274 Curves on Surfaces, Lecture 9

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10 Cluster algebras

Yesterday Zelevinsky gave a talk about cluster algebras.

Question from the audience: in a surface cluster algebra, we can show that the denominators of all the relevant rational functions don't have positive real roots (because the λ-lengths described by these rational functions exist). But the theory of cluster algebras shows that these denominators are in fact monomials (the Laurent phenomenon). Can this be shown using some notion of complex λ-lengths?

Answer: hyperbolic structures correspond to representations of the fundamental group into $\mathrm{PSL}_2(\mathbb{R})$. To complexify this, we could instead look at representations of the fundamental group into $\mathrm{PSL}_2(\mathbb{C})$. This is the group of orientation-preserving isometries of hyperbolic space $\mathbb{H}^3$.

First, some background: the isometries of $\mathbb{H}^2$ can be organized into three classes. The *elliptic* elements fix some point, which in the disk model we can arrange to be the origin. The *hyperbolic* elements fix some geodesic, which in the disk model we can arrange to be a diameter, and also fix two points on the boundary. The *parabolic* elements fix one point on the boundary.

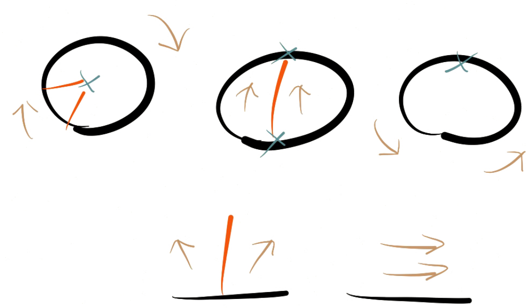

Figure 1: The three types of isometries.

This classification is obtained as follows. An element of $\mathrm{PSL}_2(\mathbb{R})$ is represented by a 2×2 real matrix with two eigenvalues λ, λ^{-1}. It could have two distinct real eigenvalues, in which case it can be conjugated to a diagonal matrix (hyperbolic). It could have two complex eigenvalues, in which case λ is on the unit circle (elliptic). Or it could have a repeated eigenvalue, so $\lambda^2 = 1$ and $\lambda = \pm 1$. We can choose a representative with $\lambda = 1$, in which case we obtain a Jordan block (parabolic).

Consider a closed surface with punctures (but no boundary components), with cusps at the boundaries. Having a cusp means that the monodromy around the puncture is parabolic. (Hyperbolic monodromy corresponds to a geodesic boundary component.)

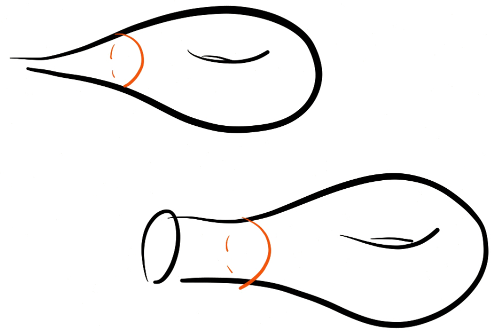

Figure 2: A cusp vs. a geodesic boundary.

Choosing a horocycle at the cusp corresponds to choosing an eigenvector of the monodromy in $\mathbb{R}^2$ up to sign, and this allows us to define λ-lengths using the determinant formula in a way which continues to work when $\mathrm{PSL}_2(\mathbb{R})$ is replaced with $\mathrm{PSL}_2(\mathbb{C})$. Unfortunately, the corresponding λ-lengths are not always defined, so this doesn't appear to prove the Laurent phenomenon.

We know from Zelevinsky's talk that (some) cluster algebras can be described using quivers (directed graphs). The quiver associated to a triangulation of a surface has vertices the edges of the triangulation and directed edges are given by clockwise adjacency. We need to check that mutation corresponds to changing the triangulation.

Mutation at a vertex occurs in three steps:

1. Add composite arrows through the vertex.

2. Reverse arrows through the vertex.

3. Delete all oriented 2-cycles that occur.

Doing this does indeed correspond to changing the triangulation (by a quadrilateral flip).

