqcup g} = T_f \otimes T_g$ and $T_{f \circ g} = T_f T_g$.
5. First locality property of the partition function: $Z_{M_1 \sqcup M_2} = Z_{M_1} \otimes Z_{M_2}$.
6. Second locality property of the partition function: suppose that $\partial M = N \sqcup \bar{N} \sqcup N'$. Then

$$(\langle \cdot, \cdot \rangle_N \otimes \mathrm{id}_{N'}) Z_M = Z_{M_N} \tag{232}$$

   where M_N is the manifold M with the two copies of N glued.
7. Gauge invariance: if $f : M_1 \to M_2$ is an isomorphism of spacetimes, it induces an isomorphism $f_\partial : \partial M_1 \to \partial M_2$. Then

$$T_{f_\partial}(Z_{M_1}) = c_{M_1}(f) Z_{M_2} \tag{233}$$

   where $c_{M_1}(f)$ is some cocycle.

We may relax any of the identities above to hold only projectively. The theory is *unitary* if $\hat{\sigma}_N$ induces a positive-definite inner product on $H(N)$.

Example Consider the spacetime category whose objects are finite ordered sets $[n] = \{1, 2, ...n\}$ and whose morphisms $[n] \to [m]$ are directed graphs Γ with vertex set containing $[n]$ and $[m]$. We define

$$H([n]) = (\mathbb{C}^2)^{\otimes n} \tag{234}$$

and write

$$Z_\Gamma = \sum_\sigma Z_\Gamma(\sigma) e_{\sigma_1} \otimes ... \otimes e_{\sigma_N} \in (\mathbb{C}^2)^{\otimes N} \tag{235}$$

where $\partial\Gamma$ has N elements and we have chosen a basis e_+, e_- of $\mathbb{C}^2$. The coefficient is

$$Z_\Gamma(\sigma) = \sum_{\tau : V(\Gamma) \to \pm 1} \prod_{e \in E(\Gamma)} \exp\left(-J_e \tau_{e^+} \tau_{e^-}\right) \prod_{e \in \partial\Gamma} \exp\left(-J_e \tau_{e^+} \sigma_{e^-}\right) \tag{236}$$

where $J : E(\Gamma) \to \mathbb{R}$ is a set of bonding energies (provided as additional data in the graph Γ) and e is an edge from e^+ to e^-. This is the Ising model on Γ.

In statistical mechanics, we fix a boundary state $\psi \in H(\partial\Gamma)$ and consider the inner product $\langle Z_\Gamma, \psi \rangle$. In fact, usually we are interested in a sequence $\psi_n \in H(\partial\Gamma_n)$ of such states on a sqeuence of graphs and we are interested in the asymptotics of $\langle Z_{\Gamma_n}, \psi_n \rangle$ as $n \to \infty$. For example Γ_n might be a sequence of grids, possibly approximating a domain in $\mathbb{R}^2$ by a given lattice spacing ϵ which goes to 0 (the thermodynamic limit).

Certain kinds of gluing are not permitted in this setting; for more general kinds of gluing we need to replace manifolds with boundary with manifolds with corners. This requires replacing categories with higher categories.

Example Consider the spacetime category whose objects are cell approximations to $(n-1)$-dimensional manifolds and whose morphisms are cell approximations to n-dimensional manifolds with boundary. (More formally we consider triples (C, N, φ) where N is a manifold, C is a cell complex, and $\varphi : C \to N$ is an embedding.)

We can define an Ising model here. To do so, we need an extra structure on spacetimes, namely a function $J : E(C) \to \mathbb{R}$ on 1-cells. We define

$$H(\partial M) = (\mathbb{C}^2)^{\otimes |V(\partial M)|} \tag{237}$$

where $V(\partial M)$ is the set of 0-cells of the boundary. This has a basis $e_{\sigma_1} \otimes ... \otimes e_{\sigma_N}$ where $N = |V(\partial M)|$ and $\sigma_i \in \{\pm 1\}$ parameterizes a basis of $\mathbb{C}^2$. We define

$$Z(M) = \sum_{\sigma} Z(M)_{\sigma} e_{\sigma_1} \otimes ... \otimes e_{\sigma_N} \tag{238}$$

where

$$Z(M)_{\sigma} = \sum_{\tau : V(M) \to \{\pm 1\}} \prod_{e \in E(M)_{\text{int}}} \exp\left(-J_e \tau_{e_+} \tau_{e_-}\right) \prod_{e \text{ connecting int. v. with boundary v.}} \exp\left(-J_e \sigma_{e_+} \tau_{e_-}\right). \tag{239}$$

The scalar product on $H(\partial M)$ is given by

$$\langle u, v \rangle = \sum_{\sigma} \overline{u(\sigma)} v(\sigma) \prod_{e \in \partial M} \exp\left(-J_e \sigma_{e_+} \sigma_{e_-}\right). \tag{240}$$

In this setting we can talk about sequences $(C^{(n)}, N, \varphi_n)$ of cell approximations to a manifold N and take the limit as $n \to \infty$ (the continuum limit).

The above definition makes no use of the embedding φ_n. One way to recover some kind of geometry is to consider how the random variables in some region are correlated to the random variables in some other region. These correlations may decay exponentially with distance or, at certain points, may decay polynomially; the latter are phase transitions. This provides a notion of length. If we take the scaling limit where $n \to \infty$ and in addition $J \to J_c$ (critical), then we expct to recover some kind of continuum field theory.

A wide-open conjecture asserts that discretizing Yang-Mills and taking the scaling limit gives a theory with mass.

Other Ising-type models include dimer models (where states are on edges) and IRF models (where states are on 2-cells). It is also interesting to introduce a discretization of the metric.

Let M be a Riemannian manifold and let $C_n^{\bullet}$ be a sequence of simplicial cochains with respect to better and better smooth triangulations of M. There is a natural map $R : \Omega^{\bullet} \to C_n^{\bullet}$ given by integrating a form against a chain, and there is also a map in the other direction $W : C_n^{\bullet} \to \Omega^{\bullet}$, the Whitney map. This induces a scalar product on $C_n^{\bullet}$ given by

$$(u, v)_W = \int_M W(u) \wedge *W(v) \tag{241}$$

which induces a notion of Hodge laplacian $\Delta_{C_n} = d_{C_n} d^*_{C_n} + d^*_{C_n} d_{C_n}$. Patodi and a coauthor showed that as $n \to \infty$ this converges in a suitable sense to the Hodge Laplacian on $\Omega^\bullet$.

Quantum mechanics may also be regarded as a quantum field theory.

