274 Curves on Surfaces, Lecture 8

Dylan Thurston
Notes by Qiaochu Yuan

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9 Teichmüller space and Markov triples

Last time there were some exercises. One was to compute λ-lengths between two horocycles in terms of Euclidean geometry in the upper half-plane. If x is the distance between the corresponding points on the boundary and Δ_1, Δ_2 are the diameters, then this is $\frac{x}{\sqrt{\Delta_1 \Delta_2}}$.

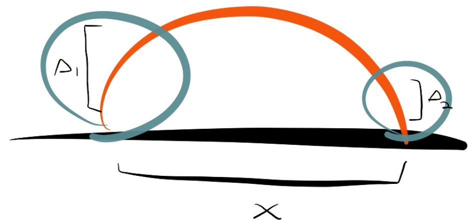

Figure 1: Euclidean distances determining the λ-length.

To check this, it suffices to check that it is invariant under hyperbolic isometries, behaves correctly with respect to scaling one of the circles, and is equal to 1 when the circles are tangent. The only nontrivial step is invariance under inversion with respect to some circle, but the circle determined by the geodesic connecting the boundary points fixes both horocycles setwise, so the conclusion follows. Alternatively, we can invert so that one of the horocycles is sent to ∞.

One was to relate the Euclidean and hyperbolic Ptolemy relations. Hint: if (x, y) is a point on the unit circle, then $x^2 + y^2 = 1$, and there is a natural way to construct a null vector from this data.

One was to relate the cross ratio of four ideal points a, b, c, d to λ-lengths. To access the cross ratio, we will work in the upper half-plane and send three of the points to $0, 1, \infty$. Let the fourth point be τ.

We want to compute τ in terms of λ-lengths, so we need to choose four horocycles (one of which is at ∞). Letting the diameters of the three finite horocycles be $\Delta_a, \Delta_b, \Delta_d$ and letting the infinite horocycle be at height y_c, we have

$$\lambda(a, b) = \frac{1}{\sqrt{\Delta_a \Delta_b}} = \lambda(B) \tag{1}$$

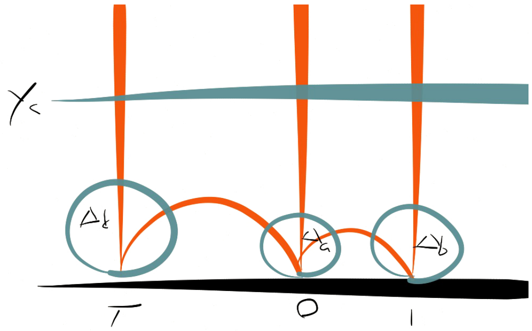

Figure 2: A decorated ideal quadrilateral in the upper half-plane.

$$\lambda(a,d) = \frac{-\tau}{\sqrt{\Delta_a \Delta_d}} = \lambda(A) \tag{2}$$

$$\lambda(c,d) = \sqrt{\frac{y_c}{\Delta_d}} = \lambda(D) \tag{3}$$

$$\lambda(b,c) = \sqrt{\frac{y_c}{\Delta_b}} = \lambda(C) \tag{4}$$

and we want some combination of these which does not depend on the choice of horocycle. This turns out to be

$$\tau = -\frac{\lambda(A)\lambda(C)}{\lambda(B)\lambda(D)}. \tag{5}$$

We should give a more careful definition of Teichmüller space. Suppose we want to classify hyperbolic metrics (complete, finite volume) on a (smooth) surface S. The first method would be to look at all hyperbolic metrics, where we declare two metrics to be equivalent if there is an isometry between them. If S is closed of genus g, this gives the moduli space $\mathcal{M}_g$. This is not a manifold but an orbifold due to metrics with additional symmetries.

A space which is easier to parameterize is to look at all hyperbolic metrics, where we declare two metrics to be equivalent if there is an isometry between them which is isotopic to the identity (in the diffeomorphism group; for surfaces, this is equivalent to being homotopic to the identity). If S is closed of genus g, this is Teichmüller space. Surprisingly, it is a finite-dimensional manifold of dimension $6g-6$ for $g > 1$ even though it is defined as the quotient of some infinite-dimensional space by some other infinite-dimensional space.

When $g = 1$ we can instead look at Euclidean metrics on the torus modulo isometries isotopic to the identity and by global scaling. By choosing an oriented basis for homology, this can be identified with oriented pairs of vectors in $\mathbb{R}^2$ up to scaling and rotation, which can be identified with the upper half-plane. If we instead quotient by all isometries, we get the upper half-plane modulo $\mathrm{PSL}_2(\mathbb{Z})$. This is the moduli space $\mathcal{M}_1$.

We can also allow n punctures with *cusps* at the punctures (we want the metric to remain complete and of finite volume). This gives moduli spaces $\mathcal{M}_{g,n}$.

To relate moduli space and Teichmüller space, we should quotient Teichmüller space by all diffeomorphisms and not just those isotopic to the identity. The additional symmetries by which we have to quotient are precisely the mapping class group $\mathrm{MCG}(S)$. For example, $\mathcal{M}_{1,1} = \mathcal{T}_{1,1}/\mathrm{MCG}(T^2, x)$, which exhibits $\mathcal{M}_{1,1}$ as the quotient of the upper half-plane by $\mathrm{SL}_2(\mathbb{Z})$.

Last time we also chose horocycles, giving us a decorated Teichmüller space $\tilde{\mathcal{T}}_{g,n}$. This has dimension $6g-6+3n$, which is the number of arcs in a triangulation. Given a triangulation, we can measure its λ-lengths to obtain a map

$$\tilde{\mathcal{T}}_{g,n} \to \mathbb{R}^{6g-6+3n} \tag{6}$$

and this is a homeomorphism; moreover, these maps coming from triangulations are related by exchange relations coming from the Ptolemy relation.

