274 Curves on Surfaces, Lecture 3

Dylan Thurston
Notes by Qiaochu Yuan

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3 More about mapping class groups

Some background reading:

1. *Primer on Mapping Class Groups*, Farb and Margalit. Available online.
2. *Papers on Group Theory and Topology*, Dehn (introduction of Dehn-Thurston coordinates). Alex will be talking about this paper.
3. *Three-Dimensional Geometry and Topology*, Thurston Sr. Begins with a nice introduction to hyperbolic geometry. Available online.

Let S be a surface with $\chi(S) < 0$ and x a marked point. The Birman exact sequence is a short exact sequence

$$1 \to \pi_1(S, x) \to \mathrm{MCG}(S, x) \to \mathrm{MCG}(S) \to 1. \tag{1}$$

It can be iterated; for example, we can write down a short exact sequence

$$1 \to \pi_1(S \setminus 5 \text{ pts}) \to \mathrm{MCG}(S, 6 \text{ pts}) \to \mathrm{MCG}(S, 5 \text{ pts}) \to 1. \tag{2}$$

The map from $\pi_1(S, x)$ is the *point-dragging map* or *push map*. Given a curve $\gamma \in \pi_1(S, x)$, we want to send it to an element of $\mathrm{MCG}(S, x)$ which is trivial in $\mathrm{MCG}(S)$, hence it needs to be isotopic to the identity. It suffices to describe this isotopy. This isotopy will drag a neighborhood of the marked point x along γ and will be trivial outside a neighborhood of γ.

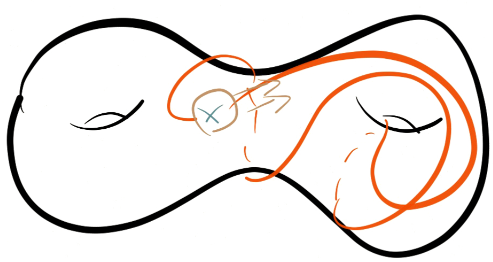

Figure 1: A marked point being pushed along a closed curve.

Why does this describe the entire kernel of the map $\mathrm{MCG}(S,x) \to \mathrm{MCG}(S)$? The general picture is as follows. For X any smooth manifold and $x \in X$ a marked point, there is a fibration

$$\mathrm{Diff}^+(X,x) \hookrightarrow \mathrm{Diff}^+(X) \to X \tag{3}$$

where the map $\mathrm{Diff}^+(X) \to X$ sends a diffeomorphism to the image of x. (A fibration behaves like a fiber bundle. The crucial property is a lifting property: in particular, any path in X lifts to a path in $\mathrm{Diff}^+(X)$.) This fibration induces a long exact sequence in homotopy

$$...\pi_1(\mathrm{Diff}^+(X)) \to \pi_1(X) \to \pi_0(\mathrm{Diff}^+(X,x)) \to \pi_0(\mathrm{Diff}^+(X)) \to \pi_0(X). \tag{4}$$

But $\pi_0(\mathrm{Diff}^+(X)) = \mathrm{MCG}(X)$ and $\pi_0(\mathrm{Diff}^+(X,x)) = \mathrm{MCG}^+(X,x)$, and $\pi_0(X)$ is a point when X is connected. The next term in the long exact sequence is a map $\pi_1(X) \to \pi_0(\mathrm{Diff}^+(X,x))$.

Theorem 3.1. *(Hamstrom) Let S be a surface with $\chi(S) < 0$. Then $\pi_1(\mathit{Diff}^+(X))$ is trivial. In fact, the connected component of the identity in $\mathit{Diff}^+(X)$ is contractible.*

This is an aspect of hyperbolic geometry. The same is true for higher-dimensional hyperbolic manifolds; this is an aspect of Mostow rigidity. (But Mostow rigidity is false for hyperbolic surfaces.)

What happens when $S = T^2$? We claimed that the map $\mathrm{MCG}(T^2,x) \to \mathrm{MCG}(T^2)$ is an isomorphism. The long exact sequence ends

$$...\pi_1(\mathrm{Diff}^+(T^2)) \to \pi_1(T^2) \to \mathrm{MCG}(T^2,x) \to \mathrm{MCG}(T^2) \to 1 \tag{5}$$

so the map $\pi_1(T^2) \to \mathrm{MCG}(T^2,x)$ needs to be trivial. There is a map $T^2 \to \mathrm{Diff}_0(T^2)$ given by T^2 acting on itself by translation, and it is a difficult theorem that this is a homotopy equivalence. (This can be proven by removing a point, which makes the Euler characteristic -1, and applying the big theorem above.) Consequently

$$\pi_1(\mathrm{Diff}^+(T^2)) = \pi_1(\mathrm{Diff}^0(T^2)) \cong \pi_1(T^2). \tag{6}$$

Similarly, T^2 admits an action by affine linear maps, and this is a homotopy equivalence to $\mathrm{Diff}(T^2)$.