Example In quantum mechanics as a QFT, spacetime has objects which are real numbers and morphisms which are segments $[t_1, t_2]$ with flat Riemannian metric dt. Quantum mechanics on a Riemannian manifold N is governed by a Hamiltonian of the form

$$\hat{H} = -\frac{\hbar^2 \Delta}{2m} + V(q) \tag{242}$$

acting on $L^2(N)$, and time evolution

$$U_t = \exp\left(\frac{i}{\hbar} \hat{H} t\right) \tag{243}$$

sends states at a given time t_1 to a new time $t_1 + t$. The corresponding quantum field theory assigns to the boundary ∂I of an interval the completed tensor product $L^2(N) \hat{\otimes} L^2(N)$ and assigns to an interval the operator $U_{t_2 - t_1} \in H(\partial I)$.

The gluing axiom in quantum mechanics can also be recovered formally from the path integral picture. (Recall that the path integral doesn't exist, but we can write down some formal object which ought to be its stationary phase approximation.) The gluing axiom here states that

$$\int_N U_t(q, q') U_s(q', q'') \, dq' = U_{s+t}(q, q'') \tag{244}$$

and substituting the path integral, the above formally is the statement that an integral over paths from q to q'' starting at some time t_1 and ending at some time t_2 can be split up based on the location q' of the path at an intermediate time t_2.

22 A brief escapade through standard quantum field theory

Let $M = [t_1, t_2] \times B$ be a cylinder (where B is n-dimensional). Assume for simplicity that $B = T^n$ (thought of as a box in $\mathbb{R}^n$ with periodic boundary conditions). We will consider N paticles on B, so

$$H_N = L^2(B^N) \tag{245}$$

is the Hilbert space of states. The Hamiltonian is

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N} \Delta_i + \sum_{i<j} V(x_i - x_j) \tag{246}$$

which describes N particles of equal mass m. The space of states has two special summands coming from symmetric and antsymmetric tensors; we will call these $H_N^{(B)}$ (bosonic or Bose statistics) and $H_N^{(F)}$ (fermionic or Fermi statistics). From here we can define ferionic Fock space

$$H^{(F)} = \bigoplus_{N \geq 0} H_N^{(F)} \tag{247}$$

and bosonic Fock space

$$H^{(B)} = \bigoplus_{N \geq 0} H_N^{(B)}. \tag{248}$$

(We should complete these spaces to get Hilbert spaces, but this is an analytic detail.)

Let's first work with fermionic Fock space. If $f \in L^2(B)$, then we define operators $\psi(f) : H_N \to H_{N-1}, \psi^\dagger(f) : H_N \to H_{N+1}$ given by

$$(\psi(f)\varphi_N)(x_1, ...x_{N-1}) = \int_{\mathbb{R}} \bar{f}(x)\varphi(x, x_1, ...x_{N-1}) \tag{249}$$

and

$$(\psi^\dagger(f)\varphi_N)(x_1, ...x_{N+1}) = f(x_1)\varphi(x_2, ...x_{N+1}) - f(x_2)\varphi(x_1, x_3, ...x_{N+1}) \pm .... \tag{250}$$

These operators satisfy

$$\psi(f)\psi(g)+\psi(g)\psi(f)=0 \tag{251}$$

$$\psi^\dagger(f)\psi^\dagger(g)+\psi^\dagger(g)\psi^\dagger(f)=0 \tag{252}$$

$$\psi(f)\psi^\dagger(g)+\psi(g)\psi^\dagger(f)=\langle f,g\rangle \tag{253}$$

and hence may be thought of as generating the Clifford algebra $\mathrm{Cl}(L^2(B))$. From here we can define creation and annihilation operators as operator-valued distributions $\psi(x), \psi^\dagger(x)$ given by

$$\psi(f)=\int_B \psi(x)\overline{f}(x)\,dx \tag{254}$$

$$\psi^\dagger(f)=\int_B \psi^\dagger(x)f(x)\,dx. \tag{255}$$

These satisfy identities coresponding to the identities above, with the last identity being

$$\psi(x)\psi^\dagger(y)+\psi^\dagger(y)\psi(x)=\delta(x-y). \tag{256}$$

Proposition 22.1. *The action of the Hamiltonian on $H_N^{(F)}$ may be identified with the action of*

$$\hat{H}=-\frac{\hbar^2}{2m}\int_B \psi^\dagger(x)\Delta_x\psi(x)\,dx+\frac{1}{2}\iint_{B\times B}\psi^\dagger(x)\psi^\dagger(y)V(x-y)\psi(x)\psi(y)\,dx\,dy \tag{257}$$

on $H^{(F)}$.

Creation and annihilation can also be defined for bosonic Fock space with anticommutators replaced by commutators; the corresponding algebra is not a Clifford algebra but a Heisenberg algebra $\mathrm{Heis}(L^2(B))$.

The above is the nonlinear Schrödinger QFT.

We know how to take the semiclassical limit $\hbar \to 0$ for fixed N; as before we consider wave functions of the form $\psi(x)=e^{i\frac{f(x)}{\hbar}}\varphi(x)$. A different semiclassical limit produces the classical nonlinear Schrödinger field theory; we do this by considering semiclassical states φ_N where $N \to \infty$. These will have the property that

$$\langle \varphi_N, \psi(x,t)\varphi_N \rangle \to \psi_{\mathrm{cl}}(x,t) \tag{258}$$

as $\hbar \to 0, N \to \infty$, where

$$\psi(x,t) = \exp\left(\frac{i}{\hbar}\hat{H}t\right)\psi(x)\exp\left(-\frac{i}{\hbar}\hat{H}t\right) \tag{259}$$

and $\psi_{\mathrm{cl}}(x,t)$ is a solution to a certain differential equation. To describe this differential equation we should consider a manifold with coordinate functions $\psi_{\mathrm{cl}}(x), \overline{\psi}_{\mathrm{cl}}(x)$ with Poisson brackets imitating the commutation relations above. Taking the classical limit of $\hat{H}$ gives the nonlinear Schrödinger equation

$$i\frac{\partial \psi_{\mathrm{cl}}(x)}{\partial t} = \frac{1}{2m}\Delta\psi_{\mathrm{cl}}(x) + \int_B V(x-y)|\psi_{\mathrm{cl}}(y)|^2\,dy\psi_{\mathrm{cl}}(x) \tag{260}$$

which is formally a Hamiltonian differential equation with Hamiltonian

$$-\frac{1}{2m}\int_B \overline{\psi}_{\mathrm{cl}}(x)\Delta\psi_{\mathrm{cl}}(x)\,dx + \frac{1}{2}\iint_{B\times B} |\psi_{\mathrm{cl}}(x)|^2|\psi_{\mathrm{cl}}(y)|^2 V(x-y)\,dx\,dy. \tag{261}$$

We constructed two QFTs (bosonic and fermionic) for $I \times B$, where

$$H(B) = \mathrm{End}(H^{(F),(B)}) \tag{262}$$

and

$$Z(I\times B) = \exp\left(\frac{i}{\hbar}(t_1 - t_2)\hat{H}\right) \in H(B). \tag{263}$$

An interesting observable here is

$$\hat{N} = \int_{B\times I} \psi^*(x)\psi(x)\,dx \tag{264}$$

which acts on N-particle states by multiplication by N. Another is the family of momentum operators

$$\hat{P}_a = \int_B \psi^* \left(-i\hbar\frac{\partial}{\partial x^a}\right)\psi\, dx \tag{265}$$

which act on N-particle states by

$$(\hat{P}_a\varphi)(x_1, ...x_N) = \sum_{j=1}^{N}\left(-i\hbar\frac{\partial}{\partial x_j^a}\right)\varphi(x_1, ...x_N). \tag{266}$$

This QFT is nonrelativistic. It is possible to construct a relativistic version of the theory with no interaction.