Moreover, the λ-lengths in the triangulation (relative to some choice of horocycles and hyperbolic structure) are cluster variables with exchange relation given by the Ptolemy relation

$$\lambda(E)\lambda(F) = \lambda(A)\lambda(C) + \lambda(B)\lambda(D). \tag{1}$$

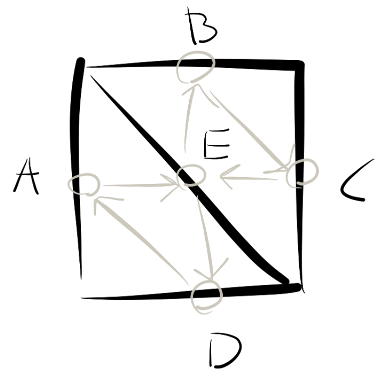

Figure 3: The quiver associated to a triangulation.

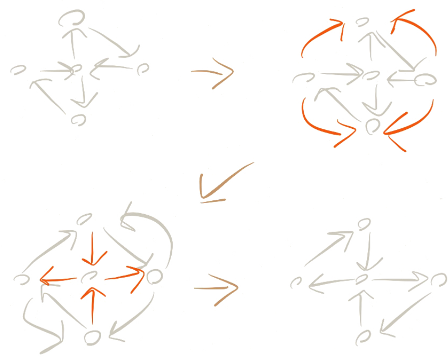

Figure 4: Mutation and changing the triangulation.

Here $\lambda(F)$ is the new cluster variable given by the length of the new diagonal.

In general, some vertices of the quiver are frozen and cannot be mutated (for a surface with boundary these are the boundary edges). We can indicate the difference by drawing frozen vertices in black and non-frozen vertices in white.

The cluster algebra associated to a pentagon is finite type of type A_2. We can get A_3 by triangulating the hexagon appropriately. If we reverse one of the arrows in the corresponding quiver, we get a different triangulation of the hexagon. More generally we can get A_n by triangulating an $(n+3)$-gon.

Similarly, we get D_n from a punctured n-gon. (To do this we needed to cancel an edge, which makes the exchange relation a little different from what one expects.)

One reason to care about cluster algebras is that they occur in many different

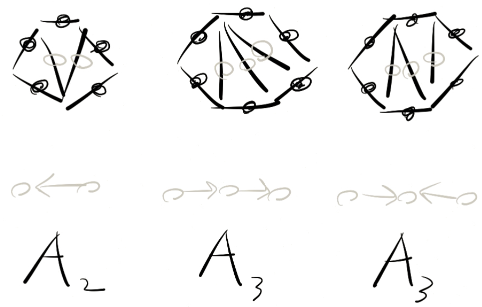

Figure 5: Cluster algebras of type A_n from polygons.

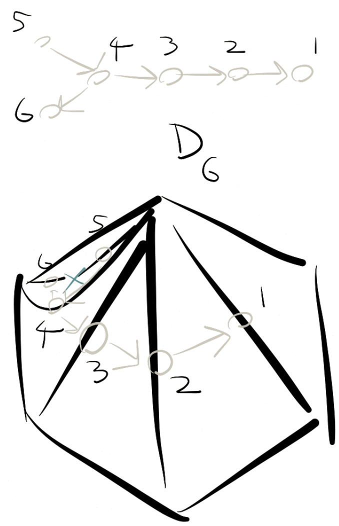

Figure 6: The cluster algebra of type D_6 from a punctured hexagon.

places for different reasons. For example, D_4, which we obtain from a punctured 4-gon, also occurs when considering a cluster algebra structure on 3×3 matrices obtained from considering minors.

Theorem 10.1. *(Felikson, Shapiro, Tumarkin) The mutation-finite cluster algebras are the following:*

1. *Surface (or orbifold) cluster algebras (including A_n, D_n),*

2. *Rank* 2 *cluster algebras,*

3. $11+7$ *additional diagrams, some of which have different possible assignments of short and long roots.*

Question from the audience: is it obvious that surface cluster algebras are mutation-finite? They may have infinitely many triangulations.

Answer: what we care about is not the number of triangulations but the number of triangulations modulo the action of the mapping class group (since this is already enough to determine the quiver). Up to the action of the mapping class group, a triangulation is specified by the combinatorial data of how to glue a fixed set of triangles together, and there are finitely many ways to do this. Alternately, there are finitely many possible B-matrices because their entries are at most 2 in absolute value.

Exercise 10.2. *Find surfaces that give the affine Dynkin diagrams.*

(Orientation of the arrows matters for $\tilde{A}_n$ since there is a cycle. But $\tilde{D}_n$ is acyclic so orientation doesn't matter there.)