In summary, the end of the long exact sequence looks like

$$\begin{array}{ccccccccccc}
\pi_1(\mathrm{Diff}^+(T^2,x)) & \longrightarrow & \pi_1(\mathrm{Diff}^+(T^2)) & \longrightarrow & \pi_1(T^2) & \longrightarrow & \mathrm{MCG}(T^2,x) & \longrightarrow & \mathrm{MCG}(T^2) & \longrightarrow & 1 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
1 & \longrightarrow & \mathbb{Z}^2 & \xrightarrow{\cong} & \mathbb{Z}^2 & \xrightarrow{0} & \mathrm{SL}_2(\mathbb{Z}) & \xrightarrow{\cong} & \mathrm{SL}_2(\mathbb{Z}) & \longrightarrow & 1
\end{array} \tag{7}$$

where $\cong$ denotes an isomorphism.

More generally, if G is a connected Lie group, we get a map $G \to \mathrm{Diff}_0(G)$ coming from the action of G on itself by translation, and we also get a map in the other direction coming from evaluation. This is not a homotopy equivalence in general. When $G = \mathrm{SU}(2)$ we know that $\mathrm{SU}(2) \cong S^3$, and $\mathrm{Diff}^+(S^3)$ is homotopy equivalent to $\mathrm{SO}(4)$ (the Smale conjecture, proved by Hatcher).

Recall that last time we skewered a torus (quotiented it by the central element $-I$ in $\mathrm{MCG}(T^2) \cong \mathrm{SL}_2(\mathbb{Z})$) to obtain a double cover $T^2 \to S^2$ branched at 4 points. The claim was that this showed

$$\mathrm{MCG}(S^2, 1 \text{ pt}, 3 \text{ pts}) \cong \mathrm{PSL}_2(\mathbb{Z}). \tag{8}$$

(The 1 point is the identity in T^2 regarded as a group and the 3 points are the non-identity points of order 2.)

What is the mapping class group of S^2 fixing four points pointwise? This is the congruence subgroup $\Gamma(2)$, which consists of the image of the kernel of the map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$ in $\mathrm{PSL}_2(\mathbb{Z})$. It is in fact the free group $\mathbb{Z} * \mathbb{Z}$ on two generators.

The relationship to the braid group B_3 comes from the map

$$(D^2, 3 \text{ pts}) \to (S^2, 3 \text{ pts}, 1 \text{ pt}) \tag{9}$$

given by identifying the boundary to a point (which becomes the fourth marked point).

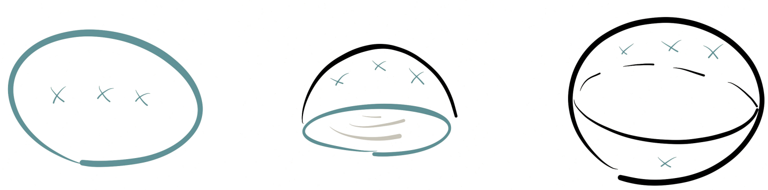

Figure 2: A 3-punctured disc getting its boundary identified to form a 4-punctured sphere.

The mapping class group B_3 of $(D^2, 3 \text{ pts})$ (fixing the boundary pointwise) has a center generated by *Dehn twist* along a boundary curve. As a braid it is given by the *full twist*. The image of Dehn twist in $\mathrm{MCG}(S^2, 3 \text{ pts}, 1 \text{ pt})$ is trivial (we can untwist). Thus we obtain an exact sequence

$$1 \to \mathbb{Z} \to \mathrm{MCG}(D^2, 3 \text{ pts}, \partial D^2) \to \mathrm{MCG}(S^2, 3 \text{ pts}, 1 \text{ pt}) \to 1 \tag{10}$$

showing that B_3 is a central extension of $\mathrm{PSL}_2(\mathbb{Z})$.