Rezakhanlou here at Berkeley is an expert on a certain limit of this field theory, which can be defined as follows. We define a family of states ρ_ε and observables

$$\mathcal{O} = \int D(x_1, ...x_n|y_1, ...y_n)\psi(x_1)^*...\psi(x_n)^*\psi(y_1)...\psi(y_n)\, dx\ , dy \tag{267}$$

where D is a differential operator, and we require that

$$\lim_{\varepsilon\to 0}\varepsilon^n\rho_\varepsilon(\mathcal{O}) = \int D(x_1, ...x_n|y_1, ...y_n)\overline{\psi}_c(x_1)...\overline{\psi}_c(x_n)\psi_c(y_1)...\psi_c(y_n)\, dx\, dy \tag{268}$$

$$\varepsilon^n\rho_\varepsilon(\mathcal{O}\hat{N}) \approx \mathcal{O}(\psi_c)\frac{1}{\varepsilon}\int_B |\psi_c|^2\, dx \tag{269}$$

$$\varepsilon^n\rho_\varepsilon(\mathcal{O}\hat{P}_a) \approx \mathcal{O}(\psi_c)\frac{1}{\varepsilon}\int_B \hat{\psi}_c\left(-i\hbar\frac{\partial}{\partial x^a}\right)\psi_c\, dx \tag{270}$$

$$\varepsilon^n\rho_\varepsilon(\hat{H}\mathcal{O}) = O(\psi_c)\left(\frac{1}{\varepsilon}\int_B \hat{\psi}_c\left(-\frac{\hbar^2}{2m}\right)\psi_c\, dx + \frac{1}{\varepsilon^2}\frac{1}{2}\iint_{B\times B} V(x-y)|\psi_c(x)|^2|\psi_c(y)|^2\, dx\, dy\right) \tag{271}$$

for some $\psi_c : B \to \mathbb{C}$. It is not obvious that such states exist. Roughly speaking ε should be one over the number of total particles, which we send to infinity. We should assume that $V(x) = \varepsilon U(x)$ for some U so that the above is homogeneous in ε. Then the Hamiltonian becomes

$$H_c = \int_B \overline{\psi}_c \left(-\frac{\hbar^2}{2m} \Delta \right) \psi_c + \frac{1}{2} \iint_{B \times B} U(x-y) |\psi_c(x)|^2 |\psi_c(y)|^2 \, dx \, dy. \tag{272}$$

Time evolution of states looks like

$$i\hbar \frac{\partial \psi_c}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi_c(x,t) + \left(\int_B U(x-y) |\psi_c(x,t)|^2 \, dy \right) \psi_c(x,t). \tag{273}$$

In the quantum case, the commutation relations between the observables we wrote down above include

$$[\hat{H}, \hat{N}] = [\hat{H}, \hat{P}_a] = [\hat{P}_a, \hat{N}] = 0. \tag{274}$$

We also had the relations

$$\psi(x)\psi^\dagger(y) - \psi^\dagger(y)\psi(x) = \delta(x-y). \tag{275}$$

As $\varepsilon \to 0$ we have $\sqrt{\varepsilon}\psi \to \psi_c$ in an appropriate sense. Dividing by suitable factors of ε the commutator becomes a Poisson bracket with respect to which

$$\{H_c, N_c\} = \{H_c, P_{a,c}\} = \{N_c, P_{a,c}\} = 0 \tag{276}$$

and

$$\{\psi_c(x), \overline{psi}_c(y)\} = \delta(x-y). \tag{277}$$

This is like a semiclassical limit except that we are taking the total number of particles to infinity rather than taking $\hbar$ to 0. We should think of this situation as a quantum-mechanical gas; the energies involved are high enough that the system behaves classically in that it is described by an effective classical field theory.

Write $\rho(x) = |\psi_c(x)|^2$ (the density of the gas) and write

$$\vec{j}(x,t) = \hat{\psi}_c(x,t) \left(-i\hbar \frac{\vec{\partial}}{\partial x} \right) \psi_c(x,t) \tag{278}$$

(the transfer current). Then

$$\frac{\partial \rho}{\partial t} = -\frac{1}{2m}\vec{\nabla}\vec{j}. \tag{279}$$

This is equivalent to the conservation law

$$\frac{\partial}{\partial t}\int_U \rho\, dV = -\frac{1}{2m}\int_{\partial U} \vec{j}d\vec{S}. \tag{280}$$

We can also write $\psi_c = \sqrt{\rho}e^{i\frac{\theta}{2\hbar}}$. Then

$$\partial_t \theta = \frac{1}{4m}(\vec{\nabla}\theta)^2 + \frac{\hbar^2}{2m}\Delta\rho - \int_B U(x-y)\rho(y)\, dy. \tag{281}$$

What we have been doing can be summarized in the commutative diagram

$$\begin{tikzcd} \mathrm{QM}_N \arrow[r, "N \text{ varies}"] \arrow[d, "\hbar\to 0"] & \mathrm{QFT} \arrow[r, "N\to\infty"] & \mathrm{EFT} \arrow[d, "\hbar\to 0"] \\ \mathrm{CM}_N \arrow[r, "N \text{ varies}"] & \sqcup_N \mathrm{CM}_N \arrow[r, "N\to\infty"] & ? \end{tikzcd} \tag{282}$$

where ? is the hydrodynamical limit, which we have not described yet.

Exercise 22.2. *We formulated the nonlinear Schrödinger equation as an infinite-dimensional Hamiltonian problem. Reformulate it using a Lagrangian.*

Consider a collection of N classical particles satisfying the equations of motion

$$\frac{m\ddot{\vec{x}}_i(t)}{2} = \sum_{j\neq i}\vec{\nabla}V(\vec{x}_i(t) - \vec{x}_j(t)). \tag{283}$$

In the limit as $N \to \infty$, we should consider the sequence of densities

$$\rho(x,t) = \frac{1}{N}\sum_{i=1}^{N}\delta(\vec{x} - \vec{x}_i(t)) \tag{284}$$

and currents

$$\vec{j}(x,t) = \frac{1}{N}\sum_{i=1}^{N}\dot{\vec{x}}_i(t)\delta(\vec{x} - \vec{x}_i(t)) \tag{285}$$

which have continuum limits $\rho_c(x,t), \vec{j}_c(x,t)$, and these limits should satisfy infinitely many differential identities. Looking at the above commutative diagram, we should be able to obtain the same setup by taking the limit as $\hbar \to 0$ in the effective field theory we described above (nonlinear Schrödinger).