Exercise 10.3. *What's a geometric surface model for D_4 in the form of a quiver with all the arrows pointing outwards? Can you see the triality symmetry?*

Previously we asked for a surface cluster algebra giving the recurrence $x_{n+1}x_{n-1} = x_n^2 + 1$. This can be obtained from a (punctured) torus one of whose side lengths is equal to 1, or alternately (by cutting the torus open) from an annulus.

Consider again the triangulation of a punctured n-gon near the puncture. We had to cancel an edge to get D_n, and this causes the corresponding mutation to disagree with the result we get from the Ptolemy relation.

Namely, changing triangulations at D, the Ptolemy relation gives

$$\lambda(E)\lambda(D) = \lambda(A)\lambda(C) + \lambda(B)\lambda(C) \tag{2}$$

but on the cluster algebra side the exchange relation is

$$x(D)x(D') = x(B) + x(A) \Rightarrow x(D') = \frac{\lambda(E)}{\lambda(C)} \tag{3}$$

where x denotes a cluster variable; this differs by a factor of $\frac{1}{\lambda(C)}$ from the expected answer $\lambda(E)$.

We can geometrically interpret the above relation as follows. For p a cusp in a hyperbolic surface and h a horocycle around p with a the length of the horocycle, the *conjugate horocycle* $\bar{h}$ is the corresponding horocycle of length $\frac{1}{a}$.

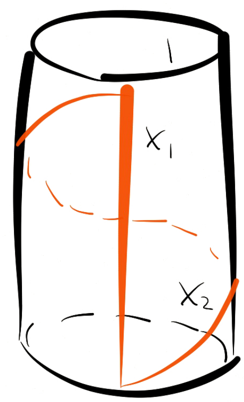

Figure 7: A cylinder / annulus giving the recurrence $x_{n+1}x_{n-1} = x_n^2 + 1$.

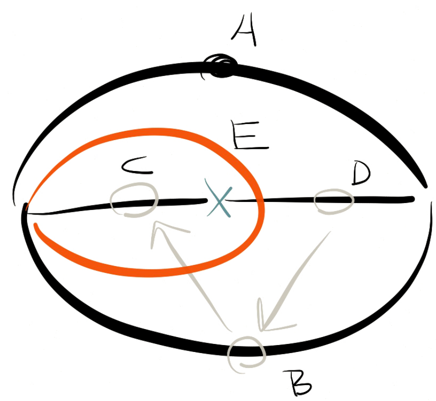

Figure 8: A disagreement between the Ptolemy relation and the exchange relation.

Lemma 10.4. *In a punctured monogon (a punctured ideal triangle with edges identified),*

$$\lambda(A)\lambda(A') = \lambda(B) \Rightarrow \lambda(A') = \frac{\lambda(B)}{\lambda(A)}. \tag{4}$$

(A' is the same geodesic as A, but the λ-length is measured with respect to the conjugate horocycle.)

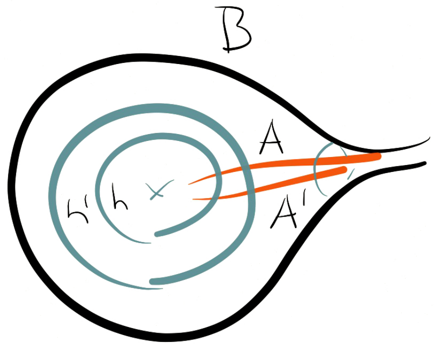

Figure 9: Conjugate horocycles.

Proof. Let a be the length of the original horocycle h. We have a formula describing this length which shows that

$$a = \frac{\lambda(B)}{\lambda(A)^2}. \tag{5}$$

The same formula applied to the conjugate horocycle $\bar{h}$ shows that

$$\frac{1}{a} = \frac{\lambda(B)}{\lambda(A')^2}. \tag{6}$$

Multiplying these gives the conclusion. □

We can now give a geometric interpretation to the cluster variable computation, which is that we are computing the λ-length of the new diagonal E with respect to a conjugate horocycle.

Exercise 10.5. *Another way of writing the relation for conjugate horocycles is as follows. Let h, h' be conjugate horocycles and let A, A' be parallel arcs, with the length of A' being measured with respect to h'. Then*

$$\lambda(A') = \lambda(A)\ell(h) \tag{7}$$

where $\ell(h)$ is the hyperbolic length of h. Give a direct geometric proof of this relationship.