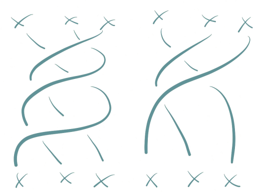

Figure 3: A full twist and a half twist.

Recall that before we were permuting curves on the thrice-punctured disc and, looking at Dehn-Thurston coordinates, we saw the Fibonacci numbers appear. This can now be explained as follows. The element of the mapping class group we were applying was a braid in B^3 whose image in $\mathrm{PSL}_2(\mathbb{Z})$ is given by the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

Exercise 3.2. *Verify this.*

Hint: look at how the braid group generators lift to the torus. They can be thought of as Dehn twists.

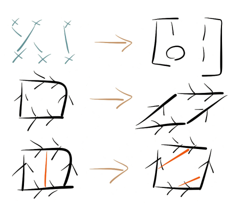

Figure 4: Some hints.

Dehn twists in general look like the following: if C is a simple closed curve on S, the Dehn twist $T_C \in \mathrm{MCG}(S)$ rotates an annular neighborhood $[0,1] \times C$ of C as follows: $\{t\} \times C$ is rotated by $2\pi t$.

Figure 5: Dehn twist around a curve C.

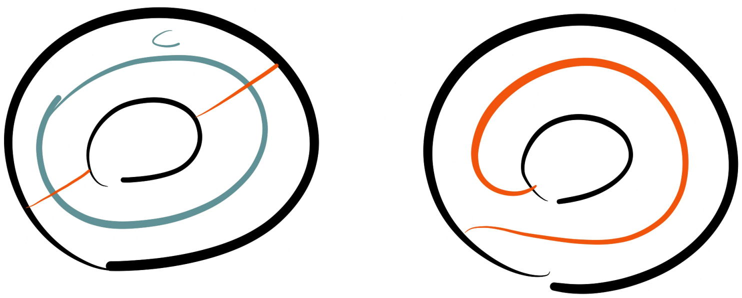

Figure 6: Another picture of a Dehn twist.

Question from the audience: is this the same as the push map?

Answer: no. The push map gives a trivial element of the mapping class group. However, there is a relationship. Let γ is a simple closed curve and C_1, C_2 curves which bound an annular neighborhood of γ.

Exercise 3.3. *Push*$(\gamma) = T_{C_1} \circ T_{C_2}^{-1}$.

Theorem 3.4. *(Lickorish, ...) Let S be a closed surface. Then $MCG^+(S)$ is generated by Dehn twists.*

Dehn twists cannot generate the mapping class group of a surface with marked points because they cannot permute the marked points. With marked points, the Dehn twists instead generate the *pure* mapping class group (the subgroup fixing the marked points pointwise).

The basic invariant of an element $M \in \mathrm{SL}_2(\mathbb{Z})$ up to conjugacy is its trace (this determines its characteristic polynomial). If $\mathrm{tr}(M) = 2$ then

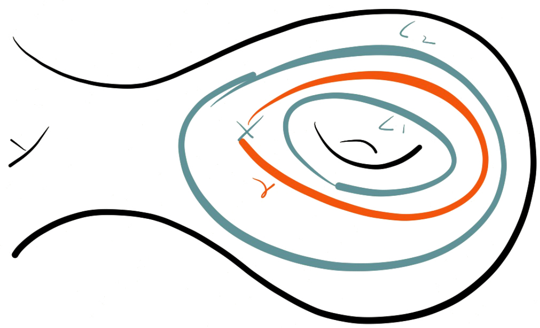

Figure 7: Dehn twists and the push map.

$$M = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \tag{11}$$

for some x, and similarly if $\mathrm{tr}(M) = -2$ then

$$M = \begin{bmatrix} -1 & x \\ 0 & -1 \end{bmatrix}. \tag{12}$$

These are the *parabolic elements*, and they look like Dehn twists when acting on the torus.

If $|\mathrm{tr}(M)| > 2$ then M has 2 distinct real eigenvalues, and iterating M we obtain exponential growth. (In the particular case above, $M = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and the eigenvalues are ϕ^2, φ^2 where ϕ, φ are the golden ratios.) These are the *hyperbolic elements*.

If $|\mathrm{tr}(M)| < 2$ then M is in fact torsion. These are the *elliptic* or *periodic* elements. The two basic possibilities are

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \tag{13}$$

and variants.

Exercise 3.5. *Which braids do these correspond to?*