23 Integrability

Consider a system of N bosons in $\mathbb{R}$ which do not interact unless they collide. The Hilbert space of states here is the space of symmetric functions in $L^2(\mathbb{R}^N)$ and the Hamiltonian is

$$H_N = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\partial_i^2 + \kappa\sum_{i<j}\delta(x_i - x_j). \tag{286}$$

By a suitable rescaling we may assume that $\frac{\hbar^2}{m} = 1$. We make sense of the above Hamiltonian as follows. If $x_i \neq x_j$, we have

$$(H_N\varphi)(x_1, ...x_N) = -\frac{1}{2}\sum_{i=1}^{N}\partial_i^2\varphi(x_1, ...x_N). \tag{287}$$

If some $x_i = x_j$, then H_N is defined on the subspace of functions continuous on the diagonal $x_i = x_j$ and satisfying the condition

$$-\frac{1}{2}\left(\partial_i\varphi - \partial_j\varphi\right)|_{x_i=x_j-0}^{x_i=x_j+0} - \kappa\varphi|_{x_i=x_j} = 0. \tag{288}$$

On such functions it is given by the same formula as above.

Theorem 23.1. *(Lieb, Liniger) A function (not necessarily square-integrable)*

$$\psi(x_1, ...x_N|k_1, ...k_N) = \sum_{\sigma\in S_N} A(\sigma)e^{i\sum_{j=1}^{N}k_{\sigma_j}x_j} \tag{289}$$

(where $x_1 > x_2 > ... > x_N$, and we extend to the remaining cases by symmetry) satisfies the equation

$$H_N\psi(x|k) = \frac{1}{2}\sum_{i=1}^{N}k_i^2\psi(x|k) \tag{290}$$

if for every simple transposition s_j we have

$$A(s_j\sigma) = \frac{i(k_{\sigma_j} - k_{\sigma_{j+1}}) + \kappa}{-i(k_{\sigma_j} - k_{\sigma_{j+1}}) + \kappa} A(\sigma). \tag{291}$$

(Project idea! Carefully prove this.)

This system is integrable in a suitable sense.

Theorem 23.2. *For each $n = 1, 2, 3, \ldots$ there exists a differential operator $D^{(n)}$ supported on the diagonals of $\mathbb{R}^N$ such that*

$$H_n = \frac{(-i)^n}{n!} \sum_{j=1}^{n} \partial_j^n + D^{(n)} \tag{292}$$

$$H_1 = -i \sum_{j=1}^{n} \partial_j \tag{293}$$

$$H_2 = -\frac{1}{2} \sum_{j=1}^{N} \partial_j^2 + \kappa \sum_{i<j} \delta(x_i - x_j) \tag{294}$$

and $[H_n, H_m] = 0$. Moreover,

$$H_n \psi(x|k) = \frac{1}{n!} \sum_{j=1}^{n} k_j \psi(x|k). \tag{295}$$

This gives a classical integrable system as $\hbar \to 0$ as well as if $N \to \infty$. (Project idea! Construct the H_n by hand.)

Consider in particular the case $N = 2$, where

$$\psi(x_1, x_2|k_2, k_2) = A\left(e^{ik_1x_1 + ik_2x_2} + \frac{i(k_1 - k_2) + \kappa}{-i(k_1 - k_2) + \kappa} e^{ik_2x_1 + ik_1x_2}\right) \tag{296}$$

(when $x_1 > x_2$). If k_1, k_2 are complex conjugates, then writing $k_1 = \frac{k}{2} + ic,_2 = \frac{k}{2} - ic$, the above becomes

$$A\left(e^{ik(x_1+x_2)} e^{c(x_1-x_2)} + (-2c + \kappa) e^{ik(x_1+x_2)} e^{-c(x_1-x_2)}\right) \tag{297}$$

(when $x_1 > x_2$). If $c = -\frac{\kappa}{2}$ then the above becomes

$$A\left(e^{ik(x_1+x_2)} e^{\frac{\kappa}{2}|x_1-x_2|}\right). \tag{298}$$

If $\kappa < 0$, then this describes a bound state eigenfunction. This construction generalizes to k-bound states $\psi_{n_1,...n_k}$ where $N = n_1 + 2n_2 + .. + kn_k$ and the direct sum of all of these states gives a Fock space. (If $\kappa > 0$ then we use a Fourier transform.)

On this Fock space we can define creation and annihilation operators as usual, and in the semiclassical limit we have weak limits $\psi(x) \to \psi_c(x), \psi^\dagger(x) \to \overline{\psi}_c(x)$ and a Hamiltonian

$$\hat{H} = -\frac{1}{2}\int_{\mathbb{R}} \psi^\dagger \partial_x^2 \psi + \frac{\kappa}{2}\int_{\mathbb{R}} (\psi^\dagger)^2 \psi^2 \, dx \tag{299}$$

acting on Fock space. Time evolution gives in the semiclassical limit (which includes the limit $\kappa \to 0$) the nonlinear Schrödinger equation

$$i\frac{\partial \psi_c}{\partial t} = -\partial_x^2 \psi_c + |\psi_c|^2 \psi_c. \tag{300}$$

This is Hamiltonian flow generated by

$$H_c = -\frac{1}{2}\int_{\mathbb{R}} \overline{\psi}_c \partial_x^2 \psi_c + \frac{1}{2}\int_{\mathbb{R}} |\psi|^4 \, dx \tag{301}$$

with respect to the Poisson bracket given by $\{\psi_c(x), \psi_c^\dagger(y)\} = \delta(x-y)$ as before.

Theorem 23.3. *(Fateev, Schwartz) For $\kappa < 0$ there exists C_κ such that*

$$\lim_{\kappa \to 0, N \to \infty} C_\kappa \int_{\mathbb{R}^N} \psi_{N+1}(x, x_1, ...x_N|k,t) \overline{\psi}_{N+1}(x, x_1, ...x_N|k,t)) \, dx_1...dx_N \tag{302}$$

is a soliton solution $\psi_c(x|k,t)$ to the nonlinear Schrödinger equation.

Soliton solutions are very special: they behave like traveling waves, and when they collide, they do not change each other's shape. (Project idea! Talk about soliton solutions to the nonlinear Schrödinger equation.)