Last time we also discussed Markov triples, the integer solutions to $x^2 + y^2 + z^2 = 3xyz$. These arise when considering lengths of the segment of a horocycle contained in an ideal triangle that it decorates.

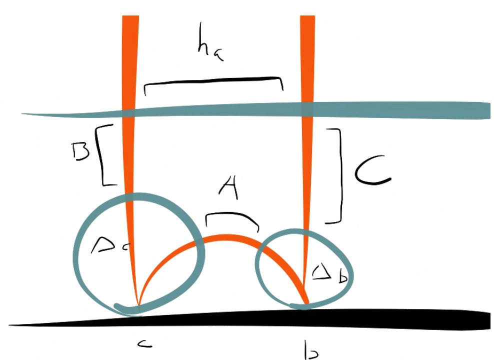

Figure 3: A segment of a horocycle at infinity in the upper half-plane.

This is necessarily some function of the λ-lengths involved, so we want to determine what this function is. Sending the corresponding horocycle to ∞ in the upper half-plane and letting its height be y_a, we compute that this length is

$$\int_0^1 \frac{dx}{y_a} = \frac{1}{y_a} = h_a. \tag{7}$$

But we want an expression in terms of λ-lengths. The relevant λ-lengths are $\lambda(A) = \frac{1}{\sqrt{\Delta_b \Delta_c}}$, $\lambda(B) = \sqrt{\frac{y_a}{\Delta_c}}$, $\lambda(C) = \sqrt{\frac{y_a}{\Delta_b}}$. This gives

$$h_a = \frac{\lambda(A)}{\lambda(B)\lambda(C)}. \tag{8}$$

Consider now a horocycle on a punctured torus with an ideal triangulation. Let the λ-lengths be x, y, z.

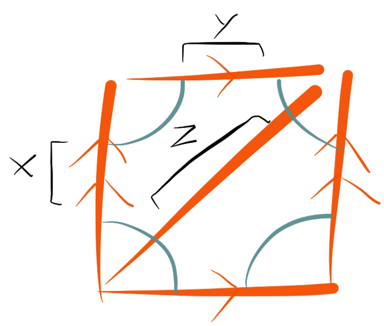

Figure 4: A horocycle on a punctured torus.

The total length of the horocycle can be computed using the above to be

$$2\left(\frac{x}{yz} + \frac{y}{zx} + \frac{z}{xy}\right) = 2\frac{x^2 + y^2 + z^2}{xyz}. \tag{9}$$

The Markov triple equation then says precisely that the total length of the corresponding horocycle is equal to 6, which occurs, for example, when $x = y = z = 1$. This total length is independent of triangulation, and changing triangulations corresponds to getting new Markov triples.

Exercise 9.1. *The above discussion was for the equilateral torus. What are λ-lengths in the square torus? What is the corresponding Diophantine equation?*

Next time we will talk about cluster algebras. These are, among other things, algebras A with sets of distinguished variables called clusters. For any cluster $C = (x_i)_{i=1}^n$, the entire algebra A is contained in the ring of Laurent polynomials $\mathbb{Q}[x_i^{\pm}]$ in the x_i. We can also move between clusters as follows: for any C and i, there is a unique cluster $C' = C \setminus \{x_i\} \cup \{x_i'\}$ satisfying an exchange relation of the form

$$x_i x_i' = P(x_1, ...\hat{x_i}, ...x_n) \tag{10}$$

where P is a polynomial with two terms. There are also rules determining which polynomials P can occur and how they are related.

The model examples to keep in mind come from surfaces. If S is a surface, possibly with boundary, and M is a set of marked points, then a choice of hyperbolic metric and horocycles allows us to associate to any triangulation a collection of λ-lengths. Changing the triangulation by replacing a diagonal in a quadrilateral changes the λ-lengths by a Ptolemy relation, which is an exchange relation. The λ-lengths associated to a given triangulation form the clusters of a cluster algebra of functions on decorated Teichmüller space.

Example Consider a pentagon. We can choose horocycles so that the λ-lengths are all equal to 1. Choosing two more diagonals gives a triangulation with two new λ-lengths x_1, x_2.

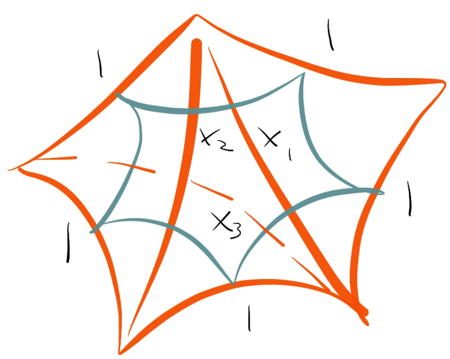

Figure 5: A pentagon and its diagonals.

Now alternately applying quadrilateral flips to get new diagonals gives a sequence of λ-lengths x_n satisfying the recurrence

$$x_{n+1} x_{n-1} = x_n + 1. \tag{11}$$

This recurrence is periodic with period 5 since there are only 5 diagonals in the pentagon. It consists of Laurent polynomials in x_1, x_2, and this is not obvious. Computing the terms gives

$$x_3 = \frac{x_2}{x_1} + \frac{1}{x_1} \tag{12}$$

$$x_4 = \frac{1}{x_1} + \frac{1}{x_1 x_2} + \frac{1}{x_2} \tag{13}$$

$$x_5 = \frac{1 + x_1}{x_2} \tag{14}$$

$$x_6 = x_1. \tag{15}$$

Exercise 9.2. *Consider the recurrence* $x_{n+1}x_{n-1} = x_n^2 + 1$. *Check a few terms to verify that we get Laurent polynomials. Can you find a surface giving rise to this recurrence as above?*