Consider now the same Hamiltonian as before, with the same constraints on diagonals as before, but with periodic boundary conditions in a box $0 \le x_i \le L$. Writing

$$\psi(x|k) = \sum_\sigma A(\sigma) \exp\left(i \sum_{j=1}^N k_{\sigma_j} x_j \right) \tag{303}$$

where $A(s_j \sigma)$ is determined from $A(\sigma)$ suitably, we have the following.

Theorem 23.4. *(Yang, Yang) $\psi(x|k)$ are eigenvectors of H if*

$$e^{ik_jL} = \prod_{\ell \neq j} \frac{k_j - k_\ell + i\kappa}{k_j - k_\ell - i\kappa}. \tag{304}$$

This describes the discrete spectrum. For sufficiently small $\kappa > 0$ this is the whole spectrum. We get an integrable system this way.

To explain what this means, let (M, ω) be a $2n$-dimensional symplectic manifold and $H \in C^\infty(M)$ be a Hamiltonian generating Hamiltonian dynamics $v_H = \omega^{-1}(dH)$.

Definition (M, ω, H) is *integrable* if there exist $I_1, ...I_n$ Poisson-commuting functions such that $\{I_j, H\} = 0$ (the I_j are *integrals of motion*) which are independent in the sense that $dI_1 \wedge ... \wedge dI_n$ is generically nonvanishing.

Geometrically, $I_1, ...I_n$ defines a map $I : M \to \mathbb{R}^n$, and the condition that the I_j Poisson-commute and are independent implies that the fibers of I are Lagrangian submanifolds. Moreover, if a trajectory begins in a given fiber, it stays in a given fiber.

Theorem 23.5. *On each fiber of I there exists an affine coordinate system $\varphi_1, ...\varphi_n$ such that $\varphi_i(t) = \varphi_i(0) + \omega_i(H)t$. Moreover, generically the fibers are diffeomorphic to $\mathbb{T}^\ell \times \mathbb{R}^{n-\ell}$.*

(Project idea! Discuss examples, possibly using Poisson-Lie groups.)

Example Let $M = \mathbb{R}^2$ and let $H = \frac{p^2+q^2}{2}$. Then $H = I_1$ is an integral of motion. The fibers of H are circles except the fiber $p = q = 0$, at which $dH = 0$.

These conditions do not imply that all orbits are periodic. These are the degenerate integrable systems.

Suppose now that M is an infinite-dimensional symplectic manifold. It is now less clear what it means to have $\frac{\dim M}{2}$ integrals of motion, but we can give the following practical definition.

Definition M is *integrable* if there are Poisson-commuting independent integrals of motion $I_1, I_2, ...$ and action-angle coordinates $\varphi_1, \varphi_2, ...$ on fibers such that $\varphi_i(t) = \varphi_i + \omega_i(H)t$ and $\{\varphi_i, I_j\} = \delta_{ij}$.

For example, let M be the space of smooth, complex-valued, rapidly decreasing functions on $\mathbb{R}$ with Poisson bracket

$$\{f, g\} = \int_{\mathbb{R}} D\psi(x) \wedge D\overline{\psi}(x)\, dx \tag{305}$$

and Hamiltonian

$$H = \frac{1}{2}\int_{\mathbb{R}} |\partial_x \psi|^2 \pm \int_{\mathbb{R}} |\psi|^4\, dx. \tag{306}$$

Theorem 23.6. *(Lax, Korteweg-de Vries) There exist local functionals*

$$I_n = \frac{(-i)^n}{n}\int_{\mathbb{R}} \overline{\psi}\partial_x^n \psi + \textit{local functionals of total degree at most } n \tag{307}$$

such that $\{I_n, I_m\} = 0, I_2 = H, I_1 = -i\int_{\mathbb{R}} \overline{\psi}\partial_x \psi\, dx$. *Furthermore, there exist angle coordinates on the level sets* φ_i *such that* $\{\varphi_i, I_j\} = \delta_{ij}$.

24 More general QFTs

The nonlinear Schrödinger QFT is not relativistic. It is possible to explicitly construct relativistic integrable QFT, e.g. Sine-Gordon, Gross-Neveu, principal chiral field theory, etc. The latter involves the representation theory of quantized universal enveloping algebras. There are also conformal field theories, which involve the representation theory of loop algebras.

A different flavor of QFT is Chern-Simons in 3 dimensions, which is topological. There is a semiclassical approach due to Witten and a combinatorial approach due to Reshetikhin and Turaev.

A general approach to writing down QFTs is to quantize classical gauge theories. If M is a spacetime with boundary ∂M and we consider fields of some kind, then there is a projection map $\pi : F_M \to F_{\partial M}$ from fields on M to boundary fields. There is also a classical action $S : F \to \mathbb{R}$ whose critical points are solutions to the Euler-Lagrange equations. Finally, we have the action of a gauge group G_M on F_M as well as the action of a boundary gauge group $G_{\partial M}$ on $F_{\partial M}$ and a restriction map $\pi : G_M \to G_{\partial M}$. For strong gauge invariance, S should be G_M-invariant.

Taking the differential of the projection map π gives a map $d\pi : T(F_M) \to T(F_{\partial M})$, giving short exact sequences

$$0 \to \mathrm{Ker}(d\pi_x) \to T_x(F_M) \to T_{\pi(x)}(F_{\partial M}) \to 0 \tag{308}$$

where $\mathrm{Ker}(d\pi) \subset T(F_M)$ is the space of leaves of the foliation induced by π.

We also have a 1-form $\alpha_{\partial M} \in \Omega^1(F_{\partial M})$ which should induce a splitting $dS = \text{bulk} + \pi^*(\alpha_{\partial M})$.

When doing quantum field theory in the presence of gauge symmetries, we need a Lagrangian fibration $p_{\partial M} : F_{\partial M} \to B_{\partial M}$, and then we would like to assign

$$H_{\partial M} = L^2(B_{\partial M}) \tag{309}$$

and

$$Z_M = \int_{(p_{\partial M} \circ \pi)^{-1}(b)} e^{i\frac{S(x)}{\hbar}} \frac{dx}{db} \in L^2(B_{\partial M}). \tag{310}$$

When the gauge group is trivial, we can formally define Z_M as a suitable sum over Feynman diagrams. This works fine as long as there are no ultraviolent divergences (e.g. in quantum mechanics) due to nondegeneracy of critical points. With nontrivial gauge group, it is no longer clear how to write down the sum over Feynman diagrams because the critical locus consists of orbits for the action of the gauge group rather than points.

To attack the general question of describing integrals of the form

$$Z_\hbar = \int_F e^{i\frac{S}{\hbar}}\, dx \tag{311}$$

where a group G acts on F, assume that S has finitely many isolated critical G-orbits and that G acts without stabilizers. Moreover, assume that F, G are compact. Then

$$Z_\hbar = |G| \int_{F/G} e^{i\frac{S[x]}{\hbar}}\, d[x] \tag{312}$$

where we integrate with respect to the pushforward $|G| d[x]$ of dx along the quotient map $\pi : F \to F/G$ (here $|G|$ is the measure of G with respect to some Haar measure). Let $s : F/G \to F$ be a section of π and let $d\sigma$ be the pushforward of $d[x]$ along s to

$S = s(F/G)$. So we want to compute

$$Z_\hbar = |G| \int_{S\subset F} e^{i\frac{S(\sigma)}{\hbar}}\, d\sigma \tag{313}$$

or more precisely its asymptotics as $\hbar \to 0$.

Theorem 24.1. *Let $S = \{\varphi(x) = 0\}$ where $\varphi : F \to \mathfrak{g}$ is a smooth function. Then*

$$Z_\hbar = |G| \int_F e^{i\frac{S(x)}{\hbar}} \det(L(x))\delta(\varphi(x))\, dx \tag{314}$$

where in coordinates

$$L_a^b(x) = \sum_i \ell_a^i(x)\partial_i\varphi^b(x) \tag{315}$$

where $\sum \ell_a^i(x)\partial_i$ is the vector field generated by a basis element $e_a \in \mathfrak{g}$ (we have fixed such a basis).

Proof. Our assumptions imply that $S \subset F$ intersects each G-orbit once. Let $\Delta(x)$ be such that

$$\Delta(x) \int_G \delta(\varphi(gx))\, dg = 1. \tag{316}$$

We compute this as follows. Writing $x_0 = Gx \cap S$, we have that the above integral can be rewritten as

$$\int_G \delta(\varphi(hx_0))\, dh \tag{317}$$

so that the distribution to be integrated is supported at the origin. Writing $h = e^{\sum t^a e_a}$ where the t^a are very small, we see that

$$\varphi^b(hx_0) = \varphi^b(x_0) + \sum_a \ell_a^i \partial_i \varphi^b(x_0) + O(|t|^2) \tag{318}$$

and so $\Delta(x)$ is the Jacobian $\det(L(x_0))$. We can therefore write

$$Z_\hbar = \int_F e^{i\frac{S(x)}{\hbar}} \det(L(x_0) \int_G \delta(\varphi(gx))\, dg\, dx \tag{319}$$

and change the order of integration to obtain

$$\int_G \left(\int_F e^{i\frac{S(x)}{\hbar}} \det(L(x_0))\delta(\varphi(gx))\,dx \right) dg. \tag{320}$$

The integrand is G-invariant, so

$$Z_\hbar = |G| \int_F e^{i\frac{S(y)}{\hbar}} \det(L(y))\delta(\varphi(y))\,dy. \tag{321}$$

$\square$

The assumptions we have been using are so strong that they are in fact never satisfied; however, this will not concern us.

25 Faddeev-Popov, BRST, BV

We will now need the notion of Grassmann integration. Let V be a finite-dimensional vector space of dimension $\dim(V) = n$. Choose an orientation on V (a basis $e \in \Lambda^n(V)$). If $f \in \Lambda(V)$ is an element of the exterior algebra, we define the Grassmann integral

$$\int_{\Lambda(V)} f = f^{\text{top}} \tag{322}$$

where $f = f^{\text{top}} e +$ other terms. If we choose a basis $a_1, ...a_n$, we have an isomorphism $\Lambda(V) \cong \mathbb{C}\langle a_1, ...a_n \rangle / \langle a_i a_j + a_j a_i \rangle$, a so-called Grassmann algebra.

Example Consider a Gaussian integral

$$\int_{\Lambda(V)} \exp\left(\frac{1}{2} \sum_{ij} a_i A_{ij} a_j \right) da_1 ... da_n \tag{323}$$

where the notation indicates that we integrate with respect to the orientation $a_1 \wedge ... \wedge a_n$ and A_{ij} is skew-symmetric. We can write this as

$$\sum_{k \geq 0} \frac{1}{2^k k!} \int_{\Lambda(V)} a_{i_1} A_{i_1 j_1} a_{j_1} ... a_{i_k} A_{i_k j_k} a_{j_k} \, da_1 ... da_n \tag{324}$$

which is equal to 0 if n is odd and

$$\frac{1}{2^{n/2}(n/2)!}\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)A_{\sigma_1\sigma_2}A_{\sigma_3\sigma_4}...A_{\sigma_{n-1}\sigma_n}=\operatorname{Pf}(A) \tag{325}$$

if n is even (the Pfaffian of A). The above naturally organizes into a sum over perfect matchings

$$\sum_{\text{matchings}}\operatorname{sgn}(\sigma)A_{\sigma_1\sigma_2}...A_{\sigma_{n-1}\sigma_n}. \tag{326}$$

Some matrices have the property that the sign of the above expression does not depend on σ, which allows Pfaffians to count combinatorial quantities related to perfect matchings. This is useful for analyzing dimer models.

Now let $W = V \oplus V^*$, let $a_1, ...a_n$ be a basis of V, let $\bar{a}^1, ...\bar{a}^n$ be the dual basis, and consider

$$\int_{\Lambda(W)}\exp\left(\sum_{i,j}\bar{a}^i A_i^j a_j\right)\, d\bar{a}^1...d\bar{a}^n\, da_1...da_n. \tag{327}$$

A similar argument to the above shows that we can write this as

$$(-1)^{\frac{n(n-1)}{2}}\det(A). \tag{328}$$

On the other hand, letting A be as before,

$$\operatorname{Pf}(A)^2=\int_{\Lambda(V)\otimes\Lambda(V)}\exp\left(\frac{1}{2}(a,Aa)+\frac{1}{2}(b,Ab)\right)\, da\, db \tag{329}$$

which, writing $c=\frac{a+b}{\sqrt{2}}, \bar{c}=\frac{a-b}{\sqrt{2}}$, gives

$$\operatorname{Pf}(A)^2=(-1)^{\frac{n(n-1)}{2}}\int\exp(\bar{c}Ac)\, d\bar{c}\, dc=\det(A). \tag{330}$$

Exercise 25.1. *Show that Pf* $\begin{bmatrix} 0 & -A^T \\ A & 0 \end{bmatrix} = \det(A)$.

We return now to the setting of a compact group G with a fixed Haar measure acting on a space F with $\varphi : F/G \to F$ a section of the quotient map $\pi : F \to F/G$. We wanted to compute the integral

$$\int_F e^{i\frac{S}{\hbar}}\,dx = |G|\int_F e^{i\frac{S(x)}{\hbar}}\det(L(x))\delta(\varphi(x))\,dx. \tag{331}$$

Now that we have Grassmann integration, we can write the determinant that appears using a Grassmann integral

$$\int_{\Lambda(\mathfrak{g}\oplus\mathfrak{g}^*)} \exp\left(\frac{1}{\hbar}\sum_{a,b} c^a L_a^b \bar{c}_b\right) d\bar{c}\,dc = \frac{1}{\hbar^n}\det(L). \tag{332}$$

Thinking of this integral as an integral over $\mathfrak{g}_{\text{odd}} \oplus \mathfrak{g}^*_{\text{odd}}$, we can write the original integral as an integral over a supermanifold

$$|G|\int_{F\times\mathfrak{g}_{\text{odd}}\times\mathfrak{g}^*_{\text{odd}}\times\mathfrak{g}^*} e^{\frac{i}{\hbar}S_{\text{FP}}(x,c,\bar{c},\lambda)}\,dx\,d\bar{c}\,dc\,d\lambda \tag{333}$$

where S_{FP} is the Faddeev-Popov action

$$S(x) - i\sum_{a,b} c^a L_a^b(x)\bar{c}_b + \sum_a \lambda_a\varphi^a(x). \tag{334}$$

Here $c, \bar{c}$ are the ghost fields and have no physical meaning, and λ is a Lagrange multiplier and also has no physical meaning. We now claim that S_{FP} has isolated critical points on $F \times \mathfrak{g}^*$; consequently, we can use Feynman diagrams to describe its semiclassical asymptotics.

Becchi, Rouet, Stora, and Tyutin extended this formalism as follows. First, consider the space of functions

$$C^\infty(F_{\text{BRST}}) = C^\infty(F\times\mathfrak{g}_{\text{odd}}\times\mathfrak{g}^*_{\text{odd}}\times\mathfrak{g}^*) = C^\infty(F\times\mathfrak{g}^*)\otimes\Lambda(\mathfrak{g})\otimes\Lambda(\mathfrak{g}^*). \tag{335}$$

Define

$$Q = Q_{\text{BRST}} = c^a\ell_a^i\partial_i - \frac{1}{2}f_{ab}^c c^a c^b\frac{\partial}{\partial c^c} + \lambda_a\frac{\partial}{\partial\bar{c}_a} \tag{336}$$

where the second two derivatives are fermionic. The first two terms give the Chevalley-Eilenberg differential for $\mathfrak{g}$ acting on $C^\infty(F)$ by vector fields, and the third term is the Koszul differential for $\mathfrak{g}^*$. Here f_{ab}^c are the structure constants for $\mathfrak{g}$. So we can write $Q = d_{\text{CE}} + d_K$.

The Chevalley-Eilenberg differential computes Lie algebra cohomology with coefficients in a module. Recall that this is the cohomology of the complex

$$C^\bullet(\mathfrak{g}, M) = \Lambda^\bullet(\mathfrak{g}) \otimes M \tag{337}$$

equipped with the differential

$$\begin{aligned} d_{\mathrm{CE}}(x_1 \wedge \ldots \wedge x_n \otimes m) &= \sum_{i<j} (-1)^{i+j+1} [x_i, x_j] \wedge x_1 \ldots \wedge \hat{x}_i \ldots \wedge \hat{x}_j \ldots \otimes m \quad (338) \\ &+ \sum_{i=1}^{n} (-1)^i x_1 \ldots \wedge \hat{x}_i \ldots \wedge x_n \otimes x_i m. \quad (339) \end{aligned}$$

Exercise 25.2. *Let e_a be a basis of $\mathfrak{g}$ and $\pi : \mathfrak{g} \to End(M)$ be the induced map. Then*

$$d_{CE} = c^a \pi(e_a) - \frac{1}{2} c^a c^b \frac{\partial}{\partial c^c}. \tag{340}$$

The Chevalley-Eilenberg and Koszul differential both square to zero and anti-commute. There is a double complex that can be built out of them, and the BRST complex is its total complex; in particular, $Q_{\mathrm{BRST}}^2 = 0$. Moreover,

$$H^0_{\mathrm{BRST}} = C^\infty(F)^G \tag{341}$$

so the BRST complex computes in some sense the quotient F/G. Moreover, $Q_{\mathrm{BRST}} S_{\mathrm{FP}} = 0$ and $S_{FP} \equiv S \bmod Q$, so Q acts as a supersymmetry of the Faddeev-Popov action and it relates the Faddeev-Popov action to the original action.

In general, we have symmetries that may not come from a group action (only from some vector fields), and we may not have isolated G-orbits. To handle this more general situation there is the Batalin-Vilkovisky formalism.

If $\varphi : F/G \to F$ is the section as above and $\Sigma = \mathrm{Im}(\varphi) \subseteq F$ is its image, then we define

$$\int_\Sigma f(s)\, ds = \int_F f(x) \partial(\varphi(x))\, dx. \tag{342}$$

Then we can write the original integral as

$$\int_{\Sigma \times \mathfrak{g}_{\mathrm{odd}} \times \mathfrak{g}^*_{\mathrm{odd}}} e^{\frac{i}{\hbar} S_{\mathrm{FP}}(s,c,\bar{c})}\, ds\, dc\, d\bar{c} \tag{343}$$

where the Faddeev-Popov action without Lagrange multipliers is

$$S_{\mathrm{FP}}(s, c, \bar{c}) = S(s) - i \sum_a c^a L_a^b(s) \bar{c}_b. \tag{344}$$

We introduce the BV-extended space

$$\mathcal{F} = T^*[-1](F') \tag{345}$$

to be the shifted cotangent bundle of $F' = F \times \mathfrak{g}$. The space of functions on this space is graded as follows (by ghost number): the coordinates x^i (fields) on F have degree 0, the coordinates c^a (ghosts) on $\mathfrak{g}$ have degree 1, the cotangent coordinates c_a^+ (antighosts) have degree 0, and the cotangent coordinates x_i^+ (antifields) have degree -1. As a cotangent bundle, we also have a symplectic form

$$\omega = \sum_n dx^n \wedge dx_n^+ = \sum_i dx^i \wedge dx_i^+ + \sum_a dc^a \wedge dc_a^+. \tag{346}$$

The BV-action is

$$S_{\mathrm{BV}} = S(x) + \sum_a c^a \ell_a^i x_i^+ - \frac{1}{2} f_{ab}^c c^a c^b c_c^+. \tag{347}$$

We can write the Faddeev-Popov integral by gauge fixing using a choice of Lagrangian submanifold L in $T^*[-1](F')$:

$$L_{\mathrm{FP}} = \{x \in \Sigma : x_j^+ = i \bar{c}_b \partial_j \varphi^b(x), c_a^+ = 0\}. \tag{348}$$

Our original integral is then

$$\int_{L_{\mathrm{FP}}} e^{\frac{i}{\hbar} S_{\mathrm{BV}}(x)} \, ds \, d\bar{c} \, dc. \tag{349}$$

The symplectic form above is odd and induces an odd Poisson bracket

$$\{F, G\} = \pm \sum_n \frac{\partial F}{\partial x^n} \frac{\partial G}{\partial x_n^+} \pm \sum_n \frac{\partial G}{\partial x^n} \frac{\partial F}{\partial x_n^+}. \tag{350}$$

Proposition 25.3. *The classical master equation*

$$\{S_{BV}, S_{BV}\} = 0$$

holds.

Proof. . We compute that

$$\{S_{\mathrm{BV}}, S_{\mathrm{BV}}\} = \left(\frac{\partial S}{\partial x^i} + \sum_a c^a \partial_i \ell_a^j x_j^+\right) c^b \ell_b^i + \left(\ell_a^i x_i^+ - f_{ab}^c c^b c_c^+\right)\left(-\frac{1}{2} f_{ed}^a c^e c^d\right). \quad (351)$$

Various combinations of terms vanish. □

This is the end of the classical BV framework: a $\mathbb{Z}$-graded manifold, an odd symplectic form, and a BV-action that Poisson-commutes with itself. It applies to manifolds without boundary.

For the quantum BV framework, we need the BV operator (BV Laplacian)

$$\Delta = \sum_n \frac{\partial^2}{\partial x_n^+ \partial x^n}. \quad (352)$$

This operator satisfies $\Delta^2 = 0$.

Exercise 25.4. *Show that* $\Delta S_{BV} = 0$.

The above is part of the quantum master equation. We fix a gauge using a submanifold

$$L_{\mathrm{FP}} = \{x \in \Sigma \subset F : x_j^+ = i\bar{c}_b \partial_j \varphi^b, c_a^+ = 0\}. \quad (353)$$

Proposition 25.5. *L is a Lagrangian submanifold of F.*

Proof. We have $\dim(F) = N, \dim(\mathfrak{g}) = n$ and $\dim(L) = (N-n)+n+n = N+n = \frac{1}{2}\dim(\mathcal{F})$, hence L is half-dimensional. The symplectic form restricted to L takes the form

$$\sum_i \left(\frac{\partial x^i}{\partial s^\alpha} ds^\alpha\right) \wedge \left(-id\bar{c}_b \partial_i \varphi^b(x) - i\bar{c}\partial_j \partial_i \varphi^b \frac{\partial x^j}{\partial s^\alpha} ds^\alpha\right) \quad (354)$$

where s^α is a local parameterization of Σ. Various combinations of terms vanish. □

We define the volume form $d\ell = ds\, d\bar{c}\, dc$ on L. Then

$$\int_{\Sigma\times\mathfrak{g}_{\mathrm{odd}}\times\mathfrak{g}^*_{\mathrm{odd}}} e^{\frac{i}{\hbar}S_{\mathrm{FP}}}\, ds\, d\bar{c}\, dc = \int_{L_{\mathrm{FP}}} e^{\frac{i}{\hbar}S_{\mathrm{BV}}}\, d\ell. \tag{355}$$

(This is because the BV-action restricts to the FP-action on L_{FP}.)

The value of this integral does not depend on the choice of L:

Proposition 25.6. *Let $L \subset \mathcal{F}$ be a Lagrangian submanifold and consider the integral*

$$Z_L = \int_L e^{i\frac{S_{BV}}{\hbar}}\, d\ell \tag{356}$$

where $d\ell$ is defined as follows: fixing dx on $\mathcal{F}$, suppose that $\ell^1, ...\ell^k, \xi_1, ...\xi_k$ are Darboux coordinates on $T^*(L)$ such that locally $L = \{\xi_i = 0\}$. If

$$dx = \rho\, d\ell^1 ... d\ell^k\, d\xi_1 ... d\xi_k \tag{357}$$

then we define

$$d\ell = \sqrt{\rho} d\ell^1 ... d\ell^k \tag{358}$$

which is a half-density and does not depend on the choice of local coordinates. Then the above integral is locally independent of L.

Proof. In Darboux coordinates,

$$\omega = \sum_n d\ell^n \wedge d\xi_n, \Delta = \sum_n \frac{\partial^2}{\partial \ell^n \partial \xi_n}. \tag{359}$$

We need the following:

1. If $\Delta F = 0$ then $\int_L F\, d\ell$ locally does not depend on L.
2. $\Delta\left(e^{\frac{i}{\hbar}S_{\mathrm{BV}}}\right) = 0$ (quantum master equation).

To prove the quantum master equation, evaluating the product it suffices to prove

$$i\hbar\Delta S - \{S, S\} = 0 \tag{360}$$

but we already showed that both of these terms vanish.

Now define $dx, d\ell$ as above. We have

$$\Delta = \frac{1}{\rho}\frac{\partial}{\partial \ell^i}\left(\rho \frac{\partial}{\partial \xi_i}\right). \tag{361}$$

We now want to consider a family of integrals of the form $I_L = \int_L F\, d\ell$. Suppose that $\Delta F = 0$. When we vary L we will not vary the variables ℓ^i but we will vary the variables ξ_i, hence write

$$\delta \xi_i = \frac{\partial \psi}{\partial \ell^i}. \tag{362}$$

Then we compute that

$$\delta I_L = \int_L \left(\frac{\partial \psi}{\partial \ell^i}\frac{\partial F}{\partial \xi_i}\right)|_{\xi=0}\, d\ell. \tag{363}$$

Integrating by parts, this is

$$\pm \int_L \psi \Delta F|_{\xi=0}\, d\ell = 0. \tag{364}$$

$\square$

Example Consider the BV-extension of Chern-Simons on a 3-manifold M (without boundary). The space of fields $\mathcal{F}$ is $\Omega^\bullet(M, \mathfrak{g})[1]$ with ghost number given by 1 minus the usual grading. The symplectic form is

$$\omega = \frac{1}{2}\int_M \operatorname{tr}(\delta A \wedge \delta A) \tag{365}$$

with ghost number -1, and the action is

$$S_M = \int_M \operatorname{tr}\left(\frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A\right) \tag{366}$$

with ghost number 0.

Exercise 25.7. $\{S_M, S_M\} = \int_M tr\left(\frac{\delta S}{\delta A} \wedge \frac{\delta S}{\delta A}\right) = 0.$

We also have a statement of the form

$$\Delta S = \int_M \operatorname{tr}\left(\frac{\delta^2 S}{\delta A \delta A}\right) = 0 \tag{367}$$

but it is unclear how to make sense of this formally.

Now write

$$QF = \{S, F\} = p(\delta S \wedge \delta F) \tag{368}$$

where p is the Poisson tensor. Then solutions to the Euler-Lagrange equation are zeroes of the vector field Q and take the form

$$\delta A + \frac{1}{2}[A \wedge A] = 0. \tag{369}$$

Taking the quotient of this space by the action of Q gives a moduli space containing the moduli space of flat connections but having other components as well.